AARMS Summer School Lecture II: The theory of Nonlocal Eignevalue Problem (NLEP)

[1] Chapter 3 of Wei-Winter's book on "Mathematical Aspects of Pattern Formation in Biological Systems", Applied Mathematical Sciences Series, Vol. 189, Springer 2014 , ISBN: 978-4471-5525-6.

In this lecture, we give a full rigorous treatment of the nonlocal eigenvalue problem (NLEP).

1 A basic theorem for $\tau = 0$

We start with the special case which arises when the time relaxation parameter τ of the inhibitor vanishes.

Next we consider our basic NLEP for $\tau = 0$.

Theorem 1 *Assume that*

$$r = 2,\ 1 < p \leq 1 + \frac{4}{N} \ \text{or}\ r = p+1,\ 1 < p < \left(\frac{N+2}{N-2}\right)_+, \tag{1.1}$$

where $\left(\frac{N+2}{N-2}\right)_+ = \frac{N+2}{N-2}$ *for* $N = 3, 4, \ldots$ *and* $\left(\frac{N+2}{N-2}\right)_+ = \infty$ *for* $N = 1, 2$. *Consider the following nonlocal eigenvalue problem*

$$\Delta\phi - \phi + pw^{p-1}\phi - \gamma_0(p-1)\frac{\int_{\mathcal{R}^N} w^{r-1}\phi}{\int_{\mathcal{R}^N} w^r} w^p = \alpha\phi, \quad \phi \in H^1(\mathcal{R}^N). \tag{1.2}$$

(1) If $\gamma_0 < 1$, then there is a positive eigenvalue to (1.2).
(2) If $\gamma_0 > 1$ and (1.1) holds then for any nonzero eigenvalue α of (1.2), we have

$$Re(\alpha) \leq -c_0 < 0.$$

(3) If $\gamma_0 \neq 1$ and $\alpha = 0$, then $\phi \in span\left\{\frac{\partial w}{\partial y_j} : j = 1, \ldots, N\right\}$.

Proof:
We first introduce a few notations and make some preparations. Set

$$L\phi := L_0\phi - \gamma_0(p-1)\frac{\int_{\mathcal{R}^N} w^{r-1}\phi}{\int_{\mathcal{R}^N} w^r} w^p, \quad \phi \in H^1(\mathcal{R}^N),$$

where $\gamma_0 = \frac{qr}{(s+1)(p-1)} > 1$ and $L_0 := \Delta - 1 + pw^{p-1}$. Note that L is not selfadjoint if $r \neq p+1$.

Let

$$X_0 := \ker(L_0) = \text{span}\left\{ \frac{\partial w}{\partial y_j} \ : \ j = 1, \ldots, N \right\}.$$

Then

$$L_0 w = (p-1)w^p, \quad L_0\left(\frac{1}{p-1}w + \frac{1}{2}x \cdot \nabla w\right) = w \tag{1.3}$$

and

$$\int_{\mathcal{R}^N} (L_0^{-1}w)w\,dy = \int_{\mathcal{R}^N} w\left(\frac{1}{p-1}w + \frac{1}{2}x \cdot \nabla w\right)\,dy = \left(\frac{1}{p-1} - \frac{N}{4}\right)\int_{\mathcal{R}^N} w^2\,dy, \tag{1.4}$$

$$\int_{\mathcal{R}^N} (L_0^{-1}w)w^p\,dy = \int_{\mathcal{R}^N} (L_0^{-1}w)\frac{1}{p-1}L_0 w\,dy = \frac{1}{p-1}\int_{\mathcal{R}^N} w^2\,dy. \tag{1.5}$$

We divide the rest of the proof into three cases.

Case 1: $r = 2, 1 < p < 1 + \frac{4}{N}$.

We introduce the following self-adjoint operator:

$$L_1\phi := L_0\phi - (p-1)\frac{\int_{\mathcal{R}^N} w\phi}{\int_{\mathcal{R}^N} w^2}w^p - (p-1)\frac{\int_{\mathcal{R}^N} w^p\phi}{\int_{\mathcal{R}^N} w^2}w + (p-1)\frac{\int_{\mathcal{R}^N} w^{p+1}\int_{\mathcal{R}^N} w\phi}{(\int_{\mathcal{R}^N} w^2)^2}w. \tag{1.6}$$

Then we have the following result:

Lemma 1.1 *(1) L_1 is selfadjoint. The kernel X_1 of L_1 satisfies*

$$X_1 = span\ \left\{ w, \frac{\partial w}{\partial y_j} \ : \ j = 1, \ldots, N \right\}.$$

(2) There exists a positive constant a_1 such that

$$L_1(\phi,\phi) := \int_{\mathcal{R}^N}\left(|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2\right) + \frac{2(p-1)\int_{\mathcal{R}^N} w\phi\int_{\mathcal{R}^N} w^p\phi}{\int_{\mathcal{R}^N} w^2}$$

$$-(p-1)\frac{\int_{\mathcal{R}^N} w^{p+1}}{(\int_{\mathcal{R}^N} w^2)^2}\left(\int_{\mathcal{R}^N} w\phi\right)^2 \geq a_1 d^2_{L^2(\mathcal{R}^N)}(\phi, X_1) \quad \text{for all } \phi \in H^1(\mathcal{R}^N),$$

where $d_{L^2(\mathcal{R}^N)}$ denotes the distance in the norm of $L^2(\mathcal{R}^n)$.

Proof: Firstly, we compute the kernel of L_1. It is easy to see that $w \in \ker(L_1)$ and $\frac{\partial w}{\partial y_j} \in \ker(L_1)$, $j = 1, \ldots, N$,. On the other hand, if $\phi \in \ker(L_1)$, then by (1.3) we get

$$L_0\phi = c_1(\phi)w + c_2(\phi)w^p = c_1(\phi)L_0\left(\frac{1}{p-1}w + \frac{1}{2}x \cdot \nabla w\right) + c_2(\phi)L_0\left(\frac{1}{p-1}w\right),$$

where

$$c_1(\phi) = (p-1)\frac{\int_{\mathcal{R}^N} w^p\phi}{\int_{\mathcal{R}^N} w^2} - (p-1)\frac{\int_{\mathcal{R}^N} w^{p+1}\int_{\mathcal{R}^N} w\phi}{(\int_{\mathcal{R}^N} w^2)^2}, \quad c_2(\phi) = (p-1)\frac{\int_{\mathcal{R}^N} w\phi}{\int_{\mathcal{R}^N} w^2}.$$

Hence

$$\phi - c_1(\phi)\left(\frac{1}{p-1}w + \frac{1}{2}x \cdot \nabla w\right) - c_2(\phi)\frac{1}{p-1}w \in \ker(L_0). \tag{1.7}$$

Thus

$$c_1(\phi) = (p-1)c_1(\phi)\frac{\int_{\mathcal{R}^N} w^p(\frac{1}{p-1}w + \frac{1}{2}x \cdot \nabla w)}{\int_{\mathcal{R}^N} w^2} - (p-1)c_1(\phi)\frac{\int_{\mathcal{R}^N} w^{p+1}\int_{\mathcal{R}^N} w(\frac{1}{p-1}w + \frac{1}{2}x \cdot \nabla w)}{(\int_{\mathcal{R}^N} w^2)^2}$$

$$= c_1(\phi) - c_1(\phi)\left(\frac{1}{p-1} - \frac{N}{4}\right)\frac{\int_{\mathcal{R}^N} w^{p+1}}{\int_{\mathcal{R}^N} w^2}$$

by (1.4) and (1.5). This implies that $c_1(\phi) = 0$. By (1.7) and Lemma **??**, this proves (1).

Secondly, we show (2). Suppose (2) is false. Then by (1) there exists (α, ϕ) such that (i) α is real and positive, (ii) $\phi \perp w$, $\phi \perp \frac{\partial w}{\partial y_j}$, $j = 1, \ldots, N$, and (iii) $L_1\phi = \alpha\phi$.

Now we show that this is impossible. From (ii) and (iii), we have

$$(L_0 - \alpha)\phi = (p-1)\frac{\int_{\mathcal{R}^N} w^p\phi}{\int_{\mathcal{R}^N} w^2}w. \tag{1.8}$$

We first claim that $\int_{\mathcal{R}^N} w^p\phi \neq 0$. In fact, if $\int_{\mathcal{R}^N} w^p\phi = 0$, then $\alpha > 0$ is an eigenvalue of L_0. By Lemma **??**, we get that $\alpha = \mu_1$ and ϕ has constant sign. This contradicts the fact that $\phi \perp w$. Therefore $\alpha \neq \mu_1, 0$, and hence $L_0 - \alpha$ is invertible in $X_0^\perp$. Thus (1.8) implies

$$\phi = (p-1)\frac{\int_{\mathcal{R}^N} w^p\phi}{\int_{\mathcal{R}^N} w^2}(L_0 - \alpha)^{-1}w.$$

Therefore we have

$$\int_{\mathcal{R}^N} w^p\phi = (p-1)\frac{\int_{\mathcal{R}^N} w^p\phi}{\int_{\mathcal{R}^N} w^2}\int_{\mathcal{R}^N}((L_0 - \alpha)^{-1}w)w^p$$

from which we get

$$\int_{\mathcal{R}^N} w^2 = (p-1)\int_{\mathcal{R}^N} ((L_0-\alpha)^{-1}w)w^p,$$

$$\int_{\mathcal{R}^N} w^2 = \int_{\mathcal{R}^N} ((L_0-\alpha)^{-1}w)((L_0-\alpha)w+\alpha w),$$

and finally

$$0 = \int_{\mathcal{R}^N} ((L_0-\alpha)^{-1}w)w. \tag{1.9}$$

Let $h_1(\alpha) = \int_{\mathcal{R}^N} ((L_0-\alpha)^{-1}w)w$. Then we compute

$$h_1(0) = \int_{\mathcal{R}^N} (L_0^{-1}w)w = \int_{\mathcal{R}^N} \left(\frac{1}{p-1}w + \frac{1}{2}x\cdot\nabla w\right) w = \left(\frac{1}{p-1} - \frac{N}{4}\right)\int_{\mathcal{R}^N} w^2 > 0$$

since $1 < p < 1 + \frac{4}{N}$. Moreover, we have

$$h_1'(\alpha) = \int_{\mathcal{R}^N} ((L_0-\alpha)^{-2}w)w = \int_{\mathcal{R}^N} ((L_0-\alpha)^{-1}w)^2 > 0.$$

This implies $h_1(\alpha) > 0$ for all $\alpha \in (0,\mu_1)$. Since $\lim_{\alpha\to+\infty} h_1(\alpha) = 0$, we get $h_1(\alpha) < 0$ for $\alpha \in (\mu_1,\infty)$. This contradicts (1.9) and (2) is proven.

Next we finish the proof of Theorem 1 in Case 1. Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp \ker(L_0)$. Then we have the following linear system for (ϕ_R,ϕ_I):

$$L_0\phi_R - (p-1)\gamma_0 \frac{\int_{\mathcal{R}^N} w\phi_R}{\int_{\mathcal{R}^N} w^2} w^p = \alpha_R\phi_R - \alpha_I\phi_I, \tag{1.10}$$

$$L_0\phi_I - (p-1)\gamma_0 \frac{\int_{\mathcal{R}^N} w\phi_I}{\int_{\mathcal{R}^N} w^2} w^p = \alpha_R\phi_I + \alpha_I\phi_R. \tag{1.11}$$

Multiplying (1.10) by ϕ_R, (1.11) by ϕ_I, integrating and adding the two resulting equations, we get

$$-\alpha_R \int_{\mathcal{R}^N} (\phi_R^2+\phi_I^2) = L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I)$$

$$+(p-1)(\gamma_0-2)\frac{\int_{\mathcal{R}^N} w\phi_R \int_{\mathcal{R}^N} w^p\phi_R + \int_{\mathcal{R}^N} w\phi_I \int_{\mathcal{R}^N} w^p\phi_I}{\int_{\mathcal{R}^N} w^2}$$

$$+(p-1)\frac{\int_{\mathcal{R}^N} w^{p+1}}{(\int_{\mathcal{R}^N} w^2)^2}\left[\left(\int_{\mathcal{R}^N} w\phi_R\right)^2 + \left(\int_{\mathcal{R}^N} w\phi_I\right)^2\right].$$

On the other hand, multiplying (1.10) by w and integrating, we have

$$(p-1)\int_{\mathcal{R}^N} w^p\phi_R - \gamma_0(p-1)\frac{\int_{\mathcal{R}^N} w\phi_R}{\int_{\mathcal{R}^N} w^2}\int_{\mathcal{R}^N} w^{p+1} = \alpha_R\int_{\mathcal{R}^N} w\phi_R - \alpha_I\int_{\mathcal{R}^N} w\phi_I. \tag{1.12}$$

Multiplying (1.11) by w and integrating, we obtain

$$(p-1)\int_{\mathcal{R}^N} w^p\phi_I - \gamma_0(p-1)\frac{\int_{\mathcal{R}^N} w\phi_I}{\int_{\mathcal{R}^N} w^2}\int_{\mathcal{R}^N} w^{p+1} = \alpha_R\int_{\mathcal{R}^N} w\phi_I + \alpha_I\int_{\mathcal{R}^N} w\phi_R. \tag{1.13}$$

Multiplying (1.12) by $\int_{\mathcal{R}^N} w\phi_R$ and (1.13) by $\int_{\mathcal{R}^N} w\phi_I$ and adding, we have

$$(p-1)\int_{\mathcal{R}^N} w\phi_R\int_{\mathcal{R}^N} w^p\phi_R + (p-1)\int_{\mathcal{R}^N} w\phi_I\int_{\mathcal{R}^N} w^p\phi_I$$

$$= \left(\alpha_R + \gamma_0(p-1)\frac{\int_{\mathcal{R}^N} w^{p+1}}{\int_{\mathcal{R}^N} w^2}\right)\left(\left(\int_{\mathcal{R}^N} w\phi_R\right)^2 + \left(\int_{\mathcal{R}^N} w\phi_I\right)^2\right).$$

Therefore we get

$$-\alpha_R\int_{\mathcal{R}^N}(\phi_R^2+\phi_I^2) = L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I)$$
$$+(p-1)(\gamma_0-2)\left(\frac{1}{p-1}\alpha_R + \gamma_0\frac{\int_{\mathcal{R}^N} w^{p+1}}{\int_{\mathcal{R}^N} w^2}\right)\frac{(\int_{\mathcal{R}^N} w\phi_R)^2 + (\int_{\mathcal{R}^N} w\phi_I)^2}{\int_{\mathcal{R}^N} w^2}$$
$$+(p-1)\frac{\int_{\mathcal{R}^N} w^{p+1}}{(\int_{\mathcal{R}^N} w^2)^2}\left[\left(\int_{\mathcal{R}^N} w\phi_R)^2 + (\int_{\mathcal{R}^N} w\phi_I\right)^2\right].$$

Decomposing

$$\phi_R = c_R w + \phi_R^\perp,\ \phi_R^\perp \perp X_1, \quad \phi_I = c_I w + \phi_I^\perp,\ \phi_I^\perp \perp X_1,$$

we get

$$\int_{\mathcal{R}^N} w\phi_R = c_R\int_{\mathcal{R}^N} w^2, \quad \int_{\mathcal{R}^N} w\phi_I = c_I\int_{\mathcal{R}^N} w^2,$$
$$d^2_{L^2(\mathcal{R}^N)}(\phi_R, X_1) = \|\phi_R^\perp\|^2_{L^2}, \quad d^2_{L^2(\mathcal{R}^N)}(\phi_I, X_1) = \|\phi_I^\perp\|^2_{L^2}.$$

Using a few some elementary computations, we derive

$$L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I)$$

$$+(\gamma_0-1)\alpha_R(c_R^2+c_I^2)\int_{\mathcal{R}^N} w^2 + (p-1)(\gamma_0-1)^2(c_R^2+c_I^2)\int_{\mathcal{R}^N} w^{p+1} + \alpha_R\left(\|\phi_R^\perp\|^2_{L^2} + \|\phi_I^\perp\|^2_{L^2}\right) = 0.$$

By Lemma 1.1 (2), we get

$$(\gamma_0-1)\alpha_R(c_R^2+c_I^2)\int_{\mathcal{R}^N} w^2$$
$$+(p-1)(\gamma_0-1)^2(c_R^2+c_I^2)\int_{\mathcal{R}^N} w^{p+1}+(\alpha_R+a_1)(\|\phi_R^\perp\|_{L^2}^2+\|\phi_I^\perp\|_{L^2}^2)\le 0.$$

Since $\gamma_0>1$, we finally have $\alpha_R<0$. This concludes the proof of Theorem 1 in Case 1.

Case 2: $r=2, p=1+\frac{4}{N}$.

We compute

$$\int_{\mathcal{R}^N}(L_0^{-1}w)w=\int_{\mathcal{R}^N} w\left(\frac{1}{p-1}w+\frac{1}{2}x\cdot\nabla w\right)=0. \tag{1.14}$$

We set

$$w_0=\frac{1}{p-1}w+\frac{1}{2}x\cdot\nabla w. \tag{1.15}$$

and follow the proof of Case 1. The following lemma is similar to Lemma 1.1 and its proof is omitted.

Lemma 1.2 *(1) The kernel of L_1 is given by*

$$X_1=span\left\{w,w_0,\frac{\partial w}{\partial y_j},\ j=1,\ldots,N\right\}.$$

(2) There exists a positive constant a_2 such that

$$L_1(\phi,\phi)=\int_{\mathcal{R}^N}(|\nabla\phi|^2+\phi^2-pw^{p-1}\phi^2)+\frac{2(p-1)\int_{\mathcal{R}^N}w\phi\int_{\mathcal{R}^N}w^p\phi}{\int_{\mathcal{R}^N}w^2}$$
$$-(p-1)\frac{\int_{\mathcal{R}^N}w^{p+1}}{(\int_{\mathcal{R}^N}w^2)^2}(\int_{\mathcal{R}^N}w\phi)^2\ge a_2 d^2_{L^2(\mathcal{R}^N)}(\phi,X_1)\quad \textit{for all } \phi\in H^1(\mathcal{R}^N).$$

Next we finish the proof of Theorem 1 in Case 2. Suppose that $\alpha_0\ne 0$ is an eigenvalue of L. Let $\alpha_0=\alpha_R+i\alpha_I$ and $\phi=\phi_R+i\phi_I$. Since $\alpha_0\ne 0$, we can choose $\phi\perp\ker(L_0)$. Then, similar to Case 1, we derive the two equations (1.10) and (1.11). We decompose

$$\phi_R=c_Rw+b_Rw_0+\phi_R^\perp,\ \phi_R^\perp\perp X_1,\quad \phi_I=c_Iw+b_Iw_0+\phi_I^\perp,\ \phi_I^\perp\perp X_1$$

and obtain

$$L_1(\phi_R,\phi_R)+L_1(\phi_I,\phi_I)$$

$$+(\gamma_0-1)\alpha_R(c_R^2+c_I^2)\int_{\mathcal{R}^N} w^2+(p-1)(\gamma_0-1)^2(c_R^2+c_I^2)\int_{\mathcal{R}^N} w^{p+1}$$

$$+\alpha_R\left[b_R^2\left(\int_{\mathcal{R}^N} w_0^2\right)^2+b_I^2\left(\int_{\mathcal{R}^N} w_0^2\right)^2+\|\phi_R^\perp\|_{L^2}^2+\|\phi_I^\perp\|_{L^2}^2\right]\le 0$$

By Lemma 1.2 (2), we have

$$(\gamma_0-1)\alpha_R(c_R^2+c_I^2)\int_{\mathcal{R}^N} w^2+(p-1)(\gamma_0-1)^2(c_R^2+c_I^2)\int_{\mathcal{R}^N} w^{p+1}$$

$$+\alpha_R(b_R^2(\int_{\mathcal{R}^N} w_0^2)^2+b_I^2(\int_{\mathcal{R}^N} w_0^2)^2)+(\alpha_R+a_2)(\|\phi_R^\perp\|_{L^2}^2+\|\phi_I^\perp\|_{L^2}^2)\le 0.$$

If $\alpha_R\ge 0$, then it follows that

$$c_R=c_I=0, \phi_R^\perp=0, \phi_I^\perp=0.$$

Hence we get $\phi_R=b_Rw_0$, $\phi_I=b_Iw_0$ and finally

$$b_RL_0w_0=(b_R-b_I)w_0,\quad b_IL_0w_0=(b_R+b_I)w_0.$$

This is impossible unless $b_R=b_I=0$. This gives the desired contradiction.

Case 3: $r=p+1, 1<p<(\frac{N+2}{N-2})_+$.

Let $r=p+1$. Then L can be written as

$$L\phi=L_0\phi-\frac{qr}{s+1}\frac{\int_{\mathcal{R}^N} w^p\phi}{\int_{\mathcal{R}^N} w^{p+1}}w^p.$$

To follow the proof of Case 1, we introduce the operator

$$L_3\phi:=L_0\phi-(p-1)\frac{\int_{\mathcal{R}^N} w^p\phi}{\int_{\mathcal{R}^N} w^{p+1}}w^p. \tag{1.16}$$

Then we have the following result:

Lemma 1.3 *(1) The operator L_3 is selfadjoint and its kernel is given by*

$$X_1=span\left\{w,\frac{\partial w}{\partial y_j}, j=1,\ldots,N\right\}.$$

(2) There exists a positive constant a_3 such that

$$L_3(\phi,\phi):=\int_{\mathcal{R}^N}(|\nabla\phi|^2+\phi^2-pw^{p-1}\phi^2)+\frac{(p-1)(\int_{\mathcal{R}^N} w^p\phi)^2}{\int_{\mathcal{R}^N} w^{p+1}}$$

$$\ge a_3d^2_{L^2(\mathcal{R}^N)}(\phi,X_3)\quad for\ all\ \phi\in H^1(\mathcal{R}^N).$$

Proof: Proving (1) is similar to showing Lemma 1.1. We omit the details. It remains to show (2). Suppose (2) is not true, then by (1) there exists (α, ϕ) such that (i) α is real and positive, (ii) $\phi \perp w, \phi \perp \frac{\partial w}{\partial y_j}$, $j = 1, \dots, N$, and (iii) $L_3\phi = \alpha\phi$.

Next we prove that this is impossible. From (ii) and (iii), we get

$$(L_0 - \alpha)\phi = \frac{(p-1)\int_{\mathcal{R}^N} w^p \phi}{\int_{\mathcal{R}^N} w^{p+1}} w^p. \tag{1.17}$$

Similar to the proof of Lemma 1.1, we have that $\int_{\mathcal{R}^N} w^p \phi \neq 0$ for $\alpha \neq \mu_1, 0$, and hence $L_0 - \alpha$ is invertible in $X_0^\perp$. Thus from (1.17) we get

$$\phi = \frac{(p-1)\int_{\mathcal{R}^N} w^p \phi}{\int_{\mathcal{R}^N} w^{p+1}} (L_0 - \alpha)^{-1} w^p.$$

Finally, we have

$$\int_{\mathcal{R}^N} w^p \phi = (p-1) \frac{\int_{\mathcal{R}^N} w^p \phi}{\int_{\mathcal{R}^N} w^{p+1}} \int_{\mathcal{R}^N} ((L_0 - \alpha)^{-1} w^p) w^p$$

and

$$\int_{\mathcal{R}^N} w^{p+1} = (p-1) \int_{\mathcal{R}^N} ((L_0 - \alpha)^{-1} w^p) w^p. \tag{1.18}$$

Letting

$$h_3(\alpha) = (p-1) \int_{\mathcal{R}^N} ((L_0 - \alpha)^{-1} w^p) w^p - \int_{\mathcal{R}^N} w^{p+1}.$$

we compute

$$h_3(0) = (p-1) \int_{\mathcal{R}^N} (L_0^{-1} w^p) w^p - \int_{\mathcal{R}^N} w^{p+1} = 0.$$

Moreover, we have

$$h_3'(\alpha) = (p-1) \int_{\mathcal{R}^N} ((L_0 - \alpha)^{-2} w^p) w^p = (p-1) \int_{\mathcal{R}^N} ((L_0 - \alpha)^{-1} w^p)^2 > 0.$$

Thus we get $h_3(\alpha) > 0$ for all $\alpha \in (0, \mu_1)$. On the other hand, we have $h_3(\alpha) < 0$ for $\alpha \in (\mu_1, \infty)$ which contradicts (1.18).

We finish the proof of Theorem 1 in Case 3. Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp \ker(L_0)$ and obtain the linear system

$$L_0 \phi_R - (p-1)\gamma_0 \frac{\int_{\mathcal{R}^N} w^p \phi_R}{\int_{\mathcal{R}^N} w^{p+1}} w^p = \alpha_R \phi_R - \alpha_I \phi_I, \tag{1.19}$$

$$L_0 \phi_I - (p-1)\gamma_0 \frac{\int_{\mathcal{R}^N} w^p \phi_I}{\int_{\mathcal{R}^N} w^{p+1}} w^p = \alpha_R \phi_I + \alpha_I \phi_R. \tag{1.20}$$

Multiplying (1.19) by ϕ_R, (1.20) by ϕ_I, integrating and adding the equations, we get

$$-\alpha_R \int_{\mathcal{R}^N} (\phi_R^2 + \phi_I^2) = L_3(\phi_R, \phi_R) + L_3(\phi_I, \phi_I)$$

$$+(p-1)(\gamma_0 - 1)\frac{(\int_{\mathcal{R}^N} w^p \phi_R)^2 + (\int_{\mathcal{R}^N} w^p \phi_I)^2}{\int_{\mathcal{R}^N} w^{p+1}}.$$

By Lemma 1.3 (2), we have

$$\alpha_R \int_{\mathcal{R}^N} (\phi_R^2 + \phi_I^2) + a_2 d_{L^2}^2(\phi, X_1) + (p-1)(\gamma_0 - 1)\frac{(\int_{\mathcal{R}^N} w^p \phi_R)^2 + (\int_{\mathcal{R}^N} w^p \phi_I)^2}{\int_{\mathcal{R}^N} w^{p+1}} \leq 0.$$

Thus we have derived $\alpha_R < 0$ since $\gamma_0 > 1$ and Theorem 1 part (2) in Case 3 is shown.

The proof of Part (1) is similar to the proof of Lemma 3.1 below. Part (3) is shown as in the argument immediately following the proof of Lemma 3.1 which implies that eigenvalues will not cross through zero.

2 The method of continuation

In our applications to the case when $\tau > 0$, we have to deal with the situation when the coefficient γ is a function of $\tau\alpha$. Now we will extend the results from the basic case $\tau = 0$ of the previous section to the case of $\tau > 0$ small enough by using a perturbation argument. The main point in ensuring that the perturbation argument works is showing that the eigenvalues remain uniformly bounded for τ small enough. Let $\gamma = \gamma(\tau\alpha)$ be a complex function of $\tau\alpha$. Let us suppose that

$$\gamma(0) \in \mathcal{R} \quad \text{and} \quad |\gamma(\tau\alpha)| \leq C \ \text{ for } \alpha_R \geq 0,\ \tau \geq 0, \tag{2.21}$$

where C is a generic constant independent of τ and α. Simple examples of $\gamma(\tau\alpha)$ satisfying (2.21) are

$$\gamma(\tau\alpha) = \frac{2}{\sqrt{1+\tau\alpha}+1} \quad \text{or} \quad \gamma(\tau\alpha) = \frac{\mu}{1+\tau\alpha}\ (\mu > 0),$$

where $\sqrt{1+\tau\alpha}$ is the principal branch of the square root. Now we have

Theorem 2 *Assume that (1.1) holds and consider the nonlocal eigenvalue problem*

$$\Delta\phi - \phi + pw^{p-1}\phi - \gamma(\tau\alpha)(p-1)\frac{\int_{\mathcal{R}} w^{r-1}\phi}{\int_{\mathcal{R}} w^r} w^p = \alpha\phi, \tag{2.22}$$

where $\gamma(\tau\alpha)$ satisfies (2.21). Then there is a small number $\tau_0 > 0$ such that for $\tau < \tau_0$,
(1) if $\gamma(0) < 1$, then there is a positive eigenvalue to (1.2);
(2) if $\gamma(0) > 1$ and (1.1) holds, then for any nonzero eigenvalue α of (2.22), we have

$$Re(\alpha) \leq -c_0 < 0.$$

Proof: Theorem 2 follows from Theorem 1 by a perturbation argument. To guarantee that the perturbation argument works, we have to show that if $\alpha_R \geq 0$ and $0 < \tau < 1$, then $|\alpha| \leq C$, where C is a generic constant (independent of τ). Multiplying (2.22) by the conjugate $\bar{\phi}$ of ϕ and integrating by parts, we get that

$$\int_{\mathcal{R}} (|\nabla\phi|^2 + |\phi|^2 - pw^{p-1}|\phi|^2) = -\alpha \int_{\mathcal{R}} |\phi|^2 - \gamma(\tau\alpha)(p-1)\frac{\int_{\mathcal{R}} w^{r-1}\phi}{\int_{\mathcal{R}} w^r} \int_{\mathcal{R}} w^p \bar{\phi}. \tag{2.23}$$

From the imaginary part of (2.23), we obtain that

$$|\alpha_I| \leq C_1 |\gamma(\tau\alpha)|,$$

where $\alpha = \alpha_R + \sqrt{-1}\alpha_I$ and C_1 is a positive constant (independent of τ). By assumption (2.21), we have $|\gamma(\tau\alpha)| \leq C$ and so $|\alpha_I| \leq C$. Taking the real part of (2.23) and noting that

$$\text{l.h.s. of (2.23)} \geq C \int_{\mathcal{R}} |\phi|^2 \quad \text{for some } C \in \mathcal{R},$$

we obtain that $\alpha_R \leq C_2$, where C_2 is a positive constant (independent of $\tau > 0$). Therefore, $|\alpha|$ is uniformly bounded and hence the perturbation argument implies the conclusion of the theorem.

3 Hopf bifurcation

Now we continue to consider the case $\tau > 0$. We relax the condition on the smallness of τ and allow τ to be any positive number. On the other hand, the function $\gamma(\tau\alpha)$ now has to be specified and the results will depend on the choice of function $\gamma(\tau\alpha)$ more explicitly than in the previous section.

In particular, we consider the following two nonlocal eigenvalue problems in the two-dimensional case:

$$L\phi := \Delta\phi - \phi + 2w\phi - \gamma\frac{\int_{\mathcal{R}^2} w\phi}{\int_{\mathcal{R}^2} w^2} w^2 = \lambda_0\phi, \ \phi \in H^1(\mathcal{R}^2), \tag{3.24}$$

where either (a) $\gamma = \dfrac{\mu}{1+\tau\lambda_0}$ with $\mu > 0$, $\tau \geq 0$, or (b) $\gamma = \dfrac{2(K+\eta_0(1+\tau\lambda_0))}{(K+\eta_0)(1+\tau\lambda_0)}$ with $\eta_0 > 0$, $\tau \geq 0$.

Case (a) will be studied in Theorem 3 and Case (b) will be considered in Theorem 4.

First we consider Case (a):

Theorem 3 *Let $\gamma = \frac{\mu}{1+\tau\lambda_0}$, where $\mu > 0$, $\tau \geq 0$ and let the operator L be defined by (3.24).*

(1) Suppose that $\mu > 1$. Then there exists a unique $\tau = \tau_1 > 0$ such that for $\tau < \tau_1$, (3.24) admits a positive eigenvalue, and for $\tau > \tau_1$, all nonzero eigenvalues of problem (3.24) satisfy $Re(\lambda) \leq -c < 0$. At $\tau = \tau_1$, L has a Hopf bifurcation.

(2) Suppose that $\mu < 1$. Then L admits a positive eigenvalue λ_0.

Proof of Theorem 3:

Theorem 3 will be proved by the following two lemmas.

Lemma 3.1 *If $\mu < 1$, then L has a positive eigenvalue λ_0.*

Proof: We conclude that ϕ is a radially symmetric function: $\phi \in H_r^2(\mathcal{R}^2) = \{u \in H^1(\mathcal{R}^2) | u = u(|y|)\}$. Let L_0 be defined as in (**??**). Then, by Lemma **??**, it follows that L_0 is invertible in $H_r^2(\mathcal{R}^2)$. We denote its inverse by L_0^{-1}. By Lemma **??**, the operator L_0 has a unique positive eigenvalue μ_1. Using $\int_{\mathcal{R}^2} w\Phi_0 > 0$, we conclude that $\lambda_0 \neq \mu_1$. Now $\lambda_0 > 0$ is an eigenvalue of (3.24) if and only if it a solution of the algebraic equation

$$\int_{\mathcal{R}^2} w^2 = \frac{\mu}{1+\tau\lambda_0} \int_{\mathcal{R}^2} [((L_0-\lambda_0)^{-1} w^2) w]. \tag{3.25}$$

The algebraic equation (3.25) can be simplified to

$$\rho(\lambda_0) := ((\mu-1)-\tau\lambda_0) \int_{\mathcal{R}^2} w^2 + \mu\lambda_0 \int_{\mathcal{R}^2} [((L_0-\lambda_0)^{-1} w) w] = 0, \tag{3.26}$$

where $\rho(0) = (\mu-1)\int_{\mathcal{R}^2} w^2 < 0$. Further, using $\lambda_0 \to \mu_1, \lambda_0 < \mu_1$, we get

$$\int_{\mathcal{R}^2} ((L_0-\lambda_0)^{-1} w) w \to +\infty$$

and thus $\rho(\lambda_0) \to +\infty$. By continuity, there is a $\lambda_0 \in (0, \mu_1)$ such that $\rho(\lambda_0) = 0$, and λ_0 is an eigenvalue of L.

Next we study the case $\mu > 1$. It suffices to restrict our attention to radially symmetric functions. By Theorem 1.4 of [**?**], for $\tau = 0$ (and by perturbation, for τ small), all eigenvalues are located on the left half of the complex plane. By [**?**], for τ large, there are unstable eigenvalues.

It is easy to see that the eigenvalues will not cross through zero: If $\lambda_0 = 0$, then we get

$$L_0\phi - \mu \frac{\int_{\mathcal{R}^2} w\phi}{\int_{\mathcal{R}^2} w^2} w^2 = 0$$

which implies that

$$L_0\left(\phi - \mu \frac{\int_{\mathcal{R}^2} w\phi}{\int_{\mathcal{R}^2} w^2} w\right) = 0$$

and by Lemma **??** we get

$$\phi - \mu \frac{\int_{\mathcal{R}^2} w\phi}{\int_{\mathcal{R}^2} w^2} w \in X_0.$$

This is impossible since ϕ is a radially symmetric function and $\phi \neq cw$ for all $c \in R$.

Hence there is a point τ_1 at which L has a Hopf bifurcation, i.e., L has a purely imaginary eigenvalue $\alpha = \sqrt{-1}\alpha_I$. To conclude the proof of Theorem 3 (1), it suffices to show that τ_1 is unique.

Lemma 3.2 *Let $\mu > 1$. Then there exists a unique $\tau_1 > 0$ such that L has a Hopf bifurcation.*

Proof:

Let $\lambda_0 = \sqrt{-1}\alpha_I$ be an eigenvalue of L. Without loss of generality, we may assume that $\alpha_I > 0$. (Note that then $-\sqrt{-1}\alpha_I$ is also an eigenvalue of L.) Letting $\phi_0 = (L_0 - \sqrt{-1}\alpha_I)^{-1} w^2$, (3.24) becomes

$$\frac{\int_{\mathcal{R}^2} w\phi_0}{\int_{\mathcal{R}^2} w^2} = \frac{1 + \tau\sqrt{-1}\alpha_I}{\mu} \tag{3.27}$$

Decomposing $\phi_0 = \phi_0^R + \sqrt{-1}\phi_0^I$, from (3.27) we derive the linear system

$$\frac{\int_{\mathcal{R}^2} w\phi_0^R}{\int_{\mathcal{R}^2} w^2} = \frac{1}{\mu}, \tag{3.28}$$

$$\frac{\int_{\mathcal{R}^2} w\phi_0^I}{\int_{\mathcal{R}^2} w^2} = \frac{\tau\alpha_I}{\mu}. \tag{3.29}$$

Note that only (3.29) depends τ, whereas (3.28) is independent of τ.

Next we compute $\int_{\mathcal{R}^2} w\phi_0^R$. Using the fact that (ϕ_0^R, ϕ_0^I) satisfies

$$L_0\phi_0^R = w^2 - \alpha_I\phi_0^I, \quad L_0\phi_0^I = \alpha_I\phi_0^R,$$

we get $\phi_0^R = \alpha_I^{-1} L_0 \phi_0^I$ and

$$\phi_0^I = \alpha_I (L_0^2 + \alpha_I^2)^{-1} w^2, \quad \phi_0^R = L_0 (L_0^2 + \alpha_I^2)^{-1} w^2. \tag{3.30}$$

Substituting (3.30) into (3.28) and (3.29), we have

$$\frac{\int_{\mathcal{R}^2}[wL_0(L_0^2+\alpha_I^2)^{-1}w^2]}{\int_{\mathcal{R}^2} w^2} = \frac{1}{\mu}, \tag{3.31}$$

$$\frac{\int_{\mathcal{R}^2}[w(L_0^2+\alpha_I^2)^{-1}w^2]}{\int_{\mathcal{R}^2} w^2} = \frac{\tau}{\mu}. \tag{3.32}$$

Setting

$$h(\alpha_I) = \frac{\int_{\mathcal{R}^2} wL_0(L_0^2+\alpha_I^2)^{-1}w^2}{\int_{\mathcal{R}^2} w^2} = h(\alpha_I) = \frac{\int_{\mathcal{R}^2} w^2(L_0^2+\alpha_I^2)^{-1}w^2}{\int_{\mathcal{R}^2} w^2},$$

we compute $h'(\alpha_I) = -2\alpha_I \frac{\int_{\mathcal{R}^2} w^2(L_0^2+\alpha_I^2)^{-2}w^2}{\int_{\mathcal{R}^2} w^2} < 0$. Since

$$h(0) = \frac{\int_{\mathcal{R}^2} w(L_0^{-1}w^2)}{\int_{\mathcal{R}^2} w^2} = 1,$$

$h(\alpha_I) \to 0$ as $\alpha_I \to \infty$ and $\mu > 1$, there exists a unique $\alpha_I > 0$ such that (3.31) holds. Substituting α_I into (3.32), we get a unique $\tau = \tau_1 > 0$ and the proof of Lemma 3.2 is finished.

Theorem 3 follows from Lemmas 3.1 and 3.2.

Finally we study Case (b) by considering the NLEP

$$\Delta\phi - \phi + 2w\phi - \frac{2(K+\eta_0(1+\tau\lambda_0))}{(K+\eta_0)(1+\tau\lambda_0)} \frac{\int_{\mathcal{R}^2} w\phi}{\int_{\mathcal{R}^2} w^2} w^2 = \lambda_0\phi, \ \phi \in H^1(\mathcal{R}^2), \tag{3.33}$$

where $0 < \eta_0 < +\infty$ and $0 \le \tau < +\infty$.

We have the following result:

Theorem 4 *(1) If $\eta_0 < K$, then for τ small enough problem (3.33) is stable and for τ large enough it is unstable.*

(2) If $\eta_0 > K$, then there exist $0 < \tau_2 \le \tau_3$ such that problem (3.33) is stable for $\tau < \tau_2$ or $\tau > \tau_3$.

Proof: Setting

$$f(\tau\lambda) = \frac{2(K+\eta_0(1+\tau\lambda))}{(K+\eta_0)(1+\tau\lambda)}, \tag{3.34}$$

we note that

$$\lim_{\tau\lambda\to+\infty} f(\tau\lambda) = \frac{2\eta_0}{K+\eta_0} =: f_\infty.$$

If $\eta_0 < K$, then by Theorem 3 (2), the problem (3.33) with $\mu = f_\infty$ possesses a positive eigenvalue α_1. Using a regular perturbation argument, this implies that for τ large enough problem (3.33) has an eigenvalue near $\alpha_1 > 0$. We conclude that for τ large enough problem (3.33) is unstable.

Next we show that problem (3.33) does not possess any nonzero eigenvalues with nonnegative real part, provided that either τ is small or $\eta_0 > K$ and τ is large. (It is immediately seen that $f(\tau\lambda) \to 2$ as $\tau\lambda \to 0$ and $f(\tau\lambda) \to \frac{2\eta_0}{\eta_0+K} > 1$ as $\tau\lambda \to +\infty$ if $\eta_0 > K$ and thus Theorem 3 should apply. However, we do not have control on $\tau\lambda$. Here we provide a rigorous proof.)

We apply the following inequality (see Lemma 1.1): For any (real-valued function) $\phi \in H_r^1(\mathcal{R}^2)$, we have

$$\int_{\mathcal{R}^2}(|\nabla\phi|^2 + \phi^2 - 2w\phi^2) + 2\frac{\int_{\mathcal{R}^2} w\phi \int_{\mathcal{R}^2} w^2\phi}{\int_{\mathcal{R}^2} w^2} - \frac{\int_{\mathcal{R}^2} w^3}{(\int_{\mathcal{R}^2} w^2)^2}(\int_{\mathcal{R}^2} w\phi)^2 \geq 0, \tag{3.35}$$

where equality holds if and only if ϕ is a multiple of w.

Let $\lambda_0 = \lambda_R + \sqrt{-1}\lambda_I$, $\phi = \phi_R + \sqrt{-1}\phi_I$ satisfy (3.33). Then we get

$$L_0\phi - f(\tau\lambda_0)\frac{\int_{\mathcal{R}^2} w\phi}{\int_{\mathcal{R}^2} w^2} w^2 = \lambda_0\phi. \tag{3.36}$$

Multiplying (3.36) by the complex conjugate $\bar{\phi}$ of the function ϕ and integrating over $\mathcal{R}^2$, we have

$$\int_{\mathcal{R}^2}(|\nabla\phi|^2 + |\phi|^2 - 2w|\phi|^2) = -\lambda_0 \int_{\mathcal{R}^2} |\phi|^2 - f(\tau\lambda_0)\frac{\int_{\mathcal{R}^2} w\phi}{\int_{\mathcal{R}^2} w^2} \int_{\mathcal{R}^2} w^2\bar{\phi}. \tag{3.37}$$

Multiplying (3.36) by w and integrating over $\mathcal{R}^2$, we obtain

$$\int_{\mathcal{R}^2} w^2\phi = (\lambda_0 + f(\tau\lambda_0)\frac{\int_{\mathcal{R}^2} w^3}{\int_{\mathcal{R}^2} w^2}) \int_{\mathcal{R}^2} w\phi. \tag{3.38}$$

Taking the complex conjugate of (3.38) gives

$$\int_{\mathcal{R}^2} w^2\bar{\phi} = (\bar{\lambda_0} + f(\tau\bar{\lambda_0})\frac{\int_{\mathcal{R}^2} w^3}{\int_{\mathcal{R}^2} w^2}) \int_{\mathcal{R}^2} w\bar{\phi}. \tag{3.39}$$

Substituting (3.39) into (3.37), it follows that

$$\int_{\mathcal{R}^2}(|\nabla\phi|^2 + |\phi|^2 - 2w|\phi|^2)$$

$$= -\lambda_0 \int_{R^2} |\phi|^2 - f(\tau\lambda_0)(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{\mathcal{R}^2} w^3}{\int_{\mathcal{R}^2} w^2})\frac{|\int_{R^2} w\phi|^2}{\int_{\mathcal{R}^2} w^2}. \tag{3.40}$$

Next we consider the real part of (3.40). Applying the inequality (3.35) and using (3.39), we get

$$-\lambda_R \geq \mathrm{Re}\left(f(\tau\lambda_0)(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{\mathcal{R}^2} w^3}{\int_{\mathcal{R}^2} w^2})\right) - 2\mathrm{Re}\left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{\mathcal{R}^2} w^3}{\int_{\mathcal{R}^2} w^2}\right) + \frac{\int_{\mathcal{R}^2} w^3}{\int_{\mathcal{R}^2} w^2},$$

where $\lambda_0 = \lambda_R + \sqrt{-1}\lambda_I$ with $\lambda_R,\ \lambda_I \in R$.

Assuming that $\lambda_R \geq 0$, we get

$$\frac{\int_{\mathcal{R}^2} w^3}{\int_{\mathcal{R}^2} w^2}|f(\tau\lambda_0) - 1|^2 + \mathrm{Re}(\bar{\lambda}_0(f(\tau\lambda_0) - 1)) \leq 0. \tag{3.41}$$

By the Pohozaev identity for (**??**) (multiplying (**??**) by $y \cdot \nabla w(y)$ and integrating by parts), we have

$$\int_{\mathcal{R}^2} w^3 = \frac{3}{2}\int_{\mathcal{R}^2} w^2. \tag{3.42}$$

Substituting (3.42) and the expression (3.34) for $f(\tau\lambda)$ into (3.41), we get

$$\frac{3}{2}|\eta_0 + K + (\eta_0 - K)\tau\lambda|^2 + \mathrm{Re}[(\eta_0 + K)(1 + \tau\bar{\lambda}_0)((\eta_0 + K)\bar{\lambda}_0 + (\eta_0 - K)\tau|\lambda_0|^2)] \leq 0.$$

This is equivalent to

$$\frac{3}{2}(1 + \mu_0\tau\lambda_R)^2 + \lambda_R + (\mu_0\tau + \tau + \mu_0\tau^2|\lambda_0|^2)\lambda_R$$

$$+(\frac{3}{2}\mu_0^2\tau^2 + \mu_0\tau - \tau)\lambda_I^2 \leq 0, \tag{3.43}$$

where $\mu_0 := \frac{\eta_0 - K}{\eta_0 + K}$.

If $\eta_0 > K$ (i.e., $\mu_0 > 0$) and τ is large, then

$$\frac{3}{2}\mu_0^2\tau^2 + \mu_0\tau - \tau \geq 0. \tag{3.44}$$

Thus (3.43) does not hold for $\lambda_R \geq 0$.

To consider the case when τ is small, we next derive an upper bound for λ_I.

By (3.37), we get

$$\lambda_I \int_{\mathcal{R}^2} |\phi|^2 = \mathrm{Im}\left(-f(\tau\lambda_0)\frac{\int_{\mathcal{R}^2} w\phi}{\int_{\mathcal{R}^2} w^2}\int_{\mathcal{R}^2} w^2\bar{\phi}\right)$$

Thus we have

$$|\lambda_I| \leq |f(\tau\lambda_0)| \sqrt{\frac{\int_{\mathcal{R}^2} w^4}{\int_{\mathcal{R}^2} w^2}} \leq C \tag{3.45}$$

where C is independent of λ_0.

Substituting (3.45) into (3.43), we conclude that (3.43) does not hold for $\lambda_R \geq 0$, if τ is small.

Remark 3.1 *The proof of Theorem 4 allows us to obtain explicit values for τ_2 and τ_3. (In fact, first from (3.44) we obtain a value for τ_3. Then from (3.43) and (3.45) we get a value for τ_2.)*