LOCAL UNIQUENESS OF THE MAGNETIC GINZBURG-LANDAU EQUATION

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Dedicated to Professor Michel Chipot on the occasion of his 70th birthday

Abstract. In this paper, we consider the magnetic Ginzburg-Landau equation:

$$\begin{cases} -\Delta_A\psi + \frac{\lambda}{2}(|\psi|^2-1)\psi = 0 \quad \text{in } \mathbb{R}^2, \\ \nabla\times\nabla\times A + \mathrm{Im}(\overline{\psi}\nabla_A\psi) = 0 \quad \text{in } \mathbb{R}^2, \\ |\psi|\to 1 \quad \text{as } |x|\to+\infty, \end{cases}$$

where $\lambda > 1$ is a coupling parameter, $\nabla_A = \nabla - iA$ and $\Delta_A = \nabla_A\cdot\nabla_A$ are, respectively, the covariant gradient and Laplacian. We prove, by perturbation arguments, that the only possible minimizer of the magnetic Ginzburg-Landau functional with degree 1 is the radial solution for λ sufficiently close to 1.

Keywords: Magnetic Ginzburg-Landau equation; Minimizers; Local uniqueness.

AMS Subject Classification 2010: 35B09; 35J47; 35J50.

1. Introduction

In this paper, we consider the magnetic Ginzburg-Landau equation:

$$\begin{cases} -\Delta_A\psi + \frac{\lambda}{2}(|\psi|^2-1)\psi = 0 \quad \text{in } \mathbb{R}^2, \\ \nabla\times\nabla\times A + \mathrm{Im}(\overline{\psi}\nabla_A\psi) = 0 \quad \text{in } \mathbb{R}^2, \\ |\psi|\to 1 \quad \text{as } |x|\to+\infty, \end{cases} \tag{1.1}$$

where $\lambda > 1$ is a coupling parameter, $\nabla_A = \nabla - iA$ and $\Delta_A = \nabla_A\cdot\nabla_A$ are, respectively, the covariant gradient and Laplacian, and $\nabla\times$ is the curl operator in $\mathbb{R}^2$ so that for a vector function A, $\nabla\times A = \partial_1A_2 - \partial_2A_1$ while for a scalar function A, $\nabla\times A = (-\partial_2A, \partial_1A)$. It is well known that (1.1) is the Euler-Lagrange equation of the following Ginzburg-Landau energy functional in $H^1_{loc}(\mathbb{R}^2;\mathbb{C})\times H^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$:

$$\mathcal{E}_\lambda(\psi, A) = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla_A\psi|^2 + |\nabla\times A|^2 + \frac{\lambda}{4}(|\psi|^2-1)^2.$$

That is, critical points of $\mathcal{E}_\lambda(\psi, A)$ in $H^1_{loc}(\mathbb{R}^2;\mathbb{C})\times H^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ are equivalent to weak solutions of (1.1).

The Ginzburg-Landau energy functional $\mathcal{E}_\lambda(\psi, A)$ has a rich physical background. It models the difference in free energy between the superconducting and normal states near the transition temperature in the Ginzburg-Landau theory. In that theory, $\psi : \mathbb{R}^2\to\mathbb{C}$ is called *the order parameter*, whose modulus (the density of

Cooper pairs of superconducting electrons in the BCS theory) indicates the local state of the material: If $|\psi| \approx 1$ then the material is in the superconducting phase while if $|\psi| \approx 0$ then the material is in the normal phase. A is the vector potential where $\nabla \times A$ is the induced magnetic field. The parameter λ is a material constant, corresponding to the ratio between characteristic lengthscales of the material: If $\lambda < 1$ then the material is of type I superconductor while if $\lambda > 1$ then the material is of type II superconductor. $\lambda = 1$ is the critical case of these two types. The Ginzburg-Landau energy functional $\mathcal{E}_\lambda(\psi, A)$ can also arise as the energy of a static configuration in the Yang-Mills-Higgs classical gauge theory on the plane, with abelian gauge group $U(1)$. In this theory, the Ginzburg-Landau energy functional $\mathcal{E}_\lambda(\psi, A)$ is often written as

$$\mathcal{E}_\lambda(\psi, A) = \frac{1}{2}\int_{\mathbb{R}^2} F_A \wedge *F_A + D_A\psi \wedge *D_A\psi + \frac{\lambda}{4} * (|\psi|^2 - 1)^2,$$

where $*$ is the Hodge duality operator, iF_A is the curvature of an $\mathcal{S}^1$ connection iA and ψ is a section of the associated complex line bundle. The induced connection couples A and ψ via the covariant derivative $D_A = d - iA$. A is a real one-form and $F_A = dA$ is a real two-form. The function $B = *F_A$ is known as the magnetic field, while ψ is called either *the order parameter* or *the Higgs field*. We refer the readers to [11,17,20] for more details of the physical backgrounds of $\mathcal{E}_\lambda(\psi, A)$.

As usual problems in the whole space $\mathbb{R}^2$, the Ginzburg-Landau energy functional $\mathcal{E}_\lambda(\psi, A)$ (and also the equation (1.1)) is invariant under translations and rotations which are given by

$$(\psi(x), A(x)) \to (\psi(g^{-1}x), gA(g^{-1}x)) \quad \text{for all } g \in \mathcal{SO}(2).$$

Besides these geometric invariance, the significant feature of the Ginzburg-Landau energy functional $\mathcal{E}_\lambda(\psi, A)$ (and the equation (1.1)), which is well known nowadays, is that they are invariant under the gauge transformations:

$$(\psi, A) \to (\psi e^{i\chi}, A + \nabla\chi) \quad \text{for all } \chi \in C^2(\mathbb{R}^2),$$

which generates an infinite dimensional symmetry group of $\mathcal{E}_\lambda(\psi, A)$ and (1.1).

For physical reasons, it is natural to consider solutions of (1.1) with finite energy values. It has been proved in [11] that these solutions satisfy the boundary condition:

$$(|\psi|, |\nabla_A\psi|, |\nabla \times A|) \to (1, 0, 0) \quad \text{as } |x| \to +\infty, \tag{1.2}$$

which leads one to define the topological degree or winding number or vortex number of ψ as follows:

$$deg(\psi) = deg\left(\frac{\psi}{|\psi|}|_{|x|=R}\right) = \frac{1}{2\pi}\int_{|x|=R} d(arg(\psi)) \quad \text{for } R \text{ sufficiently large.}$$

Applying the Stokes theorem yields that this degree of ψ satisfies

$$2\pi deg(\psi) = \int_{\mathbb{R}^2} \nabla \times A.$$

Indeed, let us rewrite $(\psi, A) = (we^{ig}, A)$, where $(w(a_j)e^{ig(a_j)}, A(a_j))$ should be understood as the limit at the zeros a_j. Since (ψ, A) has finite energy, $\int_{\mathbb{R}^2} w^2|\nabla g -$

$A|^2 < +\infty$. Thus, by (1.2) there exists $\rho_n \to +\infty$ such that $\rho_n \int_{\partial B_{\rho_n}} |\nabla g - A|^2 \to 0$ as $n \to \infty$. It follows from the Stokes theorem and the Hölder inequality that

$$\begin{aligned}\int_{\mathbb{R}^2} \nabla \times A &= \lim_{n\to\infty} \int_{\partial B_{\rho_n}} A \\ &= \lim_{n\to\infty} \int_{\partial B_{\rho_n}} \nabla g + \lim_{n\to\infty} \int_{\partial B_{\rho_n}} (\nabla g - A) \\ &= 2\pi deg(\psi).\end{aligned}$$

So that, $deg(\psi)$ is nothing but just the flux quantization of the magnetic field $B = \nabla \times A$. In the Yang-Mills-Higgs theory, the degree $\deg(\psi)$ is also known as *the first chern number* of the complex line bundle in which A is a connection and commonly, is called *charges* (cf. [19]).

It is well known that the degree $\deg(\psi)$ is an integer and is invariant under small and finite-energy perturbations. Thus, solutions of the magnetic Ginzburg-Landau equation (1.1) can be classified by the degree $\deg(\psi)$. By integrations by parts as that in [19, 20], we have

$$\begin{aligned}\mathcal{E}_\lambda(\psi, A) &= \frac{1}{2}\int_{\mathbb{R}^2} [(\partial_1\psi_1 + A_1\psi_2) \mp (\partial_2\psi_2 - A_2\psi_1)]^2 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2} [(\partial_2\psi_1 + A_2\psi_2) \pm (\partial_1\psi_2 - A_1\psi_1)]^2 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2} [\nabla \times A \pm (|\psi|^2 - 1)]^2 \pm \frac{1}{2}\int_{\mathbb{R}^2} \nabla \times A \\ &\quad + \frac{\lambda - 1}{8}\int_{\mathbb{R}^2} (|\psi|^2 - 1)^2, \end{aligned} \tag{1.3}$$

where $\psi = \psi_1 + i\psi_2$. Here, the upper sign corresponds to positive degrees and the lower sign to negative degrees. Thus, for those (ψ, A) such that the degree $\deg(\psi) \neq 0$, we must have $\mathcal{E}_\lambda(\psi, A) \geq \pi$. It follows that the global minimizers of $\mathcal{E}_\lambda(\psi, A)$ in $H^1_{loc}(\mathbb{R}^2; \mathbb{C}) \times H^1_{loc}(\mathbb{R}^2; \mathbb{R}^2)$ must have degree zero and satisfy $|\psi| \equiv 1$ in $\mathbb{R}^2$. Hence, we may assume the global minimizers to be $\psi = e^{ig}$. By (1.1), we know that

$$-\Delta g = 0, \quad \text{in } \mathbb{R}^2.$$

By the gauge invariance, we can choose $\psi \equiv 1$ and by (1.1) once more, $A \equiv 0$. It follows that all global minimizers of $\mathcal{E}_\lambda(\psi, A)$ in $H^1_{loc}(\mathbb{R}^2; \mathbb{C}) \times H^1_{loc}(\mathbb{R}^2; \mathbb{R}^2)$ are given by $\{(e^{ig}, \nabla g)\}$, where g is a harmonic function and thus, up to gauge translations, $(\psi, A) = (1, 0)$ is the *unique* global minimizer of $\mathcal{E}_\lambda(\psi, A)$ in $H^1_{loc}(\mathbb{R}^2; \mathbb{C}) \times H^1_{loc}(\mathbb{R}^2; \mathbb{R}^2)$. Clearly, these degree zero solutions are all stable. For other solutions with $\deg(\psi) \neq 0$, ψ must have zeros. These zeros are often called vortices of ψ and so that, the word "vortex" will be used to refer to a zero of ψ as well as to a solution. A multi-vortex solution refers to the case in which ψ has at least two zeros. Solutions with $\deg(\psi) \neq 0$ are only well understood in the critical case $\lambda = 1$, thanks to classical results of Jaffe and Taubes [11, 19, 20]. In this case, all solutions with $\deg(\psi) \neq 0$ can be classified by its vortices and they are all local minimizers of $\mathcal{E}_\lambda(\psi, A)$ in $H^1_{loc}(\mathbb{R}^2; \mathbb{C}) \times H^1_{loc}(\mathbb{R}^2; \mathbb{R}^2)$ and thus, they are all stable. To our best knowledge, very few is known for the cases $\lambda \neq 1$, except the so-called radial solutions:

$$\phi_{\lambda,N}(x) = f_\lambda(r)e^{iN\theta}; \quad B_{\lambda,N}(x) = Na_\lambda(r)\nabla\theta,$$

where $|N| \geq 1$ is its degree $\deg(\phi)$. We remark that the discrete symmetry $\psi \to \overline{\psi}$ and $A \to -A$ of (1.1) interchanges the negative degrees to the positive degrees. Thus, we can assume the degrees of solutions to be nonnegative in what follows. The existence of these radial solutions is established in [4] by variational arguments. The uniqueness of these radial solutions is proved in [1] and [6], respectively for $\lambda > 0$ sufficiently large and λ sufficiently close to 1 (including $\lambda = 1$). The stability of these radial solutions is also studied in the literature. It has been proved in [9] that $(\phi_{\lambda,N}, B_{\lambda,N})$ are all stable for $\lambda < 1$ while for $\lambda > 1$, $(\phi_{\lambda,1}, B_{\lambda,1})$ is stable and $(\phi_{\lambda,N}, B_{\lambda,N})$ are unstable for $N \geq 2$.

Under suitable boundary conditions, the existence of non-radial solutions was first established in bounded domains for λ sufficiently large, see, for example, [2, 3] for the non-magnetic case and [5, 18] for the magnetic case. This is due to the boundary forces which keep repelling vortices within the bounded domain. In the case of the Ginzburg-Landau equation on unbounded domains, it is conjectured in [12] by numerical evidence that for the non-magnetic Ginzburg-Landau equations on the whole plane, non-radial solutions do exist, while the studies in [10] suggest that for magnetic vortices, stationary multi-vortex configurations of degrees ± 1 occur with discrete symmetry group. The later conjecture was proved in [21] by reduction arguments for large degrees and large number vortices. It is also worth pointing out the work [14], which proved that $(\phi_{\lambda,1}, B_{\lambda,1})$ is the unique global minimizer of $\mathcal{E}_\lambda(\psi, A)$ in the degree 1 class (the definition is given below) for λ sufficiently large, while there is no global minimizers of $\mathcal{E}_\lambda(\psi, A)$ in the degree $N(N \geq 2)$ class (the definitions are also given below) for λ sufficiently large. These results partially prove a conjecture in [11].

Thus, to our best knowledge, whether the magnetic Ginzburg-Landau equation (1.1) have non-radial solutions or not is still unknown for $\lambda \neq 1$ and not sufficiently large. Since the critical case $\lambda = 1$ is well understood, it seems resonable to study the case when λ sufficiently close to 1. Therefore, the main purpose of this paper is to investigate the magnetic Ginzburg-Landau equation (1.1) for λ sufficiently close to 1 by perturbation arguments.

We shall mainly consider **stable** solutions of (1.1). For this, it is natural to consider the minimizers of $\mathcal{E}_\lambda(\psi, A)$ in the following sense: (ψ_0, A_0) is a minimizer of $\mathcal{E}_\lambda(\psi, A)$ if $\mathcal{E}_\lambda(\psi_0, A_0) \leq \mathcal{E}_\lambda(\psi_0 + \phi, A_0 + B)$ for all $(\phi, B) \in C_0^\infty(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. However, we remark that in this sense, the word "minimizer" is dependent on the degree of (ψ_0, A_0). Indeed, if (ψ_0, A_0) and (ψ_1, A_1) are two minimizers of $\mathcal{E}_\lambda(\psi, A)$ in the above sense, respectively with degrees k and l such that $k \neq l$. Then the energy values $\mathcal{E}_\lambda(\psi_0, A_0)$ and $\mathcal{E}_\lambda(\psi_1, A_1)$ are incomparable in the above sense since owing to their different degrees, one can not write $(\psi_0, A_0) = (\psi_1 + \phi, A_1 + B)$ for some $(\phi, B) \in C_0^\infty(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. Thus, a more precise definition of minimizers in the above sense is the following (cf. [14, Definition II.1]):

Definition 1.1. *We say (ψ_0, A_0) is a minimizer of $\mathcal{E}_\lambda(\psi, A)$ with degree $k \neq 0$ if deg$(\psi_0) = k$ and $\mathcal{E}_\lambda(\psi_0, A_0) \leq \mathcal{E}_\lambda(\psi_0 + \phi, A_0 + B)$ for all $(\phi, B) \in C_0^\infty(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$.*

To make the minimizers of $\mathcal{E}_\lambda(\psi, A)$ with different degrees to be comparable, we shall also introduce the following definition:

Definition 1.2. *We say (ψ_0, A_0) is a nontrivial least-energy minimizer of $\mathcal{E}_\lambda(\psi, A)$ if*

(1) *$deg(\psi_0) \neq 0$;*
(2) *$\mathcal{E}_\lambda(\psi_0, A_0) \leq \mathcal{E}_\lambda(\psi_0 + \phi, A_0 + B)$ for all $(\phi, B) \in C_0^\infty(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$;*
(3) *$\mathcal{E}_\lambda(\psi_0, A_0) \leq \mathcal{E}_\lambda(\psi_1, A_1)$ for all other minimizers (ψ_1, A_1) in the sense of Definition 1.1.*

Our main result in this paper now can be stated as follows.

Theorem 1.1. *Let $(\psi_\lambda, A_\lambda)$ be a classical solution of (1.1) which is also a nontrivial least-energy minimizer of $\mathcal{E}_\lambda(\psi, A)$. Then for λ sufficiently close to 1, $(\psi_\lambda, A_\lambda)$ must be the unique radial solution of (1.1) with degree one.*

Let us briefly sketch our strategy in proving Theorem 1.1. Our start point is the potential energy

$$\mathcal{H}(\varphi) = \int_{\mathbb{R}^2} (|\varphi|^2 - 1)^2,$$

where φ is a solution of (1.1) with $\lambda = 1$. Now, if $(\psi_\lambda, A_\lambda)$ is a classical solution of (1.1) which is also a nontrivial least-energy minimizer of $\mathcal{E}_\lambda(\psi, A)$, then the upper-bound of $(\psi_\lambda, A_\lambda)$'s energy values, generated by the potential energy $\mathcal{H}(\varphi)$, will impose the degree of $(\psi_\lambda, A_\lambda)$ to be 1 for λ sufficiently close to 1. Since the potential energy is well understood for degree one solutions (see, for example Proposition 2.1 below for more details), we can follow the plans in [5, 14] to choose suitable gauges such that $(\psi_\lambda, A_\lambda)$ could weakly converge to some (ϕ, B) in $H^1_{loc}(\mathbb{R}^2; \mathbb{C}) \times H^1_{loc}(\mathbb{R}^2; \mathbb{R}^2)$ as $\lambda \to 1$ in a suitable sense. By comparing the energy values carefully, this weak convergence could lead us to say that the weak limit (ϕ, B) must be the unique radial solution with degree 1 for $\lambda = 1$ and so that we could further say that $(\psi_\lambda, A_\lambda)$ only has one vortex for λ sufficiently close to 1. Next, we obtain some strong convergence of energy values as $\lambda \to 1$, which, together with regularity arguments in [13] and the decaying estimates in [11], implies that $(|\psi_\lambda|, |\nabla_{A_\lambda} \psi_\lambda|, |\nabla \times A_\lambda|) \to (1, 0, 0)$ exponentially as $|x| \to +\infty$, uniformly for λ sufficiently close to 1. Thus, we could say that $(\psi_\lambda, A_\lambda)$ strongly converges to (ϕ, B) in $H^1(\mathbb{R}^2; \mathbb{C}) \times H^1(\mathbb{R}^2; \mathbb{R}^2)$ as $\lambda \to 1$. After doing this, we are in the position to expand $(\psi_\lambda, A_\lambda)$ at the unique radial solution by its $H^1 \times H^1$-kernel, which is well understood (cf. [9, 16]). Then the analysis on the possible errors yields that $(\psi_\lambda, A_\lambda)$ is the unique radial solution with degree 1 for λ sufficiently close to 1.

Since the potential energy is not very clear for solutions with $\lambda = 1$ and higher degrees ≥ 2 (see [7, 16] for more discussions), our strategy seems to be invalid for discussing these cases. For example, let $(\psi_{\lambda,2}, A_{\lambda,2})$ be a classical solution of (1.1) which is also a minimizer of $\mathcal{E}_\lambda(\psi, A)$ in the sense of Definition 1.1 with degree 2. Then the energy of $(\psi_{\lambda,2}, A_{\lambda,2})$ can be bounded from above by $2\pi + \frac{\lambda-1}{8}\mathcal{H}(\varphi_a)$, where φ_a is any solution of (1.1) for $\lambda = 1$ with degree 2 and a is the distance of the two vortices. The numerical computations in [15] yield that $\mathcal{H}(\varphi_a)$ is a strictly decreasing function of $a > 0$ such that $\mathcal{H}(\varphi_a) \to 0$ as $a \to +\infty$. Thus, the minimum can not be attained, in general. It is worth pointing out that the potential energy

$\mathcal{H}(\varphi_a)$ also plays an important role in studying the magnetic Ginzburg-Landau gradient flows, see, for example [7, 16].

Notations. Throughout this paper, C and C' are indiscriminately used to denote various absolutely positive constants. $a \sim b$ means that $C'b \leq a \leq Cb$ and $a \lesssim b$ means that $a \leq Cb$.

2. Proof of Theorem 1.1

Let $(\psi_\lambda, A_\lambda)$ be a classical solution of (1.1) which is also a nontrivial least-energy minimizer of $\mathcal{E}_\lambda(\psi, A)$. For the sake of convenience, we shall sometimes write $(\psi_\lambda, A_\lambda) = (w_\lambda e^{ig_\lambda}, A_\lambda)$ in what follows. We remark that $g_\lambda(a_j)$ makes no sense when a_j is a vortex of $(\psi_\lambda, A_\lambda)$. Thus, $(w_\lambda(a_j)e^{ig_\lambda(a_j)}, A_\lambda(a_j))$ should be understood as the limit.

Lemma 2.1. *We have $0 \leq w_\lambda \leq 1$ for all $\lambda > 1$.*

Proof. The main idea of this proof comes from [20]. Under the notation $(\psi_\lambda, A_\lambda) = (w_\lambda e^{ig_\lambda}, A_\lambda)$, the magnetic Ginzburg-Landau functional can be rewritten as follows:

$$\mathcal{E}_\lambda(\psi_\lambda, A_\lambda) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla w_\lambda|^2 + w_\lambda^2|\nabla g_\lambda - A_\lambda|^2 + |\nabla \times A_\lambda|^2 + \frac{\lambda}{4}(w_\lambda^2 - 1)^2. \quad (2.1)$$

Let $\mathcal{O}_\lambda$ be the singular set of $(\psi_\lambda, A_\lambda)$, that is, $\mathcal{O}_\lambda$ is the set of vortices of the configuration $(\psi_\lambda, A_\lambda)$. Then, by the first equation in (1.1), we can write an equation of w_λ as follows:

$$\begin{cases} -\Delta w_\lambda + \frac{\lambda}{2}(w_\lambda + 1)w_\lambda(w_\lambda - 1) = -|\nabla g_\lambda - A_\lambda|^2 w_\lambda, & \text{in } \mathbb{R}^2\backslash\mathcal{O}_\lambda, \\ w_\lambda = 0, \quad \text{on } \partial\mathcal{O}_\lambda. \end{cases} \quad (2.2)$$

Since $w_\lambda \to 1$ as $|x| \to +\infty$ (cf. [11]), $\mathcal{O}_\lambda \subset B_{R_0}(0)$ with some $R_0 > 0$. Let $R > 2R_0$ and $\tau_R(s) : [0, +\infty) \to [0, 1]$ be a smooth cut-off function such that $\tau_R(s) = 1$ for $s \leq R$ and $\tau(s) = 0$ for $s \geq 2R$. Clearly, by (2.1), $w_\lambda - 1 \in H^1(\mathbb{R}^2)$ and thus, $(w_\lambda - 1)^+\tau_R \in H^1(\mathbb{R}^2)$ for all $R > 2R_0$. Since $w_\lambda \geq 0$, $\frac{\partial w_\lambda}{\partial \vec{n}} \leq 0$ on $\partial\mathcal{O}_\lambda$, where $\vec{n}$ is the unit out normal on $\partial\mathcal{O}_\lambda$. Now, multiplying (2.2) with $(w_\lambda - 1)^+\tau_R$ on both sides and integrating by parts yield that

$$\int_{B_R(0)} |\nabla(w_\lambda - 1)^+|^2 \lesssim \frac{1}{R}\left(\int_{B_{2R}(0)} |\nabla w_\lambda|^2\right)^{\frac{1}{2}}\left(\int_{B_{2R}(0)} |(w_\lambda - 1)^+|^2\right)^{\frac{1}{2}}. \quad (2.3)$$

Here, we use the fact that $(w_\lambda - 1)^+\tau_R \equiv 0$ for x sufficiently close to the closure of $\mathcal{O}_\lambda$. Since $w_\lambda - 1 \in H^1(\mathbb{R}^2)$, by letting $R \to +\infty$ in (2.3), we know that $(w_\lambda - 1)^+ = 0$ in $\mathbb{R}^2$ and thus, $w_\lambda \leq 1$ in $\mathbb{R}^2$. □

As we stated in the introduction, we shall use the perturbation argument to prove Theorem 1.1. Thus, we need the following information on the limit case $\lambda = 1$.

Proposition 2.1. *Let (φ, B) be a solution of (1.1) with $\lambda = 1$. Then its degree is N is equivalent to that it has N vortices.*

Remark 2.1. (1) *Proposition 2.1 was pointed out by Taubes in [19], without a proof. For the convenience of the readers, we would like sketch its proof in the appendix.*

(2) *Proposition 2.1 tells us that vortices and anti-vortices can not co-exist for (1.1) with $\lambda = 1$. In particular, up to gauge translations, the radial solution is the unique solution of (1.1) in the degree 1 class for $\lambda = 1$.*

In what follows, we shall denote the radial solution of (1.1) in the degree 1 class for $\lambda = 1$ by

$$(\varphi, B) = (f(r)e^{i\theta}, a(r)\nabla\theta).$$

Since it is well known that $\mathcal{E}_1(\psi, A)$ is gauge invariance (cf. [9]), $(\varphi e^{i\chi}, B + \nabla\chi)$ for all $\chi \in C^2(\mathbb{R}^2)$ are also solutions of (1.1) with $\lambda = 1$ and share the same energy value $\mathcal{E}_1(\varphi, B)$ in $H^1_{loc}(\mathbb{R}^2; \mathbb{C}) \times H^1_{loc}(\mathbb{R}^2; \mathbb{R}^2)$. We also remark that the properties of (φ, B) is well known, see, for example, [4].

Lemma 2.2. *For λ sufficiently close to 1, the degree of $(\psi_\lambda, A_\lambda)$ is equal to one. Moreover,*

$$\mathcal{E}_\lambda(\psi_\lambda, A_\lambda) \leq \frac{\lambda - 1}{8} \int_{\mathbb{R}^2} (|f|^2 - 1)^2 + \pi. \tag{2.4}$$

Proof. Since $(\psi_\lambda, A_\lambda)$ be a classical solution of (1.1) which is also a nontrivial least-energy minimizer of $\mathcal{E}_\lambda(\psi, A)$,

$$\int_{\mathbb{R}^2} \nabla \times A_\lambda \geq 2\pi.$$

Thus, $(\psi_\lambda, A_\lambda)$ has at least one vortex. Since (1.1) is invariant under translations, we may always assume that 0 is a vortex of $(\psi_\lambda, A_\lambda)$. Suppose that the degree of $(\psi_\lambda, A_\lambda)$ is equal to $k_\lambda + 1$ for some $k_\lambda \in \mathbb{N}$. Now, we rewrite $(\psi_\lambda, A_\lambda) = (w_\lambda e^{ig_\lambda}, A_\lambda)$. Since $(\psi_\lambda, A_\lambda)$ is a classical solution of (1.1) with $w_\lambda \to 1$ as $|x| \to +\infty$, for sufficiently large $R > 0$, $g_\lambda = -i\ln(\frac{\psi_\lambda}{w_\lambda})$ is C^2 in $\mathbb{R}^2 \backslash B_R(0)$. Thus, $g_\lambda - (k_\lambda + 1)\theta$ is also C^2 in $\mathbb{R}^2 \backslash B_R(0)$ and satisfies

$$g_\lambda(R, \theta + 2\pi) - (k_\lambda + 1)(\theta + 2\pi) = g_\lambda(R, \theta) - (k_\lambda + 1)\theta$$

for all θ. It follows that $g_\lambda - (k_\lambda + 1)\theta$ is a single-valued function on $\partial B_R(0)$ and so that, we can harmonically extend $g_\lambda - (k_\lambda + 1)\theta$ from $\partial B_R(0)$ into $B_R(0)$. We denote this extension by υ_λ, that is, υ_λ satisfies

$$\begin{cases} \Delta\upsilon_\lambda = 0, \quad \text{in } B_R(0), \\ \upsilon_\lambda = g_\lambda(R, \theta) - (k_\lambda + 1)\theta, \quad \text{on } \partial B_R(0). \end{cases}$$

By classical regularity theorems, $\upsilon_\lambda \in C^2(\overline{B_R})$. We extend υ_λ to $\mathbb{R}^2$ such that $\upsilon_\lambda = g_\lambda - (k_\lambda + 1)\theta$ in $\mathbb{R}^2 \backslash B_R(0)$. Then, $\upsilon_\lambda \in C^2(\mathbb{R}^2)$. Now, by gauge invariance,

$$(w_\lambda e^{i(g_\lambda - \upsilon_\lambda)}, A_\lambda - \nabla\upsilon_\lambda)$$

is also a nontrivial least-energy minimizer of $\mathcal{E}_\lambda(\psi, A)$. Let α_ρ be a smooth cut-off function such that $\alpha_\rho = 1$ for $|x| \leq \rho$ and $\alpha_\rho = 0$ for $|x| \geq 2\rho$. Then

$$\begin{aligned} (\phi_\rho, B_\rho) \quad = \quad & \Big(\alpha_\rho(f_{k_\lambda+1}e^{i(k_\lambda+1)\theta} - w_\lambda e^{i(g_\lambda - \upsilon_\lambda)}), \\ & \alpha_\rho(a_{k_\lambda+1}(k_\lambda + 1)\nabla\theta - (A_\lambda - \nabla\upsilon_\lambda))\Big) \end{aligned}$$

belong to $C_0^2(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)$, where $(f_{k_\lambda+1}e^{i(k_\lambda+1)\theta}, a_{k_\lambda+1}(k_\lambda+1)\nabla\theta)$ is the radial solution of (1.1) with degree $k_\lambda+1$ for $\lambda=1$. It follows from the density of C_0^∞ in C_0^2 and the definition (1.2) that

$$\begin{aligned}\mathcal{E}_\lambda(\psi_\lambda, A_\lambda) &= \mathcal{E}_\lambda(w_\lambda e^{i(g_\lambda-\upsilon_\lambda)}, A_\lambda-\nabla\upsilon_\lambda)\\ &\leq \mathcal{E}_\lambda(w_\lambda e^{i(g_\lambda-\upsilon_\lambda)}+\phi_\rho, A_\lambda-\nabla\upsilon_\lambda+B_\rho).\end{aligned}\tag{2.5}$$

By the constructions, for $\rho>\max\{1,R\}$,

$$\begin{aligned}&(w_\lambda e^{i(g_\lambda-\upsilon_\lambda)}+\phi_\rho, A_\lambda-\nabla\upsilon_\lambda+B_\rho)\\ =\ &(f_{k_\lambda+1}e^{i(k_\lambda+1)\theta}, a_{k_\lambda+1}\nabla(k_\lambda+1)\theta)\\ &+(1-\alpha_\rho)\Big((w_\lambda e^{i(k_\lambda+1)\theta}-f_{k_\lambda+1}e^{i(k_\lambda+1)\theta}),\\ &(A_\lambda-\nabla g_\lambda+(1-a_{k_\lambda+1})(k_\lambda+1)\nabla\theta)\Big)\\ =\ &\Big(((1-\alpha_\rho)w_\lambda+\alpha_\rho f_{k_\lambda+1})e^{i(k_\lambda+1)\theta},\\ &(1-\alpha_\rho)(A_\lambda-\nabla g_\lambda)+(1-\alpha_\rho+\alpha_\rho a_{k_\lambda+1})(k_\lambda+1)\nabla\theta\Big).\end{aligned}$$

Note that under the notation $(\psi_\lambda, A_\lambda)=(w_\lambda e^{ig_\lambda}, A_\lambda)$ and the choice of R, g_λ satisfies

$$-w_\lambda\Delta g_\lambda = 2(\nabla g_\lambda - A_\lambda)\nabla w_\lambda \quad \text{in } \mathbb{R}^2\backslash B_R(0).$$

Thus, by the results in [11], that is,

$$(|\psi_\lambda|, |\nabla_{A_\lambda}\psi_\lambda|, |\nabla\times A_\lambda|)\to(1,0,0)\quad \text{as } |x|\to+\infty,$$

exponentially (see also [7]), we have

$$\begin{aligned}&(w_\lambda-f_{k_\lambda+1})e^{i(k_\lambda+1)\theta}\in H^1(\mathbb{R}^2\backslash B_R(0),\mathbb{C}),\\ &(A_\lambda-\nabla g_\lambda+(1-a_{k_\lambda+1})(k_\lambda+1)\nabla\theta)\in L^2(\mathbb{R}^2\backslash B_R(0),\mathbb{R}^2),\\ &\nabla\times(A_\lambda-\nabla g_\lambda+(1-a_{k_\lambda+1})(k_\lambda+1)\nabla\theta)\in L^2(\mathbb{R}^2\backslash B_R(0)).\end{aligned}$$

Here, we also use the well-known facts of $(f_{k_\lambda+1}e^{i(k_\lambda+1)\theta}, a_{k_\lambda+1}(k_\lambda+1)\nabla\theta)$, that is, $1-f_{k_\lambda+1}, 1-a_{k_\lambda+1}\to 0$ exponentially as $|x|\to+\infty$ and $\frac{a'_{k_\lambda+1}}{r}\in L^2(rdr)=L^2(\mathbb{R}^2)$ (cf. [4,9]). Since under the notation $(\psi_\lambda, A_\lambda)=(w_\lambda e^{ig_\lambda}, A_\lambda)$, the magnetic Ginzburg-Landau functional $\mathcal{E}_\lambda(\psi_\lambda, A_\lambda)$ can be rewritten as

$$\mathcal{E}_\lambda(\psi_\lambda, A_\lambda)=\frac{1}{2}\int_{\mathbb{R}^2}|\nabla w_\lambda|^2+w_\lambda^2|\nabla g_\lambda-A_\lambda|^2+|\nabla\times A_\lambda|^2+\frac{\lambda}{4}(w_\lambda^2-1)^2.\tag{2.6}$$

Thus, by letting $\rho\to+\infty$ in (2.5), we have

$$\begin{aligned}\mathcal{E}_\lambda(\psi_\lambda, A_\lambda) &\leq \lim_{\rho\to+\infty}\mathcal{E}_\lambda(f_{k_\lambda+1}e^{i(k_\lambda+1)\theta}+\widetilde{\phi}_\rho, a_{k_\lambda+1}(k_\lambda+1)\nabla\theta+\widetilde{D}_\rho)\\ &= \mathcal{E}_\lambda(f_{k_\lambda+1}e^{i(k_\lambda+1)\theta}, a_{k_\lambda+1}(k_\lambda+1)\nabla\theta)\\ &= \frac{\lambda-1}{8}\int_{\mathbb{R}^2}(|f_{k_\lambda+1}|^2-1)^2+(k_\lambda+1)\pi,\end{aligned}\tag{2.7}$$

where $(\widetilde{\phi}_\rho, \widetilde{D}_\rho)=((1-\alpha_\rho)(w_\lambda-f_{k_\lambda+1})e^{i(k_\lambda+1)\theta}, (1-\alpha_\rho)(A_\lambda-\nabla g_\lambda+(1-a_{k_\lambda+1})(k_\lambda+1)\nabla\theta))$. Thus, for $k_\lambda=0$, we have $\mathcal{E}_\lambda(\psi_\lambda, A_\lambda)\leq\frac{3}{2}\pi$ for λ sufficiently close to 1. By the computations in (1.3), we know that the nontrivial least-energy minimizer of $\mathcal{E}_\lambda(\psi, A)$ must have $k_\lambda=0$ for λ sufficiently close to 1. That is, it has degree one for λ sufficiently close to 1. The estimate of (2.7) also give us an upper-bound of the energy value of the nontrivial least-energy minimizer $(\psi_\lambda, A_\lambda)$:

$$\mathcal{E}_\lambda(\psi_\lambda, A_\lambda) \le \mathcal{E}_\lambda(\varphi, B) = \frac{\lambda - 1}{8}\int_{\mathbb{R}^2}(|f|^2-1)^2 + \pi.$$

It completes the proof. □

As in [14], we also need to recall a standard terminology in functional analysis for gauge theory.

Definition 2.1. *A sequence of configurations connection-section* (ψ_n, A_n) *is said that it converges in some function space* $\mathcal{F}$ *to a limiting configuration* (ψ_0, A_0) *if there exists a change of gauge* $(\psi_n e^{i\chi_n}, A_n + \nabla\chi_n)$ *such that this sequence of pairs form-function converges to* (ψ_0, A_0) *in the function space* $\mathcal{F}$*.*

Lemma 2.3. *Up to translations,* $(\psi_\lambda, A_\lambda) \to (\varphi, B)$ *strongly in* $C^{1,\alpha}_{loc}(\mathbb{R}^2;\mathbb{C}) \times C^{1,\alpha}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ *as* $\lambda \to 1$ *in the sense of Definition 2.1 for some* $\alpha \in (0,1)$*.*

Proof. The main ideas of this proof come from [5] (see also [14]). Let us first consider the following equation:

$$\begin{cases} -\Delta\chi_{\lambda,R} = divA_\lambda, & \text{in } B_R(0); \\ \dfrac{\partial\chi_{\lambda,R}}{\partial\overrightarrow{n}} = -A_\lambda\cdot x, & \text{on } \partial B_R(0), \end{cases} \tag{2.8}$$

where $\overrightarrow{n}$ is the unit out normal of $B_R(0)$. Since A_λ is of class C^2, the above equation is unique solvable up to constants. Let $\widehat{\chi}_{\lambda,R}$ be a solution of (2.8), we choose $\chi_{\lambda,R} = \widehat{\chi}_{\lambda,R} - \frac{\int_{B_R}\widehat{\chi}_{\lambda,R}}{|B_R|}$. Then

$$\widetilde{A}_{\lambda,R} = A_\lambda + \nabla\chi_{\lambda,R}$$

satisfies $div\widetilde{A}_{\lambda,R} = 0$ in $B_R(0)$ and $\widetilde{A}_{\lambda,R}\cdot x = 0$ on $\partial B_R(0)$. In this situation, $\widetilde{A}_{\lambda,R} = (-\partial_2\xi_{\lambda,R}, \partial_1\xi_{\lambda,R})$ in $B_R(0)$, where up to constants, $\xi_{\lambda,R} \in C^2_0(B_R(0))$ and satisfies

$$\begin{cases} \Delta\xi_{\lambda,R} = \nabla\times A_\lambda, & \text{in } B_R(0); \\ \xi_{\lambda,R} = 0, & \text{on } \partial B_R(0). \end{cases} \tag{2.9}$$

Classical elliptic estimates then yield that

$$\int_{B_R(0)}(|\nabla\widetilde{A}_{\lambda,R}|^2 + |\widetilde{A}_{\lambda,R}|^2) = \int_{B_R(0)}(|\nabla^2\xi_{\lambda,R}|^2 + |\nabla\xi_{\lambda,R}|^2) \le C_R\int_{\mathbb{R}^2}|\nabla\times A_\lambda|^2,$$

which implies that $\{\widetilde{A}_{\lambda,R}\}$ is bounded in $H^1(B_R(0);\mathbb{R}^2)$ for λ sufficiently close to 1. Here, $C_R > 0$ is a constant only dependent on R. Let $\widetilde{\psi}_{\lambda,R} = \psi_\lambda e^{i\chi_{\lambda,R}}$. Then

$$\nabla_{\widetilde{A}_{\lambda,R}}\widetilde{\psi}_{\lambda,R} = (\nabla_{A_\lambda}\psi_\lambda)e^{i\chi_{\lambda,R}} \quad \text{and} \quad |\widetilde{\psi}_{\lambda,R}|^2 = |\psi_\lambda|^2 \quad \text{in } B_R(0). \tag{2.10}$$

Recall that $\{\nabla_{A_\lambda}\psi_\lambda\}$ and $\{|\psi_\lambda|^2 - 1\}$ are, respectively, bounded in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ and $L^2(\mathbb{R}^2)$ for λ sufficiently close to 1. Since

$$\nabla\widetilde{\psi}_{\lambda,R} = \nabla_{\widetilde{A}_{\lambda,R}}\widetilde{\psi}_{\lambda,R} + i\widetilde{A}_{\lambda,R}\widetilde{\psi}_{\lambda,R},$$

by (2.10) and Lemma 2.1, $\{\widetilde{\psi}_{\lambda,R}\}$ is also bounded in $H^1(B_R(0);\mathbb{C})$ for λ sufficiently close to 1. Without loss of generality, we may assume that $(\widetilde{\psi}_{\lambda,R}, \widetilde{A}_{\lambda,R}) \rightharpoonup$

$(\psi_{0,R}, A_{0,R})$ weakly in $H^1(B_R(0);\mathbb{C}) \times H^1(B_R(0);\mathbb{R}^2)$ as $\lambda \to 1$ up to a subsequence. By (2.8), for $R' > R$, the difference between $(\psi_\lambda e^{i\chi_{\lambda,R}}, A_\lambda + \nabla\chi_{\lambda,R})$ and $(\psi_\lambda e^{i\chi_{\lambda,R'}}, A_\lambda + \nabla\chi_{\lambda,R'})$ in $B_R(0)$ is a gauge $\chi_{\lambda,R'} - \chi_{\lambda,R}$ which is harmonic in $B_R(0)$. Moreover, by the boundedness of $\{A_\lambda + \nabla\chi_{\lambda,R}\}$ and $\{A_\lambda + \nabla\chi_{\lambda,R'}\}$ in $H^1(B_R(0))$ for λ sufficiently close to 1 and the Poincaré inequality, $\{\chi_{\lambda,R'} - \chi_{\lambda,R}\}$ is bounded in $H^2(B_R(0))$ for λ sufficiently close to 1 and so that $\chi_{\lambda,R'} - \chi_{\lambda,R} \rightharpoonup \widetilde{\chi}^{R,R'}$ weakly in $H^2(B_R(0))$ as $\lambda \to 1$ up to a subsequence. It follows that $(\psi_{0,R}, A_{0,R}) = (\psi_{0,R'} e^{-i\widetilde{\chi}^{R,R'}}, A_{0,R'} - \nabla\widetilde{\chi}^{R,R'})$ in $B_R(0)$ for all $R' > R$. Let us define the equivalent classes as follows:

$$[(\psi_{0,R}, A_{0,R})] = \{(\psi_{0,R} e^{i\chi}, A_{0,R} + \nabla\chi) \mid \chi \in H^2(B_R(0))\}.$$

Then in every class, we can re-choose $(\psi_{0,R}, A_{0,R})$ if necessary such that if $R' > R$ then $(\psi_{0,R'}, A_{0,R'}) = (\psi_{0,R}, A_{0,R})$ in $B_R(0)$. Hence, we can use these re-chosen $\{(\psi_{0,R}, A_{0,R})\}$ to define a configuration (ψ_0, A_0) in the whole $\mathbb{R}^2$ by setting

$$(\psi_0, A_0) \mid_{B_R(0)} = (\psi_{0,R}, A_{0,R})$$

in every $B_R(0)$. It follows that $(\psi_\lambda, A_\lambda) \rightharpoonup (\psi_0, A_0)$ weakly in $H^1_{loc}(\mathbb{R}^2;\mathbb{C}) \times H^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ as $\lambda \to 1$ up to a subsequence in the sense of Definition 2.1. That is, for every given compact set $\mathcal{K}$, we can choose $R > 0$ such that $\mathcal{K} \subset B_R(0)$ and $(\widetilde{\psi}_{\lambda,R} e^{i\chi_R}, \widetilde{A}_{\lambda,R} + \nabla\chi_R) \rightharpoonup (\psi_{0,R}, A_{0,R})$ weakly in $H^1(B_R(0);\mathbb{C}) \times H^1(B_R(0);\mathbb{R}^2)$ as $\lambda \to 1$ up to a subsequence, where $(\psi_{0,R}, A_{0,R}) = (\psi_0, A_0) \mid_{B_R(0)}$ and $\chi_R \in H^2_{loc}(\mathbb{R}^2)$. By our constructions, $div\widetilde{A}_{\lambda,3R} = 0$ in $B_{3R}(0)$ for every $R > 0$. Thus, by (2.6), for every $R > 0$,

$$-\Delta\widetilde{A}_{\lambda,3R} = w_\lambda^2(\nabla g_\lambda - A_\lambda), \quad \text{in } B_{3R}(0) \text{ in the weak sense.} \tag{2.11}$$

Here, we notice that w_λ and $\nabla g_\lambda - A_\lambda$ are independent of gauges. By Lemma 2.1, $0 \leq w_\lambda \leq 1$ in the whole $\mathbb{R}^2$. Thus, the right hand side of (2.11) belongs to $L^2(\mathbb{R}^2;\mathbb{R}^2)$ by Lemma 2.2. Applying the interior L^p-estimates and the Sobolev embedding theorem to (2.11) implies that

$$\begin{aligned} \|\widetilde{A}_{\lambda,3R}\|_{C^{1,\alpha}(B_{2R})} &\leq C_R\|\widetilde{A}_{\lambda,3R}\|_{H^2(B_{2R})} \\ &\leq C'_R(\|\nabla g_\lambda - A_\lambda\|_{L^2(B_{3R})} + \|\widetilde{A}_{\lambda,3R}\|_{L^2(B_{3R})}), \end{aligned} \tag{2.12}$$

which implies that $\{\widetilde{A}_{\lambda,3R}\}$ is uniformly bounded in $C^{1,\alpha}(B_{2R})$ for every $R > 0$ and some $\alpha \in (0,1)$. Here, $C_R, C'_R > 0$ are constants only dependent on R. Let us come back to (1.1) and write

$$\widetilde{\psi}_{\lambda,3R} = \widetilde{\psi}_{\lambda,3R,1} + i\widetilde{\psi}_{\lambda,3R,2}.$$

Since $div\widetilde{A}_{\lambda,3R} = 0$ for every $R > 0$, the first equation of (1.1) in B_{3R} can be rewritten as

$$\begin{cases} -\Delta\widetilde{\psi}_{\lambda,3R,1} + |\widetilde{A}_{\lambda,3R}|^2\widetilde{\psi}_{\lambda,3R,1} = \dfrac{\lambda}{2}(1 - |\widetilde{\psi}_{\lambda,3R}|^2)\widetilde{\psi}_{\lambda,3R,1} + 2\widetilde{A}_{\lambda,3R,2}\nabla\widetilde{\psi}_{\lambda,3R,2}, \\ -\Delta\widetilde{\psi}_{\lambda,3R,2} + |\widetilde{A}_{\lambda,3R}|^2\widetilde{\psi}_{\lambda,3R,2} = \dfrac{\lambda}{2}(1 - |\widetilde{\psi}_{\lambda,3R}|^2)\widetilde{\psi}_{\lambda,3R,2} - 2\widetilde{A}_{\lambda,3R,1}\nabla\widetilde{\psi}_{\lambda,3R,1}, \end{cases}$$

where $\widetilde{A}_{\lambda,3R} = (\widetilde{A}_{\lambda,3R,1}, \widetilde{A}_{\lambda,3R,2})$. As (2.12), applying the interior L^p-estimates and the Sobolev embedding theorem implies that

$$\begin{aligned}\|\widetilde{\psi}_{\lambda,3R,i}\|_{C^{1,\alpha}(B_R)} &\leq C_R\|\widetilde{\psi}_{\lambda,3R,i}\|_{H^2(B_R)}\\ &\leq C'_R((\|\widetilde{A}_{\lambda,3R}\|^2_{L^\infty(B_{2R})}+1)\|\widetilde{\psi}_{\lambda,3R,i}\|_{L^2(B_{2R})}\\ &\quad+\|\widetilde{A}_{\lambda,3R}\|_{L^\infty(B_{2R})}\|\nabla\widetilde{\psi}_{\lambda,3R,j}\|_{L^2(B_{2R})}).\end{aligned}$$

Thus, $\{\widetilde{\psi}_{\lambda,3R,i}\}$ is uniformly bounded in $C^{1,\alpha}(B_R)$, which, without loss of generality, implies that $\widetilde{\psi}_{\lambda,3R,i} \to \psi_{0,i}$ strongly in $C^{1,\alpha}(B_R)$ as $\lambda \to 1$ in the sense of Definition 2.1 (Here, we shall adjust α to be slightly small in the strong convergence). Since $R > 0$ is arbitrary, $(\psi_\lambda, A_\lambda) \to (\psi_0, A_0)$ strongly in $C^{1,\alpha}_{loc}(\mathbb{R}^2;\mathbb{C})\times C^{1,\alpha}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ as $\lambda \to 1$ up to subsequence in the sense of Definition 2.1. It follows from the gauge invariance of $\mathcal{E}_\lambda(\psi, A)$ that for every $R > 0$,

$$\begin{aligned}\lim_{\lambda\to1}\mathcal{E}_\lambda(\psi_\lambda, A_\lambda) &\geq \lim_{\lambda\to1}(\frac{1}{2}\int_{B_R(0)}|\nabla_{A_\lambda}\psi_\lambda|^2+|\nabla\times A_\lambda|^2+\frac{\lambda}{4}(|\psi_\lambda|^2-1)^2)\\ &= \frac{1}{2}\int_{B_R(0)}|\nabla_{A_0}\psi_0|^2+|\nabla\times A_0|^2+\frac{1}{4}(|\psi_0|^2-1)^2. \qquad (2.13)\end{aligned}$$

Letting $R \to +\infty$ in the above inequality, we have

$$\lim_{\lambda\to1}\mathcal{E}_\lambda(\psi_\lambda, A_\lambda) \geq \mathcal{E}_1(\psi_0, A_0). \qquad (2.14)$$

Since $(\psi_\lambda, A_\lambda)$ is a classical solution of (1.1) for λ sufficiently close to 1, by the gauge invariance, (ψ_0, A_0) must be a weak solution of (1.1) for $\lambda = 1$. Thanks to the results in [11] and (2.14), $(|\psi_0|, |\nabla_{A_0}\psi_0|, |\nabla \times A_0|) \to (1, 0, 0)$ as $|x| \to +\infty$. Thus, the degree of (ψ_0, A_0) is well defined and less than or equal to 1 by Lemma 2.2. Since we assume that 0 is always a vortex of $(\psi_\lambda, A_\lambda)$ for λ sufficiently close to 1, gauge translations will not change the vortices and $(\psi_\lambda, A_\lambda) \to (\psi_0, A_0)$ strongly in $C^{1,\alpha}_{loc}(\mathbb{R}^2;\mathbb{C})\times C^{1,\alpha}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ as $\lambda \to 1$ up to subsequence in the sense of Definition 2.1, (ψ_0, A_0) must be the degree 1 solution of (1.1) for $\lambda = 1$. By the uniqueness of $|\varphi|$ (cf. [19]), $(\psi_0, A_0) = (\varphi, B)$ up to gauge translations. Since the above convergence holds for every subsequence, by Proposition 2.1, we must have $(\psi_\lambda, A_\lambda) \to (\psi_0, A_0)$ strongly in $C^1_{loc}(\mathbb{R}^2;\mathbb{C}) \times C^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ as $\lambda \to 1$ in the sense of Definition 2.1. □

Using Lemmas 2.1 and 2.3 once more, we can obtain the following.

Lemma 2.4. *Up to translations,* 0 *is the only vortex of* $(\psi_\lambda, A_\lambda)$ *for* λ *sufficiently close to* 1*. Moreover, for every* $R > 0$*, there exists* $\delta_R > 0$ *independent of* λ *such that* $w_\lambda \geq \delta_R$ *for all* $|x| \geq R$ *and* λ *sufficiently close to* 1*.*

Proof. We first prove that for λ sufficiently close to 1, 0 is the only vortex of $(\psi_\lambda, A_\lambda)$ up to translations. Suppose the contrary that besides 0, $(\psi_\lambda, A_\lambda)$ still has another vortex $a_{1,\lambda}$ up to translations. Now, iny $B_R(a_{1,\lambda})$, we could run the regularity and compactness arguments as used for Lemma 2.3 to $(\psi_\lambda, A_\lambda)$. Then, $(\psi^1_\lambda, A^1_\lambda) \to (\varphi, B)$ strongly in $C^{1,\alpha}_{loc}(\mathbb{R}^2;\mathbb{C}) \times C^{1,\alpha}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ as $\lambda \to 1$ for some $\alpha \in (0, 1)$ in the sense of Definition 2.1, where $(\psi^1_\lambda, A^1_\lambda) = (\psi_\lambda(x + a_{1,\lambda}), A_\lambda(x + a_{1,\lambda}))$ for every j. Since by Lemma 2.3, $(\psi_\lambda, A_\lambda) \to (\varphi, B)$ strongly in $C^1_{loc}(\mathbb{R}^2;\mathbb{C}) \times C^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ as $\lambda \to 1$ in the sense of Definition 2.1, we have $|a_{1,\lambda}| \to +\infty$ as $\lambda \to 1$. Now, using similar arguments as used for (2.13),

$$\begin{aligned}\lim_{\lambda\to1}\mathcal{E}_\lambda(\psi_\lambda,A_\lambda) &\geq \lim_{\lambda\to1}\frac{1}{2}\int_{B_R(0)}|\nabla_{A_\lambda}\psi_\lambda|^2+|\nabla\times A_\lambda|^2+\frac{1}{4}(|\psi_\lambda|^2-1)^2\\ &\quad+\lim_{\lambda\to1}\frac{1}{2}\int_{B_R(a_{1,\lambda})}|\nabla_{A_\lambda}\psi_\lambda|^2+|\nabla\times A_\lambda|^2+\frac{1}{4}(|\psi_\lambda|^2-1)^2\\ &\geq \int_{B_R(0)}|\nabla_B\varphi|^2+|\nabla\times B|^2+\frac{1}{4}(|\varphi|^2-1)^2\\ &\geq \frac{3\pi}{2} \qquad (2.15)\end{aligned}$$

by choosing $R>0$ sufficiently large. It contradicts Lemma 2.2 for λ sufficiently close to 1, which implies that up to translations, 0 is the only vortex of (ψ_λ,A_λ) for λ sufficiently close to 1. Let us now prove the second part of this lemma. Suppose that there exists $\{x_\lambda\}$ such that $|x_\lambda|\to+\infty$ and $|\psi_\lambda(x_\lambda)|\to0$ as $\lambda\to1$. Then let us consider $(\widetilde{\psi}^0_\lambda,\widetilde{A}^0_\lambda)=(\psi_\lambda(x+x_\lambda),A_\lambda(x+x_\lambda))$. By similar arguments as used in the proof of Lemma 2.3, we can show that $(\widetilde{\psi}^0_\lambda,\widetilde{A}^0_\lambda)\to(\varphi^0,B^0)$ strongly in $C^{1,\alpha}_{loc}(\mathbb{R}^2;\mathbb{C})\times C^{1,\alpha}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ as $\lambda\to1$ for some $\alpha\in(0,1)$ in the sense of Definition 2.1 and (φ^0,B^0) is a solution of (1.1) with $\lambda=1$. As for (2.14), we can show that

$$\lim_{\lambda\to1}\mathcal{E}_\lambda(\psi^0_\lambda,A^0_\lambda)\geq\mathcal{E}_1(\varphi^0,B^0).$$

Since $\psi^0_\lambda(0)\to0$ as $\lambda\to1$, by Proposition 2.1, $\varphi^0(0)=0$ and thus by the results in [11], either $\varphi^0\equiv0$ or the degree of φ^0 is at least 1. It follows from (2.4) that $|\varphi^0|-1\in H^1(\mathbb{R}^2)$, which implies that the degree of φ^0 is at least 1 and $\mathcal{E}_1(\varphi^0,B^0)\geq\pi$. Now, since $|x_\lambda|\to+\infty$ as $\lambda\to1$ and 0 is always a vortex of ψ_λ, by using similar calculations in (2.15) in $B_R(0)$ and $B_R(x_\lambda)$ for a sufficiently large R, we will arrive at

$$\mathcal{E}_\lambda(\psi_\lambda,A_\lambda)\geq\frac{3\pi}{2}$$

for λ sufficiently close to 1, which contradicts (2.4). Thus, for every $R>0$, there exists $\delta_R>0$ independent of λ such that $|\psi_\lambda|\geq\delta_R$ for all $|x|\geq R$ and λ sufficiently close to 1. □

To continue our analysis, we need to drive some global compactness results of $\{(\psi_\lambda,A_\lambda)\}$. Let us begin with

Lemma 2.5. *Under the notation $(\psi_\lambda,A_\lambda)=(w_\lambda e^{ig_\lambda},A_\lambda)$, we have $w_\lambda-1\to f-1$ strongly in $H^1(\mathbb{R}^2)$ and $w_\lambda(\nabla g_\lambda-A_\lambda)\to f(\nabla\theta-B)$ strongly in $L^2(\mathbb{R}^2;\mathbb{R}^2)$ as $\lambda\to1$.*

Proof. Since

$$\int_{\mathbb{R}^2}|\nabla_{A_\lambda}\psi_\lambda|^2\geq\int_{B_R(0)}|\nabla_{A_\lambda}\psi_\lambda|^2 \qquad (2.16)$$

for all $R>0$, by (2.10) and Lemma 2.3, we can let $\lambda\to1$ first and $R\to+\infty$ next in (2.16). It follows that

$$\lim_{\lambda\to1}\int_{\mathbb{R}^2}|\nabla_{A_\lambda}\psi_\lambda|^2\geq\int_{\mathbb{R}^2}|\nabla_B\varphi|^2.$$

Recall that by our choice of gauges in the proof of Lemma 2.3, $div\widetilde{A}_{\lambda,R}=0$ in $B_R(0)$ for every $R>0$. Since for the radial solution (φ,B), we also have $divB=0$ in $\mathbb{R}^2$, by Lemma 2.3, $(\widetilde{\psi}_{\lambda,R},\widetilde{A}_{\lambda,R})\to(\varphi,B)$ strongly in $C^1(B_R(0);\mathbb{C})\times C^1(B_R(0);\mathbb{R}^2)$ as $\lambda\to 1$ up to a gauge χ_R which is harmonic in $B_R(0)$. Now, using elliptic estimates to (2.9), we know that

$$\begin{aligned}\int_{B_R(0)}|\nabla B|^2 &= \liminf_{\lambda\to 1}\int_{B_R(0)}|\nabla\widetilde{A}_{\lambda,R}|^2\\ &= \liminf_{\lambda\to 1}\int_{B_R(0)}|\nabla^2\xi_{\lambda,R}|^2\\ &\le \liminf_{\lambda\to 1}\int_{\mathbb{R}^2}|\nabla\times A_\lambda|^2. \end{aligned}\tag{2.17}$$

Letting $R\to+\infty$ in (2.17) and noting that $divB=0$, we could use integrating by parts and the decaying property of B at infinity to show that

$$\int_{\mathbb{R}^2}|\nabla\times B|^2=\int_{\mathbb{R}^2}|\nabla B|^2\le\liminf_{\lambda\to 1}\int_{\mathbb{R}^2}|\nabla\times A_\lambda|^2.$$

Now, by the weakly lower semi-continuity of the energy functional $\mathcal{E}_\lambda(\psi_\lambda,A_\lambda)$ and (2.4), we actually have $\nabla_{A_\lambda}\psi_\lambda\to\nabla_B\varphi$ strongly in $L^2(\mathbb{R}^2;\mathbb{C}^2)$, $w_\lambda^2-1\to f^2-1$ strongly in $L^2(\mathbb{R}^2)$ and $\nabla\times A_\lambda\to\nabla\times B$ strongly in $L^2(\mathbb{R}^2;\mathbb{R}^4)$ as $\lambda\to 1$, which implies that $w_\lambda-1\to f-1$ strongly in $H^1(\mathbb{R}^2)$ and $w_\lambda(\nabla g_\lambda-A_\lambda)\to f(\nabla\theta-B)$ strongly in $L^2(\mathbb{R}^2;\mathbb{R}^2)$ as $\lambda\to 1$. □

By Lemma 2.4, we may assume that 0 is the only vortex of (ψ_λ,A_λ) for λ sufficiently close to 1. Then for $R>0$ sufficiently large, we can act the operator $(\partial_2,-\partial_1)$ on both sides of the second equation of (1.1) to write down the following equation (cf. [11, Proposition 6.1]):

$$|-\Delta q_\lambda+w_\lambda^2 q_\lambda|\le|h_\lambda|^2\quad\text{in }\mathbb{R}^2\backslash B_R(0),\tag{2.18}$$

where $h_\lambda=\nabla_{A_\lambda}\psi_\lambda$ and $q_\lambda=\nabla\times A_\lambda$. Moreover, by [11, Corollary 6.2], we also have the following equation:

$$|h_\lambda|\Delta|h_\lambda|\ge(-2|q_\lambda|+\frac{\lambda}{2}(1-w_\lambda^2))|h_\lambda|^2+w_\lambda^2|h_\lambda|^2\quad\text{in }\mathbb{R}^2\backslash B_R(0)\tag{2.19}$$

for $R>0$ sufficiently large.

Lemma 2.6. *Under the notation $(\psi_\lambda,A_\lambda)=(w_\lambda e^{ig_\lambda},A_\lambda)$, we have that $w_\lambda-1$, $|\nabla w_\lambda|$, $|\nabla g_\lambda-A_\lambda|$ and $|\nabla(\nabla g_\lambda-A_\lambda)|$ all exponentially decays to zero as $|x|\to+\infty$, uniformly for λ sufficiently close to 1.*

Proof. Let us consider (1.1) in $B_3(y)$ with $|y|>>1$. Then by similar choices of gauges in the proof of Lemma 2.3 and running the regularity arguments as that used in the proof of Lemma 2.3, we will obtain that

$$\int_{B_3(y)}(|\nabla\widetilde{A}^y_\lambda|^2+|\widetilde{A}^y_\lambda|^2)\lesssim\int_{\mathbb{R}^2}|\nabla\times A_\lambda|^2\lesssim 1,$$

$$\|\widetilde{A}^y_\lambda\|_{C^{1,\alpha}(B_2(y))}\lesssim\|\nabla g_\lambda-A_\lambda\|_{L^2(B_2(y))}+\int_{\mathbb{R}^2}|\nabla\times A_\lambda|^2\lesssim 1.$$

and

$$\begin{aligned}\|\widetilde{\psi}^y_\lambda\|_{C^{1,\alpha}(B_1(y))} &\lesssim (\|\widetilde{A}^y_\lambda\|^2_{L^\infty(B_2(y))}+1)\|\widetilde{\psi}^y_\lambda\|_{L^2(B_2(y))}\\ &\quad+\|\widetilde{A}^y_\lambda\|_{L^\infty(B_2(y))}\|\nabla\widetilde{\psi}^y_\lambda\|_{L^2(B_2(y))}\\ &\lesssim 1+\|\nabla_{\widetilde{A}^y_\lambda}\widetilde{\psi}^y_\lambda\|_{L^2(B_2(y))}\\ &\lesssim 1\end{aligned}$$

uniformly for λ sufficiently close to 1 and $|y| \geq 2R$ with $R > 0$ sufficiently large, where $(\widetilde{\psi}_\lambda^y, \widetilde{A}_\lambda^y)$ is the correspondingly modified $(\psi_\lambda, A_\lambda)$ in $B_3(y)$, as that of $(\widetilde{\psi}_{\lambda,R}, \widetilde{A}_{\lambda,R})$ in the proof of Lemma 2.3. Here, we also use the uniform bound of w_λ obtained in Lemma 2.1, the strong convergence obtained in Lemma 2.5 and the gauge invariance of h_λ. It follows from the gauge invariance of $\nabla g_\lambda - A_\lambda$ and Lemma 2.4 that

$$|h_\lambda| \quad \leq \quad |\nabla w_\lambda| + |\nabla \widetilde{g}_\lambda^y - \widetilde{A}_\lambda^y| \lesssim \frac{1}{\delta_R}|\nabla \widetilde{\psi}_\lambda^y| + |\widetilde{A}_\lambda^y| \lesssim \frac{1}{\delta_R} + 1$$

uniformly for λ sufficiently close to 1 and $|y| \geq 2R$ with $R > 0$ sufficiently large. Therefore, $\|h_\lambda\|_{L^\infty(\mathbb{R}^2 \backslash B_R(0))} \lesssim 1$ uniformly for λ sufficiently close to 1, where $R > 0$ is sufficiently large. Now, applying the classical regularity theorems to (2.18) and using the strong convergence obtained by Lemma 2.5 yield that $\|q_\lambda\|_{L^\infty(\mathbb{R}^2 \backslash B_R(0))} << 1$ by taking $R >> 1$ uniformly for λ sufficiently close to 1. By Lemma 2.4, we can rewrite the equation (2.2) in $\mathbb{R}^2 \backslash B_R(0)$ with $R > 0$ sufficiently large as follows:

$$-\Delta w_\lambda + \frac{\lambda}{2}(w_\lambda + 1)w_\lambda(w_\lambda - 1) = -|\nabla g_\lambda - A_\lambda|^2 w_\lambda, \quad \text{in } \mathbb{R}^2 \backslash B_R(0).$$

Then by $\|h_\lambda\|_{L^\infty(\mathbb{R}^2 \backslash B_R(0))} \lesssim 1$ uniformly for λ sufficiently close to 1, the strong convergence obtained by Lemma 2.5 and the classical regularity theorems, $\|1 - w_\lambda\|_{L^\infty(\mathbb{R}^2 \backslash B_R(0))} << 1$ by taking $R >> 1$ uniformly for λ sufficiently close to 1. It follows from Lemma 2.4 that we can rewrite (2.19) as follows:

$$|h_\lambda|\Delta|h_\lambda| \geq \delta_R |h_\lambda|^2 \quad \text{in } \mathbb{R}^2 \backslash B_R(0),$$

which, together with Lemma 2.3, [11, Proposition 7.2] and the classical elliptic estimates, implies the desired conclusions. That is, $w_\lambda - 1$, $|\nabla w_\lambda|$, $|\nabla g_\lambda - A_\lambda|$ and $|\nabla(\nabla g_\lambda - A_\lambda)|$ all exponentially decays to zero as $|x| \to +\infty$, uniformly for λ sufficiently close to 1. □

With the uniform estimates in Lemma 2.6, we can obtain the following global compactness result of $\{(\psi_\lambda, A_\lambda)\}$.

Lemma 2.7. *We have* $(\psi_\lambda - \varphi, A_\lambda - B) \to 0$ *strongly in* $H^1(\mathbb{R}^2; \mathbb{R}^2) \times H^1(\mathbb{R}^2; \mathbb{C})$ *as* $\lambda \to 1$.

Proof. Since by Lemma 2.4, we may assume that $(\psi_\lambda, A_\lambda)$ only has a single vortex at 0 for λ sufficiently close to 1, $\theta - g_\lambda \in C^2(\mathbb{R}^2 \backslash B_1(0))$ and

$$g_\lambda(R, \theta + 2\pi) - (\theta + 2\pi) = g_\lambda(R, \theta) - \theta \tag{2.20}$$

for sufficiently large $R > 0$. Thus, $\theta - g_\lambda$ is a single-valued function on $\partial B_R(0)$ and so we can harmonically extend $\theta - g_\lambda$ from $\partial B_R(0)$ into $B_R(0)$. We denote this extension by υ_λ, that is, υ_λ satisfies

$$\begin{cases} \Delta \upsilon_\lambda = 0, & \text{in } B_R(0), \\ \upsilon_\lambda = \theta - g_\lambda(R, \theta), & \text{on } \partial B_R(0). \end{cases}$$

By classical regularity theorems, $\upsilon_\lambda \in C^2(\overline{B_R})$. By Lemma 2.3, we may assume that $A_\lambda \to B$ strongly in $C^{1,\alpha}(B_{3R}(0); \mathbb{R}^2)$ as $\lambda \to 1$ for some $\alpha \in (0, 1)$. We remark that in the sense of Definition 2.1, the possible changes in this strong convergence is the gauges $\{\chi_{\lambda,R}\} \subset C^2(B_{3R}(0))$, which will not change the computation (2.20). Now, by the Arzela-Ascoli theorem, $A_\lambda \to B$ uniformly in $B_{2R} \backslash B_{R/2}$ as $\lambda \to 1$ for

R sufficiently large. Since $|B| \sim \frac{1}{R}$ in $B_{2R}\backslash B_{R/2}$ (cf. [4]), we know that $|A_\lambda| \sim \frac{1}{R}$ in $B_{2R}\backslash B_{R/2}$ for λ sufficiently close to 1. It follows from Lemma 2.6 that $|\nabla g_\lambda| \sim \frac{1}{R}$ in $B_{2R}\backslash B_{R/2}$ for λ sufficiently close to 1. Since we have assumed that $A_\lambda \to B$ strongly in $C^{1,\alpha}(B_{3R}(0);\mathbb{R}^2)$ as $\lambda \to 1$ for some $\alpha \in (0,1)$, by Lemma 2.5, $w_\lambda e^{ig_\lambda} - fe^{i\theta} \to 0$ strongly in $L^2(B_{3R}(0);\mathbb{C})$ and $w_\lambda - f \to 0$ strongly in $L^2(B_{3R}(0))$ as $\lambda \to 1$, which implies $g_\lambda - \theta \to 2\pi k(x)$ a.e. in $B_{3R}(0)$ for some $k(x) \in \mathbb{Z}$ as $\lambda \to 1$. Let $x_0 \in B_{3R}(0)$ such that $g_\lambda(x_0) - \theta(x_0) \to 2\pi k_0$ for some $k_0 \in \mathbb{Z}$. Then $|g_\lambda(x)| \lesssim |g_\lambda(x_0)| + |\nabla g_\lambda| R \lesssim 1$ for all $x \in B_{2R}\backslash B_{R/2}$ and λ sufficiently close to 1. It follows from the Arzela-Ascoli theorem that $g_\lambda - \theta \to 2\pi k(x)$ uniformly in $B_{2R}\backslash B_{R/2}$ for some continuous $k(x) \in \mathbb{Z}$ as $\lambda \to 1$. By the continuity of $k(x)$, we must have that $k(x) \equiv k$. Thus, $g_\lambda - \theta \to 2\pi k$ in $B_{2R}\backslash B_{R/2}$ uniformly as $\lambda \to 1$ for some $k \in \mathbb{Z}$. By translating $(\psi_\lambda, A_\lambda)$ under the possible constant gauge $\alpha_0 = e^{-2\pi ki}$ if necessary, which will change nothing in passing to the limit, we may assume that $g_\lambda - \theta \to 0$ in $B_{2R}\backslash B_{R/2}$ uniformly as $\lambda \to 1$. Recall that we have assumed that $A_\lambda \to B$ strongly in $C^{1,\alpha}(B_{3R}(0);\mathbb{R}^2)$ as $\lambda \to 1$ for some $\alpha \in (0,1)$, by Lemma 2.5, $g_\lambda - \theta \to 0$ strongly in $H^1(B_{2R}\backslash B_{R/2})$ as $\lambda \to 1$. It follows from the Sobolev embedding theorem that $g_\lambda \to \theta$ strongly in $H^{\frac{1}{2}}(\partial B_R(0))$ as $\lambda \to 1$, which implies $\upsilon_\lambda \to 0$ strongly in $H^1(B_R(0))$ as $\lambda \to 1$. We extend υ_λ from $\overline{B_R}$ to the whole $\mathbb{R}^2$ by setting $\upsilon_\lambda = \theta - g_\lambda$ in $\mathbb{R}^2\backslash\overline{B_R}$. It follows that $\upsilon_\lambda \in C^2(\mathbb{R}^2)$. Let us consider the gauge translation $(\psi_\lambda, A_\lambda) \to (\psi_\lambda e^{i\upsilon_\lambda}, A_\lambda + \nabla\upsilon_\lambda)$ if necessary. Then

$$\psi_\lambda e^{i\upsilon_\lambda} = \begin{cases} \psi_\lambda e^{i\upsilon_\lambda}, & \text{in } B_R(0); \\ w_\lambda e^{i\theta} & \text{in } \mathbb{R}^2\backslash B_R(0) \end{cases} \tag{2.21}$$

and

$$A_\lambda + \nabla\upsilon_\lambda = \begin{cases} A_\lambda + \nabla\upsilon_\lambda, & \text{in } B_R(0); \\ A_\lambda - \nabla g_\lambda + \nabla\theta & \text{in } \mathbb{R}^2\backslash B_R(0). \end{cases} \tag{2.22}$$

Since we have proved that $\upsilon_\lambda \to 0$ strongly in $H^1(B_R(0))$ as $\lambda \to 1$, by Lemmas 2.3 and 2.5, we still have $(\psi_\lambda e^{i\upsilon_\lambda}, A_\lambda + \nabla\upsilon_\lambda) \to (\varphi, B)$ strongly in $H^1_{loc}(\mathbb{R}^2;\mathbb{R}^2) \times H^1_{loc}(\mathbb{R}^2;\mathbb{C})$ as $\lambda \to 1$ in the sense of Definition 2.1. Now, by the gauge invariance of $\mathcal{E}_\lambda(\psi_\lambda, A_\lambda)$, we may assume that $g_\lambda = \theta$ in $\mathbb{R}^2\backslash B_R(0)$ for λ sufficiently close to 1. Therefore, in $\mathbb{R}^2\backslash B_R(0)$,

$$(\psi_\lambda, A_\lambda) - (\varphi, B) \quad = \quad (w_\lambda e^{i\theta} - f(r)e^{i\theta}, A_\lambda - \nabla g_\lambda + (1 - a(r))\nabla\theta).$$

Recall that by Lemma 2.6, $w_\lambda - 1$, ∇w_λ, $|\nabla g_\lambda - A_\lambda|$ and $|\nabla(\nabla g_\lambda - A_\lambda)|$ all exponentially decay to zero as $|x| \to +\infty$, uniformly for λ sufficiently close to 1. By the strong convergence of $(\psi_\lambda, A_\lambda)$ in $H^1_{loc}(\mathbb{R}^2;\mathbb{R}^2) \times H^1_{loc}(\mathbb{R}^2;\mathbb{C})$ as $\lambda \to 1$, we know that $(\psi_\lambda - \varphi, A_\lambda - B) \to 0$ strongly in $H^1(\mathbb{R}^2;\mathbb{R}^2) \times H^1(\mathbb{R}^2;\mathbb{C})$ as $\lambda \to 1$. □

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1: Let $(\phi_\lambda, D_\lambda) = (f_\lambda(r)e^{i\theta}, d_\lambda(r)\nabla\theta)$ be a radial solution of (1.1) such that its degree is 1. By [6, Theorem 1], $(\phi_\lambda, D_\lambda)$ is unique for λ sufficiently close to 1. By the variational formula of $(\phi_\lambda, D_\lambda)$ (cf. [4]), we can use similar arguments as used for $(\psi_\lambda, A_\lambda)$ to show that $(\phi_\lambda - \varphi, D_\lambda - B) \to 0$ strongly in $H^1(\mathbb{R}^2;\mathbb{R}^2) \times H^1(\mathbb{R}^2;\mathbb{C})$ as $\lambda \to 1$. By the results of Stuart in [16], the set of translational 0-modes of the linearized equation of (1.1) for $\lambda = 1$ at (φ, B) in

$H^1(\mathbb{R}^2;\mathbb{C}) \times H^1(\mathbb{R}^2;\mathbb{R}^2)$ is given by

$$\mathcal{K}^t = \Big\{(\partial_1\varphi + i\varphi\chi_1, \partial_1 B + \nabla\chi_1), (\partial_2\varphi + i\varphi\chi_2, \partial_2 B + \nabla\chi_2)\Big\},$$

where χ_j are real functions and satisfy the following equations:

$$-\Delta\chi_j + |\varphi|^2\chi_j = -\frac{1}{2}|\varphi|^2\partial_j\theta, \quad \text{in } \mathbb{R}^2.$$

As in [9], by the uniqueness of solutions of the above equation in $H^1(\mathbb{R}^2)$ and by the fact that (φ, B) is a solution of (1.1) for $\lambda = 1$, it is easy to observe that $\chi_j = -B_j$, where $B = (B_1, B_2)$. Thus,

$$\mathcal{K}^t = \Big\{((\nabla_B\varphi)_1, (0, \nabla\times B)), ((\nabla_B\varphi)_2, (-\nabla\times B, 0))\Big\}.$$

Moreover, $\mathcal{K}^t \perp \mathcal{K}^g$ in $L^2(\mathbb{R}^2;\mathbb{C}) \times L^2(\mathbb{R}^2;\mathbb{R}^2)$, where

$$\mathcal{K}^g = \{(i\varphi\chi, \nabla\chi) \mid \chi \in H^2(\mathbb{R}^2)\}$$

is the set of gauge translational 0-modes of the linearized equation of (1.1) for $\lambda = 1$ at (φ, B). For the sake of simplicity, we shall denote

$$\begin{aligned}
\mathcal{T}_1 &= (T_{1,1}, T_{1,2}) = ((\nabla_B\varphi)_1, (0, \nabla\times B)),\\
\mathcal{T}_2 &= (T_{2,1}, T_{2,2}) = ((\nabla_B\varphi)_2, (-\nabla\times B, 0)),\\
\mathcal{G}_\chi &= (i\varphi\chi, \nabla\chi).
\end{aligned}$$

Since by Lemma 2.7, $(\phi_\lambda - \varphi, D_\lambda - B) \to 0$ and $(\psi_\lambda - \varphi, A_\lambda - B) \to 0$ both strongly in $H^1(\mathbb{R}^2;\mathbb{R}^2) \times H^1(\mathbb{R}^2;\mathbb{C})$ as $\lambda \to 1$, the difference of $(\psi_\lambda, A_\lambda)$ and $(\phi_\lambda, D_\lambda)$ strongly converges to zero in $H^1(\mathbb{R}^2;\mathbb{R}^2) \times H^1(\mathbb{R}^2;\mathbb{C})$ as $\lambda \to 1$. We claim that the projection of $(\psi_\lambda, A_\lambda)$ in $(\mathcal{K}^t \bigoplus \mathcal{K}^g)^\perp$ is unique for λ sufficiently close to 1. Indeed, let

$$\mathcal{F}(\psi, A, \lambda) = \mathcal{E}'_\lambda(\psi, A),$$

then $\mathcal{F}(\psi_\lambda, A_\lambda, \lambda) = 0$ in $L^2(\mathbb{R}^2;\mathbb{C}) \times L^2(\mathbb{R}^2;\mathbb{R}^2)$ for λ sufficiently close to 1. Let $\mathcal{L} = \mathcal{E}''_1(\varphi, B)$. Then by a direct calculation,

$$\begin{aligned}
\mathcal{L}(\eta) &= \Big(-\Delta_B\xi + \frac{1}{2}(2|\varphi|^2 - 1)\xi + \frac{1}{2}\varphi^2\overline{\xi} + i[2\nabla_B\varphi D + \varphi div(D)],\\
&\qquad [-\Delta + |\varphi|^2]D + \nabla div(D) + Im(\overline{\nabla_B\varphi}\xi - \overline{\varphi}\nabla_B\xi)\Big),
\end{aligned}$$

where $\eta = (\xi, D) \in H^2(\mathbb{R}^2;\mathbb{C}) \times H^2(\mathbb{R}^2;\mathbb{R}^2)$. By the computations in [9], in $(\mathcal{K}^g)^\perp$, the operator $\mathcal{L}$ has the form:

$$\begin{aligned}
\widetilde{\mathcal{L}}(\eta) &= \Big(-\Delta_B\xi + (\frac{1}{2} + \frac{1}{2}|\varphi|^2)\xi + (|\varphi|^2 - 1)\xi + 2i\nabla_B\varphi D,\\
&\qquad [-\Delta + |\varphi|^2]D + 2Im(\overline{\nabla_B\varphi}\xi)\Big)\\
&= \mathcal{T}\Big(\xi + \mathcal{K}(\xi, D), D + \mathcal{Y}(\xi, D)\Big),
\end{aligned}$$

where $\mathcal{T}$ is an operator from $H^2(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)$ to $L^2(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)$ and is given by

$$\begin{aligned}
\mathcal{T}(\xi, D) &= (\mathcal{T}_1(\xi, D), \mathcal{T}_2(\xi, D))\\
&= \Big(-\Delta_B\xi + (\frac{1}{2} + \frac{1}{2}|\varphi|^2)\xi, -\Delta D + |\varphi|^2 D\Big)
\end{aligned}$$

with

$$\mathcal{K}(\xi, D) = \mathcal{T}_1^{-1}\bigg((|\varphi|^2 - 1)\xi + 2i\nabla_B\varphi D\bigg), \quad \mathcal{Y}(\xi, D) = \mathcal{T}_2^{-1}\bigg(2Im(\overline{\nabla_B\varphi}\xi)\bigg)$$

two operators from $H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ to $H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. Since $\nabla_B\varphi \in L^2(\mathbb{R}^2; \mathbb{C})$ and $|\varphi|^2 - 1 \in H^1(\mathbb{R}^2)$, it is standard to check that $\mathcal{K}$ and $\mathcal{Y}$ are both compact. Moreover, since $|\varphi| \to 1$ as $|x| \to +\infty$, we also know that $\mathcal{T}$ is a bijection. Thus, by the results in [9] and the Fredholm alternative, the operator $\mathcal{L} \gtrsim 1$ in $(\mathcal{K}^t \bigoplus \mathcal{K}^g)^\perp$. It follows from the implicit function theorem that the projection of $(\psi_\lambda, A_\lambda)$ in $(\mathcal{K}^t \bigoplus \mathcal{K}^g)^\perp$ is unique for λ sufficiently close to 1. Here, the orthogonal complement is in $H^1(\mathbb{R}^2; \mathbb{R}^2) \times H^1(\mathbb{R}^2; \mathbb{C})$. Thus, if $(\psi_\lambda, A_\lambda) \neq (\phi_\lambda, D_\lambda)$, then the difference of $(\psi_\lambda, A_\lambda)$ and $(\phi_\lambda, D_\lambda)$ must lie in $\mathcal{K}^t \bigoplus \mathcal{K}^g$ for λ sufficiently close to 1. That is, for λ sufficiently close to 1, we have

$$(\psi_\lambda, A_\lambda) = (\phi_\lambda, D_\lambda) + \alpha_\lambda\mathcal{T}_1 + \beta_\lambda\mathcal{T}_2 + \mathcal{G}_{\chi_\lambda}, \tag{2.23}$$

where $\chi_\lambda \in H^2(\mathbb{R}^2)$ with $\chi_\lambda \to 0$ in $H^2(\mathbb{R}^2)$ and $\alpha_\lambda, \beta_\lambda \to 0$ as $\lambda \to 1$. By a direct calculation,

$$\mathcal{T}_1 = \bigg((f'(r)\cos\theta - if(r)\frac{1 - a(r)}{r}\sin\theta)e^{i\theta}, (0, (\frac{a(r)}{r})' + \frac{a(r)}{r^2}\bigg), \tag{2.24}$$

$$\mathcal{T}_2 = \bigg((f'(r)\sin\theta - if(r)\frac{1 - a(r)}{r}\cos\theta)e^{i\theta}, (-((\frac{a(r)}{r})' + \frac{a(r)}{r^2}), 0)\bigg). \tag{2.25}$$

Let $\tau_\lambda = \max\{\alpha_\lambda, \beta_\lambda\}$ and define

$$(\upsilon_\lambda, \vartheta_\lambda) = \frac{1}{\tau_\lambda}((\psi_\lambda, A_\lambda) - (\phi_\lambda, D_\lambda)).$$

Recall that by Lemma 2.4, 0 is the only vortex point of $(\psi_\lambda, A_\lambda)$. Moreover, it is also well known that 0 is also the only vortex point of the radial solutions $(\phi_\lambda, D_\lambda)$ and (φ, B). Thus, we always have $\upsilon_\lambda(0) = 0$. Since it is well known that $f(r) \sim r$ and $a(r) \sim r^2$ as $r \to 0$ (cf. [9]), by (2.23), (2.24) and (2.25),

$$\begin{aligned}\upsilon_\lambda(0) &= \frac{\alpha_\lambda}{\tau_\lambda}T_{1,1}(0) + \frac{\beta_\lambda}{\tau_\lambda}T_{2,1}(0) + \frac{1}{\tau_\lambda}\varphi(0)\chi_\lambda(0) \\ &= (\frac{\alpha_\lambda}{\tau_\lambda} - i\frac{\beta_\lambda}{\tau_\lambda})f'(0).\end{aligned}$$

We remark that since 0 is the vortex point of φ, $T_{j,1}(0)$ should be understood as the limit of $r \to 0$. It follows that $\alpha_\lambda = \beta_\lambda = 0$ for λ sufficiently close to 1. It remains to show $\chi_\lambda = 0$ for λ sufficiently close to 1. By Lemma 2.7, we can choose gauges (cf. (2.21) and (2.22)) such that the considered nontrivial least-energy minimizer $(\psi_\lambda, A_\lambda)$ satisfies $(\psi_\lambda, A_\lambda) \to (\varphi, B)$ strongly in $H^1(\mathbb{R}^2)$ as $\lambda \to 1$. If $\mathcal{G}_{\chi_\lambda} \neq 0$ in (2.23) for λ sufficiently close to 1, then $divA_\lambda = \Delta\chi_\lambda$ in $\mathbb{R}^2$. Since $\chi_\lambda \in H^2(\mathbb{R}^2)$ now, $divA_\lambda \in L^2(\mathbb{R}^2)$ for λ sufficiently close to 1. Let us consider the possible gauge translation $(\psi_\lambda, A_\lambda) \to (\psi_\lambda e^{i\chi_\lambda}, A_\lambda + \nabla\chi_\lambda)$. Since $\chi_\lambda \to 0$ strongly in $H^2(\mathbb{R}^2)$ as $\lambda \to 1$, $(\widehat{\psi}_\lambda, \widehat{A}_\lambda) = (\psi_\lambda e^{i\chi_\lambda}, A_\lambda + \nabla\chi_\lambda) \to (\varphi, B)$ strongly in $H^1(\mathbb{R}^2)$ as $\lambda \to 1$. Moreover, we still have the expansion (2.23) for $(\widehat{\psi}_\lambda, \widehat{A}_\lambda)$ and $(\phi_\lambda, D_\lambda)$ with $\alpha_\lambda = \beta_\lambda = 0$ and some $\chi'_\lambda \in H^2(\mathbb{R}^2)$. However, for $\widehat{A}_\lambda$, we have $div(\widehat{A}_\lambda) = 0$. Thus, without loss of generality, we may assume that $div(A_\lambda) = 0$ in $\mathbb{R}^2$ now. Since $div(A_\lambda) = 0$, we have $\Delta\chi_\lambda = 0$ in $\mathbb{R}^2$, which together with $\chi_\lambda \in H^2(\mathbb{R}^2)$, implies $\chi_\lambda = 0$ for λ sufficiently close to 1. Thus, by (2.23) once more, we must have $(\psi_\lambda, A_\lambda) = (\phi_\lambda, D_\lambda)$ for λ sufficiently close to 1 up to gauges. □

3. Appendix

In this appendix, we shall prove Proposition 2.1.

Proof of Proposition 2.1: Let (φ, B) be a solution of (1.1) with $\lambda = 1$ such that its degree is N, then under Taubes's notations and results in [19] (see also [16]), $\varphi = e^{\frac{1}{2}(u+i\theta)}$ with $\theta = \sum_{j=1}^{m} \arg(x - a_j)$ and $B = \frac{1}{2}(\partial_1\theta + \partial_2 u, \partial_2\theta - \partial_1 u)$, where $(a_1, a_2, \cdots, a_m)$ is the m vortices of (φ, B). Moreover, u also satisfies the following elliptic equation:

$$-\Delta u + (e^u - 1) = -4\pi \sum_{j=1}^{m} \delta_{a_j} \quad \text{in } \mathbb{R}^2, \tag{3.1}$$

where δ_{a_j} is the Dirac function at a_j. Now, using this information in the energy functional $\mathcal{E}_1(\psi, A)$, we have

$$\mathcal{E}_1(\psi, B) = \int_{\mathbb{R}^2} \frac{1}{4}|\nabla u|^2 e^u + \frac{1}{8}|\Delta u|^2 + \frac{1}{8}(e^u - 1)^2. \tag{3.2}$$

By Levi's monotone convergence theorem and (3.1),

$$\int_{\mathbb{R}^2} |\Delta u|^2 = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus \cup_{j=1}^{m} B_\varepsilon(a_j)} |\Delta u|^2 = \int_{\mathbb{R}^2} (e^u - 1)^2. \tag{3.3}$$

On the other hand, since $|\varphi| \le 1$ (cf. [20]), by the diamagnetic inequality (cf. [8, (2.3)]), we know that $|\varphi|^2 - 1 = e^u - 1$ in $H^1(\mathbb{R}^2)$. Moreover, since $u = -\sum_{j=1}^{m} \ln(1 + \frac{\sigma}{|x-a_j|^2}) + v$ for some $\sigma > 4m$ and smooth v which exponentially decays to zero as $|x| \to +\infty$ (cf. [19]), $|\nabla u| \sim \frac{1}{|x-a_j|}$ near each vortex point a_j and $|\nabla u| \sim \frac{1}{|x|^3}$ as $|x| \to +\infty$. Thus, we can multiply (3.1) in $\mathbb{R}^2 \setminus \cup_{j=1}^{m} B_\varepsilon(a_j)$ with $e^u - 1$ and integrate by parts, which implies that

$$\int_{\mathbb{R}^2 \setminus \cup_{j=1}^{m} B_\varepsilon(a_j)} |\nabla u|^2 e^u + \int_{\mathbb{R}^2 \setminus \cup_{j=1}^{m} B_\varepsilon(a_j)} (e^u - 1)^2 = 4m\pi.$$

Let $\varepsilon \to 0$ and applying Levi's monotone convergence theorem yield that

$$\int_{\mathbb{R}^2} |\nabla u|^2 e^u + \int_{\mathbb{R}^2} (e^u - 1)^2 = 4m\pi. \tag{3.4}$$

Inserting (3.3) and (3.4) into (3.2) and recalling that $\mathcal{E}_1(\psi, B) = N\pi$ since (φ, B)'s degree is N, we must have $m = N$. □

4. Acknowledgements

The research of J. Wei is partially supported by NSERC of Canada. The research of Y. Wu is supported by NSFC (No. 11701554, No. 11771319, No. 11971339), the Fundamental Research Funds for the Central Universities (2017XKQY091) and Jiangsu overseas visiting scholar program for university prominent young & middle-aged teachers and presidents. This paper was completed when Y. Wu was visiting University of British Columbia. He is grateful to the members in Department of Mathematics at University of British Columbia for their invitation and hospitality.

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