# Non-degeneracy and uniqueness of positive blow-up solutions to the Lane-Emden problem*

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**Abstract**

In this paper, we study the nearly critical Lane-Emden equations

$$\begin{cases} -\Delta u = u^{p-\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 3$, $p = \frac{N+2}{N-2}$ and $\varepsilon > 0$ is small. Our main result is that when $\Omega$ is a smooth bounded convex domain and the Robin function on $\Omega$ is a Morse function, then for small $\varepsilon$ the equation $(*)$ has a unique solution, which is also nondegenerate. In particular, the above conclusion holds for half balls.

In general, the solutions of $(*)$ may blow-up at multiple points $a_1, \cdots, a_k$ of $\Omega$ as $\varepsilon \to 0$. In particular, when $\Omega$ is convex, there must be a unique blow-up point (i.e., $k = 1$). In this paper, by using the local Pohozaev identities and blow-up techniques, even having multiple blow-up points (non-convex domian), we can prove that such blow-up solution is unique and nondegenerate. Combining these conclusions, we finally obtain the uniqueness and nondegeneracy of solutions to $(*)$.

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## 1 Introduction

We study the uniqueness and nondegeneracy of the solution to the following Lane-Emden problem:

$$\begin{cases} -\Delta u = u^{p-\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, and when $N \geq 3$, it holds $0 < \varepsilon < p - 1$ and $p = 2^* - 1 = \frac{N+2}{N-2}$; when $N = 2$, we assume $\varepsilon = 0$ and $1 < p < +\infty$.

It is known that if $\varepsilon > 0$ problem (1.1) has at least one solution. But the uniqueness or multiplicity are much more complicated, which is known depending on the domain $\Omega$ and the value $p - \varepsilon$.

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A longstanding and largely unsolved open question is whether (1.1) has a unique solution when $\Omega$ is convex. It was conjectured by Dancer ([14]) that the answer is affirmative. As of today, this conjecture has been verified only if $p-\varepsilon$ is close to 1 for $N \geq 2$ by Lin Changshou ([27]), and if $p-\varepsilon$ is close to $+\infty$ for $N=2$ by Grossi *etc.* ([17, 23]).

When the domain $\Omega$ possesses some kind of symmetric properties, there are some results about uniqueness. For example, when $\Omega$ is a ball, as a consequence of the famous symmetry result by Gidas, Ni and Nirenberg ([19]) it follows that any solution of (1.1) is radial, and then one gets the uniqueness of the solution to (1.1) by ODE techniques. Another cases were considered by Damascelli, Grossi and Pacella ([13] with $N=2$) and Grossi ([21] with $N \geq 3$) separately where uniqueness has been proved when the domain is both symmetric and convex with respect to $N$ orthogonal directions for small $\varepsilon$. We note that their (i.e.,[13, 21]) symmetry and special "convexity" play an essential role. Such domains do not need to be convex in common sense and the conjecture due to Dancer have not been solved before. It is well known, if the convex domain has no any symmetry, the problem becomes very challenging.

In this paper, we prove a weaker version of this conjecture when $p-\varepsilon$ is close to $2^*-1$ for $N \geq 3$. We obtain

**Theorem 1.1.** *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a smooth bounded convex domain, and the Robin function on $\Omega$ is a Morse function. Then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ problem* (1.1) *has exactly one solution, which is also nondegenerate.*

**Remark 1.1.** *(1) The Robin function $R(x)$ on $\Omega$ is given in* (2.2)*. We say a function is a Morse function, if all its critical points are nondegenerate. When $\Omega \subset \mathbb{R}^2$ is a convex domain, Caffarelli and Friedman ([9]) proved that the Robin function on $\Omega$ is a Morse function and it has only one critical point. While for convex domain $\Omega \subset \mathbb{R}^N$ with $N \geq 3$, whether the Robin function is a Morse function becomes a challenging problem. In [11], Cardaliaguet and Tahraoui proved that for convex domain, the Robin function has only one critical point, but whether it is a Morse function is still open. Specially, for domain which is sysmetric with respect to $N$ orthogonal directions, Grossi ([22]) proved that the Robin function is a Morse function. While there are still non-sysmetric domains on which the Robin function is a Morse function, f.g., the half balls $\left\{x \in \mathbb{R}^N : |x| \leq \rho, x_N > 0\right\}$. So applying our Theorem 1.1, we obtain that the same conclusion holds for half balls.*
*(2) We say a solution $u$ of* (1.1) *is nondegenerate if the linearized equation*

$$-\Delta v = (p-\varepsilon)u^{p-1-\varepsilon}v \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega$$

*has only trivial solution $v \equiv 0$. It is known that, the uniqueness and nondegeneracy of solutions to Lane-Emden equation are important for many problems. For example, in [8], under a key assumption that the domain $\Omega$ is such that the solution of*

$$-\Delta V = cV^{\frac{1}{m}} \text{ in } \Omega, \quad V = 0 \text{ on } \partial\Omega, \quad \text{for some } c > 0$$

*is nondegenerate, Bonforte and Figalli got the sharp extinction rates as $t \to T^-$ for the fast diffusion equations $u_t = \Delta u^m$ in $(0,T) \times \Omega$ for $\frac{N-2}{N+2} < m < 1$. By scaling, our Theorem 1.1 implies that this assumption is naturally true for some smooth bounded convex domains when $m$ is close to $\frac{N-2}{N+2}$ and $N \geq 3$.*

Recall that a solution sequence $u_{\varepsilon_n}$ is blow-up, if there exists some points $a_1, \cdots, a_k \in \bar{\Omega}$ and some sequences $x_{\varepsilon_n,1}, \cdots, x_{\varepsilon_n,k} \in \Omega$ such that as $\varepsilon_n \to 0$, it holds $x_{\varepsilon_n,j} \to a_j$ and $u_{\varepsilon_n}(x_{\varepsilon_n,j}) \to +\infty$. The points $a_1, \cdots, a_k$ are called blow-up points. Below in Section 2 we will see that for $N \geq 3$, any sequence $u_{\varepsilon_n}$ of solutions to (1.1) blows-up as $\varepsilon_n \to 0$, when the equation (1.1) with $\varepsilon = 0$ has no solution (it is indeed true for convex domains). Moreover, $u_{\varepsilon_n}$ satisfies (up to a subsequence)

$$|\nabla u_{\varepsilon_n}|^2 \rightharpoonup \mathcal{S}_N^{N/2} \sum_{j=1}^{k} \delta_{a_j}, \quad \text{as } \varepsilon_n \to 0, \tag{1.2}$$

where $a_j \in \Omega$, $k \in \mathbb{N}^+$, $\delta_a$ is the Dirac measure and $\mathcal{S}_N$ is the best Sobolev constant defined by

$$\mathcal{S}_N := \inf_{u \in H_0^1(\mathbb{R}^N)} \frac{\|u\|^2}{\|u\|_{p+1}^2}.$$

By [24, 11], when $\Omega$ is convex, $u_{\varepsilon_n}$ must blow-up at a single point $x_0 \in \Omega$ with $x_0$ is the unique critical point of the Robin function (defined by (2.2)). However, this fact does not imply the uniqueness of solutions to (1.1), the problem is: whether or not there are two solution sequences $u^{(1)}_{\varepsilon_n}$,$u^{(2)}_{\varepsilon_n}$ blow-up at the same point $x_0$, but the blow-up rates is different.

In this paper, using the Pohozaev identities and blow-up techniques (see [28, 23]), we prove that under some conditions, there can not be two sequences of different solutions $u^{(1)}_{\varepsilon_n}$,$u^{(2)}_{\varepsilon_n}$ blow-up at a given blow-up point $x_0$. Then combining these uniqueness results together, we obtain the uniqueness part in Theorem 1.1. Also we obtain the nondegeneracy of blow-up solutions, which implies the nondegeneracy part in Theorem 1.1.

Therefore we need to study the blow-up phenomenon firstly, i.e., the behavior of the solutions $u_\varepsilon$ as $\varepsilon \to 0$. When $\varepsilon = 0$, Pohozaev proved in [31] that if $\Omega$ is starshaped (1.1) has no solution; whereas Bahri and Coron proved in [2] that (1.1) has a solution provided that $\Omega$ has non-trivial topology. Since then a lot of attention has been paid to the limiting behavior of the solutions $u_\varepsilon$ of (1.1) as $\varepsilon \to 0$. In [1, 6], Peletier *etc.* study this problem firstly with $\Omega$ replaced by a ball. Later, Rey in [32] and Han in [25] extended the previous results to general domains separately.

Since our final result concerns the single point blow-up, we consider the simplest case $k = 1$ in (1.2) at first. Let $\varepsilon_n \to 0$ and $u_{\varepsilon_n}$ be a sequence of solutions to (1.1) with $\varepsilon = \varepsilon_n$, which satisfies

$$|\nabla u_{\varepsilon_n}|^2 \rightharpoonup \mathcal{S}_N^{N/2} \delta_{x_0}, \quad \text{as } \varepsilon_n \to 0. \tag{1.3}$$

Naturally, it stresses that the sequence $u_{\varepsilon_n}$ blows-up and the blow-up point is $x_0$. For $u_{\varepsilon_n}$ satisfying (1.3), Rey ([32]) and Han ([25]) proved that $x_0$ must be a critical point of the Robin function $R(x)$. Moreover, $u_{\varepsilon_n}$ can be writen as

$$u_{\varepsilon_n} = \alpha_{\varepsilon_n} PU_{x_{\varepsilon_n}, \mu_{\varepsilon_n}^{-1}} + w_{\varepsilon_n},$$

with $\alpha_{\varepsilon_n} \to \alpha_0 = [N(N-2)]^{\frac{N-2}{4}}$, $x_{\varepsilon_n} \to x_0$, $\mu_{\varepsilon_n} \to 0$ and $w_{\varepsilon_n} \to 0$, and they also gave some detailed asymptotic behaviors of these parameters as $\varepsilon_n \to 0$. But to get the uniqueness of such blow-up solutions, we have to prove a more sharper asymptotic description.

**Theorem 1.2.** *Suppose $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ is a smooth bounded domain (need no convex property). For $x_0 \in \Omega$, let $\varepsilon_n \to 0$ and $u^{(1)}_{\varepsilon_n}$, $u^{(2)}_{\varepsilon_n}$ be two sequences of solutions to* (1.1) *with $\varepsilon = \varepsilon_n$, which blow-up at the same point $x_0$. If $x_0$ is a nondegenerate critical point of the Robin function $R(x)$, then there exists an $n_0 > 1$ such that*

$$u^{(1)}_{\varepsilon_n} \equiv u^{(2)}_{\varepsilon_n}, \quad \text{for } n \geq n_0.$$

**Remark 1.2.** *The assumption $u^{(1)}_{\varepsilon_n}$, $u^{(2)}_{\varepsilon_n}$ blow-up at the same point $x_0$ can be formulated as*

$$|\nabla u^{(l)}_{\varepsilon_n}|^2 \rightharpoonup \mathcal{S}_N^{N/2} \delta_{x_0}, \quad \text{for } l = 1, 2, \quad \text{as } \varepsilon_n \to 0.$$

*Then according to [32, 25], $x_0$ is a critical point of the Robin function $R(x)$. The assumption that $x_0$ is nondegenerate is necessary, since if $x_0$ is degenerate, there are examples with more than one solutions which concentrates at $x_0$, see [20].*

*The existence of such solutions has been promised. One example is the least energy solution of* (1.1). *Indeed, take $u_{\varepsilon_n}$ be a sequence of least energy solutions to* (1.1) *with $\varepsilon_n \to 0$, i.e., $u_\varepsilon$ is a minimizer of*

$$J_{\varepsilon_n} = \inf \left\{ \frac{\|u_{\varepsilon_n}\|^2}{\|u_{\varepsilon_n}\|_{p+1-\varepsilon_n}^2} : \ u \in H_0^1(\Omega), u \neq 0 \right\}.$$

*It is known that $J_{\varepsilon_n} = \mathcal{S}_N + o_{\varepsilon_n}(1)$ (see [37]). Then the discussion in Section 2 shows that $u_{\varepsilon_n}$ satisfies* (1.3)*. But there are another approaches to the existence of such solutions. Actually, using the finite dimensional reduction, Rey ([33, 34]) proved that for any $x_0 \in \Omega$, if $N \geq 3$ and $x_0$ is a nondegenerate critical point of $R(x)$, then there exists a $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, equation* (1.1) *has a solution $u_\varepsilon$, which satisfies*

$$|\nabla u_\varepsilon|^2 \rightharpoonup \mathcal{S}_N^{N/2}\delta_{x_0}, \quad \textit{as } \varepsilon \to 0.$$

*To prove Theorem 1.2, we recall the Pohozaev identities,*

$$-\int_{\partial\Omega'} \langle \nabla u, \nu\rangle \, \partial_i u \mathrm{d}\sigma + \frac{1}{2}\int_{\partial\Omega'} |\nabla u|^2 \nu_i \mathrm{d}\sigma = \frac{1}{2^*-\varepsilon}\int_{\partial\Omega'} u^{2^*-\varepsilon}\nu_i \mathrm{d}\sigma,$$

*and*

$$\begin{aligned}&-\int_{\partial\Omega'} \langle \nabla u, \nu\rangle \langle \nabla u, x-y\rangle \, \mathrm{d}\sigma + \frac{1}{2}\int_{\partial\Omega'} \langle x-y, \nu\rangle \, |\nabla u|^2 \mathrm{d}\sigma - \frac{N-2}{2}\int_{\partial\Omega'} \langle \nabla u, \nu\rangle \, u\mathrm{d}\sigma \\ &= \frac{1}{2^*-\varepsilon}\int_{\partial\Omega'} \langle x-y, \nu\rangle^{2^*-\varepsilon} \mathrm{d}\sigma - \frac{N\varepsilon}{2^*(2^*-\varepsilon)}\int_{\Omega'} u^{2^*-\varepsilon}\mathrm{d}x,\end{aligned}$$

*see Lemma 2.12 for a proof. In general, for two different solutions $u^{(1)}_{\varepsilon_n}$ and $u^{(2)}_{\varepsilon_n}$, define*

$$\xi_{\varepsilon_n} = \frac{u^{(1)}_{\varepsilon_n} - u^{(2)}_{\varepsilon_n}}{\|u^{(1)}_{\varepsilon_n} - u^{(2)}_{\varepsilon_n}\|_{L^\infty(\Omega)}},$$

*we know $\|\xi_\varepsilon\|_\infty = 1$. Roughly speaking, we want to use the Pohozaev identities and blow-up techniques to prove that $\xi_\varepsilon$ is small both near and away from the blow-up points, and then we can get a contradiction. To this aim, we need to calculate carefully each surface integrals in the local Pohozaev identities.*

*Note that Theorem 1.2 holds for any smooth bounded domain. However, the uniqueness result in Theorem 1.2 does not promise the uniqueness of solutions to* (1.1)*, since there can be many different critical points $x_0$ of the Robin function $R(x)$, and there can be also solutions blowing up at more than one point.*

Now we would like to introduce some known results about uniqness of blow-up solutions. For simplicity, we call the solution in Theorem 1.2 one-peak solution, since it has only one blow-up point.

When $N \geq 3$ and $\Omega$ is both symmetric and convex with respect to $N$ orthogonal directions, Grossi ([21]) proved the uniqueness of one-peak solution to (1.1) provided $\varepsilon > 0$ small. Note that such domains do not need to be convex, so the uniqueness results in Theorem 1.2 and [21] do not contain each other.

When $N = 2$ and $\varepsilon = 0$, Marchis, Ianni and Pacella ([15, 16]) obtained the "blow-up" phenomena of the solutions to (1.1) as $p \to +\infty$. For instance, let $p_n \to +\infty$ and $u_{p_n}$ be a sequence of solutions to (1.1) with $p = p_n$, which satisfies

$$p_n \int_\Omega |\nabla u_{p_n}|^2 \to 8\pi e, \quad \text{as } p_n \to +\infty, \tag{1.4}$$

then there exists a sequence of $x_{p_n} \in \Omega$ and a critical point of the Robin function $x_\infty \in \Omega$ such that up to a subsequence

$$x_{p_n} \to x_\infty, \quad u_{p_n}(x_{p_n}) \to \sqrt{e}, \quad u_{p_n}(x) \to 0, \quad \forall\, x \in \Omega \setminus \{x_\infty\}, \quad \text{as } p_n \to +\infty.$$

Actually condition (1.4) plays the same role as (1.3) when $N = 2$. Recently, in [23], Grossi etc. proved that if $x_\infty$ is a nondegenerate critical point of the Robin function, then (1.1) has only one solution concentrating at $x_\infty \in \Omega$ satisfying (1.4).

A natural question is whether the uniqueness results in Theorem 1.2 still hold for solutions blowing-up at more than one point. The answer is affirmative.

Similar to the one-peak solution, we suppose the sequence $u_{\varepsilon_n}$ of solutions to (1.1) satisfies (1.2). Like assumption (1.3), (1.2) stress that the sequence $u_{\varepsilon_n}$ blows-up and the blow-up points are $a_1, \cdots, a_k$. For

$\vec{a} = (a_1, \cdots, a_k) \in \Omega^k$, let the matrix $M_k(\vec{a})$, the vector $\vec{\lambda}(\vec{a}) \in \mathbb{R}^k$ and the function $\Phi_k(\vec{a}, \vec{\lambda})$ be defined by (2.21), (2.22) and (2.20). In [3, 34], Bahri, Li and Rey proved that if the matrix $M_k(\vec{a})$ is positive definite, then $(\vec{a}, \vec{\lambda}(\vec{a}))$ must be a critical point of $\Phi_k(\vec{a}, \vec{\lambda})$. Moreover, $u_{\varepsilon_n}$ can be writen as

$$u_{\varepsilon_n} = \sum_{j=1}^{k} \alpha_{\varepsilon_n,j} PU_{x_{\varepsilon_n,j},\mu^{-1}_{\varepsilon_n,j}} + w_{\varepsilon_n}$$

with $\alpha_{\varepsilon_n,j} \to \alpha_0$, $x_{\varepsilon_n,j} \to a_j$, $\mu_{\varepsilon_n,j} \to 0$ and $w_{\varepsilon_n} \to 0$, and they also gave detailed asymptotic behaviors of these parameters. Then following the idea to prove Theorem 1.2, we obtain uniqueness of multi-peak solutions. But the uniqueness of multi-peak solutions is much more complicated than one-peak solutions, since we have to estimate the coupled terms of different bubbles.

**Theorem 1.3.** *Suppose $\Omega \subset \mathbb{R}^N$ with $N \geq 7$ is a smooth bounded domain (need no convex property). For different $k \geq 1$ points $a_1, \cdots, a_k \in \Omega$, let $\varepsilon_n \to 0$ and $u^{(1)}_{\varepsilon_n}$, $u^{(2)}_{\varepsilon_n}$ be two sequences of solutions to* (1.1) *with $\varepsilon = \varepsilon_n$, which concentrate at the same points $a_1, \cdots, a_k$. Suppose that $M_k(\vec{a})$ is positive definite and $(\vec{a}, \vec{\lambda}(\vec{a}))$ is a nondegenerate critical point of $\Phi_k$. Then there exists an $n_0 > 1$ such that*

$$u^{(1)}_{\varepsilon_n} \equiv u^{(2)}_{\varepsilon_n}, \quad \textit{for } n \geq n_0.$$

**Remark 1.3.** *The existence of such solutions has been given by Musso and Pistoia. In [30], Musso and Pistoia proved that for $N \geq 3$ and different $k \geq 1$ points $a_1, \cdots, a_k \in \Omega$, if $(\vec{a}, \vec{\lambda})$ is a stable critical point of $\Phi_k$ with $\vec{a} = (a_1, \cdots, a_k)$ and some $\vec{\lambda} = (\lambda_1, \cdots, \lambda_k)$, then there exists a $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, equation* (1.1) *has a solution $u_\varepsilon$, which satisfies*

$$|\nabla u_\varepsilon|^2 \rightharpoonup \sum_{j=1}^{k} \mathcal{S}_N^{N/2} \delta_{a_j}, \quad \textit{as } \varepsilon \to 0.$$

*Due to our method, we don't know whether or not Theorem 1.3 holds true for $3 \leq N \leq 6$, see Remark 4.2. The main reason is that we can proceeding as in case handling one-peak solutions to get the asymptotic estimation of critical points*

$$x^{(1)}_{\varepsilon,j} - x^{(2)}_{\varepsilon,j} = O(\bar{\mu}_\varepsilon^2).$$

*But is is not enough to get the final result. Instead, we need a more sharper estimation*

$$x^{(1)}_{\varepsilon,j} - x^{(2)}_{\varepsilon,j} = o(\bar{\mu}_\varepsilon^2),$$

*which is proved thanks to the Propositions 4.4 and 4.5. Then the limitation $N \geq 7$ comes up.*

*The assumption that $M_k(\vec{a})$ is positive and $(\vec{a}, \vec{\lambda}) \in \Omega^k \times (\mathbb{R}^+)^k$ is a non-degenerate critical point of $\Phi_k$ is necessary for our result. Recently, Bartsch, Micheletti and Pistoia give some domains such that $\Phi_k$ possesses some critical points and all these critical points are nondegenerate, see [4, 5, 29] and the reference therein. Moreover, there are also some non-convex domains such that $\Phi_k$ possesses some critical points and all these critical points are nondegenerate, see Remark 1.4 in [10]. So the uniqueness results in Theorem 1.3 make sense.*

As we said in Remark 1.3, there are non-convex domains such that $\Phi_k$ possesses some critical points and all these critical points are nondegenerate. It means that the solution to (1.1) is not unique. We want to count the number of solutions to (1.1) by using Theorem 1.4. Below denote

$$\mathcal{T}_k = \left\{ (\vec{a}, \vec{\lambda}) : \nabla_x \Phi_k(\vec{a}, \vec{\lambda}) = 0, \nabla_\lambda \Phi_k(\vec{a}, \vec{\lambda}) = 0 \right\}.$$

We give two assumptions about the domain $\Omega$,

$$\text{(A1):} \quad \Omega \text{ is such that (1.1) with } \varepsilon = 0 \text{ has no solutions,}$$

and for a $k \in \mathbb{N}^+$

$$\text{(A2)}_k\text{:} \quad M_k(\vec{a}) \text{ is positive and } (\vec{a}, \vec{\lambda}) \text{ is nondegenerate stable critical point for any } (\vec{a}, \vec{\lambda}) \in \mathcal{T}_k.$$

Then we have

**Theorem 1.4.** *Let $N \geq 7$. Suppose $\Omega$ satisfies (A1), then there exists a $k_0 \in \mathbb{N}^+$ such that if (A2)$_k$ holds for any $1 \leq k \leq k_0$, then for $\varepsilon > 0$ small,*

$$\text{the number of solutions to (1.1) is exactly } \sum_{k=1}^{k_0} |\mathcal{T}_k|,$$

*where $|\mathcal{T}_k|$ is the number of elements in $\mathcal{T}_k$.*

Finally, we study the nondegeneracy of the blow-up solutions to (1.1). It is well known that the linearized equation of (1.1) is

$$\begin{cases} -\Delta v = (p-\varepsilon)u^{p-1-\varepsilon}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Then a solution $u$ of (1.1) is nondegenerate if and only if the linearized equation (1.5) has only trivial solution $v \equiv 0$.

**Theorem 1.5.** *Suppose $\Omega \subset \mathbb{R}^N$ with $N \geq 4$ is a smooth bounded domain (need no convex property). For different $k \geq 1$ points $a_1, \cdots, a_k \in \Omega$, let $u_{\varepsilon_n}$ be a solution to* (1.1) *satisfying* (1.2)*. Let $\zeta_{\varepsilon_n}$ be a solution of* (1.5) *with $u = u_{\varepsilon_n}$. Suppose that $M_k(\vec{a})$ is positive and $(\vec{a}, \vec{\lambda}(\vec{a}))$ is a nondegenerate critical point of $\Phi_k$. Then there exists an $n_0 > 0$ such that*

$$\zeta_{\varepsilon_n} \equiv 0, \quad \text{for } n \geq n_0.$$

**Remark 1.4.** *We remark that our method is not applied for $N = 3$, see Remark 5.2.*

Note that when $k = 1$, the assumption that $M_k(\vec{a})$ is positive and $(\vec{a}, \vec{\lambda})$ is a nondegenerate critical point of $\Phi_k$ turns to be the assumption of Theorem 1.2. Moreover, when $k = 1$, the nondegeneracy still holds for $N = 3$.

**Theorem 1.6.** *Suppose $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ is a smooth bounded domain (need no convex property). For $x_0 \in \Omega$, let $u_{\varepsilon_n}$ be a solution to* (1.1) *satisfying* (1.3)*. If $x_0$ is a nondegenerate critical point of the Robin function $R(x)$, then there exists an $n_0 > 0$ such that*

$$\zeta_{\varepsilon_n} \equiv 0, \quad \text{for } n \geq n_0.$$

Before closing this section, we would like to mention the following equation for the Brezis-Nirenberg problem

$$\begin{cases} -\Delta u = u^p + \varepsilon u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

under assumption (1.3). Rey ([32]) and Han ([25]) obtained blow-up behaviors of $u_{\varepsilon_n}$ satisfying (1.6) and (1.3) as $\varepsilon_n \to 0$. Later, Glangetas ([20]) proved the uniqueness of one-peak solution for $N \geq 4$, i.e., if $x_0$ is a nondegenerate critical point of the Robin function $R(x)$, then (1.6) has only one solution satisfying (1.3) provided $\varepsilon_n > 0$ small, where the used method is to reduce into finite dimensional problems and count the local degree, which is a different method to this paper. Recently, Cao, Luo and Yan ([10]) proved the uniqueness of multi-peak solutions to (1.6) for $N \geq 6$.

This paper is organized as follows. In Section 2, we collect some results which will be used in the following sections. In Section 3, we handle the one-peak solutions: In Section 3.1, we give a sharper blow-up estimation of one-peak solutions; In Section 3.2, we apply blow-up techniques to analyse the difference $\xi_\varepsilon = \frac{u^{(1)}_{\varepsilon_n} - u^{(2)}_{\varepsilon_n}}{\|u^{(1)}_{\varepsilon_n} - u^{(2)}_{\varepsilon_n}\|_\infty}$; In Section 3.3, we use the local Pohozaev identities to show that $\xi_\varepsilon$ is 0 both near and away from the blow-up points, and then give the proof of Theorem 1.2. In Section 4, we deal with the multi-peak solutions: In Section

4.1, as in Section 3.1 we give a blow-up estimation of multi-peak solutions; In Section 4.2, we obtain a sharper estimation of $\xi_\varepsilon$, inspired by [10]; In Section 4.3, we show that $\xi_\varepsilon$ is 0 both near and away from the blow-up points, and then give the proof of Theorem 1.3. In Section 5, we study the nondegeneracy of multi-peak solutions: In Section 5.1, we use blow-up techniques to estimate the solution of $\zeta_{\varepsilon_n}$; In Section 5.2, we show that $\zeta_{\varepsilon_n}$ is 0 both near and away from the blow-up points, and then give the proof of Theorem 1.5 and 1.6. In Section 6, we consider the case $\Omega$ is convex and prove the Theorem 1.1, and also we give the proof of Theorem 1.4.

Throughout the paper, we use $C$ to denote various positive constant. We use $A = o(\varepsilon)$ and $B = O(\varepsilon)$ denote $A/\varepsilon \to 0$ and $|B/\varepsilon| \le C$ as $\varepsilon \to 0$ respectively. We use $\partial_i$ and $\nabla$ to denote the partial derivative for any function $f(x, y)$ with respect to $x$, while we use $D_i$ and $D$ to denote the partial derivative for any function $f(x, y)$ with respect to $y$. In this paper, $\|u\| = (\int_\Omega |\nabla u|^2 \mathrm{d}x)^{1/2}$ denotes the norm in $H_0^1(\Omega)$ and $\langle \cdot, \cdot \rangle$ means the inner product. For simplicity, we denote $u_\varepsilon = u_{\varepsilon_n}$.

# 2 Preliminaries

## 2.1 Green and Robin functions

The Green's function $G(x, y)$ is the solution of

$$\begin{cases} -\Delta_x G(x, y) = \delta_y & \text{in } \Omega, \\ G(x, y) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\delta_y$ is the Dirac function. It has the following form

$$G(x, y) = S(x, y) - H(x, y), \quad (x, y) \in \Omega \times \Omega,$$

where $S(x, y) = \frac{1}{(N-2)\omega_n |x-y|^{N-2}}$ is the singular part and $H(x, y)$ is the regular part of $G(x, y)$, $\omega_N$ is the measure of the unit sphere of $\mathbb{R}^N$. We recall that $H$ is a smooth function in $\Omega \times \Omega$, $G$ and $H$ are symmetric in $x$ and $y$, and

$$0 < G(x, y) < S(x, y), \quad x, y \in \Omega.$$

For any $x \in \Omega$, we denote

$$R(x) = H(x, x), \tag{2.2}$$

which is called the Robin function. Then $R(x) > 0$ in $\Omega$.

**Lemma 2.1.** *([11]). If $\Omega \subset \mathbb{R}^N$ with $N \ge 3$ is a bounded convex domain, then $R(x)$ is strictly convex and it has a unique critical point which is a strict minimum.*

**Lemma 2.2.** *([24]). Let $\Omega \subset \mathbb{R}^N$ with $N \ge 3$ be a smooth bounded convex domain and let $k \ge 2$ be an integer. Set $\Delta = \{(z_1, \cdots, z_k) : \ z_i = z_j$ for some $i \ne j\}$. Then there does not exist $(z_1, \cdots, z_k) \in \Omega^k \setminus \Delta$ such that*

$$\nabla R(z_i)\Lambda_i^2 - 2 \sum_{j=1, j\ne i}^{k} \nabla G(z_i, z_j)\Lambda_i\Lambda_j = 0,$$

*for $i = 1, \cdots, k$.*

For any point $x_* \in \Omega$, let us define the following quadric forms

$$\begin{aligned} P(u, v) := &-r \int_{\partial B_r(x_*)} \langle \nabla u, \nu \rangle \langle \nabla v, \nu \rangle \, \mathrm{d}\sigma + \frac{r}{2} \int_{\partial B_r(x_*)} \langle \nabla u, \nabla v \rangle \, \mathrm{d}\sigma \\ &- \frac{N-2}{4} \int_{\partial B_r(x_*)} \langle \nabla u, \nu \rangle v + \langle \nabla v, \nu \rangle u \mathrm{d}\sigma, \end{aligned} \tag{2.3}$$

$$Q(u,v) := -\int_{\partial B_r(x_*)} \langle \nabla u, \nu \rangle \frac{\partial v}{\partial x_i} \mathrm{d}\sigma - \int_{\partial B_r(x_*)} \langle \nabla v, \nu \rangle \frac{\partial u}{\partial x_i} \mathrm{d}\sigma + \int_{\partial B_r(x_*)} \langle \nabla u, \nabla v \rangle \nu_i \mathrm{d}\sigma, \tag{2.4}$$

where $u, v \in \mathcal{C}^2(\bar{\Omega})$ and $r > 0$ is such that $B_{2r}(x_*) \subset \Omega$. Then we have the following computations about $P, Q$ and Green's function.

**Lemma 2.3.** *For $i, h = 1, \cdots, N$, we have*

$$P\left(G(y_*, x), G(z_*, x)\right) = \begin{cases} -\frac{N-2}{2} R(x_*), & \text{if } y_* = x_*, z_* = x_*, \\ \frac{N-2}{4} G(z_*, x_*) & \text{if } y_* = x_*, z_* \neq x_*, \\ \frac{N-2}{4} G(y_*, x_*) & \text{if } y_* \neq x_*, z_* = x_*, \\ 0 & \text{if } y_* \neq x_*, z_* \neq x_*, \end{cases} \tag{2.5}$$

$$P\left(G(y_*, x), \partial_h G(z_*, x)\right) = \begin{cases} -\frac{N-1}{4} \partial_h R(x_*), & \text{if } y_* = x_*, z_* = x_*, \\ \frac{N-2}{4} \partial_h G(z_*, x_*) & \text{if } y_* = x_*, z_* \neq x_*, \\ \frac{N}{4} \partial_h G(x_*, y_*) & \text{if } y_* \neq x_*, z_* = x_*, \\ 0 & \text{if } y_* \neq x_*, z_* \neq x_*, \end{cases} \tag{2.6}$$

$$Q\left(G(y_*, x), G(z_*, x)\right) = \begin{cases} -\partial_i R(x_*), & \text{if } y_* = x_*, z_* = x_*, \\ \partial_i G(x_*, z_*) & \text{if } y_* = x_*, z_* \neq x_*, \\ \partial_i G(x_*, y_*) & \text{if } y_* \neq x_*, z_* = x_*, \\ 0 & \text{if } y_* \neq x_*, z_* \neq x_*, \end{cases} \tag{2.7}$$

$$Q\left(G(y_*, x), \partial_h G(z_*, x)\right) = \begin{cases} -\frac{1}{2} \partial^2_{ih} R(x_*), & \text{if } y_* = x_*, z_* = x_*, \\ D_i \partial_h G(z_*, x_*) & \text{if } y_* = x_*, z_* \neq x_*, \\ \partial^2_{ih} G(x_*, y_*) & \text{if } y_* \neq x_*, z_* = x_*, \\ 0 & \text{if } y_* \neq x_*, z_* \neq x_*, \end{cases} \tag{2.8}$$

*Proof.* The proof can be found in [10, Section. 5]. □

## 2.2 Asymptotic behavior of blow-up solutions

For $x \in \mathbb{R}^N$ and $\lambda > 0$, $U_{x,\lambda}$ is the function

$$U_{x,\lambda}(y) = \frac{\lambda^{\frac{N-2}{2}}}{(1 + \lambda^2 |x-y|^2)^{\frac{N-2}{2}}} \quad \text{on } \mathbb{R}^N, \tag{2.9}$$

and $PU_{x,\lambda}$ denotes the projection of $U_{x,\lambda}$ onto $H_0^1(\Omega)$, i.e.,

$$\begin{cases} -\Delta PU_{x,\lambda} = -\Delta U_{x,\lambda} & \text{in } \Omega, \\ PU_{x,\lambda} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

Writing

$$\varphi_{x,\lambda} = U_{x,\lambda} - PU_{x,\lambda}, \tag{2.11}$$

the PDE method yields

**Lemma 2.4.** *Let $x \in \Omega$ and $\lambda > 0$. We have*
*(1)* $0 \leq \varphi_{x,\lambda} \leq U_{x,\lambda}$.
*(2)* $\varphi_{x,\lambda} = \frac{1}{\lambda^{(N-2)/2}} H(x, \cdot) + f_{x,\lambda}$, *where*

$$f_{x,\lambda} = O\left(\frac{1}{\lambda^{(N+2)/2}}\right), \quad \frac{\partial f_{x,\lambda}}{\partial \lambda} = O\left(\frac{1}{\lambda^{(N+4)/2}}\right) \quad \text{and} \quad \frac{\partial f_{x,\lambda}}{\partial x_i} = O\left(\frac{1}{\lambda^{(N+2)/2}}\right),$$

*as $\lambda \to +\infty$.*

*Proof.* The proof can be found in [33]. □

For solutions $u_\varepsilon$ of (1.1), we claim

**Lemma 2.5.** *For any $N \geq 3$ there exists $\varepsilon_0 > 0$ and $S > 0$ such that*

$$\|u_\varepsilon\| \leq S, \quad \textit{for any } 0 < \varepsilon < \varepsilon_0. \tag{2.12}$$

*Proof.* The proof is implied in [24], but for reader's convenience we sketch it. Suppose to the contrary, there exists a sequence $\varepsilon_n \to 0$ such that $\|u_{\varepsilon_n}\| \to +\infty$. Since

$$\|u_{\varepsilon_n}\| = \int_\Omega u_{\varepsilon_n}^{p+1-\varepsilon_n} \mathrm{d}x \to +\infty,$$

there must be $\|u_{\varepsilon_n}\|_{L^\infty(\Omega)} \to +\infty$. Denote

$$\mathcal{B} = \left\{ x \in \bar{\Omega} : \text{ there exists } x_{\varepsilon_n} \in \Omega \text{ such that } x_{\varepsilon_n} \to x \text{ and } u_{\varepsilon_n}(x_{\varepsilon_n}) \to +\infty \right\}.$$

We know that $\mathcal{B} \neq \emptyset$. Since $\Omega$ is smooth, the moving plane method (see [18, pp.137] or [19]) implies that

$$\operatorname{dist}(\mathcal{B}, \partial\Omega) \geq \delta, \quad \text{for some } \delta > 0.$$

Then using a result by Li Yanyan ([26]), it is possible to show that any $x \in \mathcal{B}$ is isolated and simple, and it implies that $\|u_{\varepsilon_n}\| \leq C$ for some positive constant $C$. Thus we have proved (2.12). □

Then by [36], we have up to a subsequence

$$u_{\varepsilon_n} = u_0 + \sum_{j=1}^{k} \alpha_{\varepsilon_n,j} PU_{x_{\varepsilon_n,j},\mu_{\varepsilon_n,j}^{-1}} + w_{\varepsilon_n}, \tag{2.13}$$

where $u_0$ is either 0 or a solution of

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

$w_{\varepsilon_n}$ goes to 0 in $H_0^1(\Omega)$ and $k \in \mathbb{N}$. Moreover if $k \geq 1$, it holds

$$\alpha_{\varepsilon_n,j} \in \mathbb{R}, \quad \alpha_{\varepsilon_n,j} \to \alpha_0 = [N(N-2)]^{\frac{N-2}{4}},$$

$$x_{\varepsilon_n,j} \in \Omega, \quad x_{\varepsilon_n,j} \to a_j \in \bar{\Omega},$$

and $\mu_{\varepsilon_n,j} > 0$ with

$$\frac{1}{\mu_{\varepsilon_n,j}} dist(x_{\varepsilon_n,j}, \partial\Omega) \to +\infty,$$

$$\frac{\mu_{\varepsilon_n,j}}{\mu_{\varepsilon_n,i}} + \frac{\mu_{\varepsilon_n,i}}{\mu_{\varepsilon_n,j}} + \frac{1}{\mu_{\varepsilon_n,i}} \frac{1}{\mu_{\varepsilon_n,j}} |x_{\varepsilon_n,i} - x_{\varepsilon_n,j}|^2 \to +\infty, \quad i \neq j.$$

Besides these results we have the estimation

$$\|u_{\varepsilon_n}\|^2 = \|u_0\|^2 + k\mathcal{S}_N^{\frac{N}{2}} + o(1).$$

As mentioned in [3], Schoen ([35]) proved that

$$\text{either} \quad u_0 \equiv 0,\ k > 0 \quad \text{or} \quad u_0 \not\equiv 0,\ k = 0.$$

If we assume (2.14) has no solutions, then $u_{\varepsilon_n}$ must satisfy

$$u_{\varepsilon_n}=\sum_{j=1}^{k}\alpha_{\varepsilon_n,j}PU_{x_{\varepsilon_n,j},\mu_{\varepsilon_n,j}^{-1}}+w_{\varepsilon_n}, \tag{2.15}$$

which implies

$$|\nabla u_{\varepsilon_n}|^2\rightharpoonup \mathcal{S}_N^{N/2}\sum_{j=1}^{k}\delta_{a_j},\quad \text{as } \varepsilon_n\to 0,$$

for some $a_1,\cdots,a_k\in\Omega$ with $1\le k\le\frac{S}{\mathcal{S}^{N/2}}$. That is, $u_{\varepsilon_n}$ will blow-up at $k$ points. Moreover, in the blow-up case, one can get more precisions about the parameters: using the moving plane method, one can get

$$dist(x_{\varepsilon_n,j},\partial\Omega)\ge\delta,\quad \text{for some } \delta>0;$$

it follows from the results of Schoen ([35]) that there exist $\delta'>0$ and $c_0>0$ such that for $\varepsilon_n$ small enough

$$|x_{\varepsilon_n,i}-x_{\varepsilon_n,j}|\ge\delta',\quad \frac{\mu_{\varepsilon_n,j}}{\mu_{\varepsilon_n,i}}\le c_0,\quad \forall i\ne j;$$

lastly, it can be proved that (see [3])

$$\varepsilon_n\log\mu_{\varepsilon_n,j}\to 0\quad \text{as } \varepsilon_n\to 0,\quad \forall j.$$

Specially, for one-peak solutions, i.e., $k=1$, the asymptotic behavior has been done in [25, 32].

**Theorem 2.6.** *([25, 32]). For $N\ge 3$ and $x_0\in\Omega$, let $\varepsilon_n\to 0$ and $u_{\varepsilon_n}$ be a solution of (1.1) with $\varepsilon=\varepsilon_n$. Suppose $u_{\varepsilon_n}$ satisfies the assumption (1.3), then (for simplicity we denote $u_\varepsilon=u_{\varepsilon_n}$)*
*(1) $x_0$ is a critical point of $R(x)$,*
*(2) if we write*

$$u_\varepsilon=\alpha_\varepsilon PU_{x_\varepsilon,\mu_\varepsilon^{-1}}+w_\varepsilon, \tag{2.16}$$

*then*

$$\begin{cases}\mu_\varepsilon\varepsilon^{-\frac{1}{N-2}}\to c\in(0,+\infty),\\ \mu_\varepsilon^{\varepsilon}=1+O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|),\\ x_\varepsilon\in\Omega,\quad x_\varepsilon\to x_0,\\ \alpha_\varepsilon=\alpha_0+O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|),\\ \|w_\varepsilon\|_\infty=O(\mu_\varepsilon^{\frac{N-2}{2}}),\end{cases} \tag{2.17}$$

*as $\varepsilon\to 0$.*

**Lemma 2.7.** *Under the assumptions of Theorem 2.6, we have*

$$|u_\varepsilon|\le CU_{x_\varepsilon,\mu_\varepsilon^{-1}},\quad \text{in } \Omega. \tag{2.18}$$

*Proof.* See Lemma 3 in [25]. □

Also the asymptotic behavior of multi-peak solutions has been done in [3] (with $N\ge 4$) and [34] (with $N=3$). Define the constants

$$A=\int_{\mathbb{R}^N}U_{0,1}^{p+1},\quad B=\int_{\mathbb{R}^N}U_{0,1}^{p}. \tag{2.19}$$

Let $\Phi_k(\vec{x},\vec{\lambda}):\Omega^k\times(\mathbb{R}^+)^k\to\mathbb{R}$ be defined by

$$\Phi_k(\vec{x},\vec{\lambda})=\alpha_0^pB\vec{\lambda}^{\frac{N-2}{2}}M_k(\vec{x})(\vec{\lambda}^{\frac{N-2}{2}})^T-(N-2)\sum_{j=1}^{k}\log\lambda_j, \tag{2.20}$$

where $\vec{\lambda}^{\frac{N-2}{2}} = (\lambda_1^{\frac{N-2}{2}}, \cdots, \lambda_k^{\frac{N-2}{2}})$, $\vec{x} = (x_1, \cdots, x_k)$ with $x_j \in \Omega$, and the matrix $M_k(\vec{x}) = (m_{ij}(\vec{x}))_{k\times k}$ is defined by

$$m_{ii}(\vec{x}) = R(x_i), \quad m_{ij}(\vec{x}) = -G(x_i, x_j) \quad \text{for } i \neq j, i, j = 1, \cdots, k. \tag{2.21}$$

We see that if $M_k(\vec{a})$ is positive definite, then $F_{\vec{a}}(\vec{\lambda}) := \Phi_k(\vec{a}, \vec{\lambda})$ is strictly convex on $(\mathbb{R}^+)^k$, and $F_{\vec{a}}$ is infinity on the boundary. Thus there is a unique critical point $\vec{\lambda}(\vec{a})$ of $F_{\vec{a}}$, i.e.,

$$\nabla_\lambda \Phi_k(\vec{a}, \vec{\lambda}(\vec{a})) = 0. \tag{2.22}$$

For any $x \in \Omega$ and $\lambda > 0$, we define

$$E_{x,\lambda} = \left\{ u \in H_0^1(\Omega) : \langle PU_{x,\lambda}, u\rangle = \left\langle \frac{\partial PU_{x,\lambda}}{\partial \lambda}, u \right\rangle = \left\langle \frac{\partial PU_{x,\lambda}}{\partial x_i}, u \right\rangle = 0, \ \forall\, i = 1, \cdots, N \right\}.$$

**Theorem 2.8.** *([3, 34]). For $N \geq 3$ and different $k$ points $a_1, \cdots, a_k \in \Omega$, let $\varepsilon_n \to 0$ and $u_\varepsilon$ be a sequence of solutions to* (1.1) *with $\varepsilon = \varepsilon_n$. Suppose $u_{\varepsilon_n}$ satisfies the assumption* (1.2)*, then (for simplicity we denote $u_\varepsilon = u_{\varepsilon_n}$)*
*(1) the matrix $M_k(\vec{a})$ is non-negative definite with $\vec{a} = (a_1, \cdots, a_k)$. If $M_k(\vec{a})$ is positive definite, then $\nabla_x \Phi_k(\vec{a}, \vec{\lambda}(\vec{a})) = 0$, i.e., $(\vec{a}, \vec{\lambda}(\vec{a}))$ is a critical point of $\Phi_k(\vec{a}, \vec{\lambda})$.*
*(2) if $M_k(\vec{a})$ is positive, then there holds*

$$u_\varepsilon = \sum_{j=1}^{k} \alpha_{\varepsilon,j} PU_{x_{\varepsilon,j}, \mu_{\varepsilon,j}^{-1}} + w_\varepsilon, \tag{2.23}$$

*with*

$$\begin{cases} \mu_{\varepsilon,j}\varepsilon^{-\frac{1}{N-2}} \to c_j \in (0, +\infty), \\ \mu_{\varepsilon,j}^{\varepsilon} = 1 + O(\bar{\mu}_\varepsilon^{N-2}|\log \bar{\mu}_\varepsilon|), \\ x_{\varepsilon,j} \in \Omega, \quad x_{\varepsilon,j} \to a_j, \\ \alpha_{\varepsilon,j} = \alpha_0 + O(\bar{\mu}_\varepsilon^{N-2}|\log \bar{\mu}_\varepsilon|), \\ w_\varepsilon \in \bigcap_{j=1}^{k} E_{x_{\varepsilon,j}, \mu_{\varepsilon,j}^{-1}}, \quad \|w_\varepsilon\|_\infty = O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}}). \end{cases} \tag{2.24}$$

*and*

$$\|w_\varepsilon\| = \begin{cases} O(\bar{\mu}_\varepsilon^{N-2}) & \text{if } N \leq 5, \\ O(\bar{\mu}_\varepsilon^{N-2}|\log \mu_\varepsilon|^{2/3}) & \text{if } N = 6, \\ O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}}) & \text{if } N \geq 7, \end{cases} \tag{2.25}$$

*as $\varepsilon \to 0$, where $\bar{\mu}_\varepsilon = \max\{\mu_{\varepsilon,1}, \cdots, \mu_{\varepsilon,k}\}$.*

Note that when $M_k(\vec{a})$ is positive definite, it holds $\nabla_x \Phi_k(\vec{a}, \vec{\lambda}(\vec{a})) = 0$ with $\vec{\lambda} = \vec{\lambda}(\vec{a})$, which implies that

$$\nabla R(a_i)\lambda_i^{N-2} - 2 \sum_{j=1, j\neq i}^{k} \nabla G(a_i, a_j) \lambda_i^{\frac{N-2}{2}} \lambda_j^{\frac{N-2}{2}} = 0,$$

for $i = 1, \cdots, k$. So Lemma 2.2 applies. We claim that in general, we can apply Lemma 2.2.

**Lemma 2.9.** *Let $N \geq 3$ and $u_\varepsilon$ be a solution satisfying* (1.2)*. There holds*

$$\nabla R(a_i)\Lambda_i^2 - 2 \sum_{j=1, j\neq i}^{k} \nabla G(a_i, a_j) \Lambda_i \Lambda_j = 0, \tag{2.26}$$

*for $i = 1, \cdots, k$ and some $\Lambda_i > 0$.*

*Proof.* It is enough to prove (2.26) for $i = 1$. Applying (2.32) with $x_* = a_1$ and $u = u_\varepsilon$, we obtain

$$Q(u_\varepsilon, u_\varepsilon) = \frac{2}{2^* - \varepsilon} \int_{\partial B_r(a_1)} u_\varepsilon^{2^*-\varepsilon} \nu_i \mathrm{d}\sigma, \tag{2.27}$$

for small $r > 0$. Using (2.15), we can proceed as in Section 2 of [12] to prove that

$$u_\varepsilon = \sum_{j=1}^{k} C_j \mu_{\varepsilon,j}^{\frac{N-2}{2}} G(x, x_j) + o(\bar{\mu}_\varepsilon^{\frac{N-2}{2}}), \quad \text{in } \mathcal{C}^1(\Omega \setminus \cup_{j=1}^k B_d(a_j)),$$

for any $d > 0$ small and $C_j > 0$, where $\bar{\mu}_\varepsilon = \max\{\mu_{\varepsilon,1}, \cdots, \mu_{\varepsilon,k}\}$. We have $c_0^{-1} \le \frac{\mu_{\varepsilon,j}}{\bar{\mu}_\varepsilon} \le c_0$ for small $\varepsilon$. Then

$$\bar{\mu}_\varepsilon^{-(N-2)} \int_{\partial B_r(a_1)} u_\varepsilon^{2^*-\varepsilon} \nu_i \mathrm{d}\sigma = O(\bar{\mu}_\varepsilon^2),$$

and

$$\begin{aligned}\bar{\mu}_\varepsilon^{-(N-2)} Q(u_\varepsilon, u_\varepsilon) &= -2 \int_{\partial B_r(a_1)} \left\langle \bar{\mu}_\varepsilon^{-\frac{N-2}{2}} u_\varepsilon, \nu \right\rangle \partial_i (\bar{\mu}_\varepsilon^{-\frac{N-2}{2}} u_\varepsilon) \mathrm{d}\sigma + \int_{\partial B_r(a_1)} |\nabla(\bar{\mu}_\varepsilon^{-\frac{N-2}{2}} u_\varepsilon)|^2 \nu_i \mathrm{d}\sigma \\ &\to \int_{\partial B_r(a_1)} |\nabla f|^2 \nu_i - 2 \langle \nabla f, \nu \rangle \partial_i f \mathrm{d}\sigma, \quad \text{as } \varepsilon \to 0,\end{aligned}$$

with

$$f(x) = \sum_{j=1}^{k} \bar{C}_j G(x, a_j),$$

for some $\bar{C}_j > 0$. Thus, (2.27) implies, for any $r > 0$ small, that

$$\int_{\partial B_r(a_1)} |\nabla f|^2 \nu_i - 2 \langle \nabla f, \nu \rangle \partial_i f \mathrm{d}\sigma = 0. \tag{2.28}$$

Denote

$$g(x) = \bar{C}_1 H(x, a_1) + \sum_{j \ge 2}^{k} \bar{C}_j G(x, a_j),$$

then $g \in \mathcal{C}^\infty(B_r(a_1))$ and $f(x) = \bar{C}_1 S(x, a_1) - g(x)$. By direct computation, we obtain that for $x \in \partial B_r(a_1)$,

$$|\nabla f|^2 \nu_i - 2 \langle \nabla f, \nu \rangle \partial_i f = \frac{2\bar{C}_1}{(N-2)\omega_n r^{N-1}} \partial_i g - \frac{\bar{C}_1^2}{(N-2)^2 \omega_N^2 r^{2N-1}} (x - a_1)_i + O(1).$$

It follows from (2.28) that

$$\frac{2\bar{C}_1}{N-2} \partial_i g(\xi) + O(r^{N-1}) = 0,$$

where $\xi \to a_1$ as $\varepsilon \to 0$. Then letting $\varepsilon \to 0$, we obtain $\partial_i g(a_1) = 0$, i.e.,

$$\bar{C}_1^2 \nabla R(a_1) - 2 \sum_{j \ge 2}^{k} \bar{C}_1 \bar{C}_j G(a_1, a_j) = 0,$$

which finish the proof. □

**Lemma 2.10.** *Under the assumptions of Theorem 2.8, we have*

$$|u_\varepsilon| \le C \sum_{j=1}^{k} U_{x_{\varepsilon,j}, \mu_{\varepsilon,j}^{-1}}, \quad \text{in } \Omega. \tag{2.29}$$

*Proof.* See Appendix A in [12]. □

## 2.3 Other results

The next lemma is a well known characterization of the kenel of the linearized equation. We refer to [7] for a proof.

**Lemma 2.11.** *Let $U_{0,1}$ be defined by* (2.9) *and $v$ be a solution of the problem*

$$\begin{cases} -\Delta v = N(N+2)U_{0,1}^{p-1} v & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\nabla v|^2 < \infty. \end{cases} \tag{2.30}$$

*Then there exists $a_i \in \mathbb{R}$, $i = 0, 1, \cdots, N$ such that*

$$v(x) = a_0 \frac{1-|x|^2}{(1+|x|^2)^{N/2}} + \sum_{i=1}^{N} a_i \frac{x_i}{(1+|x|^2)^{N/2}}.$$

We give the Pohozaev identities.

**Lemma 2.12.** *Let $u$ be a solution of* (1.1), *$x_* \in \Omega$ and $r > 0$ is such that $B_{2r}(x_*) \subset \Omega$, then*

$$P(u,u) = \frac{r}{2^*-\varepsilon} \int_{\partial B_r(x_*)} u^{2^*-\varepsilon} \mathrm{d}\sigma - \frac{N\varepsilon}{2^*(2^*-\varepsilon)} \int_{B_r(x_*)} u^{2^*-\varepsilon} \mathrm{d}x, \tag{2.31}$$

$$Q(u,u) = \frac{2}{2^*-\varepsilon} \int_{\partial B_r(x_*)} u^{2^*-\varepsilon} \nu_i \mathrm{d}\sigma. \tag{2.32}$$

*Proof.* Multiplying $\partial_i u$ on (1.1), integrating over $\Omega' \subset \Omega$ and applying the divergence theorem and Green's indentity:

$$-\int_{\partial\Omega'} \langle \nabla u, \nu \rangle \partial_i u \mathrm{d}\sigma + \frac{1}{2} \int_{\partial\Omega'} |\nabla u|^2 \nu_i \mathrm{d}\sigma = \frac{1}{2^*-\varepsilon} \int_{\partial\Omega'} u^{2^*-\varepsilon} \nu_i \mathrm{d}\sigma.$$

Then taking $\Omega' = B_{2r}(x_*)$, we obtain (2.32).

Multiplying $(x-y)\cdot\nabla u$ on (1.1), integrating over $\Omega' \subset \Omega$ and applying the divergence theorem and Green's indentity:

$$\begin{aligned} &-\int_{\partial\Omega'} \langle \nabla u, \nu \rangle \langle \nabla u, x-y \rangle \mathrm{d}\sigma + \frac{1}{2} \int_{\partial\Omega'} \langle x-y, \nu \rangle |\nabla u|^2 \mathrm{d}\sigma \\ =& \frac{1}{2^*-\varepsilon} \int_{\partial\Omega'} \langle x-y, \nu \rangle^{2^*-\varepsilon} \mathrm{d}\sigma + \int_{\Omega'} \frac{N-2}{2} |\nabla u|^2 - \frac{N}{2^*-\varepsilon} u^{2^*-\varepsilon} \mathrm{d}x. \end{aligned} \tag{2.33}$$

Also multiplying $u$ on (1.1), integrating over $\Omega' \subset \Omega$, we obtain

$$\int_{\Omega'} |\nabla u|^2 \mathrm{d}x = \int_{\Omega'} u^{2^*-\varepsilon} \mathrm{d}x + \int_{\partial\Omega'} \langle \nabla u, \nu \rangle u \mathrm{d}\sigma. \tag{2.34}$$

Combining (2.33) with (2.34), we have

$$\begin{aligned} &-\int_{\partial\Omega'} \langle \nabla u, \nu \rangle \langle \nabla u, x-y \rangle \mathrm{d}\sigma + \frac{1}{2} \int_{\partial\Omega'} \langle x-y, \nu \rangle |\nabla u|^2 \mathrm{d}\sigma - \frac{N-2}{2} \int_{\partial\Omega'} \langle \nabla u, \nu \rangle u \mathrm{d}\sigma \\ =& \frac{1}{2^*-\varepsilon} \int_{\partial\Omega'} \langle x-y, \nu \rangle u^{2^*-\varepsilon} \mathrm{d}\sigma - \frac{N\varepsilon}{2^*(2^*-\varepsilon)} \int_{\Omega'} u^{2^*-\varepsilon} \mathrm{d}x. \end{aligned}$$

Then taking $\Omega' = B_{2r}(x_*)$ and $y = x_*$, we obtain (2.31). ∎

**Lemma 2.13.** *Let $u$ be a solution of* (1.1) *and $v$ be a solution of* (1.5)*, $x_* \in \Omega$ and $r > 0$ be such that $B_{2r}(x_*) \subset \Omega$, then*

$$P(u,v) = \frac{r}{2}\int_{\partial B_r(x_*)} u^{p-\varepsilon} v \mathrm{d}\sigma - \frac{(N-2)\varepsilon}{4}\int_{B_r(x_*)} u^{p-\varepsilon} v \mathrm{d}x, \tag{2.35}$$

$$Q(u,v) = \int_{\partial B_r(x_*)} u^{p-\varepsilon} v \nu_i \mathrm{d}\sigma. \tag{2.36}$$

*Proof.* Multiplying $\partial_i u$ on (1.5) and multiplying $\partial_i v$ on (1.1), adding them together, integrating over $\Omega' \subset \Omega$ and applying the divergence theorem and Green's indentity:

$$-\int_{\partial\Omega'} \langle \nabla u, \nu\rangle \partial_i v + \langle \nabla v, \nu\rangle \partial_i u \mathrm{d}\sigma + \int_{\partial\Omega'} \langle \nabla u, \nabla v\rangle \nu_i \mathrm{d}\sigma = \int_{\partial\Omega'} u^{p-\varepsilon} v \nu_i \mathrm{d}\sigma.$$

Then taking $\Omega' = B_{2r}(x_*)$, we obtain (2.36).

Multiplying $(x-y)\cdot\nabla u$ on (1.5) and multiplying $(x-y)\cdot\nabla v$ on (1.1), adding them together, integrating over $\Omega' \subset \Omega$ and applying the divergence theorem and Green's indentity:

$$\begin{aligned} &-\int_{\partial\Omega'} \langle \nabla u, \nu\rangle \langle \nabla v, x-y\rangle + \langle \nabla v, \nu\rangle \langle \nabla u, x-y\rangle \mathrm{d}\sigma + \int_{\partial\Omega'} \langle x-y, \nu\rangle \langle \nabla u, \nabla v\rangle \mathrm{d}\sigma \\ =& \int_{\partial\Omega'} \langle x-y, \nu\rangle u^{p-\varepsilon} v \mathrm{d}\sigma + \int_{\Omega'} (N-2)\langle \nabla u, \nabla v\rangle - N u^{p-\varepsilon} v \mathrm{d}x. \end{aligned} \tag{2.37}$$

Also multiplying $u$ on (1.5) and multiplying $v$ on (1.1), adding them together, integrating over $\Omega' \subset \Omega$, we obtain

$$\int_{\Omega'} \langle \nabla u, \nabla v\rangle \mathrm{d}x = \int_{\Omega'} u^{p-\varepsilon} v \mathrm{d}x + \int_{\partial\Omega'} \langle \nabla u, \nu\rangle v \mathrm{d}\sigma, \tag{2.38}$$

and

$$\int_{\Omega'} \langle \nabla u, \nabla v\rangle \mathrm{d}x = (p-\varepsilon)\int_{\Omega'} u^{p-\varepsilon} v \mathrm{d}x + \int_{\partial\Omega'} \langle \nabla v, \nu\rangle u \mathrm{d}\sigma. \tag{2.39}$$

Combining (2.37) with (2.38), (2.39), we have

$$\begin{aligned} &-\int_{\partial\Omega'} \langle \nabla u, \nu\rangle \langle \nabla v, x-y\rangle + \langle \nabla v, \nu\rangle \langle \nabla u, x-y\rangle \mathrm{d}\sigma + \int_{\partial\Omega'} \langle x-y, \nu\rangle \langle \nabla u, \nabla v\rangle \mathrm{d}\sigma \\ =& \int_{\partial\Omega'} \langle x-y, \nu\rangle u^{p-\varepsilon} v \mathrm{d}\sigma + \frac{N-2}{2}\int_{\partial\Omega'} \langle \nabla u, \nu\rangle v + \langle \nabla v, \nu\rangle u \mathrm{d}\sigma - \frac{(N-2)\varepsilon}{2}\int_{\Omega'} u^{p-\varepsilon} v \mathrm{d}x. \end{aligned}$$

Then taking $\Omega' = B_{2r}(x_*)$ and $y = x_*$, we obtain (2.35). □

By direct computations, we have

**Lemma 2.14.** *For any $q > 1$, we have*

$$(a+b)^q = a^q + O(a^{q-1}b + b^q),$$

$$(a+b)^q = a^q + qa^{q-1}b + O(b^q + a^{q-q^*}b^{q^*}),$$

*where $q^* = \min\{2, q\}$.*

**Lemma 2.15.** *Let $\Phi_k$ be defined by* (2.20)*.*
*(1) Denote $\vec{x} = (x_1, \cdots, x_k) = (y_1, y_2, \cdots, y_{kN})$ with $x_j \in \Omega$ and $y_i \in \mathbb{R}$, then for $i \in [(j-1)N+1, jN]$ for some $1 \le j \le k$, we have*

$$\partial_{y_i}\Phi_k(\vec{x}, \vec{\lambda}) = \alpha^p B\left(\lambda_j^{N-2}\partial_{y_i}R(x_j) - 2\sum_{l\neq j, l=1}^{k} \lambda_j^{\frac{N-2}{2}}\lambda_l^{\frac{N-2}{2}}\partial_{y_i}G(x_j, x_l)\right). \tag{2.40}$$

*(2) Denote $\vec{\lambda} = (\lambda_1, \cdots, \lambda_k)$, then for $1 \le j \le k$,*

$$\partial_{\lambda_j}\Phi_k(\vec{x}, \vec{\lambda}) = \frac{(N-2)\alpha_0^p B}{\lambda_j}\left(\lambda_j^{N-2}R(x_j) - \sum_{l\neq j, l=1}^{k} \lambda_j^{\frac{N-2}{2}}\lambda_l^{\frac{N-2}{2}}G(x_j, x_l) - \frac{1}{\alpha_0^p B}\right) \tag{2.41}$$

*(3) For $i \in [(j-1)N+1, jN]$ for some $1 \le j \le k$, we have that if $s \in [(j-1)N+1, jN]$,*

$$\partial^2_{y_i y_s}\Phi_k(\vec{x}, \vec{\lambda}) = \alpha_0^p B\left(\lambda_j^{N-2}\partial_{y_i y_s}R(x_j) - 2\sum_{l\neq j, l=1}^{k} \lambda_j^{\frac{N-2}{2}}\lambda_l^{\frac{N-2}{2}}\partial^2_{y_i y_s}G(x_j, x_l)\right); \tag{2.42}$$

*while if $s \in [(t-1)N+1, tN]$ for some $t \neq j$,*

$$\partial^2_{y_i y_s}\Phi_k(\vec{x}, \vec{\lambda}) = -2\alpha_0^p B\lambda_j^{\frac{N-2}{2}}\lambda_t^{\frac{N-2}{2}}\partial^2_{y_i y_s}G(x_j, x_t). \tag{2.43}$$

*(3) For $1 \le j \le k$, $1 \le i \le kN$, we have that if $i \in [(j-1)N+1, jN]$,*

$$\partial^2_{y_i \lambda_j}\Phi_k(\vec{x}, \vec{\lambda}) = (N-2)\lambda_0^p B\lambda_j^{\frac{N-4}{2}}\left(\lambda_j^{\frac{N-2}{2}}\partial_{y_i}R(x_j) - \sum_{l\neq j, l=1}^{k} \lambda_l^{\frac{N-2}{2}}\partial_{y_i}G(x_j, x_l)\right); \tag{2.44}$$

*while if $i \in [(t-1)N+1, tN]$ for some $t \neq j$,*

$$\partial^2_{y_i \lambda_j}\Phi_k(\vec{x}, \vec{\lambda}) = -(N-2)\lambda_0^p B\lambda_j^{\frac{N-4}{2}}\lambda_t^{\frac{N-2}{2}}\partial_{y_i}G(x_t, x_j). \tag{2.45}$$

# 3 One-peak solutions

In this section, we assume that $N \ge 3$ and $x_0 \in \Omega$.

## 3.1 Sharper estimations of one-peak solutions

We obtain some improtant estimations for solutions of (1.1) satisfying (1.3). We start with the following proposition.

**Proposition 3.1.** *Let $u_\varepsilon$ be a solution of* (1.1) *satisfying* (1.3)*, then for any small $r > 0$, it holds*

$$u_\varepsilon(x) = C_\varepsilon G(x_\varepsilon, x) + \begin{cases} O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N = 3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N \ge 4, \end{cases} \quad \text{in } \mathcal{C}^1(\Omega \setminus B_{2r}(x_\varepsilon)), \tag{3.1}$$

*where*

$$C_\varepsilon = \int_{B_r(x_\varepsilon)} u_\varepsilon^{p-\varepsilon}\mathrm{d}x = \alpha_0^p B\mu_\varepsilon^{\frac{N-2}{2}} + \begin{cases} O(\mu_\varepsilon^{\frac{3}{2}}|\log\mu_\varepsilon|), & N = 3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N = 4, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N \ge 5. \end{cases} \tag{3.2}$$

*Proof.* For $x \in \Omega \setminus B_{2r}(x_\varepsilon)$, we have

$$u_\varepsilon(x) = \int_\Omega G(x,y)u_\varepsilon^{p-\varepsilon}(y)\mathrm{d}y = \int_{\Omega\setminus B_r(x_\varepsilon)} G(x,y)u_\varepsilon^{p-\varepsilon}(y)\mathrm{d}y + \int_{B_r(x_\varepsilon)} G(x,y)u_\varepsilon^{p-\varepsilon}(y)\mathrm{d}y. \tag{3.3}$$

From Lemma 2.7, we see that $u_\varepsilon(x) = O(\mu_\varepsilon^{\frac{N-2}{2}})$ for $x \in \Omega \setminus B_r(x_\varepsilon)$. Hence

$$\int_{\Omega\setminus B_r(x_\varepsilon)} G(x,y)u_\varepsilon^{p-\varepsilon}(y)\mathrm{d}y = O(\mu_\varepsilon^{\frac{N+2}{2}})\int_{\Omega\setminus B_r(x_\varepsilon)} G(x,y)\mathrm{d}y = O(\mu_\varepsilon^{\frac{N+2}{2}}). \tag{3.4}$$

And by Taylor's expansion, we know

$$\begin{aligned}\int_{B_r(x_\varepsilon)} G(x,y)u_\varepsilon^{p-\varepsilon}(y)\mathrm{d}y = G(x,x_\varepsilon)\int_{B_r(x_\varepsilon)} u_\varepsilon^{p-\varepsilon}\mathrm{d}y + \sum_{i=1}^N \partial_i G(x,x_\varepsilon)\int_{B_r(x_\varepsilon)}(y_i - x_{\varepsilon,i})u_\varepsilon^{p-\varepsilon}\mathrm{d}y \\ + \sum_{i,j=1}^N \partial_{ij}^2 G(x,x_\varepsilon)\int_{B_r(x_\varepsilon)}(y_i - x_{\varepsilon,i})(y_j - x_{\varepsilon,j})u_\varepsilon^{p-\varepsilon}\mathrm{d}y \\ + O\left(\int_{B_r(x_\varepsilon)} |y-x_\varepsilon|^3 u_\varepsilon^{p-\varepsilon}\mathrm{d}y\right).\end{aligned} \tag{3.5}$$

We give one-by-one estimates of every term in the above equality in Lemma A.2, then

$$\begin{aligned}\int_{B_r(x_\varepsilon)} G(x,y)u_\varepsilon^{p-\varepsilon}(y)\mathrm{d}y &= C_\varepsilon G(x,x_\varepsilon) + \Delta G(x,x_\varepsilon)\mu_\varepsilon^{\frac{N+2}{2}}\frac{1}{N}\int_{|y|\le\mu_\varepsilon^{-1}r}\frac{|y|^2}{(1+|y|^2)^{\frac{N+2}{2}}}\mathrm{d}y \\ &\quad + \begin{cases} O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N=3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\ge 4, \end{cases} \\ &= C_\varepsilon \mu_\varepsilon^{\frac{N-2}{2}} G(x,x_\varepsilon) + \begin{cases} O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N=3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\ge 4, \end{cases}\end{aligned} \tag{3.6}$$

where

$$C_\varepsilon = \int_{B_r(x_\varepsilon)} u_\varepsilon^{p-\varepsilon}\mathrm{d}x = \alpha_0^p B\mu_\varepsilon^{\frac{N-2}{2}} + \begin{cases} O(\mu_\varepsilon^{\frac{3}{2}}|\log\mu_\varepsilon|), & N=3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N=4, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\ge 5. \end{cases}$$

Then (3.3) and (3.6) imply

$$u_\varepsilon(x) = C_\varepsilon G(x_\varepsilon, x) + \begin{cases} O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N=3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\ge 4, \end{cases} \quad \text{in } \Omega\setminus B_{2r}(x_\varepsilon).$$

On the other hand, for $x \in \Omega \setminus B_{2r}(x_\varepsilon)$, we have

$$\partial_h u_\varepsilon(x) = \int_\Omega \partial_h G(x,y)u_\varepsilon^{p-\varepsilon}(y)\mathrm{d}y = \int_{B_r(x_\varepsilon)} \partial_h G(x,y)u_\varepsilon^{p-\varepsilon}(y)\mathrm{d}y + O(\mu_\varepsilon^{\frac{N+2}{2}}). \tag{3.7}$$

Similar to the above estimates, for $x \in \Omega \setminus B_{2r}(x_\varepsilon)$, we can prove

$$\int_{B_r(x_\varepsilon)} \partial_h G(x,y)u_\varepsilon^{p-\varepsilon}(y)\mathrm{d}y = C_\varepsilon \partial_h G(x,x_\varepsilon) + \begin{cases} O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N=3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\ge 4. \end{cases} \tag{3.8}$$

Then (3.7) and (3.8) imply

$$\partial_h u_\varepsilon(x) = C_\varepsilon \partial_h G(x,x_\varepsilon) + \begin{cases} O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N=3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\ge 4, \end{cases} \quad \text{in } \Omega\setminus B_{2r}(x_\varepsilon). \tag{3.9}$$

□

**Proposition 3.2.** *Let $u_\varepsilon$ be a solution of* (1.1) *satisfying* (1.3). *If $x_0$ is a nondegenerate critical point of $R(x)$, then*

$$|x_\varepsilon - x_0| = \begin{cases} O(\mu_\varepsilon^2 |\log \mu_\varepsilon|), & N = 3, \\ O(\mu_\varepsilon^2), & N \geq 4, \end{cases} \tag{3.10}$$

*and*

$$|\mu_\varepsilon - \mu_0 \varepsilon^{\frac{1}{N-2}}| = \begin{cases} O(\mu_\varepsilon^2 |\log \mu_\varepsilon|), & N = 3, \\ O(\mu_\varepsilon^3 |\log \mu_\varepsilon|), & N = 4, \\ O(\mu_\varepsilon^3), & N \geq 5, \end{cases} \tag{3.11}$$

*where* $\mu_0 = \left( \frac{A}{2N^2 B R(x_0)} \right)^{\frac{1}{N-2}}$.

*Proof.* Applying (2.32) with $x_* = x_\varepsilon$ and $u = u_\varepsilon$, we obtain

$$Q(u_\varepsilon, u_\varepsilon) = \frac{2}{2^* - \varepsilon} \int_{\partial B_r(x_\varepsilon)} u_\varepsilon^{2^* - \varepsilon} \nu_i \mathrm{d}\sigma, \tag{3.12}$$

Using the expansion of $u_\varepsilon$ in Proposition 3.1, we have

$$Q(u_\varepsilon, u_\varepsilon) = C_\varepsilon^2 Q\big(G(x_\varepsilon, x), G(x_\varepsilon, x)\big) + \begin{cases} O(\mu_\varepsilon^N |\log \mu_\varepsilon|), & N = 3, \\ O(\mu_\varepsilon^N), & N \geq 4. \end{cases}$$

Since $u_\varepsilon = O(\mu_\varepsilon^{\frac{N-2}{2}})$ in $\Omega \setminus B_r(x_\varepsilon)$, we have

$$\int_{\partial B_r(x_\varepsilon)} u_\varepsilon^{2^* - \varepsilon} \nu_i \mathrm{d}\sigma = O(\mu_\varepsilon^N).$$

It follows from Lemma 2.3 and (3.12) that

$$\nabla R(x_\varepsilon) = \begin{cases} O(\mu_\varepsilon^2 |\log \mu_\varepsilon|), & N = 3, \\ O(\mu_\varepsilon^2), & N \geq 4. \end{cases} \tag{3.13}$$

Since $\nabla R(x_0) = 0$ and $\nabla^2 R(x_0)$ is nondegenerate, we have

$$\nabla R(x_\varepsilon) = \nabla^2 R(x_0)(x_\varepsilon - x_0) + o(|x_\varepsilon - x_0|)$$

which yields

$$|x_\varepsilon - x_0| = \begin{cases} O(\mu_\varepsilon^2 |\log \mu_\varepsilon|), & N = 3, \\ O(\mu_\varepsilon^2), & N \geq 4. \end{cases} \tag{3.14}$$

On the other hand, applying (2.31) with $x_* = x_\varepsilon$ and $u = u_\varepsilon$, we also obtain

$$P(u_\varepsilon, u_\varepsilon) = \frac{r}{2^* - \varepsilon} \int_{\partial B_r(x_\varepsilon)} u_\varepsilon^{2^* - \varepsilon} \mathrm{d}\sigma - \frac{N\varepsilon}{2^*(2^* - \varepsilon)} \int_{B_r(x_\varepsilon)} u_\varepsilon^{2^* - \varepsilon} \mathrm{d}x. \tag{3.15}$$

Using again the expansion of $u_\varepsilon$ in Proposition 3.1, we have

$$P(u_\varepsilon, u_\varepsilon) = C_\varepsilon^2 P\big(G(x_\varepsilon, x), G(x_\varepsilon, x)\big) + \begin{cases} O(\mu_\varepsilon^N |\log \mu_\varepsilon|), & N = 3, \\ O(\mu_\varepsilon^N), & N \geq 4. \end{cases}$$

By Lemma A.2, we have

$$\frac{N\varepsilon}{2^*(2^* - \varepsilon)} \int_{B_r(x_\varepsilon)} u_\varepsilon^{2^* - \varepsilon} \mathrm{d}x = \frac{(N-2)^2 \varepsilon}{4N} \left( \alpha_0^{p+1} A + O(\mu_\varepsilon^{N-2} |\log \mu_\varepsilon|\right).$$

It follows from Lemma 2.3 and (3.15) that

$$
\begin{aligned}
R(x_\varepsilon) &= \frac{N-2}{2N}\frac{\varepsilon}{C_\varepsilon^2}\left(\alpha_0^{p+1}A + O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|\right) + \begin{cases} O(\mu_\varepsilon^N|\log\mu_\varepsilon|), & N=3,\\ O(\mu_\varepsilon^N), & N\geq 4,\end{cases}\\
&= \frac{A}{2N^2B}\frac{\varepsilon}{\mu_\varepsilon^{N-2}} + \begin{cases} O(\mu_\varepsilon|\log\mu_\varepsilon|), & N=3,\\ O(\mu_\varepsilon^2|\log\mu_\varepsilon|), & N=4,\\ O(\mu_\varepsilon^2), & N\geq 5.\end{cases}
\end{aligned}
\tag{3.16}
$$

Let $\mu_0 = \left(\frac{A}{2N^2BR(x_0)}\right)^{\frac{1}{N-2}}$, where $A, B$ are defined in (2.19). Then

$$
\begin{aligned}
\left|\frac{\varepsilon}{\mu_\varepsilon^{N-2}} - \frac{1}{\mu_0^{N-2}}\right| &= \frac{2N^2B}{A}\left|R(x_\varepsilon) - R(x_0)\right| + \begin{cases} O(\mu_\varepsilon|\log\mu_\varepsilon|), & N=3,\\ O(\mu_\varepsilon^2|\log\mu_\varepsilon|), & N=4,\\ O(\mu_\varepsilon^2), & N\geq 5,\end{cases}\\
&= O(|\nabla R(x_\varepsilon)||x_\varepsilon - x_0|) + \begin{cases} O(\mu_\varepsilon|\log\mu_\varepsilon|), & N=3,\\ O(\mu_\varepsilon^2|\log\mu_\varepsilon|), & N=4,\\ O(\mu_\varepsilon^2), & N\geq 5,\end{cases}\\
&= \begin{cases} O(\mu_\varepsilon|\log\mu_\varepsilon|), & N=3,\\ O(\mu_\varepsilon^2|\log\mu_\varepsilon|), & N=4,\\ O(\mu_\varepsilon^2), & N\geq 5.\end{cases}
\end{aligned}
\tag{3.17}
$$

Hence

$$
\begin{aligned}
\mu_\varepsilon &= \left(\mu_0^{N-2}\varepsilon + \begin{cases} O(\mu_\varepsilon^2|\log\mu_\varepsilon|), & N=3,\\ O(\mu_\varepsilon^N|\log\mu_\varepsilon|), & N=4,\\ O(\mu_\varepsilon^N), & N\geq 5,\end{cases}\right)^{\frac{1}{N-2}}\\
&= \mu_0\varepsilon^{\frac{1}{N-2}} + \begin{cases} O(\mu_\varepsilon^2|\log\mu_\varepsilon|), & N=3,\\ O(\mu_\varepsilon^3|\log\mu_\varepsilon|), & N=4,\\ O(\mu_\varepsilon^3), & N\geq 5.\end{cases}
\end{aligned}
\tag{3.18}
$$

□

### 3.2 Blow-up analysis of one-peak solutions

In this section, we use the Pohozaev identities and blow-up techniques to estimate the difference between two solutions concentrating at the same point.

Let $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$ be solutions of (1.1) satisfying (1.3). We see that

$$
u_\varepsilon^{(l)} = \alpha_\varepsilon^{(l)}PU_{x_\varepsilon^{(l)},(\mu_\varepsilon^{(l)})^{-1}} + \omega_\varepsilon^{(l)},
$$

satisfying, for $l = 1, 2$,

$$
x_\varepsilon^{(l)} = x_0 + \begin{cases} O((\mu_\varepsilon^{(l)})^2|\log\mu_\varepsilon^{(l)}|), & N=3,\\ O((\mu_\varepsilon^{(l)})^2), & N\geq 4,\end{cases}
\tag{3.19}
$$

$$
\mu_\varepsilon^{(l)} = \mu_0\varepsilon^{\frac{1}{N-2}} + \begin{cases} O((\mu_\varepsilon^{(l)})^2|\log\mu_\varepsilon^{(l)}|), & N=3,\\ O((\mu_\varepsilon^{(l)})^3|\log\mu_\varepsilon^{(l)}|), & N=4,\\ O((\mu_\varepsilon^{(l)})^3), & N\geq 5,\end{cases}
\tag{3.20}
$$

$$\alpha_\varepsilon^{(l)} = \alpha_0 + O((\mu_\varepsilon^{(l)})^{N-2}|\log \mu_\varepsilon^{(l)}|), \tag{3.21}$$

$$\omega_\varepsilon^{(l)} \in E_{x_\varepsilon^{(l)},(\mu_\varepsilon^{(l)})^{-1}}, \quad \text{and} \quad \|\omega_\varepsilon^{(l)}\| = \begin{cases} O((\mu_\varepsilon^{(l)})^{N-2}), & \text{if } N \le 5, \\ O((\mu_\varepsilon^{(l)})^{N-2}|\log \mu_\varepsilon^{(l)}|^{2/3}) & \text{if } N = 6, \\ O((\mu_\varepsilon^{(l)})^{\frac{N+2}{2}}) & \text{if } N \ge 7. \end{cases} \tag{3.22}$$

We set

$$\xi_\varepsilon = \frac{u_\varepsilon^{(1)} - u_\varepsilon^{(2)}}{\|u_\varepsilon^{(1)} - u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}}, \tag{3.23}$$

then $\|\xi_\varepsilon\|_{L^\infty(\Omega)} = 1$ and

$$-\Delta \xi_\varepsilon = D_\varepsilon \xi_\varepsilon, \quad \text{in } \Omega, \tag{3.24}$$

where

$$D_\varepsilon(x) = (p-\varepsilon)\int_0^1 \Big(t u_\varepsilon^{(1)}(x) + (1-t)u_\varepsilon^{(2)}(x)\Big)^{p-1-\varepsilon} \mathrm{d}t. \tag{3.25}$$

**Lemma 3.3.** *For any constant $\sigma > 0$, there is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}} \mathrm{d}z \le \begin{cases} C(1+|y|)^{-\sigma}, & \sigma < N-2, \\ C|\log|y||(1+|y|)^{-(N-2)}, & \sigma \ge N-2. \end{cases} \tag{3.26}$$

*Proof.* When $\sigma < N-2$, it has been proved by Lemma B.2 in [38]. When $\sigma \ge N-2$,

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}} \mathrm{d}z &\le \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{N}} \mathrm{d}z \\ &\le C|\log|y||(1+|y|)^{-(N-2)}, \end{aligned}$$

where in the last inequality we used again Lemma B.2 in [38]. $\square$

Let $\bar\mu_\varepsilon = \max\Big\{\mu_\varepsilon^{(1)}, \mu_\varepsilon^{(2)}\Big\}$.

**Proposition 3.4.** *For $N \ge 3$ and $\xi_\varepsilon$ defined by* (3.23)*, we have*

$$|\xi_\varepsilon(x)| \le C\sum_{l=1}^{2} \frac{\Big|\log(\mu_\varepsilon^{(l)})^{-1}|x - x_\varepsilon^{(l)}|\Big|}{\Big(1 + (\mu_\varepsilon^{(l)})^{-1}|x - x_\varepsilon^{(l)}|\Big)^{N-2}}, \quad \text{in } \Omega. \tag{3.27}$$

*Hence*

$$\int_\Omega |\xi_\varepsilon| = O(\bar\mu_\varepsilon^{N-2}|\log \bar\mu_\varepsilon|) \quad \text{and} \quad \xi_\varepsilon(x) = O(\bar\mu_\varepsilon^{N-2}|\log\bar\mu_\varepsilon|) \quad \text{in } \Omega \setminus B_r(x_\varepsilon^{(1)}). \tag{3.28}$$

*Proof.* Since

$$|D_\varepsilon(y)| \le C\left(|u_\varepsilon^{(1)}|^{p-1-\varepsilon} + |u_\varepsilon^{(2)}|^{p-1-\varepsilon}\right) \le C\left(|U_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}|^{p-1} + |U_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}}|^{p-1}\right), \tag{3.29}$$

we obtain from (3.26) that

$$
\begin{aligned}
|\xi_\varepsilon(x)| &= \int_\Omega |D_\varepsilon(y)\xi_\varepsilon(y)G(x,y)|\mathrm{d}y \\
&\le C\sum_{l=1}^{2}\int_\Omega \frac{(\mu_\varepsilon^{(l)})^{-2}}{\Big(1+(\mu_\varepsilon^{(l)})^{-2}|y-x_\varepsilon^{(l)}|^2\Big)^2}\frac{1}{|y-x|^{N-2}}\mathrm{d}y \\
&\le C\sum_{l=1}^{2}\int_\Omega \frac{1}{(1+|z|)^4}\frac{\mathrm{d}z}{\Big|z-(\mu_\varepsilon^{(l)})^{-1}(x-x_\varepsilon^{(l)})\Big|^{N-2}} \\
&\le \begin{cases} C\sum_{l=1}^{2}\frac{\left|\log(\mu_\varepsilon^{(l)})^{-1}|x-x_\varepsilon^{(l)}|\right|}{\left(1+(\mu_\varepsilon^{(l)})^{-1}|x-x_\varepsilon^{(l)}|\right)^{(N-2)}}, & N\le 4, \\ C\sum_{l=1}^{2}\frac{1}{\left(1+(\mu_\varepsilon^{(l)})^{-1}|x-x_\varepsilon^{(l)}|\right)^{2}}, & N\ge 5. \end{cases}
\end{aligned}
$$

Repeating the above process, we get

$$
\begin{aligned}
|\xi_\varepsilon(x)| &\le C\sum_{l=1}^{2}\int_\Omega \frac{1}{(1+|z|)^6}\frac{\mathrm{d}z}{\Big|z-(\mu_\varepsilon^{(l)})^{-1}(x-x_\varepsilon^{(l)})\Big|^{N-2}} \\
&\le \begin{cases} C\sum_{l=1}^{2}\frac{\left|\log(\mu_\varepsilon^{(l)})^{-1}|x-x_\varepsilon^{(l)}|\right|}{\left(1+(\mu_\varepsilon^{(l)})^{-1}|x-x_\varepsilon^{(l)}|\right)^{N-2}}, & N\le 6, \\ C\sum_{l=1}^{2}\frac{1}{\left(1+(\mu_\varepsilon^{(l)})^{-1}|x-x_\varepsilon^{(l)}|\right)^{4}}, & N\ge 7. \end{cases}
\end{aligned}
$$

Then proceeding for finite number of times, we can prove

$$
|\xi_\varepsilon(x)| \le C\sum_{l=1}^{2}\frac{\Big|\log(\mu_\varepsilon^{(l)})^{-1}|x-x_\varepsilon^{(l)}|\Big|}{\Big(1+(\mu_\varepsilon^{(l)})^{-1}|x-x_\varepsilon^{(l)}|\Big)^{N-2}}, \quad \text{in } \Omega.
$$

Hence $\int_\Omega |\xi_\varepsilon| = O(\bar\mu_\varepsilon^{N-2}|\log\bar\mu_\varepsilon|)$ and $\xi_\varepsilon(x) = O(\bar\mu_\varepsilon^{N-2}|\log\bar\mu_\varepsilon|)$ in $\Omega\setminus B_r(x_\varepsilon^{(1)})$ can be deduced by (3.27). □

Now let

$$
\tilde\xi_\varepsilon(y) = \xi_\varepsilon(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y), \quad y\in\Omega_\varepsilon = \frac{\Omega - x_\varepsilon^{(1)}}{\mu_\varepsilon^{(1)}}. \tag{3.30}
$$

Then $\tilde\xi_\varepsilon$ satisfies

$$
-\Delta\tilde\xi_\varepsilon(y) = (\mu_\varepsilon^{(1)})^2 D_\varepsilon(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y)\tilde\xi_\varepsilon(y), \quad \text{in } \Omega_\varepsilon. \tag{3.31}
$$

**Proposition 3.5.** *Let $\tilde\xi_\varepsilon$ be defined by* (3.30)*. Then after taking a subsequence if necessary, we have*

$$
\tilde\xi_\varepsilon = b_0\psi_0 + \sum_{i=1}^{N} b_i\psi_i + \begin{cases} O(\bar\mu_\varepsilon|\log\bar\mu_\varepsilon|), & N=3, \\ O(\bar\mu_\varepsilon), & N\ge 4, \end{cases} \quad \textit{in } \mathcal{C}^1_{loc}(\mathbb{R}^N), \tag{3.32}
$$

*where $b_i$ are constants for $i=0,1,\cdots,N$,*

$$
\psi_0 = \frac{1-|y|^2}{(1+|y|^2)^{N/2}} \quad \textit{and} \quad \psi_i = \frac{y_i}{(1+|y|^2)^{N/2}}.
$$

*Proof.* First, we estimate $D_\varepsilon(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y)$. Let $z_\varepsilon=(\mu_\varepsilon^{(1)})^{-1}(x_\varepsilon^{(2)}-x_\varepsilon^{(1)})$ and $\lambda_\varepsilon=\frac{\mu_\varepsilon^{(1)}}{\mu_\varepsilon^{(2)}}$, then from Proposition 3.2, we know

$$|z_\varepsilon|=\begin{cases}O(\bar\mu_\varepsilon|\log\bar\mu_\varepsilon|), & N=3,\\ O(\bar\mu_\varepsilon), & N\geq 4,\end{cases}\quad\text{and}\quad |\lambda_\varepsilon-1|=\begin{cases}O(\bar\mu_\varepsilon|\log\bar\mu_\varepsilon|), & N=3,\\ O(\bar\mu_\varepsilon^2|\log\bar\mu_\varepsilon|), & N=4,\\ O(\bar\mu_\varepsilon^2), & N\geq 5.\end{cases}$$

It follows that for any fixed $R>0$,

$$\begin{aligned}
&U_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}}(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y)\\
&=(\mu_\varepsilon^{(1)})^{-\frac{N-2}{2}}U_{z_\varepsilon,\lambda_\varepsilon}(y)\\
&=(\mu_\varepsilon^{(1)})^{-\frac{N-2}{2}}\left(U_{0,1}(y)+\frac{\partial U_{0,1}(y)}{\partial\lambda}(\lambda_\varepsilon-1)+\frac{\partial U_{0,1}(y)}{\partial x}z_\varepsilon+o(|\lambda_\varepsilon-1|+|z_\varepsilon|)\right)\\
&=(\mu_\varepsilon^{(1)})^{-\frac{N-2}{2}}U_{0,1}(y)\left(1+\begin{cases}O(\bar\mu_\varepsilon|\log\bar\mu_\varepsilon|), & N=3\\ O(\bar\mu_\varepsilon), & N\geq 4\end{cases}\right),\quad\text{in } B_R(0).
\end{aligned}$$

Also

$$\begin{aligned}
u_\varepsilon^{(1)}(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y)&=\alpha_\varepsilon^{(1)}U_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y)+\omega_\varepsilon^{(1)}-\varphi_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\\
&=\alpha_\varepsilon^{(1)}(\mu_\varepsilon^{(1)})^{-\frac{N-2}{2}}U_{0,1}(y)+O(\bar\mu_\varepsilon^{\frac{N-2}{2}}),
\end{aligned}$$

and

$$u_\varepsilon^{(2)}(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y)=\alpha_\varepsilon^{(2)}(\mu_\varepsilon^{(1)})^{-\frac{N-2}{2}}U_{0,1}(y)\left(1+\begin{cases}O(\bar\mu_\varepsilon|\log\bar\mu_\varepsilon|), & N=3\\ O(\bar\mu_\varepsilon), & N\geq 4\end{cases}\right)+O(\bar\mu_\varepsilon^{\frac{N-2}{2}}).\tag{3.33}$$

Then

$$\begin{aligned}
&D_\varepsilon(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y)\\
&=(p-\varepsilon)\int_0^1\left|tu_\varepsilon^{(1)}(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y)+(1-t)u_\varepsilon^{(2)}(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y)\right|^{p-1-\varepsilon}\mathrm{d}t\\
&=(p-\varepsilon)\left|\left(1+\begin{cases}O(\bar\mu_\varepsilon|\log\bar\mu_\varepsilon|), & N=3\\ O(\bar\mu_\varepsilon), & N\geq 4\end{cases}\right)\alpha_0(\mu_\varepsilon^{(1)})^{-\frac{N-2}{2}}U_{0,1}(y)+O(\bar\mu_\varepsilon^{\frac{N-2}{2}}|\log\bar\mu_\varepsilon|)\right|^{p-1-\varepsilon}\\
&=N(N+2)U_{0,1}^{p-1}(y)(\mu_\varepsilon^{(1)})^{-2}\left(1+\begin{cases}O(\bar\mu_\varepsilon|\log\bar\mu_\varepsilon|), & N=3\\ O(\bar\mu_\varepsilon), & N\geq 4\end{cases}\right)+O(\bar\mu_\varepsilon^{N-4}|\log\bar\mu_\varepsilon|).
\end{aligned}\tag{3.34}$$

Let

$$f_\varepsilon(y):=(\mu_\varepsilon^{(1)})^2D_\varepsilon(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y).$$

Then

$$f_\varepsilon-N(N+2)U_{0,1}^{p-1}=\begin{cases}O(\bar\mu_\varepsilon|\log\bar\mu_\varepsilon|), & N=3,\\ O(\bar\mu_\varepsilon), & N\geq 4,\end{cases}\quad\text{in } B_R(0).$$

On the other hand, since

$$|D_\varepsilon(y)|\leq C\left(|U_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}|^{p-1}+|U_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}}|^{p-1}\right)\leq C|U_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}|^{p-1},\tag{3.35}$$

we have

$$\begin{aligned}
\int_{\Omega_\varepsilon}|\nabla\tilde\xi_\varepsilon|^2\mathrm{d}y&=(\mu_\varepsilon^{(1)})^2\int_{\Omega_\varepsilon}D_\varepsilon(x_\varepsilon^{(1)}+\mu_\varepsilon^{(1)}y)\tilde\xi_\varepsilon^2(y)\mathrm{d}y\\
&\leq C\int_{\mathbb{R}^N}\frac{\log|y|}{(1+|y|)^{N+2}}\mathrm{d}y<+\infty.
\end{aligned}$$

Then up to a subsequence, $\tilde{\xi}_\varepsilon \rightharpoonup \xi_0$ in $D^{1,2}(\mathbb{R}^N)$. Also by standard elliptic estimates, we have

$$\tilde{\xi}_\varepsilon \to \xi_0 \quad \text{in } \mathcal{C}^2_{loc}(\mathbb{R}^N).$$

Letting $\varepsilon \to 0$ in (3.31), we obtain

$$-\Delta \xi_0 = N(N+2)U_{0,1}^{p-1}\xi_0 \quad \text{in } \mathbb{R}^N.$$

As a reslut of Lemma 2.11, we get

$$\xi_0(y) = b_0 \frac{1-|y|^2}{(1+|y|^2)^{N/2}} + \sum_{i=1}^N b_i \frac{y_i}{(1+|y|^2)^{N/2}}.$$

Moreover, since for any $R > 0$,

$$-\Delta(\tilde{\xi}_\varepsilon - \xi_0) = N(N+2)U_{0,1}^{p-1}(\tilde{\xi}_\varepsilon - \xi_0) + g_\varepsilon, \quad \text{in } B_R(0)$$

with

$$\|g_\varepsilon\|_{L^\infty(B_R(0))} = \begin{cases} O(\bar{\mu}_\varepsilon |\log \bar{\mu}_\varepsilon|), & N = 3, \\ O(\bar{\mu}_\varepsilon), & N \geq 4, \end{cases}$$

a Brezis-Kato iteration tells that

$$\|\tilde{\xi}_\varepsilon - \xi_0\|_{L^\infty(B_R(0))} = \begin{cases} O(\bar{\mu}_\varepsilon |\log \bar{\mu}_\varepsilon|), & N = 3, \\ O(\bar{\mu}_\varepsilon), & N \geq 4, \end{cases}$$

and then the Schauder estimates implies

$$\|\tilde{\xi}_\varepsilon - \xi_0\|_{C^{1,\alpha}(B_R(0))} = \begin{cases} O(\bar{\mu}_\varepsilon |\log \bar{\mu}_\varepsilon|), & N = 3, \\ O(\bar{\mu}_\varepsilon), & N \geq 4. \end{cases}$$

Then we finish the proof of (3.32). $\square$

**Proposition 3.6.** *Let $\xi_\varepsilon$ be defined by* (3.23). *Then for small $r > 0$, there holds*

$$\xi_\varepsilon(y) = B_{\varepsilon,0} G(x_\varepsilon^{(1)}, y) + \sum_{i=1}^N B_{\varepsilon,i} \partial_i G(x_\varepsilon^{(1)}, y) + O(\bar{\mu}_\varepsilon^N |\log \bar{\mu}_\varepsilon|), \quad \text{in } \mathcal{C}^1(\Omega \setminus B_{2r}(x_\varepsilon^{(1)})), \tag{3.36}$$

*where*

$$B_{\varepsilon,0} = \int_{B_r(x_\varepsilon^{(1)})} D_\varepsilon(y)\xi_\varepsilon(y)\mathrm{d}y \quad \text{and} \quad B_{\varepsilon,i} = \int_{B_r(x_\varepsilon^{(1)})} D_\varepsilon(y)\xi_\varepsilon(y)(y - x_\varepsilon^{(1)})_i \mathrm{d}y.$$

*Moreover, for any fixed large $R > 0$,*

$$B_{\varepsilon,0} = -N(N-2)Bb_0 \left(1 + O(\frac{1}{R^2})\right) \bar{\mu}_\varepsilon^{N-2} + C \frac{\log R}{R^2} \bar{\mu}_\varepsilon^{N-2} + o(\bar{\mu}_\varepsilon^{N-2}), \tag{3.37}$$

*and*

$$B_{\varepsilon,i} = N(N+2)B_i b_i \left(1 + O(\frac{1}{R^2})\right) \bar{\mu}_\varepsilon^{N-1} + C \frac{\log R}{R} \bar{\mu}_\varepsilon^{N-1} + o(\bar{\mu}_\varepsilon^{N-1}), \tag{3.38}$$

*where $C$ are constants indenpendent of $\varepsilon, R$ and $B_i = \int_{\mathbb{R}^N} \frac{z_i^2}{(1+|z|^2)^{\frac{N+4}{2}}} \mathrm{d}z$.*

*Proof.* For any $x \in \Omega \setminus B_{2r}(x_\varepsilon^{(1)})$, from (3.27) and (3.35) we find

$$\int_{\Omega\setminus B_r(x_\varepsilon^{(1)})} D_\varepsilon(y)\xi_\varepsilon(y)G(x,y)\mathrm{d}y = O(\bar\mu_\varepsilon^N|\log\bar\mu_\varepsilon|)\int_{\Omega\setminus B_r(x_\varepsilon^{(1)})} G(x,y)\mathrm{d}y = O(\bar\mu_\varepsilon^N|\log\bar\mu_\varepsilon|),$$

and

$$\begin{aligned}
&\int_{B_r(x_\varepsilon^{(1)})} |D_\varepsilon(y)\xi_\varepsilon(y)|\,|y-x_\varepsilon^{(1)}|^2\mathrm{d}y \\
&\le C\int_{B_r(x_\varepsilon^{(1)})} |y-x_\varepsilon^{(1)}|^2 U^{p-1}_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}(y)\left(\sum_{l=1}^{2}\frac{\left|\log(\mu_\varepsilon^{(l)})^{-1}|y-x_\varepsilon^{(l)}|\right|}{\left(1+(\mu_\varepsilon^{(l)})^{-1}|y-x_\varepsilon^{(l)}|\right)^{N-2}}\right)\mathrm{d}y \\
&\le C\mu_\varepsilon^N\int_{|z|\le(\mu_\varepsilon^{(1)})^{-1}r}(\mu_\varepsilon^{(1)})^2|z|^2\frac{(\mu_\varepsilon^{(1)})^{-2}}{(1+|z|)^4}\frac{|\log|z||+|\log(|z|+O(\bar\mu_\varepsilon))|}{(1+|z|)^{N-2}}\mathrm{d}z \\
&= O(\bar\mu_\varepsilon^N|\log\bar\mu_\varepsilon|).
\end{aligned}$$

Then

$$\begin{aligned}
\xi_\varepsilon(x) &= \int_\Omega D_\varepsilon(y)\xi_\varepsilon(y)G(x,y)\mathrm{d}y \\
&= \int_{B_r(x_\varepsilon^{(1)})} D_\varepsilon(y)\xi_\varepsilon(y)G(x,y)\mathrm{d}y + \int_{\Omega\setminus B_r(x_\varepsilon^{(1)})} D_\varepsilon(y)\xi_\varepsilon(y)G(x,y)\mathrm{d}y \\
&= B_{\varepsilon,0}G(x_\varepsilon^{(1)},y) + \sum_{i=1}^N B_{\varepsilon,i}\partial_i G(x_\varepsilon^{(1)},y) \\
&\quad + O\left(\int_{B_r(x_\varepsilon^{(1)})} D_\varepsilon(y)\xi_\varepsilon(y)|y-x_\varepsilon^{(1)}|^2\mathrm{d}y\right) + O(\bar\mu_\varepsilon^N|\log\bar\mu_\varepsilon|) \\
&= B_{\varepsilon,0}G(x_\varepsilon^{(1)},y) + \sum_{i=1}^N B_{\varepsilon,i}\partial_i G(x_\varepsilon^{(1)},y) + O(\bar\mu_\varepsilon^N|\log\bar\mu_\varepsilon|).
\end{aligned}$$

For any fixed large $R>0$, we see that

$$B_{\varepsilon,0} = \int_{\left|\frac{y-x_\varepsilon^{(1)}}{\mu_\varepsilon^{(1)}}\right|\le R} D_\varepsilon(y)\xi_\varepsilon(y)\mathrm{d}y + \int_{R\le\left|\frac{y-x_\varepsilon^{(1)}}{\mu_\varepsilon^{(1)}}\right|\le\frac{r}{\mu_\varepsilon^{(1)}}} D_\varepsilon(y)\xi_\varepsilon(y)\mathrm{d}y =: K_1+K_2. \tag{3.39}$$

From (3.32) and (3.34), we obtain

$$\begin{aligned}
K_1 = \bar\mu_\varepsilon^{N-2}&\left(1+\begin{cases}O(\bar\mu_\varepsilon|\log\bar\mu_\varepsilon|), & N=3\\ O(\bar\mu_\varepsilon), & N\ge 4\end{cases}\right)\int_{|y|\le R}\frac{N(N+2)}{(1+|y|^2)^2}\left(b_0\psi_0(y)+\sum_{i=1}^N b_i\psi_i(y)\right)\mathrm{d}y \\
&+ O(\bar\mu_\varepsilon^{2N-4}|\log\bar\mu_\varepsilon|)\int_{|y|\le R}|\tilde\xi_\varepsilon(y)|\mathrm{d}y
\end{aligned}$$

By symmetry we obtain

$$\int_{|y|\le R}\frac{N(N+2)}{(1+|y|^2)^2}\left(b_0\psi_0(y)+\sum_{i=1}^N b_i\psi_i(y)\right)\mathrm{d}y = -N(N-2)Bb_0\left(1+O(\frac{1}{R^2})\right),$$

where $B$ is defined in (2.19) and we have used

$$(N+2)\int_{|y|\le R}\frac{1-|y|^2}{(1+|y|^2)^{\frac{N+4}{2}}}\mathrm{d}y = -(N-2)\int_{|y|\le R}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\mathrm{d}y + O(\frac{1}{R^2})$$

which can be proved by $-\Delta\psi_0 = N(N+2)U_{0,1}^{p-1}\psi_0$. By (3.27), we have

$$\int_{|y|\le R} |\tilde{\xi}_\varepsilon(y)|\mathrm{d}y \le C\int_{|y|\le R} \frac{|\log|y|| + |\log(|y| + O(\bar{\mu}_\varepsilon))|}{(1+|y|)^{N-2}}\mathrm{d}y = O(1).$$

As a result,

$$\begin{aligned} K_1 &= -N(N-2)Bb_0\left(1 + O(\frac{1}{R^2})\right)\bar{\mu}_\varepsilon^{N-2} + \begin{cases} O(\bar{\mu}_\varepsilon^{N-1}|\log\bar{\mu}_\varepsilon|), & N=3,\\ O(\bar{\mu}_\varepsilon^{N-1}), & N\ge 4,\end{cases} \\ &= -N(N-2)Bb_0\left(1 + O(\frac{1}{R^2})\right)\bar{\mu}_\varepsilon^{N-2} + o(\bar{\mu}_\varepsilon^{N-2}). \end{aligned} \tag{3.40}$$

On the other hand, from (3.27) and (3.35), we obtain

$$\begin{aligned} |K_2| &\le C\bar{\mu}_\varepsilon^{N-2}\int_{R\le|y|\le\frac{r}{\mu_\varepsilon^{(1)}}} \frac{|\log y|\mathrm{d}y}{(1+|y|^2)^2(1+|y|)^{N-2}} \\ &= C\frac{\log R}{R^2}\bar{\mu}_\varepsilon^{N-2} + O(\bar{\mu}_\varepsilon^{N}|\log\bar{\mu}_\varepsilon|), \end{aligned} \tag{3.41}$$

where $C$ are constants indenpendent of $\varepsilon, R$. So from (3.39), (3.40) and (3.41), we find (3.37).

Similarly, for any fixed large $R > 0$, we see that

$$\begin{aligned} B_{\varepsilon,i} &= \int_{\left|\frac{y-x_\varepsilon^{(1)}}{\mu_\varepsilon^{(1)}}\right|\le R} D_\varepsilon(y)\xi_\varepsilon(y)(y-x_\varepsilon^{(1)})_i\mathrm{d}y \\ &\quad + \int_{R\le\left|\frac{y-x_\varepsilon^{(1)}}{\mu_\varepsilon^{(1)}}\right|\le\frac{r}{\mu_\varepsilon^{(1)}}} D_\varepsilon(y)\xi_\varepsilon(y)(y-x_\varepsilon^{(1)})_i\mathrm{d}y \\ &=: K_3 + K_4. \end{aligned} \tag{3.42}$$

From (3.27), (3.32) and (3.35), we obtain

$$\begin{aligned} K_3 &= \bar{\mu}_\varepsilon^{N-1}\left(1 + \begin{cases} O(\bar{\mu}_\varepsilon|\log\bar{\mu}_\varepsilon|), & N=3\\ O(\bar{\mu}_\varepsilon), & N\ge 4\end{cases}\right)\int_{|y|\le R}\frac{N(N+2)y_i}{(1+|y|^2)^2}\left(b_0\psi_0(y) + \sum_{i=1}^{N} b_i\psi_i(y)\right)\mathrm{d}y \\ &\quad + O(\bar{\mu}_\varepsilon^{2N-4}|\log\bar{\mu}_\varepsilon|)\bar{\mu}_\varepsilon\int_{|y|\le R}|\tilde{\xi}_\varepsilon(y)||y|\mathrm{d}y \\ &= N(N+2)b_i\bar{\mu}_\varepsilon^{N-1}\int_{|y|\le R}\frac{y_i^2}{(1+|y|^2)^{\frac{N+4}{2}}}\mathrm{d}y + o(\bar{\mu}_\varepsilon^{N-1}) \\ &= N(N+2)B_i b_i\left(1 + O(\frac{1}{R^2})\right)\bar{\mu}_\varepsilon^{N-1} + o(\bar{\mu}_\varepsilon^{N-1}), \end{aligned} \tag{3.43}$$

where $B_i = \int_{\mathbb{R}^N}\frac{z_i^2}{(1+|z|^2)^{\frac{N+4}{2}}}\mathrm{d}z$. Also from (3.27) and (3.35), we obtain

$$\begin{aligned} |K_4| &\le C\bar{\mu}_\varepsilon^{N-1}\int_{R\le|y|\le\frac{r}{\mu_\varepsilon^{(1)}}}\frac{|\log y||y|\mathrm{d}y}{(1+|y|^2)(1+|y|)^{N-2}} \\ &= C\frac{\log R}{R}\bar{\mu}_\varepsilon^{N-1} + O(\bar{\mu}_\varepsilon^{N}|\log\bar{\mu}_\varepsilon|), \end{aligned} \tag{3.44}$$

where $C$ are constants indenpendent of $\varepsilon, R$. So from (3.42), (3.43) and (3.44), we find (3.38). $\square$

### 3.3 Proof of Theorem 1.2

**Proposition 3.7.** *For $l = 1, 2$ and small $r > 0$, there holds*

$$u_\varepsilon^{(l)}(x) = C_\varepsilon^{(l)} G(x_\varepsilon^{(1)}, x) + \begin{cases} O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}} |\log \bar{\mu}_\varepsilon|), & N = 3, \\ O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}}), & N \geq 4, \end{cases} \quad \textit{in } \mathcal{C}^1(\Omega \setminus B_{2r}(x_\varepsilon^{(1)})), \tag{3.45}$$

*where*

$$C_\varepsilon^{(l)} = \int_{B_r(x_\varepsilon^{(1)})} |u_\varepsilon^{(l)}|^{p-\varepsilon} \mathrm{d}x.$$

*Proof.* First, (3.1) implies that (3.45) holds for $l = 1$ and

$$u_\varepsilon^{(2)}(x) = C_\varepsilon^{(2)} G(x_\varepsilon^{(2)}, x) + \begin{cases} O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}} |\log \bar{\mu}_\varepsilon|), & N = 3, \\ O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}}), & N \geq 4, \end{cases} \quad \text{in } \mathcal{C}^1(\Omega \setminus B_{\frac{3}{2}r}(x_\varepsilon^{(2)})).$$

Also using Proposition 3.2, we calculate

$$\begin{aligned} G(x_\varepsilon^{(2)}, x) - G(x_\varepsilon^{(1)}, x) &= \nabla G(\theta x_\varepsilon^{(2)} + (1-\theta) x_\varepsilon^{(1)}, x)(x_\varepsilon^{(2)} - x_\varepsilon^{(1)}) \\ &= \begin{cases} O(\bar{\mu}_\varepsilon^2 |\log \bar{\mu}_\varepsilon|), & N = 3, \\ O(\bar{\mu}_\varepsilon^2), & N \geq 4. \end{cases} \end{aligned}$$

Since $B_{\frac{3r}{2}}(x_\varepsilon^{(2)}) \subset B_{2r}(x_\varepsilon^{(1)})$ for small $\varepsilon > 0$, we get (3.45). □

**Proposition 3.8.** *For $N \geq 3$, there holds*

$$b_0 = 0, \tag{3.46}$$

*where $b_0$ is the constant in Proposition 3.5.*

*Proof.* Applying (2.31) with $x_* = x_\varepsilon^{(1)}$ and $u = u_\varepsilon^{(l)}$, $l = 1, 2$, we obtain

$$P(u_\varepsilon^{(l)}, u_\varepsilon^{(l)}) = \frac{r}{2^* - \varepsilon} \int_{\partial B_r(x_\varepsilon^{(1)})} (u_\varepsilon^{(l)})^{2^* - \varepsilon} \mathrm{d}\sigma - \frac{N\varepsilon}{2^*(2^* - \varepsilon)} \int_{B_r(x_\varepsilon^{(1)})} (u_\varepsilon^{(l)})^{2^* - \varepsilon} \mathrm{d}x. \tag{3.47}$$

We estimate the difference between $P(u_\varepsilon^{(1)}, u_\varepsilon^{(1)})$ and $P(u_\varepsilon^{(2)}, u_\varepsilon^{(2)})$. From Proposition 3.6, Proposition 3.7 and Lemma 2.3, we get

$$\begin{aligned} &\frac{P(u_\varepsilon^{(1)}, u_\varepsilon^{(1)}) - P(u_\varepsilon^{(2)}, u_\varepsilon^{(2)})}{\|u_\varepsilon^{(1)} - u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}} \\ &= -r \int_{\partial B_r(x_\varepsilon^{(1)})} \langle \nabla \xi_\varepsilon, \nu \rangle \left\langle \nabla (u_\varepsilon^{(1)} + u_\varepsilon^{(2)}), \nu \right\rangle \mathrm{d}\sigma + \frac{r}{2} \int_{\partial B_r(x_\varepsilon^{(1)})} \left\langle \nabla (u_\varepsilon^{(1)} + u_\varepsilon^{(2)}), \nabla \xi_\varepsilon \right\rangle \mathrm{d}\sigma \\ &\quad - \frac{N-2}{2} \int_{\partial B_r(x_\varepsilon^{(1)})} \langle \nabla \xi_\varepsilon, \nu \rangle u_\varepsilon^{(1)} + \left\langle \nabla u_\varepsilon^{(2)}, \nu \right\rangle \xi_\varepsilon \mathrm{d}\sigma \\ &= (C_\varepsilon^{(1)} + C_\varepsilon^{(2)}) B_{\varepsilon,0} P(G(x_\varepsilon^{(1)}, \cdot), G(x_\varepsilon^{(1)}, \cdot)) \\ &\quad + (C_\varepsilon^{(1)} + C_\varepsilon^{(2)}) \sum_{h=1}^{N} B_{\varepsilon,h} P(G(x_\varepsilon^{(1)}, \cdot), \partial_h G(x_\varepsilon^{(1)}, \cdot)) + O(\bar{\mu}_\varepsilon^{\frac{3N-2}{2}} |\log \bar{\mu}_\varepsilon|) \\ &= -\frac{N-2}{2} (C_\varepsilon^{(1)} + C_\varepsilon^{(2)}) \left( B_{\varepsilon,0} R(x_\varepsilon^{(1)}) + \frac{N-1}{2(N-2)} \sum_{h=1}^{N} B_{\varepsilon,h} \partial_h R(x_\varepsilon^{(1)}) \right) + o(\bar{\mu}_\varepsilon^{\frac{3N-4}{2}}). \end{aligned} \tag{3.48}$$

Now we estimate the right hand side of (3.47). Let

$$\tilde{D}_\varepsilon(x) = \int_0^1 \left(tu_\varepsilon^{(1)}(x) + (1-t)u_\varepsilon^{(2)}\right)^{p-\varepsilon} \mathrm{d}t.$$

Since $u_\varepsilon^{(l)} = O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}})$ on $\partial B_r(x_{\varepsilon,j}^{(1)})$, we have

$$\begin{aligned}
&\frac{r}{2^*-\varepsilon}\int_{\partial B_r(x_\varepsilon^{(1)})} \frac{(u_\varepsilon^{(1)})^{2^*-\varepsilon} - (u_\varepsilon^{(2)})^{2^*-\varepsilon}}{\|u_\varepsilon^{(1)} - u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}}\mathrm{d}\sigma \\
&= r\int_{|y-x_\varepsilon^{(1)}|=r} \tilde{D}_\varepsilon(y)\xi_\varepsilon(y)\mathrm{d}\sigma = O(\bar{\mu}_\varepsilon^{\frac{3N-2}{2}}|\log\bar{\mu}_\varepsilon|).
\end{aligned} \tag{3.49}$$

A similar approach as (3.34), we find for any fixed $R>0$,

$$\tilde{D}_\varepsilon(x_\varepsilon^{(1)} + \mu_\varepsilon^{(1)}y) = \alpha_0^p U_{0,1}^p(y)(\mu_\varepsilon^{(1)})^{-\frac{N+2}{2}}\left(1 + \begin{cases} O(\bar{\mu}_\varepsilon|\log\bar{\mu}_\varepsilon|), & N=3\\ O(\bar{\mu}_\varepsilon), & N\geq 4\end{cases}\right) + O(\bar{\mu}_\varepsilon^{\frac{N-6}{2}}|\log\bar{\mu}_\varepsilon|),$$

in $B_R(0)$. And for fixed large $R>0$,

$$\begin{aligned}
&\frac{1}{(2^*-\varepsilon)}\int_{B_r(x_\varepsilon^{(1)})} \frac{(u_\varepsilon^{(1)})^{2^*-\varepsilon} - (u_\varepsilon^{(2)})^{2^*-\varepsilon}}{\|u_\varepsilon^{(1)} - u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}}\mathrm{d}x \\
&= \int_{\left|\frac{y-x_\varepsilon^{(1)}}{\mu_\varepsilon^{(1)}}\right|\leq R} \tilde{D}_\varepsilon(y)\xi_\varepsilon(y)\mathrm{d}y + \int_{R\leq\left|\frac{y-x_\varepsilon^{(1)}}{\mu_\varepsilon^{(1)}}\right|\leq\frac{r}{\mu_\varepsilon^{(1)}}} \tilde{D}_\varepsilon(y)\xi_\varepsilon(y)\mathrm{d}y \\
&=: K_5 + K_6.
\end{aligned}$$

Then similar to (3.40), we have

$$\begin{aligned}
K_5 &= O(\frac{1}{R^N})\bar{\mu}_\varepsilon^{\frac{N-2}{2}} + \begin{cases} O(\bar{\mu}_\varepsilon^{N/2}|\log\bar{\mu}_\varepsilon|), & N=3\\ O(\bar{\mu}_\varepsilon^{N/2}), & N\geq 4\end{cases} + O(\bar{\mu}_\varepsilon^{\frac{3N-6}{2}}|\log\bar{\mu}_\varepsilon|) \\
&= O(\frac{1}{R^N})\bar{\mu}_\varepsilon^{\frac{N-2}{2}} + o(\bar{\mu}_\varepsilon^{\frac{N-2}{2}}).
\end{aligned}$$

Since

$$\tilde{D}_\varepsilon(x) \leq CU^p_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}},$$

we have

$$\begin{aligned}
K_6 &\leq C\bar{\mu}_\varepsilon^{\frac{N-2}{2}}\int_{R\leq|y|\leq\frac{r}{\mu_\varepsilon^{(1)}}} \frac{|\log y|\mathrm{d}y}{(1+|y|^2)^{\frac{N+2}{2}}(1+|y|)^{N-2}} \\
&= O(\frac{\log R}{R^N})\bar{\mu}_\varepsilon^{\frac{N-2}{2}} + o(\bar{\mu}_\varepsilon^{\frac{N-2}{2}}).
\end{aligned}$$

Thus

$$\frac{1}{(2^*-\varepsilon)}\int_{B_r(x_\varepsilon^{(1)})} \frac{(u_\varepsilon^{(1)})^{2^*-\varepsilon} - (u_\varepsilon^{(2)})^{2^*-\varepsilon}}{\|u_\varepsilon^{(1)} - u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}}\mathrm{d}x = O(\frac{\log R}{R^N})\bar{\mu}_\varepsilon^{\frac{N-2}{2}} + o(\bar{\mu}_\varepsilon^{\frac{N-2}{2}}). \tag{3.50}$$

From (3.47)-(3.50), we get

$$\begin{aligned}
&B_{\varepsilon,0}R(x_\varepsilon^{(1)}) + \frac{N-1}{2(N-2)}\sum_{h=1}^N B_{\varepsilon,h}\partial_h R(x_\varepsilon^{(1)}) \\
&= O(\frac{\log R}{R^N})\bar{\mu}_\varepsilon^{N-2} + o(\bar{\mu}_\varepsilon^{N-2}).
\end{aligned} \tag{3.51}$$

Substituting (3.13), (3.16) and (3.38) into (3.51), we find

$$B_{\varepsilon,0} = O(\frac{\log R}{R^N})\bar{\mu}_\varepsilon^{N-2} + o(\bar{\mu}_\varepsilon^{N-2}). \tag{3.52}$$

Hence from (3.37), we obtain

$$b_0 = O(\frac{\log R}{R^2}) + o_\varepsilon(1)(1 + O(\frac{\log R}{R})),$$

then letting $\varepsilon \to 0$ and $R \to +\infty$, we have $b_0 = 0$. □

**Proposition 3.9.** *For $N \geq 3$, there holds*

$$b_i = 0, \quad i = 1, \cdots, N, \tag{3.53}$$

*where $b_i$ are the constants in Proposition 3.5.*

*Proof.* Applying (2.32) with $x_* = x_\varepsilon^{(1)}$ and $u = u_\varepsilon^{(l)}$, $l = 1, 2$, we obtain

$$Q(u_\varepsilon^{(l)}, u_\varepsilon^{(l)}) = \frac{2}{2^* - \varepsilon} \int_{\partial B_r(x_\varepsilon^{(1)})} (u_\varepsilon^{(l)})^{2^*-\varepsilon} \nu_i \mathrm{d}\sigma. \tag{3.54}$$

As in Proposition 3.8, we have

$$\begin{aligned}
&\frac{Q(u_\varepsilon^{(1)}, u_\varepsilon^{(1)}) - Q(u_\varepsilon^{(2)}, u_\varepsilon^{(2)})}{\|u_\varepsilon^{(1)} - u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}} \\
&= -2 \int_{\partial B_r(x_\varepsilon^{(1)})} \langle \nabla \xi_\varepsilon, \nu \rangle \partial_i u_\varepsilon^{(1)} + \left\langle \nabla u_\varepsilon^{(2)}, \nu \right\rangle \partial_i \xi_\varepsilon \mathrm{d}\sigma + \int_{\partial B_r(x_\varepsilon^{(1)})} \left\langle \nabla \xi_\varepsilon, \nabla(u_\varepsilon^{(1)} + u_\varepsilon^{(2)}) \right\rangle \nu_i \mathrm{d}\sigma \\
&= (C_\varepsilon^{(1)} + C_\varepsilon^{(2)}) B_{\varepsilon,0} Q(G(x_\varepsilon^{(1)}, \cdot), G(x_\varepsilon^{(1)}, \cdot)) \\
&\quad + (C_\varepsilon^{(1)} + C_\varepsilon^{(2)}) \sum_{h=1}^N B_{\varepsilon,h} Q(G(x_\varepsilon^{(1)}, \cdot), \partial_h G(x_\varepsilon^{(1)}, \cdot)) + O(\bar{\mu}_\varepsilon^{\frac{3N-2}{2}} |\log \bar{\mu}_\varepsilon|) \\
&= (C_\varepsilon^{(1)} + C_\varepsilon^{(2)}) \left( B_{\varepsilon,0} \partial_i R(x_\varepsilon^{(1)}) + \frac{1}{2} \sum_{h=1}^N B_{\varepsilon,h} \partial_{ih}^2 R(x_\varepsilon^{(1)}) \right) + o(\bar{\mu}_\varepsilon^{\frac{3N-4}{2}}).
\end{aligned} \tag{3.55}$$

Since $b_0 = 0$, we have

$$B_{\varepsilon,0} = C \frac{\log R}{R^2} \bar{\mu}_\varepsilon^{N-2} + o(\bar{\mu}_\varepsilon^{N-2}). \tag{3.56}$$

It follows from (3.52) and (3.56) that

$$B_{\varepsilon,0} = o(\bar{\mu}_\varepsilon^{N-2}). \tag{3.57}$$

Then from (3.13), (3.54), (3.55), (3.57) and (3.49), we get

$$\nabla^2 R(x_0) \cdot (B_{\varepsilon,1} \cdots, B_{\varepsilon,N})^T = o(\bar{\mu}_\varepsilon^{N-1}), \tag{3.58}$$

which implies $B_{\varepsilon,i} = o(\bar{\mu}_\varepsilon^{N-1})$. Hence from (3.38), we obtain

$$b_i = O(\frac{\log R}{R}) + o_\varepsilon(1),$$

then letting $\varepsilon \to 0$ and $R \to +\infty$, we have $b_i = 0$. □

We are now ready to show Theorem 1.2.

*Proof of Theorem 1.2.* Let $x_\varepsilon^*$ be a maximum point of $\xi_\varepsilon$, which says

$$|\xi_\varepsilon(x_\varepsilon^*)| = 1.$$

In view of Proposition 3.6, we obtain

$$x_\varepsilon^* \to x_0.$$

Let $s_\varepsilon = |x_\varepsilon^* - x_\varepsilon^{(1)}|$. By (3.46), (3.53) and Proposition 3.5, it holds

$$\tilde{\xi}_\varepsilon \to 0 \quad \text{in } \mathcal{C}^1_{loc}(\mathbb{R}^N).$$

Thus

$$\lim_{\varepsilon\to 0} \frac{s_\varepsilon}{\mu_\varepsilon^{(1)}} = +\infty. \tag{3.59}$$

Setting $\xi_\varepsilon^*(y) = \xi_\varepsilon(x_\varepsilon^{(1)} + s_\varepsilon y)$, then $\xi_\varepsilon^*$ satisfies

$$\begin{cases} -\Delta\xi_\varepsilon^*(y) = s_\varepsilon^2 D_\varepsilon(x_\varepsilon^{(1)} + s_\varepsilon y)\xi_\varepsilon^*(y), \\ |\xi_\varepsilon^*(\frac{x_\varepsilon^* - x_\varepsilon^{(1)}}{s_\varepsilon})| = 1. \end{cases}$$

From the fact

$$|s_\varepsilon^2 D_\varepsilon(x_\varepsilon^{(1)} + s_\varepsilon y)| \le C s_\varepsilon^2 \sum_{l=1}^{2} U^{p-1}_{x_\varepsilon^{(l)},(\mu_\varepsilon^{(l)})^{-1}}(x_\varepsilon^{(1)} + s_\varepsilon y) = O\left(\frac{(\mu_\varepsilon^{(1)})^2}{s_\varepsilon^2}|y|^{-4}\right),$$

for $y \neq 0$, we see that $\xi_\varepsilon^* \to \xi_0^*$ in $\mathcal{C}^1_{loc}(\mathbb{R}^N \setminus \{0\})$ with

$$-\Delta\xi_0^* = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Since $|\xi_0^*| \le 1$, we know $\xi_0^*$ is a constant, which means $\xi_0^* = -1$ or $\xi_0^* = 1$. Therefore, we obtain that

$$|\xi_\varepsilon(x)| \ge \frac{1}{2}, \quad \forall\, |x - x_\varepsilon^{(1)}| = s_\varepsilon. \tag{3.60}$$

By (3.27), we have

$$\xi_\varepsilon(x) = O(\frac{1}{R^2}), \quad \forall\, x \in \Omega \setminus B_{\mu_\varepsilon^{(1)} R}(x_\varepsilon^{(1)}).$$

Since $\mu_\varepsilon^{(1)} \ll s_\varepsilon$, there holds

$$|\xi_\varepsilon(x)| \le \frac{1}{4}, \quad \forall\, |x - x_\varepsilon^{(1)}| = s_\varepsilon,$$

which contradicts with (3.60). We finish the proof. □

# 4 Multi-peak solutions

In this section, we assume that $a_1, \cdots, a_k \in \Omega$ are $k$ different points. Let $\vec{a} := (a_1, \cdots, a_k)$, and $\vec{\lambda} = (\lambda_1, \cdots, \lambda_k)$ be the unique solution of (2.22). We also suppose that $M_k(\vec{a})$ is positive and $(\vec{a}, \vec{\lambda}) \in \Omega^k \times (\mathbb{R}^+)^k$ is a non-degenerate critical point of $\Phi_k$.

## 4.1 Sharper estimations of multi-peak solutions

In this section, we obtain some improtant estimations for solutions of (1.1) satisfying (1.2). We start with the following proposition. Let $\bar{\mu}_\varepsilon = \max\{\mu_{\varepsilon,1}, \cdots, \mu_{\varepsilon,k}\}$.

**Proposition 4.1.** *Let $u_\varepsilon$ be a solution of* (1.1) *satisfying* (1.2)*, then for any small $r > 0$, it holds*

$$u_\varepsilon(x) = \alpha_0^p B \sum_{j=1}^{k} \mu_{\varepsilon,j}^{\frac{N-2}{2}} G(x_{\varepsilon,j}, x) + \begin{cases} O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}} |\log \bar{\mu}_\varepsilon|), & N = 4, \\ O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}}), & N \geq 5, \end{cases} \quad \text{in } \mathcal{C}^1(\Omega \setminus \bigcup_{j=1}^{k} B_{2r}(x_{\varepsilon,j})). \tag{4.1}$$

*Proof.* For $x \in \Omega \setminus \bigcup_{j=1}^{k} B_{2r}(x_{\varepsilon,j})$, we have

$$\begin{aligned} u_\varepsilon(x) &= \int_\Omega G(x,y) u_\varepsilon^{p-\varepsilon}(y) \mathrm{d}y \\ &= \int_{\Omega \setminus \bigcup_{j=1}^{k} B_r(x_{\varepsilon,j})} G(x,y) u_\varepsilon^{p-\varepsilon}(y) \mathrm{d}y + \sum_{j=1}^{k} \int_{B_r(x_{\varepsilon,j})} G(x,y) u_\varepsilon^{p-\varepsilon}(y) \mathrm{d}y. \end{aligned} \tag{4.2}$$

We see that $u_\varepsilon(x) = O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}})$ for $x \in \Omega \setminus \bigcup_{j=1}^{k} B_r(x_{\varepsilon,j})$. Hence

$$\int_{\Omega \setminus \bigcup_{j=1}^{k} B_r(x_{\varepsilon,j})} G(x,y) u_\varepsilon^{p-\varepsilon}(y) \mathrm{d}y = O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}}).$$

As in Proposition 4.1, we know

$$\int_{B_r(x_{\varepsilon,j})} G(x,y) u_\varepsilon^{p-\varepsilon}(y) \mathrm{d}y = \alpha_0^p B \mu_{\varepsilon,j}^{\frac{N-2}{2}} G(x_{\varepsilon,j}, x) + \begin{cases} O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}} |\log \bar{\mu}_\varepsilon|), & N = 4, \\ O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}}), & N \geq 5, \end{cases}.$$

Then (4.1) can be easily obtained. □

**Proposition 4.2.** *Let $u_\varepsilon$ be a solution of* (1.1) *satisfying* (1.2). *Then*

$$x_{\varepsilon,j} = a_j + \begin{cases} O(\bar{\mu}_\varepsilon^2 |\log \bar{\mu}_\varepsilon|), & N = 4, \\ O(\bar{\mu}_\varepsilon^2), & N \geq 5, \end{cases} \tag{4.3}$$

*and*

$$\mu_{\varepsilon,j} = \mu_j \varepsilon^{\frac{1}{N-2}} + \begin{cases} O(\bar{\mu}_\varepsilon^3 |\log \bar{\mu}_\varepsilon|), & N = 4, \\ O(\bar{\mu}_\varepsilon^3), & N \geq 5, \end{cases} \tag{4.4}$$

*where* $\mu_j = \lambda_j \left( \frac{\alpha_0^p A}{2N^2 B} \right)^{\frac{1}{N-2}}$.

*Proof.* Applying (2.32) with $x_* = x_{\varepsilon,j}$ and $u = u_\varepsilon$, we obtain

$$Q(u_\varepsilon, u_\varepsilon) = \frac{2}{2^* - \varepsilon} \int_{\partial B_r(x_{\varepsilon,j})} u_\varepsilon^{2^* - \varepsilon} \nu_i \mathrm{d}\sigma, \tag{4.5}$$

Using Proposition 4.1 and Lemma 2.3, we have

$$\begin{aligned} &Q(u_\varepsilon, u_\varepsilon) \\ &= \alpha_0^{2p} B^2 \sum_{l,m=1}^{k} \mu_{\varepsilon,l}^{\frac{N-2}{2}} \mu_{\varepsilon,m}^{\frac{N-2}{2}} Q\big(G(x_{\varepsilon,l}, x), G(x_{\varepsilon,m}, x)\big) + \begin{cases} O(\bar{\mu}_\varepsilon^N |\log \bar{\mu}_\varepsilon|), & N = 4, \\ O(\bar{\mu}_\varepsilon^N), & N \geq 5, \end{cases} \\ &= -\alpha_0^{2p} B^2 \left( \mu_{\varepsilon,j}^{N-2} \partial_i R(x_{\varepsilon,j}) - 2 \sum_{l \neq j, l=1}^{k} \mu_{\varepsilon,j}^{\frac{N-2}{2}} \mu_{\varepsilon,l}^{\frac{N-2}{2}} \partial_i G(x_{\varepsilon,j}, x_{\varepsilon,l}) \right) \\ &\quad + \begin{cases} O(\bar{\mu}_\varepsilon^N |\log \bar{\mu}_\varepsilon|), & N = 4, \\ O(\bar{\mu}_\varepsilon^N), & N \geq 5. \end{cases} \end{aligned} \tag{4.6}$$

Also

$$\frac{2}{2^*-\varepsilon}\int_{\partial B_r(x_{\varepsilon,j})} u_\varepsilon^{2^*-\varepsilon}\nu_i \mathrm{d}\sigma = O(\bar{\mu}_\varepsilon^N). \tag{4.7}$$

Let

$$\lambda_{\varepsilon,j} = \left(\frac{\alpha_0^p A}{2N^2 B}\right)^{-\frac{1}{N-2}} \mu_{\varepsilon,j}\varepsilon^{-\frac{1}{N-2}}.$$

Then we obtain from (4.5)-(4.7) and Lemma 2.15 that

$$\nabla_x \Phi_k(\vec{x}_\varepsilon, \vec{\lambda}_\varepsilon) = \begin{cases} O(\bar{\mu}_\varepsilon^2|\log\bar{\mu}_\varepsilon|), & N=4,\\ O(\bar{\mu}_\varepsilon^2), & N\geq 5,\end{cases} \tag{4.8}$$

with

$$\vec{x}_\varepsilon = (x_{\varepsilon,1},\cdots,x_{\varepsilon,k}) \quad \text{and} \quad \vec{\lambda}_\varepsilon = (\lambda_{\varepsilon,1},\cdots,\lambda_{\varepsilon,k}).$$

On the other hand, applying (2.31) with $x_* = x_{\varepsilon,j}$ and $u = u_\varepsilon$, we also obtain

$$P(u_\varepsilon,u_\varepsilon) = \frac{r}{2^*-\varepsilon}\int_{\partial B_r(x_{\varepsilon,j})} u_\varepsilon^{2^*-\varepsilon}\mathrm{d}\sigma - \frac{N\varepsilon}{2^*(2^*-\varepsilon)}\int_{B_r(x_{\varepsilon,j})} u_\varepsilon^{2^*-\varepsilon}\mathrm{d}x. \tag{4.9}$$

Again using Proposition 4.1 and Lemma 2.3, we have

$$\begin{aligned}
&P(u_\varepsilon,u_\varepsilon)\\
&= \alpha_0^{2p}B^2\sum_{l,m=1}^k \mu_{\varepsilon,l}^{\frac{N-2}{2}}\mu_{\varepsilon,m}^{\frac{N-2}{2}} P\big(G(x_{\varepsilon,l},x), G(x_{\varepsilon,m},x)\big) + \begin{cases} O(\bar{\mu}_\varepsilon^N|\log\bar{\mu}_\varepsilon|), & N=4,\\ O(\bar{\mu}_\varepsilon^N), & N\geq 5,\end{cases}\\
&= -\frac{N-2}{2}\alpha_0^{2p}B^2\left(\mu_{\varepsilon,j}^{N-2}R(x_{\varepsilon,j}) - \sum_{l\neq j,l=1}^k \mu_{\varepsilon,j}^{\frac{N-2}{2}}\mu_{\varepsilon,l}^{\frac{N-2}{2}}G(x_{\varepsilon,j},x_{\varepsilon,l})\right)\\
&\quad + \begin{cases} O(\bar{\mu}_\varepsilon^N|\log\bar{\mu}_\varepsilon|), & N=4,\\ O(\bar{\mu}_\varepsilon^N), & N\geq 5.\end{cases}
\end{aligned} \tag{4.10}$$

Note that

$$\text{RHS of (4.9)} = O(\bar{\mu}_\varepsilon^N) - \frac{(N-2)^2}{4N}\varepsilon\left(\alpha^{p+1}A + O(\bar{\mu}_\varepsilon^{N-2}|\log\bar{\mu}_\varepsilon|)\right). \tag{4.11}$$

Thus we obtain

$$\nabla_\lambda \Phi_k(\vec{x}_\varepsilon, \vec{\lambda}_\varepsilon) = \begin{cases} O(\bar{\mu}_\varepsilon^2|\log\bar{\mu}_\varepsilon|), & N=4,\\ O(\bar{\mu}_\varepsilon^2), & N\geq 5.\end{cases} \tag{4.12}$$

Then from (4.12) we see

$$\vec{x}_\varepsilon \to \vec{a} \quad \text{and} \quad \vec{\lambda}_\varepsilon \to \vec{\lambda}(\vec{a}).$$

Moreover, since $\nabla\Phi_k(\vec{a},\vec{\lambda}) = 0$ and $\nabla^2\Phi_k(\vec{a},\vec{\lambda})$ is non-degenerate, we have

$$\nabla\Phi_k(\vec{x}_\varepsilon,\vec{\lambda}_\varepsilon) = \nabla^2\Phi_k(\vec{a},\vec{\lambda})(\vec{x}_\varepsilon - \vec{a}, \vec{\lambda}_\varepsilon - \vec{\lambda}) + o(|\vec{x}_\varepsilon - \vec{a}| + |\vec{\lambda}_\varepsilon - \vec{\lambda}|)$$

which yields

$$|\vec{x}_\varepsilon - \vec{a}| + |\vec{\lambda}_\varepsilon - \vec{\lambda}| = \begin{cases} O(\bar{\mu}_\varepsilon^2|\log\bar{\mu}_\varepsilon|), & N=4,\\ O(\bar{\mu}_\varepsilon^2), & N\geq 5.\end{cases}$$

Then (4.3) and (4.4) can be obtained easily. □

## 4.2 Blow-up analysis of multi-peak solutions

In this section, we use the Pohozaev identity and blow-up techniques to estimate the difference between two solutions concentrating at the same point.

Let $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$ be solutions of (1.1) satisfying (1.2). We see that

$$u_\varepsilon^{(l)} = \sum_{j=1}^{k} \alpha_{\varepsilon,j}^{(l)} PU_{x_{\varepsilon,j}^{(l)},(\mu_{\varepsilon,j}^{(l)})^{-1}} + \omega_\varepsilon^{(l)},$$

satisfying, for $N \geq 4$ and $l = 1, 2$, as $\varepsilon \to 0$,

$$x_{\varepsilon,j}^{(l)} = a_j + O(\bar{\mu}_\varepsilon^2), \tag{4.13}$$

$$\mu_{\varepsilon,j}^{(l)} = \mu_j \varepsilon^{\frac{1}{N-2}} + O(\bar{\mu}_\varepsilon^2), \tag{4.14}$$

$$\alpha_{\varepsilon,j}^{(l)} = \alpha_0 + O(\bar{\mu}_\varepsilon^{N-2} |\log \bar{\mu}_\varepsilon|), \tag{4.15}$$

$$\omega_\varepsilon^{(l)} \in \bigcap_{j=1}^{k} E_{x_{\varepsilon,j}^{(l)},(\mu_{\varepsilon,j}^{(l)})^{-1}} \quad \text{and} \quad \|\omega_\varepsilon^{(l)}\| = \begin{cases} O(\bar{\mu}_\varepsilon^{N-2}), & \text{if } N \leq 5, \\ O(\bar{\mu}_\varepsilon^{N-2} |\log \bar{\mu}_\varepsilon|^{2/3}), & \text{if } N = 6, \\ O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}}), & \text{if } N \geq 7, \end{cases} \tag{4.16}$$

where $\bar{\mu}_\varepsilon = \max\left\{\mu_{\varepsilon,1}^{(1)}, \cdots, \mu_{\varepsilon,k}^{(1)}, \mu_{\varepsilon,1}^{(2)}, \cdots, \mu_{\varepsilon,k}^{(2)}\right\}$.

We introduce the following operator. Let $i_\varepsilon^* : L^{p_\varepsilon}(\Omega) \to H_0^1(\Omega)$ be defined by

$$\langle i_\varepsilon^*(u), v \rangle = \int_\Omega uv \mathrm{d}x, \quad \forall\, v \in H_0^1(\Omega),$$

where $p_\varepsilon = \frac{2N-(N-2)\varepsilon}{N+2-(N-2)\varepsilon}$ is such that

$$\frac{1}{p_\varepsilon} + \frac{1}{2^* - \varepsilon} = 1.$$

Then there exists a constant $C > 0$ such that

$$\|i_\varepsilon^*(u)\| \leq C \|u\|_{p_\varepsilon}, \quad \forall\, \varepsilon > 0. \tag{4.17}$$

Define $L_\varepsilon : H_0^1(\Omega) \mapsto H_0^1(\Omega)$

$$L_\varepsilon u := u - i_\varepsilon^* \left[ f' \left( \sum_{j=1}^{k} \alpha_{\varepsilon,j}^{(1)} PU_{x_{\varepsilon,j}^{(1)},(\mu_{\varepsilon,j}^{(1)})^{-1}} \right) u \right] \tag{4.18}$$

where $f(s) = s^{p-\varepsilon}$. Moreover, we have

**Lemma 4.3.** *There exist $\varepsilon_0 > 0$ and $C > 0$ such that for any $0 < \varepsilon < \varepsilon_0$*

$$\|L_\varepsilon u\| \geq C \|u\|, \quad \forall\, u \in \bigcap_{j=1}^{k} E_{x_{\varepsilon,j}^{(1)},(\mu_{\varepsilon,j}^{(1)})^{-1}}.$$

*Proof.* See [30, Lemma 1.7] for a proof. □

**Proposition 4.4.** *For $N \geq 7$, there holds*

$$\|\omega_\varepsilon^{(1)} - \omega_\varepsilon^{(2)}\| = o(\bar{\mu}_\varepsilon^{\frac{N+2}{2}}). \tag{4.19}$$

*Proof.* For brevity, we assume $k = 1$, and the other cases $k \geq 2$ can be proved in the same way. Let $\bar{\omega}_\varepsilon = \omega_\varepsilon^{(1)} - \omega_\varepsilon^{(2)}$, and

$$\bar{\omega}_{\varepsilon,2} = \beta_\varepsilon PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}} + \beta_{\varepsilon,0}\frac{\partial PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}}{\partial\lambda} + \sum_{i=1}^{N}\beta_{\varepsilon,i}\frac{\partial PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}}{\partial x_i}$$

be such that

$$\bar{\omega}_{\varepsilon,1} := \bar{\omega}_\varepsilon - \bar{\omega}_{\varepsilon,2} \in E_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}. \tag{4.20}$$

Then

$$\|\bar{\omega}_\varepsilon\| \leq \|\bar{\omega}_{\varepsilon,1}\| + \|\bar{\omega}_{\varepsilon,2}\| \leq C\|L_\varepsilon\bar{\omega}_\varepsilon\| + C\|L_\varepsilon\bar{\omega}_{\varepsilon,2}\| + \|\bar{\omega}_{\varepsilon,2}\| \tag{4.21}$$

We estimate the last three terms one by one.

Firstly we prove $\|\bar{\omega}_{\varepsilon,2}\| = O(\bar{\mu}_\varepsilon)\|\omega_\varepsilon^{(2)}\|$. From $\left\langle \bar{\omega}_{\varepsilon,1}, PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}} \right\rangle = 0$ and Lemma A.3, we obtain

$$\begin{aligned}
&\beta_\varepsilon\|PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\|^2 \\
=& \left\langle \bar{\omega}_\varepsilon, PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}} \right\rangle - \beta_{\varepsilon,0}\left\langle PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}, \frac{\partial PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}}{\partial\lambda} \right\rangle \\
&- \sum_{i=1}^{N}\beta_{\varepsilon,i}\left\langle PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}, \frac{\partial PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}}{\partial x_i} \right\rangle \\
=& -\left\langle \omega_\varepsilon^{(2)}, PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}} - PU_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}} \right\rangle \\
&+ O((\mu_\varepsilon^{(1)})^{N-1})|\beta_{\varepsilon,0}| + O((\mu_\varepsilon^{(1)})^{N-2})\sum_{i=1}^{N}|\beta_{\varepsilon,i}| \\
=& O\left( \|\frac{\partial PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}}{\partial\lambda}\| \cdot |(\mu_\varepsilon^{(1)})^{-1} - (\mu_\varepsilon^{(2)})^{-1}| + \|\frac{\partial PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}}{\partial x}\| \cdot |x_\varepsilon^{(1)} - x_\varepsilon^{(2)}| \right)\|\omega_\varepsilon^{(2)}\| \\
&+ O((\mu_\varepsilon^{(1)})^{N-1})|\beta_{\varepsilon,0}| + O((\mu_\varepsilon^{(1)})^{N-2})\sum_{i=1}^{N}|\beta_{\varepsilon,i}|.
\end{aligned}$$

Also from Proposition 3.2, we know

$$|(\mu_\varepsilon^{(1)})^{-1} - (\mu_\varepsilon^{(2)})^{-1}| = O(\bar{\mu}_\varepsilon) \quad \text{and} \quad |x_\varepsilon^{(1)} - x_\varepsilon^{(2)}| = O(\bar{\mu}_\varepsilon^2).$$

Then

$$\beta_\varepsilon(\mu_\varepsilon^{(1)})^{-1} = O(1)\|\omega_\varepsilon^{(2)}\| + O((\mu_\varepsilon^{(1)})^{N-2})|\beta_{\varepsilon,0}| + O((\mu_\varepsilon^{(1)})^{N-3})\sum_{i=1}^{N}|\beta_{\varepsilon,i}|. \tag{4.22}$$

Similarly, from $\left\langle \bar{\omega}_{\varepsilon,1}, \frac{\partial PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}}{\partial\lambda} \right\rangle = 0$ and $\left\langle \bar{\omega}_{\varepsilon,1}, \frac{\partial PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}}{\partial x_i} \right\rangle = 0$ ,we find

$$\beta_{\varepsilon,0} = O(1)\|\omega_\varepsilon^{(2)}\| + O((\mu_\varepsilon^{(1)})^{N-3})|\beta_\varepsilon| + O((\mu_\varepsilon^{(1)})^{N-3})\sum_{i=1}^{N}|\beta_{\varepsilon,i}|. \tag{4.23}$$

and

$$\beta_{\varepsilon,i}(\mu_\varepsilon^{(1)})^{-2} = O(1)\|\omega_\varepsilon^{(2)}\| + O((\mu_\varepsilon^{(1)})^{N-2})|\beta_\varepsilon| + O((\mu_\varepsilon^{(1)})^{N-1})|\beta_{\varepsilon,0}| + O((\mu_\varepsilon^{(1)})^{N-2})\sum_{j\neq i}|\beta_{\varepsilon,j}|. \tag{4.24}$$

Hence it follows from (4.22)-(4.24) that

$$|\beta_\varepsilon|(\mu_\varepsilon^{(1)})^{-1} + |\beta_{\varepsilon,0}| + \sum_{i=1}^{N} |\beta_{\varepsilon,i}|(\mu_\varepsilon^{(1)})^{-2} = O(1)\|\omega_\varepsilon^{(2)}\|.$$

As a consequence,

$$\|\bar{\omega}_{\varepsilon,2}\| = O(|\beta_\varepsilon| + |\beta_{\varepsilon,0}|\mu_\varepsilon^{(1)} + \sum_{i=1}^{N} |\beta_{\varepsilon,i}|(\mu_\varepsilon^{(1)})^{-1}) = O(\bar{\mu}_\varepsilon)\|\omega_\varepsilon^{(2)}\|. \tag{4.25}$$

Using (4.17) and the Hölder inequality, we know that

$$\begin{aligned} \|L_\varepsilon \bar{\omega}_{\varepsilon,2}\| &\le \|\bar{\omega}_{\varepsilon,2}\| + \|i_\varepsilon^*[f'(\alpha_\varepsilon^{(1)} PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}})\bar{\omega}_{\varepsilon,2}]\| \\ &\le \|\bar{\omega}_{\varepsilon,2}\| + C\|PU^{p-1-\varepsilon}_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\bar{\omega}_{\varepsilon,2}\|_{p_\varepsilon} \\ &\le \|\bar{\omega}_{\varepsilon,2}\| + C\|PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\|^{p-1-\varepsilon}\|\bar{\omega}_{\varepsilon,2}\| \\ &\le C\|\bar{\omega}_{\varepsilon,2}\|. \end{aligned} \tag{4.26}$$

For the remainder $\|L_\varepsilon\bar{\omega}_\varepsilon\|$, we see that

$$\begin{aligned} L_\varepsilon\bar{\omega}_\varepsilon &= \bar{\omega}_\varepsilon - i_\varepsilon^*[f'(\alpha_\varepsilon^{(1)} PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}})\bar{\omega}_\varepsilon] \\ &= \left(u_\varepsilon^{(1)} - \alpha_\varepsilon^{(1)} PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\right) - \left(u_\varepsilon^{(2)} - \alpha_\varepsilon^{(2)} PU_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}}\right) \\ &\quad - \left(i_\varepsilon^*[f'(\alpha_\varepsilon^{(1)} PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}})\omega_\varepsilon^{(1)}] - i_\varepsilon^*[f'(\alpha_\varepsilon^{(1)} PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}})\omega_\varepsilon^{(2)}]\right) \\ &= R_\varepsilon^{(1)} - R_\varepsilon^{(2)} - S_\varepsilon + T_\varepsilon, \end{aligned} \tag{4.27}$$

where

$$\begin{aligned} R_\varepsilon^{(l)} &= i_\varepsilon^*\left[f(u_\varepsilon^{(l)}) - f(\alpha_\varepsilon^{(l)} PU_{x_\varepsilon^{(l)},(\mu_\varepsilon^{(l)})^{-1}}) - f'(\alpha_\varepsilon^{(l)} PU_{x_\varepsilon^{(l)},(\mu_\varepsilon^{(l)})^{-1}})\omega_\varepsilon^{(l)}\right], \\ S_\varepsilon &= \left(i_\varepsilon^*[f(\alpha_\varepsilon^{(1)} PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}})] - \alpha_\varepsilon^{(1)} PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\right) \\ &\quad - \left(i_\varepsilon^*[f(\alpha_\varepsilon^{(2)} PU_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}})] - \alpha_\varepsilon^{(2)} PU_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}}\right), \\ T_\varepsilon &= i_\varepsilon^*\left[\left(f'(\alpha_\varepsilon^{(1)} PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}) - f'(\alpha_\varepsilon^{(2)} PU_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}})\right)\omega_\varepsilon^{(2)}\right]. \end{aligned}$$

Using Lemma 2.14, we have that, for $l = 1, 2$,

$$\begin{aligned} \|R_\varepsilon^{(l)}\| &\le C\|f(u_\varepsilon^{(l)}) - f(\alpha_\varepsilon^{(l)} PU_{x_\varepsilon^{(l)},(\mu_\varepsilon^{(l)})^{-1}}) - f'(\alpha_\varepsilon^{(l)} PU_{x_\varepsilon^{(l)},(\mu_\varepsilon^{(l)})^{-1}})\omega_\varepsilon^{(l)}\|_{p_\varepsilon} \\ &\le C\|(\omega_\varepsilon^{(l)})^{p-\varepsilon}\|_{p_\varepsilon} \le C\|\omega_\varepsilon^{(l)}\|^p. \end{aligned} \tag{4.28}$$

Also from Lemma 2.14, Lemma 2.4 and the facts that

$$U_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}} = (1 + O(\bar{\mu}_\varepsilon))U_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}},$$

$$\varphi_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}} = (1 + O(\bar{\mu}_\varepsilon^2))\varphi_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}},$$

we obtain

$$
\begin{aligned}
\|S_\varepsilon\| &= \sup_{\|v\|=1}\int_\Omega \Big[\Big(f(\alpha_\varepsilon^{(1)}PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}) - \alpha_\varepsilon^{(1)}\alpha_0^{p-1}U^{p-1}_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\Big)\\
&\qquad - \Big(f(\alpha_\varepsilon^{(2)}PU_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}}) - \alpha_\varepsilon^{(2)}\alpha_0^{p-1}U^{p-1}_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}}\Big)\Big] v\mathrm{d}y\\
&= O(\bar\mu_\varepsilon^{N-2}|\log\bar\mu_\varepsilon|)\|U^p_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\|_{p'_\varepsilon} + \sup_{\|v\|=1}\alpha_0^p\int_\Omega\Big[\Big(PU^p_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}} - U^p_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\Big)\\
&\qquad - \Big(PU^p_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}} - U^p_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}}\Big)\Big] v\mathrm{d}y\\
&= O(\bar\mu_\varepsilon^{N-2}|\log\bar\mu_\varepsilon|) + \sup_{\|v\|=1}O(\bar\mu_\varepsilon)\int_\Omega U^{p-1}_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\varphi_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}v + \varphi^p_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}v\mathrm{d}y\\
&= O(\bar\mu_\varepsilon^{N-2}|\log\bar\mu_\varepsilon|) + o(\bar\mu_\varepsilon^{\frac{N+2}{2}}).
\end{aligned}
\tag{4.29}
$$

Since $N \geq 7$, we obtain from (4.29) that

$$
\|S_\varepsilon\| = o(\bar\mu_\varepsilon^{\frac{N+2}{2}}). \tag{4.30}
$$

Finally, it holds

$$
\begin{aligned}
\|T_\varepsilon\| &\leq C\|\Big(\alpha_\varepsilon^{(1)}PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}} - \alpha_\varepsilon^{(2)}PU_{x_\varepsilon^{(2)},(\mu_\varepsilon^{(2)})^{-1}}\Big)PU^{p-2-\varepsilon}_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\omega_\varepsilon^{(2)}\|_{p_\varepsilon}\\
&\leq C\Bigg(\|\frac{\partial PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}}{\partial\lambda}\||(\mu_\varepsilon^{(1)})^{-1} - (\mu_\varepsilon^{(2)})^{-1}|\\
&\qquad + \|\frac{\partial PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}}{\partial x}\||x_\varepsilon^{(1)} - x_\varepsilon^{(2)}|\Bigg)\|PU_{x_\varepsilon^{(1)},(\mu_\varepsilon^{(1)})^{-1}}\|^{p-2-\varepsilon}\|\omega_\varepsilon^{(2)}\|\\
&= O(\bar\mu_\varepsilon)\|\omega_\varepsilon^{(2)}\|.
\end{aligned}
\tag{4.31}
$$

Combining (4.28), (4.30), (4.31) with (4.27), we obtain

$$
\|L_\varepsilon\bar\omega_\varepsilon\| = O\left(\sum_{l=1}^2\|\omega_\varepsilon^{(l)}\|^p + \bar\mu_\varepsilon\|\omega_\varepsilon^{(2)}\|\right) + o(\bar\mu_\varepsilon^{\frac{N+2}{2}}). \tag{4.32}
$$

Then from (4.25), (4.26) and (4.32), we see

$$
\|\bar\omega_\varepsilon\| = O\left(\sum_{l=1}^2\|\omega_\varepsilon^{(l)}\|^p + \bar\mu_\varepsilon\|\omega_\varepsilon^{(2)}\|\right) + o(\bar\mu_\varepsilon^{\frac{N+2}{2}}),
$$

which and (4.16) give (4.19). □

**Remark 4.1.** *When $N = 6$, we see from* (4.29) *that*

$$
\|S_\varepsilon\| = O(\bar\mu_\varepsilon^4|\log\bar\mu_\varepsilon|),
$$

*and hence $\|\bar\omega_\varepsilon\| = O(\bar\mu_\varepsilon^4|\log\bar\mu_\varepsilon|)$.*

**Proposition 4.5.** *For $N \geq 7$ and $j = 1, \cdots, k$, there holds*

$$
|x_{\varepsilon,j}^{(1)} - x_{\varepsilon,j}^{(2)}| = o(\bar\mu_\varepsilon^2), \tag{4.33}
$$

$$
|\mu_{\varepsilon,j}^{(1)} - \mu_{\varepsilon,j}^{(2)}| = o(\bar\mu_\varepsilon^2). \tag{4.34}
$$

*Proof.* First, from Proposition 4.2, we see

$$|x_{\varepsilon,j}^{(1)} - x_{\varepsilon,j}^{(2)}| = O(\bar{\mu}_\varepsilon^2) \quad \text{and} \quad |\mu_{\varepsilon,j}^{(1)} - \mu_{\varepsilon,j}^{(2)}| = O(\bar{\mu}_\varepsilon^3).$$

Applying (2.32) with $x_* = x_{\varepsilon,j}^{(1)}$ and $u = u_\varepsilon^{(l)}$, we obtain

$$Q(u_\varepsilon^{(l)}, u_\varepsilon^{(l)}) = \frac{2}{2^* - \varepsilon} \int_{\partial B_r(x_{\varepsilon,j}^{(1)})} (u_\varepsilon^{(l)})^{2^*-\varepsilon} \nu_i \mathrm{d}\sigma. \tag{4.35}$$

We want to estimate the difference between $Q(u_\varepsilon^{(1)}, u_\varepsilon^{(1)})$ and $Q(u_\varepsilon^{(2)}, u_\varepsilon^{(2)})$. By direct calculations as in Proposition 4.1, we find

$$PU_{x_{\varepsilon,j}^{(l)}, (\mu_{\varepsilon,j}^{(l)})^{-1}}(y) = N(N-2)B(\mu_{\varepsilon,j}^{(l)})^{\frac{N-2}{2}} G(x_{\varepsilon,j}^{(l)}, y) + O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}}), \quad \text{in } \mathcal{C}^1(\Omega \setminus B_d(x_{\varepsilon,j}^{(l)})).$$

Let

$$G_{\varepsilon,j}^{(l)} := N(N-2)B(\mu_{\varepsilon,j}^{(l)})^{\frac{N-2}{2}} G(x_{\varepsilon,j}^{(l)}, y)$$

and

$$\delta_{\varepsilon,j}^{(l)} := PU_{x_{\varepsilon,j}^{(l)}, (\mu_{\varepsilon,j}^{(l)})^{-1}}(y) - G_{\varepsilon,j}^{(l)},$$

which satisfy

$$\delta_{\varepsilon,j}^{(2)} = (1 + O(\bar{\mu}_\varepsilon))\delta_{\varepsilon,j}^{(1)}, \quad \text{in } \mathcal{C}^1(\Omega \setminus B_d(x_{\varepsilon,j}^{(l)})).$$

As a consequence,

$$\begin{aligned}
& Q\left(\sum_{j=1}^k \alpha_{\varepsilon,j}^{(1)} PU_{x_{\varepsilon,j}^{(1)}, (\mu_{\varepsilon,j}^{(1)})^{-1}}, \sum_{j=1}^k \alpha_{\varepsilon,j}^{(1)} PU_{x_{\varepsilon,j}^{(1)}, (\mu_{\varepsilon,j}^{(1)})^{-1}}\right) \\
& \quad - Q\left(\sum_{j=1}^k \alpha_{\varepsilon,j}^{(2)} PU_{x_{\varepsilon,j}^{(2)}, (\mu_{\varepsilon,j}^{(2)})^{-1}}, \sum_{j=1}^k \alpha_{\varepsilon,j}^{(2)} PU_{x_{\varepsilon,j}^{(2)}, (\mu_{\varepsilon,j}^{(2)})^{-1}}\right) \\
&= Q\left(\sum_{j=1}^k \alpha_{\varepsilon,j}^{(1)} G_{\varepsilon,j}^{(1)}, \sum_{j=1}^k \alpha_{\varepsilon,j}^{(1)} G_{\varepsilon,j}^{(1)}\right) - Q\left(\sum_{j=1}^k \alpha_{\varepsilon,j}^{(2)} G_{\varepsilon,j}^{(2)}, \sum_{j=1}^k \alpha_{\varepsilon,j}^{(2)} G_{\varepsilon,j}^{(2)}\right) \\
& \quad + 2Q\left(\sum_{j=1}^k \alpha_{\varepsilon,j}^{(1)} \delta_{\varepsilon,j}^{(1)}, \sum_{j=1}^k \alpha_{\varepsilon,j}^{(1)} G_{\varepsilon,j}^{(1)}\right) - 2Q\left(\sum_{j=1}^k \alpha_{\varepsilon,j}^{(2)} \delta_{\varepsilon,j}^{(2)}, \sum_{j=1}^k \alpha_{\varepsilon,j}^{(2)} G_{\varepsilon,j}^{(2)}\right) + O(\bar{\mu}_\varepsilon^{N+2}) \\
&= \alpha_0^{2p} B^2 \Bigg[ (\mu_{\varepsilon,j}^{(1)})^{N-2} \partial_i R(x_{\varepsilon,j}^{(1)}) - 2 \sum_{l \neq j, l=1}^k (\mu_{\varepsilon,j}^{(1)})^{\frac{N-2}{2}} (\mu_{\varepsilon,l}^{(1)})^{\frac{N-2}{2}} \partial_i G(x_{\varepsilon,j}^{(1)}, x_{\varepsilon,l}^{(1)}) \\
& \quad - \left( (\mu_{\varepsilon,j}^{(2)})^{N-2} \partial_i R(x_{\varepsilon,j}^{(2)}) - 2 \sum_{l \neq j, l=1}^k (\mu_{\varepsilon,j}^{(2)})^{\frac{N-2}{2}} (\mu_{\varepsilon,l}^{(2)})^{\frac{N-2}{2}} \partial_i G(x_{\varepsilon,j}^{(2)}, x_{\varepsilon,2}^{(1)}) \right) \Bigg] + o(\bar{\mu}_\varepsilon^N),
\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}
& Q\left(\sum_{j=1}^k \alpha_{\varepsilon,j}^{(1)} PU_{x_{\varepsilon,j}^{(1)}, (\mu_{\varepsilon,j}^{(1)})^{-1}}, \omega_\varepsilon^{(1)}\right) - Q\left(\sum_{j=1}^k \alpha_{\varepsilon,j}^{(2)} PU_{x_{\varepsilon,j}^{(2)}, (\mu_{\varepsilon,j}^{(2)})^{-1}}, \omega_\varepsilon^{(2)}\right) \\
&= Q\left(\sum_{j=1}^k \alpha_{\varepsilon,j}^{(1)} PU_{x_{\varepsilon,j}^{(1)}, (\mu_{\varepsilon,j}^{(1)})^{-1}}, \bar{\omega}_\varepsilon\right) + Q\left(\sum_{j=1}^k \alpha_{\varepsilon,j}^{(1)} PU_{x_{\varepsilon,j}^{(1)}, (\mu_{\varepsilon,j}^{(1)})^{-1}} - \alpha_{\varepsilon,j}^{(2)} PU_{x_{\varepsilon,j}^{(2)}, (\mu_{\varepsilon,j}^{(2)})^{-1}}, \omega_\varepsilon^{(2)}\right) \\
&= O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}} \|\bar{\omega}_\varepsilon\|) + O(\bar{\mu}_\varepsilon^{\frac{N}{2}} \|\omega_\varepsilon^{(2)}\|) = o(\bar{\mu}_\varepsilon^N).
\end{aligned} \tag{4.37}$$

Also

$$Q(\omega_\varepsilon^{(l)},\omega_\varepsilon^{(l)}) = O(\|\omega_\varepsilon^{(l)}\|^2) = o(\bar\mu_\varepsilon^N). \tag{4.38}$$

On the other hand, we have

$$\begin{aligned}&\int_{\partial B_r(x^{(1)}_{\varepsilon,j})}\left[(u_\varepsilon^{(1)})^{2^*-\varepsilon} - (u_\varepsilon^{(2)})^{2^*-\varepsilon}\right]\nu_i \mathrm{d}\sigma\\ =\ & O(1)\int_{\partial B_r(x^{(1)}_{\varepsilon,j})}(u_\varepsilon^{(1)})^{p-\varepsilon}|u_\varepsilon^{(1)}-u_\varepsilon^{(2)}|\mathrm{d}\sigma = o(\bar\mu_\varepsilon^N).\end{aligned} \tag{4.39}$$

Let

$$\lambda^{(l)}_{\varepsilon,j} = \left(\frac{\alpha_0^p A}{2N^2B}\right)^{-\frac{1}{N-2}}\mu^{(l)}_{\varepsilon,j}\varepsilon^{-\frac{1}{N-2}} \quad \text{for } l=1,2.$$

Then from (4.35) and (4.36)-(4.38), we obtain

$$\nabla_x\Phi_k(\vec{x}_\varepsilon^{(1)},\vec{\lambda}_\varepsilon^{(1)}) - \nabla_x\Phi_k(\vec{x}_\varepsilon^{(2)},\vec{\lambda}_\varepsilon^{(2)}) = o(\bar\mu_\varepsilon^2). \tag{4.40}$$

with

$$\vec{x}_\varepsilon^{(l)} = (x^{(l)}_{\varepsilon,1},\cdots,x^{(l)}_{\varepsilon,k}) \quad \text{and} \quad \vec{\lambda}_\varepsilon^{(l)} = (\lambda^{(l)}_{\varepsilon,1},\cdots,\lambda^{(l)}_{\varepsilon,k}).$$

While applying (2.31) with $x_* = x^{(1)}_{\varepsilon,j}$ and $u = u_\varepsilon^{(l)}$, we can prove similarly that

$$\nabla_\lambda\Phi_k(\vec{x}_\varepsilon^{(1)},\vec{\lambda}_\varepsilon^{(1)}) - \nabla_\lambda\Phi_k(\vec{x}_\varepsilon^{(2)},\vec{\lambda}_\varepsilon^{(2)}) = o(\bar\mu_\varepsilon^2). \tag{4.41}$$

Then (4.33)-(4.34) follows by (4.40) and (4.41). □

Let $\xi_\varepsilon$ be defined by (3.23).

**Proposition 4.6.** *For $N\geq 4$ and $\xi_\varepsilon$ defined by* (3.23)*, we have*

$$|\xi_\varepsilon(x)| \leq C\sum_{l=1}^2\sum_{j=1}^k \frac{\left|\log(\mu^{(l)}_{\varepsilon,j})^{-1}|x-x^{(l)}_{\varepsilon,j}|\right|}{\left(1+(\mu^{(l)}_{\varepsilon,j})^{-1}|x-x^{(l)}_{\varepsilon,j}|\right)^{N-2}}, \quad \textit{in } \Omega. \tag{4.42}$$

*Hence*

$$\int_\Omega|\xi_\varepsilon| = O(\bar\mu_\varepsilon^{N-2}|\log\bar\mu_\varepsilon|) \quad \textit{and} \quad \xi_\varepsilon(x) = O(\bar\mu_\varepsilon^{N-2}|\log\bar\mu_\varepsilon|) \quad \textit{in } \Omega\setminus\bigcup_{j=1}^k B_r(x^{(1)}_{\varepsilon,j}). \tag{4.43}$$

*Proof.* Let $\Omega^{(l)}_{\varepsilon,j} = \frac{\Omega - x^{(l)}_{\varepsilon,j}}{\mu^{(l)}_{\varepsilon,j}}$. Since

$$|D_\varepsilon(y)| \leq C\left(|u_\varepsilon^{(1)}|^{p-1-\varepsilon} + |u_\varepsilon^{(2)}|^{p-1-\varepsilon}\right) \leq C\sum_{l=1}^2\sum_{j=1}^k U^{p-1}_{x^{(l)}_{\varepsilon,j},(\mu^{(l)}_{\varepsilon,j})^{-1}},$$

we obtain that

$$\begin{aligned}|\xi_\varepsilon(x)| &= \int_\Omega |D_\varepsilon(y)\xi_\varepsilon(y)G(x,y)|\mathrm{d}y\\ &\leq C\sum_{l=1}^2\sum_{j=1}^k\int_\Omega \frac{(\mu^{(l)}_{\varepsilon,j})^{-2}}{\left(1+(\mu^{(l)}_{\varepsilon,j})^{-2}|y-x^{(l)}_{\varepsilon,j}|^2\right)^2}\frac{1}{|y-x|^{N-2}}\mathrm{d}y\\ &\leq C\sum_{l=1}^2\sum_{j=1}^k\int_{\Omega^{(l)}_{\varepsilon,j}}\frac{1}{(1+|z|)^4}\frac{\mathrm{d}z}{\left|z-(\mu^{(l)}_{\varepsilon,j})^{-1}(x-x^{(l)}_{\varepsilon,j})\right|^{N-2}}\\ &\leq \begin{cases} C\sum_{l=1}^2\sum_{j=1}^k \frac{\left|\log(\mu^{(l)}_{\varepsilon,j})^{-1}|x-x^{(l)}_{\varepsilon,j}|\right|}{\left(1+(\mu^{(l)}_{\varepsilon,j})^{-1}|x-x^{(l)}_{\varepsilon,j}|\right)^2}, & N=4,\\ C\sum_{l=1}^2\sum_{j=1}^k \frac{1}{\left(1+(\mu^{(l)}_{\varepsilon,j})^{-1}|x-x^{(l)}_{\varepsilon,j}|\right)^2}, & N\geq 5.\end{cases}\end{aligned}$$

For $N \geq 5$, repeating the above process, we get

$$|\xi_\varepsilon(x)| \leq \int_\Omega |D_\varepsilon(y)\xi_\varepsilon(y)G(x,y)|\mathrm{d}y \leq C\sum_{l,m=1}^{2}\sum_{j,s=1}^{k} I_{j,s}^{l,m}$$

with

$$I_{j,s}^{l,m} = \int_\Omega \frac{(\mu_{\varepsilon,j}^{(l)})^{-2}}{\Big(1+(\mu_{\varepsilon,j}^{(l)})^{-2}|y-x_{\varepsilon,j}^{(l)}|^2\Big)^2}\frac{1}{\Big(1+(\mu_{\varepsilon,s}^{(m)})^{-1}|x-x_{\varepsilon,s}^{(m)}|\Big)^2}\frac{1}{|y-x|^{N-2}}\mathrm{d}y.$$

For $j=s$,

$$\begin{aligned} I_{j,j}^{l,m} &= \int_{\Omega_{\varepsilon,j}^{(l)}} \frac{1}{(1+|z|)^4}\frac{1}{\Big(1+(\mu_{\varepsilon,j}^{(m)})^{-1}\Big|\mu_{\varepsilon,j}^{(l)}z+x_{\varepsilon,j}^{(l)}-x_{\varepsilon,j}^{(m)}\Big|\Big)^2}\frac{\mathrm{d}z}{\Big|z-(\mu_{\varepsilon,j}^{(l)})^{-1}(x-x_{\varepsilon,j}^{(l)})\Big|^{N-2}} \\ &\leq C\int_{\Omega_{\varepsilon,j}^{(l)}}\frac{1}{(1+|z|)^6}\frac{\mathrm{d}z}{\Big|z-(\mu_{\varepsilon,j}^{(l)})^{-1}(x-x_{\varepsilon,j}^{(l)})\Big|^{N-2}}. \end{aligned}$$

While for $j\neq s$, take $0<r<\frac{1}{2}|x_{\varepsilon,j}^{(l)}-x_{\varepsilon,s}^{(m)}|$, since

$$\begin{aligned} \Big|\mu_{\varepsilon,j}^{(l)}z+x_{\varepsilon,j}^{(l)}-x_{\varepsilon,s}^{(m)}\Big| \geq r \Longrightarrow \left|\frac{x_{\varepsilon,j}^{(l)}-x_{\varepsilon,s}^{(m)}}{\mu_{\varepsilon,s}^{(m)}}\right| &\leq C\left|\frac{\mu_{\varepsilon,j}^{(l)}z+x_{\varepsilon,j}^{(l)}-x_{\varepsilon,s}^{(m)}}{\mu_{\varepsilon,s}^{(m)}}\right|, \\ \Longrightarrow \left|\frac{\mu_{\varepsilon,j}^{(l)}z+x_{\varepsilon,j}^{(l)}-x_{\varepsilon,s}^{(m)}}{\mu_{\varepsilon,s}^{(m)}}\right| &\geq C|z|, \end{aligned}$$

we have

$$\begin{aligned} I_{j,s}^{l,m} &= \int_{|y-x_{\varepsilon,s}^{(m)}|\geq r}\frac{(\mu_{\varepsilon,j}^{(l)})^{-2}}{\Big(1+(\mu_{\varepsilon,j}^{(l)})^{-2}|y-x_{\varepsilon,j}^{(l)}|^2\Big)^2}\frac{1}{\Big(1+(\mu_{\varepsilon,s}^{(m)})^{-1}|x-x_{\varepsilon,s}^{(m)}|\Big)^2}\frac{1}{|y-x|^{N-2}}\mathrm{d}y \\ &+\int_{|y-x_{\varepsilon,j}^{(l)}|\geq r}\frac{(\mu_{\varepsilon,j}^{(l)})^{-2}}{\Big(1+(\mu_{\varepsilon,j}^{(l)})^{-2}|y-x_{\varepsilon,j}^{(l)}|^2\Big)^2}\frac{1}{\Big(1+(\mu_{\varepsilon,s}^{(m)})^{-1}|x-x_{\varepsilon,s}^{(m)}|\Big)^2}\frac{1}{|y-x|^{N-2}}\mathrm{d}y \\ &\leq C\int_{\Big|\mu_{\varepsilon,j}^{(l)}z+x_{\varepsilon,j}^{(l)}-x_{\varepsilon,s}^{(m)}\Big|\geq r}\frac{1}{(1+|z|)^4}\frac{1}{\left(1+\left|\frac{\mu_{\varepsilon,j}^{(l)}z+x_{\varepsilon,j}^{(l)}-x_{\varepsilon,s}^{(m)}}{\mu_{\varepsilon,s}^{(m)}}\right|\right)^2}\frac{\mathrm{d}z}{\Big|z-(\mu_{\varepsilon,j}^{(l)})^{-1}(x-x_{\varepsilon,j}^{(l)})\Big|^{N-2}} \\ &+C\int_{\Big|\mu_{\varepsilon,s}^{(m)}z+x_{\varepsilon,s}^{(m)}-x_{\varepsilon,j}^{(l)}\Big|\geq r}\frac{1}{\left(1+\left|\frac{\mu_{\varepsilon,s}^{(m)}z+x_{\varepsilon,s}^{(m)}-x_{\varepsilon,j}^{(l)}}{\mu_{\varepsilon,j}^{(l)}}\right|\right)^4}\frac{1}{(1+|z|)^2}\frac{\mathrm{d}z}{\Big|z-(\mu_{\varepsilon,s}^{(m)})^{-1}(x-x_{\varepsilon,s}^{(m)})\Big|^{N-2}} \\ &\leq C\int_{\mathbb{R}^N}\frac{1}{(1+|z|)^6}\left(\frac{1}{\Big|z-(\mu_{\varepsilon,j}^{(l)})^{-1}(x-x_{\varepsilon,j}^{(l)})\Big|^{N-2}}+\frac{1}{\Big|z-(\mu_{\varepsilon,s}^{(m)})^{-1}(x-x_{\varepsilon,s}^{(m)})\Big|^{N-2}}\right)\mathrm{d}z. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} |\xi_\varepsilon(x)| &\leq C\sum_{l=1}^{2}\sum_{j=1}^{k}\int_{\mathbb{R}^N}\frac{1}{(1+|z|)^6}\frac{\mathrm{d}z}{\Big|z-(\mu_{\varepsilon,j}^{(l)})^{-1}(x-x_{\varepsilon,j}^{(l)})\Big|^{N-2}} \\ &\leq \begin{cases} C\sum_{l=1}^{2}\sum_{j=1}^{k}\frac{\Big|\log(\mu_{\varepsilon,j}^{(l)})^{-1}|x-x_{\varepsilon,j}^{(l)}|\Big|}{\Big(1+(\mu_{\varepsilon,j}^{(l)})^{-1}|x-x_{\varepsilon,j}^{(l)}|\Big)^{N-2}}, & N\leq 6, \\ C\sum_{l=1}^{2}\sum_{j=1}^{k}\frac{1}{\Big(1+(\mu_{\varepsilon,j}^{(l)})^{-1}|x-x_{\varepsilon,j}^{(l)}|\Big)^4}, & N\geq 7. \end{cases} \end{aligned}$$

Then proceeding for finite number of times, we can prove (4.42), and (4.43) can be deduced by (4.42) easily. $\square$

Now let

$$\tilde{\xi}_{\varepsilon,j}(y) = \xi_\varepsilon(x^{(1)}_{\varepsilon,j} + \mu^{(1)}_{\varepsilon,j} y), \quad y \in \Omega_{\varepsilon,j} = \frac{\Omega - x^{(1)}_{\varepsilon,j}}{\mu^{(1)}_{\varepsilon,j}}. \tag{4.44}$$

Then $\tilde{\xi}_{\varepsilon,j}$ satisfies

$$-\Delta \tilde{\xi}_{\varepsilon,j}(y) = (\mu^{(1)}_{\varepsilon,j})^2 D_\varepsilon(x^{(1)}_{\varepsilon,j} + \mu^{(1)}_{\varepsilon,j} y)\tilde{\xi}_{\varepsilon,j}(y), \quad \text{in } \Omega_{\varepsilon,j}. \tag{4.45}$$

**Proposition 4.7.** *Let $N \geq 7$ and $\tilde{\xi}_{\varepsilon,j}$ be defined by* (4.44). *Then after taking a subsequence if necessary, we have*

$$\tilde{\xi}_{\varepsilon,j} = b_{j,0}\psi_0 + \sum_{j=1}^{k}\sum_{i=1}^{N} b_{j,i}\psi_i + o(\bar{\mu}_\varepsilon), \quad \mathcal{C}^1_{loc}(\mathbb{R}^N), \tag{4.46}$$

*where $b_{j,i}$ are constants for $j = 1, \cdots, k$, $i = 0, 1, \cdots, N$, $\psi_0 = \frac{1-|y|^2}{(1+|y|^2)^{N/2}}$ and $\psi_i = \frac{y_i}{(1+|y|^2)^{N/2}}$.*

*Proof.* First we need to estimate $D_\varepsilon(x^{(1)}_{\varepsilon,j} + \mu^{(1)}_{\varepsilon,j} y)$. For any $R > 0$, $y \in B_R(0)$ and $l \neq j$,

$$0 \leq PU_{x^{(1)}_{\varepsilon,l},(\mu^{(1)}_{\varepsilon,l})^{-1}}(x^{(1)}_{\varepsilon,l} + \mu^{(1)}_{\varepsilon,l} y) \leq U_{x^{(1)}_{\varepsilon,l},(\mu^{(1)}_{\varepsilon,l})^{-1}}(x^{(1)}_{\varepsilon,l} + \mu^{(1)}_{\varepsilon,l} y) = O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}}),$$

and

$$0 \leq PU_{x^{(2)}_{\varepsilon,l},(\mu^{(2)}_{\varepsilon,l})^{-1}}(x^{(2)}_{\varepsilon,l} + \mu^{(2)}_{\varepsilon,l} y) \leq U_{x^{(2)}_{\varepsilon,l},(\mu^{(2)}_{\varepsilon,l})^{-1}}(x^{(2)}_{\varepsilon,l} + \mu^{(2)}_{\varepsilon,l} y) = O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}}).$$

So

$$\begin{aligned} u^{(1)}_\varepsilon(x^{(1)}_{\varepsilon,j} + \mu^{(1)}_{\varepsilon,j} y) &= \alpha^{(1)}_{\varepsilon,j} U_{x^{(1)}_{\varepsilon,j},(\mu^{(1)}_{\varepsilon,j})^{-1}}(x^{(1)}_{\varepsilon,j} + \mu^{(1)}_{\varepsilon,j} y) - \phi_{x^{(1)}_{\varepsilon,j},(\mu^{(1)}_{\varepsilon,j})^{-1}}(x^{(1)}_{\varepsilon,j} + \mu^{(1)}_{\varepsilon,j} y) \\ &\quad + \sum_{l\neq j, l=1}^{k} PU_{x^{(1)}_{\varepsilon,l},(\mu^{(1)}_{\varepsilon,l})^{-1}}(x^{(1)}_{\varepsilon,l} + \mu^{(1)}_{\varepsilon,l} y) + w^{(1)}_\varepsilon(x^{(1)}_{\varepsilon,l} + \mu^{(1)}_{\varepsilon,l} y) \\ &= \alpha^{(1)}_{\varepsilon,j}(\mu^{(1)}_{\varepsilon,j})^{-\frac{N-2}{2}} U_{0,1}(y) + O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}}), \quad \text{in } B_R(0), \end{aligned}$$

and

$$\begin{aligned} u^{(2)}_\varepsilon(x^{(1)}_{\varepsilon,j} + \mu^{(1)}_{\varepsilon,j} y) &= \alpha^{(2)}_{\varepsilon,j} U_{x^{(2)}_{\varepsilon,j},(\mu^{(2)}_{\varepsilon,j})^{-1}}(x^{(1)}_{\varepsilon,j} + \mu^{(1)}_{\varepsilon,j} y) - \phi_{x^{(2)}_{\varepsilon,j},(\mu^{(2)}_{\varepsilon,j})^{-1}}(x^{(1)}_{\varepsilon,j} + \mu^{(1)}_{\varepsilon,j} y) \\ &\quad + \sum_{l\neq j, l=1}^{k} PU_{x^{(2)}_{\varepsilon,l},(\mu^{(2)}_{\varepsilon,l})^{-1}}(x^{(1)}_{\varepsilon,l} + \mu^{(1)}_{\varepsilon,l} y) + w^{(2)}_\varepsilon(x^{(1)}_{\varepsilon,l} + \mu^{(1)}_{\varepsilon,l} y) \\ &= \alpha^{(2)}_{\varepsilon,j}(\mu^{(1)}_{\varepsilon,j})^{-\frac{N-2}{2}} U_{0,1}(y) o(\bar{\mu}_\varepsilon) + O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}}), \quad \text{in } B_R(0), \end{aligned}$$

where in the last we used

$$U_{x^{(2)}_{\varepsilon,j},(\mu^{(2)}_{\varepsilon,j})^{-1}}(x^{(1)}_{\varepsilon,j} + \mu^{(1)}_{\varepsilon,j} y) = (\mu^{(1)}_{\varepsilon,j})^{-\frac{N-2}{2}} U_{0,1}(y) o(\bar{\mu}_\varepsilon).$$

Then as in (3.34)

$$D_\varepsilon(x^{(1)}_{\varepsilon,j} + \mu^{(1)}_{\varepsilon,j} y) = N(N+2)U^{p-1}_{0,1}(y)(\mu^{(1)}_{\varepsilon,j})^{-2}(1 + o(\bar{\mu}_\varepsilon)) + O(\bar{\mu}_\varepsilon^{N-4}|\log \bar{\mu}_\varepsilon|).$$

Hence

$$-\Delta \tilde{\xi}_{\varepsilon,j} = N(N+2)U^{p-1}_{0,1}\tilde{\xi}_{\varepsilon,j} + o(\bar{\mu}_\varepsilon) \quad \text{in } B_R(0).$$

So (4.46) follows as in Proposition 3.5. $\square$

**Proposition 4.8.** *Let $N \geq 7$ and $\xi_\varepsilon$ be defined by* (3.23)*. Then for small $r>0$, there holds*

$$\xi_\varepsilon(y) = \sum_{j=1}^{k} B_{\varepsilon,j,0} G(x_{\varepsilon,j}^{(1)}, y) + \sum_{j=1}^{k}\sum_{i=1}^{N} B_{\varepsilon,j,i}\partial_i G(x_{\varepsilon,j}^{(1)}, y) + O(\bar{\mu}_\varepsilon^N |\log \bar{\mu}_\varepsilon|), \quad in\ \mathcal{C}^1(\Omega \setminus \bigcup_{j=1}^{k} B_{2r}(x_{\varepsilon,j}^{(1)})), \tag{4.47}$$

*where*

$$B_{\varepsilon,j,0} = \int_{B_r(x_{\varepsilon,j}^{(1)})} D_\varepsilon(y)\xi_\varepsilon(y)\mathrm{d}y \quad and \quad B_{\varepsilon,j,i} = \int_{B_r(x_{\varepsilon,j}^{(1)})} D_\varepsilon(y)\xi_\varepsilon(y)(y - x_{\varepsilon,j}^{(1)})_i \mathrm{d}y.$$

*Moreover, for any fixed large $R > 0$,*

$$B_{\varepsilon,j,0} = -N(N-2)Bb_{j,0}\left(1 + O(\frac{1}{R^2})\right)\bar{\mu}_\varepsilon^{N-2} + C\frac{\log R}{R^2}\bar{\mu}_\varepsilon^{N-2} + o(\bar{\mu}_\varepsilon^{N-1}), \tag{4.48}$$

*and*

$$B_{\varepsilon,j,i} = N(N+2)B_i b_{j,i}\left(1 + O(\frac{1}{R^2})\right)\bar{\mu}_\varepsilon^{N-1} + C\frac{\log R}{R}\bar{\mu}_\varepsilon^{N-1} + o(\bar{\mu}_\varepsilon^{N-1}), \tag{4.49}$$

*where $C$ are constants indenpendent of $\varepsilon$, $R$ and $B_i = \int_{\mathbb{R}^N} \frac{z_i^2}{(1+|z|^2)^{\frac{N+4}{2}}}\mathrm{d}z$.*

*Proof.* For any $x \in \Omega \setminus \bigcup_{j=1}^{k} B_{2r}(x_{\varepsilon,j}^{(1)})$, since

$$\begin{aligned}
&\int_{B_r(x_{\varepsilon,j}^{(1)})} |D_\varepsilon(y)\xi_\varepsilon(y)|\,|y - x_{\varepsilon,j}^{(1)}|^2 \mathrm{d}y \\
&\leq C\int_{B_r(x_{\varepsilon,j}^{(1)})} |y - x_{\varepsilon,j}^{(1)}|^2 \left(\sum_{j=1}^{k} U^{p-1}_{x_{\varepsilon,j}^{(1)},(\mu_{\varepsilon,j}^{(1)})^{-1}}(y)\right)\left(\sum_{l=1}^{2}\sum_{j=1}^{k} \frac{\left|\log(\mu_{\varepsilon,j}^{(l)})^{-1}|x - x_{\varepsilon,j}^{(l)}|\right|}{\left(1 + (\mu_{\varepsilon,j}^{(l)})^{-1}|x - x_{\varepsilon,j}^{(l)}|\right)^{N-2}}\right)\mathrm{d}y \\
&\leq C\bar{\mu}_\varepsilon^N \int_{|z| \leq (\mu_{\varepsilon,j}^{(1)})^{-1}r} (\mu_{\varepsilon,j}^{(1)})^2|z|^2\left(\frac{(\mu_{\varepsilon,j}^{(1)})^{-2}}{(1+|z|)^4} + O(\bar{\mu}_\varepsilon^2)\right) \\
&\qquad\qquad \left(\frac{|\log|z|| + |\log(|z| + O(\bar{\mu}_\varepsilon))|}{(1+|z|)^{N-2}} + O(\bar{\mu}_\varepsilon^{N-2}|\log\bar{\mu}_\varepsilon|)\right)\mathrm{d}z \\
&= O(\bar{\mu}_\varepsilon^N |\log \bar{\mu}_\varepsilon|),
\end{aligned}$$

we find

$$\begin{aligned}
\xi_\varepsilon(x) &= \int_\Omega D_\varepsilon(y)\xi_\varepsilon(y)G(x,y)\mathrm{d}y \\
&= \sum_{j=1}^{k}\int_{B_r(x_{\varepsilon,j}^{(1)})} D_\varepsilon(y)\xi_\varepsilon(y)G(x,y)\mathrm{d}y + \int_{\Omega\setminus\bigcup_{j=1}^{k} B_r(x_{\varepsilon,j}^{(1)})} D_\varepsilon(y)\xi_\varepsilon(y)G(x,y)\mathrm{d}y \\
&= \sum_{j=1}^{k} B_{\varepsilon,j,0}G(x_{\varepsilon,j}^{(1)}, y) + \sum_{j=1}^{k}\sum_{i=1}^{N} B_{\varepsilon,j,i}\partial_i G(x_{\varepsilon,j}^{(1)}, y) \\
&\quad + O\left(\sum_{j=1}^{k}\int_{B_r(x_{\varepsilon,j}^{(1)})} D_\varepsilon(y)\xi_\varepsilon(y)|y - x_{\varepsilon,j}^{(1)}|^2\mathrm{d}y\right) + O(\bar{\mu}_\varepsilon^N|\log\bar{\mu}_\varepsilon|) \\
&= \sum_{j=1}^{k} B_{\varepsilon,j,0}G(x_{\varepsilon,j}^{(1)}, y) + \sum_{j=1}^{k}\sum_{i=1}^{N} B_{\varepsilon,j,i}\partial_i G(x_{\varepsilon,j}^{(1)}, y) + O(\bar{\mu}_\varepsilon^N|\log\bar{\mu}_\varepsilon|).
\end{aligned}$$

Using Proposition 4.7, we can prove the estimations of $B_{\varepsilon,j,i}$ as in Proposition 3.6. □

## 4.3 Proof of Theorem 1.3

**Proposition 4.9.** *Let $N \geq 5$. For $l = 1, 2$ and small $r > 0$, there holds*

$$u_\varepsilon^{(l)}(x) = \alpha_0^p B \sum_{j=1}^{k} (\mu_{\varepsilon,j}^{(1)})^{\frac{N-2}{2}} G(x_{\varepsilon,j}^{(1)}, x) + O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}}), \quad \text{in } \mathcal{C}^1(\Omega \setminus \bigcup_{j=1}^{k} B_{2r}(x_{\varepsilon,j}^{(1)})).$$

*Proof.* It can be proved as Proposition 3.7. □

**Proposition 4.10.** *For $N \geq 7$, there holds*

$$b_{j,0} = 0, \quad j = 1, \cdots, k, \tag{4.50}$$

*where $b_{j,0}$ is the constant in Proposition 4.7.*

*Proof.* Applying (2.31) with $x_* = x_{\varepsilon,j}^{(1)}$ and $u = u_\varepsilon^{(l)}$, we obtain

$$P(u_\varepsilon^{(l)}, u_\varepsilon^{(l)}) = \frac{r}{2^* - \varepsilon} \int_{\partial B_r(x_{\varepsilon,j}^{(1)})} (u_\varepsilon^{(l)})^{2^*-\varepsilon} \mathrm{d}\sigma - \frac{N\varepsilon}{2^*(2^* - \varepsilon)} \int_{B_r(x_{\varepsilon,j}^{(1)})} (u_\varepsilon^{(l)})^{2^*-\varepsilon} \mathrm{d}x, \tag{4.51}$$

We estimate the difference between $P(u_\varepsilon^{(1)}, u_\varepsilon^{(1)})$ and $P(u_\varepsilon^{(2)}, u_\varepsilon^{(2)})$. From Proposition 4.8 and Proposition 4.9, we get

$$\begin{aligned}
&\frac{P(u_\varepsilon^{(1)}, u_\varepsilon^{(1)}) - P(u_\varepsilon^{(2)}, u_\varepsilon^{(2)})}{\|u_\varepsilon^{(1)} - u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}} \\
&= -r \int_{\partial B_r(x_{\varepsilon,j}^{(1)})} \langle \nabla \xi_\varepsilon, \nu \rangle \left\langle \nabla (u_\varepsilon^{(1)} + u_\varepsilon^{(2)}), \nu \right\rangle \mathrm{d}\sigma + \frac{r}{2} \int_{\partial B_r(x_{\varepsilon,j}^{(1)})} \left\langle \nabla (u_\varepsilon^{(1)} + u_\varepsilon^{(2)}), \nabla \xi_\varepsilon \right\rangle \mathrm{d}\sigma \\
&\quad - \frac{N-2}{2} \int_{\partial B_r(x_{\varepsilon,j}^{(1)})} \langle \nabla \xi_\varepsilon, \nu \rangle u_\varepsilon^{(1)} + \left\langle \nabla u_\varepsilon^{(2)}, \nu \right\rangle \xi_\varepsilon \mathrm{d}\sigma \\
&= 2\alpha_0^p B \sum_{l,m=1}^{k} B_{\varepsilon,l,0} (\mu_{\varepsilon,m}^{(1)})^{\frac{N-2}{2}} P\left(G(x_{\varepsilon,m}^{(1)}, \cdot), G(x_{\varepsilon,l}^{(1)}, \cdot)\right) \\
&\quad + 2\alpha_0^p B \sum_{l,m=1}^{k} \sum_{h=1}^{N} B_{\varepsilon,l,h} (\mu_{\varepsilon,m}^{(1)})^{\frac{N-2}{2}} P\left(G(x_{\varepsilon,m}^{(1)}, \cdot), \partial_h G(x_{\varepsilon,l}^{(1)}, \cdot)\right) + O(\bar{\mu}_\varepsilon^{\frac{3N-2}{2}} |\log \bar{\mu}_\varepsilon|) \\
&=: I + II + O(\bar{\mu}_\varepsilon^{\frac{3N-2}{2}} |\log \bar{\mu}_\varepsilon|)
\end{aligned} \tag{4.52}$$

By Lemma 2.3, we find

$$\begin{aligned}
I &= -\frac{N-2}{2} \alpha_o^p B \left[ 2B_{\varepsilon,j,0} (\mu_{\varepsilon,j}^{(1)})^{\frac{N-2}{2}} R(x_{\varepsilon,j}^{(1)}) - B_{\varepsilon,j,0} \sum_{m \neq j, m=1}^{k} (\mu_{\varepsilon,m}^{(1)})^{\frac{N-2}{2}} G(x_{\varepsilon,m}^{(1)}, x_{\varepsilon,j}^{(1)}) \right] \\
&\quad + \frac{N-2}{2} \alpha_o^p B \sum_{l \neq j, l=1}^{k} (\mu_{\varepsilon,j}^{(1)})^{\frac{N-2}{2}} B_{\varepsilon,l,0} G(x_{\varepsilon,l}^{(1)}, x_{\varepsilon,j}^{(1)}) \\
&= -\frac{N-2}{2} \alpha_o^p B \sum_{l=1}^{k} m_{\varepsilon,j,l} d_{\varepsilon,l}
\end{aligned} \tag{4.53}$$

where

$$d_{\varepsilon,l} = B_{\varepsilon,l,0} (\mu_{\varepsilon,l}^{(1)})^{-\frac{N-2}{2}}. \tag{4.54}$$

and

$$
m_{\varepsilon,j,l}=\begin{cases}2(\mu_{\varepsilon,j}^{(1)})^{N-2}R(x_{\varepsilon,j}^{(1)})-\displaystyle\sum_{m\neq j,m=1}^{k}(\mu_{\varepsilon,j}^{(1)})^{\frac{N-2}{2}}(\mu_{\varepsilon,m}^{(1)})^{\frac{N-2}{2}}G(x_{\varepsilon,m}^{(1)},x_{\varepsilon,j}^{(1)}), & \text{for } l=j,\\ -(\mu_{\varepsilon,j}^{(1)})^{\frac{N-2}{2}}(\mu_{\varepsilon,l}^{(1)})^{\frac{N-2}{2}}G(x_{\varepsilon,l}^{(1)},x_{\varepsilon,j}^{(1)}), & \text{for } l\neq j.\end{cases}\tag{4.55}
$$

Also, from Lemma 2.3, Lemma 2.15, Proposition 4.2 and (4.6), we get

$$
\begin{aligned}
II=&-\frac{N-2}{2}\alpha_0^pB\sum_{h=1}^{N}B_{\varepsilon,j,h}\left[(\mu_{\varepsilon,j}^{(1)})^{\frac{N-2}{2}}\partial_hR(x_{\varepsilon,j}^{(1)})-\sum_{m\neq j,m=1}^{k}(\mu_{\varepsilon,m}^{(1)})^{\frac{N-2}{2}}\partial_hG(x_{\varepsilon,j}^{(1)},x_{\varepsilon,m}^{(1)})\right]\\
&-\frac{1}{2}\alpha_0^pB\sum_{h=1}^{N}B_{\varepsilon,j,h}\left[(\mu_{\varepsilon,j}^{(1)})^{\frac{N-2}{2}}\partial_hR(x_{\varepsilon,j}^{(1)})-2\sum_{m\neq j,m=1}^{k}(\mu_{\varepsilon,m}^{(1)})^{\frac{N-2}{2}}\partial_hG(x_{\varepsilon,j}^{(1)},x_{\varepsilon,m}^{(1)})\right]\\
&+\frac{N-2}{2}\alpha_0^pB\sum_{l\neq j,l=1}^{k}\sum_{h=1}^{N}B_{\varepsilon,l,h}(\mu_{\varepsilon,j}^{(1)})^{\frac{N-2}{2}}\partial_hG(x_{\varepsilon,l}^{(1)},x_{\varepsilon,j}^{(1)})\\
=&-\frac{1}{2}\left(\frac{\alpha_0^pA}{2N^2B}\right)^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\lambda_j^{-\frac{N-4}{2}}\sum_{l=1}^{k}\sum_{h=1}^{N}B_{\varepsilon,l,h}\frac{\partial^2\Phi_k(\vec{a},\vec{\lambda})}{\partial y_{(l-1)N+h}\partial\lambda_j}+O(\bar{\mu}_\varepsilon^{\frac{3N}{2}}),
\end{aligned}\tag{4.56}
$$

where

$$
\vec{\lambda}=\vec{\lambda}(\vec{a})=(\lambda_1,\cdots,\lambda_k).\tag{4.57}
$$

Thus from (4.52), (4.53) and (4.56), we obtain

$$
\begin{aligned}
\frac{P(u_\varepsilon^{(1)},u_\varepsilon^{(1)})-P(u_\varepsilon^{(2)},u_\varepsilon^{(2)})}{\|u_\varepsilon^{(1)}-u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}}=&-\frac{N-2}{2}\alpha_o^pB\sum_{l=1}^{k}m_{\varepsilon,j,l}d_{\varepsilon,l}\\
&-\frac{1}{2}\left(\frac{\alpha_0^pA}{2N^2B}\right)^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\lambda_j^{-\frac{N-4}{2}}\sum_{l=1}^{k}\sum_{h=1}^{N}B_{\varepsilon,l,h}\frac{\partial^2\Phi_k(\vec{a},\vec{\lambda})}{\partial y_{(l-1)N+h}\partial\lambda_j}+o(\bar{\mu}_\varepsilon^{\frac{3N-4}{2}}).
\end{aligned}\tag{4.58}
$$

Now let

$$
\tilde{D}_\varepsilon(x)=\int_0^1\left(tu_\varepsilon^{(1)}(x)+(1-t)u_\varepsilon^{(2)}\right)^{p-\varepsilon}\mathrm{d}t.\tag{4.59}
$$

Since $u_\varepsilon^{(l)}=O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}})$ on $\partial B_r(x_{\varepsilon,j}^{(1)})$, we have

$$
\begin{aligned}
&\frac{r}{2^*-\varepsilon}\int_{\partial B_r(x_\varepsilon^{(1)})}\frac{(u_\varepsilon^{(1)})^{2^*-\varepsilon}-(u_\varepsilon^{(2)})^{2^*-\varepsilon}}{\|u_\varepsilon^{(1)}-u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}}\mathrm{d}\sigma\\
&=r\int_{|y-x_\varepsilon^{(1)}|=r}\tilde{D}_\varepsilon(y)\xi_\varepsilon(y)\mathrm{d}\sigma=O(\bar{\mu}_\varepsilon^{\frac{3N-2}{2}}|\log\bar{\mu}_\varepsilon|).
\end{aligned}\tag{4.60}
$$

By a similar approach as (3.34), we also find

$$
\tilde{D}_\varepsilon(x_{\varepsilon,j}^{(1)}+\mu_{\varepsilon,j}^{(1)}y)=\alpha_0^pU_{0,1}^p(y)(\mu_{\varepsilon,j}^{(1)})^{-\frac{N+2}{2}}(1+o(\bar{\mu}_\varepsilon))+O(\bar{\mu}_\varepsilon^{\frac{N-6}{2}}|\log\bar{\mu}_\varepsilon|).
$$

Then for fixed large $R>0$,

$$
\begin{aligned}
&\frac{1}{(2^*-\varepsilon)}\int_{B_r(x_\varepsilon^{(1)})}\frac{(u_\varepsilon^{(1)})^{2^*-\varepsilon}-(u_\varepsilon^{(2)})^{2^*-\varepsilon}}{\|u_\varepsilon^{(1)}-u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}}\mathrm{d}x\\
&=\int_{\left|\frac{x-x_\varepsilon^{(1)}}{\mu_\varepsilon^{(1)}}\right|\leq R}\tilde{D}_\varepsilon(y)\xi_\varepsilon(y)\mathrm{d}y+\int_{R\leq\left|\frac{x-x_\varepsilon^{(1)}}{\mu_\varepsilon^{(1)}}\right|\leq\frac{r}{\mu_\varepsilon^{(1)}}}\tilde{D}_\varepsilon(y)\xi_\varepsilon(y)\mathrm{d}y\\
&=:K_5+K_6.
\end{aligned}
$$

Similar to (3.40) and (3.41), we have

$$K_5 = O(\frac{1}{R^N})\bar{\mu}_\varepsilon^{\frac{N-2}{2}} + o(\bar{\mu}_\varepsilon^{\frac{N}{2}})$$

and

$$K_6 = O(\frac{\log R}{R^N})\bar{\mu}_\varepsilon^{\frac{N-2}{2}} + o(\bar{\mu}_\varepsilon^{\frac{N}{2}}).$$

Thus

$$\frac{1}{(2^*-\varepsilon)}\int_{B_r(x_\varepsilon^{(1)})}\frac{(u_\varepsilon^{(1)})^{2^*-\varepsilon}-(u_\varepsilon^{(2)})^{2^*-\varepsilon}}{\|u_\varepsilon^{(1)}-u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}}\mathrm{d}x = O(\frac{\log R}{R^N})\bar{\mu}_\varepsilon^{\frac{N-2}{2}} + o(\bar{\mu}_\varepsilon^{\frac{N}{2}}). \tag{4.61}$$

From (4.51), (4.58), (4.60) and (4.61), we get

$$M_{\varepsilon,k}\vec{d}_\varepsilon + \varepsilon^{\frac{1}{2}}D_k\nabla^2_{x\lambda}\Phi_k(\vec{a},\vec{\lambda})\bar{B}_{\varepsilon,k} = O(\frac{\log R}{R^N})\bar{\mu}_\varepsilon^{\frac{3N-6}{2}} + o(\bar{\mu}_\varepsilon^{\frac{3N-4}{2}}), \tag{4.62}$$

where the matix $M_{k,\varepsilon} = (m_{\varepsilon,j,l})_{k\times k}$, the matrix $D_k = diag(C\lambda_1^{-\frac{N-4}{2}},\cdots,C\lambda_k^{-\frac{N-4}{2}})$ with constant $C>0$, and the vectors

$$\vec{d}_\varepsilon = \Big(B_{\varepsilon,1,0}(\mu_{\varepsilon,1}^{(1)})^{-\frac{N-2}{2}},\cdots,B_{\varepsilon,k,0}(\mu_{\varepsilon,k}^{(1)})^{-\frac{N-2}{2}}\Big), \tag{4.63}$$

$$\bar{B}_{\varepsilon,k} = (\bar{B}_{\varepsilon,1},\cdots,\bar{B}_{\varepsilon,kN}), \quad \text{with } \bar{B}_{\varepsilon,i} = B_{\varepsilon,j,h}, i=(j-1)N+h. \tag{4.64}$$

From (4.10) and (4.11), we see that for $\varepsilon$ small enough, $M_{\varepsilon,k}$ is the mainly diagonally dominant matrix, and the eigenvalues

$$C_1\varepsilon \le \rho_{min}(M_{\varepsilon,k}) \le \rho_{max}(M_{\varepsilon,k}) \le C_2\varepsilon.$$

So (4.62) means

$$B_{\varepsilon,j,0} = O(\frac{\log R}{R^N})\bar{\mu}_\varepsilon^{N-2} + O(\bar{\mu}_\varepsilon^{N-1}).$$

Then from (4.48), we obtain that $b_{j,0}=0$ and $B_{\varepsilon,j,0} = o(\bar{\mu}_\varepsilon^{N-1})$. Therefore, we find

$$\nabla^2_{x\lambda}\Phi_k(\vec{a},\vec{\lambda})\bar{B}_{\varepsilon,k} = C\frac{\log R}{R^N}\bar{\mu}_\varepsilon^{N-2} + o(\bar{\mu}_\varepsilon^{N-1}),$$

where $C$ is independent of $R,\varepsilon$. It follows from (4.49) that $C=0$ and hence

$$\nabla^2_{x\lambda}\Phi_k(\vec{a},\vec{\lambda})\bar{B}_{\varepsilon,k} = o(\bar{\mu}_\varepsilon^{N-1}). \tag{4.65}$$

□

**Proposition 4.11.** *For $N\ge 7$, there holds*

$$b_{j,i} = 0, \quad j=1,\cdots,k,\ i=1,\cdots,N, \tag{4.66}$$

*where $b_{j,i}$ are the constants in Proposition 4.7.*

*Proof.* Applying (2.32) with $x_* = x_{\varepsilon,j}^{(1)}$ and $u = u_\varepsilon^{(l)}$, we obtain

$$Q(u_\varepsilon^{(l)},u_\varepsilon^{(l)}) = \frac{2}{2^*-\varepsilon}\int_{\partial B_r(x_{\varepsilon,j}^{(1)})}(u_\varepsilon^{(l)})^{2^*-\varepsilon}\nu_i\mathrm{d}\sigma. \tag{4.67}$$

As in Proposition 4.10, we can prove that

$$
\begin{aligned}
&\frac{Q(u_\varepsilon^{(1)},u_\varepsilon^{(1)})-Q(u_\varepsilon^{(2)},u_\varepsilon^{(2)})}{\|u_\varepsilon^{(1)}-u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}}\\
=&\,\alpha_0^pB\sum_{l,m=1}^{k}B_{\varepsilon,l,0}(\mu_{\varepsilon,m}^{(1)})^{\frac{N-2}{2}}Q(G(x_{\varepsilon,m}^{(1)},\cdot),G(x_{\varepsilon,l}^{(1)},\cdot))\\
&+\alpha_0^pB\sum_{l,m=1}^{k}\sum_{h=1}^{N}B_{\varepsilon,l,h}(\mu_{\varepsilon,m}^{(1)})^{\frac{N-2}{2}}Q(G(x_{\varepsilon,m}^{(1)},\cdot),\partial_hG(x_{\varepsilon,l}^{(1)},\cdot))+o(\bar\mu_\varepsilon^{\frac{3n-4}{2}})\\
=&-\sum_{h=1}^{N}\alpha_0^pBB_{\varepsilon,j,h}\left(\frac{1}{2}(\mu_{\varepsilon,j}^{(1)})^{\frac{N-2}{2}}\partial_{ih}^2R(x_{\varepsilon,j}^{(1)})-\sum_{l\neq j,l=1}^{k}(\mu_{\varepsilon,m}^{(1)})^{\frac{N-2}{2}}\partial_{ih}^2G(x_{\varepsilon,j}^{(1)},x_{\varepsilon,l}^{(1)})\right)\\
&+\sum_{h=1}^{N}\sum_{l\neq j,l=1}^{k}\alpha_0^pBB_{\varepsilon,l,h}(\mu_{\varepsilon,j}^{(1)})^{\frac{N-2}{2}}\partial_hD_iG(x_{\varepsilon,j}^{(1)},x_{\varepsilon,l}^{(1)})+o(\bar\mu_\varepsilon^{\frac{3N-4}{2}})\\
=&-\frac{1}{2}\left(\frac{\alpha_0^pA}{2N^2B}\right)^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\lambda_j^{-\frac{N-2}{2}}\sum_{l=1}^{k}\sum_{h=1}^{N}\frac{\partial^2\Phi_k(\vec a,\vec\lambda)}{\partial_{y_{(l-1)N+h}}\partial_{y_{(j-1)+i}}}B_{\varepsilon,l,h}+o(\bar\mu_\varepsilon^{\frac{3N-4}{2}}).
\end{aligned}
\tag{4.68}
$$

Then from (4.60), (4.67) and (4.68), we obtain

$$\nabla^2_{xx}\Phi_k(\vec a,\vec\lambda)\bar B_{\varepsilon,k}=o(\bar\mu_\varepsilon^{N-1}). \tag{4.69}$$

Noting that $(\vec a,\vec\lambda)$ is a nondegenerate critical point of $\Phi_k$, we see

$$\text{Rank}\left(\nabla^2_{(x,\lambda)x}\Phi_k(\vec a,\vec\lambda)\right)=kN. \tag{4.70}$$

Hence, (4.65), (4.69) and (4.70) imply that

$$B_{\varepsilon,j,i}=o(\bar\mu_\varepsilon^{N-1}).$$

It follows from (4.49) that $b_{j,i}=0$. □

*Proof of Theorem 1.3.* It can be proved just like Theorem 1.2. □

**Remark 4.2.** *When $N=6$, we see from (4.29) that $\|S_\varepsilon\|=O(\bar\mu_\varepsilon^4|\log\bar\mu_\varepsilon|)$ and hence $\|\bar\omega_\varepsilon\|=O(\bar\mu_\varepsilon^4|\log\bar\mu_\varepsilon|)$. As a result, we are unable to obtain $|x_{\varepsilon,j}^{(1)}-x_{\varepsilon,j}^{(2)}|=o(\bar\mu_\varepsilon^4)$, instead, we only have $|x_{\varepsilon,j}^{(1)}-x_{\varepsilon,j}^{(2)}|=O(\bar\mu_\varepsilon^4)$. Then similarly as in Proposition 4.50, we have*

$$\nabla^2_{x\lambda}\Phi_k(\vec a,\vec\lambda)\bar B_{\varepsilon,k}=O(\bar\mu_\varepsilon^{N-1}),$$

*which is not enough to prove $b_{j,i}=0$.*

# 5 Nondegeneracy of positive blow-up solutions

## 5.1 Blow-up analysis of the solutions to linearized equation

In this section, we assume that $a_1,\cdots,a_k\in\Omega$ are $k$ different points. Let $\vec a:=(a_1,\cdots,a_k)$, and $\vec\lambda=(\lambda_1,\cdots,\lambda_k)$ be the unique solution of (2.22). We also suppose that $M_k(\vec a)$ is positive and $(\vec a,\vec\lambda)\in\Omega^k\times(\mathbb{R}^+)^k$ is a non-degenerate critical point of $\Phi_k$.

Let $u_\varepsilon$ be solutions of (1.1) satisfying (1.2). We see that

$$u_\varepsilon = \sum_{j=1}^{k} \alpha_{\varepsilon,j} PU_{x_{\varepsilon,j},\mu_{\varepsilon,j}^{-1}} + w_\varepsilon,$$

satisfying, for $N \geq 4$, as $\varepsilon \to 0$,

$$x_{\varepsilon,j} = a_j + O(\bar{\mu}_\varepsilon^2) \tag{5.1}$$

$$\mu_{\varepsilon,j} = \mu_j \varepsilon^{\frac{1}{N-2}} + O(\bar{\mu}_\varepsilon^2) \tag{5.2}$$

$$\alpha_{\varepsilon,j} = \alpha_0 + O(\bar{\mu}_\varepsilon^{N-2}|\log \bar{\mu}_\varepsilon|), \tag{5.3}$$

$$w_\varepsilon \in \bigcap_{j=1}^{k} E_{x_{\varepsilon,j},\mu_{\varepsilon,j}^{-1}}, \quad \text{and} \quad \|w_\varepsilon\| = \begin{cases} O(\bar{\mu}_\varepsilon^{N-2}), & \text{if } N \leq 5, \\ O(\bar{\mu}_\varepsilon^{N-2}|\log \bar{\mu}_\varepsilon|^{2/3}) & \text{if } N = 6, \\ O(\bar{\mu}_\varepsilon^{\frac{N+2}{2}}) & \text{if } N \geq 7, \end{cases} \tag{5.4}$$

where $\bar{\mu}_\varepsilon = \max\{\mu_{\varepsilon,1}, \cdots, \mu_{\varepsilon,k}\}$.

Suppose $\zeta_\varepsilon$ is a solution of

$$\begin{cases} -\Delta v = (p-\varepsilon) u_\varepsilon^{p-1-\varepsilon} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \|v\|_{L^\infty(\Omega)} = 1. \end{cases} \tag{5.5}$$

Then

**Proposition 5.1.** *For $N \geq 4$, we have*

$$|\zeta_\varepsilon(x)| \leq C \sum_{j=1}^{k} \frac{\left|\log \mu_{\varepsilon,j}^{-1}|x - x_{\varepsilon,j}|\right|}{\left(1 + \mu_{\varepsilon,j}^{-1}|x - x_{\varepsilon,j}|\right)^{N-2}}, \quad \textit{in } \Omega. \tag{5.6}$$

*Hence*

$$\int_\Omega |\zeta_\varepsilon| = O(\bar{\mu}_\varepsilon^{N-2}|\log \bar{\mu}_\varepsilon|) \quad \textit{and} \quad \zeta_\varepsilon(x) = O(\bar{\mu}_\varepsilon^{N-2}|\log \bar{\mu}_\varepsilon|) \quad \textit{in } \Omega \setminus \bigcup_{j=1}^{k} B_r(x_{\varepsilon,j}). \tag{5.7}$$

*Proof.* It can be proved as in Proposition 4.6. □

Now let $\tilde{\zeta}_{\varepsilon,j}$ be defined by

$$\tilde{\zeta}_{\varepsilon,j}(y) = \zeta_\varepsilon(x_{\varepsilon,j} + \mu_{\varepsilon,j} y), \quad y \in \Omega_{\varepsilon,j} = \frac{\Omega - x_{\varepsilon,j}}{\mu_{\varepsilon,j}}. \tag{5.8}$$

Then $\tilde{\zeta}_{\varepsilon,j}$ satisfies

$$-\Delta \tilde{\zeta}_{\varepsilon,j} = \mu_{\varepsilon,j}^2 (p-\varepsilon) u_{\varepsilon,j}^{p-1-\varepsilon} \tilde{\zeta}_{\varepsilon,j} \tag{5.9}$$

with

$$u_{\varepsilon,j}(y) = u_\varepsilon(x_{\varepsilon,j} + \mu_{\varepsilon,j} y). \tag{5.10}$$

**Proposition 5.2.** *Let $N \geq 4$ and $\tilde{\zeta}_{\varepsilon,j}$ be defined by* (5.8)*. Then after taking a subsequence if necessary, we have*

$$\tilde{\zeta}_{\varepsilon,j} = \tilde{b}_{j,0}\psi_0 + \sum_{j=1}^{k}\sum_{i=1}^{N} \tilde{b}_{j,i}\psi_i + o(\bar{\mu}_\varepsilon), \quad \mathcal{C}^1_{loc}(\mathbb{R}^N), \tag{5.11}$$

*where $\tilde{b}_{j,i}$ are constants for $j = 1, \cdots, k$, $i = 0, 1, \cdots, N$ , $\psi_0 = \frac{1-|y|^2}{(1+|y|^2)^{N/2}}$ and $\psi_i = \frac{y_i}{(1+|y|^2)^{N/2}}$.*

*Proof.* First we need to estimate $(p-\varepsilon)u_{\varepsilon,j}^{p-1-\varepsilon}$. For any $R>0$, $y\in B_R(0)$ and $l\neq j$,

$$PU_{x_{\varepsilon,l},\mu_{\varepsilon,l}^{-1}}(x_{\varepsilon,l}+\mu_{\varepsilon,l}y)=O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}}).$$

Then

$$\begin{aligned}u_\varepsilon(x_{\varepsilon,j}+\mu_{\varepsilon,j}y)&=\alpha_{\varepsilon,j}U_{x_{\varepsilon,j},\mu_{\varepsilon,j}^{-1}}(x_{\varepsilon,j}+\mu_{\varepsilon,j}y)-\phi_{x_{\varepsilon,j},\mu_{\varepsilon,j}^{-1}}(x_{\varepsilon,j}+\mu_{\varepsilon,j}y)\\&\quad+\sum_{l\neq j,l=1}^{k}PU_{x_{\varepsilon,l},\mu_{\varepsilon,l}^{-1}}(x_{\varepsilon,j}+\mu_{\varepsilon,j}y)+w_\varepsilon(x_{\varepsilon,j}+\mu_{\varepsilon,j}y)\\&=\alpha_{\varepsilon,j}\mu_{\varepsilon,j}^{-\frac{N-2}{2}}U_{0,1}(y)+O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}}),\quad\text{in } B_R(0),\end{aligned}\tag{5.12}$$

and

$$\begin{aligned}(p-\varepsilon)u_{\varepsilon,j}^{p-1-\varepsilon}(y)&=(p-\varepsilon)\left(\alpha_{\varepsilon,j}\mu_{\varepsilon,j}^{-\frac{N-2}{2}}U_{0,1}(y)+O(\bar{\mu}_\varepsilon^{\frac{N-2}{2}})\right)^{p-1-\varepsilon}\\&=N(N+2)U_{0,1}^{p-1}(y)\mu_{\varepsilon,j}^{-2}(1+o(\bar{\mu}_\varepsilon))+O(\bar{\mu}_\varepsilon^{N-4}|\log\bar{\mu}_\varepsilon|)\quad\text{in } B_R(0).\end{aligned}\tag{5.13}$$

Hence

$$-\Delta\tilde{\zeta}_{\varepsilon,j}=N(N+2)U_{0,1}^{p-1}\tilde{\zeta}_{\varepsilon,j}+o(\bar{\mu}_\varepsilon)\quad\text{in } B_R(0).\tag{5.14}$$

So (4.46) follows as in Proposition 3.5. □

**Remark 5.1.** *When $N=3$, we actually have*

$$\tilde{\zeta}_{\varepsilon,j}=\tilde{b}_{j,0}\psi_0+\sum_{j=1}^{k}\sum_{i=1}^{N}\tilde{b}_{j,i}\psi_i+O(\bar{\mu}_\varepsilon|\log\bar{\mu}_\varepsilon|),\quad\mathcal{C}^1_{loc}(\mathbb{R}^N).\tag{5.15}$$

**Proposition 5.3.** *Let $N\geq 4$. Then for small $r>0$, there holds*

$$\zeta_\varepsilon(y)=\sum_{j=1}^{k}\tilde{B}_{\varepsilon,j,0}G(x_{\varepsilon,j},y)+\sum_{j=1}^{k}\sum_{i=1}^{N}\tilde{B}_{\varepsilon,j,i}\partial_iG(x_{\varepsilon,j},y)+O(\bar{\mu}_\varepsilon^N|\log\bar{\mu}_\varepsilon|),\quad\text{in } \mathcal{C}^1(\Omega\setminus\bigcup_{j=1}^{k}B_{2r}(x_{\varepsilon,j})),\tag{5.16}$$

*where*

$$\tilde{B}_{\varepsilon,j,0}=(p-\varepsilon)\int_{B_r(x_{\varepsilon,j})}u_\varepsilon^{p-1-\varepsilon}(y)\zeta_\varepsilon(y)\mathrm{d}y\quad\text{and}\quad\tilde{B}_{\varepsilon,j,i}=(p-\varepsilon)\int_{B_r(x_{\varepsilon,j})}(y-x_{\varepsilon,j})_iu_\varepsilon^{p-1-\varepsilon}(y)\zeta_\varepsilon(y)\mathrm{d}y.$$

*Moreover, for any fixed large $R>0$,*

$$\tilde{B}_{\varepsilon,j,0}=-N(N-2)B\tilde{b}_{j,0}\left(1+O(\frac{1}{R^2})\right)\bar{\mu}_\varepsilon^{N-2}+C\frac{\log R}{R^2}\bar{\mu}_\varepsilon^{N-2}+o(\bar{\mu}_\varepsilon^{N-1}),\tag{5.17}$$

*and*

$$\tilde{B}_{\varepsilon,j,i}=N(N+2)B_i\tilde{b}_{j,i}\left(1+O(\frac{1}{R^2})\right)\bar{\mu}_\varepsilon^{N-1}+C\frac{\log R}{R}\bar{\mu}_\varepsilon^{N-1}+o(\bar{\mu}_\varepsilon^{N-1}),\tag{5.18}$$

*where $C$ are constants indenpendent of $\varepsilon$, $R$ and $B_i=\int_{\mathbb{R}^N}\frac{z_i^2}{(1+|z|^2)^{\frac{N+4}{2}}}\mathrm{d}z$.*

*Proof.* For any $x\in\Omega\setminus\bigcup_{j=1}^{k}B_{2r}(x_{\varepsilon,j})$, since

$$\begin{aligned}&\int_{B_r(x_{\varepsilon,j})}\left|u_\varepsilon^{p-1-\varepsilon}(y)\zeta_\varepsilon(y)\right||y-x_{\varepsilon,j}|^2\mathrm{d}y\\&\leq C\bar{\mu}_\varepsilon^N\int_{B_r(x_{\varepsilon,j})}|y-x_{\varepsilon,j}|^2\left(\sum_{j=1}^{k}U^{p-1}_{x_{\varepsilon,j},\mu_{\varepsilon,j}^{-1}}(y)\right)\left(\sum_{j=1}^{k}\frac{\left|\log\mu_{\varepsilon,j}^{-1}|x-x_{\varepsilon,j}|\right|}{\left(1+\mu_{\varepsilon,j}^{-1}|x-x_{\varepsilon,j}|\right)^{N-2}}\right)\mathrm{d}y\\&=O(\bar{\mu}_\varepsilon^N|\log\bar{\mu}_\varepsilon|),\end{aligned}\tag{5.19}$$

we find

$$
\begin{aligned}
\zeta_\varepsilon(x) &= (p-\varepsilon)\int_\Omega u_\varepsilon^{p-1-\varepsilon}\zeta_\varepsilon(y)G(x,y)\mathrm{d}y \\
&= (p-\varepsilon)\left(\sum_{j=1}^k\int_{B_r(x_{\varepsilon,j})} u_\varepsilon^{p-1-\varepsilon}\zeta_\varepsilon(y)G(x,y)\mathrm{d}y + \int_{\Omega\setminus\bigcup_{j=1}^k B_r(x_{\varepsilon,j})} u_\varepsilon^{p-1-\varepsilon}\zeta_\varepsilon(y)G(x,y)\mathrm{d}y\right) \\
&= \sum_{j=1}^k \tilde{B}_{\varepsilon,j,0}G(x_{\varepsilon,j},y) + \sum_{j=1}^k\sum_{i=1}^N \tilde{B}_{\varepsilon,j,i}\partial_i G(x_{\varepsilon,j},y) \\
&\quad + O\left(\sum_{j=1}^k\int_{B_r(x_{\varepsilon,j})} u_\varepsilon^{p-1-\varepsilon}\zeta_\varepsilon(y)|y-x_{\varepsilon,j}|^2\mathrm{d}y\right) + O(\bar{\mu}_\varepsilon^N|\log\bar{\mu}_\varepsilon|) \\
&= \sum_{j=1}^k \tilde{B}_{\varepsilon,j,0}G(x_{\varepsilon,j},y) + \sum_{j=1}^k\sum_{i=1}^N \tilde{B}_{\varepsilon,j,i}\partial_i G(x_{\varepsilon,j},y) + O(\bar{\mu}_\varepsilon^N|\log\bar{\mu}_\varepsilon|).
\end{aligned}
\tag{5.20}
$$

Using Proposition 5.2, we can prove the estimations of $\tilde{B}_{\varepsilon,j,i}$ as in Proposition 3.6. □

## 5.2 Proof of Theorem 1.5 and Theorem 1.6

**Proposition 5.4.** *For $N \geq 4$, there holds*

$$\tilde{b}_{j,0} = 0, \quad j = 1,\cdots,k, \tag{5.21}$$

*where $\tilde{b}_{j,0}$ is the constant in Proposition 5.2.*

*Proof.* Applying (2.35) with $x_* = x_{\varepsilon,j}$ and $u = u_\varepsilon$, $v = \zeta_\varepsilon$, we obtain

$$P(u_\varepsilon,\zeta_\varepsilon) = \frac{r}{2}\int_{\partial B_r(x_{\varepsilon,j})} u_\varepsilon^{p-\varepsilon}\zeta_\varepsilon\mathrm{d}\sigma - \frac{(N-2)\varepsilon}{4}\int_{B_r(x_{\varepsilon,j})} u_\varepsilon^{p-\varepsilon}\zeta_\varepsilon\mathrm{d}x, \tag{5.22}$$

From Proposition 5.3 and Proposition 4.1, we get as in Proposition 4.10 that

$$
\begin{aligned}
P(u_\varepsilon,\zeta_\varepsilon) &= \alpha_0^p B\sum_{l,m=1}^k \tilde{B}_{\varepsilon,l,0}\mu_{\varepsilon,m}^{\frac{N-2}{2}} P\left(G(x_{\varepsilon,m},\cdot),G(x_{\varepsilon,l},\cdot)\right) \\
&\qquad + \alpha_0^p B\sum_{l,m=1}^k\sum_{h=1}^N \tilde{B}_{\varepsilon,l,h}\mu_{\varepsilon,m}^{\frac{N-2}{2}} P\left(G(x_{\varepsilon,m},\cdot),\partial_h G(x_{\varepsilon,l},\cdot)\right) + O(\bar{\mu}_\varepsilon^{\frac{3N-2}{2}}|\log\bar{\mu}_\varepsilon|) \\
&= -\frac{N-2}{4}\alpha_o^p B\sum_{l=1}^k \tilde{m}_{\varepsilon,j,l}\tilde{d}_{\varepsilon,l} \\
&\qquad - \frac{1}{4}\left(\frac{\alpha_0^p A}{2N^2B}\right)^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\lambda_j^{-\frac{N-4}{2}}\sum_{l=1}^k\sum_{h=1}^N \tilde{B}_{\varepsilon,l,h}\frac{\partial^2\Phi_k(\vec{a},\vec{\lambda})}{\partial y_{(l-1)N+h}\partial\lambda_j} + o(\bar{\mu}_\varepsilon^{\frac{3N-4}{2}}),
\end{aligned}
\tag{5.23}
$$

where

$$\tilde{d}_{\varepsilon,l} = \tilde{B}_{\varepsilon,l,0}\mu_{\varepsilon,l}^{-\frac{N-2}{2}}. \tag{5.24}$$

and

$$
\tilde{m}_{\varepsilon,j,l} = \begin{cases} 2\mu_{\varepsilon,j}^{N-2}R(x_{\varepsilon,j}) - \displaystyle\sum_{m\neq j,m=1}^k \mu_{\varepsilon,j}^{\frac{N-2}{2}}\mu_{\varepsilon,m}^{\frac{N-2}{2}}G(x_{\varepsilon,m},x_{\varepsilon,j}), & \text{for } l = j, \\ -\mu_{\varepsilon,j}^{\frac{N-2}{2}}\mu_{\varepsilon,l}^{\frac{N-2}{2}}G(x_{\varepsilon,l},x_{\varepsilon,j}), & \text{for } l \neq j. \end{cases}
\tag{5.25}
$$

Moreover, we also have

$$\int_{\partial B_r(x_{\varepsilon,j})} u_\varepsilon^{p-\varepsilon}\zeta_\varepsilon \mathrm{d}\sigma = O(\bar{\mu}_\varepsilon^{\frac{3N-2}{2}}|\log\bar{\mu}_\varepsilon|).,\tag{5.26}$$

and

$$\int_{B_r(x_{\varepsilon,j})} u_\varepsilon^{p-\varepsilon}\zeta_\varepsilon \mathrm{d}x = O(\frac{\log R}{R^N})\bar{\mu}_\varepsilon^{\frac{N-2}{2}} + o(\bar{\mu}_\varepsilon^{\frac{N}{2}}).\tag{5.27}$$

From (5.22), (5.23), (5.26) and (5.27), we get

$$\tilde{M}_{\varepsilon,k}\vec{d}_\varepsilon + \varepsilon^{\frac{1}{2}}D_k\nabla^2_{x\lambda}\Phi_k(\vec{a},\vec{\lambda})\bar{B}_{\varepsilon,k} = O(\frac{\log R}{R^N})\bar{\mu}_\varepsilon^{\frac{3N-6}{2}} + o(\bar{\mu}_\varepsilon^{\frac{3N-4}{2}}),\tag{5.28}$$

where the matix $\tilde{M}_{k,\varepsilon} = (\tilde{m}_{\varepsilon,j,l})_{k\times k}$, the matrix $D_k = diag(C\lambda_1^{-\frac{N-4}{2}},\cdots,C\lambda_k^{-\frac{N-4}{2}})$ with constant $C>0$, and the vectors

$$\vec{d}_\varepsilon = \left(\tilde{B}_{\varepsilon,1,0}\mu_{\varepsilon,1}^{-\frac{N-2}{2}},\cdots,\tilde{B}_{\varepsilon,k,0}\mu_{\varepsilon,k}^{-\frac{N-2}{2}}\right),\tag{5.29}$$

$$\bar{B}_{\varepsilon,k} = (\bar{B}_{\varepsilon,1},\cdots,\bar{B}_{\varepsilon,kN}),\quad \text{with } \bar{B}_{\varepsilon,i} = \tilde{B}_{\varepsilon,j,h}, i=(j-1)N+h.\tag{5.30}$$

From (4.10) and (4.11), we see that for $\varepsilon$ small enough, $\tilde{M}_{\varepsilon,k}$ is the mainly diagonally dominant matrix, and the eigenvalues

$$C_1\varepsilon \le \rho_{min}(\tilde{M}_{\varepsilon,k}) \le \rho_{max}(\tilde{M}_{\varepsilon,k}) \le C_2\varepsilon.$$

So (5.28) means

$$\tilde{B}_{\varepsilon,j,0} = O(\frac{\log R}{R^N})\bar{\mu}_\varepsilon^{N-2} + O(\bar{\mu}_\varepsilon^{N-1}).$$

Then from (4.48), we obtain that $\tilde{b}_{j,0}=0$ and $\tilde{B}_{\varepsilon,j,0} = o(\bar{\mu}_\varepsilon^{N-1})$. Therefore, we find

$$\nabla^2_{x\lambda}\Phi_k(\vec{a},\vec{\lambda})\bar{B}_{\varepsilon,k} = C\frac{\log R}{R^N}\bar{\mu}_\varepsilon^{N-2} + o(\bar{\mu}_\varepsilon^{N-1}),\tag{5.31}$$

where $C$ is nondependent of $R,\varepsilon$. It follows from (5.18) that $C=0$ and hence

$$\nabla^2_{x\lambda}\Phi_k(\vec{a},\vec{\lambda})\bar{B}_{\varepsilon,k} = o(\bar{\mu}_\varepsilon^{N-1}).\tag{5.32}$$

□

**Proposition 5.5.** *For $N\ge 4$, there holds*

$$\tilde{b}_{j,i} = 0,\quad j=1,\cdots,k,\ i=1,\cdots,N,\tag{5.33}$$

*where $\tilde{b}_{j,i}$ are the constants in Proposition 5.2.*

*Proof.* Applying (2.36) with $x_* = x_{\varepsilon,j}$ and $u=u_\varepsilon$, $v=\zeta_\varepsilon$, we obtain

$$Q(u_\varepsilon,\zeta_\varepsilon) = \int_{\partial B_r(x_{\varepsilon,j})} u_\varepsilon^{p-\varepsilon}\zeta_\varepsilon\nu_i \mathrm{d}\sigma.\tag{5.34}$$

As in Proposition 4.66, we can prove that

$$
\begin{aligned}
Q(u_\varepsilon,\zeta_\varepsilon) &= \alpha_0^p B\sum_{l,m=1}^{k}\tilde B_{\varepsilon,l,0}\mu_{\varepsilon,m}^{\frac{N-2}{2}}Q(G(x_{\varepsilon,m},\cdot),G(x_{\varepsilon,l},\cdot))\\
&\qquad+\alpha_0^p B\sum_{l,m=1}^{k}\sum_{h=1}^{N}\tilde B_{\varepsilon,l,h}\mu_{\varepsilon,m}^{\frac{N-2}{2}}Q(G(x_{\varepsilon,m},\cdot),\partial_h G(x_{\varepsilon,l},\cdot))+o(\bar\mu_\varepsilon^{\frac{3N-4}{2}})\\
&= -\sum_{h=1}^{N}\alpha_0^p B\tilde B_{\varepsilon,j,h}\left(\frac12\mu_{\varepsilon,j}^{\frac{N-2}{2}}\partial^2_{ih}R(x_{\varepsilon,j})-\sum_{l\neq j,l=1}^{k}\mu_{\varepsilon,m}^{\frac{N-2}{2}}\partial^2_{ih}G(x_{\varepsilon,j},x_\varepsilon^{(l)})\right) \qquad (5.35)\\
&\qquad+\sum_{h=1}^{N}\sum_{l\neq j,l=1}^{k}\alpha_0^p B\tilde B_{\varepsilon,l,h}\mu_{\varepsilon,j}^{\frac{N-2}{2}}\partial_h D_i G(x_{\varepsilon,j},x_{\varepsilon,l})+o(\bar\mu_\varepsilon^{\frac{3N-4}{2}})\\
&= -\frac12\left(\frac{\alpha_0^p A}{2N^2B}\right)^{\frac12}\varepsilon^{\frac12}\lambda_j^{-\frac{N-2}{2}}\sum_{l=1}^{k}\sum_{h=1}^{N}\frac{\partial^2\Phi_k(\vec a,\vec\lambda)}{\partial_{y_{(l-1)N+h}}\partial_{y_{(j-1)+i}}}\tilde B_{\varepsilon,l,h}+o(\bar\mu_\varepsilon^{\frac{3N-4}{2}}).
\end{aligned}
$$

Then from (5.34), (5.35) and (5.26), we obtain

$$\nabla^2_{xx}\Phi_k(\vec a,\vec\lambda)\bar B_{\varepsilon,k}=o(\bar\mu_\varepsilon^{N-1}), \tag{5.36}$$

with $\bar B_{\varepsilon,k}$ defined by (5.30). Hence, (5.32), (5.36) and (4.70) imply that

$$\tilde B_{\varepsilon,j,i}=o(\bar\mu_\varepsilon^{N-1}). \tag{5.37}$$

It follows from (5.18) that $\tilde b_{j,i}=0$. □

**Remark 5.2.** *When $N=3$, we can only get from* (5.15) *that*

$$\nabla^2_{x\lambda}\Phi_k(\vec a,\vec\lambda)\bar B_{\varepsilon,k}=O(\bar\mu_\varepsilon^{N-1}), \tag{5.38}$$

*which is not enough to prove $\tilde b_{j,i}=0$.*

*Proof of Theorem 1.5.* We prove it by contradiction. Suppose $\zeta_{\varepsilon_n}\not\equiv 0$ for a sequence $\varepsilon_n\to 0$. Without loss of generality, we can assume $\|\zeta_{\varepsilon_n}\|_{L^\infty(\Omega)}=1$ for every $n\geq 1$. Then just like in the proof of Theorem 1.2, we can get a contradiction. □

*Proof of Theorem 1.6.* When $N\geq 4$, it has been proved in Theorem 1.5. When $N=3$, proceeding exactly as in Section 3, we can actually prove

$$\tilde\zeta_{\varepsilon,j}=O(\bar\mu_\varepsilon|\log\bar\mu_\varepsilon|),\quad \mathcal{C}^1_{loc}(\mathbb{R}^N).$$

Then a contradiction disscussion gives the final conclusion. □

# 6 Proof of Theorem 1.1 and Theorem 1.4

*Proof of Theorem 1.1.* The existence of a solution to (1.1) can be abtained easily by using the Nehari manifold.

We prove the uniqueness by contradiction. Suppose that there exists a sequence $\varepsilon_n\to 0$ such that equation (1.1) with $\varepsilon=\varepsilon_n$ have two different solutions $u_n^{(1)}$, $u_n^{(2)}$ for every $n\geq 1$. Since the domain $\Omega$ is convex, $u_n^{(1)}$ and $u_n^{(2)}$ must blow-up at $k$ different points $a_1,\cdots,a_k$. By Lemma 2.9, and Lemma 2.2, we see that there must be $k=1$, i.e., $u_n^{(l)}$ satisfy

$$|\nabla u_n^{(l)}|^2\rightharpoonup \mathcal{S}^{N/2}\delta_{x^{(l)}}$$

for two critical points $x^{(1)}, x^{(2)} \in \Omega$ of $R(x)$. On the other hand, according to Lemma 2.1, when the domain $\Omega$ is convex, it holds $x^{(1)} = x^{(2)}$. Since $x^{(1)}$ is nondegenerate, Theorem 1.2 applies. Then we obtain $u_n^{(1)} = u_n^{(2)}$ for $n$ large, which is a contradiction.

As for the nondegeneracy, suppose to the contrary, there exists a sequence $\varepsilon_n \to 0$ such that the solution $u_n$ to (1.1) with $\varepsilon = \varepsilon_n$ is degenerate. That is, for any $n \geq 1$, there exists a nontrivial solution $\zeta_n$ of (1.5) with $u = u_{\varepsilon_n}$, $\varepsilon = \varepsilon_n$. However, from Theorem 1.5, we see that $\zeta_n \equiv 0$ for $n$ large, which is a contradiction. $\square$

*Proof of Theorem 1.4.* From Lemma 2.5, we know that

$$\|u_\varepsilon\| \leq S, \quad \text{for any } 0 < \varepsilon < \varepsilon_0.$$

Denote $k_0 = \frac{S}{\mathcal{S}_n^{N/2}}$.

For any $1 \leq k \leq k_0$, since (A2)$_k$ holds, we obtain from [30] that

$$\text{the number of solutions to (1.1) } \geq \sum_{k=1}^{k_0} |\mathcal{T}_k|,$$

for $0 < \varepsilon < \varepsilon_0'$, where $|\mathcal{T}_k|$ is the number of elements in $\mathcal{T}_k$ and $0 < \varepsilon_0' < \varepsilon_0$. Suppose that there exists a sequence $\varepsilon_n \to 0$ such that

$$\text{the number of solutions to (1.1) with } \varepsilon = \varepsilon_n \ > \sum_{k=1}^{k_0} |\mathcal{T}_k|,$$

for any $n \geq 1$. Since $\Omega$ satisfies (A1), we have a solutions $u_{\varepsilon_n}$ to (1.1) with $\varepsilon = \varepsilon_n$ satisfies (up to a subsequence)

$$u_{\varepsilon_n} = \sum_{j=1}^{k} \alpha_{\varepsilon_n,j} PU_{x_{\varepsilon_n,j}, \mu^{-1}_{\varepsilon_n,j}} + w_{\varepsilon_n},$$

for some $1 \leq k \leq k_0$, where $x_{\varepsilon_n,j} \to a_j$ with $(\vec{a}, \vec{\lambda}(\vec{a})) \in \mathcal{T}_k$. Then there must exist a $1 \leq k \leq k_0$ and two sequences of solutions $u_{\varepsilon_n}^{(1)}$, $u_{\varepsilon_n}^{(2)}$ to (1.1) with $\varepsilon = \varepsilon_n$ such that they blow-up at the same points $a_1, \cdots, a_k$ and $u_{\varepsilon_n}^{(1)} \not\equiv u_{\varepsilon_n}^{(2)}$ for $n \geq 1$. However, from Theorem 1.3, we see that $u_{\varepsilon_n}^{(1)} \equiv u_{\varepsilon_n}^{(2)}$ for $n$ large, which is a contradiction. That is

$$\text{the number of solutions to (1.1) } = \sum_{k=1}^{k_0} |\mathcal{T}_k|,$$

for $0 < \varepsilon < \varepsilon_0'$. $\square$

# Appendix A  Some Computations

In this section, we give some important estimations.

**Lemma A.1.** *Let $N \geq 4$. There hold*

$$U^\varepsilon_{x_\varepsilon, \mu_\varepsilon^{-1}} = 1 + O(\mu_\varepsilon^{N-2} |\log \mu_\varepsilon|), \tag{A.1}$$

$$PU^\varepsilon_{x_\varepsilon, \mu_\varepsilon^{-1}} = 1 + O(\mu_\varepsilon^{N-2} |\log \mu_\varepsilon|), \tag{A.2}$$

$$\int_\Omega |PU_{x_\varepsilon, \mu_\varepsilon^{-1}}|^{p+1-\varepsilon} = A + O(\mu_\varepsilon^{N-2} |\log \mu_\varepsilon|), \tag{A.3}$$

$$\int_\Omega |PU_{x_\varepsilon, \mu_\varepsilon^{-1}}|^{p} = \mu_\varepsilon^{\frac{N-2}{2}} B + O(\mu_\varepsilon^{\frac{N+2}{2}}), \tag{A.4}$$

$$\int_\Omega |PU_{x_\varepsilon,\mu_\varepsilon^{-1}}|^{p-\varepsilon} = \mu_\varepsilon^{\frac{N-2}{2}} B + \begin{cases} O(\mu_\varepsilon^{\frac{3N-6}{2}}|\log\mu_\varepsilon|), & N=4, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\geq 5, \end{cases} \tag{A.5}$$

$$\int_\Omega |PU_{x_\varepsilon,\mu_\varepsilon^{-1}}|^{p-1-\varepsilon} = O(\mu_\varepsilon^2), \tag{A.6}$$

$$\int_\Omega |PU_{x_\varepsilon,\mu_\varepsilon^{-1}}|^{p-2-\varepsilon} = O(\mu_\varepsilon^{\frac{6-N}{2}}), \quad \textit{for } N=4,5, \tag{A.7}$$

*Proof.* The equation (A.1) was proved in [32]. Then

$$PU^\varepsilon_{x_\varepsilon,\mu_\varepsilon^{-1}} = (U_{x_\varepsilon,\mu_\varepsilon^{-1}} - \varphi_{x_\varepsilon,\mu_\varepsilon^{-1}})^\varepsilon = U^\varepsilon_{x_\varepsilon,\mu_\varepsilon^{-1}} e^{\varepsilon\log(1-\frac{\varphi_{x_\varepsilon,\mu_\varepsilon^{-1}}}{U_{x_\varepsilon,\mu_\varepsilon^{-1}}})} = 1 + O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|),$$

and hence

$$\begin{aligned}\int_\Omega |PU_{x_\varepsilon,\mu_\varepsilon^{-1}}|^{p+1-\varepsilon} &= (1+O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|))\left(\int_\Omega U^{p+1}_{x_\varepsilon,\mu_\varepsilon^{-1}} + O(\int_\Omega U^p_{x_\varepsilon,\mu_\varepsilon^{-1}})\varphi_{x_\varepsilon,\mu_\varepsilon^{-1}}\right) \\ &= A + O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|).\end{aligned}$$

From Lemma 2.14, we have

$$\begin{aligned}\int_\Omega |PU_{x_\varepsilon,\mu_\varepsilon^{-1}}|^p &= \int_\Omega U^p_{x_\varepsilon,\mu_\varepsilon^{-1}} + pU^{p-1}_{x_\varepsilon,\mu_\varepsilon^{-1}}\varphi_{x_\varepsilon,\mu_\varepsilon^{-1}} + O(\varphi^p_{x_\varepsilon,\mu_\varepsilon^{-1}} + U^{p-p^*}_{x_\varepsilon,\mu_\varepsilon^{-1}}\varphi^{p^*}_{x_\varepsilon,\mu_\varepsilon^{-1}}) \\ &= \mu_\varepsilon^{\frac{N-2}{2}} B + O(\mu_\varepsilon^{\frac{N+2}{2}}).\end{aligned}$$

Then,

$$\begin{aligned}\int_\Omega |PU_{x_\varepsilon,\mu_\varepsilon^{-1}}|^{p-\varepsilon} &= (1+O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|))\int_\Omega |PU_{x_\varepsilon,\mu_\varepsilon^{-1}}|^p \\ &= \mu_\varepsilon^{\frac{N-2}{2}} B + \begin{cases} O(\mu_\varepsilon^{\frac{3N-6}{2}}|\log\mu_\varepsilon|), & N=4, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\geq 5. \end{cases}\end{aligned}$$

Finally,

$$\int_\Omega |PU_{x_\varepsilon,\mu_\varepsilon^{-1}}|^{p-1-\varepsilon} = O(\int_\Omega U^{p-1-\varepsilon}_{x_\varepsilon,\mu_\varepsilon^{-1}} + \varphi^{p-1-\varepsilon}_{x_\varepsilon,\mu_\varepsilon^{-1}}) = O(\mu_\varepsilon^2),$$

and

$$\int_\Omega |PU_{x_\varepsilon,\mu_\varepsilon^{-1}}|^{p-2-\varepsilon} = O(\mu_\varepsilon^{\frac{6-N}{2}}), \quad \text{for } N=4,5.$$

□

**Lemma A.2.** *Let $N\geq 3$ and $u_\varepsilon$ be as in Section 3.1. There hold*

$$\int_{B_r(x_\varepsilon)} |u_\varepsilon|^{p+1-\varepsilon} = \alpha_0^{p+1} A + O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|), \tag{A.8}$$

$$\int_{B_r(x_\varepsilon)} |u_\varepsilon|^{p-\varepsilon} = \alpha_0^p B \mu_\varepsilon^{\frac{N-2}{2}} + \begin{cases} O(\mu_\varepsilon^{\frac{3}{2}}|\log\mu_\varepsilon|), & N=3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N=4, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\geq 5, \end{cases} \tag{A.9}$$

$$\int_{B_r(x_\varepsilon)} (x-x_\varepsilon)_i |u_\varepsilon|^{p-\varepsilon} = \begin{cases} O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N=3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\geq 4, \end{cases} \tag{A.10}$$

$$\int_{B_r(x_\varepsilon)} (x-x_\varepsilon)_i(x-x_\varepsilon)_j|u_\varepsilon|^{p-\varepsilon} = \delta_{ij}\mu_\varepsilon^{\frac{N+2}{2}}\frac{1}{N}\int_{|y|\le\mu_\varepsilon^{-1}r}\frac{|y|^2}{(1+|y|^2)^{\frac{N+2}{2}}}\mathrm{d}y + O(\mu_\varepsilon^{\frac{N+2}{2}}), \tag{A.11}$$

$$\int_{B_r(x_\varepsilon)} |x-x_\varepsilon|^3|u_\varepsilon|^{p-\varepsilon} = O(\mu_\varepsilon^{\frac{N+2}{2}}). \tag{A.12}$$

*Proof.* Denote

$$u_\varepsilon = \alpha_0 U_{x_\varepsilon,\mu_\varepsilon^{-1}} + v_\varepsilon$$

with

$$v_\varepsilon = (\alpha_\varepsilon-\alpha_0)PU_{x_\varepsilon,\mu_\varepsilon^{-1}} - \alpha_0\phi_{x_\varepsilon,\mu_\varepsilon^{-1}} + w_\varepsilon.$$

Then

$$|v_\varepsilon| \le O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|)U_{x_\varepsilon,\mu_\varepsilon^{-1}} + O(\mu_\varepsilon^{\frac{N-2}{2}}),$$

uniformly in $\Omega$. We prove the equations one-by-one.

(1) By Lemma 2.14, we have

$$\begin{aligned}\int_{B_r(x_\varepsilon)} u_\varepsilon^{p+1-\varepsilon}\mathrm{d}x &= \int_{B_r(x_\varepsilon)}(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}} + v_\varepsilon)^{p+1-\varepsilon}\mathrm{d}x\\ &= \int_{B_r(x_\varepsilon)}(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}})^{p+1-\varepsilon} + O(1)U^{p-\varepsilon}_{x_\varepsilon,\mu_\varepsilon^{-1}}v_\varepsilon + O(1)v_\varepsilon^{p+1-\varepsilon}\mathrm{d}x.\end{aligned}$$

Since

$$\begin{aligned}\int_{B_r(x_\varepsilon)}(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}})^{p+1-\varepsilon}\mathrm{d}x &= \alpha_0^{p+1}(1+O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|))\int_{|y|\le\mu_\varepsilon^{-1}r}\frac{\mathrm{d}y}{(1+|y|^2)^N}\\ &= \alpha_0^{p+1}(1+O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|))(A+O(\mu_\varepsilon^N))\\ &= \alpha_0^{p+1}A + O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|),\end{aligned}$$

$$\begin{aligned}\int_{B_r(x_\varepsilon)} U^{p-\varepsilon}_{x_\varepsilon,\mu_\varepsilon^{-1}}v_\varepsilon\mathrm{d}x &= O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|)\int_{B_r(x_\varepsilon)}U^{p+1}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x + O(\mu_\varepsilon^{\frac{N-2}{2}})\int_{B_r(x_\varepsilon)}U^p_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x\\ &= O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|) + O(\mu_\varepsilon^{N-2}),\end{aligned}$$

and

$$\begin{aligned}\int_{B_r(x_\varepsilon)} v_\varepsilon^{p+1-\varepsilon}\mathrm{d}x &= O(\mu_\varepsilon^{2N}|\log\mu_\varepsilon|^{p+1})\int_{B_r(x_\varepsilon)}U^{p+1}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x + O(\mu_\varepsilon^N)\\ &= O(\mu_\varepsilon^n),\end{aligned}$$

we obtain

$$\int_{B_r(x_\varepsilon)}|u_\varepsilon|^{p+1-\varepsilon} = \alpha_0^{p+1}A + O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|).$$

(2) By Lemma 2.14, we have

$$\begin{aligned}&\int_{B_r(x_\varepsilon)} u_\varepsilon^{p-\varepsilon}\mathrm{d}x = \int_{B_r(x_\varepsilon)}(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}} + v_\varepsilon)^{p-\varepsilon}\mathrm{d}x\\ &= \int_{B_r(x_\varepsilon)}(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}})^{p-\varepsilon} + O(1)U^{p-1-\varepsilon}_{x_\varepsilon,\mu_\varepsilon^{-1}}v_\varepsilon + O(1)v_\varepsilon^{p-\varepsilon} + O(1)U^{p-\varepsilon-(p-\varepsilon)^*}_{x_\varepsilon,\mu_\varepsilon^{-1}}v_\varepsilon^{(p-\varepsilon)^*}\mathrm{d}x.\end{aligned}$$

Since

$$\begin{aligned}\int_{B_r(x_\varepsilon)}(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}})^{p-\varepsilon}\mathrm{d}x &= \alpha_0^{p}(1+O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|))\int_{|y|\le\mu_\varepsilon^{-1}r}\frac{\mu_\varepsilon^{\frac{N-2}{2}}}{(1+|y|^2)^{\frac{N+2}{2}}}\mathrm{d}y\\ &= \alpha_0^{p}(1+O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|))\mu_\varepsilon^{\frac{N-2}{2}}(B+O(\mu_\varepsilon^2))\\ &= \alpha_0^{p}B\mu_\varepsilon^{\frac{N-2}{2}} + O(\mu_\varepsilon^{\frac{N+2}{2}} + \mu_\varepsilon^{\frac{3N-6}{2}}|\log\mu_\varepsilon|),\end{aligned}$$

$$
\begin{aligned}
\int_{B_r(x_\varepsilon)} U^{p-1-\varepsilon}_{x_\varepsilon,\mu_\varepsilon^{-1}} v_\varepsilon \mathrm{d}x &= O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|)\int_{B_r(x_\varepsilon)} U^{p}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x + O(\mu_\varepsilon^{\frac{N-2}{2}})\int_{B_r(x_\varepsilon)} U^{p-1}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x \\
&= O(\mu_\varepsilon^{\frac{3N-6}{2}}|\log\mu_\varepsilon| + \mu_\varepsilon^{\frac{N+2}{2}}), \\
\int_{B_r(x_\varepsilon)} v_\varepsilon^{p-\varepsilon}\mathrm{d}x &= O(\mu_\varepsilon^{N+2}|\log\mu_\varepsilon|^p)\int_{B_r(x_\varepsilon)} U^{p}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x + O(\mu_\varepsilon^{\frac{N+2}{2}}) \\
&= O(\mu_\varepsilon^{\frac{N+2}{2}}),
\end{aligned}
$$

and for $N \le 5$

$$
\begin{aligned}
\int_{B_r(x_\varepsilon)} U^{p-\varepsilon-2}_{x_\varepsilon,\mu_\varepsilon^{-1}} v_\varepsilon^2 \mathrm{d}x &= O(\mu_\varepsilon^{2N-4}|\log\mu_\varepsilon|^2)\int_{B_r(x_\varepsilon)} U^{p}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x + O(\mu_\varepsilon^{N-2})\int_{B_r(x_\varepsilon)} U^{p-2}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x \\
&= \begin{cases} O(\mu_\varepsilon^{\frac{5}{2}}|\log\mu_\varepsilon|^2), & N=3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\ge 4, \end{cases}
\end{aligned}
$$

we obtain

$$
\int_{B_r(x_\varepsilon)} |u_\varepsilon|^{p-\varepsilon} = \alpha_0^p B \mu_\varepsilon^{\frac{N-2}{2}} + \begin{cases} O(\mu_\varepsilon^{\frac{3}{2}}|\log\mu_\varepsilon|), & N=3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N=4, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\ge 5. \end{cases}
$$

(3) By Lemma 2.14, we have

$$
\begin{aligned}
&\int_{B_r(x_\varepsilon)} (x-x_\varepsilon)_i u_\varepsilon^{p-\varepsilon}\mathrm{d}x = \int_{B_r(x_\varepsilon)} (x-x_\varepsilon)_i (\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}} + v_\varepsilon)^{p-\varepsilon}\mathrm{d}x \\
&= \int_{B_r(x_\varepsilon)} (x-x_\varepsilon)_i \left[ (\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}})^{p-\varepsilon} + O(1) U^{p-1-\varepsilon}_{x_\varepsilon,\mu_\varepsilon^{-1}} v_\varepsilon + O(1) v_\varepsilon^{p-\varepsilon} + O(1) U^{p-\varepsilon-(p-\varepsilon)^*}_{x_\varepsilon,\mu_\varepsilon^{-1}} v_\varepsilon^{(p-\varepsilon)^*} \right]\mathrm{d}x.
\end{aligned}
$$

Since

$$
\int_{B_r(x_\varepsilon)} (x-x_\varepsilon)_i (\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}})^{p-\varepsilon}\mathrm{d}x = 0,
$$

$$
\begin{aligned}
&\int_{B_r(x_\varepsilon)} |x-x_\varepsilon| U^{p-1-\varepsilon}_{x_\varepsilon,\mu_\varepsilon^{-1}} v_\varepsilon \mathrm{d}x \\
&= O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|)\int_{B_r(x_\varepsilon)} |x-x_\varepsilon| U^{p}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x + O(\mu_\varepsilon^{\frac{N-2}{2}})\int_{B_r(x_\varepsilon)} |x-x_\varepsilon| U^{p-1}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x \\
&= \begin{cases} O(\mu_\varepsilon^{\frac{5}{2}}|\log\mu_\varepsilon|), & N=3, \\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\ge 4, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\int_{B_r(x_\varepsilon)} |x-x_\varepsilon| v_\varepsilon^{p-\varepsilon}\mathrm{d}x &= O(\mu_\varepsilon^{N+2}|\log\mu_\varepsilon|^p)\int_{B_r(x_\varepsilon)} |x-x_\varepsilon| U^{p}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x + O(\mu_\varepsilon^{\frac{N+2}{2}}) \\
&= O(\mu_\varepsilon^{\frac{N+2}{2}}),
\end{aligned}
$$

and for $N \le 5$

$$
\begin{aligned}
&\int_{B_r(x_\varepsilon)} |x-x_\varepsilon| U^{p-\varepsilon-2}_{x_\varepsilon,\mu_\varepsilon^{-1}} v_\varepsilon^2 \mathrm{d}x \\
&= O(\mu_\varepsilon^{2N-4}|\log\mu_\varepsilon|^2)\int_{B_r(x_\varepsilon)} |x-x_\varepsilon| U^{p}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x + O(\mu_\varepsilon^{N-2})\int_{B_r(x_\varepsilon)} |x-x_\varepsilon| U^{p-2}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x \\
&= O(\mu_\varepsilon^{\frac{N+2}{2}}),
\end{aligned}
$$

we obtain

$$\int_{B_r(x_\varepsilon)}(x-x_\varepsilon)_i|u_\varepsilon|^{p-\varepsilon}=\begin{cases}O(\mu_\varepsilon^{\frac{N+2}{2}}|\log\mu_\varepsilon|), & N=3,\\ O(\mu_\varepsilon^{\frac{N+2}{2}}), & N\ge 4.\end{cases}$$

(4) By Lemma 2.14, we have

$$\begin{aligned}&\int_{B_r(x_\varepsilon)}(x-x_\varepsilon)_i(x-x_\varepsilon)_j u_\varepsilon^{p-\varepsilon}\mathrm{d}x=\int_{B_r(x_\varepsilon)}(x-x_\varepsilon)_i(x-x_\varepsilon)_j(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}}+v_\varepsilon)^{p-\varepsilon}\mathrm{d}x\\
&=\int_{B_r(x_\varepsilon)}(x-x_\varepsilon)_i(x-x_\varepsilon)_j\big[(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}})^{p-\varepsilon}+O(1)U_{x_\varepsilon,\mu_\varepsilon^{-1}}^{p-1-\varepsilon}v_\varepsilon\\
&\quad+O(1)v_\varepsilon^{p-\varepsilon}+O(1)U_{x_\varepsilon,\mu_\varepsilon^{-1}}^{p-\varepsilon-(p-\varepsilon)^*}v_\varepsilon^{(p-\varepsilon)^*}\big]\mathrm{d}x.\end{aligned}$$

Since

$$\begin{aligned}&\int_{B_r(x_\varepsilon)}(x-x_\varepsilon)_i(x-x_\varepsilon)_j(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}})^{p-\varepsilon}\mathrm{d}x\\
&=\delta_{ij}\int_{B_r(x_\varepsilon)}(x-x_\varepsilon)_i^2(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}})^{p-\varepsilon}\mathrm{d}x\\
&=\alpha_0^p(1+O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|))\mu_\varepsilon^{\frac{N+2}{2}}\frac{1}{N}\int_{|y|\le\mu_\varepsilon^{-1}r}\frac{|y|^2}{(1+|y|^2)^{\frac{N+2}{2}}}\mathrm{d}y\\
&=\alpha_0^p\mu_\varepsilon^{\frac{N+2}{2}}\frac{1}{N}\int_{|y|\le\mu_\varepsilon^{-1}r}\frac{|y|^2}{(1+|y|^2)^{\frac{N+2}{2}}}\mathrm{d}y+O(\mu_\varepsilon^{\frac{3N-2}{2}}|\log\mu_\varepsilon|),\end{aligned}$$

$$\begin{aligned}&\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^2U_{x_\varepsilon,\mu_\varepsilon^{-1}}^{p-1-\varepsilon}v_\varepsilon\mathrm{d}x\\
&=O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|)\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^2U_{x_\varepsilon,\mu_\varepsilon^{-1}}^{p}\mathrm{d}x+O(\mu_\varepsilon^{\frac{N-2}{2}})\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^2U_{x_\varepsilon,\mu_\varepsilon^{-1}}^{p-1}\mathrm{d}x\\
&=O(\mu_\varepsilon^{\frac{N+2}{2}}),\end{aligned}$$

$$\begin{aligned}\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^2v_\varepsilon^{p-\varepsilon}\mathrm{d}x&=O(\mu_\varepsilon^{N+2}|\log\mu_\varepsilon|^p)\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^2U_{x_\varepsilon,\mu_\varepsilon^{-1}}^{p}\mathrm{d}x+O(\mu_\varepsilon^{\frac{N+2}{2}})\\
&=O(\mu_\varepsilon^{\frac{N+2}{2}}),\end{aligned}$$

and for $N\le 5$

$$\begin{aligned}&\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^2U_{x_\varepsilon,\mu_\varepsilon^{-1}}^{p-\varepsilon-2}v_\varepsilon^2\mathrm{d}x\\
&=O(\mu_\varepsilon^{2N-4}|\log\mu_\varepsilon|^2)\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^2U_{x_\varepsilon,\mu_\varepsilon^{-1}}^{p}\mathrm{d}x+O(\mu_\varepsilon^{N-2})\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^2U_{x_\varepsilon,\mu_\varepsilon^{-1}}^{p-2}\mathrm{d}x\\
&=O(\mu_\varepsilon^{\frac{N+2}{2}}),\end{aligned}$$

we obtain

$$\int_{B_r(x_\varepsilon)}(x-x_\varepsilon)_i(x-x_\varepsilon)_j|u_\varepsilon|^{p-\varepsilon}=\delta_{ij}\mu_\varepsilon^{\frac{N+2}{2}}\frac{1}{N}\int_{|y|\le\mu_\varepsilon^{-1}r}\frac{|y|^2}{(1+|y|^2)^{\frac{N+2}{2}}}\mathrm{d}y+O(\mu_\varepsilon^{\frac{N+2}{2}}).$$

(5) By Lemma 2.14, we have

$$\begin{aligned}&\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^3u_\varepsilon^{p-\varepsilon}\mathrm{d}x=\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^3(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}}+v_\varepsilon)^{p-\varepsilon}\mathrm{d}x\\
&=\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^3\big[(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}})^{p-\varepsilon}+O(1)U_{x_\varepsilon,\mu_\varepsilon^{-1}}^{p-1-\varepsilon}v_\varepsilon\\
&\quad+O(1)v_\varepsilon^{p-\varepsilon}+O(1)U_{x_\varepsilon,\mu_\varepsilon^{-1}}^{p-\varepsilon-(p-\varepsilon)^*}v_\varepsilon^{(p-\varepsilon)^*}\big]\mathrm{d}x.\end{aligned}$$

Since

$$\begin{aligned}
&\int_{B_r(x_\varepsilon)} |x-x_\varepsilon|^3(\alpha_o U_{x_\varepsilon,\mu_\varepsilon^{-1}})^{p-\varepsilon}\mathrm{d}x\\
&=(1+O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|))O(\mu_\varepsilon^{\frac{N+4}{2}})\int_{|y|\le\mu_\varepsilon^{-1}r}\frac{|y|^3}{(1+|y|^2)^{\frac{N+2}{2}}}\mathrm{d}y\\
&=O(\mu_\varepsilon^{\frac{N+2}{2}}),
\end{aligned}$$

$$\begin{aligned}
&\int_{B_r(x_\varepsilon)} |x-x_\varepsilon|^3 U^{p-1-\varepsilon}_{x_\varepsilon,\mu_\varepsilon^{-1}} v_\varepsilon\mathrm{d}x\\
&=O(\mu_\varepsilon^{N-2}|\log\mu_\varepsilon|)\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^3U^p_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x+O(\mu_\varepsilon^{\frac{N-2}{2}})\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^3U^{p-1}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x\\
&=O(\mu_\varepsilon^{\frac{N+2}{2}}),
\end{aligned}$$

$$\begin{aligned}
\int_{B_r(x_\varepsilon)} |x-x_\varepsilon|^3 v_\varepsilon^{p-\varepsilon}\mathrm{d}x&=O(\mu_\varepsilon^{N+2}|\log\mu_\varepsilon|^p)\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^3U^p_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x+O(\mu_\varepsilon^{\frac{N+2}{2}})\\
&=O(\mu_\varepsilon^{\frac{N+2}{2}}),
\end{aligned}$$

and for $N\le 5$

$$\begin{aligned}
&\int_{B_r(x_\varepsilon)} |x-x_\varepsilon|^3 U^{p-\varepsilon-2}_{x_\varepsilon,\mu_\varepsilon^{-1}} v_\varepsilon^2\mathrm{d}x\\
&=O(\mu_\varepsilon^{2N-4}|\log\mu_\varepsilon|^2)\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^3U^p_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x+O(\mu_\varepsilon^{N-2})\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^3U^{p-2}_{x_\varepsilon,\mu_\varepsilon^{-1}}\mathrm{d}x\\
&=O(\mu_\varepsilon^{\frac{N+2}{2}}),
\end{aligned}$$

we obtain

$$\int_{B_r(x_\varepsilon)}|x-x_\varepsilon|^3|u_\varepsilon|^{p-\varepsilon}=O(\mu_\varepsilon^{\frac{N+2}{2}}).$$

□

**Lemma A.3.** *Let $N\ge 4$. There hold*

$$\left\langle PU_{x_\varepsilon,\mu_\varepsilon^{-1}},\frac{\partial PU_{x_\varepsilon,\mu_\varepsilon^{-1}}}{\partial\lambda}\right\rangle=O(\mu_\varepsilon^{N-1}),\tag{A.13}$$

$$\left\langle PU_{x_\varepsilon,\mu_\varepsilon^{-1}},\frac{\partial PU_{x_\varepsilon,\mu_\varepsilon^{-1}}}{\partial x_i}\right\rangle=O(\mu_\varepsilon^{N-2}),\tag{A.14}$$

$$\left\langle \frac{\partial PU_{x_\varepsilon,\mu_\varepsilon^{-1}}}{\partial\lambda},\frac{\partial PU_{x_\varepsilon,\mu_\varepsilon^{-1}}}{\partial x_i}\right\rangle=O(\mu_\varepsilon^{N-1}),\tag{A.15}$$

$$\left\langle \frac{\partial PU_{x_\varepsilon,\mu_\varepsilon^{-1}}}{\partial x_i},\frac{\partial PU_{x_\varepsilon,\mu_\varepsilon^{-1}}}{\partial x_j}\right\rangle=O(\mu_\varepsilon^{N-1}),\quad \textit{where}\quad i\ne j,\tag{A.16}$$

$$\|PU_{x_\varepsilon,\mu_\varepsilon^{-1}}\|^2=N(N-2)A+O(\mu_\varepsilon^{N-2}),\tag{A.17}$$

$$\|\frac{\partial PU_{x_\varepsilon,\mu_\varepsilon^{-1}}}{\partial\lambda}\|^2=N(N+2)B\mu_\varepsilon^2+O(\mu_\varepsilon^N),\tag{A.18}$$

$$\|\frac{\partial PU_{x_\varepsilon,\mu_\varepsilon^{-1}}}{\partial x_i}\|=N(N+2)C\mu_\varepsilon^{-2}+O(\mu_\varepsilon^{N-2}),\quad \textit{where}\quad C=\int_{\mathbb{R}^N}U_{0,1}^{p+1}(y)\left|\frac{\partial U_{0,1}(y)}{\partial x_i}\right|\mathrm{d}y,\tag{A.19}$$

$$\|\frac{\partial^2 PU_{x_\varepsilon,\mu_\varepsilon^{-1}}}{\partial\lambda^2}\| = O(\mu_\varepsilon^2), \tag{A.20}$$

$$\|\frac{\partial^2 PU_{x_\varepsilon,\mu_\varepsilon^{-1}}}{\partial\lambda\partial x_i}\| = O(1), \tag{A.21}$$

$$\|\frac{\partial^2 PU_{x_\varepsilon,\mu_\varepsilon^{-1}}}{\partial x_i^2}\| = O(\mu_\varepsilon^{-2}). \tag{A.22}$$

*Proof.* The proof can be found in [33, Appendix. B]. □

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