BUBBLING SOLUTIONS FOR THE LIOUVILLE EQUATION WITH A SINGULAR SOURCE: NON-SIMPLE BLOW-UP

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ABSTRACT. We are concerned with the existence of blowing-up solutions to the following boundary value problem

$$-\Delta u = \lambda e^u - 4\pi N_\lambda \boldsymbol{\delta}_0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a smooth and bounded domain in $\mathbb{R}^2$ such that $0 \in \Omega$, N_λ is a positive number close to an integer N ($N \geq 1$) from the right side, $\boldsymbol{\delta}_0$ defines the Dirac measure with pole at 0, and $\lambda > 0$ is a small parameter. We assume that Ω is $(N+1)$-symmetric and the regular part of the Green's function satisfies a nondegeneracy condition (both assumptions are verified if Ω is the unit ball) and we find a solution which exhibits a non-simple blow-up profile as $\lambda \to 0^+$.

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1. Introduction

Given Ω a smooth and bounded domain in $\mathbb{R}^2$ containing the origin, consider the following Liouville equation with Dirac mass measure

$$\begin{cases} -\Delta u = \lambda e^u - 4\pi N \boldsymbol{\delta}_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here λ is a positive small parameter, $\boldsymbol{\delta}_0$ denotes Dirac mass supported at 0 and N is a positive integer.

Problem (1.1) is motivated by its links with the modeling of physical phenomena. In particular, (1.1) arises in the study of vortices in a planar model of Euler flows (see [12], [31]). In vortex theory the interest in constructing *blowing-up* solutions is related to relevant physical properties, in particular the presence of vortices with a strongly localised electromagnetic field.

The asymptotic behaviour of family of blowing up solutions can be referred to the papers [2], [6], [19], [20], [22], [24] for the regular problem, i.e. when $N = 0$. An extension to the singular case $N > 0$ is contained in [3]-[4].

The analysis of the blowing-up behaviour at points away from 0 actually is very similar to the asymptotic analysis arising in the regular case, which has been pursued with success and, at the present time, an accurate description of the concentration phenomenon is available. Precisely, the analysis in the above works yields that if u_λ is an unbounded family of solutions of (1.1) for which $\lambda \int_\Omega e^{u_\lambda}$ is uniformly bounded and u_λ is uniformly bounded in a neighborhood of 0, then, up to a subsequence, there is an integer $m \geq 1$ such that

$$\lambda \int_\Omega e^{u_\lambda} dx \to 8\pi m \text{ as } \lambda \to 0^+. \tag{1.2}$$

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Moreover there are points $\xi_1^\lambda,\dots,\xi_m^\lambda\in\Omega$ which remain uniformly distant from the boundary $\partial\Omega$, from 0 and from one another such that

$$\lambda e^{u_\lambda}-8\pi\sum_{j=1}^m\boldsymbol{\delta}_{\xi_j^\lambda}\to 0 \tag{1.3}$$

in the measure sense. Also the location of the blowing-up points is well understood when concentration occurs away from 0. Indeed, in [22] and [24] it is established that the m-tuple $(\xi_1^\lambda,\dots,\xi_m^\lambda)$ converges, up to a subsequence, to a critical point of the functional

$$\frac12\sum_{j=1}^m H(\xi_j,\xi_j)+\frac12\sum_{\substack{j,h=1\\ j\neq h}}^m G(\xi_j,\xi_h)-\frac N2\sum_{j=1}^m G(\xi_j,0). \tag{1.4}$$

Here $G(x,y)$ is the Green's function of $-\Delta$ over Ω under Dirichlet boundary conditions and $H(x,y)$ denotes its regular part:

$$H(x,y)=G(x,y)-\frac{1}{2\pi}\log\frac{1}{|x-y|}.$$

The above description of blowing-up behaviour continues to work if we are in the presence of multiples singularities $\sum_i N_i\boldsymbol{\delta}_{p_i}$ in (1.1), provided that we substitute the term $\frac N2\sum_j G(\xi_j,0)$ by $\sum_i\frac{N_i}{2}\sum_j G(\xi_j,p_i)$ in (1.4).

The reciprocal issue, namely the existence of positive solutions with the property (1.3), has been addressed for the regular case $N=0$ first in [30] in the case of a single point of concentration (i.e. $m=1$), later generalised to the case of multiple concentration associated to any nondegenerate critical point of the functional (1.4) ([2], [8]) or, more generally, to any *topologically nontrivial* critical point ([13]-[15]). In particular, still for $N=0$, a family of solutions u_λ concentrating at m-tuple of points as $\lambda\to 0^+$ has been found in some special cases: for any $m\ge 1$, provided that Ω is not simply connected ([13]), and for $m\in\{1,\dots,h\}$ if Ω is a h-dumbell with thin handles ([15]). In the singular case $N>0$ solutions which concentrate in the measure sense at m distinct points away from 0 have been built in [13] provided that $m<N+1$. This result has been extended in [10] to the case of multiple singular sources. Moreover a degree formula has been obtained in [8] assuming that N is not a positive integer.

We point out that in all the above results concentration occurs at points different from the location of the source. The problem of finding solutions with additional concentration around the source is of different nature. In case they exist, the blowing-up at the singularity provides an additional contribution of $8\pi(N+1)$ in the limit (1.2), see [3], [4], [14], [25], [26]. More precisely the asymptotic analysis in the general case can be formulated as follows: if u_λ is an unbounded family of solutions of (1.1) for which $\lambda\int_\Omega e^{u_\lambda}$ is uniformly bounded and u_λ is unbounded in any neighborhood of 0, then, up to a subsequence, there is an integer $m\ge 0$ such that

$$\lambda\int_\Omega e^{u_\lambda}dx\to 8\pi m+8\pi(N+1)\text{ as }\lambda\to 0^+.$$

Moreover there are m distinct points $\xi_1,\dots,\xi_m\in\Omega\setminus\{0\}$ such that, up to subsequence,

$$\lambda e^{u_\lambda}\to 8\pi\sum_{j=1}^m\boldsymbol{\delta}_{\xi_j}+8\pi(N+1)\boldsymbol{\delta}_0 \tag{1.5}$$

in the measure sense. We mention that also in this case the analysis can be generalized to any number of sources. Moreover, under some extra assumptions it is possible to define a functional which replaces (1.4) in locating the points ξ_j where the concentration occurs, anyway to avoid technicalities we will not go into any further detail (see [14]).

The question on the existence of solutions to (1.1) concentrating at 0 is far from being completely settled. Indeed only partial results are known: in [14] the construction of solutions concentrating at 0 is carried out provided that $N \in (0, +\infty) \setminus \mathbb{N}$. To our knowledge, the only paper dealing with the case $N \in \mathbb{N}$ is [12], where, for any fixed positive integer N, it is proved the existence of a solution to (1.1), where $\boldsymbol{\delta}_0$ is replaced by $\boldsymbol{\delta}_{p_\lambda}$ for a suitable $p_\lambda \in \Omega$, with $N+1$ blowing up points at the vertices of a sufficiently tiny regular polygon centered in p_λ; moreover p_λ lies uniformly away from the boundary $\partial\Omega$ but its location is determined by the geometry of the domain in a $\lambda-$dependent way and does not seem possible to be prescribed arbitrarily as in [14]. Finally in [9] bubbling solutions blowing-up at 0 have been found under the effect of an anisotropic potential.

The case $N \in \mathbb{N}$ is more difficult to treat, and at the same time the most relevant to physical applications. Indeed, in vortex theory the number N represents vortex multiplicity, so that in that context the most interesting case is precisely when it is a positive integer. The difference between the case $N \in \mathbb{N}$ and $N \notin \mathbb{N}$ is also analytically essential. Indeed, as usual in problems involving small parameters and concentration phenomena like (1.1), after suitable rescaling of the blowing-up around a concentration point one sees a limiting equation. More specifically, as we will see in Section 2, we can associate to (1.1) the limiting problem of Liouville type (2.4) which will play a crucial role in the construction of solutions blowing up at 0 as $\lambda \to 0^+$; if $N \in \mathbb{N}$, (2.4) admits a larger class of finite mass solutions with respect to the case $N \notin \mathbb{N}$ since the family of all solutions extends to one carrying an extra parameter $b \in \mathbb{R}^2$ (see [23]). This suggests that if $N \in \mathbb{N}$ and the blow-up point happens to be the singular source, then solutions may exhibit non-simple blow-up phenomenon. In this case, it is shown in [5, 18] that there are $N+1$ local maximum points of the bubbling solutions which, after suitable rescaling, lie on a regular polygon. In [28] Harnack inequalities and second order vanishing conditions for non-simple blow-ups are obtained.

In this paper we investigate the existence of non-simple blow-up solutions when the weight of the source approaches an integer N in a $\lambda-$dependent way. More precisely we consider the following perturbed problem

$$\begin{cases} -\Delta u = \lambda e^u - 4\pi N_\lambda \delta_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

where N_λ is close to an integer $N \geq 1$.

Let us pass to enumerate the hypotheses on the domain Ω that will be steadily used throughout the paper: first of all

(A1) Ω is $(N+1)$-symmetric, i.e.

$$x \in \Omega \Longleftrightarrow x e^{\mathrm{i}\frac{2\pi}{N+1}} \in \Omega.$$

In order to state the second crucial assumption on Ω, let us fix some notation: for any b in a small neighborhood of 0 and let us denote by $\beta_0, \ldots, \beta_N$ the $(N+1)$-roots of b, i.e., $\beta_i^{N+1} = b$ and $\beta_i \neq \beta_j$ for $i \neq j$. Then, we will assume that

(A2) the function

$$b \longmapsto \sum_{i,j=0}^{N} H(\beta_i, \beta_j) - N \sum_{i=0}^{N} H(\beta_i, 0) \tag{1.7}$$

(which is well defined for b in a neighborhood of 0) has a nondegenerate maximum at 0.

We point out that the function (1.7) actually can be rewritten in terms of the regular part of the Green's function associated to the domain $\{x^{N+1} \,|\, x \in \Omega\}$, which is smooth thanks to the $(N+1)$-symmetry of Ω; we refer to Appendix B for more details.

Remark 1.1. *If Ω is the unit ball $B(0,1)$, as shown in Remark B.1, the function* (1.7) *coincides with a multiple of the Robin function $H(b,b)$:*

$$\sum_{i,j=0}^{N} H(\beta_i,\beta_j) - N\sum_{i=0}^{N} H(\beta_i,0) = (N+1)H(b,b) = \frac{N+1}{2\pi}\log(1-|b|^2)$$

which admits a nondegenerate maximum at 0. *So assumptions* (A1) − (A2) *are satisfied in the case of the unit ball $B(0,1)$.*

In order to state our results, we have to impose N_λ converging to N with a sufficient rate:

(A3) there exists $\eta > 0$ such that

$$N_\lambda - N = O(\lambda^\eta).$$

As we will see in next theorem the smallness of parameter λ in (1.6) will yield the existence of solutions blowing up at 0 and this holds for every family N_λ satisfying (A3), even in the limit case $N_\lambda = N$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth and bounded domain such that $0 \in \Omega$ and assume that hypotheses* (A1) − (A2) − (A3) *hold. Then, for λ sufficiently small the problem* (1.6) *has a family of solutions u_λ blowing up at the origin as $\lambda \to 0^+$. More precisely u_λ satisfies*

$$\lambda e^{u_\lambda} \to 8\pi(N+1)\boldsymbol{\delta}_0$$

in the measure sense.

Theorem 1.1 does not give any information to distinguish if the blowing up is radially symmetric or not in the first approximation. For instance, in Section 7.2 we show that if $N_\lambda = N$ and if Ω is the unit ball then the solution found in Theorem 1.1 actually blows up at the origin with a radially symmetric limiting profile.

The main purpose of this paper is to construct an example of non-simple blow-up exhibiting a non-symmetric scenarios. More precisely, next theorem provides a solution which develops a branch of $N+1$ bubbles centered at vertices β_i, namely $N+1$ functions exhibiting a peaked behavior of logarithmic type; and since the rate of convergence $\beta_i \to 0$ is lower than the speed of the concentration parameter $\delta \to 0$ (see estimate (1.8)), then the solution splits into a cluster of peaks concentrating at 0 which are arranged as *satellites* at the vertices of a regular $(N+1)$-polygon. The exact formulation of the result is the following.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth and bounded domain such that $0 \in \Omega$ and assume that hypotheses* (A1) − (A2) − (A3) *hold. Suppose, in addition, that*

$$N_\lambda - N \geq c\lambda \log^2 \lambda$$

for some $c > 0$. Then, for λ sufficiently small the problem (1.6) *has a family of solutions u_λ blowing up at the origin as $\lambda \to 0^+$:*

$$\lambda e^{u_\lambda} \to 8\pi(N+1)\boldsymbol{\delta}_0 \quad \text{in the measure sense.}$$

More precisely there exist $\delta = \delta(\lambda) > 0$ and $b = b(\lambda) \in \Omega$ in a neighborhood of 0 such that u_λ satisfies

$$u_\lambda + 4\pi N_\lambda G(x,0) = -2\log\left(\delta^{2(N+1)} + |x^{N+1} - b|^2\right) + 8\pi\sum_{i=0}^{N} H(x,\beta_i) + o(1)$$

in H^1-sense, where[1]

$$\delta^{2(N+1)} \sim \lambda, \quad \sqrt{\lambda|\log\lambda|} \leq |b| \leq \lambda^{\frac{\eta}{2}}\sqrt{|\log\lambda|}. \tag{1.8}$$

[1]We use the notation $\sim$ to denote quantities which in the limit $\lambda \to 0^+$ are of the same order.

The analysis reveals that the configuration of the limiting clustered peaks is determined by two crucial aspects: the rate of convergence $N_\lambda \to N$ and the shape of Ω, described in terms of the function (1.7). This kind of non-simple blow-up is new even in the case of the ball, for which both assumptions (A1) $-$ (A2) are satisfied as observed in Remark 1.1. On the other hand, the existence of such a phenomenon for $N_\lambda \to N^-$ or when $N_\lambda = N$ is still open (even in the case of the ball, see Section 7.2). However Theorem 1.2 does suggest that non-simple blow-up should not occur for $N_\lambda \to N^-$ or $N_\lambda = N$. In particular in Section 7.2 it is proved that if $N_\lambda = N$ and if Ω is the unit ball then the solution we find in Theorem 1.1 exhibits a simple blow-up. In view of this result, Theorem 1.2, the first order estimate of [5, 18] and the second order estimates of [28], it seems reasonable to raise the following conjecture:

Conjecture: *When $\Omega = B_1(0)$ and $N_\lambda = N$, there are no non-simple blow-up phenomena for problem* (1.6)*.*

Previous known examples of non-simple blow up solutions are available for other models: we recall, for instance, the regular Liouville equation on a disk in [7] (without boundary condition), the Liouville equation with anisotropic coefficients in [29], the Toda system in [1], or the sinh-Poisson equation allowing also negative bubbling ([16]).

The proofs use singular perturbation methods which combine the variational approach with a Lyapunov-Schmidt type procedure. Roughly speaking, the first step consists in the construction of an approximate solution, which should turn out to be precise enough. In view of the expected asymptotic behavior, the shape of such approximate solution will resemble a *bubble* of the form (2.5) with a suitable choice of the parameter $\delta = \delta(\lambda, b)$. Then we look for a solution to (1.6) in a small neighborhood of the first approximation. As quite standard in singular perturbation theory, a crucial ingredient is nondegeneracy of the explicit family of solutions of the limiting Liouville problem (2.4), in the sense that all bounded elements in the kernel of the linearization correspond to variations along the parameters of the family, as established in [12]. This allows us to study the invertibility of the linearized operator associated to the problem (1.6) under suitable orthogonality conditions. Next we introduce an intermediate problem and a fixed point argument will provide a solution for an auxiliary equation, which turns out to be solvable for any choice of b. Finally we test the auxiliary equation on the elements of the kernel of the linearized operator and we find out that, in order to find an *exact* solution of (1.6), the location the asymptotic peaks, which is detected by the parameter b, should be a critical point for the *reduced* finite dimensional functional. So, after this reduction process, solving (1.6) is equivalent to solving a finite-dimensional optimization problem. We point out that the two scales of concentration of b and $N_\lambda - N$ appear coupled at almost every point of the proof, so if $N_\lambda \leq N$ the method breaks down since we are unable to catch a critical point for the reduced energy.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results, notation, and the definition of the approximating solution. Moreover, a more general version of Theorems 1.1-1.2 is stated there (see Theorems 2.1-2.2). In Section 3 we prove the solvability of the linearized problem. The error up to which the approximating solution solves problem (1.6) is estimated in Section 4. Section 5 considers the solvability of an auxiliary problem by a contraction argument. In Section 6 we reduce the problem to finite dimension by the Liapunov-Schmidt reduction method and we compute the reduced energy. Finally in Section 7 we complete the proof of Theorems 1.1-1.2. In Appendix A, B, C we collect some results, most of them well-known, which are usually referred to throughout the paper.

NOTATION: In our estimates throughout the paper, we will frequently denote by $C > 0$, $c > 0$ fixed constants, that may change from line to line, but are always independent of the variables under consideration. We also use the notations $O(1)$, $o(1)$, $O(\lambda)$, $o(\lambda)$ to describe the asymptotic behaviors of quantities in a standard way.

2. Preliminaries and statement of the main results

We are going to provide an equivalent formulation of problem (1.6) and Theorems 1.1-1.2. Indeed, let us set

$$\alpha := N + 1 \geq 2$$

and let us observe that, setting v the regular part of u, namely

$$v = u + 4\pi(\alpha_\lambda - 1)G(x, 0), \quad \alpha_\lambda = N_\lambda + 1, \tag{2.1}$$

problem (1.6) is then equivalent to solving the following (regular) boundary value problem

$$\begin{cases} -\Delta v = \lambda V(x)|x|^{2(\alpha_\lambda - 1)} e^v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}, \tag{2.2}$$

where $V(x)$ is the function

$$V(x) = e^{-4\pi(\alpha_\lambda - 1)H(x,0)}. \tag{2.3}$$

Here G and H are the Green's function and its regular part as defined in the introduction. Theorems 1.1-1.2 will be a consequence of more general results concerning Liouville-type problem (2.2). In order to provide such results for (2.2), we now give a construction of a suitable approximate solution for (2.2). In what follows, we identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 + \mathrm{i}x_2 \in \mathbb{C}$ and we denote by $x\,y$ the multiplication of the complex numbers x, y and, analogously, by x^α the power of the complex number x.

For any $\alpha \in \mathbb{N}$, we can associate to (2.2) a limiting problem of Liouville type which will play a crucial role in the construction of solutions blowing up at 0 as $\lambda \to 0^+$:

$$-\Delta w = |x|^{2(\alpha-1)} e^w \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} |x|^{2(\alpha-1)} e^{w(x)} dx < +\infty. \tag{2.4}$$

All solutions of this problem are given, in complex notation, by the three-parameter family of functions

$$w^\alpha_{\delta,b}(x) := \log \frac{8\alpha^2 \delta^{2\alpha}}{(\delta^{2\alpha} + |x^\alpha - b|^2)^2} \quad \delta > 0,\ b \in \mathbb{C}. \tag{2.5}$$

The following quantization property holds:

$$\int_{\mathbb{R}^2} |x|^{2(\alpha-1)} e^{w^\alpha_{\delta,b}(x)} dx = 8\pi\alpha. \tag{2.6}$$

In the following we agree that

$$W_\lambda = w^\alpha_{\delta,b}(x),$$

where the value $\delta = \delta(\lambda, b)$ is defined as

$$\delta^{2\alpha} := \frac{\lambda}{8\alpha^2} e^{8\pi \mathcal{H}_\alpha(b,b) - 4\pi \frac{\alpha_\lambda - 1}{\alpha} \mathcal{H}_\alpha(b,0)} \tag{2.7}$$

and the function $\mathcal{H}_\alpha$ has been introduced in Appendix B.

To obtain a better first approximation, we need to modify the function W_λ in order to satisfy the zero boundary condition. Precisely, we consider the projection PW_λ onto the space $H^1_0(\Omega)$, where the projection $P : H^1(\mathbb{R}^N) \to H^1_0(\Omega)$ is defined as the unique solution of the problem

$$\Delta P v = \Delta v \quad \text{in } \Omega, \qquad P v = 0 \quad \text{on } \partial\Omega.$$

We recall by Appendix B that $\mathcal{H}_\alpha(x^\alpha, b)$ is harmonic in Ω and satisfies $\mathcal{H}_\alpha(x^\alpha, b) = \frac{1}{2\pi}\log|x^\alpha - b|$ on $\partial\Omega$. A straightforward computation gives that for any $x \in \partial\Omega$

$$\left|PW_\lambda - W_\lambda + \log\left(8\alpha^2\delta^{2\alpha}\right) - 8\pi\mathcal{H}_\alpha(x^\alpha, b)\right| = \left|W_\lambda - \log\left(8\alpha^2\delta^{2\alpha}\right) + 4\log|x^\alpha - b|\right| \le C\delta^{2\alpha}.$$

Since the expressions considered inside the absolute values are harmonic in Ω, then the maximum principle applies and implies the following asymptotic expansion

$$\begin{aligned} PW_\lambda =& W_\lambda - \log\left(8\alpha^2\delta^{2\alpha}\right) + 8\pi\mathcal{H}_\alpha(x^\alpha, b) + O(\delta^{2\alpha}) \\ =& -2\log\left(\delta^{2\alpha} + |x^\alpha - b|^2\right) + 8\pi\mathcal{H}_\alpha(x^\alpha, b) + O(\delta^{2\alpha}) \end{aligned} \tag{2.8}$$

uniformly for $x \in \overline{\Omega}$ and b in a small neighborhood of 0.

We point out that, in order to simplify the notation, in our estimates throughout the paper we will describe the asymptotic behaviors of quantities under considerations in terms of $\delta = \delta(\lambda, b)$ instead of the parameter λ of the equation. Clearly according to (2.7) δ has the same rate as $\lambda^{\frac{1}{2\alpha}}$, so at each step we can easily pass to the analogous asymptotic in terms of λ: for instance, in (2.8) the error term "$O(\delta^{2\alpha})$" can be equivalently replaced by "$O(\lambda)$".

We shall look for a solution to (2.2) in a small neighborhood of the first approximation, namely a solution of the form

$$v_\lambda = PW_\lambda + \phi_\lambda,$$

where the rest term ϕ_λ is small in $H^1(\Omega)$-norm.

In order to state the two main theorems for problem (2.2), let us reformulate the two assumptions (A1) – (A2) in an equivalent way according to the new framework in terms of α instead of N and the function $\mathcal{H}_\alpha$ in the place of H (see (B.1)):

(A1)* Ω is α-symmetric with respect to the origin, i.e.

$$x \in \Omega \Longleftrightarrow xe^{\mathrm{i}\frac{2\pi}{\alpha}} \in \Omega;$$

(A2)* the function

$$b \longmapsto \alpha\mathcal{H}_\alpha(b, b) - (\alpha - 1)\mathcal{H}_\alpha(b, 0)$$

has a nondegenerate maximum at 0;

(A3)* there exists $\eta > 0$ such that

$$\alpha_\lambda - \alpha = O(\delta^{2\alpha\eta}).$$

Observe that, since $\mathcal{H}_\alpha$ is symmetric in the two variables, we have $\nabla_b(\alpha\mathcal{H}_\alpha(b, b) - (\alpha-1)\mathcal{H}_\alpha(b, 0))\big|_{b=0} = \frac{\alpha+1}{2}\nabla_b(\mathcal{H}_\alpha(b, b))\big|_{b=0}$, so assumption (A2)* implies that 0 is a critical point of $b \mapsto \mathcal{H}_\alpha(b, b)$.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth and bounded domain such that $0 \in \Omega$ and suppose that Ω satisfies assumptions* (A1)* – (A2)* – (A3)*. *Then, for λ sufficiently small the problem* (2.2) *has a family of solutions v_λ satisfying*

$$v_\lambda = -2\log\left(\delta^{2\alpha} + |x^\alpha - b_\lambda|^2\right) + 8\pi\mathcal{H}_\alpha(x^\alpha, b_\lambda) + o(1)$$

in H^1-sense, where

$$|b_\lambda| \le \max\{\delta^{\frac{\alpha}{2}}, \delta^{\frac{\alpha\eta}{2}}\} \quad \textit{for } \lambda \textit{ sufficiently small.}$$

In particular, if $\alpha_\lambda = \alpha$, $|b_\lambda| \le \delta^{\frac{\alpha}{2}}$.

Theorem 2.1 provides no lower bound on $|b_\lambda|$, so we have no way to rule out the case when b_λ is zero, which corresponds to a radially symmetric first approximation of the blowing up. For instance, in the case of the ball one can use the space of radially symmetric functions, which is a natural constraint for our problem, and find a solution with exactly $b_\lambda = 0$. In particular, if $\alpha_\lambda = \alpha$, in Section 7.2 it is shown that in the unit ball the solution of Theorem 2.1 approaches, after some rescaling, a radial bubbling profile.

The phenomenon of non-simple blow-up occurs for suitable rate of $\alpha_\lambda - \alpha$, as stated in the next theorem.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth and bounded domain such that $0 \in \Omega$ and suppose that Ω satisfies assumptions* (A1)* − (A2)* − (A3)*. *Assume, in addition, that*

$$\alpha_\lambda - \alpha \geq c\delta^{2\alpha} \log^2 \delta$$

for some $c > 0$. Then, for λ sufficiently small the problem (2.2) *has a family of solutions v_λ satisfying*

$$v_\lambda = -2 \log \left(\delta^{2\alpha} + |x^\alpha - b_\lambda|^2\right) + 8\pi \mathcal{H}_\alpha(x^\alpha, b_\lambda) + o(1)$$

in H^1-sense, where

$$c\delta^\alpha |\log \delta|^{2/3} \leq |b_\lambda| \leq C\delta^{\alpha\eta} |\log \delta|^{1/3}.$$

In the remaining part of this paper we will prove Theorems 2.1-2.2 and at the end of Section 7 we shall see how Theorems 1.1-1.2 follow quite directly as a corollary according to the change of variable (2.1).

We end up this section by setting notation and basic well-known facts which will be of use in the rest of the paper. We denote by $\|\cdot\|$ and $\|\cdot\|_p$ the norms in the space $H^1_0(\Omega)$ and $L^p(\Omega)$, respectively, namely

$$\|u\| := \|u\|_{H^1_0(\Omega)}, \qquad \|u\|_p := \|u\|_{L^p(\Omega)} \quad \forall u \in H^1_0(\Omega).$$

In next lemma we recall the well-known Moser-Trudinger inequality ([21, 27]).

Lemma 2.3. *There exists $C > 0$ such that for any bounded domain Ω in $\mathbb{R}^2$*

$$\int_\Omega e^{\frac{4\pi u^2}{\|u\|^2}} dx \leq C|\Omega| \quad \forall u \in H^1_0(\Omega),$$

where $|\Omega|$ stands for the measure of the domain Ω. In particular, for any $q \geq 1$

$$\|e^u\|_q \leq C^{\frac{1}{q}} |\Omega|^{\frac{1}{q}} e^{\frac{q}{16\pi}\|u\|^2} \quad \forall u \in H^1_0(\Omega).$$

For any $\alpha \geq 1$ we will make use of the Hilbert spaces

$$\mathrm{L}_\alpha(\mathbb{R}^2) := \left\{ u \in L^2_{loc}(\mathbb{R}^2) \ : \ \left\| \frac{|y|^{\alpha-1}}{1+|y|^{2\alpha}} u \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\}$$

and

$$\mathrm{H}_\alpha(\mathbb{R}^2) := \left\{ u \in \mathrm{W}^{1,2}_{loc}(\mathbb{R}^2) \ : \ \|\nabla u\|_{L^2(\mathbb{R}^2)} + \left\| \frac{|y|^{\alpha-1}}{1+|y|^{2\alpha}} u \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\},$$

endowed with the norms

$$\|u\|_{\mathrm{L}_\alpha} := \left\| \frac{|y|^{\alpha-1}}{1+|y|^{2\alpha}} u \right\|_{L^2(\mathbb{R}^2)} \text{ and } \|u\|_{\mathrm{H}_\alpha} := \left(\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \left\| \frac{|y|^{\alpha-1}}{1+|y|^{2\alpha}} u \right\|^2_{L^2(\mathbb{R}^2)} \right)^{1/2}.$$

Proposition 2.4. *For any $\alpha \geq 1$ the embedding $\mathrm{H}_\alpha(\mathbb{R}^2) \hookrightarrow \mathrm{L}_\alpha(\mathbb{R}^2)$ is compact.*

Proof. See [17, Proposition 6.1]. □

As commented in the introduction, our proof uses the singular perturbation methods. For that, the nondegeneracy of the functions that we use to build our approximating solution is essential. Next proposition is devoted to the nondegeneracy of the finite mass solutions of the Liouville equation (regular and singular).

Proposition 2.5. *Assume that $\xi \in \mathbb{R}^2$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ solves the problem*

$$-\Delta \phi = 8\alpha^2 \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha - \xi|^2)^2}\phi \ \ in\ \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla \phi(y)|^2 dy < +\infty. \tag{2.9}$$

Then there exist c_0, c_1, $c_2 \in \mathbb{R}$ such that

$$\phi(y) = c_0 Z_0 + c_1 Z_1 + c_2 Z_2.$$

$$Z_0(y) := \frac{1 - |y^\alpha - \xi|^2}{1 + |y^\alpha - \xi|^2}, \quad Z_1(y) := \frac{\mathrm{Re}(y^\alpha - \xi)}{1 + |y^\alpha - \xi|^2}, \quad Z_2(y) := \frac{\mathrm{Im}(y^\alpha - \xi)}{1 + |y^\alpha - \xi|^2}.$$

Proof. In [17, Theorem 6.1] it was proved that any solution ϕ of (2.9) is actually a bounded solution. Therefore we can apply the result in [11] to conclude that $\phi = c_0 Z_0 + c_1 Z_1 + c_2 Z_2$ for some $c_0, c_1, c_2 \in \mathbb{R}$.

□

3. Analysis of the linearized operator

According to Proposition 2.5, by the change of variable $x = \delta y$, we immediately get that all solutions $\psi \in \mathrm{H}_\alpha(\mathbb{R}^2)$ of

$$-\Delta \psi = 8\alpha^2 \frac{\delta^{2\alpha}|x|^{2(\alpha-1)}}{(\delta^{2\alpha} + |x^\alpha - b|^2)^2}\psi = |x|^{2(\alpha-1)} e^{W_\lambda}\psi \quad \text{in} \quad \mathbb{R}^2$$

are linear combinations of the functions

$$Z^0_{\delta,b}(x) = \frac{\delta^{2\alpha} - |x^\alpha - b|^2}{\delta^{2\alpha} + |x^\alpha - b|^2},\ Z^1_{\delta,b}(x) = \frac{\delta^\alpha \mathrm{Re}(x^\alpha - b)}{\delta^{2\alpha} + |x^\alpha - b|^2},\ Z^2_{\delta,b}(x) = \frac{\delta^\alpha \mathrm{Im}(x^\alpha - b)}{\delta^{2\alpha} + |x^\alpha - b|^2}.$$

We introduce the projections $PZ^j_{\delta,b}$ onto $H^1_0(\Omega)$. It is immediate that

$$PZ^0_{\delta,b}(x) = Z^0_{\delta,b}(x) + 1 + O\left(\delta^{2\alpha}\right) = \frac{2\delta^{2\alpha}}{\delta^{2\alpha} + |x^\alpha - b|^2} + O(\delta^{2\alpha}) \tag{3.1}$$

and

$$PZ^j_{\delta,b}(x) = Z^j_{\delta,b}(x) + O(\delta^\alpha) \text{ for } j = 1, 2 \tag{3.2}$$

uniformly with respect to $x \in \overline{\Omega}$ and b in a small neighborhood of 0.

We agree that $Z^j_\lambda := Z^j_{\delta,b}$ for any $j = 0, 1, 2$, where δ is defined in terms of λ and b according to (2.7). Motivated by the symmetry of the domain in assumption (A1)*, we consider the subspaces of $H^1_\star(\Omega)$, $L^p_\star(\Omega)$ made up of α-symmetric functions:

$$H^1_{0,\star}(\Omega) = \{u \in H^1_0(\Omega) \,|\, u(xe^{\mathrm{i}\frac{2\pi}{\alpha}}) = u(x)\}, \qquad L^p_\star(\Omega) = \{u \in L^p(\Omega) \,|\, u(xe^{\mathrm{i}\frac{2\pi}{\alpha}}) = u(x)\}.$$

Clearly $PW_\lambda, PZ^j_\lambda \in H^1_{0,\star}(\Omega)$. Let us consider the following linear problem: given $h \in H^1_{0,\star}(\Omega)$, find a function $\phi \in H^1_{0,\star}(\Omega)$ satisfying

$$\begin{cases} -\Delta\phi - \lambda V(x)|x|^{2(\alpha_\lambda - 1)} e^{PW_\lambda}\phi = \Delta h \\ \displaystyle\int_\Omega \nabla\phi \nabla PZ^j_\lambda = 0 \ \ j = 1, 2 \end{cases}. \tag{3.3}$$

Before going on, we recall the following identities which follow by straightforward computations: for every $\xi \in \mathbb{R}^2$

$$\int_{\mathbb{R}^2} \log(1 + |y|^2)\frac{1 - |y|^2}{(1 + |y|^2)^3} dy = -\frac{\pi}{2}, \tag{3.4}$$

$$\int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^3} dy = 0, \tag{3.5}$$

$$\int_{\mathbb{R}^2} \frac{y_1^2}{(1+|y|^2)^4} dy = \int_{\mathbb{R}^2} \frac{y_2^2}{(1+|y|^2)^4} dy = \frac{1}{2} \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4} dy = \frac{\pi}{12}. \tag{3.6}$$

Proposition 3.1. *There exist $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda \in (0, \lambda_0)$, any $b \in \mathbb{R}^2$ in a small neighborhood of 0 and any $h \in H^1_{0,\star}(\Omega)$, if $\phi \in H^1_{0,\star}(\Omega)$ solves (3.3), then the following holds*

$$\|\phi\| \le C|\log\delta|\|h\|.$$

Proof. We argue by contradiction. Assume that there exist sequences $\lambda_n \to 0$, $h_n \in H^1_{0,\star}(\Omega)$, b_n in a small neighborhood of 0 and $\phi_n \in H^1_{0,\star}(\Omega)$ which solves (3.3) and

$$\|\phi_n\| = 1, \qquad |\log\delta_n|\|h_n\| \to 0. \tag{3.7}$$

We define $\widetilde{\Omega}_n := \frac{\Omega}{\delta_n}$ and

$$\tilde{\phi}_n(y) := \begin{cases} \phi_n(\delta_n y) & \text{if } y \in \widetilde{\Omega}_n \\ 0 & \text{if } y \in \mathbb{R}^2 \setminus \widetilde{\Omega}_n \end{cases}.$$

We split the remaining argument into five steps. In what follows at many steps of the reasoning we will pass to a subsequence, without further notice.

Step 1. Using the polar coordinates (ρ, θ) let us set, according to (C.2),

$$\tilde{\Phi}_n(\rho e^{\mathrm{i}\theta}) = \tilde{\phi}_n\big(\rho^{\frac{1}{\alpha}} e^{\mathrm{i}\frac{\theta}{\alpha}}\big) \qquad \rho \ge 0,\ \theta \in [-\pi, \pi).$$

We will show that

$$\tilde{\Phi}_n \in \mathrm{H}_1(\mathbb{R}^2) \quad \textit{and} \quad \tilde{\Phi}_n(\cdot + \delta_n^{-\alpha} b_n) \textit{ is bounded in } \mathrm{H}_1(\mathbb{R}^2).$$

It is immediate to check that

$$\int_{\mathbb{R}^2} |\nabla\tilde{\phi}_n|^2 dy = \int_{\Omega} |\nabla\phi_n|^2 dx = 1. \tag{3.8}$$

Next, we multiply the equation in (3.3) by ϕ_n; then we integrate over Ω to obtain

$$\lambda_n \int_{\Omega} V(x)|x|^{2(\alpha_{\lambda_n}-1)} e^{PW_{\lambda_n}} \phi_n^2 dx = \int_{\Omega} |\nabla\phi_n|^2 dx + \int_{\Omega} \nabla h_n \nabla\phi_n dx \le C$$

by (3.7). So Proposition 4.2 (taking p sufficiently close to 1) gives $\int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n^2 \le C$ or, equivalently,

$$\int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{\big(1+|y^\alpha - \delta_n^{-\alpha} b_n|^2\big)^2} \tilde{\phi}_n^2 dy \le C. \tag{3.9}$$

We deduce that $\tilde{\phi}_n$ belongs to $\mathrm{H}_\alpha(\mathbb{R}^2)$ and satisfies (3.8)-(3.9). Thanks to Lemma C.2 we get $\tilde{\Phi}_n \in \mathrm{H}_1(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} |\nabla\tilde{\Phi}_n|^2 dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} |\nabla\tilde{\phi}_n|^2 dy = \frac{1}{\alpha}$$

and

$$\int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^2} |\tilde{\Phi}_n(y + \delta_n^{-\alpha} b_n)|^2 dy = \int_{\mathbb{R}^2} \frac{1}{(1+|y-\delta_n^{-\alpha} b_n|^2)^2} |\tilde{\Phi}_n|^2 dy$$
$$= \alpha \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha - \delta_n^{-\alpha} b_n|^2)^2} |\tilde{\phi}_n|^2 dy \le C.$$

We have thus proved that the sequence $\tilde{\Phi}_n(y + \delta_n^{-\alpha} b_n)$ $(n \in \mathbb{N})$ is bounded in $\mathrm{H}_1(\mathbb{R}^2)$.

Step 2. We will show that, for some $\gamma_0 \in \mathbb{R}$:

$$\tilde{\Phi}_n(y+\delta_n^{-\alpha}b_n) \to \gamma_0 \frac{1-|y|^2}{1+|y|^2} \text{ weakly in } \mathrm{H}_1(\mathbb{R}^2) \text{ and strongly in } \mathrm{L}_1(\mathbb{R}^2).$$

Step 1 and Proposition 2.4 give

$$\tilde{\Phi}_n(y+\delta_n^{-\alpha}b_n) \to f \text{ weakly in } \mathrm{H}_1(\mathbb{R}^2) \text{ and strongly in } \mathrm{L}_1(\mathbb{R}^2) \tag{3.10}$$

for some $f \in \mathrm{H}_1(\mathbb{R}^2)$. Let $\psi \in C_c^\infty(\mathbb{R}^2)$ and set

$$\psi_n(x) = \psi\Big(\frac{x^\alpha - b_n}{\delta_n^\alpha}\Big) \qquad \tilde{\psi}_n(y) = \psi_n(\delta_n y).$$

Setting, according to (C.2), in polar coordinates

$$\tilde{\Psi}_n(\rho e^{\mathrm{i}\theta}) = \tilde{\psi}_n\big(\rho^{\frac{1}{\alpha}} e^{\mathrm{i}\frac{\theta}{\alpha}}\big) \qquad \rho \geq 0,\ \theta \in [-\pi, \pi),$$

we immediately get

$$\tilde{\Psi}_n(y) = \psi(y - \delta_n^{-\alpha}b_n).$$

We have $\psi_n \in C_c^\infty(\Omega)$, for large n. Then by applying Corollary C.3 we get

$$\begin{aligned}\int_\Omega \nabla\phi_n \cdot \nabla\psi_n dx &= \int_{\mathbb{R}^2} \nabla\tilde{\phi}_n \cdot \nabla\tilde{\psi}_n dy = \alpha \int_{\mathbb{R}^2} \nabla\tilde{\Phi}_n \cdot \nabla\tilde{\Psi}_n dy \\ &= \alpha \int_{\mathbb{R}^2} \nabla\tilde{\Phi}_n(y+\delta_n^{-\alpha}b_n) \cdot \nabla\psi dy \\ &= \alpha \int_{\mathbb{R}^2} \nabla f \nabla\psi dy + o(1).\end{aligned} \tag{3.11}$$

Similarly we compute

$$\begin{aligned}\int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n\psi_n dx &= 8\alpha^2 \int_\Omega \frac{\delta_n^{2\alpha}|x|^{2(\alpha-1)}}{(\delta_n^{2\alpha} + |x^\alpha - b_n|^2)^2} \phi_n\psi_n dx \\ &= 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha - \delta_n^{-\alpha}b_n|^2)^2} \tilde{\phi}_n(y)\tilde{\psi}_n(y) dy \\ &= 8\alpha \int_{\mathbb{R}^2} \frac{1}{(1+|y-\delta_n^{-\alpha}b_n|^2)^2} \tilde{\Phi}_n\tilde{\Psi}_n dy \\ &= 8\alpha \int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^2} \tilde{\Phi}_n(y+\delta_n^{-\alpha}b_n)\psi dy \\ &= 8\alpha \int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^2} f\psi dy + o(1),\end{aligned}$$

by which, using Proposition 4.2 we deduce

$$\begin{aligned}\lambda_n \int_\Omega |x|^{2(\alpha_{\lambda_n}-1)} V(x) e^{PW_n} \phi_n\psi_n dx &= \int_\Omega |x|^{2(\alpha-1)} e^{W_\lambda} \phi_n\psi_n dx + o(1) \\ &= 8\alpha \int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^2} f\psi dy + o(1).\end{aligned} \tag{3.12}$$

Finally we estimate

$$\int_\Omega \nabla h_n \nabla\psi_n dx = O(\|h_n\|) = o(1). \tag{3.13}$$

We multiply the equation in (3.3) by ψ_n, we integrate over Ω and and pass to the limit as $n \to +\infty$; combining (3.11)-(3.12)-(3.13) we obtain

$$\int_{\mathbb{R}^2} \nabla f \nabla \psi dy - \int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} f \psi dy = 0.$$

Thus, we deduce that the function $f \in \mathrm{H}_1(\mathbb{R}^2)$ is a solution of the equation

$$-\Delta f = \frac{8}{(1+|y|^2)^2} f \quad \text{in } \mathbb{R}^2.$$

Proposition 2.5 gives

$$f = \gamma_0 \frac{1-|y|^2}{1+|y|^2} + \sum_{j=1,2} \gamma_j \frac{y_j}{1+|y|^2}$$

for suitable constants $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}$. Now we use the orthogonality $\int_\Omega \nabla \phi_n \nabla P Z^1_{\lambda_n} = 0$ in (3.3) to obtain

$$\begin{aligned}
0 = \int_\Omega \nabla \phi_n \nabla P Z^1_{\lambda_n} dx &= \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n Z^1_{\lambda_n} dx \\
&= 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha - \delta_n^{-\alpha} b_n|^2)^2} \tilde{\phi}_n(y) \frac{\mathrm{Re}(y^\alpha - \delta_n^{-\alpha} b_n)}{1+|y^\alpha - \delta_n^{-\alpha} b_n|^2} dy \\
&= 8\alpha \int_{\mathbb{R}^2} \frac{y_1 - \delta_n^{-\alpha} b_n}{(1+|y-\delta_n^{-\alpha} b_n|^2)^3} \tilde{\Phi}_n(y) dy \\
&= 8\alpha \int_{\mathbb{R}^2} \frac{y_1}{(1+|y|^2)^3} \tilde{\Phi}_n(y + \delta_n^{-\alpha} b_n) dy.
\end{aligned}$$

Then we pass to the limit when $n \to +\infty$ and we obtain

$$0 = \int_{\mathbb{R}^2} \frac{y_1}{(1+|y|^2)^3} f dy = \gamma_1 \int_{\mathbb{R}^2} \frac{y_1^2}{(1+|y|^2)^3} dy.$$

So $\gamma_1 = 0$ and, similarly, $\gamma_2 = 0$.

Step 3. We will show that

$$\int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n dx == o\Big(\frac{1}{|\log \delta_n|}\Big).$$

We multiply the equation in (3.3) by $PZ^0_{\lambda_n}$, we integrate over Ω and we get

$$\int_\Omega \nabla \phi_n \nabla P Z^0_{\lambda_n} dx - \lambda_n \int_\Omega V(x) |x|^{2(\alpha_{\lambda_n}-1)} e^{PW_{\lambda_n}} \phi_n P Z^0_{\lambda_n} dx = -\int_\Omega \nabla h_n \nabla P Z^0_{\lambda_n} dx. \tag{3.14}$$

We are now concerned with the estimates of each term of the above expression.

First, we compute

$$\int_\Omega \nabla \phi_n \nabla P Z^0_{\lambda_n} dx = \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n Z^0_{\lambda_n} dx. \tag{3.15}$$

Using Proposition 4.2 and (3.1), we obtain

$$\begin{aligned}
\lambda_n \int_\Omega V(x) |x|^{2(\alpha_{\lambda_n}-1)} e^{PW_{\lambda_n}} \phi_n P Z^0_{\lambda_n} dx &= \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n (Z^0_{\lambda_n} + 1) dx + o\Big(\frac{1}{|\log \delta_n|}\Big) \\
= \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n Z^0_{\lambda_n} dx &+ \int_{\mathbb{R}^2} |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n dx + o\Big(\frac{1}{|\log \delta_n|}\Big).
\end{aligned} \tag{3.16}$$

Finally, since $PZ^0_\lambda = O(1)$, we have $\int_\Omega |\nabla PZ^0_\lambda|^2 = \int_\Omega |x|^{2(\alpha-1)} e^{W_\lambda} PZ^0_\lambda = O(1)$, by which, owing to (3.7),

$$\int_\Omega |\nabla h_n|\,|\nabla PZ^0_{\lambda_n}|dx \le \|h_n\|\,\|PZ^0_{\lambda_n}\| = o\Big(\frac{1}{|\log\delta_n|}\Big). \tag{3.17}$$

We now multiply (3.14) by $\log\delta_n$ and pass to the limit: inserting (3.15), (3.16), (3.17), we obtain the thesis of the step.

Step 4. We will show that $\gamma_0 = 0$.

We multiply the equation in (3.3) by PW_{λ_n}, we integrate over Ω and we get

$$\int_\Omega \nabla\phi_n \nabla PW_{\lambda_n} dx - \lambda_n \int_\Omega V(x)|x|^{2(\alpha_{\lambda_n}-1)} e^{PW_{\lambda_n}} \phi_n PW_{\lambda_n} dx = -\int_\Omega \nabla h_n \nabla PW_{\lambda_n} dx. \tag{3.18}$$

Let us estimate each of the terms above. Let us begin with:

$$\int_\Omega \nabla\phi_n \nabla PW_{\lambda_n} dx = \int_\Omega \phi_n |x|^{2(\alpha-1)} e^{W_{\lambda_n}} dx = o(1) \tag{3.19}$$

by step 3. By Proposition 4.2 and (3.7), using that $|PW_\lambda| = O(|\log\delta|)$, we get

$$\lambda_n \int_\Omega V(x)|x|^{2(\alpha_{\lambda_n}-1)} e^{PW_{\lambda_n}} \phi_n PW_{\lambda_n} dx = \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n PW_{\lambda_n} dx + o(1). \tag{3.20}$$

Observe that by (2.8) and (B.2) we have

$$\begin{aligned} PW_\lambda(x) &= -2\log(\delta^{2\alpha} + |x^\alpha - b|^2) + 8\pi\mathcal{H}_\alpha(b,b) + O(|x^\alpha - b|) + O(\delta^{2\alpha}) \\ &= -4\alpha\log\delta - 2\log(1 + \delta^{-2\alpha}|x^\alpha - b|^2) + 8\pi\mathcal{H}_\alpha(b,b) + O(|x^\alpha - b|). \end{aligned}$$

Recalling Step 3 and Lemma 4.1 we obtain

$$\begin{aligned} \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n PW_{\lambda_n} dx &= -2\int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n \log(1 + \delta_n^{-2\alpha}|x^\alpha - b_n|^2) dx \\ &\quad + 8\pi\mathcal{H}_\alpha(b_n,b_n) \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n dx + o(1) \\ &= -16\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2)^2} \tilde\phi_n \log(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2) dy \\ &\quad + 64\pi\alpha^2 \mathcal{H}_\alpha(b_n,b_n) \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2)^2} \tilde\phi_n dy + o(1) \end{aligned}$$

and, using Corollary C.3,

$$\begin{aligned} \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n PW_{\lambda_n} dx &= -16\alpha \int_{\mathbb{R}^2} \frac{1}{(1 + |y - \delta_n^{-\alpha} b_n|^2)^2} \tilde\Phi_n \log(1 + |y - \delta_n^{-\alpha} b_n|^2) dy \\ &\quad + 64\pi\alpha \mathcal{H}_\alpha(b_n,b_n) \int_{\mathbb{R}^2} \frac{1}{(1 + |y - \delta_n^{-\alpha} b_n|^2)^2} \tilde\Phi_n dy + o(1). \end{aligned}$$

Now, by Step 2,

$$\begin{aligned} \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n PW_{\lambda_n} dx &= -16\alpha\gamma_0 \int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^2} \frac{1-|y|^2}{1+|y|^2} \log(1+|y|^2) dy \\ &\quad + 64\pi\alpha\gamma_0 \mathcal{H}_\alpha(b_n,b_n) \int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^2} \frac{1-|y|^2}{1+|y|^2} dy + o(1) \\ &= 8\alpha\gamma_0\pi + o(1) \end{aligned} \tag{3.21}$$

by (3.4)-(3.5). Finally, since $PW_\lambda = O(|\log\delta|)$, then we have $\int_\Omega |\nabla PW_\lambda|^2 = \int_\Omega |x|^{2(\alpha-1)}e^{W_\lambda}PW_\lambda = O(|\log\delta|)$, by which, owing to (3.7),

$$\int_\Omega |\nabla h_n|\,|\nabla PW_{\lambda_n}|dx \le \|h_n\|\,\|PW_{\lambda_n}\| = o(1). \tag{3.22}$$

By inserting (3.19), (3.20), (3.21) and (3.22) into (3.18) and passing to the limit we deduce $\gamma_0 = 0$.

Step 5. End of the proof.

We will show that a contradiction arises. According to Step 2 and Step 4 we have

$$\tilde\Phi_n(y+\delta_n^{-\alpha}b_n) \to 0 \quad \text{weakly in } \mathrm{H}_1(\mathbb{R}^2) \text{ and strongly in } \mathrm{L}_1(\mathbb{R}^2).$$

By Proposition 4.2 and (3.7)

$$\lambda_n\int_\Omega V(x)|x|^{2(\alpha_{\lambda_n}-1)}e^{PW_{\lambda_n}}\phi_n^2dx = \int_\Omega |x|^{2(\alpha-1)}e^{W_{\lambda_n}}\phi_n^2dx + o(1).$$

Now, using Lemma C.2,

$$\begin{aligned}\lambda_n\int_\Omega |x|^{2(\alpha-1)}e^{W_{\lambda_n}}\phi_n^2dx &= 8\alpha^2\int_{\mathbb{R}^2}\frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha-\delta_n^{-\alpha}b_n|^2)^2}\tilde\phi_n^2dy\\ &= 8\alpha\int_{\mathbb{R}^2}\frac{1}{(1+|y-\delta_n^{-\alpha}b_n|^2)^2}\tilde\Phi_n^2dy = 8\alpha\|\tilde\Phi_n(\cdot+\delta_n^{-\alpha}b_n)\|_{\mathrm{L}_1}^2 = o(1).\end{aligned}$$

Moreover, by (3.7),

$$\int_\Omega \nabla h_n\nabla\phi_n dx = o(1).$$

We multiply the equation in (3.3) by ϕ_n, we integrate over Ω and we obtain

$$\int_\Omega |\nabla\phi_n|^2dx = \lambda_n\int_\Omega V(x)|x|^{2(\alpha_{\lambda_n}-1)}e^{PW_{\lambda_n}}\phi_n^2dx - \int_\Omega \nabla h_n\nabla\phi_n dx = o(1),$$

in contradiction with (3.7). This concludes the proof of the proposition. □

In addition to (3.3), let us consider the following linear problem: given $h \in H^1_{0,\star}(\Omega)$, find a function $\phi \in H^1_{0,\star}(\Omega)$ and constants $c_1, c_2 \in \mathbb{R}$ satisfying

$$\begin{cases} -\Delta\phi - \lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\phi = \Delta h + \sum\limits_{j=1,2} c_j Z_\lambda^j|x|^{2(\alpha-1)}e^{W_\lambda} \\ \int_\Omega \nabla\phi\nabla PZ_\lambda^j = 0 \;\; j=1,2 \end{cases}. \tag{3.23}$$

In order to solve problem (3.23), we need to establish an a priori estimate analogous to that of Proposition 3.1.

Proposition 3.2. *There exist $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda \in (0,\lambda_0)$, any $b \in \mathbb{R}^2$ in a small neighborhood of 0 and any $h \in H^1_{0,\star}(\Omega)$, if $(\phi, c_1, c_2) \in H^1_{0,\star}(\Omega)\times\mathbb{R}^2$ solves (3.23), then the following holds*

$$\|\phi\| \le C|\log\delta|\|h\|.$$

Proof. First observe that by (3.2)

$$\begin{aligned}\int_\Omega \nabla PZ_\lambda^1\nabla PZ_\lambda^2dx &= \int_\Omega |x|^{2(\alpha-1)}e^{W_\lambda}Z_\lambda^1PZ_\lambda^2dx = \int_{\mathbb{R}^2}|x|^{2(\alpha-1)}e^{W_\lambda}Z_\lambda^1Z_\lambda^2dx + o(1)\\ &= \int_{\mathbb{R}^2}\nabla Z_\lambda^1\nabla Z_\lambda^2dx + o(1) = o(1) = o(1).\end{aligned} \tag{3.24}$$

Similarly

$$\begin{aligned}\|PZ_\lambda^1\|^2 &= \int_\Omega |x|^{2(\alpha-1)}e^{W_\lambda}Z_\lambda^1 PZ_\lambda^1 dx = \int_\Omega |x|^{2(\alpha-1)}e^{W_\lambda}(Z_\lambda^1)^2 dx + o(1)\\ &= 8\alpha^2\int_{\mathbb{R}^2}\frac{|y|^{2(\alpha-1)}|\mathrm{Re}(y^\alpha-\delta^{-\alpha}b)|^2}{(1+|y^\alpha-\delta^{-\alpha}b|^2)^4}dy + o(1)\\ &= 8\alpha\int_{\mathbb{R}^2}\frac{y_1^2}{(1+|y|^2)^4}dy = \frac{2}{3}\pi\alpha + o(1)\end{aligned} \tag{3.25}$$

where we have used Lemma C.1 and (3.6). Analogously $\|PZ_\lambda^2\|^2 = \frac{2}{3}\pi\alpha + o(1)$.

Then, taking into account that $-\Delta PZ_\lambda^j = |x|^{2(\alpha-1)}e^{W_\lambda}Z_\lambda^j$, according to Proposition 3.1 we have

$$\|\phi\| \le C|\log\delta|\big(\|h\| + |c_1| + |c_2|\big). \tag{3.26}$$

Hence it suffices to estimate the values of the constants c_j. We multiply the equation in (3.23) by PZ_λ^1 and we find

$$\int_\Omega \phi|x|^{2(\alpha-1)}e^{W_\lambda}Z_\lambda^1 dx - \lambda\int_\Omega V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\phi PZ_\lambda^1 dx = \frac{2}{3}\pi\alpha c_1 + o(c_1) + o(c_2) + O(\|h\|). \tag{3.27}$$

Let us fix $p \in (1,+\infty)$ sufficiently close to 1. Then, by (3.2) and (4.1) we may estimate

$$\begin{aligned}\int_\Omega |x|^{2(\alpha-1)}e^{W_\lambda}|\phi||PZ_\lambda^1 - Z_\lambda^1|dx &\le C\delta^\alpha\int_\Omega |x|^{2(\alpha-1)}e^{W_\lambda}|\phi|dx \le C\delta^\alpha\|\phi\|\,\||x|^{2(\alpha-1)}e^{W_\lambda}\|_p\\ &\le C\delta^{\alpha-2\alpha\frac{p-1}{p}}\|\phi\| \le \delta^{\frac{\alpha}{2}}\|\phi\|\end{aligned}$$

and, since $PZ_\lambda^1 = O(1)$, using Proposition 4.2,

$$\begin{aligned}\int_\Omega \big||x|^{2(\alpha-1)}e^{W_\lambda} - \lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\big||\phi||PZ_\lambda^1|dx &\le C\int_\Omega |R_\lambda||\phi|dx\\ &\le C\delta^{-2\alpha\frac{p-1}{p}}(\delta^{2\alpha}+\delta^\alpha|b|)\|\phi\| \le \delta^{\frac{\alpha}{2}}\|\phi\|.\end{aligned}$$

By inserting the above estimates into (3.27) we obtain

$$|c_1| + o(c_2) \le C\|h\| + C\delta^{\frac{\alpha}{2}}\|\phi\|.$$

We multiply the equation in (3.23) by PZ_λ^2 and, by a similar argument as above, we find

$$|c_2| + o(c_1) \le C\|h\| + C\delta^{\frac{\alpha}{2}}\|\phi\|,$$

and so

$$|c_1| + |c_2| \le C\|h\| + C\delta^{\frac{\alpha}{2}}\|\phi\|.$$

Combining this with (3.26) we obtain the thesis. □

4. Estimate of the error term

The goal of this section is to provide an estimate of the error up to which the approximate solution PW_λ solves problem (2.2). First of all, we perform the following estimates.

Lemma 4.1. *Let $\gamma = 0,1,2$ and $p > 1$ be fixed. The following holds:*

$$\||x|^{2(\alpha-1)}|x^\alpha - b|^\gamma e^{W_\lambda}\|_p \le C\delta^{\gamma\alpha}\delta^{-2\alpha\frac{p-1}{p}}, \tag{4.1}$$

$$\|(|x|^{2(\alpha_\lambda-1)} - |x|^{2(\alpha-1)})|x^\alpha - b|^\gamma e^{W_\lambda}\|_p \le C|\alpha_\lambda - \alpha|\delta^{\gamma\alpha}\delta^{-2\alpha\frac{p-1}{p}} \tag{4.2}$$

uniformly for b in a small neighborhood of 0. As a corollary,

$$\||x|^{2(\alpha_\lambda-1)}|x^\alpha - b|^\gamma e^{W_\lambda}\|_p \le C\delta^{\gamma\alpha}\delta^{-2\alpha\frac{p-1}{p}} \tag{4.3}$$

uniformly for b in a small neighborhood of 0.

Proof. By Lemma C.1 we compute

$$\begin{aligned}\||x|^{2(\alpha-1)}|x^\alpha-b|^\gamma e^{W_\lambda}\|_p^p &= (8\alpha^2)^p\delta^{2\alpha p}\int_\Omega \frac{|x|^{2(\alpha-1)p}|x^\alpha-b|^{\gamma p}}{(\delta^{2\alpha}+|x^\alpha-b|^2)^{2p}}dx\\ &\le C(8\alpha^2)^p\delta^{2\alpha p}\int_\Omega \frac{|x|^{2(\alpha-1)}|x^\alpha-b|^{\gamma p}}{(\delta^{2\alpha}+|x^\alpha-b|^2)^{2p}}dx\\ &= C(8\alpha^2)^p\delta^{\alpha\gamma p-2\alpha(p-1)}\int_{\mathbb{R}^2}\frac{|y|^{2(\alpha-1)}|y^\alpha-\delta^{-\alpha}b|^{\gamma p}}{(1+|y^\alpha-\delta^{-\alpha}b|^2)^{2p}}dy\\ &= C\frac{(8\alpha^2)^p}{\alpha}\delta^{\alpha\gamma p-2\alpha(p-1)}\int_{\mathbb{R}^2}\frac{|y|^{\gamma p}}{(1+|y|^2)^{2p}}dy.\end{aligned}$$

Taking into account that the last integral is finite for $\gamma=0,1,2$ and $p>1$ we deduce (4.1).

In order to prove (4.2) first we will show that

$$\big||x|^{2(\alpha_\lambda-1)}-|x|^{2(\alpha-1)}\big|\le C|\alpha_\lambda-\alpha||x|^{2(\alpha-1)}|x|^{-2|\alpha_\lambda-\alpha|}|\log|x|| \tag{4.4}$$

uniformly for $x\in\Omega$. Indeed, to this aim we are going to use the following immediate inequality

$$|t^\varepsilon-1|\le\varepsilon|\log t|\qquad \forall t>0,\ \ \forall\varepsilon>0.$$

First assume $\alpha_\lambda>\alpha$: for $x\in\Omega$ we compute

$$\begin{aligned}\big||x|^{2(\alpha_\lambda-1)}-|x|^{2(\alpha-1)}\big| &= |x|^{2(\alpha-1)}\big||x|^{2(\alpha_\lambda-\alpha)}-1\big|\\ &\le C(\alpha_\lambda-\alpha)|x|^{2(\alpha-1)}|\log|x||\end{aligned}$$

and the thesis of (4.4) follows; on the other hand, if $\alpha_\lambda<\alpha$, for $x\in\Omega$ we have

$$\begin{aligned}\big||x|^{2(\alpha_\lambda-1)}-|x|^{2(\alpha-1)}\big| &= |x|^{2(\alpha_\lambda-1)}\big||x|^{2(\alpha-\alpha_\lambda)}-1\big|\\ &\le C(\alpha-\alpha_\lambda)|x|^{2(\alpha_\lambda-1)}|\log|x||\\ &= C(\alpha-\alpha_\lambda)|x|^{2(\alpha-1)}|x|^{-2(\alpha-\alpha_\lambda)}|\log|x||\end{aligned}$$

and (4.4) is completely proved.

Let us now pass to the second estimate (4.2): since $|x|^{2p(\alpha-1)}|x|^{-2p|\alpha_\lambda-\alpha|}|\log|x||^p\le C|x|^{2(\alpha-1)}$ in Ω for $p>1$,

$$\begin{aligned}\||x|^{2(\alpha-1)}|x|^{-2|\alpha_\lambda-\alpha|}|x^\alpha-b|^\gamma|\log|x||e^{W_\lambda}\|_p^p &\le C(8\alpha^2)^p\delta^{2\alpha p}\int_\Omega\frac{|x|^{2(\alpha-1)}|x^\alpha-b|^{\gamma p}}{(\delta^{2\alpha}+|x^\alpha-b|^2)^{2p}}dx\\ &= C\frac{(8\alpha^2)^p}{\alpha}\delta^{\alpha\gamma p-2\alpha(p-1)}\int_{\mathbb{R}^2}\frac{|y|^{\gamma p}}{(1+|y|^2)^{2p}}dy\end{aligned}$$

and (4.2) follows by (4.4).

□

Proposition 4.2. *Define*

$$R_\lambda := -\Delta PW_\lambda-\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}=|x|^{2(\alpha-1)}e^{W_\lambda}-\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}.$$

For any fixed $p>1$ the following holds

$$\|R_\lambda\|_p\le C\delta^{-2\alpha\frac{p-1}{p}}\Big(\delta^{2\alpha}+\delta^\alpha|b|+|\alpha_\lambda-\alpha|\Big)\le C\delta^{-2\alpha\frac{p-1}{p}}\Big(\delta^{2\alpha}+\delta^\alpha|b|+\delta^{2\alpha\eta}\Big)$$

uniformly for b in a small neighborhood of 0*. Consequently, by* (4.1)*, if* $p>1$

$$\|\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\|_p=\||x|^{2(\alpha-1)}e^{W_\lambda}\|_p+O(\delta^{-2\alpha\frac{p-1}{p}})=O(\delta^{-2\alpha\frac{p-1}{p}}) \tag{4.5}$$

uniformly for b in a small neighborhood of 0*.*

Proof. By (2.8) and the choice of δ in (2.7) we derive

$$\begin{aligned}&\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\\&=\frac{\lambda}{8\alpha^2\delta^{2\alpha}}V(x)|x|^{2(\alpha_\lambda-1)}e^{W_\lambda+8\pi\mathcal{H}_\alpha(x^\alpha,b)+O(\delta^{2\alpha})}\\&=|x|^{2(\alpha_\lambda-1)}e^{W_\lambda}e^{-4\pi(\alpha_\lambda-1)(H(x,0)-\frac{\mathcal{H}_\alpha(b,0)}{\alpha})+8\pi(\mathcal{H}_\alpha(x^\alpha,b)-\mathcal{H}_\alpha(b,b))+O(\delta^{2\alpha})}\\&=|x|^{2(\alpha_\lambda-1)}e^{W_\lambda}e^{-4\pi\frac{\alpha_\lambda-1}{\alpha}(\mathcal{H}_\alpha(x^\alpha,0)-\mathcal{H}_\alpha(b,0))+8\pi(\mathcal{H}_\alpha(x^\alpha,b)-\mathcal{H}_\alpha(b,b))+O(\delta^{2\alpha})}.\end{aligned} \tag{4.6}$$

Thanks to assumption (A2)* by (B.2)-(B.3)

$$-4\pi\frac{\alpha_\lambda-1}{\alpha}(\mathcal{H}_\alpha(x^\alpha,0)-\mathcal{H}_\alpha(b,0))+8\pi(\mathcal{H}_\alpha(x^\alpha,b)-\mathcal{H}_\alpha(b,b))=O(|b||x^\alpha-b|)+O(|x^\alpha-b|^2).$$

So (4.6) reduces to

$$\begin{aligned}&\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\\&=|x|^{2(\alpha_\lambda-1)}e^{W_\lambda}+\Big(O(|b||x^\alpha-b|)+O(|x^\alpha-b|^2)+O(\delta^{2\alpha})\Big)|x|^{2(\alpha_\lambda-1)}e^{W_\lambda}.\end{aligned} \tag{4.7}$$

The thesis follows by Lemma 4.1. □

5. The nonlinear problem: a contraction argument

In order to solve (1.6), let us consider the following intermediate problem:

$$\begin{cases}-\Delta(PW_\lambda+\phi)-\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda+\phi}=\sum_{j=1,2}c_jZ^j_\lambda|x|^{2(\alpha-1)}e^{W_\lambda},\\ \phi\in H^1_{0,\star}(\Omega),\quad \int_\Omega\nabla\phi\nabla PZ^j_\lambda dx=0\ \ j=1,2.\end{cases} \tag{5.1}$$

Then it is convenient to solve as a first step the problem for ϕ as a function of b. To this aim, first let us rewrite problem (5.1) in a more convenient way.

For any $p>1$, let

$$i^*_p:L^p_\star(\Omega)\to H^1_{0,\star}(\Omega) \tag{5.2}$$

be the adjoint operator of the embedding $i_p:H^1_{0,\star}(\Omega)\hookrightarrow L_\star^{\frac{p}{p-1}}(\Omega)$, i.e. $u=i^*_p(v)$ if and only if $-\Delta u=v$ in Ω, $u=0$ on $\partial\Omega$. We point out that i^*_p is a continuous mapping, namely

$$\|i^*_p(v)\|\le c_p\|v\|_p,\ \text{for any } v\in L^p_\star(\Omega), \tag{5.3}$$

for some constant c_p which depends on Ω and p. Next let us set

$$K:=\text{span}\left\{PZ^1_\lambda,\ PZ^2_\lambda\right\}$$

and

$$K^\perp:=\left\{\phi\in H^1_{0,\star}(\Omega)\ :\ \int_\Omega\nabla\phi\nabla PZ^1_\lambda dx=\int_\Omega\nabla\phi\nabla PZ^2_\lambda dx=0\right\}$$

and denote by

$$\Pi:H^1_{0,\star}(\Omega)\to K,\qquad \Pi^\perp:H^1_{0,\star}(\Omega)\to K^\perp$$

the corresponding projections. Let $L : K^\perp \to K^\perp$ be the linear operator defined by

$$L(\phi) := \Pi^\perp\Big(i_p^*\big(\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\phi\big)\Big) - \phi. \tag{5.4}$$

Notice that problem (3.23) reduces to

$$L(\phi) = \Pi^\perp h, \quad \phi \in K^\perp.$$

As a consequence of Proposition 3.2 we derive the invertibility of L.

Proposition 5.1. *For any $p > 1$ there exist $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda \in (0, \lambda_0)$, any $b \in \mathbb{R}^2$ in a small neighborhood of 0 and any $h \in K^\perp$ there is a unique solution $\phi \in K^\perp$ to the problem*

$$L(\phi) = h.$$

In particular, L is invertible; moreover,

$$\|L^{-1}\| \le C|\log\delta|.$$

Proof. Observe that the operator $\phi \mapsto \Pi^\perp\big(i_p^*\big(\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\phi\big)\big)$ is a compact operator in $K^\perp$. Let us consider the case $h = 0$, and take $\phi \in K^\perp$ with $L(\phi) = 0$. In other words, ϕ solves the system (3.23) with $h = 0$ for some $c_1, c_2 \in \mathbb{R}$. Proposition 3.2 implies $\phi \equiv 0$. Then, Fredholm's alternative implies the existence and uniqueness result.

Once we have existence, the norm estimate follows directly from Proposition 3.2. □

Now we come back to our goal of finding a solution to problem (5.1). In what follows we denote by $N : H^1_{0,\star} \to K^\perp$ the nonlinear operator

$$N(\phi) = \Pi^\perp\Big(i_p^*\big(\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}(e^\phi - 1 - \phi)\big)\Big)$$

Therefore problem (5.1) turns out to be equivalent to the problem

$$L(\phi) + N(\phi) = \tilde{R}, \quad \phi \in K^\perp \tag{5.5}$$

where, recalling Lemma 4.1,

$$\tilde{R} = \Pi^\perp\big(i_p^*(R_\lambda)\big) = \Pi^\perp\Big(PW_\lambda - i_p^*\big(\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\big)\Big).$$

We need the following auxiliary lemma.

Lemma 5.2. *For any $p > 1$ and any $\phi_1, \phi_2 \in H^1_{0,\star}(\Omega)$ with $\|\phi\|_1$, $\|\phi_2\| < 1$ the following holds*

$$\|e^{\phi_1} - \phi_1 - e^{\phi_2} + \phi_2\|_p \le C(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\|, \tag{5.6}$$

$$\|N(\phi_1) - N(\phi_2)\| \le C\delta^{-2\alpha\frac{p^2-1}{p^2}}(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\| \tag{5.7}$$

uniformly for b in a small neighborhood of 0.

Proof. A straightforward computation gives that the inequality $|e^a - a - e^b + b| \le e^{|a|+|b|}(|a|+|b|)|a-b|$ holds for all $a, b \in \mathbb{R}$. Then, by applying Hölder's inequality with $\frac{1}{q} + \frac{1}{r} + \frac{1}{t} = 1$, we derive

$$\|e^{\phi_1} - \phi_1 - e^{\phi_2} + \phi_2\|_p \le C\|e^{|\phi_1|+|\phi_2|}\|_{pq}(\|\phi_1\|_{pr} + \|\phi_2\|_{pr})\|\phi_1 - \phi_2\|_{pt}$$

and (5.6) follows by using Lemma 2.3 and the continuity of the embeddings $H^1_0(\Omega) \subset L^{pr}(\Omega)$ and $H^1_0(\Omega) \subset L^{pt}(\Omega)$. Let us prove (5.7). According to (5.3) we get

$$\|N(\phi_1) - N(\phi_2)\| \le C\|\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}(e^{\phi_1} - \phi_1 - e^{\phi_2} + \phi_2)\|_p,$$

and by Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$ we derive

$$\begin{aligned}\|N(\phi_1)-N(\phi_2)\| &\le C\|\lambda|x|^{2(\alpha_\lambda-1)}V(x)e^{PW_\lambda}\|_{p^2}\|e^{\phi_1}-\phi_1-e^{\phi_2}+\phi_2|\|_{pq}\\ &\le C\|\lambda|x|^{2(\alpha_\lambda-1)}V(x)e^{PW_\lambda}\|_{p^2}(\|\phi_1\|+\|\phi_2\|)\|\phi_1-\phi_2\|\end{aligned}$$

by (5.6), and the conclusion follows recalling (4.5). □

Problem (5.1) or, equivalently, problem (5.5) turns out to be solvable for any choice of point b in a small neighborhood of 0, provided that λ is sufficiently small. Indeed we have the following result.

Proposition 5.3. *Let $\varepsilon>0$ be a fixed small number. Then there exists $\lambda_0>0$ such that for any $\lambda\in(0,\lambda_0)$ and any $b\in\mathbb{R}^2$ in a small neighborhood of 0 there is a unique $\phi_\lambda=\phi_{\lambda,b}\in K^\perp$ satisfying (5.1) for some $c_1,c_2\in\mathbb{R}$ and*

$$\|\phi_\lambda\|\le\delta^{-\varepsilon}\Big(\delta^{2\alpha}+\delta^\alpha|b|+|\alpha_\lambda-\alpha|\Big)\le C\delta^{-\varepsilon}\Big(\delta^{2\alpha}+\delta^\alpha|b|+\delta^{2\alpha\eta}\Big).$$

Moreover the map $b\mapsto\phi_{\lambda,b}\in H^1_{0,\star}(\Omega)$ is C^1.

Proof. Since problem (5.5) is equivalent to problem (5.1), we will show that problem (5.5) can be solved via a contraction mapping argument. Indeed, in virtue of Proposition 5.1, let us introduce the map

$$T:=L^{-1}(\tilde R-N(\phi)),\quad \phi\in K^\perp.$$

Let us fix $p>1$ sufficiently close to 1. According to (5.3) and Proposition 4.2 we have

$$\|\tilde R\|\le C\delta^{-\frac{\varepsilon}{2}}\Big(\delta^{2\alpha}+\delta^\alpha|b|+|\alpha_\lambda-\alpha|\Big). \tag{5.8}$$

Similarly, by (5.7),

$$\|N(\phi_1)-N(\phi_2)\|\le C\delta^{-\frac{\varepsilon}{2}}(\|\phi_1\|+\|\phi_2\|)\|\phi_1-\phi_2\|\quad\forall\phi_1,\phi_2\in H^1_{0,\star}(\Omega),\|\phi_1\|,\|\phi_2\|<1. \tag{5.9}$$

In particular, by taking $\phi_2=0$,

$$\|N(\phi)\|\le C\delta^{-\frac{\varepsilon}{2}}\|\phi\|^2\quad\forall\phi\in H^1_{0,\star}(\Omega),\|\phi\|<1. \tag{5.10}$$

We claim that T is a contraction map over the ball

$$\mathcal{B}:=\Big\{\phi\in K^\perp\,\Big|\,\|\phi\|\le\delta^{-\varepsilon}\Big(\delta^{2\alpha}+\delta^\alpha|b|+|\alpha_\lambda-\alpha|\Big)\Big\}$$

provided that λ is small enough. Indeed, combining Proposition 5.1, (5.8), (5.9), (5.10), for any $\phi\in\mathcal{B}$ we have

$$\begin{aligned}\|T(\phi)\|&\le C|\log\delta|\big(\|\tilde R\|+\|N(\phi)\|\big)\le C|\log\delta|\delta^{-\frac{\varepsilon}{2}}\big(\delta^{2\alpha}+\delta^\alpha|b|+|\alpha_\lambda-\alpha|+\|\phi\|^2\big)\\ &<\delta^{-\varepsilon}\big(\delta^{2\alpha}+\delta^\alpha|b|+|\alpha_\lambda-\alpha|\big),\end{aligned}$$

provided that $\varepsilon<\frac{\alpha}{2}$ and $\varepsilon<\alpha\eta$ (see assumption (A3)*). Similarly, for any $\phi_1,\phi_2\in\mathcal{B}$

$$\|T(\phi_1)-T(\phi_2)\|\le C|\log\delta|\|N(\phi_1)-N(\phi_2)\|\le C\delta^{-\frac{\varepsilon}{2}}|\log\delta|(\|\phi_1\|+\|\phi_2\|)\|\phi_1-\phi_2\|\le\frac{1}{2}\|\phi_1-\phi_2\|.$$

We now consider the dependence of $\phi_{\lambda,b}$ on b. In order to prove that the map $b\to\phi_{\lambda,b}$ is C^1, we apply the Implicit Function Theorem to the function

$$\Phi(b,\phi)=\phi+\Pi^\perp\Big(PW_\lambda-i_p^*\big(\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda+\Pi^\perp\phi}\big)\Big),\ \phi\in H^1_{0,\star}(\Omega).$$

Indeed $\Phi(b,\phi_{\lambda,b})=0$ and the linear operator: $\frac{\partial\Phi}{\partial\phi}(b,\phi_{\lambda,b}):H^1_{0,\star}(\Omega)\to H^1_{0,\star}(\Omega)$ is given by

$$\frac{\partial\Phi}{\partial\phi}(b,\phi_{\lambda,b})(\psi)=\psi-\Pi^{\perp}\left(i_p^*\big(\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda+\phi_{\lambda,b}}\Pi^{\perp}\psi\big)\right).$$

We observe that $\frac{\partial\Phi}{\partial\phi}(b,\phi_{\lambda,b})$ is a Fredholm's operator. By comparing $\frac{\partial\Phi}{\partial\phi}(b,\phi_{\lambda,b})$ with the definition of L in (5.4), we have that

$$\frac{\partial\Phi}{\partial\phi}(b,\phi_{\lambda,b})(\psi)=\Pi(\psi)-L(\Pi^{\perp}\psi)-\Pi^{\perp}\left(i_p^*\big(\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}(e^{\phi_{\lambda,b}}-1)\Pi^{\perp}\psi\big)\right)$$

by which, using Proposition 5.1,

$$\begin{aligned}\left\|\frac{\partial\Phi}{\partial\phi}(b,\phi_{\lambda,b})(\psi)\right\|&\geq\sqrt{\|\Pi(\psi)\|^2+\|L(\Pi^{\perp}\psi)\|^2}-\left\|\Pi^{\perp}\left(i_p^*\big(\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}(e^{\phi_{\lambda,b}}-1)\Pi^{\perp}\psi\big)\right)\right\|\\&\geq\frac{c}{|\log\delta|}\|\psi\|-c\|\lambda|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}(e^{\phi_{\lambda,b}}-1)\Pi^{\perp}\psi\|_p.\end{aligned}\tag{5.11}$$

Now, using Hölder's inequality with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$ we compute

$$\|\lambda|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}(e^{\phi_{\lambda,b}}-1)\Pi^{\perp}\psi\|_p\leq\|\lambda|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\|_{p^2}\|e^{\phi_{\lambda,b}}-1\|_{pq}\|\psi\|_{pr}.$$

Observe that $\|e^{\phi_{\lambda,b}}-1\|_{pq}\leq\|e^{\phi_{\lambda,b}}-1-\phi_{\lambda,b}\|_{pq}+\|\phi_{\lambda,b}\|_{pq}\leq C\|\phi_{\lambda,b}\|\leq C(\delta^{\frac{\alpha}{2}}+\delta^{\alpha\eta})$ by Lemma 5.2. Hence, using (4.5) we get

$$\|\lambda|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}(e^{\phi_{\lambda,b}}-1)\Pi^{\perp}\psi\|_p\leq C(\delta^{\frac{\alpha}{2}}+\delta^{\alpha\eta})\delta^{-2\alpha\frac{p^2-1}{p^2}}\|\psi\|\leq C(\delta^{\frac{\alpha}{4}}+\delta^{\frac{\alpha\eta}{2}})\|\psi\|.$$

Inserting this into (5.11), we conclude that

$$\left\|\frac{\partial\Phi}{\partial\phi}(b,\phi_{\lambda,b})(\psi)\right\|\geq\frac{c}{|\log\delta|}\|\psi\|$$

which guarantees the invertibility of the operator $\frac{\partial\Phi}{\partial\phi}(b,\phi_{\lambda,b})$.

□

6. The finite dimensional reduction

After problem (5.1) has been solved according to Proposition 5.3, then we find a solution to the original problem (2.2) if $b\in\mathbb{R}^2$ is such that

$$c_j=0\text{ for }j=1,2.$$

Let us find the condition on b in order to get the c_j's equal to zero. This problem is actually variational; more precisely, it is equivalent to find a critical point of a function of b.

Indeed, let us consider the following energy functional associated with (2.2):

$$I_\lambda(v)=\frac{1}{2}\int_\Omega|\nabla v|^2dx-\lambda\int_\Omega V(x)|x|^{2(\alpha_\lambda-1)}e^v\,dx,\quad v\in H^1_{0,\star}(\Omega).\tag{6.1}$$

Solutions of (2.2) correspond to critical points of I_λ. Now we introduce the new functional

$$J_\lambda(b)=I_\lambda(PW_\lambda+\phi_\lambda)\tag{6.2}$$

defined in a small neighborhood of 0, where $\phi_\lambda=\phi_{\lambda,b}$ has been constructed in Proposition 5.3. The next proposition reduces the problem (2.2) to the one of finding critical points of the functional J_λ.

Proposition 6.1. *If b in a small neighborhood of 0 is a critical point of J_λ, then the corresponding function $v_\lambda=PW_\lambda+\phi_\lambda$ is a solution of (2.2).*

Proof. Let b be a critical point of J_λ:

$$\frac{\partial J_\lambda}{\partial b_1}(b) = \frac{\partial J_\lambda}{\partial b_2}(b) = 0. \tag{6.3}$$

Using Proposition 5.3 we can differentiate directly $I_\lambda(PW_\lambda + \phi_\lambda)$ under the integral sign, so that

$$\int_\Omega \nabla(PW_\lambda + \phi_\lambda)\nabla\frac{\partial(PW_\lambda + \phi_\lambda)}{\partial b_i}dx - \lambda\int_\Omega V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda+\phi_\lambda}\frac{\partial(PW_\lambda + \phi_\lambda)}{\partial b_i} = 0, \ \ i = 1, 2.$$

Taking into account that ϕ_λ solves problem (5.1), this is equivalent to

$$\sum_{j=1,2} c_j \int_\Omega Z_\lambda^j |x|^{2(\alpha-1)}e^{W_\lambda}\frac{\partial(PW_\lambda + \phi_\lambda)}{\partial b_i}dx = 0, \quad i = 1, 2. \tag{6.4}$$

Let us fix $p > 1$ sufficiently close to 1. Next let $1 < q < +\infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. It is easily checked that $|\frac{\partial(Z_\lambda^j|x|^{2(\alpha-1)}e^{W_\lambda})}{\partial b_i}| \le C\delta^{-\alpha}|x|^{2(\alpha-1)}e^{W_\lambda}$, so by Lemma 4.1

$$\left\|\frac{\partial(Z_\lambda^j|x|^{2(\alpha-1)}e^{W_\lambda})}{\partial b_i}\right\|_p = O(\delta^{-\alpha-2\alpha\frac{p-1}{p}}).$$

Then, since $\int_\Omega \phi_\lambda Z_\lambda^j|x|^{2(\alpha-1)}e^{W_\lambda}dx = 0$, in view of Proposition 5.3 by differentiating we get

$$\int_\Omega \frac{\partial\phi_\lambda}{\partial b_i}Z_\lambda^j|x|^{2(\alpha-1)}e^{W_\lambda}dx = -\int_\Omega \phi_\lambda\frac{\partial(Z_\lambda^j|x|^{2(\alpha-1)}e^{W_\lambda})}{\partial b_i}dx = O(\delta^{-\alpha-2\alpha\frac{p-1}{p}}\|\phi_\lambda\|_q) = o(\delta^{-\alpha})$$

provided that p is chosen sufficiently close to 1.

Observe that $\frac{\partial W_\lambda}{\partial b_i} = 4\delta^{-\alpha}Z_\lambda^i + O(1)$, by which $\frac{\partial PW_\lambda}{\partial b_i} = 4\delta^{-\alpha}PZ_\lambda^i + O(1)$, therefore the system (6.4) can be rewritten as

$$\sum_{j=1,2} c_j \int_\Omega Z_\lambda^j|x|^{2(\alpha-1)}e^{W_\lambda}PZ_\lambda^i dx + o(c_j) = 0, \quad i = 1, 2. \tag{6.5}$$

By (3.24)-(3.25) we deduce that the system (6.5) is diagonal dominant and then we achieve $c_1 = c_2 = 0$. □

Next purpose of this section is to provide an asymptotic expansion of the energy $I_\lambda(PW_\lambda)$, where I_λ is the energy functional in (6.1).

Proposition 6.2. *The following asymptotic expansions hold:*

$$\begin{aligned} I_\lambda(PW_\lambda) = &-8\pi\alpha(2 + \log\lambda - \log(8\alpha^2)) - 32\pi^2\alpha\mathcal{H}_\alpha(b, b) + 32\pi^2(\alpha_\lambda - 1)\mathcal{H}_\alpha(b, 0) \\ &+ O(\delta^\alpha|b|) + O(\delta^{2\alpha}|\log\delta|) + o(\delta^{\alpha\eta}) \end{aligned}$$

uniformly for b in a small neighborhood of 0, and

$$\begin{aligned} I_\lambda(PW_\lambda) = &-8\pi\alpha(1 + \log\lambda - \log(8\alpha^2)) - 32\pi^2\alpha\mathcal{H}_\alpha(b, b) + 32\pi^2(\alpha_\lambda - 1)\mathcal{H}_\alpha(b, 0) - 8\pi\alpha|b|^{2\frac{\alpha_\lambda-\alpha}{\alpha}} \\ &+ O(\delta^\alpha|b|) + O(\delta^{2\alpha}|\log\delta|) + |\alpha_\lambda - \alpha|O\left(\frac{\delta^\alpha}{|b|} + \frac{\delta^{2\alpha}|\log\delta|}{|b|^2} + \frac{\delta^{2\alpha}|\log|b||}{|b|^2}\right) \end{aligned}$$

uniformly for b in a small neighborhood of 0 with $|b| \ge |\alpha_\lambda - \alpha|$.

Proof. First by Lemma C.1 we compute

$$\begin{aligned}
&\int_\Omega |x|^{2(\alpha-1)}e^{W_\lambda}\log(\delta^{2\alpha}+|x^\alpha-b|^2)dx\\
&=8\alpha^2\int_{\frac{\Omega}{\delta}}\frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha-\delta^{-\alpha}b|^2)^2}\Big(\log(\delta^{2\alpha})+\log(1+|y^\alpha-\delta^{-\alpha}b|^2)\Big)dy\\
&=8\alpha\int_{\frac{\Omega^\alpha}{\delta^\alpha}}\frac{\log(\delta^{2\alpha})+\log(1+|y-\delta^{-\alpha}b|^2)}{(1+|y-\delta^{-\alpha}b|^2)^2}dy\\
&=8\alpha\int_{\mathbb{R}^2}\frac{\log(\delta^{2\alpha})+\log(1+|y|^2)}{(1+|y|^2)^2}dy+O(\delta^{2\alpha}|\log\delta|)\\
&=8\pi\alpha(1+\log(\delta^{2\alpha}))+O(\delta^{2\alpha}|\log\delta|)
\end{aligned}$$

since $\int_{\mathbb{R}^2}\frac{1}{(1+|x|^2)^2}=\int_{\mathbb{R}^2}\frac{\log(1+|y|^2)}{(1+|x|^2)^2}=\pi$. On the other hand, by (B.2), recalling (2.6) and Lemma A.2, we get

$$\begin{aligned}
\int_\Omega |x|^{2(\alpha-1)}e^{W_\lambda}\mathcal{H}(x^\alpha,b)dx&=\mathcal{H}_\alpha(b,b)\int_\Omega|x|^{2(\alpha-1)}e^{W_\lambda}dx+\int_\Omega|x|^{2(\alpha-1)}e^{W_\lambda}O(|b||x^\alpha-b|)dx\\
&\quad+\int_\Omega|x|^{2(\alpha-1)}e^{W_\lambda}O(|x^\alpha-b|^2)dx\\
&=8\pi\alpha\mathcal{H}_\alpha(b,b)+O(\delta^\alpha|b|)+O(\delta^{2\alpha}|\log\delta|).
\end{aligned}$$

So by (2.8), combining the above computations, we can write:

$$\begin{aligned}
&\frac12\int_\Omega|\nabla PW_\lambda|^2dx\\
&=\frac12\int_\Omega|x|^{2(\alpha-1)}e^{W_\lambda}PW_\lambda dx\\
&=-\int_\Omega|x|^{2(\alpha-1)}e^{W_\lambda}\log(\delta^{2\alpha}+|x^\alpha-b|^2)dx+4\pi\int_\Omega|x|^{2(\alpha-1)}e^{W_\lambda}\mathcal{H}(x^\alpha,b)dx+O(\delta^{2\alpha}) \qquad (6.6)\\
&=-8\alpha\pi(1+\log(\delta^{2\alpha}))+32\pi^2\alpha\mathcal{H}_\alpha(b,b)+O(\delta^\alpha|b|)+O(\delta^{2\alpha}|\log\delta|)\\
&=-8\pi\alpha(1+\log\lambda-\log(8\alpha^2))-32\pi^2\alpha\mathcal{H}_\alpha(b,b)+32\pi^2(\alpha_\lambda-1)\mathcal{H}_\alpha(b,0)\\
&\quad+O(\delta^\alpha|b|)+O(\delta^{2\alpha}|\log\delta|)
\end{aligned}$$

by (2.7). Now, the expansion in (4.7) gives

$$\begin{aligned}
&\lambda\int_\Omega V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}dx\\
&=\int_\Omega|x|^{2(\alpha_\lambda-1)}e^{W_\lambda}dx+\int_\Omega\Big(O(|b||x^\alpha-b|)+O(|x^\alpha-b|^2)+O(\delta^{2\alpha})\Big)|x|^{2(\alpha_\lambda-1)}e^{W_\lambda}dx \qquad (6.7)\\
&=\int_\Omega|x|^{2(\alpha_\lambda-1)}e^{W_\lambda}dx+O(\delta^\alpha|b|)+O(\delta^{2\alpha}|\log\delta|)
\end{aligned}$$

where we have used Lemma A.2 and Lemma A.3 (with $\varepsilon<\alpha\eta$). Next we are going to estimate the above integral $\int_\Omega|x|^{2(\alpha_\lambda-1)}e^{W_\lambda}dx$:

$$\begin{aligned}
\int_\Omega|x|^{2(\alpha_\lambda-1)}e^{W_\lambda}dx&=\int_\Omega|x|^{2(\alpha-1)}e^{W_\lambda}dx-\int_\Omega(|x|^{2(\alpha-1)}-|x|^{2(\alpha_\lambda-1)})e^{W_\lambda}dx\\
&=8\pi\alpha+O(\delta^{2\alpha})+o(\delta^{\alpha\eta})
\end{aligned}$$

by Lemma A.3 with $\varepsilon < \alpha\eta$. So by inserting this into (6.7)

$$\lambda \int_\Omega V(x)|x|^{2(\alpha_\lambda-1)} e^{PW_\lambda} dx = 8\pi\alpha + O(\delta^\alpha |b|) + O(\delta^{2\alpha}|\log\delta|) + o(\delta^{\alpha\eta}).$$

Combing this with (6.6) we get

$$\begin{aligned} I_\lambda(PW_\lambda) &= \frac{1}{2}\int_\Omega |\nabla PW_\lambda|^2 dx - \lambda \int_\Omega |x|^{2(\alpha-1)} V(x) e^{PW_\lambda}\, dx \\ &= -8\pi\alpha(2+\log\lambda - \log(8\alpha^2)) - 32\pi^2\alpha\mathcal{H}_\alpha(b,b) + 32\pi^2(\alpha_\lambda - 1)\mathcal{H}_\alpha(b,0) \\ &\quad + O(\delta^\alpha|b|) + O(\delta^{2\alpha}|\log\delta|) + o(\delta^{\alpha\eta}) \end{aligned}$$

and the first part of the thesis follows.

Now let us suppose $|b| \geq |\alpha_\lambda - \alpha|$: let us deal with the integral

$$\int_\Omega |x|^{2(\alpha_\lambda-1)} e^{W_\lambda} dx = \int_\Omega |x|^{2(\alpha-1)} |x|^{2(\alpha_\lambda-\alpha)} e^{W_\lambda} dx$$

and observe that

$$\begin{aligned} |b|^{-2\frac{\alpha_\lambda-\alpha}{\alpha}} |x|^{2(\alpha_\lambda-\alpha)} &= |b|^{-2\frac{\alpha_\lambda-\alpha}{\alpha}} \left(|x^\alpha - b + b|^2\right)^{\frac{\alpha_\lambda-\alpha}{\alpha}} \\ &= |b|^{-2\frac{\alpha_\lambda-\alpha}{\alpha}} \left(|x^\alpha - b|^2 + |b|^2 + 2\mathrm{Re}(\bar b\cdot(x^\alpha - b))\right)^{\frac{\alpha_\lambda-\alpha}{\alpha}} \\ &= \left(1 + \frac{|x^\alpha - b|^2}{|b|^2} + 2\frac{\mathrm{Re}(\bar b\cdot(x^\alpha-b))}{|b|^2}\right)^{\frac{\alpha_\lambda-\alpha}{\alpha}}. \end{aligned}$$

Let us notice that for $|x|^\alpha \geq \frac{|b|}{2}$ we have

$$\frac{|x^\alpha - b|^2}{|b|^2} + 2\frac{\mathrm{Re}(\bar b\cdot(x^\alpha-b))}{|b|^2} = \frac{|x|^{2\alpha} - |b|^2}{|b|^2} \geq -\frac{1}{2}.$$

Therefore we use Lemma 6.4 and for $|x|^\alpha \geq \frac{|b|}{2}$ we compute:

$$\begin{aligned} |b|^{-2\frac{\alpha_\lambda-\alpha}{\alpha}} |x|^{2(\alpha_\lambda-\alpha)} &= \left(1 + \frac{|x^\alpha - b|^2}{|b|^2} + 2\frac{\mathrm{Re}(\bar b\cdot(x^\alpha-b))}{|b|^2}\right)^{\frac{\alpha_\lambda-\alpha}{\alpha}} \\ &= 1 + |\alpha_\lambda - \alpha| O\left(\frac{|x^\alpha - b|}{|b|}\right) + |\alpha_\lambda-\alpha| O\left(\frac{|x^\alpha-b|^2}{|b|^2}\right). \end{aligned}$$

So, using Lemma A.2, and recalling that $|b| \geq |\alpha_\lambda - \alpha|$ (so that $|b|^{\alpha_\lambda-\alpha} = O(1)$), we obtain

$$\begin{aligned} &\int_{x\in\Omega,\, |x|^\alpha \geq \frac{|b|}{2}} |x|^{2(\alpha_\lambda-1)} e^{W_\lambda} dx \\ &= |b|^{2\frac{\alpha_\lambda-\alpha}{\alpha}} \int_{x\in\Omega,\, |x|^\alpha\geq\frac{|b|}{2}} |x|^{2(\alpha-1)} e^{W_\lambda} dx + |\alpha_\lambda-\alpha| O\left(\frac{\delta^\alpha}{|b|} + \frac{\delta^{2\alpha}|\log\delta|}{|b|^2}\right) \end{aligned} \tag{6.8}$$

which implies, combining with Lemma A.1,

$$\begin{aligned} \int_\Omega |x|^{2(\alpha_\lambda-1)} e^{W_\lambda} dx &= |b|^{2\frac{\alpha_\lambda-\alpha}{\alpha}} \int_\Omega |x|^{2(\alpha-1)} e^{W_\lambda} dx + |\alpha_\lambda - \alpha| O\left(\frac{\delta^\alpha}{|b|} + \frac{\delta^{2\alpha}|\log\delta|}{|b|^2} + \frac{\delta^{2\alpha}|\log|b||}{|b|^2}\right) \\ &= 8\pi\alpha|b|^{2\frac{\alpha_\lambda-\alpha}{\alpha}} + O(\delta^{2\alpha}) + |\alpha_\lambda-\alpha| O\left(\frac{\delta^\alpha}{|b|} + \frac{\delta^{2\alpha}|\log\delta|}{|b|^2} + \frac{\delta^{2\alpha}|\log|b||}{|b|^2}\right). \end{aligned} \tag{6.9}$$

uniformly for b in a small neighborhood of 0 with $|b| \geq |\alpha_\lambda - \alpha|$.

By inserting (6.9) into (6.7), we arrive at

$$\lambda\int_\Omega V(x)|x|^{2(\alpha_\lambda-1)}e^{W_\lambda}=8\pi\alpha|b|^{2\frac{\alpha_\lambda-\alpha}{\alpha}}+O(\delta^\alpha|b|)+O(\delta^{2\alpha}|\log\delta|)$$
$$+|\alpha_\lambda-\alpha|O\Big(\frac{\delta^\alpha}{|b|}+\frac{\delta^{2\alpha}|\log\delta|}{|b|^2}+\frac{\delta^{2\alpha}|\log|b||}{|b|^2}\Big) \tag{6.10}$$

uniformly for b in a small neighborhood of 0, $|b|\geq|\alpha_\lambda-\alpha|$. Finally combining (6.10) with (6.6) we conclude the proof of the second part of the thesis.

□

Finally we describe an expansion for the functional J_λ defined in (6.2); a key step is its expected closeness to the functional $I_\lambda(W_\lambda)$ analyzed in the previous proposition.

Proposition 6.3. *The following expansion holds:*

$$J_\lambda(b)=I_\lambda(PW_\lambda)+O(\delta^{3\alpha})+O(\delta^\alpha|b|^2)+O(|\alpha_\lambda-\alpha|)$$

uniformly for b in a small neighborhood of 0.

Proof. We compute:

$$J_\lambda(b)=I_\lambda(PW_\lambda+\phi_\lambda)=\frac{1}{2}\int_\Omega|\nabla(PW_\lambda+\phi_\lambda)|^2-\lambda\int_\Omega V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda+\phi_\lambda}dx$$
$$=I_\lambda(PW_\lambda)+\frac{1}{2}\int_\Omega|\nabla\phi_\lambda|^2dx+\int_\Omega\nabla\phi_\lambda\nabla PW_\lambda dx-\lambda\int_\Omega V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}(e^{\phi_\lambda}-1)dx$$
$$=I_\lambda(PW_\lambda)+\frac{1}{2}\int_\Omega|\nabla\phi_\lambda|^2dx+\int_\Omega R_\lambda\phi_\lambda dx-\lambda\int_\Omega V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}(e^{\phi_\lambda}-1-\phi_\lambda)dx \tag{6.11}$$

where R_λ is the error term defined in Proposition 4.2. Let us fix $\varepsilon>0$ sufficiently small and $p>1$ sufficiently close to 1. Next let $1<q<\infty$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Then, by Proposition 4.2 and Proposition 5.3

$$\int_\Omega|\nabla\phi_\lambda|^2dx=\|\phi_\lambda\|^2\leq C\delta^{-2\varepsilon}(\delta^{4\alpha}+\delta^{2\alpha}|b|^2+|\alpha_\lambda-\alpha|^2). \tag{6.12}$$

$$\int_\Omega|R_\lambda\phi_\lambda|dx\leq\|R_\lambda\|_p\|\phi_\lambda\|_q\leq C\|R_\lambda\|_p\|\phi_\lambda\|\leq C\delta^{-\varepsilon-2\alpha\frac{p-1}{p}}(\delta^{4\alpha}+\delta^{2\alpha}|b|^2+|\alpha_\lambda-\alpha|^2). \tag{6.13}$$

Then, (5.6) with $\phi_2=0$ gives

$$\|e^{\phi_\lambda}-1-\phi_\lambda\|_q\leq C\|\phi_\lambda\|^2\leq C\delta^{-2\varepsilon}(\delta^{4\alpha}+\delta^{2\alpha}|b|^2+|\alpha_\lambda-\alpha|^2). \tag{6.14}$$

and, consequently,

$$\|e^{\phi_\lambda}-1\|_q\leq C\|\phi_\lambda\|\leq C\delta^{-\varepsilon}(\delta^{2\alpha}+\delta^\alpha|b|+|\alpha_\lambda-\alpha|). \tag{6.15}$$

Therefore, (4.5) implies

$$\lambda\int_\Omega V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\big|e^{\phi_\lambda}-1-\phi_\lambda\big|dx=O(\|\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda}\|_p\|e^{\phi_\lambda}-1-\phi_\lambda\|_q)$$
$$=O\big(\delta^{-2\alpha\frac{p-1}{p}}\delta^{-2\varepsilon}(\delta^{4\alpha}+\delta^{2\alpha}|b|^2+|\alpha_\lambda-\alpha|^2)\big). \tag{6.16}$$

The thesis follows by inserting (6.12), (6.13) and (6.16) into (6.11).

□

Lemma 6.4. *The following holds:*

$$(1+t)^{\frac{\alpha_\lambda-\alpha}{\alpha}} = 1 + O(|\alpha_\lambda - \alpha||t|)$$

uniformly for $t \geq -\frac{1}{2}$.

Proof. We use the Taylor expansion: for every $t > -1$

$$(1+t)^{\frac{\alpha_\lambda-\alpha}{\alpha}} = 1 + f_\lambda(t)$$

where, according to the Lagrange reminder,

$$f_\lambda(t) = \frac{\alpha_\lambda - \alpha}{\alpha}(1+\xi)^{\frac{\alpha_\lambda-\alpha}{\alpha}-1}t \qquad \xi \in (0,t).$$

Therefore, if $t \geq -\frac{1}{2}$, we get $|(1+\xi)^{\frac{\alpha_\lambda-\alpha}{\alpha}-1}| \leq 2^{1-\frac{\alpha_\lambda-\alpha}{\alpha}}$ and the thesis follows. □

7. Proof of Theorem 1.1 and Theorem (1.2)

According to Proposition 6.1, we find a solution to the original problem (2.2) if the functional J_λ has a critical point $b \in \mathbb{R}^2$ in a small neighborhood of 0. More precisely, in Theorem 2.1 and Theorem 2.2 we will find two families of solutions associated to local minima of J_λ. To this aim let us study the following two minimization problems.

Proposition 7.1. *The functional* J_λ *admits a minimum in the ball*

$$\{b \in \mathbb{R}^2 \mid |b| < m_\lambda\},$$

where

$$m_\lambda := \max\{\delta^{\frac{\alpha}{2}}, \delta^{\frac{\alpha\eta}{2}}\}.$$

Proof. Combining Proposition 6.2 with Proposition 6.3, and taking into account of the definition of m_λ, we have

$$\begin{aligned} J_\lambda(b) &= -8\pi\alpha(2 + \log\lambda - \log(8\alpha^2)) - 32\pi^2\alpha\mathcal{H}_\alpha(b,b) + 32\pi^2(\alpha_\lambda - 1)\mathcal{H}_\alpha(b,0) \\ &\quad + O(\delta^\alpha|b|) + O(\delta^{2\alpha}\log\delta) + o(\delta^{\alpha\eta}) \\ &= -8\pi\alpha(2 + \log\lambda - \log(8\alpha^2)) - 32\pi^2\alpha\mathcal{H}_\alpha(b,b) + 32\pi^2(\alpha - 1)\mathcal{H}_\alpha(b,0) + o(m_\lambda^2) \end{aligned}$$

uniformly in the ball $|b| < m_\lambda$. So we get

$$J_\lambda(0) = -8\pi\alpha(2 + \log\lambda - \log(8\alpha^2)) - 32\pi^2\mathcal{H}_\alpha(0,0) + o(m_\lambda^2). \tag{7.1}$$

Now, using hypothesis (A2)*, $\alpha\mathcal{H}_\alpha(b,b) - (\alpha-1)\mathcal{H}_\alpha(b,0) \leq \mathcal{H}_\alpha(0,0) - c|b|^2$ in a neighborhood of 0, which gives

$$J_\lambda(b) \geq -8\pi\alpha(2 + \log\lambda - \log(8\alpha^2)) - 32\pi^2\mathcal{H}_\alpha(0,0) + cm_\lambda^2 + o(m_\lambda^2) \quad \text{if } |b| = m_\lambda.$$

Combining this with (7.1) we deduce

$$\inf_{|b|<m_\lambda} J_\lambda(b) < \inf_{|b|=m_\lambda} J_\lambda(b)$$

and the thesis follows. □

Proposition 7.2. *Assume that*

$$\alpha_\lambda - \alpha \geq c\delta^{2\alpha}\log^2\delta \tag{7.2}$$

for some $c > 0$. *Then, setting*

$$m_{1,\lambda} := \frac{\sqrt{\alpha_\lambda - \alpha}}{|\log(\alpha_\lambda - \alpha)|^{1/3}}, \qquad m_{2,\lambda} := \sqrt{\alpha_\lambda - \alpha}\,|\log(\alpha_\lambda - \alpha)|^{1/3},$$

the functional J_λ admits a minimum in the annulus

$$\{b\,|\,m_{1,\lambda}\le |b|\le m_{2,\lambda}\}.$$

Proof. Combining the expansion in Proposition 6.2 with Proposition 6.3, we have

$$\begin{aligned}J_\lambda(b)=&-8\pi\alpha(1+\log\lambda-\log(8\alpha^2))-32\pi^2\alpha\mathcal{H}_\alpha(b,b)+32\pi^2(\alpha_\lambda-1)\mathcal{H}_\alpha(b,0)-8\pi\alpha|b|^{2\frac{\alpha_\lambda-\alpha}{\alpha}}\\&+O(\delta^\alpha|b|)+O(\delta^{2\alpha}|\log\delta|)+|\alpha_\lambda-\alpha|O\bigg(1+\frac{\delta^\alpha}{|b|}+\frac{\delta^{2\alpha}|\log\delta|}{|b|^2}+\frac{\delta^{2\alpha}|\log|b||}{|b|^2}\bigg)\end{aligned}$$

uniformly in the annulus $\{m_{1,\lambda}\le |b|\le m_{2,\lambda}\}$. Observe that the definition of $m_{1,\lambda}$, $m_{2\lambda}$ implies that the following expansion holds:

$$|b|^{2\frac{\alpha_\lambda-\alpha}{\alpha}}=1+2\frac{\alpha_\lambda-\alpha}{\alpha}\log|b|+O(|\alpha-\alpha_\lambda|^2|\log^2|b|)=1+2\frac{\alpha_\lambda-\alpha}{\alpha}\log|b|+o(\alpha_\lambda-\alpha)$$

uniformly in the annulus $\{b\,|\,m_{1,\lambda}\le |b|\le m_{2,\lambda}\}$. Moreover, recalling (7.2),

$$\delta^\alpha m_{2,\lambda}=o(\alpha_\lambda-\alpha),\qquad \delta^{2\alpha}|\log\delta|=o(\alpha_\lambda-\alpha),$$

and $m_{1,\lambda}\ge c\delta^\alpha|\log\delta|^{2/3}$, which implies

$$\frac{\delta^\alpha}{m_{1,\lambda}}+\frac{\delta^{2\alpha}|\log\delta|}{m_{1,\lambda}^2}+\frac{\delta^{2\alpha}|\log m_{1,\lambda}|}{m_{1,\lambda}^2}=o(1).$$

Therefore the above expression of $J_\lambda(b)$ can be rewritten as

$$\begin{aligned}J_\lambda(b)=&-8\pi\alpha(2+\log\lambda-\log(8\alpha^2))-32\pi^2\alpha\mathcal{H}_\alpha(b,b)+32\pi^2(\alpha-1)\mathcal{H}_\alpha(b,0)\\&-16\pi(\alpha_\lambda-\alpha)\log|b|+O(\alpha_\lambda-\alpha)\end{aligned}$$

uniformly in the annulus $\{m_{1,\lambda}\le |b|\le m_{2,\lambda}\}$. Let b_λ be such that $|b_\lambda|=\sqrt{\alpha_\lambda-\alpha}$. Then b_λ belongs to the annulus $\{m_{1,\lambda}\le |b|\le m_{2,\lambda}\}$ and the evaluation of J_λ at b_λ gives

$$\begin{aligned}J_\lambda(b_\lambda)=&-8\pi\alpha(2+\log\lambda-\log(8\alpha^2))-32\pi^2\alpha\mathcal{H}_\alpha(b_\lambda,b_\lambda)+32\pi^2(\alpha-1)\mathcal{H}_\alpha(b_\lambda,0)\\&-8\pi(\alpha_\lambda-\alpha)\log(\alpha_\lambda-\alpha)+O(\alpha_\lambda-\alpha)\\=&-8\pi\alpha(2+\log\lambda-\log(8\alpha^2))-32\pi^2\mathcal{H}_\alpha(0,0)-8\pi(\alpha_\lambda-\alpha)\log(\alpha_\lambda-\alpha)+O(\alpha_\lambda-\alpha)\end{aligned}$$

where we have used that $|\alpha\mathcal{H}_\alpha(b,b)-(\alpha-1)\mathcal{H}_\alpha(b,0)-\mathcal{H}_\alpha(0,0)|\le C|b|^2$ in a neighborhood of 0 thanks to assumption (A2)*. On the other hand, if $|b|=m_{2,\lambda}:=\sqrt{\alpha_\lambda-\alpha}\,|\log(\alpha_\lambda-\alpha)|^{1/3}$, using now that $\alpha\mathcal{H}_\alpha(b,b)-(\alpha-1)\mathcal{H}_\alpha(b,0)\le\mathcal{H}_\alpha(0,0)-c|b|^2=\mathcal{H}_\alpha(0,0)-c(\alpha_\lambda-\alpha)|\log(\alpha_\lambda-\alpha)|^{2/3}$ again by assumption (A2)*, we compute

$$\begin{aligned}J_\lambda(b)=&-8\pi\alpha(2+\log\lambda-\log(8\alpha^2))-32\pi^2\alpha\mathcal{H}_\alpha(b,b)+32\pi^2(\alpha-1)\mathcal{H}_\alpha(b,0)\\&-8\pi(\alpha_\lambda-\alpha)\log(\alpha_\lambda-\alpha)+O((\alpha_\lambda-\alpha)\log|\log(\alpha_\lambda-\alpha)|)\\\ge&-8\pi\alpha(1+\log\lambda-\log(8\alpha^2))-32\pi^2\mathcal{H}_\alpha(0,0)-8\pi(\alpha_\lambda-\alpha)\log(\alpha_\lambda-\alpha)\\&+c(\alpha_\lambda-\alpha)|\log(\alpha_\lambda-\alpha)|^{2/3}+O((\alpha_\lambda-\alpha)\log|\log(\alpha_\lambda-\alpha)|).\end{aligned}\tag{7.3}$$

Similarly, if $|b|=m_{1,\lambda}:=\frac{\sqrt{\alpha_\lambda-\alpha}}{|\log(\alpha_\lambda-\alpha)|^{1/3}}$ we have

$$\begin{aligned} J_\lambda(b) &= -8\pi\alpha(2+\log\lambda-\log(8\alpha^2)) - 32\pi^2\alpha\mathcal{H}_\alpha(b,b) + 32\pi^2(\alpha-1)\mathcal{H}_\alpha(b,0) \\ &\quad - 8\pi(\alpha_\lambda-\alpha)\log(\alpha_\lambda-\alpha) + \frac{16}{3}\pi(\alpha_\lambda-\alpha)\log|\log(\alpha_\lambda-\alpha)| + O(\alpha_\lambda-\alpha) \\ &\geq -8\pi\alpha(2+\log\lambda-\log(8\alpha^2)) - 32\pi^2\mathcal{H}_\alpha(0,0) - 8\pi(\alpha_\lambda-\alpha)\log(\alpha_\lambda-\alpha) \\ &\quad + \frac{16}{3}\pi(\alpha_\lambda-\alpha)\log|\log(\alpha_\lambda-\alpha)| + O(\alpha_\lambda-\alpha). \end{aligned} \tag{7.4}$$

Combining (7.3)-(7.4) we deduce

$$\inf_{|b|=m_{2,\lambda}} J_\lambda(b) \geq J_\lambda(b_\lambda) + c(\alpha_\lambda-\alpha)|\log(\alpha_\lambda-\alpha)|^{2/3},$$

$$\inf_{|b|=m_{1,\lambda}} J_\lambda(b) \geq J_\lambda(b_\lambda) + c(\alpha_\lambda-\alpha)\log|\log(\alpha_\lambda-\alpha)|,$$

and, recalling that $\alpha_\lambda > \alpha$, the thesis follows.

□

7.1. **Proof of Theorems 2.1-2.2 and 1.1-1.2.** Combining Proposition 5.3 and Proposition 6.1 with Propositions 7.1, under the assumptions of Theorem 2.1 for λ sufficiently small we provide a solution to the problem (2.2) of the form $v_\lambda = PW_\lambda + \phi_\lambda$ for some $b = b_\lambda$ with $\|\phi_\lambda\| = o(1)$ and

$$|b_\lambda| \leq \max\{\delta^{\frac{\alpha}{2}}, \delta^{\frac{\alpha\eta}{2}}\}.$$

Similarly, combining Proposition 5.3 and Proposition 6.1 with Propositions 7.2, under the assumptions of Theorem 2.2 for λ sufficiently small we get a solution to the problem (2.2) of the form $v_\lambda = PW_\lambda + \phi_\lambda$ for some $b = b_\lambda$ with $\|\phi_\lambda\| = o(1)$ and

$$c\delta^\alpha|\log\delta|^{2/3} \leq m_{1,\lambda} \leq |b_\lambda| \leq m_{2,\lambda} \leq C\delta^{\alpha\eta}|\log\delta|^{1/3}. \tag{7.5}$$

Theorems 2.1-2.2 are thus proved. Clearly, by (2.1),

$$u_\lambda = v_\lambda - 4\pi(\alpha_\lambda-1)G(x,0)$$

solves equation (1.6). Moreover, using (4.5) and (6.15), by Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$ we get

$$\begin{aligned} \lambda\||x|^{2(\alpha_\lambda-1)}V(x)(e^{v_\lambda}-e^{PW_\lambda})\|_1 &= \lambda\||x|^{2(\alpha_\lambda-1)}V(x)e^{PW_\lambda}(e^{\phi_\lambda}-1)\|_1 \\ &\leq \lambda\||x|^{2(\alpha_\lambda-1)}V(x)e^{PW_\lambda}\|_p\|e^{\phi_\lambda}-1\|_q \\ &= O(\delta^{-2\alpha\frac{p-1}{p}-\varepsilon}(\delta^{2\alpha}+\delta^\alpha|b_\lambda|+|\alpha_\lambda-\alpha|)) = o(1), \end{aligned}$$

if p is chosen sufficiently close to 1 and ε sufficiently close to 0. Similarly, by Proposition 4.2,

$$\|\lambda V(x)|x|^{2(\alpha_\lambda-1)}e^{PW_\lambda} - |x|^{2(\alpha-1)}e^{W_\lambda}\|_1 = \|R_\lambda\|_1 = O(\delta^{-2\alpha\frac{p-1}{p}}(\delta^{2\alpha}+\delta^\alpha|b_\lambda|+\delta^{2\alpha\eta})) = o(1).$$

Therefore

$$\|\lambda e^{u_\lambda} - |x|^{2(\alpha-1)}e^{W_\lambda}\|_1 = \|\lambda|x|^{2(\alpha_\lambda-1)}V(x)e^{v_\lambda} - |x|^{2(\alpha-1)}e^{W_\lambda}\|_1 = o(1) \tag{7.6}$$

by which, recalling (2.6),

$$\lambda\int_\Omega e^{u_\lambda}dx = \int_{\mathbb{R}^2}|x|^{2(\alpha-1)}e^{W_\lambda}dx + o(1) = 8\pi\alpha + o(1).$$

Similarly for every neighborhood U of 0

$$\lambda\int_U e^{u_\lambda}dx \to 8\pi\alpha.$$

Theorem 1.1 and 1.2 are thus completely proved.

7.2. **The case of the ball $\Omega = B(0,1)$.** Let us consider problem (2.2) in the unit ball $\Omega = B(0,1)$ (so that $H(x,0)=0$) with $\alpha_\lambda = \alpha$:

$$\begin{cases} -\Delta v = \lambda |x|^{2(\alpha-1)} e^v & \text{in } B(0,1), \\ v = 0 & \text{on } \partial B(0,1). \end{cases} \tag{7.7}$$

The object of this section is to prove that the solution $v_\lambda = W_\lambda + \phi_\lambda$ we have constructed in Theorem 2.1 for problem (7.7) via Lyaponuv-Schmidt reduction approaches a symmetric profile, after suitable rescaling, so the non-simple blow-up scenario does not occur in this case.

Observe that in $B(0,1)$ we have $\mathcal{H}_\alpha = H$, moreover up to a rotation we may assume $b_\lambda \in \mathbb{R}$, $b_\lambda > 0$ and the expansion (2.8) can be refined as follows:

$$PW_\lambda = W_\lambda - \log(8\alpha^2\delta^{2\alpha}) + 8\pi H(x^\alpha, b_\lambda) + \frac{2\delta^{2\alpha}}{1+b_\lambda^2} + O(b_\lambda \delta^{2\alpha}) + O(\delta^{4\alpha}).$$

We multiply the equation in (7.7) by PZ_λ^1 and integrate over $B(0,1)$:

$$\int_{B(0,1)} \nabla(PW_\lambda + \phi_\lambda)\nabla PZ_\lambda^1 dx - \lambda \int_{B(0,1)} |x|^{2(\alpha-1)} e^{PW_\lambda + \phi_\lambda} PZ_\lambda^1 dx = 0. \tag{7.8}$$

Let us begin by observing that the orthogonality in (5.1) gives

$$\int_{B(0,1)} \nabla\phi_\lambda \nabla PZ_\lambda^1 dx = \int_{B(0,1)} |x|^{2(\alpha-1)} e^{W_\lambda}\phi_\lambda Z_\lambda^1 dx = 0. \tag{7.9}$$

By the choice of δ in (2.7) we derive

$$\begin{aligned} \lambda|x|^{2(\alpha-1)} e^{PW_\lambda} &= \frac{\lambda}{8\alpha^2\delta^{2\alpha}} |x|^{2(\alpha-1)} e^{W_\lambda + 8\pi H(x^\alpha, b_\lambda) + \frac{2\delta^{2\alpha}}{1+b_\lambda^2} + O(b_\lambda\delta^{2\alpha}) + O(\delta^{4\alpha})} \\ &= |x|^{2(\alpha-1)} e^{W_\lambda} e^{8\pi(H(x^\alpha, b_\lambda) - H(b_\lambda, b_\lambda)) + \frac{2\delta^{2\alpha}}{1+b_\lambda^2} + O(b_\lambda\delta^{2\alpha}) + O(\delta^{4\alpha})} \\ &= |x|^{2(\alpha-1)} e^{W_\lambda} e^{8\pi(H(x^\alpha, b_\lambda) - H(b_\lambda, b_\lambda))} \Big(1 + \frac{2\delta^{2\alpha}}{1+b_\lambda^2} + O(b_\lambda\delta^{2\alpha}) + O(\delta^{4\alpha})\Big). \end{aligned} \tag{7.10}$$

Using the expression of H given in Remark B.1 we compute

$$\begin{aligned} e^{8\pi(H(x^\alpha, b_\lambda) - H(b_\lambda, b_\lambda))} = e^{4\log(b_\lambda |x^\alpha - \frac{1}{b_\lambda}|) - 4\log(1-b_\lambda^2)} &= \frac{|b_\lambda(x^\alpha - b_\lambda) - (1-b_\lambda^2)|^4}{(1-b_\lambda^2)^4} \\ &= 1 - 4b_\lambda \frac{\mathrm{Re}(x^\alpha - b_\lambda)}{1-b_\lambda^2} + O\Big(b_\lambda^2 |x^\alpha - b_\lambda|^2\Big). \end{aligned}$$

Consequently (7.10) becomes

$$\begin{aligned} \lambda|x|^{2(\alpha-1)} e^{PW_\lambda} = \frac{1 + b_\lambda^2 + 2\delta^{2\alpha}}{1+b_\lambda^2} |x|^{2(\alpha-1)} e^{W_\lambda} \Big(1 - 4b_\lambda \frac{\mathrm{Re}(x^\alpha - b_\lambda)}{1-b_\lambda^2} + O\Big(b_\lambda^2|x^\alpha - b_\lambda|^2\Big)\Big) \\ + (O(b_\lambda\delta^{2\alpha}) + O(\delta^{4\alpha}))|x|^{2(\alpha-1)} e^{W_\lambda}. \end{aligned}$$

Combining the above expansion with Lemma A.2 we get

$$\begin{aligned}
&\int_{B(0,1)} \nabla PW_\lambda \nabla PZ^1_\lambda dx - \lambda \int_{B(0,1)} |x|^{2(\alpha-1)} e^{PW_\lambda} PZ^1_\lambda dx \\
&= \int_{B(0,1)} |x|^{2(\alpha-1)} e^{W_\lambda} PZ^j_\lambda dx - \lambda \int_{B(0,1)} |x|^{2(\alpha-1)} e^{PW_\lambda} PZ^1_\lambda dx \\
&= -16\pi\alpha b_\lambda \delta^\alpha \frac{1+b_\lambda^2+2\delta^{2\alpha}}{1-b_\lambda^4} + O(\delta^{3\alpha}) + O(b_\lambda\delta^{2\alpha}) + O(\delta^{2\alpha}|\log\delta| b_\lambda^2) \\
&= -16\pi\alpha b_\lambda\delta^\alpha + O(b_\lambda^3\delta^\alpha) + O(\delta^{3\alpha}) + O(b_\lambda\delta^{2\alpha}) + O(\delta^{2\alpha}|\log\delta| b_\lambda^2).
\end{aligned} \tag{7.11}$$

Next, by using (6.13), (6.16) and the orthogonality (7.9)

$$\begin{aligned}
&\lambda \int_{B(0,1)} |x|^{2(\alpha-1)} e^{PW_\lambda}(e^{\phi_\lambda}-1) Z^j_\lambda dx \\
&= \lambda \int_{B(0,1)} |x|^{2(\alpha-1)} e^{PW_\lambda}(e^{\phi_\lambda}-1-\phi_\lambda) Z^j_\lambda dx - \int_{B(0,1)} R_\lambda \phi_\lambda Z^1_\lambda dx \\
&= O(\delta^{-2\alpha\frac{p-1}{p}-2\varepsilon}\delta^{2\alpha}(\delta^{2\alpha}+b_\lambda^2)) = o(\delta^{3\alpha}) + o(b_\lambda^2\delta^\alpha)
\end{aligned} \tag{7.12}$$

provided that ε is chosen sufficiently close to 0 and p sufficiently close to 1. Finally by (3.2), (4.5) and (6.15)

$$\begin{aligned}
\lambda \int_{B(0,1)} |x|^{2(\alpha-1)} e^{W_\lambda}(e^{\phi_\lambda}-1)(PZ^1_\lambda - Z^1_\lambda) dx &= O(\delta^\alpha) \int_{B(0,1)} \lambda |x|^{2(\alpha-1)} e^{PW_\lambda} |e^{\phi_\lambda}-1| dx \\
&= O(\|\delta^\alpha e^{W_\lambda}|x|^{2(\alpha-1)}\|_p \|e^{\phi_\lambda}-1\|_q) \\
&= O(\delta^{\alpha-2\alpha\frac{p-1}{p}}\delta^{-\varepsilon}\delta^\alpha(\delta^\alpha + b_\lambda)) \\
&= o(\delta^{2\alpha}) + o(b_\lambda\delta^\alpha)
\end{aligned} \tag{7.13}$$

provided that p is sufficiently close to 1 and ε is sufficiently small. By inserting (7.9), (7.11), (7.12) and (7.13) into (7.8) we arrive at

$$b_\lambda = o(b_\lambda) + o(\delta^\alpha)$$

by which, if $b_\lambda \to 0$,

$$b_\lambda = o(\delta^\alpha)$$

This implies that, after some rescalation, the limiting profile of the solution is radial at its first-order approximation:

$$\begin{aligned}
PW_\lambda &= -2\log\left(\delta^{2\alpha} + |x^\alpha - b_\lambda|^2\right) + 4\log|b_\lambda x^\alpha - 1| + o(\delta^\alpha) \\
&= -2\log\left(\delta^{2\alpha} + |x|^{2\alpha}\right) + o(\delta^\alpha)
\end{aligned}$$

uniformly in Ω.

Appendix A

In this appendix we derive some crucial integral estimates which arise in the asymptotic expansion of the energy of approximate solution W_λ.

Lemma A.1. *The following holds:*

$$\int_{|x|^\alpha \le \frac{|b|}{2}} |x|^{2(\alpha_\lambda-1)} e^{W_\lambda} = |b|^{2\frac{\alpha_\lambda-\alpha}{\alpha}} \int_{|x|^\alpha \le \frac{|b|}{2}} |x|^{2(\alpha-1)} e^{W_\lambda} dx + |\alpha_\lambda - \alpha| O\left(\frac{\delta^{2\alpha}|\log|b||}{|b|^2}\right)$$

uniformly for b *in a small neighborhood of* 0, $|b| \ge |\alpha_\lambda - \alpha|$.

Proof. Observe that $e^{W_\lambda} \le C\delta^{2\alpha}|b|^{-4}$ for $|x|^\alpha \le \frac{|b|}{2}$. Therefore we compute

$$\int_{|x|^\alpha \le \frac{|b|}{2}} |x|^{2(\alpha-1)} e^{W_\lambda} dx \le C\frac{\delta^{2\alpha}}{|b|^4} \int_{|x|^\alpha \le \frac{|b|}{2}} |x|^{2(\alpha-1)} dx \le C\frac{\delta^{2\alpha}}{|b|^2} \tag{A.1}$$

and, by (4.4),

$$\begin{aligned}\int_{|x|^\alpha \le \frac{|b|}{2}} \Big||x|^{2(\alpha_\lambda-1)} - |x|^{2(\alpha-1)}\Big| e^{W_\lambda} dx &\le C|\alpha_\lambda - \alpha| \int_{|x|^\alpha \le \frac{|b|}{2}} |x|^{2(\alpha-1)}|x|^{-2|\alpha_\lambda-\alpha|}|\log|x||e^{W_\lambda} dx \\ &\le C|\alpha_\lambda - \alpha|\frac{\delta^{2\alpha}}{|b|^4} \int_{|x|^\alpha \le \frac{|b|}{2}} |x|^{2(\alpha-1)}|x|^{-2|\alpha_\lambda-\alpha|}|\log|x||dx \\ &\le C|\alpha_\lambda - \alpha|\frac{\delta^{2\alpha}}{|b|^{2+2\frac{|\alpha_\lambda-\alpha|}{\alpha}}}|\log|b||\end{aligned}$$

and these two estimates hold uniformly for b in a small neighborhood of 0, $b \neq 0$. By the last estimate, taking into account that $|b|^{-2\frac{|\alpha_\lambda-\alpha|}{\alpha}} = O(1)$ for $|b| \ge |\alpha_\lambda - \alpha|$, we get

$$\int_{|x|^\alpha \le \frac{|b|}{2}} |x|^{2(\alpha_\lambda-1)} e^{W_\lambda} dx = \int_{|x|^\alpha \le \frac{|b|}{2}} |x|^{2(\alpha-1)} e^{W_\lambda} dx + |\alpha_\lambda - \alpha| O\bigg(\frac{\delta^{2\alpha}|\log|b||}{|b|^2}\bigg)$$

uniformly for b in a small neighborhood of 0, $|b| \ge |\alpha_\lambda - \alpha|$. Finally, observing that $|b|^{2\frac{\alpha_\lambda-\alpha}{\alpha}} = 1 + O(|\alpha_\lambda - \alpha||\log|b|)$ for $|b| \ge |\alpha_\lambda - \alpha|$ and using (A.1) we get the thesis. □

Lemma A.2. *The following holds:*

$$\int_\Omega |x|^{2(\alpha-1)} e^{W_\lambda}|x^\alpha - b|dx = O(\delta^\alpha), \qquad \int_\Omega |x|^{2(\alpha-1)} e^{W_\lambda}|x^\alpha - b|^2 dx = O(\delta^{2\alpha}|\log\delta|)$$

$$\int_\Omega |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda dx = O(\delta^\alpha), \qquad \int_\Omega |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda \mathrm{Re}(x^\alpha - b) dx = 4\pi\alpha\delta^\alpha + O(\delta^{2\alpha})$$

uniformly for b in a small neighborhood of 0.

Proof. By Lemma C.1 we compute

$$\begin{aligned}\int_\Omega |x|^{2(\alpha-1)} e^{W_\lambda}|x^\alpha - b|dx &\le 8\alpha^2\delta^\alpha \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha - \delta^{-\alpha}b|^2)^2}|y^\alpha - \delta^{-\alpha}b|dy \\ &= 8\alpha\delta^\alpha \int_{\mathbb{R}^2} \frac{|y|}{(1+|y|^2)^2} dy.\end{aligned}$$

and the first estimate follows. In order to prove the second estimate, let $R > 0$ be sufficiently large such that $\Omega \subset B(0, \frac{R}{2})$. Using again Lemma C.1 we compute:

$$\begin{aligned}\int_\Omega |x|^{2(\alpha-1)} e^{W_\lambda}|x^\alpha - b|^2 dx &= 8\alpha^2\delta^{2\alpha} \int_{\frac{\Omega}{\delta}} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha - \delta^{-\alpha}b|^2)^2}|y^\alpha - \delta^{-\alpha}b|^2 dy \\ &= 8\alpha\delta^{2\alpha} \int_{\frac{\Omega^\alpha}{\delta^\alpha}} \frac{1}{(1+|y - \delta^{-\alpha}b|^2)^2}|y - \delta^{-\alpha}b|^2 dy \\ &\le 8\alpha\delta^{2\alpha} \int_{B(0,\frac{R^\alpha}{\delta^\alpha})} \frac{|y|^2}{(1+|y|^2)^2} dy \\ &\le C\delta^{2\alpha}|\log\delta|.\end{aligned}$$

Next by (3.2)

$$\begin{aligned}
&\int_\Omega |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda dx\\
&= 8\alpha^2 \int_{\frac{\Omega}{\delta}} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha-\delta^{-\alpha}b|^2)^3}\mathrm{Re}(y^\alpha-\delta^{-\alpha}b)dy + O(\delta^\alpha)\\
&= 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha-\delta^{-\alpha}b|^2)^3}\mathrm{Re}(y^\alpha-\delta^{-\alpha}b)dy + O(\delta^\alpha)\\
&= 8\alpha \int_{\mathbb{R}^2} \frac{y_1}{(1+|y|^2)^3} dy + O(\delta^\alpha)\\
&= O(\delta^\alpha)
\end{aligned}$$

since $\int_{\mathbb{R}^2} \frac{y_1}{(1+|y|^2)^2} dy = 0$ by oddness. Similarly we compute

$$\begin{aligned}
&\int_\Omega |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda \mathrm{Re}(x^\alpha - b) dx\\
&= 8\alpha^2\delta^\alpha \int_{\frac{\Omega}{\delta}} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha-\delta^{-\alpha}b|^2)^3}(\mathrm{Re}(y^\alpha-\delta^{-\alpha}b))^2dy + O(\delta^{2\alpha})\\
&= 8\alpha^2\delta^\alpha \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha-\delta^{-\alpha}b|^2)^3}(\mathrm{Re}(y^\alpha-\delta^{-\alpha}b))^2dy + O(\delta^{2\alpha})\\
&= 8\alpha\delta^\alpha \int_{\mathbb{R}^2} \frac{y_1^2}{(1+|y|^2)^3} dy + O(\delta^{2\alpha})\\
&= 4\pi\alpha\delta^\alpha + O(\delta^{2\alpha}).
\end{aligned}$$

where we have used the identity $\int_{\mathbb{R}^2} \frac{(y_1)^2}{(1+|y|^2)^3} dy = \frac{1}{2}\int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^3} dy = \frac{\pi}{2}$.

□

Lemma A.3. *Let $\varepsilon > 0$ be a small fixed number and let $\gamma = 0, 1, 2$. Then the following holds:*

$$\int_\Omega \big||x|^{2(\alpha-1)} - |x|^{2(\alpha_\lambda-1)}\big||x^\alpha - b|^\gamma e^{W_\lambda} dx = O(|\alpha_\lambda - \alpha|\delta^{\alpha\gamma-\varepsilon}),$$

uniformly for b in a small neighborhood of 0.

Proof. By Hölder's inequality for any $p > 1$

$$\begin{aligned}
\int_\Omega |x|^{2(\alpha-1)}|x|^{-2|\alpha_\lambda-\alpha|}|x^\alpha-b|^\gamma|\log|x||e^{W_\lambda}dx &\le \||x|^{2(\alpha-1)}|x^\alpha-b|^\gamma e^{W_\lambda}\|_p \||x|^{-2|\alpha_\lambda-\alpha|}\log|x|\|_{\frac{p-1}{p}}\\
&\le C\||x|^{2(\alpha-1)}|x^\alpha-b|^\gamma e^{W_\lambda}\|_p\\
&\le C\delta^{\alpha\gamma}\delta^{-2\alpha\frac{p-1}{p}}
\end{aligned}$$

by estimate (4.1). Then the thesis follows by using the inequality (4.4) and taking p sufficiently close to 1.

□

Appendix B

In this appendix we carry out some asymptotic expansions involving the regular part $H(x, y)$ of the Green's function in the case of symmetric domains. According to hypothesis (A1)*, we assume that Ω is α-symmetric:

$$x \in \Omega \Longleftrightarrow xe^{\mathrm{i}\frac{2\pi}{\alpha}} \in \Omega,$$

and this implies that the new domain

$$\Omega_\alpha := \{x^\alpha \,|\, x \in \Omega\}$$

is smooth. Let us denote by $\mathcal{H}_\alpha(z,b)$ the regular part of the Green's function of $-\Delta$ in Ω_α: for any fixed $b\in\Omega_\alpha$ the function $\mathcal{H}_\alpha(\cdot,b)$ satisfies

$$\Delta_z\mathcal{H}_\alpha(z,b)=0 \text{ in } \Omega_\alpha, \quad \mathcal{H}_\alpha(z,b)=2\pi\log|z-b| \text{ on } \partial\Omega_\alpha.$$

Now, for any fixed $b\in\Omega_\alpha$ we have that the function $x\in\Omega\mapsto\mathcal{H}_\alpha(x^\alpha,b)$ is harmonic in Ω and satisfies $\mathcal{H}_\alpha(x^\alpha,y)=2\pi\log|x^\alpha-b|=2\pi\sum_{i=0}^{\alpha-1}\log|x-\beta_i|$ on $\partial\Omega$, which implies

$$\sum_{i=0}^{\alpha-1}H(x,\beta_i)=\mathcal{H}_\alpha(x^\alpha,b) \text{ in } \Omega.$$

In particular, the function considered in assumption (A2) (where $\alpha=N+1$) coincides with:

$$\sum_{i,j=0}^{\alpha-1}H(\beta_i,\beta_j)-(\alpha-1)\sum_{i=0}^{\alpha-1}H(\beta_i,0)=\alpha\mathcal{H}_\alpha(b,b)-(\alpha-1)\mathcal{H}_\alpha(b,0) \quad \forall b\in\Omega_\alpha. \tag{B.1}$$

Thanks to the symmetry of $\mathcal{H}_\alpha(z,b)$, we get[2]

$$\nabla_b\Big(\alpha\mathcal{H}_\alpha(b,b)-(\alpha-1)\mathcal{H}_\alpha(b,0)\Big)\Big|_{b=0}=(\alpha+1)\nabla_z\mathcal{H}(0,0),$$

so, if assumption (A2)* holds we get $\nabla_z\mathcal{H}(0,0)=0$ and, by symmetry $\nabla_b\mathcal{H}(0,0)=0$; consequently,

$$\begin{aligned}\mathcal{H}_\alpha(x^\alpha,b)-\mathcal{H}_\alpha(b,b)&=\langle\nabla_z\mathcal{H}_\alpha(b,b),x^\alpha-b\rangle+O(|x^\alpha-b|^2)\\&=O(|b||x^\alpha-b|)+O(|x^\alpha-b|^2)\end{aligned} \tag{B.2}$$

and

$$\begin{aligned}\mathcal{H}_\alpha(x^\alpha,0)-\mathcal{H}_\alpha(b,0)&=\langle\nabla_z\mathcal{H}_\alpha(b,0),x^\alpha-b\rangle+O(|x^\alpha-b|^2)\\&=O(|b||x^\alpha-b|)+O(|x^\alpha-b|^2)\end{aligned} \tag{B.3}$$

uniformly for $x\in\Omega$ and b in a small neighborhood of 0.

Remark B.1. *If Ω is the unit ball $\Omega=B(0,1)$, then $\Omega_\alpha=\Omega=B(0,1)$ and so*

$$\mathcal{H}_\alpha(z,b)=H(z,b)=\frac{1}{2\pi}\log\Big(|b|\Big|z-\frac{b}{|b|^2}\Big|\Big).$$

Consequently by (B.1), *for any $b\in B(0,1)$*

$$\sum_{i,j=0}^{\alpha-1}H(\beta_i,\beta_j)-(\alpha-1)\sum_{i=0}^{\alpha-1}H(\beta_i,0)=\alpha\mathcal{H}_\alpha(b,b)-(\alpha-1)\mathcal{H}_\alpha(b,0)=\alpha H(b,b)=\frac{\alpha}{2\pi}\log(1-|b|^2).$$

Appendix C

This appendix is devoted to deduce some integral identities associated to the change of variable $x\mapsto x^\alpha$ which appears frequently when dealing with functions in spaces $\mathrm{H}_\alpha(\mathbb{R}^2)$, $\mathrm{L}_\alpha(\mathbb{R}^2)$ introduced in Section 2.

[2]Here $\nabla_z\mathcal{H}_\alpha$, $\nabla_b\mathcal{H}_\alpha$ denote the gradient of the function $\mathcal{H}_\alpha(\cdot,\cdot)$ with respect to the first and the second variable, respectively.

Lemma C.1. *Let $\xi \in \mathbb{R}^2$. For any $f \in \mathrm{L}_1(\mathbb{R}^2)$ we have that $f(y^\alpha) \in \mathrm{L}_\alpha(\mathbb{R}^2)$ and*

$$\int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha-\xi|^2)^2}|f(y^\alpha)|^2dy = \frac{1}{\alpha}\int_{\mathbb{R}^2} \frac{1}{(1+|y-\xi|^2)^2}|f(y)|^2dy. \tag{C.1}$$

Moreover, if $f \in \mathrm{H}_1(\mathbb{R}^2)$, then $f(y^\alpha) \in \mathrm{H}_\alpha(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} |\nabla(f(y^\alpha))|^2dy = \alpha\int_{\mathbb{R}^2}|\nabla f|^2dy.$$

Proof. It is sufficient to prove the thesis for a smooth function f. Using the polar coordinates (ρ,θ) and then applying the change of variables $(\rho',\theta') = (\rho^\alpha, \alpha\theta)$

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha-\xi|^2)^2}|f(y^\alpha)|^2dy &= \int_0^{+\infty}d\rho\int_0^{2\pi}\frac{\rho^{2\alpha-1}}{(1+|\rho^\alpha e^{\mathrm{i}\alpha\theta}-\xi|^2)^2}|f(\rho^\alpha e^{\mathrm{i}\alpha\theta})|^2d\theta\\ &= \frac{1}{\alpha^2}\int_0^{+\infty}d\rho'\int_0^{2\alpha\pi}\frac{\rho'}{(1+|\rho' e^{\mathrm{i}\theta'}-\xi|^2)^2}|f(\rho' e^{\mathrm{i}\theta'})|^2d\theta'\\ &= \frac{1}{\alpha}\int_0^{+\infty}d\rho'\int_0^{2\pi}\frac{\rho'}{(1+|\rho' e^{\mathrm{i}\theta'}-\xi|^2)^2}|f(\rho' e^{\mathrm{i}\theta'})|^2d\theta'\\ &= \frac{1}{\alpha}\int_{\mathbb{R}^2}\frac{1}{(1+|y-\xi|^2)^2}|f(y)|^2dy.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla(f(y^\alpha))|^2dy &= \int_0^{+\infty}d\rho\int_0^{2\pi}\rho\Big(\Big|\frac{\partial(f(\rho^\alpha e^{\mathrm{i}\alpha\theta}))}{\partial\rho}\Big|^2+\frac{1}{\rho^2}\Big|\frac{\partial(f(\rho^\alpha e^{\mathrm{i}\alpha\theta})}{\partial\theta}\Big|^2\Big)d\theta\\ &= \alpha^2\int_0^{+\infty}d\rho\int_0^{2\pi}\rho\Big(\rho^{2(\alpha-1)}\Big|\frac{\partial f}{\partial\rho'}(\rho^\alpha e^{\mathrm{i}\alpha\theta})\Big|^2+\frac{1}{\rho^2}\Big|\frac{\partial f}{\partial\theta'}(\rho^\alpha e^{\mathrm{i}\alpha\theta})\Big|^2\Big)d\theta\\ &= \alpha\int_0^{+\infty}d\rho'\int_0^{2\pi\alpha}\rho'^{\frac{2}{\alpha}-1}\Big(\rho'^{\frac{2(\alpha-1)}{\alpha}}\Big|\frac{\partial f}{\partial\rho'}(\rho' e^{\mathrm{i}\theta'})\Big|^2+\frac{1}{\rho'^{2/\alpha}}\Big|\frac{\partial f}{\partial\theta'}(\rho' e^{\mathrm{i}\theta'})\Big|^2\Big)d\theta'\\ &= \alpha\int_0^{+\infty}d\rho'\int_0^{2\pi\alpha}\rho'\Big(\Big|\frac{\partial f}{\partial\rho'}(\rho' e^{\mathrm{i}\theta'})\Big|^2+\frac{1}{\rho'^2}\Big|\frac{\partial f}{\partial\theta'}(\rho' e^{\mathrm{i}\theta'})\Big|^2\Big)d\theta'\\ &= \alpha\int_{\mathbb{R}^2}|\nabla f|^2dy.\end{aligned}$$

□

Now we are going to obtain a sort of counterpart of Lemma C.1 which converts a α-symmetric function in $\mathrm{L}_\alpha(\mathbb{R}^2)$ (in $\mathrm{H}_\alpha(\mathbb{R}^2)$ respectively) into a function in $\mathrm{L}_1(\mathbb{R}^2)$ (in $\mathrm{H}_1(\mathbb{R}^2)$ respectively) by a suitable change of variables.

Lemma C.2. *Let $\xi \in \mathbb{R}^2$ and let $f \in \mathrm{L}_\alpha(\mathbb{R}^2)$ be α-symmetric, i.e.*

$$f(xe^{\mathrm{i}\frac{\pi}{\alpha}}) = f(x) \quad \forall x \in \mathbb{R}^2$$

and set

$$F:\mathbb{R}^2\to\mathbb{R},\quad F(\rho) = f\big(\rho^{\frac{1}{\alpha}}e^{\mathrm{i}\frac{\theta}{\alpha}}\big) \qquad \rho \geq 0,\ \theta\in[-\pi,\pi). \tag{C.2}$$

Then $F \in \mathrm{L}_1(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2}\frac{1}{(1+|y-\xi|^2)^2}|F(y)|^2dy = \alpha\int_{\mathbb{R}^2}\frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha-\xi|^2)^2}|f(y)|^2dy. \tag{C.3}$$

Moreover, if $f \in \mathrm{H}_\alpha(\mathbb{R}^2)$*, then* $F \in \mathrm{H}_1(\mathbb{R}^2)$ *and*

$$\int_{\mathbb{R}^2} |\nabla F|^2 dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} |\nabla f|^2 dy. \tag{C.4}$$

Proof. Taking into account that as single point has capacity 0 in $\mathbb{R}^2$, it is sufficient to prove the thesis for a smooth function f such that $f = 0$ in a neighborhood of 0. Since by definition

$$f(y) = F(y^\alpha) \quad \text{if } y \in \mathbb{R}^2,$$

then the thesis follows by applying Lemma C.1. □

An analogous identity holds for the scalar product associated to (C.3)-(C.4) as stated in the following corollary.

Corollary C.3. *Let* $\xi \in \mathbb{R}^2$*.*

- *For any* $f, g \in \mathrm{L}_\alpha(\mathbb{R}^2)$ *we have that*

$$\int_{\mathbb{R}^2} \frac{1}{(1+|y-\xi|^2)^2} FG dy = \alpha \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1+|y^\alpha-\xi|^2)^2} fg dy;$$

- *for any* $f, g \in \mathrm{H}_\alpha(\mathbb{R}^2)$ *we have that*

$$\int_{\mathbb{R}^2} \nabla F \nabla G dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} \nabla f \nabla g dy.$$

where F, G *are the functions defined according to* (C.2) *starting from* f*,* g*, respectively.*

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