EXISTENCE AND STABILITY OF SYMMETRIC AND ASYMMETRIC PATTERNS FOR THE HALF-LAPLACIAN GIERER-MEINHARDT SYSTEM IN ONE-DIMENSIONAL DOMAIN

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ABSTRACT. In this paper, we study the existence and stability of multiple spikes pattern to the fractional Gierer-Meinhardt model with periodic boundary conditions and the fractional power $s = \frac{1}{2}$. Specifically, we rigorously establish the existence of symmetric multiple spikes and asymmetric two-spikes solutions by the classical Lyapunov-Schmidt reduction method. We also investigate the stability of the constructed solution by studying its associated large and small eigenvalue problems, where we need to consider two nonlocal eigenvalue problems in their fractional versions. In the study of the large eigenvalue problem, the quantity $D_K(\varepsilon) = \frac{2}{\pi K} \log \frac{1}{\varepsilon}$ is the critical threshold which determines the stability of K-peaked solutions. For the symmetric two-spikes pattern we obtain the asymptotic expansion for the critical threshold $D_K(\varepsilon)$ up to the second order. Moreover, we provide some elementary properties of the Green's function, including the first and second derivatives, they are linked to the location of the spikes and the stability. Among these properties on the Green's function, we find out that the polygamma function $\phi(x) = \frac{d}{dx} \log \Gamma(x)$ plays a crucial role.

Keywords: Gierer-Meinhardt system; eigenvalue; stability; fractional laplacian, localized solutions.

1. INTRODUCTION

In mathematical biology, many models have been proposed and analyzed to explore the so-called Turing instability since the work [33] in 1952. One of the most famous in biological pattern formation is the Gierer-Meinhardt system proposed by Gierer and Meinhardt in 1972, see [8], which reads as follows

$$\begin{cases} u_t = \varepsilon^2 \Delta u - u + \frac{u^2}{v}, \quad u > 0 & \text{in } \Omega, \\ \tau v_t = D \Delta v - v + u^2, \quad v > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Equation (1.1) is used as a prototype of deterministic reaction-diffusion system to explain the results of experiments on head regeneration and transplantation in the freshwater polyp *hydra*. In (1.1), the unknowns $u = u(x,t)$ and $v = v(x,t)$ represent the concentrations of the activator and inhibitor at a point $x \in \Omega$ and at a time $t > 0$, where Ω is a bounded and smooth domain and $\nu = \nu(x)$ is the outer normal at $x \in \partial\Omega$.

In past decades, there have been many works concerning the system (1.1) which focus on the analysis, both rigorous and formal, of the existence, structure, and linear stability of such localized solutions. In a 1-D domain, the Gierer-Meinhardt system has been particularly well studied by using pure PDE methods and formal asymptotic analysis see [18, 34, 35, 38]. In the 2-D case, the rigorous analysis on the existence and stability of multiple-peaked patterns that are far from spatial homogeneity for the singularly perturbed Gierer-Meinhardt system has also been investigated, see [36, 37]. There are also some works concerning the extended 1-D and 2-D cases which involve the effection of the precusors [39, 41, 42], bulk-membrane-coupling [11], and anomalous diffusion [26, 27, 43]. For more background of this model and other related reaction diffusion systems, we refer the readers to the book [25, 40].

Gierer-Meinhardt systems have widespread applications in the modelling of biological phenomena for which distinct agents diffuse while simultaneously undergoing prescribed reaction kinetics. While these models have typically assumed a normal (or Brownian) diffusion process for which the mean-squared-displacement (MSD) is proportional to the elapsed time, a growing body of literature has considered the alternative of *anomalous diffusion* which may be better suited for biological processes in complex environments [24, 29, 31] (see also §7.1 in [5]). It has been shown that both superdiffusion and subdiffusion can reduce the threshold for Turing instabilities when compared to the same systems with normal diffusion

[10, 16]. Fractional reaction diffusion systems have also found a diverse range of applications from population dynamics [1, 20, 23] to economics [2, 3]. Nonlocal, anomalous diffusuion can allow systems to exhibit novel behavior that cannot be modelled in local systems. For example [23, Section 4] emphasizes that their results about fractional species competition have no local analouge, and [22] utilizes a fractional diffusion model to resolve controversy arising from a local model of polymer transport. The study of fractional generalizations of local systems has proven productive in enhancing our models of real-world phenomena. Therefore, understanding the pattern formation and its linear stability of the fractional case is quite necessary in studying the extended Gierer-Meinhardt system.

As a continuation work of [12, 43], we shall study the existence and stability of localized multi-spike solutions to the one-dimensional Gierer-Meinhardt model with periodic boundary condition and the fractional power $s = \frac{1}{2}$, i.e.,

$$\begin{cases} u_t + \varepsilon(-\Delta)^{\frac{1}{2}} u + u - \frac{u^2}{v} = 0, & \text{for } x \in (-1,1), \\ \tau v_t + D(-\Delta)^{\frac{1}{2}} v + v - u^2 = 0, & \text{for } x \in (-1,1), \\ u(x) = u(x+2),\ v(x) = v(x+2), & \text{for } x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $0 < \varepsilon \ll 1$ and the parameters $0 < D < \infty$ and $\tau \geq 0$ are independent of ε. The (nonlocal) fractional Laplacian $(-\Delta)^{\frac{1}{2}}$ replaces the classical Laplacian as the infinitesimal generator of the underlying Lévy process for $s = \frac{1}{2}$ and is defined by

$$(-\Delta)^s \phi(x) \equiv C_s \int_{-\infty}^{\infty} \frac{\phi(x) - \phi(y)}{|x-y|^{1+2s}} dy, \quad \text{where} \quad C_s \equiv \frac{2^{2s} s \Gamma(s+1/2)}{\sqrt{\pi}\Gamma(1-s)}.$$

Due to the periodicity of the funciton, we could also write

$$(-\Delta)^{\frac{1}{2}} \phi(x) = C_s \int_{-1}^{1} (\phi(x) - \phi(y)) K_s(x,y) dy,$$

with

$$K_s(x,y) = \frac{1}{|x-y|^2} + \sum_{m=1}^{\infty} \left(\frac{1}{|x-y+2m|^2} + \frac{1}{|x-y-2m|^2} \right).$$

In previous work [12], the last two authors of this paper and Gomez analyzed (1.2) in the case where the fractional power $s = \frac{1}{2}$ in the equation of v is replaced by $s \in (\frac{1}{2}, 1)$. Specifically, it has been rigorously proven that the symmetric and asymmetric two-spikes solutions exists and the linear stability of these solutions is determined by the eigenvalues of a certain 2×2 matrix. As [12], we prove the existence of symmetric multiple spikes and asymmetric two spikes solutions for (1.2) The only issue is that the decay of the ansatz is not good enough; we use a symmetry property to deal with this difficulty and thus the existence part can be obtained similarly. The stability turns to be more complicated for the case $s = \frac{1}{2}$. On one side, in the study of the large eigenvalue problem there are several cases need to be considered, whereas we only need to study a single case when $s > \frac{1}{2}$. On the other side, in the study of the small eigenvalue problem we have to figure out the sign on the second derivative of the Green's function. Due to the conditional convergence of the series for the case of $s = \frac{1}{2}$, it does not seem possible to handle using elementary computation. Through a further analysis on the corresponding series we find that such a function is closely related to the polygamma function $\phi(z) := \frac{d}{dz} \log \Gamma(z)$, and using its properties we are able to determine the sign of the second derivative the Green's function thereby solving the stability part.

To state the main results of this paper, we write the steady problem of (1.2) as

$$\begin{cases} \varepsilon(-\Delta)_x^{\frac{1}{2}} u + u - \frac{u^2}{v} = 0, & x \in (-1,1), \\ D(-\Delta)^{\frac{1}{2}} v + v - u^2 = 0, & x \in (-1,1), \\ u(x) = u(x+2),\ v(x) = v(x+2), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

Let $D = \frac{1}{\beta^2}$ and the Green's function $G_\beta(x,z)$ be the function satisfying

$$\begin{cases} (-\Delta)^{\frac{1}{2}} G_\beta(x,z) + \beta^2 G_\beta(x,z) = \delta(x-z), & x \in (-1,1), \\ G_\beta(x,z) = G_\beta(x+2,z), & x \in \mathbb{R}. \end{cases}$$

It is not difficult to verify that $G_\beta(x,z)$ admits the following Fourier series expansion

$$G_\beta(x,z) = \frac{1}{2}\sum_{\ell=-\infty}^{\infty} \frac{e^{i\ell\pi(x-z)}}{\beta^2+\ell\pi} = \frac{1}{2}\beta^{-2} + \sum_{\ell=1}^{\infty}\frac{\cos(\ell\pi(x-z))}{\beta^2+\ell\pi}.$$

Let $G_0(x,z)$ be the Green's function given by

$$\begin{cases} (-\Delta)^{\frac{1}{2}} G_0(x,z) = \delta(x-z), & x \in (-1,1), \\ G_0(x,z) = G_0(x+2,z), & x \in \mathbb{R}. \end{cases} \tag{1.4}$$

2.green

Then it is not difficult to check that

$$G_\beta(x,z) = \frac{1}{2\beta^2} + G_0(x,z) + O(\beta^2). \tag{1.5}$$

2.rel-g

The singular part of $G_\beta(x,z)$ behaves as $\frac{1}{\pi}\log\frac{1}{|x-z|}$ and we decompose $G_\beta(x,z)$ as

$$G_\beta(x,z) = \frac{1}{\pi}\log\frac{1}{|x-z|} - H_\beta(x,z) = K_\beta(x,z) - H_\beta(x,z),$$

where $K_\beta(x,z) = \frac{1}{\pi}\log\frac{1}{|x-z|}$ and $H_\beta(x,z)$ denote the singular part and the regular part of the Green function respectively.

To describe the location of spikes, we denote $\mathbf{p} \in (-1,1)^K$, where $\mathbf{p}$ is arranged such that

$$\mathbf{p} \in B_\sigma(\mathbf{p}^0) = \left\{ \mathbf{q} = (q_1,\cdots,q_K) \mid \sum_{j=1}^{K} |q_j - p_j^0|^2 \le \sigma^2 \right\}, \quad \text{where} \quad p_j^0 = \frac{2j-1-K}{K}, \quad j = 1,\cdots,K. \tag{1.6}$$

2.p

For $\mathbf{p} \in B_\sigma(\mathbf{p}^0)$, we define

$$F(\mathbf{p}) = \sum_{j=1}^{K} H_\beta(p_j,p_j) - \sum_{i\neq j} G_\beta(p_i,p_j), \tag{1.7}$$

2.f

and $M(\mathbf{p}) = \nabla^2_{\mathbf{p}} F(\mathbf{p})$. Here $M(\mathbf{p})$ is a $K\times K$ matrix and one can easily see that it is a circulant matrix at $\mathbf{p}^0$. In addition, we have $\mathrm{rank}(M(\mathbf{p}^0)) \le K-1$ due to the fact that the summation of each row is 0.

Our first theorem concerns the existence of symmetric multiple spikes solutions.

th1.exist **Theorem 1.1.** *Let $\mathbf{p}^0$ be defined as in (1.6). Suppose $M(\mathbf{p}^0)$ is a matrix of $\mathrm{rank}(M(\mathbf{p}^0)) = K-1$. Moreover, we assume that the following technical condition holds:*

$$\text{if } K > 1, \text{ and } \eta_0 := \lim_{\varepsilon\to 0}\frac{2\beta^2}{\pi}\log\frac{1}{\varepsilon} \neq K. \tag{1.8}$$

2.tech

Then for ε sufficiently small and $D = \frac{1}{\beta^2}$ sufficiently large, problem (1.3) has a solution $u_\varepsilon, v_\varepsilon$ such that

$$u_\varepsilon \sim \xi_\varepsilon\left(\sum_{j=1}^{K} w\left(\frac{x-p_j^\varepsilon}{\varepsilon}\right) + O(h(\varepsilon,\beta))\right), \quad v_\varepsilon(p_j^\varepsilon) \sim \xi_\varepsilon, \tag{1.9}$$

2.conclusic

where w is the unique solution of

$$(-\Delta)^{\frac{1}{2}} w + w - w^2 = 0, \quad w(x) = w(-x), \tag{1.10}$$

1.ground

and ξ_ε and $h(\varepsilon,\beta)$ are given by

$$\xi_\varepsilon = \begin{cases} \frac{1}{\varepsilon K\pi}, & \text{if } \eta_\varepsilon \to 0, \\ \frac{1}{\varepsilon\eta_\varepsilon\pi}, & \text{if } \eta_\varepsilon \to +\infty, \\ \frac{1}{\varepsilon(\eta_0+K)\pi}, & \text{if } \eta_\varepsilon \to \eta_0, \end{cases} \tag{1.11}$$

3.spike-h

and

$$h(\varepsilon,\beta) = \begin{cases} \eta_\varepsilon, & \text{if } \eta_\varepsilon \to 0, \\ \eta_\varepsilon^{-1}, & \text{if } \eta_\varepsilon \to \infty, \\ \beta^2, & \text{if } \eta_\varepsilon \to \eta_0. \end{cases} \tag{1.12}$$

Furthermore, $p_j^\varepsilon \to p_j^0$ *as* $\varepsilon \to 0$ *for* $j = 1, \cdots, K$.

Remark: For D sufficiently large or $K = 2, 3, 4$ one can verify that $\text{rank}(M(\mathbf{p}^0)) = K - 1$. In addition, under these conditions one can show that all the non-zero eigenvalues of $M(\mathbf{p}^0)$ are negative. This part is left to section 4.

Next we study the stability and instability of the symmetric multiple spikes solution construted in Theorem 1.1. Writing the eigenvalue problem for the fractional Gierer-Meinhardt system as

$$\begin{cases} \varepsilon(-\Delta)^{\frac{1}{2}}\phi_\varepsilon + \phi_\varepsilon - 2\frac{u_\varepsilon}{v_\varepsilon}\phi_\varepsilon + \frac{u_\varepsilon^2}{v_\varepsilon^2}\psi_\varepsilon + \lambda_\varepsilon\phi_\varepsilon = 0, \\ D(-\Delta)^{\frac{1}{2}}\psi_\varepsilon + \psi_\varepsilon - 2u_\varepsilon\phi_\varepsilon + \tau\lambda_\varepsilon\psi_\varepsilon = 0, \end{cases} \tag{1.13}$$

where $(u_\varepsilon, v_\varepsilon)$ is the solution constructed in Theorem 1.1 and $\lambda_\varepsilon \in \mathbb{C}$. Here we say $(u_\varepsilon, v_\varepsilon)$ is linearly stable if the eigenvalue $\lambda_\varepsilon < 0$, while $(u_\varepsilon, v_\varepsilon)$ is called linearly unstable if there exists a eigenvalue λ_ε such that its real part $\Re(\lambda_\varepsilon) > 0$.

Theorem 1.2. *Suppose* $M(\mathbf{p}^0)$ *is a semi-negative matrix of rank* $K-1$, *and for* ε *sufficiently small and* $D = \frac{1}{\beta^2}$ *is sufficiently large. Let* $\eta_\varepsilon = \frac{2\beta^2}{\pi}\log\frac{1}{\varepsilon}$ *and* $(u_\varepsilon, v_\varepsilon)$ *be the K-peaked solutions constructed in Theorem 1.1 with the center of peaks approaching* $\mathbf{p}^0$. *Then*

(i). $\eta_\varepsilon \to 0$. *If* $K = 1$, *then there exists an unique* $\tau_1 > 0$ *such that for* $\tau < \tau_1$, $(u_\varepsilon, v_\varepsilon)$ *is a linearly stable, while for* $\tau > \tau_1$, $(u_\varepsilon, v_\varepsilon)$ *is linearly unstable; while if* $K > 1$, $(u_\varepsilon, v_\varepsilon)$ *is linearly unstable for any* $\tau \geq 0$.

(ii). $\eta_\varepsilon \to +\infty$. $(u_\varepsilon, v_\varepsilon)$ *is linearly stable for any* $\tau > 0$.

(iii). $\eta_\varepsilon \to \eta_0$. *If* $K > 1$ *and* $\eta_0 < K$, *then* $(u_\varepsilon, v_\varepsilon)$ *is linearly unstable for any* $\tau > 0$. *If* $\eta_0 > K$, *then there exist* $0 < \tau_2 \leq \tau_3$ *such that* $(u_\varepsilon, v_\varepsilon)$ *is linearly stable for any* $\tau < \tau_2$ *and* $\tau > \tau_3$. *If* $K = 1$ *and* $\eta_0 < 1$, *then there exists* $0 < \tau_4 \leq \tau_5$ *such that* $(u_\varepsilon, v_\varepsilon)$ *is linearly stable for any* $\tau < \tau_4$ *and linearly unstable* $\tau > \tau_5$.

Concerning the (iii) of Theorem 1.2, when $K \geq 1$ or τ is large, $D_K(\varepsilon) = \frac{2}{\pi K}\log\frac{1}{\varepsilon}$ is the critical threshold for the asymptotic behavior of the diffusion coefficient of the inhibitor which determines the stability of 2-peaked solutions. This number also appears in the study of classical 1-D and 2-D Gierer-Meinhardt systems. For 1-D classical Gierer-Meinhardt system, it has been shown [18] for $K \geq 2$ that the leading order of the critical thresholds $D_K(\varepsilon) = D_K$ are independent of ε. Moreover, the critical thresholds arise in the computation of the small eigenvalues. While in the classical 2-D case, $D_K(\varepsilon)$ is obtained in the study of the large eigenvalues. In fact, system (1.3) is more like the classical Gierer-Meinhardt system in 2-D case. The quantity $D_K(\varepsilon)$ also appears in the study of large eigenvalue problem. In addition, by the formal asymptotic computation, we obtain the next order term in the asymptotic expansion of $D_K(\varepsilon)$, which is very useful in pratice.

Proposition 1.3. *Consider the symmetric two spikes pattern of the Gierer-Meinhardt system (1.3), where* $D = O(\log\frac{1}{\varepsilon})$. *If*

$$D \sim \frac{1}{\pi}\log\frac{1}{\varepsilon} + \frac{1}{\pi}\mu_1,$$

where μ_1 *and* $\hat{u}$ *are*

$$\mu_1 = \frac{1}{2\pi}\int_{\mathbb{R}} w\hat{u}dy - \log\frac{\pi}{4},$$

and

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\hat{u} + \hat{u} - 2w\hat{u} + w^2\hat{v} = 0, & \hat{u}(y) \to 0 \ \text{ as } \ |y| \to \infty, \\ (-\Delta)^{\frac{1}{2}}\hat{v} - \frac{1}{2}w^2 = 0, & \hat{v}(y) \to -\log|y| \ \text{ as } \ |y| \to \infty, \end{cases}$$

and w *is the solution to (1.10). Then the portion of the conitinuous spectrum of the linearized problem (1.13) lies within an* $O(\frac{1}{\log\frac{1}{\varepsilon}})$ *neighborhood of the origin* $\lambda = 0$ *is given by*

$$\lambda = \frac{1}{\log\frac{1}{\varepsilon}}\left(\mu_1 + \log\frac{\pi}{4} - \frac{1}{2\pi}\int_{\mathbb{R}} w\hat{u}dy\right).$$

In Theorem 1.1 and Theorem 1.2, we provide the existence and stability results for the symmetric multiple spikes. In fact, when $\eta_\varepsilon \to 0$ or $+\infty$ as $\varepsilon \to 0$, we could only see the symmetric pattern. While as η_ε tends to some positive constant η_0. The spike height may be the same or different yielding, respectively, symmetric and asymmetric patterns. Specifically, in the following result we shall see that the existence of asymmetric pattern for two spikes and such a solution is not stable

[th1.asy] **Theorem 1.4.** *Let* $\mathbf{p}^0 = (p_1^0, p_2^0) = (-\frac{1}{2}, \frac{1}{2})$. *Suppose that*

$$\eta_0 := \lim_{\varepsilon \to 0} \frac{2\beta^2}{\pi} \log \frac{1}{\varepsilon} > 2,$$

then for ε *sufficiently small and* $D = \frac{1}{\beta^2}$ *sufficiently large, problem* (1.3) *has a solution* $u_\varepsilon, v_\varepsilon$ *such that*

$$u_\varepsilon \sim \sum_{j=1}^{2} \xi_{\varepsilon,j} w\left(\frac{x - p_j^\varepsilon}{\varepsilon}\right), \quad v_\varepsilon(p_j^\varepsilon) \sim \xi_{\varepsilon,j}, \tag{1.14}$$ [1.asy-concl

where

$$\xi_{\varepsilon,1} = \frac{1}{\pi\varepsilon}\left(\frac{1}{2\eta_0} + \frac{\sqrt{1-\frac{4}{\eta_0^2}}}{4+2\eta_0}\right)(1+O(\beta^2)), \quad \xi_{\varepsilon,2)} = \frac{1}{\pi\varepsilon}\left(\frac{1}{2\eta_0} - \frac{\sqrt{1-\frac{4}{\eta_0^2}}}{4+2\eta_0}\right)(1+O(\beta^2)). \tag{1.15}$$ [1.asy-heigh

Furthermore, the solution $(u_\varepsilon, v_\varepsilon)$ *is linearly unstable for any* $\tau > 0$.

Remark: From Theorem 1.4 we have seen that when $\eta_0 > 2$ there exists asymmetric patterns. Besides, such a solution is always unstable due to the large eigenvalue is always positive. This shows a striking difference to the symmetric pattern.

Before we end the introduction, we would like to give some remarks on our proof for the results of the symmetric and asymmetric patterns. Since the proof of the existence part for both cases are almost the same, we shall only focus on the symmetric case and state the different points if necessary for asymmetric case. While for the stability, as we shall see, one of the spectrums for the large eigenvalue problem of the asymmetric case is always positive, it leads to the instability of the asymmetric pattern. So, in the small eigenvalue problem we shall always consider the symmetric case.

The paper is organized as follows: in section 2, we shall present some preliminary results, including the study of two nonlocal eigenvalue problems and the calculations on the height of the spikes. In section 3, we rigorously prove the existence of the symmetric and asymmetric patterns. In section 4, we consider the stability for the constructed solutions by studying the associated large and small eigenvalue problems. We also derive some properties on the Green's function $G_\beta(x,z)$, and these properties are useful in our study on the small eigenvalue problem. In section 5 we give the proof of Proposition 1.3, and this part has independent interest. Some numerical explanation is given in the Appendix.

[sec:prelim] 2. PRELIMINARIES

In this section we collect several key preliminary results needed for the existence and stability proofs in §3 and §4. Let w be the ground state solution satisfying

$$\begin{cases} (-\Delta)^{\frac{1}{2}} w + w - w^2 = 0, & \text{in} \quad \mathbb{R}, \\ w(x) \to 0 & \text{as} \quad |x| \to \infty, \end{cases} \tag{2.1}$$ [eq:core-pro

we have the following result [7] (also see Proposition 4.1 in [43] and the references therein)

[pr3.1] **Proposition 2.1.** *Equation* (2.1) *admits a positive, radially symmetric solution satisfying the following properties:*

(a) *The solution* w *and its derivative have the following expression*

$$w(x) = \frac{2}{1+x^2} \quad \text{and} \quad w'(x) = -\frac{4x}{(1+x^2)^2}.$$

(b) *Let* $L_0 = (-\Delta)^{\frac{1}{2}} + 1 - 2w$ *be the linearized operator. Then we have*

$$\mathrm{Ker}(L_0) = \mathrm{span}\left\{\frac{\partial w}{\partial x}\right\}.$$

(c) *Considering the following eigenvalue problem*

$$(-\Delta)^s\phi + \phi - 2w\phi + \mu_1\phi = 0.$$

There is an unique positive eigenvalue $\mu_1 > 0$.

For the linearized operator L_0, one can easily derive the following useful indentities

$$L_0 w = -w^2, \quad L_0\left(w + x\cdot\partial_x w\right) = -w.$$

Hence

$$\int_{\mathbb{R}}(L_0^{-1}w)w dx = \int_{\mathbb{R}}\left(-x\cdot\partial_x w - w\right)w dx = -\frac{1}{2}\int_{\mathbb{R}} w^2 dx,$$

and

$$\int_{\mathbb{R}}(L_0^{-1}w)w^2 dx = -\int_{\mathbb{R}} L_0^{-1}wL_0 w dx = -\int_{\mathbb{R}} w^2 dx.$$

Next we recall two stability results of the nonlocal eigenvalue problem. The reader can find the proof in [12, Theorem 3.2, Theorem 3.3].

th3.stability **Theorem 2.2.** *Consider the following nonlocal eigenvalue problem*

$$(-\Delta)^s\phi + \phi - 2w\phi + \gamma\frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx}w^2 + \alpha\phi = 0. \tag{2.2}$$ 3.2

(1) *If* $\gamma < 1$, *then there is a eigenvalue* α *to* (2.2) *such that* $\Re(\alpha) > 0$.
(2) *If* $\gamma > 1$ *and* $s > \frac{1}{4}$, *then for any nonzero eigenvalue* α *of* (2.2), *we have*

$$\Re(\alpha) \le -c_0 < 0.$$

(3) *If* $\gamma \neq 1$ *and* $\alpha = 0$, *then* $\phi = c_0\partial_x w$ *for some constant* c_0.

th3.2 **Theorem 2.3.** *Consider the following nonlocal eigenvalue problem*

$$(-\Delta)^s\phi + \phi - 2w\phi + \gamma(\tau\alpha)\frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx}w^2 + \alpha\phi = 0, \tag{2.3}$$ 3.4

where $\gamma(\tau\alpha)$ *is a complex function of* $\tau\alpha$ *and satisfies that*

$$\gamma(0) \in \mathbb{R}, \quad |\gamma(\tau\alpha)| \le C \text{ for } \alpha_R \ge 0,\ \tau \ge 0. \tag{2.4}$$ 3.3

Then there is a small number $\tau_0 > 0$ *such that for* $\tau < \tau_0$,

(1) *if* $\gamma(0) < 1$, *then there is a positive eigenvalue to* (2.3);
(2) *if* $\gamma(0) > 1$ *and* $s > \frac{1}{4}$, *then for any nonzero eigenvalue* α *of* (2.3), *we have*

$$\Re(\alpha) \le -c_0 < 0.$$

Based on Theorems 2.2 and 2.3 we shall study the following two nonlocal eigenvalue problems:

$$L\phi := (-\Delta)^{\frac{1}{2}}\phi + \phi - 2w\phi + \gamma\frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx}w^2 + \lambda_0\phi = 0, \quad \phi \in H^1(\mathbb{R}), \tag{2.5}$$ A.1

where

(a). $\gamma = \frac{\mu}{1+\tau\lambda_0}$, where $\mu > 0,\ \tau \ge 0$.
(b). $\gamma = \frac{2(K+\eta_0(1+\tau\lambda_0))}{(K+\eta_0)(1+\tau\lambda_0)}$, where $\eta_0 > 0,\ \tau \ge 0$.

First, we study the problem (2.5) in case (a).

tha.1 **Theorem 2.4.** *Let* $\gamma = \frac{\mu}{1+\tau\lambda_0}$ *where* $\mu > 0$, $\tau \ge 0$ *and let* L *be defined in* (2.5).

(1). *If $\mu > 1$, then there exists a unique $\tau_1 > 0$ such that for $\tau > \tau_1$, equation (2.5) admits a positive eigenvalue, and for $\tau < \tau_1$, all nonzero eigenvalues of problem (2.5) satisfy $\Re(\lambda) < 0$. At $\tau = \tau_1$, the eigenvalue problem (2.5) has a hopf bifurcation.*

(2). *If $\mu < 1$, then L admits a positive eigenvalue $\lambda_0 > 0$.*

We prove Theorem 2.4 by the following two lemmas.

lea.1 **Lemma 2.5.** *If $\mu < 1$, then L has a positive eigenvalue $\lambda_0 > 0$.*

Proof. We may assume that ϕ is an even positive function, namely,

$$\phi \in H^1_e(\mathbb{R}) = \left\{ u \in H^1(\mathbb{R}) \mid u(y) = u(-y) \right\}.$$

Let L_0 be given in Proposition 2.1. Then by the second conclusion, L_0 is invertible in $H^1_e(\mathbb{R})$. Let us denote the inverse as L_0^{-1}. By Proposition 2.1, L_0 has a unique positive eigenvalue μ_1. It is easy to see that $\lambda_0 \neq \mu_1$ since we have $\int_{\mathbb{R}} w\phi_0 dx > 0$.

Then λ_0 is a eigenvalue of (2.5) if and only if it satisfies the following algebraic equation:

$$\int_{\mathbb{R}} w^2 dx = -\frac{\mu}{1+\tau\lambda_0} \int_{\mathbb{R}} ((L_0+\lambda_0)^{-1} w^2) w dx. \tag{2.6}$$ a.p-1

We can rewrite (2.6) as

$$\rho(\lambda_0) := (\mu - 1 - \tau\lambda_0) \int_{\mathbb{R}} w^2 dx - \mu\lambda_0 \int_{\mathbb{R}} \left((L_0+\lambda_0)^{-1} w \right) w dx = 0.$$

We notice that $\rho(0) = (\mu-1)\int_{\mathbb{R}} w^2 dx < 0$. On the other hand, as $\lambda_0 \to \mu_1$ from left, we have $\int_{\mathbb{R}}((L_0 + \lambda_0)^{-1} w) w dx \to -\infty$, and hence $\rho(\lambda_0) \to +\infty$. By continuity, there exists a $\lambda_0 \in (0, \mu_1)$ such that $\rho(\lambda_0) = 0$. Such a positive λ_0 will be a eigenvalue of L. □

When $\mu > 1$ we notice that the eigenvalues will not cross through zero: Indeed, if $\lambda_0 = 0$, then we have

$$L_0\phi + \mu \frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx} w^2 = 0,$$

which implies that

$$L_0 \left(\phi - \mu \frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx} w \right) = 0,$$

and hence, by Proposition 2.1,

$$\phi - \mu \frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx} w \in \mathrm{Ker}(L_0).$$

This is impossible since ϕ is radially symmetric and $\phi \neq cw$ for all $c \in \mathbb{R}$. As a consequence, there must be a point τ_1 at which L has a Hopf bifurcation, i.e., L has a purely imaginary eigenvalue $\alpha = \sqrt{-1}\alpha_I$. To prove Theorem 2.4, all we need to show that τ_1 is unique, that is,

lea.3 **Lemma 2.6.** *Let $\mu > 1$. There there exists an unique $\tau_1 > 0$ such that L has Hopf bifurcation.*

Proof. Let $\lambda_0 = \sqrt{-1}\alpha_I$ be a eigenvalue of L. We notice that $\sqrt{-1}\alpha_I$ is a eigenvalue of L then $-\sqrt{-1}\alpha_I$ is also a eigenvalue of L. Therefore, in the following we shall assume that $\alpha_I > 0$. Let $\phi_0 = -(L_0 + \sqrt{-1}\alpha_I)^{-1} w^2$. Then (2.5) becomes

$$\frac{\int_{\mathbb{R}} w\phi_0 dx}{\int_{\mathbb{R}} w^2 dx} = \frac{1+\tau\sqrt{-1}\alpha_I}{\mu}. \tag{2.7}$$ 3.11

Let $\phi_0 = \phi_0^R + \sqrt{-1}\phi_0^I$. Then from (2.7), we obtain the two equations

$$\frac{\int_{\mathbb{R}} w\phi_0^R dx}{\int_{\mathbb{R}} w^2 dx} = \frac{1}{\mu}, \quad \frac{\int_{\mathbb{R}} w\phi_0^I dx}{\int_{\mathbb{R}} w^2 dx} = \frac{\tau\alpha_I}{\mu}. \tag{2.8}$$ 3.12

We write (2.5) into its real and imaginary part. Then

$$-L_0\phi_0^R = w^2 - a_I\phi_0^I, \quad -L_0\phi_0^I = \alpha_I\phi_0^R. \tag{2.9}$$ 3.13

So $\phi_0^R = -\alpha_I^{-1} L_0 \phi_0^I$ and

$$\phi_0^I = \alpha_I (L_0^2 + \alpha_I^2)^{-1} w^2, \quad \phi_0^R = -L_0 (L_0^2 + \alpha_I^2)^{-1} w^2. \tag{2.10}$$

Substituting (2.10) into (2.8), we obtain

$$\frac{\int_{\mathbb{R}} w L_0 (L_0^2 + \alpha_I^2)^{-1} w^2 dx}{\int_{\mathbb{R}} w^2 dx} = -\frac{1}{\mu}, \quad \frac{\int_{\mathbb{R}} w (L_0^2 + \alpha_I^2)^{-1} w^2 dx}{\int_{\mathbb{R}} w^2 dx} = \frac{\tau}{\mu}. \tag{2.11}$$

Let $h(\alpha_I) = -\frac{\int_{\mathbb{R}} w L_0 (L_0^2+\alpha_I^2)^{-1} w^2 dx}{\int_{\mathbb{R}} w^2 dx}$. Then intgeration by parts gives $h(\alpha_I) = \frac{\int_{\mathbb{R}} w^2 (L_0^2+\alpha_I^2)^{-1} w^2 dx}{\int_{\mathbb{R}} w^2 dx}$. Note that $h'(\alpha_I) = -2\alpha_I \frac{\int_{\mathbb{R}} w^2 (L_0^2+\alpha_I^2)^{-2} w^2 dx}{\int_{\mathbb{R}} w^2 dx} < 0$. It is known that

$$h(0) = -\frac{\int_{\mathbb{R}} w (L_0^{-1} w^2) dx}{\int_{\mathbb{R}} w^2 dx} = 1,$$

$h(\alpha_I) \to 0$ as $\alpha_I \to +\infty$ and $\mu > 1$, there exists an unique $\alpha_I > 0$ such that the first equation of (2.11) holds. Substituting this unique α_I into the second one of (2.11), we obtain an unique $\tau = \tau_1 > 0$. Then Lemma 2.6 is proved. □

Next, we study the following NLEP:

$$(-\Delta)^{\frac{1}{2}} \phi + \phi - 2w\phi + \frac{2(K + \eta_0(1+\tau\lambda_0))}{(K+\eta_0)(1+\tau\lambda_0)} \frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx} w^2 + \lambda_0 \phi = 0, \quad \phi \in H^1(\mathbb{R}), \tag{2.12}$$

where $\eta_0 \in (0, +\infty)$ and $\tau \in [0, +\infty)$. Then we have

Theorem 2.7. *Consider the eigenvalue problem (2.12), we have:*

(1) *If $\eta_0 < K$, then for τ small, problem (2.12) is stable, while for τ large it is unstable.*
(2) *If $\eta_0 > K$, then there exists $0 < \tau_2 \le \tau_3$ such that problem (2.12) is stable for $\tau < \tau_2$ or $\tau > \tau_3$.*

Proof. Let us set

$$f(\tau\lambda) = \frac{2(K + \eta_0(1+\tau\lambda))}{(K+\eta_0)(1+\tau\lambda)}. \tag{2.13}$$

We note that

$$\lim_{\tau\lambda \to +\infty} f(\tau\lambda) = \frac{2\eta_0}{K+\eta_0} := f_\infty.$$

If $\eta_0 < K$, then by Theorem 2.2, problem (2.2) with $\gamma = f_\infty$ has a positive eigenvalue α_1. Now by perturbation arguments, for τ large, problem (2.12) has a eigenvalue near $\alpha_1 > 0$. This implies that for τ large, problem (2.12) is unstable.

Now we show that problem (2.12) has no nonzero eigenvalues with nonnegative real part, provided that either τ is small or $\eta_0 > K$ and τ is large. We apply the following inequality ([12, Lemma A.2]): For any real-valued function $\phi \in H_e^1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} (|(-\Delta)^{\frac{1}{4}} \phi|^2 + |\phi|^2 - 2w\phi^2) dx + 2 \frac{\int_{\mathbb{R}} w\phi dx \int_{\mathbb{R}} w^2 \phi dx}{\int_{\mathbb{R}} w^2 dx} - \frac{\int_{\mathbb{R}} w^3 dx}{\left(\int_{\mathbb{R}} w^2 dx\right)^2} \left(\int_{\mathbb{R}} w\phi dx \right)^2 \ge 0, \tag{2.14}$$

where equality holds if and only if ϕ is a multiple of w.

In (2.12) we set $\lambda_0 = \lambda_R + \sqrt{-1}\lambda_I$ and $\phi = \phi_R + \sqrt{-1}\phi_I$, we get

$$L_0 \phi + f(\tau\lambda_0) \frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx} w^2 + \lambda_0 \phi = 0. \tag{2.15}$$

Multiplying the above equation by $\bar{\phi}$, the conjugate function of ϕ and integrating over $\mathbb{R}$, we have

$$\int_{\mathbb{R}} (|(-\Delta)^{\frac{1}{4}} \phi|^2 + |\phi|^2 - 2w\phi^2) dx = -\lambda_0 \int_{\mathbb{R}} |\phi|^2 dx - f(\tau\lambda_0) \frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx} \int_{\mathbb{R}} w^2 \bar{\phi} dx. \tag{2.16}$$

Multiplying (2.15) by w and integrating over $\mathbb{R}$, we get that

$$\int_{\mathbb{R}} w^2 \phi dx = \left(\lambda_0 + f(\tau\lambda_0) \frac{\int_{\mathbb{R}} w^3 dx}{\int_{\mathbb{R}} w^2 dx} \right) \int_{\mathbb{R}} w\phi dx. \tag{2.17}$$

Taking the conjugate of (2.17), we have

$$\int_{\mathbb{R}} w^2 \bar{\phi} dx = \left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0) \frac{\int_{\mathbb{R}} w^3 dx}{\int_{\mathbb{R}} w^2 dx} \right) \int_{\mathbb{R}} w\bar{\phi} dx. \tag{2.18}$$

Substituting (2.18) into (2.16), we have that

$$\int_{\mathbb{R}} \left(|(-\Delta)^{\frac{1}{4}}\phi|^2 + |\phi|^2 - 2w|\phi|^2 \right) dx = -\lambda_0 \int_{\mathbb{R}} |\phi|^2 dx - f(\tau\lambda_0) \left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0) \frac{\int_{\mathbb{R}} w^3 dx}{\int_{\mathbb{R}} w^2 dx} \right) \frac{|\int_{\mathbb{R}} w\phi dx|^2}{\int_{\mathbb{R}} w^2 dx}. \tag{2.19}$$

Consider the real part of (2.19). By (2.14) and (2.18), we get

$$-\lambda_R \geq \Re \left(f(\tau\lambda_0) \left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0) \frac{\int_{\mathbb{R}} w^3 dx}{\int_{\mathbb{R}} w^2 dx} \right) \right) - 2\Re \left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0) \frac{\int_{\mathbb{R}} w^3 dx}{\int_{\mathbb{R}} w^2 dx} \right) + \frac{\int_{\mathbb{R}} w^3 dx}{\int_{\mathbb{R}} w^2 dx}, \tag{2.20}$$

where we used $\lambda_0 = \lambda_R + \sqrt{-1}\lambda_I$ with $\lambda_R,\ \lambda_I \in \mathbb{R}$.

Assuming that $\lambda_R \geq 0$, then we have

$$\frac{\int_{\mathbb{R}} w^3 dx}{\int_{\mathbb{R}} w^2 dx} |f(\tau\lambda_0) - 1|^2 + \Re(\bar{\lambda}_0 (f(\tau\lambda_0) - 1)) \leq 0. \tag{2.21}$$

By direct computation, we see that

$$\int_{\mathbb{R}} w^3 dx = \frac{3}{2} \int_{\mathbb{R}} w^2 dx = 3\pi. \tag{2.22}$$

Substituting (2.22) and the expression (2.13) for $f(\tau\lambda)$ into (2.21), we have

$$\frac{3}{2} |\eta_0 + K + (\eta_0 - K)\tau\lambda_0|^2 + \Re \left((\eta_0 + K)(1 + \tau\bar{\lambda}_0)((\eta_0 + K)\bar{\lambda}_0 + (\eta_0 - K)\tau|\lambda_0|^2) \right) \leq 0,$$

which is equivalent to

$$\frac{3}{2}(1 + \mu_0\tau\lambda_R)^2 + \lambda_R + (\mu_0\tau\lambda_R + \tau\lambda_R + \mu_0\tau^2|\lambda_0|^2)\lambda_R + \left(\frac{3}{2}\mu_0^2\tau^2 + \mu_0\tau - \tau \right) \lambda_I^2 \leq 0, \tag{2.23}$$

where we have introduced that $\mu_0 := \dfrac{\eta_0 - K}{\eta_0 + K}$.

If $\eta_0 > K$ (i.e., $\mu_0 > 0$) and τ is large, then

$$\frac{3}{2}\mu_0^2\tau^2 + \mu_0\tau - \tau \geq 0.$$

So (2.23) does not hold for $\lambda_R \geq 0$. To consider the case when τ is small, we have now derived an upper bound for λ_I. From (2.16), we have

$$\lambda_I \int_{\mathbb{R}} |\phi|^2 dx = \Im \left(-f(\tau\lambda_0) \frac{\int_{\mathbb{R}} w\phi dx}{\int_{\mathbb{R}} w^2 dx} \int_{\mathbb{R}} w^2 \bar{\phi} dx \right).$$

Hence,

$$|\lambda_I| \leq |f(\tau\lambda_0)| \sqrt{\frac{\int_{\mathbb{R}} w^4 dx}{\int_{\mathbb{R}} w^2 dx}} \leq C, \tag{2.24}$$

where C is independent of λ_0. Substituting (2.24) into (2.23), we see that (2.23) cannot hold for $\lambda_R \geq 0$, if τ is small. Thus we have proved that the (2) point of Theorem 2.7. □

2.1. **Calculating the Height of the spikes.** Let χ be a smooth cut-off function which is equal to 1 in $B_1(0)$ and equals to 0 in $\mathbb{R}\setminus\overline{B_2(0)}$. We also assume that a multiple spike solution $(u_\varepsilon, v_\varepsilon)$ of (1.3) is given by the following ansatz:

$$u_\varepsilon \sim \sum_{j=1}^{K} \xi_{\varepsilon,j} w\left(\frac{x-p_j}{\varepsilon}\right)\chi\left(\frac{x-p_j}{r_0}\right), \quad v_\varepsilon(p_j)\sim \xi_{\varepsilon,j}, \tag{2.25}$$

where $w=\frac{2}{1+|x|^2}$ is the unique solution to (2.1), $\xi_{\varepsilon,j},\ j=1,\cdots,K$ are the height of the peaks, to be determined later, $\mathbf{p}=(p_1,\cdots,p_K)$ are the location of the points and satisfy

$$\mathbf{p}=(p_1,\cdots,p_K)\in B_\sigma(\mathbf{p}^0),\ \ p_j^0=\frac{2j-1-K}{K},\ \ j=1,\cdots,K, \quad \sigma\ll 1.$$

Now we shall derive a relation between each $\xi_{\varepsilon,j}$. We write the second equation of (1.3) as

$$(-\Delta)^{\frac{1}{2}}v_\varepsilon+\beta^2 v_\varepsilon-\beta^2 u_\varepsilon^2=0, \tag{2.26}$$

we get by using (1.5) and (2.26),

$$\begin{aligned}
v_\varepsilon(p_j) &= \beta^2\int_{-1}^{1} G_\beta(p_j,z)u_\varepsilon^2(z)dz\\
&= \beta^2\int_{-1}^{1}\left(\frac{\beta^{-2}}{2}+G_0(p_j,z)+O(\beta^2)\right)\left(\sum_{\ell=1}^{K}\xi_{\varepsilon,\ell}^2 w^2\left(\frac{z-p_\ell}{\varepsilon}\right)+O(\varepsilon^2)\right)dz\\
&= \int_{-1}^{1}\left(\frac{1}{2}+\beta^2 G_0(p_j,z)+O(\beta^4)\right)\left(\sum_{\ell=1}^{K}\xi_{\varepsilon,\ell}^2 w^2\left(\frac{z-p_\ell}{\varepsilon}\right)+O(\varepsilon^2)\right)dz.
\end{aligned}$$

Thus,

$$\xi_{\varepsilon,j}=\sum_{\ell=1}^{K}\frac{1}{2}\varepsilon\xi_{\varepsilon,\ell}^2\int_{\mathbb{R}}w^2(y)dy+\xi_{\varepsilon,j}^2\beta^2\int_{-1}^{1}G_0(p_j,z)w^2\left(\frac{z-p_j}{\varepsilon}\right)dz+O(\varepsilon\beta^2)\sum_{\ell=1}^{K}\xi_{\varepsilon,\ell}^2. \tag{2.27}$$

Then we get

$$\begin{aligned}
\xi_{\varepsilon,j} &= \sum_{\ell=1}^{K}\frac{1}{2}\varepsilon\xi_{\varepsilon,\ell}^2\int_{\mathbb{R}}w^2(y)dy+\frac{1}{\pi}\xi_{\varepsilon,j}^2\beta^2\int_{-1}^{1}\log\frac{1}{|z-p_j|}w^2\left(\frac{z-p_j}{\varepsilon}\right)dz+O(\varepsilon\beta^2)\sum_{\ell=1}^{K}\xi_{\varepsilon,\ell}^2\\
&= \sum_{\ell=1}^{K}\frac{1}{2}\varepsilon\xi_{\varepsilon,\ell}^2\int_{\mathbb{R}}w^2(y)dy+\frac{1}{\pi}\varepsilon\xi_{\varepsilon,j}^2\beta^2\log\frac{1}{\varepsilon}\int_{\mathbb{R}}w^2(y)dy+O(\varepsilon\beta^2)\sum_{\ell=1}^{K}\xi_{\varepsilon,\ell}^2.
\end{aligned} \tag{2.28}$$

Define

$$\xi_{\varepsilon,j}=\frac{2\hat{\xi}_{\varepsilon,j}}{\varepsilon\int_{\mathbb{R}}w^2(y)dy}.$$

Then (2.28) is equivalent to

$$\hat{\xi}_{\varepsilon,j}=\sum_{\ell=1}^{K}\hat{\xi}_{\varepsilon,\ell}^2+\eta_\varepsilon\hat{\xi}_{\varepsilon,j}^2+O(\beta^2)\sum_{\ell=1}^{K}\hat{\xi}_{\varepsilon,\ell}^2, \quad j=1,\cdots,K, \tag{2.29}$$

where

$$\eta_\varepsilon=\frac{2\beta^2}{\pi}\log\frac{1}{\varepsilon}.$$

Next, we shall divide our discussion on (2.29) into three cases according to the limit value of η_ε,

Case 1. $\eta_\varepsilon\to 0$. We always get the symmetric pattern

$$\hat{\xi}_{\varepsilon,j}=\frac{1}{K}+O(\eta_\varepsilon), \quad j=1,\cdots,K.$$

This implies that

$$\xi_{\varepsilon,j}=\frac{1}{\varepsilon K\pi}(1+O(\eta_\varepsilon)), \quad j=1,\cdots,K. \tag{2.30}$$

Case 2. $\eta_\varepsilon \to \infty$. As Case 1 we only get the symmetric pattern. From (2.29) we have

$$\hat{\xi}_{\varepsilon,j} = \eta_\varepsilon \hat{\xi}_{\varepsilon,j}^2 + O(1) \sum_{\ell=1}^{K} \hat{\xi}_{\varepsilon,\ell}^2.$$

Then we could get

$$\xi_{\varepsilon,j} = \frac{1}{\varepsilon \eta_\varepsilon \pi} \left(1 + O\left(\frac{1}{\eta_\varepsilon} \right) \right), \quad j = 1, \cdots, K. \tag{2.31}$$

3.h-2

Case 3. $\eta_\varepsilon \to \eta_0$. $(0 < \eta_0 < \infty)$. Then from (2.29) we get

$$\hat{\xi}_{\varepsilon,j} = (1+\eta_0)\hat{\xi}_{\varepsilon,j}^2 + \sum_{\ell \neq j} \hat{\xi}_{\varepsilon,\ell}^2 + O(\beta^2) \sum_{\ell=1}^{K} \hat{\xi}_{\varepsilon,\ell}^2.$$

For the *symmetric pattern* we have

$$\hat{\xi}_{\varepsilon,1} = \cdots = \hat{\xi}_{\varepsilon,K} = \frac{1}{K+\eta_0} \left(1 + O(\beta^2) \right),$$

or equivalently,

$$\xi_{\varepsilon,j} = \frac{1}{\varepsilon(K+\eta_0)\pi} (1 + O(\beta^2)), \quad j = 1, \cdots, K. \tag{2.32}$$

3.h-3

While in the asymmetric case, we take two spikes into consideration and obtain the following system

$$\begin{cases} \hat{\xi}_{\varepsilon,1} = (1+\eta_0)\hat{\xi}_{\varepsilon,1}^2 + \hat{\xi}_{\varepsilon,2}^2 + O(\beta^2) \sum_{j=1}^{2} \hat{\xi}_{\varepsilon,j}^2, \\ \hat{\xi}_{\varepsilon,2} = (1+\eta_0)\hat{\xi}_{\varepsilon,2}^2 + \hat{\xi}_{\varepsilon,1}^2 + O(\beta^2) \sum_{j=1}^{2} \hat{\xi}_{\varepsilon,j}^2. \end{cases} \tag{2.33}$$

3.asy-2

From (2.33) we derive that

$$\hat{\xi}_{\varepsilon,1} + \hat{\xi}_{\varepsilon,2} = \frac{1}{\eta_0} (1 + O(\beta^2)).$$

As a consequence, we have

$$(2+\eta_0)\hat{\xi}_{\varepsilon,j}^2 - \left(\frac{2}{\eta_0} + 1 \right) \hat{\xi}_{\varepsilon,j} + \frac{1}{\eta_0^2} + O(\beta^2) \sum_{\ell=1}^{2} \hat{\xi}_{\varepsilon,\ell}^2 = 0, \quad j = 1, 2.$$

Solving the above quadratic equation we have

$$\hat{\xi}_{\varepsilon,j} = \frac{1}{2\eta_0} \pm \frac{\sqrt{1 - \frac{4}{\eta_0^2}}}{4 + 2\eta_0} + O(\beta^2), \quad j = 1, 2. \tag{2.34}$$

3.asy

Then

$$\xi_{\varepsilon,i} = \frac{1}{\pi\varepsilon} \left(\frac{1}{2\eta_0} \pm \frac{\sqrt{1 - \frac{4}{\eta_0^2}}}{4 + 2\eta_0} \right) (1 + \beta^2), \quad j = 1, 2. \tag{2.35}$$

3.h-4

For the symmetric pattern, we notice that in all three cases the heights satisfy the relation

$$\xi_{\varepsilon,j} = \xi_\varepsilon (1 + O(h(\varepsilon, \beta))), \quad j = 1, \cdots, K, \tag{2.36}$$

3.spike-h-1

where

$$\xi_\varepsilon = \begin{cases} \frac{1}{\varepsilon K \pi}, & \text{if} \quad \eta_\varepsilon \to 0, \\ \frac{1}{\varepsilon \eta_\varepsilon \pi}, & \text{if} \quad \eta_\varepsilon \to +\infty, \\ \frac{1}{\varepsilon(\eta_0 + K)\pi}, & \text{if} \quad \eta_\varepsilon \to \eta_0, \end{cases} \tag{2.37}$$

3.spike-h

and

$$h(\varepsilon, \beta) = \begin{cases} \eta_\varepsilon, & \text{if} \quad \eta_\varepsilon \to 0, \\ \eta_\varepsilon^{-1}, & \text{if} \quad \eta_\varepsilon \to \infty, \\ \beta^2, & \text{if} \quad \eta_\varepsilon \to \eta_0. \end{cases}$$

While for the asymmetric pattern, we have

$$\xi_{\varepsilon,1} = \frac{1}{\pi\varepsilon}\left(\frac{1}{2\eta_0} + \frac{\sqrt{1-\frac{4}{\eta_0^2}}}{4+2\eta_0}\right)(1+\beta^2), \quad \xi_{\varepsilon,2} = \frac{1}{\pi\varepsilon}\left(\frac{1}{2\eta_0} - \frac{\sqrt{1-\frac{4}{\eta_0^2}}}{4+2\eta_0}\right)(1+\beta^2). \tag{2.38}$$

3. Rigorous proof of the existence results

In this section we shall prove the existence theorem, i.e., Theorem 1.1. We divide the discussion into three sections. First of all, we give an approximate solution. Then we apply the classical Liapunov-Schmidt reduction method to reduce the infinite dimensional problem to a finite dimensional problem in second subsection. In last subsection we solve the finite dimensional problem and thereby prove the Theorem 1.1. As we pointed out in the introduction, the proof for the symmetric and asymmetric patterns are almost the same, we shall only focus on the symmetric case and state the difference for the asymmetric case in the end of this section.

3.1. **Study of the Approximate Solutions.** From the discussion in last section, we rescale

$$\begin{cases} \hat{u}(y) = \frac{1}{\xi_\varepsilon} u(\varepsilon y), & y \in \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), \\ \hat{v}(x) = \frac{1}{\xi_\varepsilon} v(x), & x \in (-1,1), \end{cases}$$

where ξ_ε is given in (2.37). The equilibrium solution $(\hat{u}, \hat{v})$ solves the following rescaled Gierer-Meinhardt system

$$\begin{cases} (-\Delta)_y^{\frac{1}{2}} \hat{u} + \hat{u} - \frac{\hat{u}^2}{\hat{v}} = 0, & y \in \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), \\ (-\Delta)_x^{\frac{1}{2}} \hat{v} + \beta^2 \hat{v} - \xi_\varepsilon \beta^2 \hat{u}^2 = 0, & x \in (-1,1). \end{cases} \tag{3.1}$$

For a function $\hat{u} \in H^1\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right)$, let $T[\hat{u}]$ be the unique solution of the following problem:

$$\begin{cases} (-\Delta)^{\frac{1}{2}} T[\hat{u}] + \beta^2 T[\hat{u}] - \xi_\varepsilon \beta^2 \hat{u}^2 = 0 & x \in (-1,1), \\ T[\hat{u}](x) = T[\hat{u}](x+2) & x \in \mathbb{R}. \end{cases}$$

By Green representation formula, we have

$$T[\hat{u}](x) = \xi_\varepsilon \beta^2 \int_{-1}^{1} G_\beta(x,\zeta)\left(\hat{u}\left(\frac{\zeta}{\varepsilon}\right)\right)^2 d\zeta.$$

System (3.1) is equivalent to the following equation in operator form:

$$S_\varepsilon(\hat{u},\hat{v}) = \begin{pmatrix} S_1(\hat{u},\hat{v}) \\ S_2(\hat{u},\hat{v}) \end{pmatrix} = 0, \quad H^1\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times H^1(-1,1) \to L^2\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times L^2(-1,1), \tag{3.2}$$

where

$$\begin{cases} S_1(\hat{u},\hat{v}) = (-\Delta)_y^{\frac{1}{2}} \hat{u} + \hat{u} - \frac{\hat{u}^2}{\hat{v}}, \\ S_2(\hat{u},\hat{v}) = (-\Delta)_x^{\frac{1}{2}} \hat{v} + \beta^2 \hat{v} - \xi_\varepsilon \beta^2 \hat{u}^2. \end{cases} \tag{3.3}$$

For $\mathbf{p} \in B_\sigma(\mathbf{p}^0)$ we set

$$w_j(y) = w\left(y - \frac{p_j}{\varepsilon}\right)\chi\left(\frac{\varepsilon y - p_j}{r_0}\right),$$

where $w(y) = \frac{2}{1+y^2}$ is the ground state solution of (2.1), it is then straightforward to check that

$$(-\Delta)_y^{\frac{1}{2}} w_j(y) + w_j(y) - w_j^2(y) = h.o.t.,$$

where $h.o.t.$ refers to terms of order ε^2 in $L^\infty\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right)$. [1]

---

[1]More specifically, $h.o.t.$ means term which can be composed into two parts, the leading order is of ε^2 and even symmetric with respect to p_i, while the order of the left part is $o(\varepsilon^2)$.

We choose the approximate solutions as follows:

$$u_{\varepsilon,\mathbf{p}}(y)=\sum_{j=1}^{K} w_j(y), \quad v_{\varepsilon,\mathbf{p}}(x)=T[u_{\varepsilon,\mathbf{p}}](x), \quad x=\varepsilon y\in(-1,1). \tag{3.4}$$

4.inh

Notice that $v_{\varepsilon,\mathbf{p}}$ satisfies

$$\begin{aligned}0&=(-\Delta)_x^{\frac{1}{2}} v_{\varepsilon,\mathbf{p}}+\beta^2 v_{\varepsilon,\mathbf{p}}-\xi_\varepsilon\beta^2 u_{\varepsilon,\mathbf{p}}^2\\&=(-\Delta)_x^{\frac{1}{2}} v_{\varepsilon,\mathbf{p}}+\beta^2 v_{\varepsilon,\mathbf{p}}-\xi_\varepsilon\beta^2\sum_{j=1}^{K} w_j^2-2\xi_\varepsilon\beta^2\sum_{\ell\neq j} w_\ell w_j.\end{aligned}$$

Hence,

$$v_{\varepsilon,\mathbf{p}}(p_j)=\xi_\varepsilon\beta^2\int_{-1}^{1} G_\beta(p_j,\zeta)\sum_{\ell=1}^{K} w_\ell^2\left(\frac{\zeta}{\varepsilon}\right)d\zeta+O(\xi_\varepsilon\beta^2\varepsilon^2).$$

Similar to the computation as in section, we otain

$$v_{\varepsilon,\mathbf{p}}(p_j)=1+O(h(\varepsilon,\beta)).$$

Substituting the ansatz (3.4) into (3.3) we get

$$S_2(u_{\varepsilon,\mathbf{p}},v_{\varepsilon,\mathbf{p}})=0,$$

To compute $S_1(u_{\varepsilon,\mathbf{p}},v_{\varepsilon,\mathbf{p}})$, we calculate for $x=p_j+\varepsilon z,\ |\varepsilon z|<\rho$ with $j=1,\cdots,K$ and ρ small

$$\begin{aligned}v_{\varepsilon,\mathbf{p}}(p_j+\varepsilon z)-v_{\varepsilon,\mathbf{p}}(p_j)&=\xi_\varepsilon\beta^2\int_{-1}^{1}\left(G_\beta(p_j+\varepsilon z,\zeta)-G_\beta(p_j,\zeta)\right)u_{\varepsilon,\mathbf{p}}^2 d\zeta\\&=\xi_\varepsilon\beta^2\int_{-1}^{1}\left(G_\beta(p_j+\varepsilon z,\zeta)-G_\beta(p_j,\zeta)\right)w_j^2 d\zeta\\&\quad+\xi_\varepsilon\beta^2\int_{-1}^{1}\left(G_\beta(p_j+\varepsilon z,\zeta)-G_\beta(p_j,\zeta)\right)\sum_{\ell\neq j} w_\ell^2 d\zeta+O(\xi_\varepsilon\beta^2\varepsilon^2) \qquad (3.5)\\&=\xi_\varepsilon\beta^2\varepsilon\int_{\mathbb{R}}\frac{1}{\pi}\log\frac{|\zeta|}{|z-\zeta|}w^2(\zeta)d\zeta-\xi_\varepsilon\beta^2\varepsilon\left(\frac{\partial F(\mathbf{p})}{\partial p_j}\varepsilon z\int_{\mathbb{R}} w^2(\zeta)d\zeta\right)\\&\quad+o(\xi_\varepsilon\beta^2\varepsilon^2|z|),\end{aligned}$$

4.poten

where

$$F(\mathbf{p})=\sum_{j=1}^{K} H_\beta(p_j,p_j)-\sum_{i\neq j} G_\beta(p_i,p_j).$$

For convenience, in the following discussion we shall denote the first term on the right-hand side of (3.5) by $P_j(z)$. It is not difficult to verify that $P_j(z)$ is even symmetric in z. Substituting (3.5) into $S_1(u_{\varepsilon,\mathbf{p}},v_{\varepsilon,\mathbf{p}})$ we have

$$\begin{aligned}S_1(u_{\varepsilon,\mathbf{p}},v_{\varepsilon,\mathbf{p}})&=(-\Delta)^{\frac{1}{2}}u_{\varepsilon,\mathbf{p}}+u_{\varepsilon,\mathbf{p}}-\frac{u_{\varepsilon,\mathbf{p}}^2}{v_{\varepsilon,\mathbf{p}}}\\&=\sum_{j=1}^{K}\chi\left(\frac{\varepsilon y-p_j}{r_0}\right)(-\Delta)^{\frac{1}{2}}w\left(y-\frac{p_j}{\varepsilon}\right)+\sum_{j=1}^{K}\chi\left(\frac{\varepsilon y-p_j}{r_0}\right)w\left(y-\frac{p_j}{\varepsilon}\right)-\sum_{j=1}^{K}\frac{w_j^2}{v_{\varepsilon,\mathbf{p}}}+O(\varepsilon^2)\\&=E_1+E_2+O(\varepsilon^2)\quad\text{in}\quad L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right),\end{aligned}$$

where

$$E_1=\sum_{j=1}^{K}\chi\left(\frac{\varepsilon y-p_j}{r_0}\right)w^2\left(y-\frac{p_j}{\varepsilon}\right)-\sum_{j=1}^{K} w_j^2,\quad\text{and}\quad E_2=\sum_{j=1}^{K} w_j^2-\frac{\sum_{j=1}^{K} w_j^2}{v_{\varepsilon,\mathbf{p}}}.$$

According to the setting of cut-off function $\chi(x)$, we have

$$E_1 = O(\varepsilon^4),$$

and one can easily check that

$$\|E_1\|_{L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)} = O(\varepsilon^{7/2}). \tag{3.6}$$ 4.11

In addition, for $x - p_j = \varepsilon z$ with $|\varepsilon z| < \rho$ with ρ small, we calculate

$$\begin{aligned}
E_2 &= \frac{w_j^2}{v_{\varepsilon,\mathbf{p}}^2(p_j)}(v_{\varepsilon,\mathbf{p}}(x) - v_{\varepsilon,\mathbf{p}}(p_j))\left(1 + \sum_{n=1}^{\infty}\left(\frac{v_{\varepsilon,\mathbf{p}}(p_j) - v_{\varepsilon,\mathbf{p}}(x)}{v_{\varepsilon,\mathbf{p}}(p_j)}\right)^n\right) + O(h(\varepsilon,\beta))w_j^2 + O(\varepsilon^4) \\
&= \frac{w_j^2}{v_{\varepsilon,\mathbf{p}}^2(p_j)}P_j(z)\left(1 + \sum_{n=1}^{\infty}\left(\frac{-P_j(z)}{v_{\varepsilon,\mathbf{p}}(p_j)}\right)^n\right) + O(h(\varepsilon,\beta))w_j^2 - \frac{w_j^2}{v_{\varepsilon,\mathbf{p}}^2(p_j)}\xi_\varepsilon\beta^2\varepsilon^2\frac{\partial F(\mathbf{p})}{\partial p_j}z\int_{\mathbb{R}} w^2(\zeta)d\zeta \\
&\quad + o(\xi_\varepsilon\beta^2\varepsilon^2) \\
&= E_{21} + E_{22} + o(\xi_\varepsilon\beta^2\varepsilon^2),
\end{aligned} \tag{3.7}$$ 4.12

where

$$E_{21} = O(\xi_\varepsilon\beta^2\varepsilon) + O(h(\varepsilon,\beta)) \text{ is symmetry in } x - p_j, \quad \text{and} \quad \|E_{22}\|_{L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)} = O(\xi_\varepsilon\beta^2\varepsilon^2). \tag{3.8}$$ 4.13

Thus, we have thus established the following lemma

le4.1 **Lemma 3.1.** *For $x = p_j + \varepsilon z$, $|\varepsilon z| < \rho$, we have the decomposition for $S[u_{\varepsilon,\mathbf{p}}](x)$,*

$$S_1\left(u_{\varepsilon,\mathbf{p}}, v_{\varepsilon,\mathbf{p}}\right) = S_{1,1} + S_{1,2},$$

where

$$S_{1,1}(z) = -\frac{w_j^2}{v_{\varepsilon,\mathbf{p}}^2(p_j)}\xi_\varepsilon\beta^2\varepsilon^2\frac{\partial F(\mathbf{p})}{\partial p_j}z\int_{\mathbb{R}} w^2(\zeta)\zeta + o(\xi_\varepsilon\beta^2\varepsilon^2),$$

and

$$S_{1,2}(z) = \xi_\varepsilon\beta^2\varepsilon R_{j1}(z) + h(\varepsilon,\beta)R_{j2}(z) + o(\xi_\varepsilon\beta^2\varepsilon^2),$$

where $R_{j1}(z)$, $R_{j2}(z)$ are even in z satisfying that $R_{j1}(|z|) = O(\log(1+|z|))$ and $R_{j2}(z) = O\left(\frac{1}{1+|z|^2}\right)$. Furthermore,

$$S_1\left(u_{\varepsilon,\mathbf{p}}, v_{\varepsilon,\mathbf{p}}\right) = O(\varepsilon^2) \quad \text{for} \quad |x - p_j| \geq \rho,\ \forall j = 1,\cdots,K.$$

3.2. **The Liapunov-Schmidt Reduction Method.** In this subsection, we use the Liapunov-Schmidt reduction method to solve the problem

$$S[u_{\varepsilon,\mathbf{p}} + \phi] := S_1\left(u_{\varepsilon,\mathbf{p}} + \phi, v_{\varepsilon,\mathbf{p}} + \psi\right) = \sum_{j=1}^{K} c_j\frac{\partial w_j}{\partial y} \tag{3.9}$$ 5.1

for real constants c_j and a perturbation $\phi \in H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)$ which is small in the corresponding norm. To proceed we study the linearized operator defined by

$$\tilde{L}_{\varepsilon,\mathbf{p}} := S'_\varepsilon\begin{pmatrix} u_{\varepsilon,\mathbf{p}} \\ v_{\varepsilon,\mathbf{p}} \end{pmatrix},$$

where

$$\tilde{L}_{\varepsilon,\mathbf{p}} : H_T^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right) \times H_T^1(-1,1) \to L_T^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right) \times L_T^2(-1,1),$$

where $\varepsilon > 0$ is small and $H^1_T\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)$ and $L^2_T(-1,1)$ denote the periodic functions in $H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)$ and $L^2(-1,1)$ respectively, $\mathbf{p} \in B_\delta(\mathbf{p}^0)$. The approximate kernel and co-kernel are respectively defined by

$$\mathbf{K}_{\varepsilon,\mathbf{p}} := \text{Span}\left\{\frac{\partial w_j}{\partial y}\Big|\, j=1,\cdots,K\right\} \subset H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right),$$

$$\mathbf{C}_{\varepsilon,\mathbf{p}} := \text{Span}\left\{\frac{\partial w_j}{\partial y}\Big|\, j=1,\cdots,K\right\} \subset L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right).$$

It is not difficult to see that $\tilde{L}_{\varepsilon,\mathbf{p}}$ is not invertible in ε and β due to the approximate kernel,

$$\mathcal{K}_{\varepsilon,\mathbf{p}} := \mathbf{K}_{\varepsilon,\mathbf{p}} \oplus \{0\} \subset H^1_T\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right) \times H^1_T(-1,1).$$

The approximate cokernel is defined as follows:

$$\mathcal{C}_{\varepsilon,\mathbf{p}} = \mathbf{C}_{\varepsilon,\mathbf{p}} \oplus \{0\} \subset L^2_T\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right) \times L^2_T(-1,1).$$

We then define

$$\mathcal{K}^\perp_{\varepsilon,\mathbf{p}} := \mathbf{K}^\perp_{\varepsilon,\mathbf{p}} \oplus H^1_T(-1,1) \subset H^1_T\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right) \times H^1_T(-1,1),$$

$$\mathcal{C}^\perp_{\varepsilon,\mathbf{p}} := \mathbf{C}^\perp_{\varepsilon,\mathbf{p}} \oplus L^2_T(-1,1) \subset L^2_T\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right) \times L^2_T(-1,1),$$

where $\mathbf{C}^\perp_{\varepsilon,\mathbf{p}}$ and $\mathbf{K}^\perp_{\varepsilon,\mathbf{p}}$ denote the orthogonal complement with the scalar product of $L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)$ in $H^1_T\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)$ and $L^2_T\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)$ respectively.

Let $\pi_{\varepsilon,\mathbf{p}}$ denote the projection in $L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right) \times L^2(-1,1)$ onto $\mathcal{C}^\perp_{\varepsilon,\mathbf{p}}$. Next, we shall prove that the equation

$$\pi_{\varepsilon,\mathbf{p}} \circ S_\varepsilon \begin{pmatrix} u_{\varepsilon,\mathbf{p}} + \Phi_{\varepsilon,\mathbf{p}} \\ v_{\varepsilon,\mathbf{p}} + \Psi_{\varepsilon,\mathbf{p}} \end{pmatrix} = 0$$

has unique solution $\Sigma_{\varepsilon,\mathbf{p}} = \begin{pmatrix} \Phi_{\varepsilon,\mathbf{p}} \\ \Psi_{\varepsilon,\mathbf{p}} \end{pmatrix} \in \mathcal{K}^\perp_{\varepsilon,\mathbf{p}}$ if ε, β are small enough. Set

$$\mathcal{L}_{\varepsilon,\mathbf{p}} = \pi_{\varepsilon,\mathbf{p}} \circ \tilde{L}_{\varepsilon,\mathbf{p}} : \mathcal{K}^\perp_{\varepsilon,\mathbf{p}} \to \mathcal{C}^\perp_{\varepsilon,\mathbf{p}}. \tag{3.10}$$ 4.defl

Now we show the invertibility of the corresponding linearized operator $\mathcal{L}_{\varepsilon,\mathbf{p}}$.

pr5.1 **Proposition 3.2.** *Let $\mathcal{L}_{\varepsilon,\mathbf{p}}$ be defined in (3.10). Then there exist positive ε_0, β_0, C such that for all $\varepsilon \in (0,\varepsilon_0)$, $\beta \in (0,\beta_0)$,*

$$\|\mathcal{L}_{\varepsilon,\mathbf{p}}\Sigma\|_{L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)\times L^2(-1,1)} \geq C\|\Sigma\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)\times H^1(-1,1)},$$

for arbitrary $\mathbf{p} \in B_\sigma(\mathbf{p}^0)$, $\Sigma \in \mathcal{K}^\perp_{\varepsilon,\mathbf{p}}$.

Proof. The proof follows the standard method of Liaypunov-Schmidt reduction which was also used in [13, 14, 36, 37, 38]. Suppose the proposition is not true. Then there exist sequences $\{\varepsilon_k\}$, $\{\beta_k\}$, $\{\mathbf{p}^k\}$ and Σ_k with

$$\varepsilon_k > 0,\ \varepsilon_k \to 0,\ \beta_k > 0,\ \beta_k \to 0,\ \mathbf{p}^k \in B_\delta(\mathbf{p}^0),$$

and

$$\Sigma_k = \begin{pmatrix} \phi_k(y) \\ \psi_k(x) \end{pmatrix} \in \mathcal{K}^\perp_{\varepsilon,\mathbf{p}}$$

such that

$$\|\Sigma_k\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)\times H^1(-1,1)} = 1, \qquad \|L_{\varepsilon_k,\mathbf{p}^k}\Sigma_k\|_{L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)\times L^2(-1,1)} \to 0, \quad \text{as} \quad k \to \infty.$$

That is

$$\begin{cases} (-\Delta)_y^{\frac{1}{2}}\phi_k + \phi_k - 2u_{\varepsilon_k,\mathbf{p}^k} v_{\varepsilon_k,\mathbf{p}^k}^{-1}\phi_k + v_{\varepsilon_k,\mathbf{p}^k}^{-2} u_{\varepsilon_k,\mathbf{p}^k}^2 \psi_k = f_k^1 + f_k^2, \\ (-\Delta)_x^{\frac{1}{2}}\psi_k - \beta_k^2\psi_k + 2\xi_{\varepsilon_k}\beta_k^2 u_{\varepsilon_k,\mathbf{p}^k}\phi_k = g_k, \\ \|f_k^1\|_{L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)} \to 0,\ f_k^2 \in \mathbf{C}^{\perp}_{\varepsilon_k,\mathbf{p}^k},\ \phi_k \in \mathbf{K}^{\perp}_{\varepsilon_k,\mathbf{p}^k}, \\ \|\phi_k\|^2_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)} + \|\psi_k\|^2_{H^1(-1,1)} = 1. \end{cases} \tag{3.11}$$

5.system-li

We shall show this is impossible. To simplify our notation, we set $u_k = u_{\varepsilon_k,\mathbf{p}^k}$ and $\Omega_k = \left(-\frac{1}{\varepsilon_k}, \frac{1}{\varepsilon_k}\right)$. We cut off ϕ_k as follows: introduce

$$\phi_{k,j}(y) = \phi_k(y)\chi\left(\frac{\varepsilon_k y - p_j}{\varepsilon_k}\right)$$

and decompose ϕ_k into

$$\phi_k = \sum_{j=1}^{K} \phi_{k,j} + \phi_{k,K+1},$$

it is easy to see that $\phi_{k,K+1} = o(1)$ in $H^1(\Omega_k)$ due to it satisfies the equation

$$(-\Delta)_y^{\frac{1}{2}}\phi_{k,K+1} + \phi_{k,K+1} = o(1) \quad \text{in} \quad H^1(\Omega_k).$$

We then define $\psi_{k,j}$ for $j = 1, \cdots, K+1$ by

$$(-\Delta)_x^{\frac{1}{2}}\psi_{k,j} + \beta_k^2\psi_{k,j} - 2\xi_{\varepsilon_k} u_k \phi_{k,j} = 0.$$

Note that as $\|g_k\|_{L^2(-1,1)} \to 0$ we have

$$\|\psi_k - \sum_{j=1}^{K+1}\psi_{k,j}\|_{L^2(-1,1)} \to 0.$$

Since $\phi_{k,K+1} = o_{\varepsilon_k}(1)$ in $H^1(\Omega_k)$, we also we have $\psi_{k,K+1} = o_{\varepsilon_k}(1)$ in $H^1(-1,1)$. Sending $k \to \infty$, we can see that

$$\phi_{k,j} \to \phi_j \quad \text{in} \quad H^1(\mathbb{R})$$

with

$$\phi_j \in \left\{\phi \in H^1(\mathbb{R}) \middle| \int_{\mathbb{R}} \phi \frac{\partial w}{\partial y} dy = 0\right\} = K_0^{\perp}.$$

In addition, ϕ_i verifies the following nonlocal problem

Case 1 : $\eta_{\varepsilon_k} \to 0$,

$$(-\Delta)^{\frac{1}{2}}\phi_j + \phi_j - 2w\phi_j + 2\frac{\sum_{\ell=1}^{K}\int_{\mathbb{R}} w\phi_\ell dy}{K\int_{\mathbb{R}} w^2(y)dy} w^2 \in C_0^{\perp}. \tag{3.12}$$

4.limit-1

Case 2 : $\eta_{\varepsilon_k} \to \infty$,

$$(-\Delta)^{\frac{1}{2}}\phi_j + \phi_j - 2w\phi_j + 2\frac{\int_{\mathbb{R}} w\phi_j dy}{\int_{\mathbb{R}} w^2(y)dy} w^2 \in C_0^{\perp}. \tag{3.13}$$

4.limit-2

Case 3 : $\eta_{\varepsilon_k} \to \eta_0$,

$$(-\Delta)^{\frac{1}{2}}\phi_j + \phi_j - 2w\phi_j + 2\frac{(1+\eta_0)\int_{\mathbb{R}} w\phi_j dy + \sum_{\ell\neq j}^{K}\int_{\mathbb{R}} w\phi_\ell dy}{(K+\eta_0)\int_{\mathbb{R}} w^2 dy} w^2 \in C_0^{\perp}, \tag{3.14}$$

4.limit-3

where

$$K_0 = C_0 = \text{Span}\left\{\frac{\partial w}{\partial y}\right\},$$

and $K_0^\perp, C_0^\perp$ denotes the orthogonal complement with respect to the scalar product of $L^2(\mathbb{R})$ in the sapce $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$ respectively.

After linear transformation, we could write the equation (3.12)-(3.14) as (still denoted by ϕ_j):

$$(-\Delta)_y^{\frac{1}{2}}\phi_j + \phi_j - 2w\phi_j + 2\lambda_i \frac{\int_{\mathbb{R}} w\phi_j dy}{\int_{\mathbb{R}} w^2 dy} w^2 \in C_0^\perp, \tag{3.15}$$ 4.limlin

where

$$\lambda_j = \begin{cases} 0, \cdots, 0, K, & \text{for case 1,} \\ 1, \cdots, 1, & \text{for case 2,} \\ \frac{\eta_0}{K+\eta_0}, \cdots, \frac{\eta_0}{K+\eta_0}, 1, & \text{for case 3.} \end{cases}$$

It is known that

$$(-\Delta)^{\frac{1}{2}} w + w - 2w^2 = -w^2.$$

Therefore, equation (3.15) can be written as

$$((-\Delta)^{\frac{1}{2}} + 1 - 2w)\left(\phi_j - 2\lambda_j \frac{\int_{\mathbb{R}} w\phi_j dy}{\int_{\mathbb{R}} w^2 dy} w\right) \in C_0^\perp.$$

Since the operator

$$(-\Delta)^{\frac{1}{2}} + 1 - 2w: \ K_0^\perp \to C_0^\perp$$

is one-to-one map with bounded inverse. As a consequence,

$$\phi_j - 2\lambda_j \frac{\int_{\mathbb{R}} w\phi_j dy}{\int_{\mathbb{R}} w^2 dy} w = 0.$$

Mutiplying by w and after integration we get

$$(1 - 2\lambda_j)\int_{\mathbb{R}} w\phi_j dy = 0.$$

If $\lambda_j \neq \frac{1}{2}$ we derive that $\int_{\mathbb{R}} w\phi_j dy = 0$ and it implies that

$$((-\Delta)^{\frac{1}{2}} + 1 - 2w)\phi_j = 0, \quad j = 1, \cdots, K,$$

and by Proposition 2.1 we have $\phi_j \in K_0,\ j = 1, \cdots, K$. Then it implies that $\phi_j = 0,\ j = 1, \cdots, K$. By taking the limit equation in ψ_k we see that $\psi_k \to 0$ in $H^1(-1,1)$. On the other hand, from the fourth equation in (3.11) we have

$$\sum_{j=1}^{K}(\|\phi_j\|^2_{H^1(\mathbb{R})} + \|\psi_j\|^2_{H^1(-1,1)}) = 1.$$

Contradiction arises and the proof is complete. □

As a consequence of Proposition 3.2 we have

pr5.2 **Proposition 3.3.** *There exist positive constants ε_1, β_1 such that for all $\varepsilon \in (0, \varepsilon_1)$ and $\beta \in (0, \beta_1)$, the map $\mathcal{L}_{\varepsilon,\mathbf{p}}$ is surjective for arbitrary $\mathbf{p} \in B_\sigma(\mathbf{p}^0)$.*

Now we are in position to solve the problem

$$\pi_{\varepsilon,\mathbf{p}} \circ S_\varepsilon \begin{pmatrix} u_{\varepsilon,\mathbf{p}} + \phi \\ v_{\varepsilon,\mathbf{p}} + \psi \end{pmatrix} = 0.$$

Since $\mathcal{L}_{\varepsilon,\mathbf{p}} \mid_{\mathcal{K}^\perp_{\varepsilon,\mathbf{p}}}$ is invertible (call the inverse $\mathcal{L}^{-1}_{\varepsilon,\mathbf{p}}$) we can rewrite the above problem as

$$\Sigma = -(\mathcal{L}^{-1}_{\varepsilon,\mathbf{p}} \circ \pi_{\varepsilon,\mathbf{p}}) S_\varepsilon \begin{pmatrix} u_{\varepsilon,\mathbf{p}} + \phi \\ v_{\varepsilon,\mathbf{p}} + \psi \end{pmatrix} - (\mathcal{L}^{-1}_{\varepsilon,\mathbf{p}} \circ \pi_{\varepsilon,\mathbf{p}}) N_{\varepsilon,\mathbf{p}}(\Sigma) \equiv M_{\varepsilon,\mathbf{p}}(\Sigma), \tag{3.16}$$ 5.10

where

$$N_{\varepsilon,\mathbf{p}}(\Sigma) = S_\varepsilon \begin{pmatrix} u_{\varepsilon,\mathbf{p}} + \phi \\ v_{\varepsilon,\mathbf{p}} + \psi \end{pmatrix} - S_\varepsilon \begin{pmatrix} u_{\varepsilon,\mathbf{p}} \\ v_{\varepsilon,\mathbf{p}} \end{pmatrix} - S'_\varepsilon \begin{pmatrix} u_{\varepsilon,\mathbf{p}} \\ v_{\varepsilon,\mathbf{p}} \end{pmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix}$$

and the operator $M_{\varepsilon,\mathbf{p}}$ is defined for $\Sigma \in H^1_T\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times H^1_T(-1,1)$. We are going to show that the operator $M_{\varepsilon,\mathbf{p}}$ is a contraction map on

$$B_{\varepsilon,\delta} := \left\{ \Sigma \in H^1_T\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times H^1_T(-1,1) \Big| \, \|\Sigma\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)\times H^1(-1,1)} < \delta \right\}, \qquad (3.17)$$ 5.11

if σ and ε are small enough. By Proposition 3.2 we have

$$\begin{aligned} \|M_{\varepsilon,\mathbf{p}}(\Sigma)\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)\times H^1(-1,1)} &\le C\left(\|\pi_{\varepsilon,\mathbf{p}} \circ N_{\varepsilon,\mathbf{p}}(\Sigma)\|_{L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)\times L^2(-1,1)} + \|\pi_{\varepsilon,\mathbf{p}} \circ S_\varepsilon \begin{pmatrix} u_{\varepsilon,\mathbf{p}} \\ v_{\varepsilon,\mathbf{p}} \end{pmatrix} \|_{L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)\times L^2(-1,1)} \right) \\ &\le C(c(\delta)\delta + \xi_\varepsilon \beta^2 \varepsilon + h(\varepsilon,\beta)), \end{aligned}$$

where $C > 0$ is a constant independent of $\delta > 0$, $\varepsilon > 0$ and $c(\delta) \to 0$ as $\delta \to 0$. Similarly we show that

$$\|M_{\varepsilon,\mathbf{p}}(\Sigma_1) - M_{\varepsilon,\mathbf{p}}(\Sigma_2)\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)\times H^1(-1,1)} \le C(c(\delta)\delta)\|\Sigma_1 - \Sigma_2\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)\times H^1(-1,1)},$$

where $c(\delta) \to 0$ as $\delta \to 0$. If we choose δ sufficiently small then $M_{\varepsilon,\mathbf{p}}$ is a contraction map on $B_{\varepsilon,\delta}$. The existence then follows by the standard fixed point theorem and $\Sigma_{\varepsilon,\mathbf{p}}$ is a solution to (3.16). We thus proved

le5.2 **Lemma 3.4.** *There exists $\bar{\varepsilon} > 0$, $\overline{\beta} > 0$ such that for every pair of $\varepsilon, \mathbf{p}$ with $0 < \varepsilon < \bar{\varepsilon}$ and $\mathbf{p} \in B_\sigma(\mathbf{p}^0)$ there is a unique $(\phi_{\varepsilon,\mathbf{p}}, \psi_{\varepsilon,\mathbf{p}}) \in \mathcal{K}^\perp_{\varepsilon,\mathbf{p}}$ satisfying $S_\varepsilon \begin{pmatrix} u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}} \\ v_{\varepsilon,\mathbf{p}} + \psi_{\varepsilon,\mathbf{p}} \end{pmatrix} \in \mathcal{C}_{\varepsilon,\mathbf{p}}$. Furthermore, we have the estimate*

$$\|(\phi_{\varepsilon,\mathbf{p}}, \psi_{\varepsilon,\mathbf{p}})\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)\times H^1(-1,1)} \le C\left(\xi_\varepsilon \beta^2 \varepsilon + h(\varepsilon,\beta)\right).$$

More refined estimates for $\phi_{\varepsilon,\mathbf{p}}$ are needed. We recall from the discussion in last section that $S_1(u_{\varepsilon,\mathbf{p}}, v_{\varepsilon,\mathbf{p}})$ can be decomposed into the two parts $S_{1,1}$ and $S_{1,2}$ if x is close to the center of spike, where $S_{1,1}$ is in leading order an odd function and $S_{1,2}$ is in leading order an even function. We can similarly decompose $\phi_{\varepsilon,\mathbf{p}}$ as in the following lemma.

le5.3 **Lemma 3.5.** *Let $\phi_{\varepsilon,\mathbf{p}}$ be defined in Lemma 3.4. Then for $x = p_j + \varepsilon z$, $|\varepsilon z| < \rho$, $j = 1, \cdots, K$, we have the decomposition*

$$\phi_{\varepsilon,\mathbf{p}} = \phi_{\varepsilon,\mathbf{p},1} + \phi_{\varepsilon,\mathbf{p},2}, \qquad (3.18)$$ 5.13

where $\phi_{\varepsilon,\mathbf{p},2}$ is an even function in z which satisfies

$$\phi_{\varepsilon,\mathbf{p},1} = O(\xi_\varepsilon \beta^2 \varepsilon^2) \quad \text{in} \quad H^1\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), \qquad (3.19)$$ 5.14

and

$$\phi_{\varepsilon,\mathbf{p},2} = O(\xi_\varepsilon \beta^2 \varepsilon + h(\varepsilon,\beta)) \quad \text{in} \quad H^1\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right). \qquad (3.20)$$ 5.15

Proof. We first solve

$$S[u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p},2}] - S[u_{\varepsilon,\mathbf{p}}] - \sum_{j=1}^K S_{1,2}\left(y - \frac{p_j}{\varepsilon}\right) \in \mathbf{C}_{\varepsilon,\mathbf{p}}, \qquad (3.21)$$ 5.16

for $\phi_{\varepsilon,\mathbf{p},2} \in \mathbf{K}^\perp_{\varepsilon,\mathbf{p}}$. Then we solve

$$S[u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p},2} + \phi_{\varepsilon,\mathbf{p},1}] - S[u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p},2}] - \sum_{j=1}^K S_{1,1}\left(y - \frac{p_j}{\varepsilon}\right) \in \mathbf{C}_{\varepsilon,\mathbf{p}}, \qquad (3.22)$$ 5.17

for $\phi_{\varepsilon,\mathbf{p},1} \in \mathbf{K}^\perp_{\varepsilon,\mathbf{p}}$. Using the same proof as in Proposition 3.2, both equations (3.21) and (3.22) have unique solution provided $\varepsilon, \beta \ll 1$. By uniqueness, $\phi_{\varepsilon,\mathbf{p}} = \phi_{\varepsilon,\mathbf{p},1} + \phi_{\varepsilon,\mathbf{p},2}$, and it is easy to see that $\phi_{\varepsilon,\mathbf{p},1}$ and $\phi_{\varepsilon,\mathbf{p},2}$ have the required properties. □

3.3. **The Reduced Problem.** In this subsection, we solve the reduced problem which will will complete the proof of Theorem 1.1. By Proposition 3.2 for every $\mathbf{p} \in B_\sigma(\mathbf{p}^0)$ there exists an unique solution $(\phi_{\varepsilon,\mathbf{p}}, \psi_{\varepsilon,\mathbf{p}}) \in \mathcal{K}^\perp_{\varepsilon,\mathbf{p}}$ such that

$$S_\varepsilon \begin{pmatrix} u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}} \\ v_{\varepsilon,\mathbf{p}} + \psi_{\varepsilon,\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \Xi_{\varepsilon,\mathbf{p}} \\ 0 \end{pmatrix} \in \mathcal{C}_{\varepsilon,\mathbf{p}}.$$

To complete the proof of Theorem 1.1 we need to determine $\mathbf{p}^\varepsilon = (p_1, p_2, \cdots, p_K)$ near $\mathbf{p}^0$ such that

$$S_\varepsilon \begin{pmatrix} u_{\varepsilon,\mathbf{p}^\varepsilon} + \phi_{\varepsilon,\mathbf{p}^\varepsilon} \\ v_{\varepsilon,\mathbf{p}^\varepsilon} + \psi_{\varepsilon,\mathbf{p}^\varepsilon} \end{pmatrix} \perp \mathcal{C}_{\varepsilon,\mathbf{p}^\varepsilon},$$

which in turn implies that $S_\varepsilon \begin{pmatrix} u_{\varepsilon,\mathbf{p}^\varepsilon} + \phi_{\varepsilon,\mathbf{p}^\varepsilon} \\ v_{\varepsilon,\mathbf{p}^\varepsilon} + \psi_{\varepsilon,\mathbf{p}^\varepsilon} \end{pmatrix} = 0$. To this end, let

$$W_\varepsilon(\mathbf{p}) := (W_{\varepsilon,1}(\mathbf{p}), W_{\varepsilon,2}(\mathbf{p}), \cdots, W_{\varepsilon,K}(\mathbf{p})) : \ B_\sigma(\mathbf{p}^0) \to \mathbb{R}^K$$

where

$$W_{\varepsilon,j}(\mathbf{p}) := \frac{1}{\xi_\varepsilon \beta^2 \varepsilon} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} S_1\left(u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}}, v_{\varepsilon,\mathbf{p}} + \psi_{\varepsilon,\mathbf{p}}\right) \frac{\partial w_j}{\partial p_j} dy, \qquad j = 1, \cdots, K.$$

Then $W_\varepsilon(\mathbf{p})$ is a map which is continuous in $\mathbf{p}$ and our problem is reduced to finding a zero of the vector field $W_\varepsilon(\mathbf{p})$. Let us now calculate $W_\varepsilon(\mathbf{p})$

$$\begin{aligned} W_{\varepsilon,j}(\mathbf{p}) &= \frac{1}{\xi_\varepsilon \beta^2 \varepsilon} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} S_1\left(u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}}, v_{\varepsilon,\mathbf{p}} + \psi_{\varepsilon,\mathbf{p}}\right) \frac{\partial w_j}{\partial p_j} dy \\ &= \frac{1}{\xi_\varepsilon \beta^2 \varepsilon} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[(-\Delta)^{\frac{1}{2}} (u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}}) + (u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}}) - \frac{(u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}})^2}{v_{\varepsilon,\mathbf{p}} + \psi_{\varepsilon,\mathbf{p}}} \right] \frac{\partial w_j}{\partial p_j} dy \\ &= \frac{1}{\xi_\varepsilon \beta^2 \varepsilon} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[(-\Delta)^{\frac{1}{2}} (u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}}) + (u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}}) - \frac{(u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}})^2}{v_{\varepsilon,\mathbf{p}}} \right] \frac{\partial w_j}{\partial p_j} dy \\ &\quad - \frac{1}{\xi_\varepsilon \beta^2 \varepsilon} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[\frac{(u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}})^2}{v_{\varepsilon,\mathbf{p}} + \psi_{\varepsilon,\mathbf{p}}} - \frac{(u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}})^2}{v_{\varepsilon,\mathbf{p}}} \right] \frac{\partial w_j}{\partial p_j} dy \\ &= I_1 + I_2, \end{aligned} \tag{3.23}$$

where $I_1, \ I_2$ are defined by the last equality and $\psi_{\varepsilon,\mathbf{p}}$ satisifies

$$D(-\Delta)^{\frac{1}{2}} \psi_{\varepsilon,\mathbf{p}} + \psi_{\varepsilon,\mathbf{p}} - 2\xi_\varepsilon u_{\varepsilon,\mathbf{p}} \phi_{\varepsilon,\mathbf{p}} - \xi_\varepsilon \phi^2_{\varepsilon,\mathbf{p}} = 0. \tag{3.24}$$

For I_1, we have by Lemma 3.5

$$\begin{aligned} I_1 &= \frac{1}{\xi_\varepsilon \beta^2 \varepsilon} \left(\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[(-\Delta)^{\frac{1}{2}} (u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}}) + (w_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}}) - \frac{(u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}})^2}{v_{\varepsilon,\mathbf{p}}(p_j)} \right] \frac{\partial w_j}{\partial p_j} dy \right. \\ &\quad \left. + \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{(u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}})^2}{v^2_{\varepsilon,\mathbf{p}}(p_j)} (v_{\varepsilon,\mathbf{p}}(p_j + \varepsilon y) - v_{\varepsilon,\mathbf{p}}(p_j)) \frac{\partial w_j}{\partial p_j} dy \right) + o(1) \\ &= -\frac{1}{\xi_\varepsilon \beta^2 \varepsilon^2} \left(\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[(-\Delta)^{\frac{1}{2}} (w_j + \phi_{\varepsilon,\mathbf{p}}) + (w_j + \phi_{\varepsilon,\mathbf{p}}) - \frac{(w_j + \phi_{\varepsilon,\mathbf{p}})^2}{v_{\varepsilon,\mathbf{p}}(p_j)} \right] \frac{\partial w_j}{\partial y} dy \right) \\ &\quad - \frac{1}{\xi_\varepsilon \beta^2 \varepsilon^2} \left(\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{(w_j + \phi_{\varepsilon,\mathbf{p},2})^2}{v^2_{\varepsilon,\mathbf{p}}(p_j)} (v_{\varepsilon,\mathbf{p}}(p_j + \varepsilon y) - v_{\varepsilon,\mathbf{p}}(p_j)) \frac{\partial w_j}{\partial y} dy \right) + o(1). \end{aligned} \tag{3.25}$$

Note that, by Lemma 3.5, we have

$$\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} [(-\Delta)^{\frac{1}{2}} \phi_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}} - 2w_j \phi_{\varepsilon,\mathbf{p}}] \frac{\partial w_j}{\partial y} dy = \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \phi_{\varepsilon,\mathbf{p},1} \frac{\partial}{\partial y} \left((-\Delta)^{\frac{1}{2}} w_j + w_j - w_j^2 \right) dy + o(\xi_\varepsilon \beta^2 \varepsilon^2) = o(\xi_\varepsilon \beta^2 \varepsilon^2), \tag{3.26}$$

and

$$\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \phi_{\varepsilon,\mathbf{p}}^2 \frac{\partial w_j}{\partial y} dy = 2\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \phi_{\varepsilon,\mathbf{p},1}\phi_{\varepsilon,\mathbf{p},2} \frac{\partial w_j}{\partial y} dy = o(\xi_\varepsilon \beta^2 \varepsilon^2). \tag{3.27}$$

Now by Lemma 3.5 and equations (3.26) and (3.27) we have

$$\begin{aligned} I_1 &= -\frac{1}{\xi_\varepsilon \beta^2 \varepsilon^2} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} w_j^2 (v_{\varepsilon,\mathbf{p}}(p_j+\varepsilon y) - v_{\varepsilon,\mathbf{p}}(p_j)) \frac{\partial w_j}{\partial y} dy + o(1) \\ &= -\frac{1}{\varepsilon} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} w_j^2 \left(P_j(z) - \varepsilon y \partial_{p_j} F(\mathbf{p}) \right) \frac{\partial w_j}{\partial y} dy + o(1) \\ &= -\frac{1}{3} \int_{\mathbb{R}} w^3(y) dy \partial_{p_j} F(\mathbf{p}) + o(1). \end{aligned} \tag{3.28}$$

Similarly, we calculate

$$\begin{aligned} I_2 &= -\frac{1}{\xi_\varepsilon \beta^2 \varepsilon^2} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left[\frac{(u_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{v_{\varepsilon,\mathbf{p}}+\psi_{\varepsilon,\mathbf{p}}} - \frac{(u_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{v_{\varepsilon,\mathbf{p}}} \right] \frac{\partial w_j}{\partial y} dy \\ &= \frac{1}{\xi_\varepsilon \beta^2 \varepsilon^2} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{(u_{\varepsilon,\mathbf{p}}+\phi_{\varepsilon,\mathbf{p}})^2}{v_{\varepsilon,\mathbf{p}}^2} \psi_{\varepsilon,\mathbf{p}} \frac{\partial w_j}{\partial y} dy + o(1) \\ &= \frac{1}{\xi_\varepsilon \beta^2 \varepsilon^2} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{1}{3} \frac{\partial w_j^3}{\partial y} (\psi_{\varepsilon,\mathbf{p}} - \psi_{\varepsilon,\mathbf{p}}(p_j)) dy + o(1). \end{aligned} \tag{3.29}$$

Since $\psi_{\varepsilon,\mathbf{p}}$ satisifies (3.24), a similar argument to that used in Lemma 3.5 gives

$$\begin{aligned} \psi_{\varepsilon,\mathbf{p}}(p_j+\varepsilon z) - \psi_{\varepsilon,\mathbf{p}}(p_j) &= \xi_\varepsilon \int_{-1}^{1} \left(G_D(p_j+\varepsilon z,\zeta) - G_D(p_j,\zeta) \right) \left(2u_{\varepsilon,\mathbf{p}}\left(\frac{\zeta}{\varepsilon}\right) \phi_{\varepsilon,\mathbf{p}}\left(\frac{\zeta}{\varepsilon}\right) + \phi_{\varepsilon,\mathbf{p}}^2\left(\frac{\zeta}{\varepsilon}\right) \right) d\zeta \\ &= o\left(\xi_\varepsilon \beta^2 \varepsilon^2 |\partial_{p_j} F(\mathbf{p})||z| \right) + \hat{P}_j(z) + h.o.t., \end{aligned} \tag{3.30}$$

where $\hat{P}_j(z)$ is an even function in $z = y - \frac{p_j}{\varepsilon}$. Substituting (3.30) into (3.29) we obtain that

$$I_2 = o(1). \tag{3.31}$$

Combining the estimates for I_1 and I_2, we obtain

$$W_\varepsilon(\mathbf{p}) = -\pi \nabla_p F(\mathbf{p}) + o(1), \tag{3.32}$$

where $F(\mathbf{p})$ is defined in (1.7) and we have used that $\int_{\mathbb{R}} w^3(y)dy = 3\pi$, and $o(1)$ is continuous function of $\mathbf{p}$ which goes to 0 as $\varepsilon \to 0$. At $\mathbf{p}^0$, we have $\nabla_{\mathbf{p}} F(\mathbf{p}^0) = 0$. On the other hand, we have assumed that $\nabla_{\mathbf{p}}^2 F(\mathbf{p}^0)$ is a matrix of rank $K-1$. [2]

It is known that $(1,\cdots,1)^t \in \mathrm{Ker}(\nabla_{\mathbf{p}}^2 F(\mathbf{p}^0))$ and we can choose $\mathbf{p}$ such that $W_\varepsilon(\mathbf{p}) \perp (1,\cdots,1)^t$. Next, we can apply Brouwer's fixed point theorem to show that for $\varepsilon \ll 1$ there exists a point $\mathbf{p}$ such that $W_\varepsilon(\mathbf{p}) = 0$ and $\mathbf{p} \in B_\sigma(\mathbf{p}^0)$. Thus we have proved the following proposition

Proposition 3.6. *For ε sufficiently small there exist points $\mathbf{p}^\varepsilon$ with $\mathbf{p}^\varepsilon \to \mathbf{p}^0$ such that $W_\varepsilon(\mathbf{p}^\varepsilon) = 0$.*

Proof of Theorem 1.1. By above Proposition, there exists $\mathbf{p}^\varepsilon \to \mathbf{p}^0$ such that $W_\varepsilon(\mathbf{p}^\varepsilon) = 0$. In other words, $S[u_{\varepsilon,\mathbf{p}^\varepsilon} + \phi_{\varepsilon,\mathbf{p}^\varepsilon}] = 0$. Let $u_\varepsilon = \xi_\varepsilon u_{\varepsilon,\mathbf{p}}$, $v_\varepsilon = \xi_\varepsilon v_{\varepsilon,\mathbf{p}}$. By Maximum principle, $u_\varepsilon > 0$ and $v_\varepsilon > 0$. Moreover $(u_\varepsilon, v_\varepsilon)$ satisfies all the properties of Theorem 1.1. □

Remark: In the asymmetric case, instead of considering the system (3.1) we study the original system (1.3) directly. Then, the ansatz is given by

$$u_{\varepsilon,\mathbf{p}} = \sum_{j=1}^{2} \xi_{\varepsilon,j} w_j(y), \quad v_{\varepsilon,\mathbf{p}} = T[u_{\varepsilon,\mathbf{p}}](x).$$

[2] When D is large or $K = 2,3,4$, we are able to show that $M(\mathbf{p}^0)$ is semi-negative and $\mathrm{rank}(M(\mathbf{p}^0)) = K-1$. The proof is given in next section.

After the standard procedure as we did for the symmetric case, we reduce the original problem to a same finite dimensional problem (3.32) with $K = 2$. By the same proof we are able to establish the existence for the asymmetric two spikes pattern.

f-stability

4. Rigorous proof of the stability analysis

In this section, we shall consider the large and small eigenvalues respectively. From which we are able to characterize the linear stability of the multi-spikes constructed in last section.

4.1. **Stability Analysis: Large Eigenvalues.** Linearizing around the equilibrium states $(u_\varepsilon, v_\varepsilon)$, we obtain the following eigenvalue problem

$$\begin{cases} (-\Delta)_y^{\frac{1}{2}}\phi_\varepsilon + \phi_\varepsilon - 2\frac{u_\varepsilon}{v_\varepsilon}\phi_\varepsilon + \frac{u_\varepsilon^2}{v_\varepsilon^2}\psi_\varepsilon + \lambda_\varepsilon\phi_\varepsilon = 0, \\ \frac{1}{\beta^2}(-\Delta)_x^{\frac{1}{2}}\psi_\varepsilon + \psi_\varepsilon - 2u_\varepsilon\phi_\varepsilon + \tau\lambda_\varepsilon\psi_\varepsilon = 0, \end{cases} \tag{4.1}$$

7.1

where λ_ε is some complex number and

$$\phi_\varepsilon \in H^1\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), \quad \psi_\varepsilon \in H^1(-1, 1).$$

In this subection, we study the large eigenvalues, i.e. those for which we may assume that there exists $c > 0$ such that $|\lambda_\varepsilon| \geq c > 0$ for ε small. If $\Re(\lambda_\varepsilon) < -c$ then we are done (since these eigenvalues are always stable) and we therefore assume that $\Re(\lambda_\varepsilon) \geq -c$. For a subsequence $\varepsilon \to 0$ and $\lambda_\varepsilon \to \lambda_0$ we shall derive a limiting NLEP satisfied by λ_0. In the following we shall divide our discussion into two cases: *symmetric pattern* and *asymmetric pattern*. First, we study the symmetric case.

Symmetric pattern. Let

$$\hat{u}_\varepsilon = \xi_\varepsilon^{-1}u_\varepsilon = u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}}, \quad \hat{v}_\varepsilon = \xi_\varepsilon^{-1}v_\varepsilon = v_{\varepsilon,\mathbf{p}} + \psi_{\varepsilon,\mathbf{p}}.$$

Then (4.1) becomes

$$\begin{cases} (-\Delta)_y^{\frac{1}{2}}\phi_\varepsilon + \phi_\varepsilon - 2\frac{\hat{u}_\varepsilon}{\hat{v}_\varepsilon}\phi_\varepsilon + \frac{\hat{u}_\varepsilon^2}{\hat{v}_\varepsilon^2}\psi_\varepsilon + \lambda_\varepsilon\phi_\varepsilon = 0, \\ \frac{1}{\beta^2}(-\Delta)_x^{\frac{1}{2}}\psi_\varepsilon + \psi_\varepsilon - 2\xi_\varepsilon\hat{u}_\varepsilon\phi_\varepsilon + \tau\lambda_\varepsilon\psi_\varepsilon = 0. \end{cases} \tag{4.2}$$

5.2

The second equation in (4.2) is equivalent to

$$(-\Delta)_x^{\frac{1}{2}}\psi_\varepsilon + \beta^2(1+\tau\lambda_\varepsilon)\psi_\varepsilon - 2\beta^2\xi_\varepsilon\hat{u}_\varepsilon\phi_\varepsilon = 0. \tag{4.3}$$

5.3

We introduce the following:

$$\beta_{\lambda_\varepsilon} = \beta\sqrt{1+\tau\lambda_\varepsilon},$$

where in $\sqrt{1+\tau\lambda_\varepsilon}$ we take the principal part of the square root. Let us assume that

$$\|\phi_\varepsilon\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)} = 1.$$

We cut off ϕ_ε as follows: Introduce

$$\phi_{\varepsilon,j}(\varepsilon y - p_j) = \phi_\varepsilon\chi\left(\frac{\varepsilon y - p_j}{r_0}\right),$$

where $\chi(x)$ was introduced in (2.25). Using (4.2), Lemma 3.4 and $\Re(\lambda_\varepsilon) \geq -c$ and the algebraic decay of w, we get that

$$\phi_\varepsilon = \sum_{j=1}^{K}\phi_{\varepsilon,j} + o_\varepsilon(1) \quad \text{in} \quad H^1\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right).$$

Then by a standard procedure, we extend $\phi_{\varepsilon,j}$ to a function defined on $\mathbb{R}$ such that

$$\|\phi_{\varepsilon,j}\|_{H^1(\mathbb{R})} \leq C\|\phi_{\varepsilon,j}\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)}, \quad j = 1, \cdots, K.$$

Since $\|\phi_\varepsilon\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)} = 1$, $\|\phi_{\varepsilon,j}\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)} \le C$. By taking a subsequence of ε, we may assume that $\phi_{\varepsilon,j} \to \phi_j$ as $\varepsilon \to 0$ in $L^2 \cap L^\infty(\mathbb{R})$ for $j = 1, \cdots, K$.

We have by (4.3)

$$\psi_\varepsilon(x) = 2\beta^2\xi_\varepsilon \int_{-1}^{1} G_{\beta_{\lambda_\varepsilon}}(x,\zeta)\hat{u}_\varepsilon\left(\frac{\zeta}{\varepsilon}\right)\phi_\varepsilon\left(\frac{\zeta}{\varepsilon}\right)d\zeta. \tag{4.4}$$

At $x = p_j^\varepsilon$, $j = 1, \cdots, K$, we calculate

$$\begin{aligned}
\psi(p_j^\varepsilon) &= 2\beta^2\xi_\varepsilon \int_{-1}^{1} G_{\beta_{\lambda_\varepsilon}}(p_j^\varepsilon,\zeta)\sum_{\ell=1}^{K} w\left(\frac{\zeta - p_\ell^\varepsilon}{\varepsilon}\right)\phi_{\varepsilon,\ell}\left(\frac{\zeta - p_\ell^\varepsilon}{\varepsilon}\right)d\zeta + O(\xi_\varepsilon|\beta_{\lambda_\varepsilon}^2|\varepsilon)\\
&= 2\beta^2\xi_\varepsilon \int_{-1}^{1}\left(\frac{(\beta_{\lambda_\varepsilon})^{-2}}{2} + G_0(p_j^\varepsilon,\zeta) + O(|\beta_{\lambda_\varepsilon}|^2)\right)\sum_{\ell=1}^{K} w\left(\frac{\zeta - p_\ell^\varepsilon}{\varepsilon}\right)\phi_{\varepsilon,\ell}\left(\frac{\zeta - p_\ell^\varepsilon}{\varepsilon}\right)d\zeta + O(\xi_\varepsilon|\beta_{\lambda_\varepsilon}^2|\varepsilon)\\
&= 2\xi_\varepsilon \int_{-1}^{1}\left(\frac{1}{2(1+\tau\lambda_\varepsilon)} + \beta^2 G_0(p_j^\varepsilon,\zeta) + O(|\beta_{\lambda_\varepsilon}|^4)\right) w\left(\frac{\zeta - p_j^\varepsilon}{\varepsilon}\right)\phi_{\varepsilon,j}\left(\frac{\zeta - p_j^\varepsilon}{\varepsilon}\right)d\zeta\\
&\quad + 2\xi_\varepsilon \sum_{\ell\neq j}\int_{-1}^{1}\left(\frac{1}{2(1+\tau\lambda_\varepsilon)} + \beta^2 G_0(p_j^\varepsilon,\zeta) + O(|\beta_{\lambda_\varepsilon}|^4)\right) w\left(\frac{\zeta - p_\ell^\varepsilon}{\varepsilon}\right)\phi_{\varepsilon,\ell}\left(\frac{\zeta - p_\ell^\varepsilon}{\varepsilon}\right)d\zeta\\
&= \sum_{\ell=1}^{K}\frac{\varepsilon\xi_\varepsilon}{(1+\tau\lambda_\varepsilon)}\int_{\mathbb{R}} w(y)\phi_{\varepsilon,\ell}(y)dy(1+o(1)) + 2\xi_\varepsilon\frac{\beta^2}{\pi}\varepsilon\log\frac{1}{\varepsilon}\int_{\mathbb{R}} w(y)\phi_{\varepsilon,j}(y)dy + O(\xi_\varepsilon|\beta_{\lambda_\varepsilon}|^2\varepsilon).
\end{aligned} \tag{4.5}$$

Let $\eta_\varepsilon = \frac{2\beta^2}{\pi}\log\frac{1}{\varepsilon}$ and we separate our discussion into three cases.

Case 1: $\eta_\varepsilon \to 0$, we get from (4.5):

$$\psi_\varepsilon(p_j^\varepsilon) = \sum_{\ell=1}^{K}\frac{\varepsilon\xi_\varepsilon}{(1+\tau\lambda_\varepsilon)}\int_{\mathbb{R}} w\phi_{\varepsilon,\ell}dy(1+o(1)). \tag{4.6}$$

Substituting (4.6) into the first equation (4.2), sending $\varepsilon \to 0$ and using (2.30), we derive the following nonlocal eigenvalue problem (NLEP):

$$(-\Delta)^{\frac{1}{2}}\phi_j + \phi_j - 2w\phi_j + \frac{2\sum_{\ell=1}^{K}\int_{\mathbb{R}} w\phi_\ell dy}{K(1+\tau\lambda_0)\int_{\mathbb{R}} w^2(y)dy}w^2 + \lambda_0\phi_j = 0, \quad j = 1, \cdots, K. \tag{4.7}$$

If $K = 1$, by Theorem 2.4, the above problem is stable if $\tau < \tau_1$, which implies that the large eigenvalues are stable. If $\tau > \tau_1$, by Theorem 2.4, problem (4.7) has a eigenvalue λ_0 with $\Re(\lambda_0) \ge a_0 > 0$ for some a_0. By Theorem 4.1 below, we have problem (4.2) also admits a eigenvalue λ_ε with $\lambda_0 + o(1)$ which implies that the problem (4.2) is unstable. If $K > 1$, problem (4.2) admits a positive eigenvalue: We can choose, for example,

$$\phi_1 = -\phi_2 = \Phi_0, \quad \phi_3 = \cdots = \phi_K = 0, \; \lambda_0 = \mu_1,$$

where Φ_0 is the principal eigenfunction of L_0 given in Proposition 2.1. Repeating the above arguments for $K = 1$ and by Theorem 4.1 again, we conclude that there is a eigenvalue of (4.2) with eigenvalue whose real part is positive. Thus all multiple-peaked solutions are unstable.

Case 2. $\eta_\varepsilon \to \infty$. In this case, similar to Case 1, we get from (4.5) that

$$\psi_\varepsilon(p_j^\varepsilon) = \varepsilon\xi_\varepsilon\eta_\varepsilon\int_{\mathbb{R}} w\phi_{\varepsilon,j}dy(1+o(1)), \quad j = 1, \cdots, K. \tag{4.8}$$

and for any $\tau \ge 0$, in the limit $\varepsilon \to 0$ we obtain the following NLEP:

$$(-\Delta)^{\frac{1}{2}}\phi_j + \phi_j - 2w\phi_j + 2\frac{\int_{\mathbb{R}} w\phi_j dy}{\int_{\mathbb{R}} w^2 dy}w^2 + \lambda_0\phi_j = 0, \quad j = 1, \cdots, K. \tag{4.9}$$

By Theorem 2.2, (4.9) has only stable eigenvalues. Therefore, if $\eta_\varepsilon \to \infty$, then the large eigenvalues of a K-peaked solutions are all stable.

Case 3. $\eta_\varepsilon \to \eta_0$. Similar as above, we get from (4.5) that

$$\psi_\varepsilon(p_j^\varepsilon) = \left(\sum_{\ell=1}^{K} \frac{1}{1+\tau\lambda_0} \varepsilon\xi_\varepsilon \int_{\mathbb{R}} w\phi_{\varepsilon,\ell} dy + \varepsilon\xi_\varepsilon\eta_0 \int_{\mathbb{R}} w\phi_{\varepsilon,j} dy \right) (1+o(1)). \tag{4.10}$$

Sending $\varepsilon \to 0$, we obtain the following nonlocal eigenvalue problem

$$(-\Delta)^{\frac{1}{2}}\phi_j + \phi_j - 2w\phi_j + \frac{2\left[(1+\eta_0(1+\tau\lambda_0))\int_{\mathbb{R}} w\phi_j dy + \sum_{\ell\neq j} w\phi_\ell dy\right]}{(K+\eta_0)(1+\tau\lambda_0)\int_{\mathbb{R}} w^2(y)dy} w^2 + \lambda_0\phi_j = 0, \quad j = 1, \cdots, K. \tag{4.11}$$

5.phi-fin

Let

$$\mathcal{G} = \begin{pmatrix} 1+\eta_0(1+\tau\lambda_0) & 1 & \cdots & 1 \\ 1 & 1+\eta_0(1+\tau\lambda_0) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+\eta_0(1+\tau\lambda_0) \end{pmatrix},$$

$\mathcal{G}$ is sysmmetric and eigenvalues of $\mathcal{G}$ are given by

$$\lambda_1 = \cdots = \lambda_{K-1} = \eta_0(1+\tau\lambda_0), \quad \lambda_K = K + \eta_0(1+\tau_0\lambda_0).$$

Let P be an orthogonal matrix such that

$$P\mathcal{G}P^{-1} = \begin{pmatrix} \eta_0(1+\tau\lambda_0) & 0 & \cdots & 0 \\ 0 & \eta_0(1+\tau\lambda_0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K+\eta_0(1+\tau\lambda_0) \end{pmatrix}.$$

From (4.11), using the notation,

$$\Phi = (\phi_1, \cdots, \phi_K)^T,$$

we get

$$(-\Delta)^{\frac{1}{2}}\Phi + \Phi - 2w\Phi + \frac{\mathcal{G}\int_{\mathbb{R}} \Phi w dy}{(K+\eta_0)(1+\tau\lambda_0)\int_{\mathbb{R}} w^2(y)dy} w^2 + \lambda_0\Phi = 0.$$

Let $P\Phi = \bar{\Phi}$, then we get

$$(-\Delta)^{\frac{1}{2}}\bar{\Phi} + \bar{\Phi} - 2w\bar{\Phi} + \frac{2}{(K+\eta_0)(1+\tau\lambda_0)\int_{\mathbb{R}} w^2(y)dy} P\mathcal{G}P^{-1}\left(\int_{\mathbb{R}} w\bar{\Phi}\right) w^2 + \lambda_0\bar{\Phi} = 0,$$

and it can be written in components

$$(-\Delta)^{\frac{1}{2}}\bar{\Phi}_j + \bar{\Phi}_j - 2w\bar{\Phi}_j + \frac{\lambda_j}{(K+\eta_0)(1+\tau\lambda_0)\int_{\mathbb{R}} w^2(y)dy}\left(\int_{\mathbb{R}} w\bar{\Phi}_j(y)dy\right) w^2 + \lambda_0\bar{\Phi}_j = 0, \quad j = 1, \cdots, K. \tag{4.12}$$

5.fin-t

For $j = 1, \cdots, K-1$, (4.12) becomes

$$(-\Delta)^{\frac{1}{2}}\bar{\Phi}_j + \bar{\Phi}_j - 2w\bar{\Phi}_j + \frac{2\eta_0}{(K+\eta_0)\int_{\mathbb{R}} w^2(y)dy}\left(\int_{\mathbb{R}} w\bar{\Phi}_j(y)dy\right) w^2 + \lambda_0\bar{\Phi}_j = 0, \quad j = 1, \cdots, K-1, \tag{4.13}$$

5.fin-t-1

while for $j = K$, (4.12) becomes

$$(-\Delta)^{\frac{1}{2}}\bar{\Phi}_K + \bar{\Phi}_K - 2w\bar{\Phi}_K + \frac{2(K+\eta_0(1+\tau\lambda_0))}{(K+\eta_0)(1+\tau\lambda_0)\int_{\mathbb{R}} w^2(y)dy}\left(\int_{\mathbb{R}} w\bar{\Phi}_K(y)dy\right) w^2 + \lambda_0\bar{\Phi}_K = 0. \tag{4.14}$$

5.fin-k

If $K > 1$ and $\frac{2\eta_0}{K+\eta_0} < 1$ (i.e., $\eta_0 < K$), then by Theorem 2.2, problem (4.13) is unstable for all $\tau \geq 0$, which implies that problem (4.2) is linearly unstable for all $\tau \geq 0$. If $K \geq 1$ and $\frac{2\eta_0}{K+\eta_0} > 1$ or what is equivalent, $\eta_0 > K$, then by Theorem 2.2, problem (4.13) is stable. While for problem (4.14), by Theorem 2.7 we get that it is stable if $0 \leq \tau < \tau_2$ or $\tau > \tau_3$ for suitable $\tau_2 \leq \tau_3$.

If $K = 1$ and $\eta_0 < 1$, we see that the problem can be written in the form as (4.14). By Theorem 2.7, prolem (4.14) is stable if $0 \le \tau < \tau_4$ and unstable for $\tau > \tau_5$, for some suitable $\tau_4 < \tau_5$. Then we finish the whole proof for the large eigenvalue of symmetric pattern.

Asymmetric pattern. In the asymmetric case, we only consider the problem with two spikes. Using the Green's representation for the second equation of (4.1) we get

$$\begin{aligned}\psi(p_j) &= 2\beta^2 \int_{-1}^{1} G_{\beta_{\lambda_\varepsilon}}(x,\zeta) \sum_{\ell=1}^{2} \xi_{\varepsilon,\ell}\phi_{\varepsilon,\ell} w_\ell d\zeta \\ &= \sum_{\ell=1}^{2} \frac{\varepsilon \xi_{\varepsilon,\ell}}{1+\tau\lambda_\varepsilon} \int_{\mathbb{R}} w\phi_\ell(y)dy + 2\xi_{\varepsilon,j}\frac{\beta^2}{\pi}\varepsilon \log\frac{1}{\varepsilon}\int_{\mathbb{R}} w\phi_j dy, \quad j=1,2,\end{aligned} \tag{4.15}$$

The eigenvalue problem turns to be

$$(-\Delta)^{\frac{1}{2}}\phi_j + \phi_j - 2w\phi_j + \varepsilon\xi_{\varepsilon,j}\eta_0 \int_{\mathbb{R}} w\phi_j dy + \sum_{\ell=1}^{2} \frac{\varepsilon\xi_{\varepsilon,\ell}}{1+\tau\lambda_\varepsilon}\int_{\mathbb{R}} w\phi_\ell dy + \lambda_\varepsilon \phi_j = 0, \quad j=1,2, \tag{4.16}$$

where

$$\xi_{\varepsilon,1} = \frac{2(1+O(\beta^2))}{\varepsilon\int_{\mathbb{R}} w^2(y)dy}\left(\frac{1}{2\eta_0} + \frac{\sqrt{1-\frac{4}{\eta_0^2}}}{4+2\eta_0}\right), \ \xi_{\varepsilon,2} = \frac{2(1+O(\beta^2))}{\varepsilon\int_{\mathbb{R}} w^2(y)dy}\left(\frac{1}{2\eta_0} - \frac{\sqrt{1-\frac{4}{\eta_0^2}}}{4+2\eta_0}\right), \text{ where } \eta_0 > 2.$$

The associated two by two matrix of (4.18) is given by

$$\begin{pmatrix} \left(\eta_0 + \frac{1}{1+\tau\lambda_\varepsilon}\right)\left(\frac{1}{\eta_0} + \frac{\sqrt{1-\frac{4}{\eta_0^2}}}{2+\eta_0}\right) & \frac{1}{1+\tau\lambda_\varepsilon}\left(\frac{1}{\eta_0} - \frac{\sqrt{1-\frac{4}{\eta_0^2}}}{2+\eta_0}\right) \\ \frac{1}{1+\tau\lambda_\varepsilon}\left(\frac{1}{\eta_0} + \frac{\sqrt{1-\frac{4}{\eta_0^2}}}{2+\eta_0}\right) & \left(\eta_0 + \frac{1}{1+\tau\lambda_\varepsilon}\right)\left(\frac{1}{\eta_0} - \frac{\sqrt{1-\frac{4}{\eta_0^2}}}{2+\eta_0}\right) \end{pmatrix} \tag{4.17}$$

After simple calculation we get the eigenvalues of the above matrix are

$$\lambda_{1,2} = 1 + \frac{1}{1+\tau\lambda_\varepsilon}\frac{1}{\eta_0} \pm \sqrt{1 + \frac{1}{(1+\tau\lambda_\varepsilon)^2}\frac{1}{\eta_0^2} + \frac{2}{1+\tau\lambda_\varepsilon}\frac{1}{\eta_0} - \frac{4}{2+\eta_0} - \frac{8}{\eta_0(2+\eta_0)(1+\tau\lambda_\varepsilon)}}. \tag{4.18}$$

Next, we claim that

$$\lambda_2 = 1 + \frac{1}{1+\tau\lambda_\varepsilon}\frac{1}{\eta_0} - \sqrt{1 + \frac{1}{(1+\tau\lambda_\varepsilon)^2}\frac{1}{\eta_0^2} + \frac{2}{1+\tau\lambda_\varepsilon}\frac{1}{\eta_0} - \frac{4}{2+\eta_0} - \frac{8}{\eta_0(2+\eta_0)(1+\tau\lambda_\varepsilon)}} < 1. \tag{4.19}$$

It is equivalent to show that

$$1 + \frac{2}{1+\tau\lambda_\varepsilon}\frac{1}{\eta_0} - \frac{4}{2+\eta_0} - \frac{8}{\eta_0(2+\eta_0)(1+\tau\lambda_\varepsilon)} > 0. \tag{4.20}$$

Using $\eta_0 > 2$ we have $1 > \frac{4}{2+\eta_0}$ and it implies that (4.20). Thus (4.19) holds. By Theorem 2.2 we conclude that the system (4.18) admits an unstable eigenvalue and it proves that the asymmetric two spikes pattern is always unstable.

In the end of this subsection, we give the following result which establishes the relation between the corresponding limit eigenvalue problem of each case and the original eigenvalue problem (4.2)

Theorem 4.1. *Let λ_ε be a eigenvalue of (4.2) such that $\Re(\lambda_\varepsilon) > -c$ for some $c > 0$.*

(1) *Suppose that for suitable sequences $\varepsilon_n \to 0$ we have $\lambda_{\varepsilon_n} \to \lambda_0 \neq 0$. Then λ_0 is a eigenvalue of the problem given in (4.7) ((4.9), (4.11) and (4.18) for the other three cases).*
(2) *Let $\lambda_0 \neq 0$ with $\Re(\lambda_0) > 0$ be a eigenvalue of the problem given in (4.7) ((4.9), (4.11) and (4.18) for the other three cases). Then for ε sufficiently small, there is a eigenvalue λ_ε of (4.2) with $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$.*

Proof. One can see [12, Theorem 6.1] for the proof. □

4.2. **Stability Analysis: Small Eigenvalues.** We now study the problem (4.1) for small eigenvalues of the symmetric pattern. As in last subsection, we set

$$\hat{u}_\varepsilon = \xi_\varepsilon^{-1} u_\varepsilon = u_{\varepsilon,\mathbf{p}} + \phi_{\varepsilon,\mathbf{p}}, \quad \hat{v}_\varepsilon = \xi_\varepsilon^{-1} v_\varepsilon = v_{\varepsilon,\mathbf{p}} + \psi_{\varepsilon,\mathbf{p}}.$$

In the following discussion, we take $\tau = 0$ for simplicity. As $\lambda_\varepsilon \ll 1$ the results in this section are also valid for τ finite, this is due to the fact that the small eigenvalue are of the order $O(\varepsilon^2)$, we shall prove it in this subsection.

We cut off $\hat{u}_\varepsilon$ as follows

$$\tilde{u}_{\varepsilon,j}(y) = \chi\left(\frac{\varepsilon y - p_j^\varepsilon}{r_0}\right)\hat{u}_\varepsilon(y), \quad j = 1, \cdots, K, \tag{4.21}$$ 8.3

where $\chi(x)$ and r_0 are given in §2. Similarly to the §3 we define

$$\mathcal{K}_{\varepsilon,\mathbf{p},new} := \text{ Span}\left\{\frac{\partial \tilde{u}_{\varepsilon,j}}{\partial y} \mid i = 1, \cdots, K\right\} \subset H^1\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right),$$
$$\mathcal{C}_{\varepsilon,\mathbf{p},new} := \text{ Span}\left\{\frac{\partial \tilde{u}_{\varepsilon,j}}{\partial y} \mid i = 1, \cdots, K\right\} \subset L^2\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right).$$

Then it is easy to see that

$$\hat{u}_\varepsilon(y) = \sum_{j=1}^K \tilde{u}_{\varepsilon,j}(y) + O(\varepsilon^2). \tag{4.22}$$ 8.4

Note that

$$\tilde{u}_{\varepsilon,j}(y) \sim w\left(y - \frac{p_j^\varepsilon}{\varepsilon}\right) \quad \text{in} \quad H^1(-1,1)$$

and $\tilde{u}_{\varepsilon,j}$ satisfies

$$(-\Delta)^{\frac{1}{2}}\tilde{u}_{\varepsilon,j} + \tilde{u}_{\varepsilon,j} - \frac{\tilde{u}_{\varepsilon,j}^2}{\hat{v}_\varepsilon} + O(\varepsilon^2) = 0. \tag{4.23}$$ 8.5

Thus $\tilde{u}'_{\varepsilon,j} := \frac{\partial \tilde{u}_{\varepsilon,j}}{\partial y}$ satisfies

$$(-\Delta)_y^{\frac{1}{2}}\tilde{u}'_{\varepsilon,j} + \tilde{u}'_{\varepsilon,j} - 2\frac{\tilde{u}_{\varepsilon,j}}{\hat{v}_\varepsilon}\tilde{u}'_{\varepsilon,j} + \varepsilon\frac{\tilde{u}_{\varepsilon,j}^2}{\hat{v}_\varepsilon^2}\hat{v}'_\varepsilon + h.o.t. = 0, \tag{4.24}$$ 8.6

and we have

$$\tilde{u}'_{\varepsilon,j} = \frac{\partial w}{\partial y}\left(y - \frac{p_j^\varepsilon}{\varepsilon}\right)(1 + o(1)).$$

Let us now decompose

$$\phi_\varepsilon = \sum_{j=1}^K a_j^\varepsilon \tilde{u}'_{\varepsilon,j} + \phi_\varepsilon^\perp, \tag{4.25}$$ 8.7

where a_j^ε are complex numbers and $\phi_\varepsilon^\perp \perp \mathcal{K}_{\varepsilon,\mathbf{p},new}$. Similarly, we can decompose

$$\psi_\varepsilon = \sum_{j=1}^K a_j^\varepsilon \psi_{\varepsilon,j} + \psi_\varepsilon^\perp, \tag{4.26}$$ 8.8

where $\psi_{\varepsilon,j}$ satisfies

$$D(-\Delta)^{\frac{1}{2}}\psi_{\varepsilon,j} + \psi_{\varepsilon,j} - 2\xi_\varepsilon\hat{u}_\varepsilon\tilde{u}'_{\varepsilon,j} = 0, \quad j = 1, \cdots, K, \tag{4.27}$$ 8.9

and $\psi_\varepsilon^\perp$ satisfies

$$D(-\Delta)^{\frac{1}{2}}\psi_\varepsilon^\perp + \psi_\varepsilon^\perp - 2\xi_\varepsilon\hat{u}_\varepsilon\phi_\varepsilon^\perp = 0. \tag{4.28}$$ 8.10

We impose periodic boundary conditions for (4.27) and (4.28).

Suppose that $\|\phi_\varepsilon\|_{H^1(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} = 1$. Then $|a_j^\varepsilon| \le C$ since

$$a_j^\varepsilon = \frac{\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \phi_\varepsilon \frac{\partial \tilde{u}_{\varepsilon,j}}{\partial y} dy}{\int_{\mathbb{R}} w^2 dy} + o(1).$$

Substituting the decompositions of ϕ_ε and ψ_ε into (4.2) we have

$$(-\Delta)_y^{\frac{1}{2}}\phi_\varepsilon^\perp + \phi_\varepsilon^\perp - \frac{2\hat{u}_\varepsilon}{\hat{v}_\varepsilon}\phi_\varepsilon^\perp + \frac{\hat{u}_\varepsilon^2}{\hat{v}_\varepsilon^2}\psi_\varepsilon^\perp + \lambda_\varepsilon\phi_\varepsilon^\perp - \varepsilon\sum_{j=1}^K a_j^\varepsilon\left(\frac{\tilde{u}_{\varepsilon,j}^2}{\hat{v}_\varepsilon^2}\frac{\partial \hat{v}_\varepsilon}{\partial x} - \frac{1}{\varepsilon}\frac{\hat{u}_\varepsilon^2}{\hat{v}_\varepsilon^2}\psi_{\varepsilon,j}\right) + h.o.t. = -\lambda_\varepsilon\sum_{j=1}^K a_j^\varepsilon \tilde{u}_{\varepsilon,j}'. \quad (4.29)$$ 8.11

Set

$$J_1 := \varepsilon\sum_{j=1}^K a_j^\varepsilon \frac{\tilde{u}_{j,\varepsilon}^2}{\hat{v}_\varepsilon^2}\left(-\frac{1}{\varepsilon}\psi_{\varepsilon,j} + \frac{\partial \hat{v}_\varepsilon}{\partial x}\right),$$

and

$$J_2 := (-\Delta)_y^{\frac{1}{2}}\phi_\varepsilon^\perp + \phi_\varepsilon^\perp - 2\frac{\hat{u}_\varepsilon}{\hat{v}_\varepsilon}\phi_\varepsilon^\perp + \frac{\hat{u}_\varepsilon^2}{\hat{v}_\varepsilon^2}\psi_\varepsilon^\perp + \lambda_\varepsilon\phi_\varepsilon^\perp.$$

We divide the proof into two steps.

Step 1. In this step we shall use equation (4.29) to give a good error bounds for $\phi_\varepsilon^\perp$. Since $\phi_\varepsilon^\perp \perp \mathcal{K}_{\varepsilon,\mathbf{p},new}$, then similar to the proof of Proposition 3.2, it follows that

$$\|\phi_\varepsilon^\perp\|_{H^1(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})} \le C\|J_0\|_{L^2(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})}.$$

Let us now compute J_1. Let ξ_ε be the same as Theorem 1.1 and $k(\varepsilon,\beta) = \xi_\varepsilon\beta^2\varepsilon$, then we calculate that for $x \in B_\delta(p_j^\varepsilon)$:

$$\begin{aligned}\frac{\partial \hat{v}_\varepsilon}{\partial x} &= \xi_\varepsilon\beta^2\int_{-1}^1 \partial_x G_{\beta_{\lambda_\varepsilon}}(x,\zeta)\left(\hat{u}_\varepsilon\left(\frac{\zeta}{\varepsilon}\right)\right)^2 d\zeta\\ &= \xi_\varepsilon\beta^2\int_{-1}^1 \frac{\partial}{\partial x}G_\beta(x,\zeta)\left(\left(\tilde{u}_{\varepsilon,j}\left(\frac{\zeta}{\varepsilon}\right)\right)^2 + \sum_{\ell\ne j}\left(\tilde{u}_{\varepsilon,\ell}\left(\frac{\zeta}{\varepsilon}\right)\right)^2 + O(\varepsilon^2)\right)d\zeta,\end{aligned}$$

and by (4.27),

$$\psi_{\varepsilon,j} = 2\xi_\varepsilon\beta^2\int_{-1}^1 G_\beta\tilde{u}_{\varepsilon,j}\frac{\partial \tilde{u}_{\varepsilon,j}}{\partial y}d\zeta = \varepsilon\xi_\varepsilon\beta^2\int_{-1}^1 (K_\beta(|x-\zeta|) - H_\beta(x,\zeta))\frac{\partial}{\partial\zeta}(\tilde{u}_{\varepsilon,j})^2 d\zeta,$$

Thus for $x \in B_\delta(p_j^\varepsilon)$, we have

$$\begin{aligned}\frac{\partial \hat{v}_\varepsilon}{\partial x} - \frac{1}{\varepsilon}\psi_{\varepsilon,j} = \xi_\varepsilon\beta^2\Bigg[&\left(\int_{-1}^1\left[\frac{\partial}{\partial x}K_\beta(|x-\zeta|)\left(\tilde{u}_{\varepsilon,j}\left(\frac{\zeta}{\varepsilon}\right)\right)^2 - K_\beta(|x-\zeta|)\frac{\partial}{\partial\zeta}\left(\tilde{u}_{\varepsilon,j}\left(\frac{\zeta}{\varepsilon}\right)\right)^2\right]d\zeta\right)\\ &-\int_{-1}^1\left[\frac{\partial}{\partial x}H_\beta(x,\zeta)\left(\tilde{u}_{\varepsilon,j}\left(\frac{\zeta}{\varepsilon}\right)\right)^2 - H_\beta(x,\zeta)\frac{\partial}{\partial\zeta}\left(\tilde{u}_{\varepsilon,j}\left(\frac{\zeta}{\varepsilon}\right)\right)^2\right]d\zeta\\ &+\int_{-1}^1\sum_{\ell\ne j}\frac{\partial}{\partial x}G_\beta(x,\zeta)\left(\tilde{u}_{\varepsilon,\ell}\left(\frac{\zeta}{\varepsilon}\right)\right)^2 d\zeta + O(\varepsilon^2)\Bigg].\end{aligned}$$

Using the fact that

$$\frac{\partial}{\partial x}K_\beta(|x-\zeta|) + \frac{\partial}{\partial\zeta}K_\beta(|x-\zeta|) = 0, \quad \forall x \ne \zeta,$$

and using integration by parts, we get

$$\frac{\partial \hat{v}_\varepsilon}{\partial x} - \frac{1}{\varepsilon}\psi_{\varepsilon,j} = k(\varepsilon,\beta)\int_{\mathbb{R}} w^2\left(-\frac{\partial}{\partial x}F_j(x) + o(\varepsilon)\right)dy, \quad (4.30)$$ 8.16

where

$$F_j(x) = H_\beta(x,p_j^\varepsilon) - \sum_{\ell\ne j}G_\beta(x,p_\ell^\varepsilon).$$

Observe that

$$\frac{\partial}{\partial x}F_j(x)\mid x=p_j^\varepsilon=o_\varepsilon(1),$$

since $\mathbf{p}^\varepsilon\to\mathbf{p}^0$ and $\mathbf{p}^0$ is a critical point of $F(\mathbf{p})$ (see (1.7) for the definition of $F(\mathbf{p})$). Hence we have

$$\|J_1\|_{L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)}=o\left(\varepsilon k(\varepsilon,\beta)\sum_{j=1}^K|a_j^\varepsilon|\right) \tag{4.31}$$

8.18

and

$$\|\phi_\varepsilon^\perp\|_{H^1\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)}\le C\|J_1\|_{L^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)}=o\left(\varepsilon k(\varepsilon,\beta)\sum_{j=1}^K|a_j^\varepsilon|\right). \tag{4.32}$$

8.19

Using the equation for $\psi_\varepsilon^\perp$ and (4.32), we obtain that $\psi_\varepsilon^\perp=o\left(\varepsilon^2k(\varepsilon,\beta)\sum_{j=1}^K|a_j^\varepsilon|\right)$. We calculate

$$\begin{aligned}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}\left(J_2\frac{\partial\tilde u_{\varepsilon,j}}{\partial y}\right)dy=&-\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}\left(\frac{\tilde u_{\varepsilon,j}^2}{\hat v_\varepsilon^2}\left(\varepsilon\frac{\partial\hat v_\varepsilon}{\partial x}\phi_\varepsilon^\perp-\frac{\partial\tilde u_{\varepsilon,j}}{\partial y}\psi_\varepsilon^\perp\right)\right)dy+\lambda_\varepsilon\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}\phi_\varepsilon^\perp\frac{\partial\tilde u_{\varepsilon,j}}{\partial x}dy\\ =&-\int_{I_{\varepsilon,p_j^\varepsilon}}\frac{\tilde u_{\varepsilon,j}^2}{\hat v_\varepsilon^2}\left(\varepsilon\frac{\partial\hat v_\varepsilon}{\partial x}(p_j^\varepsilon+\varepsilon\zeta)-\varepsilon\frac{\partial\hat v_\varepsilon}{\partial x}(p_j^\varepsilon)\right)\phi_\varepsilon^\perp d\zeta-\int_{I_{\varepsilon,p_j^\varepsilon}}\frac{\tilde u_{\varepsilon,j}^2}{\hat v_\varepsilon^2}\left(\varepsilon\frac{\partial\hat v_\varepsilon}{\partial x}(p_j^\varepsilon)\right)\phi_\varepsilon^\perp d\zeta\\ &+\int_{I_{\varepsilon,p_i^\varepsilon}}\frac{\tilde u_{\varepsilon,j}^2}{\hat v_\varepsilon^2}\frac{\partial\tilde u_{\varepsilon,j}}{\partial y}\left(\psi_\varepsilon^\perp(p_j^\varepsilon+\varepsilon\zeta)-\psi_\varepsilon^\perp(p_j^\varepsilon)\right)d\zeta+\lambda_\varepsilon\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}\phi_\varepsilon^\perp\frac{\partial\tilde u_{\varepsilon,j}}{\partial x}dy\\ =&\,o\left(\varepsilon^2k(\varepsilon,\beta)+\varepsilon\lambda_\varepsilon k(\varepsilon,\beta)\right)\sum_{j=1}^K|a_j^\varepsilon|.\end{aligned} \tag{4.33}$$

8.20

where

$$I_{\varepsilon,p_j^\varepsilon}=\{y\mid p_j^\varepsilon+\varepsilon y\in(-1,1)\},$$

and we have used (4.28) and $\frac{\partial\hat v_\varepsilon}{\partial x}=O(1)$.

Step 2. In this step we shall derive an algebraic equation for a_j^ε. Multiplying both sides of (4.29) by $\frac{\partial\tilde u_{\ell,\varepsilon}}{\partial y}$ and integrating over $\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)$, we obtain

$$r.h.s.=-\lambda_\varepsilon\sum_{j=1}^K a_j^\varepsilon\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}\frac{\partial\tilde u_{\varepsilon,j}}{\partial y}\frac{\partial\tilde u_{\varepsilon,\ell}}{\partial y}d\zeta=-\lambda_\varepsilon(1+o(1))a_\ell^\varepsilon\int_{\mathbb{R}}\left(\frac{\partial w}{\partial y}\right)^2dy=-\lambda_\varepsilon\pi(1+o(1))a_\ell^\varepsilon, \tag{4.34}$$

8.right

where we have used

$$\int_{\mathbb{R}}\left(\frac{\partial w}{\partial y}\right)^2dy=\int_{\mathbb{R}}\frac{16y^2}{(1+y^2)^4}dy=\pi.$$

By (4.30) and (4.33)

$$\begin{aligned}l.h.s.=&-\varepsilon\sum_{j=1}^K a_j^\varepsilon\int_{I_{\varepsilon,p_\ell^\varepsilon}}\frac{(\tilde u_{\varepsilon,j})^2}{(\hat v_\varepsilon)^2}\left(-\frac{1}{\varepsilon}\psi_{\varepsilon,j}+\frac{\partial\hat v_\varepsilon}{\partial x}\right)\frac{\partial\tilde u_{\varepsilon,\ell}}{\partial y}dy+\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}\left(J_2\frac{\partial\tilde u_{\varepsilon,\ell}}{\partial y}\right)dy\\ =&\,\varepsilon k(\varepsilon,\beta)\sum_{j=1}^K a_j^\varepsilon\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}\frac{\tilde u_{\varepsilon,j}^2}{\hat v_\varepsilon^2}\left(\frac{\partial}{\partial x}F_j(x)\right)\frac{\partial\tilde u_{\varepsilon,\ell}}{\partial y}dy+o\left(\varepsilon^2k(\varepsilon,\beta)+\varepsilon\lambda_\varepsilon k(\varepsilon,\beta)\right)\sum_{j=1}^K|a_j^\varepsilon|\\ =&\,\varepsilon^2k(\varepsilon,\beta)\int_{\mathbb{R}}w^2\frac{\partial w}{\partial y}y\sum_{j=1}^K a_j^\varepsilon\left(\frac{\partial^2}{\partial p_j^\varepsilon\partial p_\ell^\varepsilon}F(\mathbf{p}^\varepsilon)\right)dy+o\left(\varepsilon^2k(\varepsilon,\beta)+\varepsilon\lambda_\varepsilon k(\varepsilon,\beta)\right)\sum_{j=1}^K|a_j^\varepsilon|\\ =&-\varepsilon^2k(\varepsilon,\beta)\pi\sum_{j=1}^K a_j^\varepsilon\frac{\partial^2}{\partial p_j^\varepsilon\partial p_\ell^\varepsilon}F(\mathbf{p}^\varepsilon)+o\left(\varepsilon^2k(\varepsilon,\beta)+\varepsilon\lambda_\varepsilon k(\varepsilon,\beta)\right)\sum_{j=1}^K|a_j^\varepsilon|,\end{aligned} \tag{4.35}$$

8.left

where we have used that

$$\int_{\mathbb{R}} w^2 \frac{\partial w}{\partial y} y dy = -\frac{1}{3} \int_{\mathbb{R}} w^3(y) dy = -\pi. \tag{4.36}$$

Combining (4.34) and (4.35) we have

$$\varepsilon^2 k(\varepsilon,\beta)\pi \sum_{j=1}^{K} a_j^\varepsilon \left(\frac{\partial^2}{\partial p_j^\varepsilon \partial p_\ell^\varepsilon} F(\mathbf{p}^\varepsilon) \right) + o\left(\varepsilon^2 k(\varepsilon,\beta) + \varepsilon \lambda_\varepsilon k(\varepsilon,\beta) \right) \sum_{j=1}^{K} |a_j^\varepsilon| = \lambda_\varepsilon \pi a_\ell^\varepsilon (1+o(1)). \tag{4.37}$$

5.balance

From (4.37), we see that the small eigenvalues with $\lambda_\varepsilon \to 0$ satisfying $|\lambda_\varepsilon| \sim \varepsilon^2 k(\varepsilon,\beta)$. Furthermore,

$$\frac{\lambda_\varepsilon}{\varepsilon^2 k(\varepsilon,\beta)} \to \sigma_0, \tag{4.38}$$

as $\varepsilon \to 0$, where σ_0 is a eigenvalue of the matrix $M(\mathbf{p}^0)$ and $\mathbf{p}^\varepsilon \to \mathbf{p}^0$ as $\varepsilon \to 0$. The vector $\vec{a}^\varepsilon = (a_1^\varepsilon, \cdots, a_K^\varepsilon)^T$ approaches an eigenvector of $M(\mathbf{p}^0)$ corresponding to the eigenvalue σ_0. In the following subsection, we shall show that if anyone of the following two conditions holds

(1) D is sufficiently large.

(2) $K = 2,3,4$ and D is arbitrary positive constant.

Then $\mathrm{rank}(M(\mathbf{p}^0)) = K-1$ and all the nonzero eigenvalues are negative. It implies that the small eigenvalue is always stable when $2 \le K \le 4$ or D is sufficiently large.

4.3. **Eigenvalue of the circulant matrix.** In the following, we shall compute the eigenvalue of the matrix $M(\mathbf{p}^0)$, defined by

$$M(\mathbf{p}^0) = \begin{pmatrix} -\sum_{j\neq 1} G_\beta''(p_1,p_j) & G_\beta''(p_1,p_2) & \cdots & G_\beta''(p_1,p_K) \\ G_\beta''(p_2,p_1) & -\sum_{j\neq 2} G_\beta''(p_2,p_i) & \cdots & G_\beta''(p_2,p_K) \\ \vdots & \vdots & \ddots & \vdots \\ G_\beta''(p_K,p_1) & G_\beta''(p_K,p_2) & \cdots & -\sum_{j\neq K} G_\beta''(p_K,p_j) \end{pmatrix},$$

where the Green's function $G_\beta(x,z)$ admits the following expression

$$G_\beta(x,z) = \frac{1}{2\beta^2} + \sum_{k=1}^{\infty} \frac{\cos(k\pi(x-z))}{\beta^2 + k\pi} = \frac{1}{\beta^2}\left(\frac{1}{2} + \sum_{k=1}^{\infty} \frac{\cos(k\pi(x-z))}{k\pi D} - \frac{1}{\pi D} \sum_{k=1}^{\infty} \frac{\cos(k\pi(x-z))}{k(1+k\pi D)} \right), \quad D = \frac{1}{\beta^2}.$$

It is known that (the left-hand side of (4.39) is the Fourier expansion of the right-hand side of (4.39) in $(-1,1)\setminus\{0\}$)

$$\sum_{k=1}^{\infty} \frac{\cos(k\pi x)}{k} = -\log\sin\left(\frac{\pi|x|}{2}\right) - \log 2 \quad \text{for} \quad x \in (-1,1)\setminus\{0\}. \tag{4.39}$$

9.fou

After straightforward computations we have

$$G_\beta'(x,0) = -\frac{1}{2}\cot\frac{\pi x}{2} + \frac{1}{\pi D}\frac{\pi - \pi x}{2} - \frac{1}{\pi D}\sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k(1+k\pi D)}, \quad x > 0, \tag{4.40}$$

9.green-1

and

$$G_\beta''(x,0) = \frac{\pi}{4}\csc^2\frac{\pi x}{2} + \frac{1}{\pi D^2}\left(\log\sin\left(\frac{\pi|x|}{2}\right) + \log 2 - \frac{\pi D}{2} + \sum_{k=1}^{\infty} \frac{\cos(k\pi x)}{k(1+k\pi D)} \right). \tag{4.41}$$

9.green-2

Since $p_1^0, \cdots, p_K^0$ are equally distributed, then it is easy to see that $M(\mathbf{p}^0)$ is a circulant matrix and all the eigenvalues can be written as (see [42, section 6])

$$\begin{aligned} \lambda_\ell = {} & \frac{1}{\pi D^2} \sum_{j=1}^{K-1} \left[\log\sin\left(\frac{j\pi}{K}\right) + \log 2 - \frac{\pi D}{2} + \sum_{k=1}^{\infty} \frac{1}{k(1+k\pi D)} \cos\left(\frac{2kj\pi}{K}\right) \right] \left(\cos\left(\frac{2(\ell-1)j\pi}{K}\right) - 1 \right) \\ & + \sum_{j=1}^{K-1} \frac{\pi}{4}\csc^2\left(\frac{j\pi}{K}\right) \left(\cos\left(\frac{2(\ell-1)j\pi}{K}\right) - 1 \right), \quad \ell = 1, \cdots, K. \end{aligned} \tag{4.42}$$

a.eig

Obviously, one can easily verify that $\lambda_1 = 0$ and it corresponds to the summation of each row of $M(\mathbf{p}^0)$ vanishes. To compute (4.42), we recall the following identities (see [4] and [6])

$$\sum_{j=1}^{K-1} \csc^2\left(\frac{j\pi}{K}\right) = K - 1 + \sum_{j=1}^{K-1} \cot^2\left(\frac{j\pi}{K}\right) = \frac{K^2-1}{3}, \tag{4.43}$$

9.id-1

and

$$\begin{aligned}\sum_{j=1}^{K-1} \csc^2\left(\frac{j\pi}{K}\right)\cos\left(\frac{2(\ell-1)j\pi}{K}\right) &= \sum_{j=1}^{K-1} \cot^2\left(\frac{j\pi}{K}\right)\cos\left(\frac{2(\ell-1)j\pi}{K}\right) + \sum_{j=1}^{K-1}\cos\left(\frac{2(\ell-1)j\pi}{K}\right) \\ &= \frac{1}{2}\sum_{\alpha=0}^{1}\sum_{\beta=0}^{2}\binom{2}{2\alpha}\binom{2}{\beta} B_{2\alpha}^{(1)}\left(\frac{\ell-1}{K}\right) B_{2-2\alpha}^{(2)}(\beta)K^{2\alpha} - 1,\end{aligned} \tag{4.44}$$

9.id-2

where $\binom{m}{n}$ denotes the Binomial coefficient and $B_n^{(m)}$ denotes the Bernoulli polynomial of order m and degree n defined using the generating functions

$$\left(\frac{t}{e^t-1}\right)^m e^{e^{tx}} = \sum_{n=0}^{\infty} B_n^{(m)}\frac{t^n}{n!}.$$

After a tedious computation we have

$$\sum_{j=1}^{K-1} \csc^2\left(\frac{j\pi}{K}\right)\left(\cos\left(\frac{2(\ell-1)j\pi}{K}\right) - 1\right) = 2(\ell-1)^2 - 2(\ell-1)K. \tag{4.45}$$

a.sum-1

Substituting (4.45) into (4.42) we get

$$\begin{aligned}\lambda_\ell = \frac{1}{\pi D^2}\sum_{j=1}^{K-1}&\left[\log\sin\left(\frac{j\pi}{K}\right) + \log 2 - \frac{\pi D}{2} + \sum_{k=1}^{\infty}\frac{1}{k(1+k\pi D)}\cos\left(\frac{2kj\pi}{K}\right)\right]\left(\cos\left(\frac{2(\ell-1)j\pi}{K}\right) - 1\right) \\ &+ \frac{\pi}{2}((\ell-1)^2 - (\ell-1)K), \quad \ell = 1, \cdots, K.\end{aligned} \tag{4.46}$$

a.eig-1

Concerning (4.46), we see that if D is sufficiently large then the sign of λ_ℓ is decided by $(\ell-1)(\ell-1-K)$ and it is easy to see that $\lambda_\ell < 0$ for $\ell = 2, \cdots, K$.

Next, we shall show when $K = 2, 3, 4$, $\mathrm{rank}(M(\mathbf{p}^0)) = K - 1$ and non-zero eigenvalues are negative for all D. Using (4.46) we have

$$\begin{aligned}\lambda_2 &= -\frac{2}{\pi D^2}\left(-\frac{\pi D}{2} - \sum_{k=1}^{\infty}\frac{\pi D}{1+k\pi D}\cos(k\pi)\right) - \frac{\pi}{2} \\ &= \frac{1}{D}\left(1 - \frac{\pi D}{2} - \sum_{m=1}^{\infty}\frac{2\pi D}{(1+(2m-1)\pi D)(1+2m\pi D)}\right), \quad K = 2,\end{aligned} \tag{4.47}$$

9.2-e-1

$$\lambda_2(\lambda_3) = \frac{3}{2D}\left(1 - \frac{2}{3}\pi D - \sum_{m=1}^{\infty}\left(\frac{2\pi D}{(1+(3m-2)\pi D)(1+3m\pi D)} + \frac{\pi D}{(1+(3m-1)\pi D)(1+3m\pi D)}\right)\right), \quad K = 3, \tag{4.48}$$

9.2-e-2

and

$$\begin{aligned}\lambda_2(\lambda_4) &= \frac{1}{D}\left(2 - \frac{3}{2}\pi D - \sum_{m=1}^{\infty}\frac{4\pi D}{(1+(4m-2)\pi D)(1+4m\pi D)} - \sum_{m=1}^{\infty}\frac{2\pi D}{(1+(2m-1)\pi D)(1+2m\pi D)}\right), \\ \lambda_3 &= \frac{2}{D}\left(1 - \pi D - \sum_{m=1}^{\infty}\frac{4\pi D}{(1+(4m-2)\pi D)(1+4m\pi D)}\right), \quad K = 4.\end{aligned} \tag{4.49}$$

9.2-e-3

To determine the sign of λ, defined in (4.47), (4.48) and (4.49), we need the following lemma le9.sign **Lemma 4.2.** *Consider the following function*

$$\mathcal{F}_1(x) = 1 - \frac{x}{2} - \sum_{m=1}^{\infty} \frac{2x}{(1+(2m-1)x)(1+2mx)},$$

and

$$\mathcal{F}_2(x) = 1 - \frac{2}{3}x - \sum_{m=1}^{\infty} \left(\frac{2x}{(1+(3m-2)x)(1+3mx)} + \frac{x}{(1+(3m-1)x)(1+3mx)} \right)$$

Then $\mathcal{F}_1(x)$ and $\mathcal{F}_2(x)$ are negative when x is positive.

Proof. Since the proof is almost the same, we shall only focus on the function $\mathcal{F}_2(x)$. Recall that the digamma function $\psi(z) = \frac{d}{dz}\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ has the following series representation

$$\psi(z) = \sum_{m=0}^{\infty} \left(\frac{1}{m+1} - \frac{1}{m+x} \right) - \gamma, \tag{4.50}$$

where γ is the Euler constant. Therefore, we can deduce that

$$\psi\left(1+\frac{1}{3x}\right) - \psi\left(\frac{1}{3}+\frac{1}{3x}\right) = \sum_{m=0}^{\infty} \left(\frac{1}{\frac{1}{3}+\frac{1}{3x}+m} - \frac{1}{1+\frac{1}{3x}+m} \right) = \sum_{m=1}^{\infty} \frac{6x^2}{(1+(3m-2)x)(1+3mx)}, \tag{4.51}$$ gamma-1

and

$$\psi\left(1+\frac{1}{3x}\right) - \psi\left(\frac{2}{3}+\frac{1}{3x}\right) = \sum_{m=0}^{\infty} \left(\frac{1}{\frac{2}{3}+\frac{1}{3x}+m} - \frac{1}{1+\frac{1}{3x}+m} \right) = \sum_{m=1}^{\infty} \frac{3x^2}{(1+(3m-2)x)(1+3mx)}, \tag{4.52}$$ gamma-2

Using (4.51) and (4.52), we derive that

$$\mathcal{F}_2(x) = 1 - \frac{2}{3}x - \frac{1}{3x}\left(2\psi\left(1+\frac{1}{3x}\right) - \psi\left(\frac{2}{3}+\frac{1}{3x}\right) - \psi\left(\frac{1}{3}+\frac{1}{3x}\right) \right).$$

To show $\mathcal{F}_2$ is negative for $x > 0$, it is enough to prove that

$$2\psi(1+t) - \psi\left(\frac{1}{3}+t\right) - \psi\left(\frac{2}{3}+t\right) > \frac{1}{t} - \frac{2}{9t^2}, \quad t > 0, \tag{4.53}$$ gamma-e

where $t = \frac{1}{3x}$. By the expansion of $\log\Gamma(t+a)$ for $t > 0$ and $a \in [0,1]$ (See (25) in [28]), we have

$$\log\Gamma(t+a) = \left(t+a-\frac{1}{2}\right)\log t - t + \frac{1}{2}\log 2\pi + \sum_{j=2}^{2n+1} \frac{(-1)^j B_j(a)}{j(j-1)t^{j-1}} + R_{2n+1}^{(a)}(t), \tag{4.54}$$ gamma-exp

where $n \geq 0$ and

$$\begin{aligned} R_{2n+1}^{(a)}(z) = & \frac{(-1)^{n+1}}{2\pi t^{2n+1}} \int_0^{+\infty} \frac{t^2}{t^2+s^2} s^{2n} \log(1-2e^{-2\pi s}\cos(2\pi a) + e^{-4\pi s}) ds \\ & + \frac{(-1)^{n+1}}{\pi t^{2n+2}} \int_0^{+\infty} \frac{t^2}{t^2+s^2} s^{2n+1} \arctan\left(\frac{\sin(2\pi a)}{e^{2\pi s} - \cos(2\pi a)} \right) ds \end{aligned} \tag{4.55}$$ gamma-rem

and $B_j(a)$ is the j-th Bernoulli polynomial. Then, we have

$$\begin{aligned} & 2\log\Gamma(1+t) - \log\Gamma\left(\frac{1}{3}+t\right) - \log\Gamma\left(\frac{2}{3}+t\right) \\ = & \log t + \frac{2}{9t} + 2R_3^{(1)}(t) - R_3^{(\frac{2}{3})}(t) - R_3^{(\frac{1}{3})}(t) \\ = & \log t + \frac{2}{9t} + \frac{1}{\pi}\int_0^{\infty} \frac{s^2}{t(t^2+s^2)} \log \frac{1-2e^{-2\pi s}+e^{-4\pi s}}{1+e^{-2\pi s}+e^{-4\pi s}} ds. \end{aligned} \tag{4.56}$$ gamma-eq

Differentiating both sides of (4.56) gives

$$2\psi(1+t)-\psi\left(\frac{1}{3}+t\right)-\psi\left(\frac{2}{3}+t\right)=\frac{1}{t}-\frac{2}{9t^2}+\frac{1}{\pi}\int_0^\infty\frac{s^2(3t^2+s^2)}{t^2(t^2+s^2)^2}\log\frac{1+e^{-2\pi s}+e^{-4\pi s}}{1-2e^{-2\pi s}+e^{-4\pi s}}ds$$
$$>\frac{1}{t}-\frac{2}{9t^2},\quad \forall t>0. \tag{4.57}$$

Hence (4.53) is proved and we finish the proof. □

With Lemma 4.2, we are able to show that $\lambda_2<0$ by taking $x=\pi D$ in $\mathcal{F}_1$ when $K=2$. If $K=3$, we take $x=\pi D$ in $\mathcal{F}_2$ and it proves that $\lambda_3<0$. While when $K=4$, to show $\lambda_2(\lambda_4)<0$ we could write the terms in the bracket of $\lambda_2(\lambda_4)$ (see (4.49)) as $\mathcal{F}_1(2\pi D)+\mathcal{F}_1(\pi D)$ and we get that it is negative. While for λ_3, we could prove it is negative just by taking $x=2\pi D$ in $\mathcal{F}_1$.

5. Proof of Proposition 1.3.

In this section we shall analyze the linear stability of the fractional Gierer-Meinhardt system with two spikes and give the proof for Proposition 1.3. Consider the following system

$$\begin{cases}\varepsilon(-\Delta)^{\frac{1}{2}}u_\varepsilon+u_\varepsilon-\frac{u_\varepsilon^2}{v_\varepsilon}=0,\\ D(-\Delta)^{\frac{1}{2}}v_\varepsilon+v_\varepsilon-\varepsilon^{-1}u_\varepsilon^2=0.\end{cases} \tag{5.1}$$

In the inner region near the $j-$th spike, centered at p_j, $j=1,2$, we set $u_\varepsilon=Du_j$, $v_\varepsilon=Dv_j$ and $y=\varepsilon^{-1}(x-p_j)$, then

$$\begin{cases}(-\Delta)^{\frac{1}{2}}u_j+u_j-\frac{u_j^2}{v_j}=0, & u_j(y)\to 0 \text{ as } |y|\to+\infty,\\ (-\Delta)^{\frac{1}{2}}v_j-u_j^2=0, & v_j(y)\sim -S_j\log r+C_j+o(1) \text{ as } |y|\to+\infty,\end{cases} \tag{5.2}$$

where $S_j=\frac{1}{\pi}\int_{\mathbb{R}}u_j^2dx$. In (5.1), since u_ε is algebraically small away from p_j, we have in the sense of distribution that $\varepsilon^{-1}u_\varepsilon^2\to\pi D^2\sum_{j=1}^2S_j\delta_{p_j}$, therefore, from the second equation of (5.1) we see that the limit function v satisfies

$$(-\Delta)^{\frac{1}{2}}v+\frac{1}{D}v=\pi D\sum_{j=1}^2S_j\delta_{p_j},\quad v(x)=-DS_j\log|x-p_j|+D\left(-\frac{S_j}{\sigma}+C_j\right)\text{ as } x\to p_j, \tag{5.3}$$

where $\sigma=-\frac{1}{\log\varepsilon}$. We define the Green function $G_D(x,0)$ and its regular part $R_D(x,0)$ by

$$(-\Delta)^{\frac{1}{2}}G_D(x,0)+\frac{1}{D}G_D(x,0)=\delta_0,\quad G_D(x,0)=-\frac{1}{\pi}\log|x|+R_D(x,0)\text{ as } x\to 0,$$

where R_D is the regular part of G_D and

$$R_D(0,0)=\frac{D}{2}-\frac{1}{\pi}\log\pi+O\left(\frac{1}{D}\right).$$

The solution to (5.3) is $v(x)=\pi D\sum_{j=1}^2S_jG_D(x,p_j)$. Comparing with the local behavior of $v(x)$ in (5.3) we derive that S_j satisfies

$$S_j+\pi\sigma S_jR_D(p_j,p_j)+\pi\sigma S_iG_D(p_i,p_j)=\sigma C_j. \tag{5.4}$$

Since the two spikes are equally distributed on $(-1,1)$, we have $G_D(p_i,p_j)=\frac{D}{2}-\frac{1}{\pi}\log 2+O\left(\frac{1}{D}\right)$. In the stability threshold we require that $D=O(\sigma^{-1})\gg 1$, we expand (5.4) to

$$\begin{cases}S_1-\sigma S_1\log\pi-\sigma S_2\log 2+\frac{D\pi\sigma}{2}\sum_{\ell=1}^2S_\ell=\sigma C_1,\\ S_2-\sigma S_2\log\pi-\sigma S_1\log 2+\frac{D\pi\sigma}{2}\sum_{\ell=1}^2S_\ell=\sigma C_2.\end{cases} \tag{5.5}$$

To determine what the appropriate scaling for S_j in terms of $\sigma \ll 1$ for the above equation (5.5) we use $C_j = O(S_j^{\frac{1}{2}})$ as $S_j \to 0$. Indeed, we set $u_j = \mathcal{U}_j S_j^p$ and $v_j = \mathcal{V}_j S_j^p$, where $\mathcal{U}_j$ and $\mathcal{V}_j$ are $O(1)$ as $S_j \to 0$, we obtain that the first equation in (5.2) is kept the same but that the equation for v_j becomes

$$(-\Delta)^{\frac{1}{2}}\mathcal{V}_j - S_j^p \mathcal{U}_j^2 = 0, \quad \mathcal{V}_j = -S_j^{1-p}\log r + S_j^{-p}C_j \ \text{ as } \ r \to 0.$$

Comparing the powers of S_j we see that $p = 1-p$ and it gives that $p = \frac{1}{2}$. Then, to ensure that $\mathcal{U}_j = O(1)$ we need $C_j = O(S_j^{\frac{1}{2}})$. This shows that if $S_j \sim S_{j0}\sigma^2$, the appropriate scaling for u_j, v_j and C_j are all $O(\sigma)$. To obtain a two-term expansion for the inner problem, we set

$$(u_j, v_j, C_j) = \sigma(u_{j0}, v_{j0}, C_{j0}) + \sigma^2(u_{j1}, v_{j1}, C_{j1}) + \sigma^3(u_{j2}, v_{j2}, C_{j2}) + \cdots,$$

and

$$S_j = S_{j0}\sigma^2 + S_{j1}\sigma^3 + \cdots.$$

Substituting these expansions into (5.2) and collecting powers of σ we derive that

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u_{j0} + u_{j0} - \frac{u_{j0}^2}{v_{j0}} = 0, & u_{j0}(y) \to 0 \ \text{ as } \ |y| \to +\infty, \\ (-\Delta)^{\frac{1}{2}}v_{j0} = 0, & v_{j0}(y) \to C_{j0} \ \text{ as } \ |y| \to +\infty, \end{cases} \tag{5.6}$$

At next order, u_{j1} and v_{j1} satisfy

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u_{j1} + u_{j1} - \frac{2u_{j0}}{v_{j0}}u_{j1} + \frac{u_{j0}^2}{v_{j0}^2}v_{j1} = 0, & u_{j1}(y) \to 0 \ \text{ as } \ |y| \to +\infty, \\ (-\Delta)^{\frac{1}{2}}v_{j1} - u_{j0}^2 = 0, & v_{j1}(y) \to -S_{j0}\log|y| + C_{j1} \ \text{ as } \ |y| \to +\infty. \end{cases} \tag{5.7}$$

Then at one higher order, we obtain that v_{j2} verifies that

$$(-\Delta)^{\frac{1}{2}}v_{j2} - 2u_{j0}u_{j1} = 0, \quad v_{j2}(y) \to -S_{j1}\log|y| + C_{j2} \ \text{ as } \ |y| \to +\infty. \tag{5.8}$$

The solution to (5.6) is simply

$$u_{j0} = C_{j0}w, \quad v_{j0} = C_{j0},$$

where $w(x) = \frac{2}{1+|x|^2}$ is the radially symmetric ground-state solution to $(-\Delta)^{\frac{1}{2}}w + w - w^2 = 0$. Using the Green function of $(-\Delta)^{\frac{1}{2}}$ in $\mathbb{R}$ and representation formula (5.7) we derive that

$$S_{j0} = \frac{1}{\pi}C_{j0}^2\int_{\mathbb{R}} w^2 dy = 2C_{j0}^2.$$

It is convenient to decompose u_{j1} and v_{j1} in terms of new variables $\hat{u}$ and $\hat{v}$ by

$$u_{j1} = C_{j1}w + S_{j0}\hat{u}_j, \quad v_{j1} = C_{j1} + S_{j0}\hat{v}_j,$$

then it is easy to check that $\hat{u}_j$ and $\hat{v}_j$ are the unique radially symmetric solutions to

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\hat{u}_j + \hat{u}_j - 2w\hat{u}_j + w^2\hat{v}_j = 0, & \hat{u}_j(y) \to 0 \ \text{ as } \ |y| \to +\infty, \\ (-\Delta)^{\frac{1}{2}}\hat{v}_j - \frac{1}{2}w^2 = 0, & \hat{v}_j \to -\log|y| \ \text{ as } \ |y| \to +\infty. \end{cases} \tag{5.9}$$

Concerning (5.8), integrating both sides we see that

$$\begin{aligned} S_{j1}\pi &= 2\int_{\mathbb{R}} C_{j0}w(C_{j1}w + S_{j0}\hat{u}_j)dy = 4C_{j0}C_{j1}\pi + 2C_{j0}S_{j0}\int_{\mathbb{R}} w\hat{u}_j dy \\ &= 4C_{j0}C_{j1}\pi + 2C_{j0}S_{j0}\int_{\mathbb{R}}(w + xw')w^2\hat{v}_j dy \approx 4C_{j0}C_{j1}\pi + 11.4482C_{j0}S_{j0}, \end{aligned}$$

where we used

$$\int_{\mathbb{R}}(w + zw')w^2\hat{v}_j dz = \frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\log\frac{1}{|z-y|}\left(\frac{2}{1+z^2}\right)^2\left(\frac{2}{1+z^2} - \frac{4z^2}{(1+z^2)^2}\right)\frac{2}{(1+y^2)^2}dzdy \approx 5.7241.$$

Summarizing the above computation, we have the following lemma

Lemma 5.1. *For $S_j = S_{j0}\sigma^2 + S_{j1}\sigma^3 + \cdots$, where $\sigma = -1/\log\varepsilon \ll 1$, the asymptotic solution to the core problem (5.2) is*

$$u_j \sim \sigma(u_{j0} + \sigma u_{j1} + \cdots), \quad v_j \sim \sigma(v_{j0} + \sigma v_{j1} + \sigma^2 v_{j2} + \cdots), \quad C_j \sim \sigma(C_{j0} + \sigma C_{j1} + \cdots),$$

where u_{j0}, u_{j1}, v_{j0}, v_{j1} are define by

$$u_{j0} = C_{j0}w, \quad u_{j1} = C_{j1}w + S_{j0}\hat{u}_j, \quad v_{j0} = C_{j0}, \quad v_{j1} = C_{j1} + S_{j0}\hat{v}_j,$$

with $(\hat{u}_j, \hat{v}_j)$ verifying (5.9). Finally, C_{j0} and C_{j1} are related to S_{j0} and S_{j1} by

$$C_{j0} = \sqrt{\frac{S_{j0}}{2}}, \quad C_{j1} = \frac{S_{j1}}{4C_{j0}} - \frac{S_{j0}}{2\pi}\int_{\mathbb{R}} w\hat{u}_j dy. \tag{5.10}$$

With Lemma 5.1 we are able to prove Proposition 1.3.

Proof of Proposition 1.3. Consider the linearized problem

$$\begin{cases} \varepsilon(-\Delta)^{\frac{1}{2}}\phi + \phi - 2\frac{u_\varepsilon}{v_\varepsilon}\phi + \frac{u_\varepsilon^2}{v_\varepsilon^2}\psi + \lambda\phi = 0, \\ D(-\Delta)^{\frac{1}{2}}\psi + \psi - 2\varepsilon^{-1}u_\varepsilon\phi + \lambda\psi = 0. \end{cases} \tag{5.11}$$

In the inner region near the center p_j, we introduce the local variables $\Phi_j(y)$ and $\Psi_j(y)$ by

$$\phi(x) = \Phi_j(y), \ \psi(x) = \Psi_j(y), \ y = \varepsilon^{-1}(x - p_j).$$

Upon substituting the above relation into (5.11), and using $u_\varepsilon = Du_j$ and $v_\varepsilon = Dv_j$ near p_j, where u_j and v_j satisfy the core problem (5.2), we obtain that

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\Phi_j + \Phi_j - \frac{2u_j}{v_j}\Phi_j + \frac{u_j^2}{v_j^2}\Psi_j + \lambda\Phi_j = 0, & \Phi_j(y) \to 0 \ \text{as} \ |y| \to \infty, \\ (-\Delta)^{\frac{1}{2}}\Psi_j - 2u_j\Phi_j = 0, & \Psi_j(y) \sim -\theta_j \log|y| + B_j \ \text{as} \ |y| \to \infty, \end{cases} \tag{5.12}$$

where B_j depends on S_j and λ. One can easily check that $\theta_j\pi = 2\int_{\mathbb{R}} u_j\Phi_j dy$. To determine θ_j, we must match the behavior of the core solution to an outer problem for ψ. Since u_ε is localized near the center, we have $2\varepsilon^{-1}u_\varepsilon\phi \to 2D\sum_{j=1}^2 \left(\int_{\mathbb{R}} u_j\Phi_j dy\right)\delta_{p_j} = \pi D\sum_{j=1}^2 \theta_j\delta_{p_j}$. Using this expression we obtain that the outer problem for ψ is

$$(-\Delta)^{\frac{1}{2}}\psi + \beta_\lambda^2\psi = \pi\sum_{j=1}^2 \theta_j\delta_{p_j}, \quad \psi(x) \sim -\theta_j\log|x - p_j| - \frac{\theta_j}{\sigma} + B_j \text{ as } x \to p_j, \tag{5.13}$$

where $\beta_\lambda = \sqrt{(1+\tau\lambda)/D}$. The solution to (5.13) is $\psi = \pi\sum_{j=1}^2 \theta_j G_{D_\lambda}(x, p_j)$ with G_{D_λ} satisfying

$$(-\Delta)^{\frac{1}{2}}G_{D_\lambda}(x,0) + \beta_\lambda^2 G_{D_\lambda}(x,0) = \delta_0, \quad G_{D_\lambda}(x,0) \sim -\frac{1}{\pi}\log|x| + R_{D_\lambda} \ \text{as} \ |x| \to 0.$$

From the above discussion, we conclude that

$$\theta_j + \pi\sigma\theta_j R_{D_\lambda}(p_j, p_j) + \pi\sigma\theta_i G_{D_\lambda}(p_i, p_j) = \sigma B_j.$$

Using $R_{D_\lambda}(0) \sim \frac{D}{2(1+\tau\lambda)} - \frac{1}{\pi}\log\pi + O(\sigma)$ we have

$$\begin{cases} \theta_1 - \theta_1\sigma\log\pi - \theta_2\sigma\log 2 + \sum_{\ell=1}^2 \frac{\pi D\sigma}{2(1+\tau\lambda)}\theta_\ell = \sigma B_1, \\ \theta_2 - \theta_2\sigma\log\pi - \theta_1\sigma\log 2 + \sum_{\ell=1}^2 \frac{\pi D\sigma}{2(1+\tau\lambda)}\theta_\ell = \sigma B_2. \end{cases} \tag{5.14}$$

Using Lemma 5.1 we first calculate the coefficients in (5.12) as

$$\frac{u_j}{v_j} = \frac{C_{j0}w + (C_{j1}w + S_{j0}\hat{u}_j)\sigma + \cdots}{C_{j0} + (C_{j1} + S_{j0}\hat{v}_j)\sigma + \cdots} = w + \frac{\sigma S_{j0}}{C_{j0}}(\hat{u}_j - w\hat{v}_j) + \cdots,$$

$$\frac{u_j^2}{v_j^2} = \left(\frac{C_{j0}w + (C_{j1}w + S_{j0}\hat{u}_j)\sigma + \cdots}{C_{j0} + (C_{j1} + S_{j0}\hat{v}_j)\sigma + \cdots}\right)^2 = w^2 + \frac{2\sigma S_{j0}}{C_{j0}}w(\hat{u}_j - w\hat{v}_j) + \cdots,$$

So that the local problem becomes

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\Phi_j + \Phi_j - \left[2w + \frac{2\sigma S_{j0}}{C_{j0}}(\hat{u}_j - w\hat{v}_j) + \cdots\right]\Phi_j + \left[w^2 + \frac{2\sigma S_{j0}}{C_{j0}} w(\hat{u}_j - w\hat{v}_j) + \cdots\right]\Psi_j + \lambda\Phi_j = 0, \\ (-\Delta)^{\frac{1}{2}}\Psi_j - 2\sigma\left[C_{j0}w + \sigma(C_{j1}w + S_{j0}\hat{u}_j) + \cdots\right]\Phi_j = 0, \\ \Phi_j(y) \to 0,\ \Psi_j(y) \sim -\theta_j \log|y| + B_j \quad \text{as} \quad |y| \to +\infty, \end{cases} \tag{5.15}$$

To analyze (5.15) together with (5.14), we substitute the appropriate expansions

$$\begin{cases} \Phi_j = \frac{1}{\sigma}\left(\Phi_{j0} + \sigma\Phi_{j1} + \cdots\right), \quad \Psi_j = \frac{1}{\sigma}\left(\Psi_{j0} + \sigma\Psi_{j1} + \cdots\right), \quad B_j = \frac{1}{\sigma}\left(B_{j0} + \sigma B_{j1} + \cdots\right), \\ \theta_j = \theta_{j0} + \sigma\theta_{j1} + \cdots, \quad \lambda = \lambda_0 + \sigma\lambda_1 + \cdots. \end{cases} \tag{5.16}$$

The leading order is

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\Phi_{j0} + \Phi_{j0} - 2w\Phi_{j0} + w^2\Psi_{j0} + \lambda_0\Phi_{j0} = 0, & \Phi_{j0}(y) \to 0 \text{ as } |y| \to +\infty, \\ (-\Delta)^{\frac{1}{2}}\Psi_{j0} = 0, & \Psi_{j0}(y) \to B_{j0} \text{ as } |y| \to +\infty, \end{cases} \tag{5.17}$$

then we conclude $\Psi_{j0} = B_{j0}$. At next order, we have

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\Phi_{j1} + \Phi_{j1} - 2w\Phi_{j1} + w^2\Psi_{j1} - \frac{2S_{j0}}{C_{j0}}(\hat{u}_j - w\hat{v}_j)\Phi_{j0} + \frac{2S_{j0}}{C_{j0}} w(\hat{u}_j - w\hat{v}_j)\Psi_{j0} + \lambda_1\Phi_{j0} + \lambda_0\Phi_{j1} = 0, \\ (-\Delta)^{\frac{1}{2}}\Psi_{j1} - 2C_{j0}w\Phi_{j0} = 0, \\ \Phi_{j1}(y) \to 0,\ \Psi_{j1}(y) \to -\theta_{j0}\log|y| + B_{j1} \quad \text{as} \quad y \to \infty. \end{cases} \tag{5.18}$$

At one more higher order, the problem for Ψ_{j2} is

$$(-\Delta)^{\frac{1}{2}}\Psi_{j2} - 2C_{j0}w\Phi_{j1} - 2(C_{j1}w + S_{j0}\hat{u}_j)\Phi_{j0} = 0, \quad \Psi_{j2}(y) \to -\theta_{j1}\log|y| + B_{j2} \quad \text{as} \quad |y| \to +\infty. \tag{5.19}$$

In addition, substituting (5.16) into (5.14)

$$\begin{cases} \theta_{j0} + \sum\limits_{\ell=1}^{2} \frac{\pi\sigma D}{2(1+\lambda\tau)}\theta_{\ell 0} = B_{j0}, \quad j = 1, 2, \\ \theta_{11} - \theta_{10}\log\pi - \theta_{20}\log 2 + \sum\limits_{\ell=1}^{2} \frac{\pi\sigma D}{2(1+\lambda\tau)}\theta_{\ell 1} = B_{11}, \\ \theta_{21} - \theta_{20}\log\pi - \theta_{10}\log 2 + \sum\limits_{\ell=1}^{2} \frac{\pi\sigma D}{2(1+\lambda\tau)}\theta_{\ell 1} = B_{21}. \end{cases} \tag{5.20}$$

Next, we solve (5.17)-(5.19). First we notice that

$$\theta_{j0} = \frac{2C_{j0}}{\pi}\int_{\mathbb{R}} w\Phi_{j0}dy.$$

To identify C_{j0} we use the expansion of C_j, S_j and (5.5). Since we consider the symmetric case, i.e.,

$$\hat{u}_1 = \hat{u}_2 = \hat{u},\ \hat{v}_1 = \hat{v}_2 = \hat{v},\ S_1 = S_2 = S,\ C_1 = C_2 = C,\ S_{1l} = S_{2l} = S_l,\ C_{1l} = C_{2l} = C_l,\ l = 0, 1, 2, \cdots.$$

We set

$$\mu = \pi D\sigma. \tag{5.21}$$

Collecting the power of σ we get

$$C_0 = S_0(1+\mu) = \sqrt{\frac{S_0}{2}}, \quad S_0 = \frac{1}{2(1+\mu)^2}, \quad \theta_{j0} = \frac{2C_0}{\pi}\int_{\mathbb{R}} w\Phi_{j0}dy = \frac{2}{1+\mu}\frac{\int_{\mathbb{R}} w\Phi_{j0}dy}{\int_{\mathbb{R}} w^2dy}.$$

In the following we consider $\hat{\Phi} = \Phi_1 - \Phi_2$, from (5.17) and the fact that $\theta_{10} - \theta_{20} = B_{10} - B_{20}$ we see that

$$(-\Delta)^{\frac{1}{2}}\hat{\Phi}_0 + \hat{\Phi}_0 - 2w\hat{\Phi}_0 + \frac{2}{1+\mu}\frac{\int_{\mathbb{R}} w\hat{\Phi}_0 dy}{\int_{\mathbb{R}} w^2dy}w^2 + \lambda_0\hat{\Phi}_0 = 0, \quad \hat{\Phi}_0 \to 0 \text{ as } |y| \to +\infty. \tag{5.22}$$

For (5.22) we have seen that $\Re(\lambda_0) < 0$ if and only if $2/(1+\mu) > 1$. Therefore, the stability threshold where $\lambda_0 = 0$, $\hat{\Phi}_0 = w$ occurs and $\mu = 1$. We derive that

$$C_0 = \frac{1}{4}, \quad S_0 = \frac{1}{8}, \quad \hat{\Phi}_0 = w, \quad \theta_{10} - \theta_{20} = B_{10} - B_{20} = \Psi_{10} - \Psi_{20} = 1.$$

Upon substituting the above equation into (5.18) we obtain at $\lambda_0 = 0$ that $\Psi_{11} - \Psi_{21}$ verifies

$$(-\Delta)^{\frac{1}{2}}(\Psi_{11} - \Psi_{21}) - \frac{1}{2}w^2 = 0, \quad \Psi_{11} - \Psi_{21} \sim -\log r + B_{11} - B_{21} \ \text{as} \ |x| \to \infty. \tag{5.23}$$
`a.psi1`

Compared with (5.9) we conclude that

$$\Psi_{11} - \Psi_{21} = \hat{v} + B_{11} - B_{21}. \tag{5.24}$$
`a.psi1-v`

In the following we are going to analyze the effect of the higher-order terms. To this aim, we set

$$\lambda = \sigma\lambda_1 + \cdots, \quad \mu = 1 + \sigma\mu_1 + \cdots, \tag{5.25}$$
`a.exp-mu`

and we shall derive an expression for λ_1 in terms of the parameter μ_1. We first use (5.5), Lemma 5.1 and the asymptotic behavior of $R_D(0,0)$ to obtain

$$[1 + (1 + \sigma\mu_1) - \sigma\log\pi + \cdots](\sigma^2 S_0 + \sigma^3 S_1 + \cdots) = \sigma^2(C_0 + \sigma C_1 + \cdots). \tag{5.26}$$
`a.eq`

From the $O(\sigma^3)$ we obtain that

$$C_1 = \mu_1 S_0 + 2S_1 - S_0\log\pi.$$

Combined with (5.10) we derive that

$$C_1 = -\frac{\mu_1}{8} - \frac{1}{8\pi}\int_{\mathbb{R}} w\hat{u}dy + \frac{\log\pi}{8}. \tag{5.27}$$
`a.eq-1`

Next, using (5.18), (5.19) and (5.23) we have

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\hat{\Phi}_1 + \hat{\Phi}_1 - 2w\hat{\Phi}_1 + w^2(B_{11} - B_{21}) + w^2\hat{v} + \lambda_1 w = 0, \\ (-\Delta)^{\frac{1}{2}}\hat{\Psi}_2 - \frac{1}{2}w\hat{\Phi}_1 - 2(C_1 w + \frac{1}{8}\hat{u})w = 0, \\ \hat{\Phi}_1(y) = \Phi_{11}(y) - \Phi_{21}(y) \to 0 \ \text{as} \ |y| \to +\infty, \\ \hat{\Psi}_2(y) = \Psi_{12}(y) - \Psi_{22}(y) \sim -(\theta_{11} - \theta_{21})\log r + B_{12} - B_{22} \ \text{as} \ |y| \to +\infty. \end{cases} \tag{5.28}$$
`a.eq-3`

Using the asymptotic behavior we obtain that

$$(\theta_{11} - \theta_{21}) = \frac{1}{2\pi}\int_{\mathbb{R}} w\hat{\Phi}_1 dy + 4C_1 + \frac{1}{4\pi}\int_{\mathbb{R}} w\hat{u}dy. \tag{5.29}$$
`a.eq-5`

Using (5.20) we have

$$B_{11} - B_{21} = \theta_{11} - \theta_{21} - \log\frac{\pi}{2}.$$

Combined with equation (5.28)

$$(-\Delta)^{\frac{1}{2}}\hat{\Phi}_1 + \hat{\Phi}_1 - 2w\hat{\Phi}_1 + \frac{\int_{\mathbb{R}} w\hat{\Phi}_1 dy}{\int_{\mathbb{R}} w^2 dy}w^2 + \lambda_1 w = \mathcal{R}_1, \tag{5.30}$$
`a.eq-6`

where

$$\mathcal{R}_1 = w^2\log\frac{\pi}{2} - w^2\hat{v} - 4C_1 w^2 - \frac{1}{4\pi}w^2\int_{\mathbb{R}} w\hat{u}dx.$$

In order to make equation (5.30) has solution, we need

$$\lambda_1\int_{\mathbb{R}} w(w + yw')dy = \int_{\mathbb{R}}\left(\log\frac{\pi}{2} - \hat{v} - 4C_1 - \frac{1}{4\pi}\int_{\mathbb{R}} w\hat{u}dz\right)w^2(w + yw')dy,$$

it implies that

$$\begin{aligned} \pi\lambda_1 &= 2\pi\log\frac{\pi}{2} - 8\pi C_1 - \frac{1}{2}\int_{\mathbb{R}} w\hat{u}dy - \int_{\mathbb{R}} w\hat{u}dy \\ &= \pi\mu_1 + 2\pi\log\frac{\pi}{2} - \pi\log\pi + \int_{\mathbb{R}} w\hat{u}dy - \frac{3}{2}\int_{\mathbb{R}} w\hat{u}dy \\ &= \pi\mu_1 + \pi\log\frac{\pi}{4} - \frac{1}{2}\int_{\mathbb{R}} w\hat{u}dy. \end{aligned}$$

So the threshold for μ_1 is

$$\mu_1 = \frac{1}{2\pi}\int_{\mathbb{R}} w\hat{u}dy - \log\frac{\pi}{4} \approx 0.911019 + 0.241564 = 1.15258.$$

Substituting it into (5.25), and using (5.21) we derive that

$$D = \frac{1}{\pi}\log\frac{1}{\varepsilon} + \frac{1}{\pi}\left(\frac{1}{2\pi}\int_{\mathbb{R}} w\hat{u}dy - \log\frac{\pi}{4}\right).$$

Thus we finish the proof. □

6. Overview of Numerical Calculations

In this section we outline the numerical solutions to the time-dependent fractional GM system with periodic boundary conditions (1.2). Our methodology is based upon the simulations performed in appendix B of [12]. To approximate the fractional laplacian we discretize over $[-1,1]$ using the finite difference quadrature discretization developed in [17] and perfrom time stepping using an implicit-explicit semi-backwards difference scheme as in [32].

Let $x_i = -1 + ih$ for $i = 0, \dots N-1$ discretize the interval $[-1,1]$ into N uniformly distributed points. Noting that $C_{1/2} = \frac{1}{\pi}$, the quadratic interpolant weights of [17] for $\alpha = 1$ become

$$w_j = \frac{1}{\pi h}\begin{cases} 4 - \frac{5}{2}\log(3) & j = \pm 1 \\ -4 + 2x\log\left(\frac{x+1}{x-1}\right) & j = \pm 2, \pm 4, \pm 6, \dots \\ 4 + 3\log(x) - \left(x + \frac{3}{2}\right)\log(x+2) + \left(x - \frac{3}{2}\right)\log(x-2) & j = \pm 3, \pm 5, \pm 7, \dots \end{cases} \tag{6.1}$$

(the value of w_0 is irrelevant to the computation). Let ϕ be a 2-periodic funcition discretized over $[-1,1]$ as $\phi_i = \phi(-1 + 2i/N)$ for $i = 0, \dots, N-1$. By periodicity, the discretizaton provided by (FL$_h$) in [17] simplifies to

$$(-\Delta)^{1/2}\phi(x_i) \approx (-\Delta_h)^{1/2}\phi_i = \sum_{j=0}^{N-1} W_{i-j}(\phi_i - \phi_j) \tag{6.2}$$

where

$$W_\sigma = w_\sigma + \sum_{k=1}^{\infty}(w_{\sigma+Nk} + w_{\sigma-Nk}).$$

In our computations we truncate this series to 5000 terms. To simulate the full system (1.2) we use an identical time-stepping as in [12] which we summarize here. Let $\Phi(t) = (u_0(t), \dots, u_{N-1}(t), v_0(t), \dots, v_{N-1}(t))^T$, $\mathcal{A} = \mathrm{diag}(\epsilon(-\Delta_h)^{1/2}), \tau^{-1}D(-\Delta_h)^{1/2})$ and $\mathcal{N}(\Phi)$ be a function which computes the nonlinearities of the system. Now (1.2) is approximated as

$$\frac{d\Phi}{dt} + \mathcal{A}\Phi + \mathcal{N}(\Phi) = 0.$$

Fix a timestep $\Delta t > 0$ and denote $\Phi_n = \Phi(n\Delta t)$. The 2-SBDF scheme [32] uses an implicit second-order backwards time-stepping for the fractional laplace term, and explicit time-stepping for the nonlinear terms. In particular, we compute the next time-step by solving

$$(3\mathcal{I} - 2\Delta t\mathcal{A})\Phi_{n+1} = 4\Phi_n - \Phi_{n-1} + 4\Delta t\mathcal{N}(\Phi_n) - 2\Delta t\mathcal{N}(\Phi_{n-1}).$$

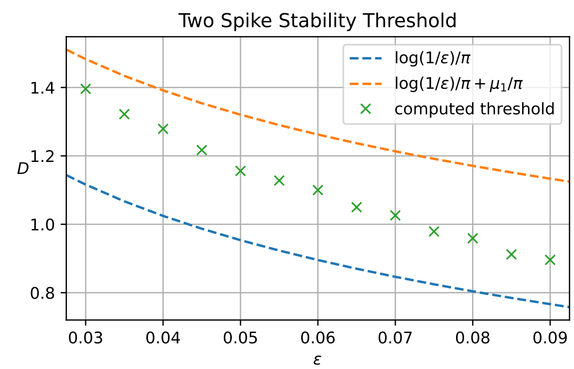
To attain Φ_1 we perform five steps of size $\Delta t/5$ using the first order 1-SBDF scheme

$$(\mathcal{I} - \Delta t\mathcal{A})\Phi_{n+1} = \Phi_n + \Delta t\mathcal{N}(\Phi_{n-1}). \tag{6.3}$$

In our computations, we use a mesh size of $N = 2000$ and timesteps of size $\Delta t = 0.01$. For the initial conditions we set the ansatz as (2.25) with the spike heights are (2.30), (2.31), and (2.32) for the η tends to 0, ∞, η_0 respectively in the symmetric case and (2.35) in the asymmetric case.

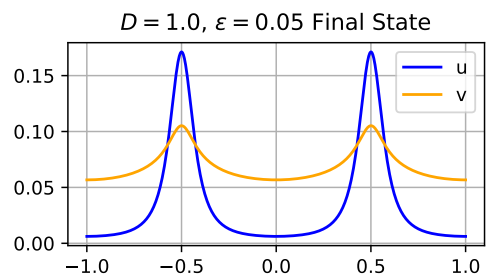
Based on our numerical simulations, we attach the following three figures to explain what we have done:

(1). In Figure 1 we have ploted three curves: the first order approximation of the threshold $\frac{1}{\pi}\log\frac{1}{\varepsilon}$, the second order approximation of the threshold (established in Proposition 1.3), and the computed threshold. They are represented by the blue dotted curve, the orange dotted curve and the X marks respectively. The computed threshold is attained by simulating the two-spike system for several initial values of ϵ and D. As a we have seen in the figure, the difference between the first-order approximation of the threshold and the computed threshold is approximately $\frac{1}{2|\log(1/\epsilon)|}$, and the



ike_Forming

FIGURE 1. Two spike stability threshold



Spike_Final

FIGURE 2. Two Spike Final State

second-order approximation of of the threshold is approximately $\frac{1}{|\log(1/\epsilon)|}$. As this simulation can become expensive, it is economical to determine a coarse estimation of the critical thereshold by simulating the system for a short time and observing the trend in spike height differences $|u(1/2) - u(-1/2)|$ since unstable two spike solutions near the threshold degenerate into solutions of a single bump at $\pm 1/2$. Solutions for which the small errors do not grow or decay exponentially are further simulated to attain a more precise value of the threshold.

(2). The activator and inhibitor of one such stable state is pictured in Figure fig:Two_Spike_Final 2. In this simulation, the final two spike state for $\varepsilon = 0.05$, $D = 1.0$ and $\tau = 0.02$ is shown. This value is attained at time $T = 500$ with the difference in spike heights is on the order 10^{-8} and decreasing. The analagous simulations starting with the asymmetric initial conditions did not yield such any stable states.

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