AARMS Summer School Lecture I: The study of the profile function

[1] Appendix of Wei-Winter's book on "Mathematical Aspects of Pattern Formation in Biological Systems", Applied Mathematical Sciences Series, Vol. 189, Springer 2014 , ISBN: 978-4471-5525-6.

In this lecture, we give a self-contained proof of the existence, nondegeneracy and spectrum of the spike profile function w. These are basic properties which we shall use throughout this course.

First, we have the following existence result.

Lemma 0.1 *Let $1 < p < \frac{n+2}{n-2}$. Consider the following minimisation problem*

$$c_p := \inf_{u \in H^1(\mathcal{R}^n)\setminus\{0\}} \frac{\int_{\mathcal{R}^n} (|\nabla u|^2 + u^2)\, dx}{\left(\int_{\mathcal{R}^n} u^{p+1}\, dx\right)^{2/(p+1)}}. \tag{0.1}$$

Then c_p can be attained by a radially symmetric function $w = w(r)$ which satisfies

$$\Delta w - w + w^p = 0, \quad w > 0, \quad w \in H^1(\mathcal{R}^n). \tag{0.2}$$

Further, we have that $w > 0$ and $w'(r) < 0$ for $r > 0$.

Proof: By Sobolev's embedding theorem, $0 < c_p < \infty$. Let $\{u_k\}$ be a minimising sequence. By scaling invariance, we may assume that $\int_{\mathcal{R}^n} u_k^{p+1}\, dx = 1$ and hence $\int_{\mathcal{R}^n} (|\nabla u_k|^2 + u_k^2)\, dx \to c_p$. By the rearrangement inequality we have

$$\int_{\mathcal{R}^n} (u_k^*)^{p+1}\, dx = \int_{\mathcal{R}^n} (u_k)^{p+1}\, dx = 1$$

and

$$\int_{\mathcal{R}^n} \left(|\nabla u_k^*|^2 + (u_k^*)^2\right)\, dx \leq \int_{\mathcal{R}^n} \left(|\nabla u_k|^2 + u_k^2\right)\, dx,$$

where u_k^* is the Schwarz rearrangement of u_k. Moreover, it holds that

$$\int_{\mathcal{R}^n} \left(|\nabla |u_k||^2 + |u_k|^2\right)\, dx = \int_{\mathcal{R}^n} \left(|\nabla u_k|^2 + u_k^2\right)\, dx\, .$$

Because of these two facts, we may assume that the minimising sequence $\{u_k\}$ is radially symmetric and strictly decreasing. By Strauss's lemma, for $u = u(r)$, $u'(r) < 0$, there holds

$$|u(r)| \leq c r^{-(n-1)/2} \|u\|_{H^1(\mathcal{R}^n)}. \tag{0.3}$$

From (0.3), we deduce that the space of radially symmetric function in $H^1(\mathcal{R}^n)$, denoted by $H_r^1(\mathcal{R}^n)$, is continuously embedded into $L^{p+1}(\mathcal{R}^n)$. Thus $\{u_k(r)\}$ contains a convergent subsequence $\{u_k(r)\}$ in $L^{p+1}(\mathcal{R}^n)$. Assume that its limit is w, then $\int_{\mathcal{R}^n} w^{p+1}\,dx = 1$. By Fatou's Lemma, we have

$$\int_{\mathcal{R}^n} (|\nabla w|^2 + w^2)\,dx \le \liminf_{k\to\infty} \int_{\mathcal{R}^n} (|u_k|^2 + u_k^2)\,dx = c_p.$$

On the other hand, $w \in H^1(\mathcal{R}^n)$ and hence

$$\int_{\mathcal{R}^n} (|\nabla w|^2 + w^2)\,dx \ge c_p \left(\int_{\mathcal{R}^n} w^{p+1}\,dx\right)^{1/(p+1)} = c^p.$$

This shows that w is in fact a minimiser of the problem (0.1).

Denote

$$E[u] = \frac{\int_{\mathcal{R}^n}(|\nabla u|^2 + u^2)\,dx}{(\int_{\mathcal{R}^n} u^{p+1}\,dx)^{2/(p+1)}}.$$

Then

$$E[w + t\phi] \ge 0 \text{ for all } t \in \mathcal{R} \text{ and for all } \phi \in C_0^\infty(\mathcal{R}^n).$$

The Euler-Lagrange equation of w implies that w is a weak solution of (0.2). Since $w^{p-1} \in L_{loc}^{n/2+\epsilon_0}(\mathcal{R}^n)$ for some $\epsilon_0 > 0$, the elliptic regularity theorem in Gilbarg-Trudinger, Theorem 8.17 yields that w is bounded. By L^p and Schauder estimates, we get $w \in C_{loc}^{2,\alpha}(\mathcal{R}^n)$ for some $\alpha > 0$. Thus w is a classical solution of (0.2). By the strong Maximum Principle, we finally have $w > 0$ in $\mathcal{R}^n$.

Next, we prove the nondegeneracy of w.

Lemma 0.2 *Let ϕ be a bounded solution of*

$$\Delta\phi - \phi + pw^{p-1}\phi = 0, \quad |\phi| \le 1. \tag{0.4}$$

Then

$$\phi = \sum_{j=1}^n c_j \frac{\partial w}{\partial x_j} \quad \textit{for some real constants } c_j,\ j = 1, \dots, n.$$

Proof: We divide the proof into four steps.

Step 1. ϕ decays exponentially to 0 at infinity, more precisely

$$|\phi(x)| \le Ce^{(1-\delta)|x|} \quad \text{for some } C, \delta > 0. \tag{0.5}$$

In fact, let ψ be the unique solution of

$$\Delta\psi - \psi + pw^{p-1}\psi = 0.$$

Since $w^{p-1}\psi$ decays exponentially, so does ψ, both in the sense of (0.5). Then the difference $u = \phi - \psi$ satisfies

$$\Delta u - u = 0, \quad u \text{ is bounded.} \tag{0.6}$$

Thus $u \equiv 0$ and so $\phi \equiv \psi$.

Step 2. Assume that $\phi = \phi(r)$. Then $\phi \equiv 0$.

This is the key step.

First, we show that $\lambda_2 \geq 0$, where λ_2 denotes the second eigenvalue of the operator

$$\Delta\phi - \phi + pw^{p-1}\phi + \lambda\phi = 0, \quad |\phi| \leq 1.$$

In fact, expanding the minimality condition

$$\left.\frac{d^2}{dt^2}\right|_{t=0} (E(w + t\phi) - E(w)) \geq 0,$$

we obtain that

$$\int_{\mathcal{R}^n} \left(|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2\right) dx + (p-1)\frac{\left(\int_{\mathcal{R}^n} w^p\phi\, dx\right)^2}{\int_{\mathcal{R}^n} w^{p+1}\, dx} \geq 0. \tag{0.7}$$

By the Courant-Fischer theorem, we deduce that $\lambda_2 \geq 0$. In particular, $\lambda_{2,r} \geq 0$, where $\lambda_{2,r}$ is the second eigenvalue in the radial class.

Since

$$\Delta w - w + pw^{p-1}w = (p-1)w^p,$$

we see that $\int_{\mathcal{R}^n} w^p\phi\, dx = 0$. Thus ϕ can change sign only once. Without loss of generality, we may assume that for some $r_0 > 0$ we have $\phi \leq 0$ for $r \leq r_0$ and $\phi \geq 0$ for $r \geq r_0$.

Now we consider the function

$$\eta(r) = rw' - \beta w.$$

Then η satisfies

$$\Delta\eta - \eta + pw^{p-1}\eta = 2w - (2 + \beta(p-1))w^p. \tag{0.8}$$

We may choose β such that $1 = \left(1 + \frac{\beta(p-1)}{2}\right) w^{p-1}(r_0)$ so that

$$2w - (2 + \beta(p-1))w^p \leq 0 \text{ for } r \leq r_0 \text{ and } 2w - (2 + \beta(p-1))w^p \geq 0 \text{ for } r \geq r_0.$$

Multiplying (0.8) by ϕ and (0.4) by η, we arrive at

$$\int_{\mathcal{R}^n} \phi(2w - (2 + \beta(p-1))w^p)\, dx = 0$$

which is impossible by the properties of ϕ unless $\phi \equiv 0$. This proves Step 2.

Step 3. Finally, we decompose ϕ into Fourier modes

$$\phi(x) = \sum_{j=1}^{\infty} \phi_j(r)\psi_j(\theta),$$

where $\psi_0 = 1$ and $\psi_j(\theta)$ are the normalised eigenfunctions on S^{n-1} with eigenvalues λ_j. Thus $\lambda_1 = 1$, $\lambda_2 = \ldots = \lambda_{n+1} = n-1$, $\lambda_{n+2} > n-1, \ldots$ Then ϕ_j satisfies

$$\Delta\phi_j - \phi_j + pw^{p-1}\phi_j = \frac{\lambda_j}{r^2}\phi_j. \tag{0.9}$$

We claim that

$$\phi_j \equiv 0 \quad \text{for } j \geq n+2. \tag{0.10}$$

Proof of (0.10): For $j \geq n+2$, we have $\phi_j(0) = 0$. Let r_0 be the first positive zero of ϕ_j (which may be $+\infty$) and we may assume, without loss of generality, that $\phi_j > 0$ for $r \in (0, r_0)$. Multiplying (0.9) by w' and integrating by parts, we obtain

$$\int_{B_{r_0}} \frac{\lambda_j - (n-1)}{r^2}\phi_j w'\, dx = \int_{\partial B_{r_0}} \left(w' \frac{\partial \phi_j}{\partial r} - \phi_j w'' \right) ds = \int_{\partial B_{r_0}} w' \frac{\partial \phi_j}{\partial r}\, ds.$$

Now note that the l.h.s is strictly negative while the r.h.s. is strictly positive unless $\phi_j \equiv 0$. Step 3 follows.

Finally, we prove

Step 4. For $j = 2, \ldots, n+1$, $\phi_j = c_j w'$ for some $c_j \neq 0$.

Note that ϕ_j satisfies

$$\Delta\phi_j - \phi_j + pw^{p-1}\phi_j = \frac{n-1}{r^2}\phi_j, \quad \phi_j(0) = 0. \tag{0.11}$$

Then by the uniqueness of the solutions of ordinary differential equations, we have

$$\phi_j = \frac{\phi_j'(0)}{w''(0)} w'$$

and Step 4 is completed.

Combining Steps 1–4, we have proved the lemma.

A corollary of Lemma 0.2 is the following result on the spectrum of w.

Lemma 0.3 *Let w be the least energy solution given in Lemma 0.2. Then we have*

(1) There holds

$$\int_{\mathcal{R}^n} \left(|\nabla \phi|^2 + \phi^2 - p w^{p-1} \phi^2 \right) dx + (p-1) \frac{\left(\int_{\mathcal{R}^n} w^p \phi \, dx \right)^2}{\int_{\mathcal{R}^n} w^{p+1} \, dx} \geq 0$$

for all $\phi \in H^1(\mathcal{R}^n)$.

(2) The eigenvalue problem

$$\Delta \phi - \phi + p w^{p-1} \phi + \lambda \phi = 0, \quad \phi \in H^1(\mathcal{R}^n)$$

satisfies

$$\lambda_1 < 0, \, \lambda_2 = \ldots = \lambda_{n+1} = 0, \, \lambda_{n+2} > 0,$$

where the eigenfunction corresponding to λ_1 is simple, radially symmetric and it can be made positive.

Proof: We just need to prove the statement on λ_1 and its eigenfunction. In fact, we consider

$$\lambda_1 = \inf_{\phi \in H^1(\mathcal{R}^n) \setminus \{0\}} \frac{\int_{\mathcal{R}^n} (|\nabla \phi|^2 + \phi^2 - p w^{p-1} \phi^2) \, dx}{\int_{\mathcal{R}^n} \phi^2 \, dx}.$$

Then we have

$$\lambda_1 < \frac{\int_{\mathcal{R}^n} (|\nabla w|^2 + w^2 - p w^{p+1}) \, dx}{\int_{\mathcal{R}^n} w^2 \, dx} < 0$$

and the corresponding eigenfunction is simple, radially symmetric and it can be made positive. The rest follows from Lemma 0.2 and its proof, using that by (0.7) we have $\lambda_2 \geq 0$.