EXISTENCE AND STABILITY OF INFINITE TIME BLOW-UP IN THE KELLER-SEGEL SYSTEM

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ABSTRACT. Perhaps the most classical diffusion model for chemotaxis is the Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^2 \times (0,\infty), \\ v = (-\Delta_{\mathbb{R}^2})^{-1} u := \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \log \frac{1}{|x-z|}\, u(z,t)\, dz, \\ u(\cdot,0) = u_0 \geq 0 \quad \text{in } \mathbb{R}^2. \end{cases} \tag{$*$}$$

We consider the *critical mass case* $\int_{\mathbb{R}^2} u_0(x)\,dx = 8\pi$ which corresponds to the exact threshold between finite-time blow-up and self-similar diffusion towards zero. We find a radial function u_0^* with mass 8π such that for any initial condition u_0 sufficiently close to u_0^* the solution $u(x,t)$ of ($*$) is globally defined and blows-up in infinite time. As $t \to +\infty$ it has the approximate profile

$$u(x,t) \approx \frac{1}{\lambda^2} U\left(\frac{x-\xi(t)}{\lambda(t)}\right), \quad U(y) = \frac{8}{(1+|y|^2)^2},$$

where $\lambda(t) \approx \frac{c}{\sqrt{\log t}}$, $\xi(t) \to q$ for some $c > 0$ and $q \in \mathbb{R}^2$. This result answers affirmatively the nonradial stability conjecture raised in [26].

1. Introduction

This paper deals with the classical Keller-Segel problem in $\mathbb{R}^2$,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^2 \times (0,\infty), \\ v = (-\Delta_{\mathbb{R}^2})^{-1} u := \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \log \frac{1}{|x-z|}\, u(z,t)\, dz, \\ u(\cdot,0) = u_0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

which is a well-known model for the dynamics of a population density $u(x,t)$ evolving by diffusion with a chemotactic drift. We consider positive solutions which are well defined, unique and smooth up to a maximal time $0 < T \leq +\infty$. This problem formally preserves mass, in the sense that

$$\int_{\mathbb{R}^2} u(x,t)dx = \int_{\mathbb{R}^2} u_0(x)\,dx =: M \quad \text{for all} \quad t \in (0,T).$$

An interesting feature of (1.1) is the connection between the second moment of the solution and its mass which is precisely given by

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2\, u(x,t)\, dx = 4M - \frac{M^2}{2\pi},$$

provided that the second moments are finite. If $M > 8\pi$, the negative rate of production of the second moment and the positivity of the solution implies finite blow-up time. If $M < 8\pi$ the solution lives at all times and diffuses to zero with a self similar profile according to [5]. When $M = 8\pi$ the solution is globally defined in time. If the initial second moment is finite, it is preserved in time, and there is *infinite time blow-up* for the solution, as was shown in [4].

Globally defined in time solutions of (1.1) are of course its positive finite mass steady states, which consist of the family

$$U_{\lambda,\xi}(x) = \frac{1}{\lambda^2} U\left(\frac{x-\xi}{\lambda}\right), \quad U(y) = \frac{8}{(1+|y|^2)^2}, \quad \lambda > 0,\ \xi \in \mathbb{R}^2. \tag{1.2}$$

We observe that all these steady states have the exact mass 8π and infinite second moment

$$\int_{\mathbb{R}^2} U_{\lambda,\xi}(x)\, dx \;=\; 8\pi, \quad \int_{\mathbb{R}^2} |x|^2\, U_{\lambda,\xi}(x)\, dx \;=\; +\infty.$$

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As a consequence, if a solution of (1.1) is attracted by the family $(U_{\lambda,\xi})$, its mass must be larger than 8π and if the initial second moment is finite, then blow-up occurs in a singular limit corresponding to $\lambda \to 0_+$.

In the critical mass $M = 8\pi$ case, the infinite-time blow-up in (1.1) when the second moment is finite, takes place in the form of a bubble in the form (1.2) with $\lambda = \lambda(t) \to 0$ according to [2, 4]. Formal rates and precise profiles were derived in [12, 8] to be

$$\lambda(t) \sim \frac{c}{\sqrt{\log t}} \quad \text{as } t \to +\infty.$$

A radial solution with this rate was built by Ghoul and Masmoudi in [26] and its stability within the radial class was established. The framework of the construction in [26] was actually fully nonradial, but for stability a spectral gap inequality only known in the radial case was used. Numerical evidence for this inequality was obtained in [7], and stability for general nonradial perturbation was conjectured in [26]. A related spectral estimate, useful in the analysis of finite time blow-up was found in [15].

In this paper we construct an infinite-time blow-up solution with a different method to that in [26], which in particular leads to a proof of the stability assertion among non-radial functions. The following is our main result.

Theorem 1.1. *There exists a nonnegative, radially symmetric function $u_0^*(x)$ with critical mass $\int_{\mathbb{R}^2} u_0^*(x)\,dx = 8\pi$ and finite second moment $\int_{\mathbb{R}^2} |x|^2\, u_0^*(x)\,dx < +\infty$ such that for every $u_1(x)$ sufficiently close (in suitable sense) to u_0^* with $\int_{\mathbb{R}^2} u_1\,dx = 8\pi$, we have that the solution $u(x,t)$ of system (1.1) with initial condition $u(x,0) = u_1(x)$ has the form*

$$u(x,t) \;=\; \frac{1}{\lambda(t)^2} U\Big(\frac{x-\xi(t)}{\lambda(t)}\Big)(1+o(1)), \quad U(y) = \frac{8}{(1+|y|^2)^2} \tag{1.3}$$

uniformly on bounded sets of $\mathbb{R}^2$, and

$$\lambda(t) \;=\; \frac{c}{\sqrt{\log t}}\,(1+o(1)), \quad \xi(t) \to q \quad \text{as } t \to +\infty,$$

for some number $c > 0$ and some $q \in \mathbb{R}^2$.

Sufficiently close for the perturbation $u_1(x) := u_0^*(x) + \varphi(x)$ in this result is measured in the C^1-weighted norm for some $\sigma > 1$

$$\|\varphi\|_{*\sigma} := \|(1+|\cdot|^{4+\sigma})\varphi\|_{L^\infty(\mathbb{R}^2)} + \|(1+|\cdot|^{5+\sigma})\nabla\varphi(x)\|_{L^\infty(\mathbb{R}^2)} < +\infty.$$

The perturbation φ must have zero mass too.

"Uniformly on bounded sets" of $\mathbb{R}^2$ in (1.3) means that for any bounded $K \subset \mathbb{R}^2$

$$\lim_{t\to\infty} \sup_{x\in K} \lambda(t)^2 U\Big(\frac{x-\xi(t)}{\lambda(t)}\Big)^{-1} \left| u(x,t) - \frac{1}{\lambda(t)^2} U\Big(\frac{x-\xi(t)}{\lambda(t)}\Big) \right| = 0.$$

The expansion of $u(x,t)$ can be made more precise though, and this is explained along the proof of theorem.

The scaling parameter is rather simple to find at main order from the approximate conservation of second moment, see Section 2. The center $\xi(t)$ actually obeys a relatively simple system of nonlocal ODEs.

We devote the rest of this paper to the proof of Theorem 1.1. Our approach borrows elements of constructions in the works [16, 21, 18, 17] based on the so-called *inner-outer gluing scheme*, where a system is derived for an inner equation defined near the blow-up point and expressed in the variable of the blowing-up bubble, and an outer problem that sees the whole picture in the original scale. The result of Theorem 1.1 has already been announced in [20] in connection with [16, 21, 18].

There is a vast literature on chemotaxis in biology and in mathematics. The Patlak-Keller-Segel model [44, 35] is used in mathematical biology to describe the motion of mono-cellular organisms, like Dictyostelium Discoideum, which move randomly but experience a drift in presence of a chemo-attractant. Under certain circumstances, these cells are able to emit the chemo-attractant themselves. Through the chemical signal, they coordinate their motion and eventually aggregate. Such a self-organization scenario is at the basis of many models of chemotaxis and is considered as a fundamental mechanism in biology. Of course, the aggregation induced by the drift competes with the noise associated with the random motion so that aggregation occurs only if the chemical signal is strong enough. A classical survey of the mathematical problems in chemotaxis models can be found in [31, 32]. After a proper adimensionalization, it turns out that all coefficients in the Patlak-Keller-Segel model studied in this paper can be taken equal to 1 and that the only free parameter left is the total mass. For further considerations on chemotaxis, we shall refer to [30] for biological models and to [11] for physics backgrounds.

In many situations of interest, cells are moving on a substrate. The two-dimensional case is therefore of special interest in biology, but also turns out to be particularly interesting from the mathematical point of view as well, because of scaling properties, at least in the simplest versions of the Keller-Segel model. Boundary conditions induce various additional difficulties. In the idealized situation of the Euclidean plane $\mathbb{R}^2$, it is known since the early work of W. Jäger and S. Luckhaus in [33] that solutions globally exist if the mass M is small and blow-up in finite time if M is large. The blow-up in a bounded domain is studied in [33, 1, 39, 40, 46]. The precise threshold for blow-up, $M = 8\pi$, has been determined in [23, 5], with sufficient conditions for global existence if $M \le 8\pi$ in [5] (also see [22] in the radial case). The key estimate is the boundedness of the free energy, which relies on the logarithmic Hardy-Littlewood-Sobolev inequality established in optimal form in [9]. We refer to [3] for a review of related results. If $M < 8\pi$, diffusion dominates: intermediate asymptotic profiles and exact rates of convergence have been determined in [7]. Also see [41, 25]. In the supercritical case $M > 8\pi$, various formal expansions are known for many years, starting with [27, 28, 49] which were later justified in [45, 38], in the radial case, and in [14], in the non-radially symmetric regime. This latter result is based on the analysis of the spectrum of a linearized operator done in [15], based on the earlier work [19], and relies on a scalar product already considered in [45] and similar to the one used in [6, 7] in the subcritical mass regime. An interesting subproduct of the blow-up mechanism in [45, 29] is that the blow-up takes the form of a concentration in the form of a Dirac distribution with mass exactly 8π at blow-up time, as was expected from [29, 24], but it is still an open question to decide whether this is, locally in space, the only mechanism of blow-up.

The critical mass case $M = 8\pi$ is more delicate. If the second moment is infinite, there is a variety of behaviors as observed for instance in [36, 37, 43]. For solutions with finite second moment, blow-up is expected to occur as $t \to +\infty$: see [34] for grow-up rates in $\mathbb{R}^2$, and [48] for the higher-dimensional radial case. The existence in $\mathbb{R}^2$ of a global radial solution and first results of large time asymptotics were established in [2] using cumulated mass functions. In [4], the infinite time blow-up was proved without symmetry assumptions using the free energy and an assumption of boundedness of the second moment. Also see [42, 43] for an existence result under weaker assumptions, and further estimates on the solutions. Asymptotic stability of the family of steady states determined by (1.2) under the mass constraint $M = 8\pi$ has been determined in [10]. The blow-up rate $\lambda(t)$ and the shape of the limiting profile U were identified in formal asymptotic expansions in [50, 51, 47, 12, 13] and also in [8, Chapter 8]. As already mentioned, a radial solution with rate $\lambda(t) \sim (\log t)^{-1/2}$ was built and its stability within the radial class was established in [26].

2. Formal derivation of the behavior of the parameters

We consider here a first approximation to a solution $u(x,t)$ of (1.1), globally defined in time, such that on bounded sets in x,

$$u(x,t) = \frac{1}{\lambda(t)^2} U\left(\frac{x-\xi(t)}{\lambda(t)}\right)(1+o(1)) \quad \text{as } t \to +\infty \tag{2.1}$$

for certain functions $0 < \lambda(t) \to 0$ and $\xi(t) \to q \in \mathbb{R}^2$, where we recall that

$$U(y) = \frac{8}{(1+|y|^2)^2}.$$

We know that (2.1) can only happen in the critical mass, finite second moment case:

$$\int_{\mathbb{R}^2} u(x,t)dx = 8\pi, \qquad \int_{\mathbb{R}^2} |x|^2 u(x,t)dx < +\infty,$$

which according to the results in [4, 26, 12] is consistent with a behavior of the form (2.1). Since the second moment of U is infinite, we do not expect the approximation (2.1) be uniform in $\mathbb{R}^2$ but sufficiently far, a faster decay in x should take place as we shall see next. We will find an approximate asymptotic expression for the scaling parameter $\lambda(t)$ that matches with this behavior.

Let us introduce the function $\Gamma_0 := (-\Delta)^{-1}U$. We directly compute

$$\Gamma_0(y) = \log \frac{8}{(1+|y|^2)^2}$$

and hence Γ_0 solves the Liouville equation

$$-\Delta\Gamma_0 = e^{\Gamma_0} = U \quad \text{in } \mathbb{R}^2.$$

Then $\nabla\Gamma_0(y) \approx -\frac{4y}{|y|^2}$ for all large y, and hence we get, away from $x = \xi$,

$$-\nabla\cdot(u\nabla(-\Delta)^{-1}u) \approx 4\nabla u \cdot \frac{x-\xi}{|x-\xi|^2}.$$

Therefore, defining

$$\mathcal{E}(u) := \Delta u - \nabla\cdot(u\nabla(-\Delta)^{-1}u) \tag{2.2}$$

and writing in polar coordinates

$$u(r,\theta,t) = u(x,t), \quad x = \xi(t) + re^{i\theta},$$

we find $\mathcal{E}(u) \approx \partial_r^2 u + \frac{5}{r}\partial_r u$. Hence, assuming that $\dot\xi(t) \to 0$ sufficiently fast, equation (1.1) approximately reads

$$\partial_t u = \partial_r^2 u + \frac{5}{r}\partial_r u,$$

which can be idealized as a homogeneous heat equation in $\mathbb{R}^6$ for radially symmetric functions. It is therefore reasonable to believe that beyond the self-similar region $r \gg \sqrt{t}$ the behavior changes into a function of $r/\sqrt{t}$ with fast decay at $+\infty$ that yields finiteness of the second moment. To obtain a first global approximation, we simply cut-off the bubble (2.1) beyond the self-similar zone. We introduce a further parameter $\alpha(t)$ and set

$$\bar u(x,t) = \frac{\alpha(t)}{\lambda^2} U\Big(\frac{x-\xi}{\lambda}\Big)\chi(x,t), \tag{2.3}$$

where

$$\chi(x,t) = \chi_0\Big(\frac{x-\xi}{\sqrt{t}}\Big) \tag{2.4}$$

with χ_0 a smooth radial cut-off function such that

$$\chi_0(z) = \begin{cases} 1 & \text{if } |z| \le 1, \\ 0 & \text{if } |z| \ge 2. \end{cases} \tag{2.5}$$

We introduce the parameter $\alpha(t)$ because the total mass of the actual solution should equal 8π for all t. But

$$\frac{1}{\lambda^2}\int_{\mathbb{R}^2} U\Big(\frac{x-\xi}{\lambda}\Big)\chi(x,t)\,dx = 8\pi + 16\pi\Upsilon\frac{\lambda^2}{t} + O\Big(\frac{\lambda^4}{t^2}\Big), \tag{2.6}$$

as $t \to \infty$, where

$$\Upsilon = \int_0^\infty (\tilde\chi_0(s) - 1)s^{-3}ds < 0, \tag{2.7}$$

and $\chi_0(x) = \tilde\chi_0(|x|)$. To achieve $\int_{\mathbb{R}^2}\bar u(x,t)\,dx = 8\pi$ we set $\alpha = \bar\alpha$ where

$$\bar\alpha(t) = 1 - 2\Upsilon\frac{\lambda^2}{t} + O\Big(\frac{\lambda^4}{t^2}\Big).$$

Next we will obtain an approximate value of the scaling parameter $\lambda(t)$ that is consistent with the existence of a solution $u(x,t) \approx \bar u(x,t)$ where $\bar u$ is the function in (2.3) with $\alpha = \bar\alpha$. Let us consider the "error operator"

$$S(u) = -u_t + \mathcal{E}(u), \tag{2.8}$$

where $\mathcal{E}(u)$ is defined in (2.2). We have the following well-known identities, valid for an arbitrary function $\omega(x)$ of class $C^2(\mathbb{R}^2)$ with finite mass and $D^2\omega(x) = O(|x|^{-4-\sigma})$ for large $|x|$. We have

$$\int_{\mathbb{R}^2}|x|^2\mathcal{E}(\omega)\,dx = 4M - \frac{M^2}{2\pi}, \quad M = \int_{\mathbb{R}^2}\omega(x)dx \tag{2.9}$$

and

$$\int_{\mathbb{R}^2} x\mathcal{E}(\omega)\,dx = 0, \quad \int_{\mathbb{R}^2}\mathcal{E}(\omega)\,dx = 0. \tag{2.10}$$

Let us recall the simple proof of (2.9). Integrating by parts on finite balls with large radii and using the behavior of the boundary terms we get the identities

$$\int_{\mathbb{R}^2} |x|^2 \Delta\omega\, dx = 4M,$$

$$\begin{aligned}\int_{\mathbb{R}^2} |x|^2 \nabla\cdot(\omega\nabla(-\Delta)^{-1})\omega)\, dx &= -2\int_{\mathbb{R}^2} x\cdot\omega\nabla(-\Delta)^{-1}\omega\, dx \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\omega(x)\omega(y)\frac{x\cdot(x-y)}{|x-y|^2}dx\,dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\omega(x)\omega(y)\frac{(x-y)\cdot(x-y)}{|x-y|^2}dx\,dy \\ &= \frac{M^2}{2\pi} \end{aligned} \tag{2.11}$$

and then (2.9) follows. The proof of (2.10) is even simpler. For a solution $u(x,t)$ of (1.1) we then get

$$\frac{d}{dt}\int_{\mathbb{R}^2} u(x,t)|x|^2 dx = 4M - \frac{M^2}{2\pi}, \quad M = \int_{\mathbb{R}^2} u(x,t)dx.$$

In particular, if $u(x,t)$ is sufficiently close to $\bar u(x,t)$ and since $\int_{\mathbb{R}^2}\bar u(x,t)dx = 8\pi$, we get the approximate validity of the identity

$$\frac{d}{dt}\int_{\mathbb{R}^2}\bar u(x,t)|x|^2 dx = 0.$$

This means

$$aI(t) := \int_{\mathbb{R}^2}\frac{\bar\alpha}{\lambda^2}U\left(\frac{x-\xi}{\lambda}\right)\chi_0\left(\frac{x-\xi}{\sqrt t}\right)|x|^2 dx \;=\; constant.$$

We readily check that for some constant κ

$$I(t) = 16\pi\lambda^2\int_0^{\frac{\sqrt t}{\lambda}}\frac{\rho^3 d\rho}{(1+\rho^2)^2} + \kappa + o(1) = 16\pi\lambda^2\log\frac{\sqrt t}{\lambda} + \kappa + o(1) \quad \text{as } \lambda\to 0.$$

Then we conclude that $\lambda(t)$ approximately satisfies

$$\lambda^2\log t \;=\; c^2 \;=\; constant$$

and hence we get at main order

$$\lambda(t) = \frac{c}{\sqrt{\log t}}.$$

We also notice that the center of mass is preserved for a true solution, thanks to (2.10):

$$\frac{d}{dt}\int_{\mathbb{R}^2} xu(x,t)dx = 0.$$

Since the center of mass of $\bar u(x,t)$ is exactly $\xi(t)$ we then get that approximately

$$\xi(t) = constant = q.$$

3. The approximations u_0 and u_1

From now on we to consider the Keller-Segel system starting at a large t_0:

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) & \text{in } \mathbb{R}^2\times(t_0,\infty), \\ v = (-\Delta_{\mathbb{R}^2})^{-1}u := \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2}\log\frac{1}{|x-z|}u(z,t)\,dz, \\ u(\cdot,t_0) = u_0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{3.1}$$

which is equivalent to (1.1). We do this so that some expansions for t large take a simpler form.

In this section we will define a basic approximation to a solution of the Keller-Segel system (3.1). Let us consider parameter functions

$$0<\lambda(t)\to 0, \quad \xi(t)\to q, \quad \alpha(t)\to 1 \quad \text{as } t\to+\infty$$

that we will later specify. Let us consider the functions

$$U(y) = \frac{8}{(1+|y|^2)^2}, \quad \Gamma_0(y) = \log U(y)$$

and define the approximate solution $u_0(x,t)$ as

$$u_0(x,t) = \frac{\alpha}{\lambda^2} U\Big(\frac{x-\xi}{\lambda}\Big)\chi(x,t), \qquad v_0(x,t) = (-\Delta_x)^{-1}u_0 = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x-\bar{x}|}\, u_0(\bar{x},t)\, d\bar{x}, \tag{3.2}$$

where χ is the cut-off function (5.3). We consider the error operator

$$S(u) = -\partial_t u + \mathcal{E}(u),$$

where

$$\mathcal{E}(u) = \Delta_x u - \nabla_x\cdot(u\nabla_x v), \quad v = (-\Delta_x)^{-1}u.$$

and next measure the error of approximation $S(u_0)$.

We have

$$\begin{aligned} -\partial_t u_0(x,t) = &-\frac{\dot\alpha}{\lambda^2}U(y)\chi_0(z) + \alpha\frac{\dot\lambda}{\lambda^3}Z_0\chi_0(z) + \frac{\alpha}{\lambda^3}\dot\xi\cdot\nabla_y U(y)\,\chi_0(z)\\ &+ \frac{\alpha}{\lambda^2\sqrt{t}}U(y)\dot\xi\cdot\nabla_z\chi_0(z) + \frac{\alpha}{2\lambda^2 t}U(y)\nabla_z\chi_0(z)\cdot z, \end{aligned} \tag{3.3}$$

$$z = \frac{x-\xi}{\sqrt{t}}$$

where

$$Z_0(y) = 2U(y) + y\cdot\nabla_y U(y), \quad y = \frac{x-\xi}{\lambda}. \tag{3.4}$$

We also have

$$\begin{aligned} \mathcal{E}(u_0) &= \Delta_x u_0 - \nabla_x\cdot(u_0\nabla_x v_0)\\ &= \frac{2\alpha}{\lambda^3 t^{1/2}}\nabla_z\chi_0(z)\cdot\nabla_y U(y) + \frac{\alpha}{t}\frac{1}{\lambda^2}\Delta_z\chi_0(z)U(y) - \frac{\alpha}{\lambda^2\sqrt{t}}U(y)\nabla_z\chi_0(z)\cdot\nabla_x v_0\\ &+ \frac{\alpha\chi_0(z)}{\lambda^4}\Big[(\chi_0(z)\alpha - 1)U^2(y) - \nabla_y U(y)\cdot(\nabla_y v_0 - \nabla_y\Gamma_0)\Big]\ . \end{aligned}$$

Let us decompose

$$v_0(y) = \alpha\Gamma_0(y) + \mathcal{R}(y). \tag{3.5}$$

For the term $\mathcal{R}$ in (3.5) we directly estimate

$$|\nabla_y\mathcal{R}(y)| \le \begin{cases} \frac{\lambda^2}{t}\frac{1}{|y|} & |y| \ge \frac{\sqrt{t}}{\lambda},\\ 0 & |y| \le \frac{\sqrt{t}}{\lambda}. \end{cases} \tag{3.6}$$

Then

$$\begin{aligned} \mathcal{E}(u_0) = &\frac{2\alpha}{\lambda^3 t^{1/2}}\nabla_z\chi_0(z)\cdot\nabla_y U(y) + \frac{\alpha}{t}\frac{1}{\lambda^2}\Delta_z\chi_0(z)U(y) - \frac{\alpha}{\lambda^2\sqrt{t}}U(y)\nabla_z\chi_0(z)\nabla_x v_0\\ &+ \frac{\alpha\chi_0(z)}{\lambda^4}\Big[(\alpha-1)U^2(y) - (\alpha-1)\nabla_y U(y)\cdot\nabla_y\Gamma_0(y) + \alpha(\chi_0(z)-1)U^2(y)\\ &- \nabla_y U(y)\cdot\nabla_y\mathcal{R}(y)\Big]. \end{aligned}$$

and thus

$$\begin{aligned} S(u_0) = &-\frac{\dot\alpha}{\lambda^2}U(y)\chi_0(z) + \alpha\frac{\dot\lambda}{\lambda^3}Z_0\chi_0(z) + \frac{\alpha}{\lambda^3}\dot\xi\cdot\nabla_y U(y)\,\chi_0(z)\\ &+ \frac{\alpha}{\lambda^2\sqrt{t}}U(y)\dot\xi\cdot\nabla_z\chi_0(z) + \frac{\alpha}{2\lambda^2 t}U(y)\nabla_z\chi_0(z)\cdot z\\ &+ \frac{2\alpha}{\lambda^3 t^{1/2}}\nabla_z\chi_0(z)\cdot\nabla_y U(y) + \frac{\alpha}{t}\frac{1}{\lambda^2}\Delta_z\chi_0(z)U(y) - \frac{\alpha}{\lambda^2\sqrt{t}}U(y)\nabla_z\chi_0(z)\cdot\nabla_x v_0\\ &- \frac{\alpha(\alpha-1)\chi_0(z)}{\lambda^4}\nabla_y\cdot(U(y)\nabla_y\Gamma_0(y))\\ &+ \frac{\alpha\chi_0(z)}{\lambda^4}\Big[\alpha(\chi-1)U^2(y) - \nabla_y U(y)\cdot\nabla_y\mathcal{R}(y)\Big]. \end{aligned} \tag{3.7}$$

For a function $v(\zeta)$ defined for $\zeta \in \mathbb{R}^2$ consider the operator

$$\Delta_6 v(\zeta) = \Delta v(\zeta) + 4\frac{\zeta}{|\zeta|^2} \cdot \nabla_\zeta v(\zeta). \tag{3.8}$$

The reason for the notation is that for radial functions $v = v(r)$, $r = |\zeta|$, we have

$$\Delta_6 v = \partial_r^2 v + \frac{5}{r}\partial_r v,$$

which corresponds to Laplace's operator in $\mathbb{R}^6$ on radial functions.

Let $\tilde\varphi_\lambda(\zeta, t)$ be the (radial) solution to

$$\begin{cases} \partial_t \tilde\varphi_\lambda = \Delta_6 \tilde\varphi_\lambda + E(\zeta, t) \quad \text{in } \mathbb{R}^2 \times (\frac{t_0}{2}, \infty), \\ \tilde\varphi_\lambda(\cdot, \frac{t_0}{2}) = 0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{3.9}$$

given by Duhamel's formula, where $E(\zeta, t)$ is the radial function

$$E(\zeta, t; \lambda) = \frac{\dot\lambda}{\lambda^3} Z_0\Big(\frac{\zeta}{\lambda}\Big)\chi_0\Big(\frac{\zeta}{\sqrt{t}}\Big) + \frac{1}{2\lambda^2 t} U\Big(\frac{\zeta}{\lambda}\Big)\nabla_z \chi_0(z)\cdot z + \tilde E(x, t), \tag{3.10}$$

and

$$\begin{aligned} \tilde E(\zeta, t; \lambda) = {}& \frac{2}{\lambda^3 t^{1/2}} \nabla_z \chi_0(z) \cdot \nabla_y U(y) + \frac{1}{\lambda^2 t}\Delta_z \chi_0(z) U(y) \\ & - \frac{1}{\lambda^3 t^{1/2}} U(y) \nabla_z \chi_0(z)\cdot \nabla_y \Gamma_0(y), \end{aligned} \tag{3.11}$$

with $z = \frac{\zeta}{\sqrt{t}}$, $y = \frac{\zeta}{\lambda}$.

We then define

$$\varphi_\lambda(x, t) = \tilde\varphi_\lambda(x - \xi(t), t). \tag{3.12}$$

The reason to define φ_λ for $t > \frac{t_0}{2}$ is that it gives better properties for the first approximation of λ constructed in Section 7. Since $\lambda(t)$ is defined naturally for $t > t_0$, we will need to define $\lambda(t)$ for $\frac{t_0}{2} < t < t_0$ in an appropriate way (see Proposition 5.1 and Section 7). We will write $\lambda = \lambda_0 + \lambda_1$ where both of these functions are constructed so that they are defined for $t > \frac{t_0}{2}$. The construction of λ_0 is given in Proposition 5.1. In particular $\lambda_0(t) = \frac{c_0}{\sqrt{\log t}}(1 + o(1))$ as $t \to \infty$. Note that $\varphi_\lambda(\cdot, t_0)$ is not zero.

We define the approximate solution

$$u_1 := u_0 + \varphi_\lambda \tag{3.13}$$

which depends on the parameter functions $\alpha(t)$, $\xi(t)$, $\lambda(t)$. Correspondingly, we write

$$v_1 \;:=\; (-\Delta_x)^{-1}(u_1)\,.$$

We will establish in the next sections that a suitable choice of these functions makes it possible to find an actual solution of (3.1) as a lower order perturbation of u_1.

4. The first error of approximation

We will assume the following conditions on λ, α, ξ

$$\begin{cases} |\lambda(t)| + t\log(t)|\dot\lambda(t)| \le \dfrac{C}{\sqrt{\log(t)}} \\ |\dot\xi(t)| \le \dfrac{C}{t^\gamma} \\ |\alpha(t) - 1| \le \dfrac{C}{t\log t}, \quad |\dot\alpha(t)| \le \dfrac{C}{t^2\,\log t}, \end{cases} \tag{4.1}$$

where $\frac{3}{2} < \gamma < 2$.

We compute

$$S(u_1) = S(u_0 + \varphi_\lambda) = S(u_0) - \partial_t \varphi_\lambda + \mathcal{L}_{u_0}[\varphi_\lambda] - \nabla\cdot(\varphi_\lambda \nabla\psi_\lambda).$$

where

$$\mathcal{L}_{u_0}[\varphi] = \Delta\varphi - \nabla\cdot(\varphi\nabla v_0) - \nabla\cdot(u_0\nabla\psi),$$
$$\psi_\lambda = (-\Delta)^{-1}\varphi_\lambda, \quad v_0 = (-\Delta)^{-1}u_0.$$

Then

$$\begin{aligned} S(u_1) = &-\frac{\dot\alpha}{\lambda^2}U(y)\chi + (\alpha-1)\frac{\dot\lambda}{\lambda^3}Z_0\chi + \frac{\alpha}{\lambda^3}\dot\xi\cdot\nabla_y U(y)\,\chi + \frac{\alpha}{\lambda^2\sqrt{t}}U(y)\dot\xi\cdot\nabla\chi_0 \\ &+ \frac{(\alpha-1)}{2t}\frac{1}{\lambda^2}U\nabla_z\chi_0\cdot\frac{x-\xi}{\sqrt t} + \frac{2(\alpha-1)}{\lambda^3 t^{1/2}}\nabla_z\chi_0\cdot\nabla_y U \\ &+ \frac{(\alpha-1)}{t}\Delta\chi_0\frac{1}{\lambda^2}U - \frac{\alpha^2-1}{\lambda^3\sqrt t}U\nabla_z\chi_0\cdot\nabla_y\Gamma_0 - \frac{\alpha}{\lambda^3\sqrt t}U\nabla_z\chi_0\cdot\nabla_y\mathcal{R} \\ &- \frac{\alpha(\alpha-1)\chi}{\lambda^4}\nabla_y\cdot(U\nabla_y\Gamma_0) + \frac{\alpha^2\chi(1-\chi)}{\lambda^4}U^2 - \frac{\alpha\chi}{\lambda^4}\nabla_y U\cdot\nabla_y\mathcal{R} \\ &+ \nabla\varphi_\lambda\cdot\dot\xi - \frac{4}{r}\partial_r\varphi_\lambda - \nabla\cdot(\varphi_\lambda\nabla v_0) - \nabla\cdot(u_0\nabla\psi_\lambda) - \nabla\cdot(\varphi_\lambda\nabla\psi_\lambda), \end{aligned} \tag{4.2}$$

where $\mathcal{R}$ is defined in the decomposition (3.5).

Lemma 4.1. *Let φ_λ be defined by (3.12)-(3.9) with λ satisfying (4.1). Then*

$$|\varphi_\lambda(x,t)| + (|x-\xi|+\lambda)|\nabla\varphi_\lambda(x,t)| \le C\frac{1}{t\log t}\begin{cases}\frac{1}{\lambda^2+|x-\xi|^2} & |x-\xi|\le\sqrt t\\ \frac1t e^{-\frac{|x-\xi|^2}{4t}} & |x-\xi|\ge\sqrt t.\end{cases} \tag{4.3}$$

We also have

$$|\nabla\varphi_\lambda(x,t)| \le \frac{C}{t\log t}\frac{|x-\xi|}{(\lambda+|x-\xi|)^4}, \quad |x-\xi|\le\sqrt t. \tag{4.4}$$

Proof. In terms of the function $\tilde\varphi_\lambda$ defined in (3.9), with $r=|x-\xi|$ we claim that

$$|\tilde\varphi_\lambda(r,t)| \le C\frac{1}{t\log t}\begin{cases}\frac{1}{\lambda^2+r^2} & r\le\sqrt t,\\ \frac1t e^{-\frac{r^2}{4t}} & r\ge\sqrt t.\end{cases}$$

For the proof of this we use barriers. Consider

$$\psi_1(r,t) = \frac{1}{t\log t}\frac{1}{\lambda^2+r^2}$$

and note that

$$\partial_t\psi_1 - \Big(\partial_{rr}+\frac5r\partial_r\Big)\psi_1 \ge c\frac{\lambda^{-4}}{t\log t(1+r/\lambda)^4}, \quad r\le 2\delta\sqrt t$$

for some $c>0$, $\delta>0$.

Let $\chi_{\delta\sqrt t}(r,t) = \tilde\chi_0(\frac{r}{\delta\sqrt t})$ where $\tilde\chi_0\in C^\infty(\mathbb{R})$ is such that $\tilde\chi_0(s)=1$ for $s\le1$ and $\tilde\chi_0(s)=0$ for $s\ge2$. Consider

$$\psi(r,t) = \psi_1(r,t)\chi_{\delta\sqrt t}(r,t) + \frac{C_1}{t^2\log t}e^{-\frac{r^2}{4t}}.$$

The function $\tilde E$ (3.11) can be estimated by

$$|\tilde E(\zeta,t)| \le \frac{1}{\lambda^2 t^3}h_1\Big(\frac{\zeta}{\sqrt t}\Big)$$

where $h_1(z)$ is a smooth function with compact support. Then E (3.10) has the estimate

$$|\tilde E(\zeta,t)| \le C\frac{|\lambda\dot\lambda|}{(r^2+\lambda^2)^2} + \frac{1}{\lambda^2t^3}h_2\Big(\frac{\zeta}{\sqrt t}\Big)$$

where $h_2(z)$ is a smooth function with compact support.

Then for C_1 sufficiently large

$$\partial_t\psi - \Big(\partial_{rr}+\frac5r\partial_r\Big)\psi \ge c|E(r,t)|,$$

where $c>0$.

By the comparison principle,

$$|\tilde\varphi_\lambda(r,t)| \le C\psi(r,t),$$

for some uniform constant C. After a suitable scaling, from standard parabolic estimates we also get

$$(\lambda + r)|\nabla_x \tilde\varphi_\lambda(r,t)| \le C\psi(r,t).$$

With these two inequalities we obtain (4.3).

To prove (4.4) we change variables $y = \frac{x-\xi}{\lambda}$ in the equation (3.9) and define

$$\tilde\varphi_\lambda(r,t) = \frac{1}{\lambda^2}\hat\varphi_\lambda\Big(\frac{r}{\lambda}, t\Big).$$

We get the equation, after interpreting $\rho = |y|$, $y \in \mathbb{R}^6$

$$\lambda^2 \partial_t \hat\varphi = \Delta_{\mathbb{R}^6}\hat\varphi + \lambda\dot\lambda(2\hat\varphi_\lambda + y\cdot\nabla_y\hat\varphi_\lambda) + \lambda^4 E(\lambda y, t),$$

where E is defined in (3.10). Differentiating with respect to y and using the bound we already have for $\nabla_y\hat\varphi_\lambda$ from (4.4), and using standard parabolic estimates, we get

$$|D_y^2\hat\varphi_\lambda(y,t)| \le \frac{C}{t\log t}\frac{1}{(1+|y|)^4}, \quad |y| \le \sqrt{t\log t}.$$

Using that $\nabla\hat\varphi_\lambda(0,t) = 0$ we deduce that

$$|\nabla_y\hat\varphi_\lambda(y,t)| \le \frac{C}{t\log t}\frac{|y|}{(1+|y|)^4}, \quad |y| \le \sqrt{t\log t},$$

which readily gives (4.4).

□

Lemma 4.2. *Assuming* (4.1) *we have*

$$\lambda^4|S(u_1)|\chi(x,t) \le C\frac{1}{t\,\log t}\frac{\log(2+|y|)}{1+|y|^6}, \quad y = \frac{x-\xi}{\lambda}, \tag{4.5}$$

and

$$|S(u_1)|(1-\chi) \le C\frac{1}{t^4\log t}e^{-c\frac{|x|^2}{t}}, \tag{4.6}$$

for some $c \in (0, \frac14)$.

Proof. Let us analyze the terms involving φ_λ. We estimate, using Lemma 4.1,

$$\big|\lambda^2 U(y)\varphi_\lambda(\xi+\lambda y)\big| \le C\frac{1}{t\log t}\frac{1}{(1+|y|)^6}, \quad |y| \le \sqrt{t\log t}.$$

Similarly, by (3.5)

$$\begin{aligned} -\frac{4}{r}\partial_r\tilde\varphi_\lambda - \nabla\tilde\varphi_\lambda\cdot\nabla v_0 &= -\frac{4}{r}\partial_r\tilde\varphi_\lambda - \nabla\tilde\varphi_\lambda\cdot\nabla\Gamma_0 - (\alpha-1)\nabla\tilde\varphi_\lambda\cdot\nabla\Gamma_0 - \nabla\tilde\varphi_\lambda\cdot\nabla\mathcal{R} \\ &= 4\Big(\frac{r}{r^2+\lambda^2} - \frac{1}{r}\Big)\partial_r\tilde\varphi_\lambda - (\alpha-1)\nabla\tilde\varphi_\lambda\cdot\nabla\Gamma_0 - \nabla\tilde\varphi_\lambda\cdot\nabla\mathcal{R}. \end{aligned} \tag{4.7}$$

By (4.4)

$$\Big|\lambda^4 4\Big(\frac{r}{r^2+\lambda^2} - \frac{1}{r}\Big)\partial_r\varphi_\lambda\Big| \le \frac{C}{t\log t}\frac{1}{(1+|y|)^6}, \quad |y| \le \sqrt{t\log t}.$$

The other terms in (4.7) are estimated similarly, using the hypotheses on α and the estimate on $\mathcal{R}$ (3.6), and we get

$$\Big|-\frac{4}{r}\partial_r\tilde\varphi_\lambda - \nabla\tilde\varphi_\lambda\cdot\nabla v_0\Big| \le \frac{C}{t\log t}\frac{1}{(1+|y|)^6}, \quad |y| \le \sqrt{t\log t}.$$

The terms involving $\psi_\lambda = (-\Delta)^{-1}\varphi_\lambda$ are estimated using the formula

$$\partial_r\psi_\lambda(r,t) = \frac{1}{r}\int_0^r \varphi_\lambda(s,t)s ds.$$

In $\lambda^4 S(u_1)$ we have also the term $-\dot\alpha\lambda^2 U(y)\chi$, which thanks to (4.1) can be estimated as

$$\Big|\lambda^2\dot\alpha U(y)\chi\Big| \le \frac{C\lambda^2}{t^2\log t}\frac{1}{(1+|y|)^4}\chi(y,t) \le \frac{C}{t\log t}\frac{1}{(1+|y|)^6}\chi(y,t).$$

The remaining terms are estimated similarly, and we obtain (4.5).

The stated inequality (4.6) follows from the Gaussian decay of φ_λ in Lemma 4.1. □

5. The inner-outer gluing system

Let us consider the initial approximation

$$u_1(x,t) = u_0(x,t) + \varphi_\lambda(x,t)$$

built in Section 3 for a given choice of the parameter functions $\lambda(t)$, $\alpha(t)$, $\xi(t)$ satisfying (4.1). Here u_0 is the function defined in (3.2) and φ_λ that in (3.12). We look for a solution of the Keller-Segel equation (3.1) in the form of a small perturbation of u_1, namely

$$u(x,t) = u_1(x,t) + \Phi(x,t). \tag{5.1}$$

We write the perturbation Φ as a sum of an "inner" contribution, better expressed in the scale of u_0, and a remote effect that takes into consideration the "outer" regime. Precisely, we write

$$\Phi(x,t) = \frac{1}{\lambda^2}\phi^i(y,t)\chi(x,t) + \varphi^o(x,t), \quad y = \frac{x-\xi}{\lambda}, \tag{5.2}$$

where χ is the smooth cut-off

$$\chi(x,t) = \chi_0\Big(\frac{x-\xi}{\sqrt{t}}\Big) \tag{5.3}$$

with χ_0 a smooth radial cut-off function such that $\chi_0(z) = 1$ if $|z| \le 1$, $\chi_0(z) = 1$ if $|z| \ge 2$. (The same as defined in (2.4).)

Recall $S(u)$ given by

$$S(u) = -\partial_t u + \Delta u - \nabla\cdot(u\nabla v), \quad v = (-\Delta)^{-1}u,$$

where the operators act on the original variable x unless otherwise indicated. In the computations that follow we will express the equation

$$S(u_1+\Phi) = 0$$

for Φ given by (5.2), as a parabolic system in its inner and outer contributions ϕ^i and φ^o. The coupling in that system will be small if $\phi^i(y,t)$ decays sufficiently fast in space and time. That can only be achieved for suitable choices of the parameters α, λ, ξ that yield certain solvability conditions satisfied. The set of all these relations is what we call the inner-outer gluing system. Next we formulate this system. It will be necessary to successively refine its original expression by further decomposing ϕ^i into two contributions with separate space decay, finally arriving at the equations (5.48), (5.49), (5.50) and (5.52) which are the ones we will actually solve.

Let us observe that

$$\begin{aligned} S(u_1+\Phi) = S(u_1) &- \partial_t\Big(\frac{1}{\lambda^2}\phi^i\chi\Big) - \partial_t\varphi^o + \mathcal{L}_{u_1}\Big[\frac{1}{\lambda^2}\phi^i\chi\Big] + \mathcal{L}_{u_1}[\varphi^o] \\ &- \nabla\cdot(\Phi\nabla(-\Delta)^{-1}\Phi), \end{aligned}$$

where

$$\mathcal{L}_{u_1}[\varphi] = \Delta\varphi - \nabla\cdot(\varphi\nabla v_1) - \nabla\cdot(u_1\nabla(-\Delta)^{-1}\varphi), \qquad v_1 = (-\Delta)^{-1}u_1.$$

We use the notation

$$\psi = \frac{1}{\lambda^2}(-\Delta)^{-1}\phi^i, \quad \hat\psi = \frac{1}{\lambda^2}(-\Delta)^{-1}(\phi^i\chi),$$

in the expressions that follow. We expand

$$\mathcal{L}_{u_1}[\frac{1}{\lambda^2}\phi^i\chi] = \chi\frac{1}{\lambda^2}\Delta\phi^i + \frac{2}{\lambda^2}\nabla\chi\cdot\nabla\phi^i + \frac{1}{\lambda^2}\phi^i\Delta\chi - \nabla\cdot(\frac{1}{\lambda^2}\phi^i\chi\nabla v_1) - \nabla\cdot(u_1\nabla\hat\psi).$$

We have

$$\begin{aligned} \nabla\cdot(u_1\nabla\hat\psi) = \nabla\cdot(\frac{\alpha}{\lambda^2}U\nabla\psi)\chi + \nabla\cdot(\frac{\alpha}{\lambda^2}U\nabla(\hat\psi-\psi))\chi + \frac{\alpha}{\lambda^2}U\nabla\chi\cdot\nabla\hat\psi \\ + \nabla\cdot(\varphi_\lambda\nabla\psi) + \nabla\cdot(\varphi_\lambda\nabla(\hat\psi-\psi)) \end{aligned}$$

and

$$\nabla\cdot(\frac{1}{\lambda^2}\phi^i\chi\nabla v_1) = \nabla\cdot(\frac{1}{\lambda^2}\phi^i\nabla v_1)\chi + \frac{1}{\lambda^2}\phi^i\nabla\chi\cdot\nabla v_1.$$

Recall the notation

$$v_1 = v_0 + \psi_\lambda, \quad v_0 = \frac{\alpha}{\lambda^2}(-\Delta)^{-1}(U\chi), \quad \psi_\lambda = (-\Delta)^{-1}\varphi_\lambda,$$

and also (3.5)

$$v_0 = \alpha\Gamma_0 + \mathcal{R}, \quad \mathcal{R} = \frac{\alpha}{\lambda^2}(-\Delta)^{-1}\big(U(\chi - 1)\big).$$

Then

$$\begin{aligned}\nabla\cdot(\frac{1}{\lambda^2}\phi^i\chi\nabla v_1) &= \nabla\cdot(\frac{1}{\lambda^2}\phi^i\nabla v_0)\chi + \nabla\cdot(\frac{1}{\lambda^2}\phi^i\nabla\psi_\lambda)\chi + \frac{1}{\lambda^2}\phi^i\nabla\chi\cdot\nabla v_0\\ &\quad + \frac{1}{\lambda^2}\phi^i\nabla\chi\cdot\nabla\psi_\lambda\\ &= \frac{\alpha}{\lambda^2}\nabla\cdot(\phi^i\nabla\Gamma_0)\chi + \nabla\cdot(\frac{1}{\lambda^2}\phi^i\nabla\mathcal{R})\chi + \nabla\cdot(\frac{1}{\lambda^2}\phi^i\nabla\psi_\lambda)\chi\\ &\quad + \frac{\alpha}{\lambda^2}\phi^i\nabla\chi\cdot\nabla\Gamma_0 + \frac{1}{\lambda^2}\phi^i\nabla\chi\cdot\nabla\mathcal{R} + \frac{1}{\lambda^2}\phi^i\nabla\chi\cdot\nabla\psi_\lambda.\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{L}_{u_1}[\frac{1}{\lambda^2}\phi^i\chi] &= \chi\frac{1}{\lambda^2}\Delta\phi^i + \frac{2}{\lambda^2}\nabla\chi\cdot\nabla\phi^i + \frac{1}{\lambda^2}\phi^i\Delta\chi\\ &\quad - \Big[\nabla\cdot(\frac{\alpha}{\lambda^2}\phi^i\nabla\Gamma_0)\chi + \nabla\cdot(\frac{1}{\lambda^2}\phi^i\nabla\mathcal{R})\chi + \nabla\cdot(\frac{1}{\lambda^2}\phi^i\nabla\psi_\lambda)\chi\\ &\quad + \frac{\alpha}{\lambda^2}\phi^i\nabla\chi\cdot\nabla\Gamma_0 + \frac{1}{\lambda^2}\phi^i\nabla\chi\cdot\nabla\mathcal{R} + \frac{1}{\lambda^2}\phi^i\nabla\chi\cdot\nabla\psi_\lambda\Big]\\ &\quad - \Big[\nabla\cdot(\frac{\alpha}{\lambda^2}U\nabla\psi)\chi + \nabla\cdot(\frac{\alpha}{\lambda^2}U\nabla(\hat\psi - \psi))\chi + \frac{\alpha}{\lambda^2}U\nabla\chi\cdot\nabla\hat\psi\\ &\quad + \nabla\cdot(\varphi_\lambda\nabla\psi) + \nabla\cdot(\varphi_\lambda\nabla(\hat\psi - \psi))\Big].\end{aligned}$$

Next we expand

$$\mathcal{L}_{u_1}[\varphi^o] = \Delta\varphi^o - \nabla\cdot(\varphi^o\nabla v_1) - \nabla\cdot(u_1\nabla\psi^o), \quad \psi^o = (-\Delta)^{-1}\varphi^o.$$

We have

$$\begin{aligned}\nabla\cdot(u_1\nabla\psi^o) &= \nabla\cdot(\frac{\alpha}{\lambda^2}U\chi\nabla\psi^o) + \nabla\cdot(\varphi_\lambda\nabla\psi^o)\\ &= \nabla\cdot(\frac{\alpha}{\lambda^2}U\nabla\psi^o)\chi + \frac{\alpha}{\lambda^2}U\nabla\chi\cdot\nabla\psi^o + \nabla\cdot(\varphi_\lambda\nabla\psi^o)\chi\\ &\quad + \nabla\cdot(\varphi_\lambda\nabla\psi^o)(1-\chi),\end{aligned}$$

and

$$\begin{aligned}\nabla\cdot(\varphi^o\nabla v_1) &= \nabla\cdot(\varphi^o\nabla v_0) + \nabla\cdot(\varphi^o\nabla\psi_\lambda)\\ &= \alpha\nabla\cdot(\varphi^o\nabla\Gamma_0) + \nabla\cdot(\varphi^o\nabla\mathcal{R}) + \nabla\cdot(\varphi^o\nabla\psi_\lambda)\\ &= \nabla\varphi^o\cdot\nabla\Gamma_0 - \frac{1}{\lambda^2}U\varphi^o + (\alpha-1)\nabla\cdot(\varphi^o\nabla\Gamma_0)\\ &\quad + \nabla\cdot(\varphi^o\nabla\mathcal{R}) + \nabla\cdot(\varphi^o\nabla\psi_\lambda).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}_{u_1}[\varphi^o] &= \Delta\varphi^o - \Big[\nabla\cdot(\frac{\alpha}{\lambda^2}U\nabla\psi^o)\chi + \frac{\alpha}{\lambda^2}U\nabla\chi\cdot\nabla\psi^o + \nabla\cdot(\varphi_\lambda\nabla\psi^o)\chi\\ &\quad + \nabla\cdot(\varphi_\lambda\nabla\psi^o)(1-\chi)\Big]\\ &\quad - \Big[\nabla\varphi^o\cdot\nabla\Gamma_0 - \frac{1}{\lambda^2}U\varphi^o + (\alpha-1)\nabla\cdot(\varphi^o\nabla\Gamma_0)\\ &\quad + \nabla\cdot(\varphi^o\nabla\mathcal{R}) + \nabla\cdot(\varphi^o\nabla\psi_\lambda)\Big].\end{aligned}$$

Based on the previous formulas we formulate the inner equation

$$\begin{aligned}\lambda^4\partial_t(\frac{1}{\lambda^2}\phi^i) &= L[\phi^i] - (\alpha-1)\nabla_y\cdot(U\nabla_y\psi) - (\alpha-1)\nabla_y\cdot(\phi^i\nabla\Gamma_0) + \lambda^4 S(u_1)\\ &\quad - \lambda^2\nabla_y\cdot(\varphi_\lambda\nabla_y\psi^o) - \lambda^2\nabla_y\cdot(\varphi^o\nabla_y\psi_\lambda) + \lambda^2 U\varphi^o - \alpha\nabla_y\cdot(U\nabla_y\psi^o)\\ &\quad - \lambda^2\nabla_y\cdot(\varphi_\lambda\nabla_y\psi) - \nabla_y\cdot(\phi^i\nabla_y\psi_\lambda) - (\alpha-1)\lambda^2\nabla\cdot(\varphi^o\nabla\Gamma_0)\\ &\quad - \alpha\nabla_y\cdot(U\nabla_y(\hat\psi-\psi)) - \lambda^2\nabla_y\cdot(\varphi_\lambda\nabla_y(\hat\psi-\psi)) - \nabla_y\cdot((\phi^i\chi + \lambda^2\varphi^o)\nabla_y(\hat\psi + \psi^o)),\end{aligned}$$

where

$$L[\phi] = \Delta_y\phi - \nabla_y\cdot(U\nabla_y\psi) - \nabla_y\cdot(\phi\nabla\Gamma_0). \tag{5.4}$$

We slightly modify the inner equation into the form

$$\lambda^2\partial_t\phi^i = L[\phi^i] + B_0[\phi^i] + E_1\tilde\chi + F(\phi^i,\varphi^o,\mathbf{p})\tilde\chi \tag{5.5}$$

where

$$\mathbf{p} = (\lambda,\alpha,\xi),$$

$$E_1(y,t) = \lambda^4 S(u_1(\mathbf{p}))(x,t), \quad y = \frac{x-\xi}{\lambda},$$

$$\begin{aligned} F(\phi^i,\varphi^o,\mathbf{p}) = &-\lambda^2\nabla_y\cdot(\varphi_\lambda\nabla_y\psi^o) - \lambda^2\nabla_y\cdot(\varphi^o\nabla_y\psi_\lambda) + \lambda^2 U\varphi^o \\ &- (\alpha-1)\lambda^2\nabla_y\cdot(\varphi^o\nabla_y\Gamma_0) - \alpha\nabla_y\cdot(U\nabla_y\psi^o) \\ &+ \lambda\dot\xi\cdot\nabla_y\phi^i - \lambda^2\nabla_y\cdot(\varphi_\lambda\nabla_y\psi) - \nabla_y\cdot(\phi^i\nabla_y\psi_\lambda) \\ &- (\alpha-1)\nabla_y\cdot(U\nabla_y\psi) - (\alpha-1)\nabla_y\cdot(\phi^i\nabla_y\Gamma_0) \\ &- \alpha\nabla_y\cdot(U\nabla_y(\hat\psi-\psi)) - \lambda^2\nabla_y\cdot(\varphi_\lambda\nabla_y(\hat\psi-\psi)) \\ &- \nabla_y\cdot((\phi^i\chi+\lambda^2\varphi^o)\nabla_y(\hat\psi+\psi^o)), \qquad \hat\psi = (-\Delta_y)^{-1}(\phi^i\chi), \end{aligned} \tag{5.6}$$

$$B_0[\phi^i] = \lambda\dot\lambda(2\phi^i + y\cdot\nabla_y\phi^i), \tag{5.7}$$

and

$$\tilde\chi(y,t) = \chi_0\Big(\frac{\lambda y}{2\sqrt{t}}\Big), \tag{5.8}$$

with χ_0 as in (2.5). Similarly we formulate the outer equation as

$$\partial_t\varphi^o = \Delta\varphi^o - \nabla\Gamma_0\cdot\nabla\varphi^o + G(\phi^i,\varphi^o,\mathbf{p}) \tag{5.9}$$

where

$$\begin{aligned} G(\phi^i,\varphi^o,\mathbf{p}) = {}& S(u_1,\mathbf{p})(1-\chi) + \frac{2}{\lambda^2}\nabla\chi\cdot\nabla\phi^i + \frac{1}{\lambda^2}\phi^i\Delta\chi - \frac{1}{\lambda^2}\phi^i\partial_t\chi - \frac{\alpha}{\lambda^2}\phi^i\nabla\chi\cdot\nabla\Gamma_0 \\ &+ \frac{1}{\lambda^2}U\varphi^o(1-\chi) - \alpha\lambda^2 U\nabla\chi\cdot\nabla\psi^o - \nabla\cdot(\varphi_\lambda\nabla\psi^o)(1-\chi) \\ &- (\alpha-1)\nabla\cdot(\varphi^o\nabla\Gamma_0)(1-\chi) - \nabla\cdot(\varphi^o\nabla\mathcal{R}) - \nabla\cdot(\varphi^o\nabla\psi_\lambda)(1-\chi) \\ &- \frac{1}{\lambda^2}\nabla\cdot(\phi^i\nabla\mathcal{R})\chi - \frac{1}{\lambda^2}\phi^i\nabla\chi\cdot\nabla\mathcal{R} - \frac{1}{\lambda^2}\phi^i\nabla\chi\cdot\nabla\psi_\lambda \\ &- \frac{\alpha}{\lambda^2}U\nabla\chi\cdot\nabla\hat\psi - \nabla\cdot(\varphi_\lambda\nabla(\hat\psi-\psi))(1-\chi) \\ &- \nabla(\varphi_\lambda\nabla\psi)(1-\chi) - \nabla\cdot((\frac{1}{\lambda^2}\phi^i\chi+\varphi^o)\nabla(\hat\psi+\psi^o))(1-\chi). \end{aligned} \tag{5.10}$$

If ϕ^i, φ^o is a solution to system (5.5), (5.9), then u given by (5.1), (5.2) satisfies the Keller-Segel system (3.1).

5.1. **Choice of λ_0 and α_0.** We explain the choice of λ_0 in the context of the elliptic equation

$$L[\phi] = h \quad \text{in } \mathbb{R}^2, \tag{5.11}$$

where h is radial.

Lemma 5.1. *Let $h(y)$ be a radial function such that*

$$\|(1+|y|)^\gamma h(y)\|_{L^\infty(\mathbb{R}^2)} < \infty,$$

for some $\gamma > 4$ and satisfying

$$\int_{\mathbb{R}^2} h(y)dy = 0 \tag{5.12}$$

$$\int_{\mathbb{R}^2} h(y)|y|^2 dy = 0. \tag{5.13}$$

Then there exists a radial solution $\phi(y)$ of equation (5.11) such that

$$|\phi(y)| \le C\|(1+|y|)^\gamma h(y)\|_{L^\infty(\mathbb{R}^2)}\frac{1}{(1+|y|)^{\gamma-2}}, \quad \text{if } \gamma \ne 6 \tag{5.14}$$

$$|\phi(y)| \le C\|(1+|y|)^\gamma h(y)\|_{L^\infty(\mathbb{R}^2)} \frac{\log(1+|y|)}{(1+|y|)^4}, \quad \text{if } \gamma = 6, \tag{5.15}$$

and

$$\int_{\mathbb{R}^2} \phi(y)dy = 0. \tag{5.16}$$

Proof. Defining $g = \frac{\phi}{U} - (-\Delta)^{-1}\phi$ we obtain the equation

$$\nabla \cdot (U\nabla g) = h. \tag{5.17}$$

Assuming $\gamma > 6$ we choose the radial function g defined by

$$g(\rho) = -\int_\rho^\infty \frac{1}{rU(r)} \int_0^r h(s)sdsdr, \quad \rho = |y|,$$

and using (5.12) we get

$$|g(\rho)| \le C\|(1+|y|)^\gamma h\|_{L^\infty(\mathbb{R}^2)} \frac{1}{(1+|y|)^{\gamma-6}}.$$

Now we solve Liouville's equation

$$-\Delta\psi - U\psi = Ug \quad \text{in } \mathbb{R}^2, \quad \psi(\rho) \to 0 \quad \text{as } \rho \to \infty. \tag{5.18}$$

Multiplying (5.17) by $|y|^2$ and using (5.13) we see that

$$\int_{\mathbb{R}^2} gZ_0 dy = \frac{1}{2}\int_{\mathbb{R}^2} h(y)|y|^2 dy = 0,$$

with Z_0 defined in (3.4). Then by the variations of parameter formula we find that (5.18) has a unique solution ψ, which satisfies

$$|\psi(y)| + (1+|y|)|\nabla\psi(y)| \le \|(1+|y|)^\gamma h\|_{L^\infty(\mathbb{R}^2)} \frac{1}{(1+|y|)^{\gamma-4}}. \tag{5.19}$$

Then we see that ϕ defined by $\phi = Ug + U\psi$ satisfies (5.11), (5.14) and (5.16) because $\phi = -\Delta\psi$ and ψ has the decay (5.19).

If $4 < \gamma \le 6$ we do almost the same, except that we define

$$g(\rho) = \int_0^\rho \frac{1}{rU(r)} \int_0^r h(s)sdsdr.$$

□

Remark 5.1. *We observe that $L[Z_0] = 0$. This can also be seen in the context of the Lemma 5.1, where $\phi = Z_0$ which corresponds to g being constant. Indeed, suppose $g \equiv 1$. Then from (5.18) $\psi = -1 - \frac{1}{2}z_0$, where z_0 is defined in (9.2). This gives $\phi = Ug + U\psi = -\frac{1}{U}z_0 = -\frac{1}{2}Z_0$. This shows that $L[Z_0] = 0$.*

If h doesn't satisfy the zero second moment condition (5.13), then a solution still exists but with worse decay and non-zero mass. More precisely, if h is radial, $\|(1+|y|)^\gamma h(y)\|_{L^\infty(\mathbb{R}^2)} < \infty$ for some $\gamma > 6$, and satisfies only (5.12), then one can construct a solution ϕ to (5.11), but any such solution has the estimate

$$|\phi(y)| \le C\|(1+|y|)^\gamma h(y)\|_{L^\infty(\mathbb{R}^2)} \frac{\log(1+|y|)}{(1+|y|)^4},$$

so worse decay than the one in (5.14). Moreover, the mass of ϕ becomes

$$\int_{\mathbb{R}^2} \phi = -\int_{\mathbb{R}^2} \Delta\psi = -\int_{\mathbb{R}^2} gZ_0 = -\frac{1}{2}\int_{\mathbb{R}^2} h(y)|y|^2 dy.$$

For the inner equation (5.5) it is then natural to impose that the first error $S(u_1)\chi$ satisfies the second moment condition

$$\int_{\mathbb{R}^2} S(u_1)\chi|y|^2 dy = 0, \quad \text{for all } t > t_0.$$

The next lemma gives a way of expressing the second moment of u_1.

Lemma 5.2. *Let u_1 be defined in (3.13). Then*

$$\begin{aligned}\int_{\mathbb{R}^2} S(u_1)|x-\xi|^2 dx &= 4\int_{\mathbb{R}^2}\varphi_\lambda dx - \alpha\int_{\mathbb{R}^2}\tilde E(x-\xi,t;\lambda)|x-\xi|^2dx\\ &\quad + \int_{\mathbb{R}^2}\nabla\varphi_\lambda\,dx\cdot\dot\xi - \frac{\dot\alpha}{\lambda^2}\int_{\mathbb{R}^2}U\chi|x-\xi|^2dx - (1-\alpha)\int_{\mathbb{R}^2}E(x-\xi,t;\lambda)|x-\xi|^2dx\\ &\quad + 4\Big(\int_{\mathbb{R}^2}u_0 + \int_{\mathbb{R}^2}\varphi_\lambda\Big)\Big(1-\frac{1}{8\pi}\int_{\mathbb{R}^2}u_0 - \frac{1}{8\pi}\int_{\mathbb{R}^2}\varphi_\lambda\Big).\end{aligned} \tag{5.20}$$

where E, $\tilde E$ are defined in (3.10), (3.11).

Proof of Lemma 5.2. Using (2.11) we see that

$$\begin{aligned}\int_{\mathbb{R}^2} S(u_1)|x-\xi|^2dx = &-\int_{\mathbb{R}^2}\partial_t u_0|x-\xi|^2dx - \int_{\mathbb{R}^2}\partial_t\varphi_\lambda|x-\xi|^2dx\\ &+4\Big(\int_{\mathbb{R}^2}u_0+\int_{\mathbb{R}^2}\varphi_\lambda\Big)\Big(1-\frac{1}{8\pi}\int_{\mathbb{R}^2}u_0-\frac{1}{8\pi}\int_{\mathbb{R}^2}\varphi_\lambda\Big).\end{aligned}$$

But recall that $\varphi_\lambda(x,t)=\tilde\varphi_\lambda(x-\xi(t),t)$ where $\tilde\varphi_\lambda$ satisfies (3.9). Multiplying that equation by $|\zeta|^2$ and integrating on $\mathbb{R}^2$ results in

$$\int_{\mathbb{R}^2}\partial_t\tilde\varphi_\lambda|\zeta|^2\,d\zeta = -4\int_{\mathbb{R}^2}\tilde\varphi_\lambda\,d\zeta + \int_{\mathbb{R}^2}E(\zeta,t)|\zeta|^2\,d\zeta.$$

Therefore

$$\int_{\mathbb{R}^2}\partial_t\varphi_\lambda|x-\xi|^2\,dx = -4\int_{\mathbb{R}^2}\varphi_\lambda\,dx - \frac{(x-\xi)\cdot\dot\xi}{|x-\xi|}\int_{\mathbb{R}^2}\partial_r\tilde\varphi_\lambda + \int_{\mathbb{R}^2}E(\zeta,t)|\zeta|^2\,d\zeta$$

and then

$$\begin{aligned}\int_{\mathbb{R}^2}S(u_1)|x-\xi|^2dx = &-\int_{\mathbb{R}^2}\partial_t u_0|x-\xi|^2dx + 4\int_{\mathbb{R}^2}\varphi_\lambda dx + \int_{\mathbb{R}^2}\nabla\varphi_\lambda\,dx\cdot\dot\xi - \int_{\mathbb{R}^2}E(x-\xi,t)|x-\xi|^2dx\\ &+4\Big(\int_{\mathbb{R}^2}u_0+\int_{\mathbb{R}^2}\varphi_\lambda\Big)\Big(1-\frac{1}{8\pi}\int_{\mathbb{R}^2}u_0-\frac{1}{8\pi}\int_{\mathbb{R}^2}\varphi_\lambda\Big).\end{aligned} \tag{5.21}$$

But from the formula for $\partial_t u_0$ (3.3) and the definitions of E and $\tilde E$ (3.10), (3.11) we get

$$-\partial_t u_0(x,t) = -\frac{\dot\alpha}{\lambda^2}U(y)\chi_0(z) + \alpha E(x-\xi,t) - \alpha\tilde E(x-\xi,t).$$

Hence

$$\begin{aligned}&\int_{\mathbb{R}^2}(\partial_t u_0 + E(x-\xi,t))|x-\xi|^2dx\\ &= \int_{\mathbb{R}^2}(\partial_t u_0 + \alpha E(x-\xi))|x-\xi|^2dx + (1-\alpha)\int_{\mathbb{R}^2}E(x-\xi,t)|x-\xi|^2dx\\ &= \frac{\dot\alpha}{\lambda^2}\int_{\mathbb{R}^2}U\chi|x-\xi|^2dx + \alpha\int_{\mathbb{R}^2}\tilde E(x-\xi,t)|x-\xi|^2dx + (1-\alpha)\int_{\mathbb{R}^2}E(x-\xi,t)|x-\xi|^2dx.\end{aligned}$$

Replacing this in (5.21) we obtain (5.20). □

In the definition (3.13) of u_1 we will stress the dependence on the parameters by writing $\mathbf{p}=(\lambda,\alpha,\xi)$ and $u_1=u_1(\mathbf{p})$. At this point we would like to construct λ_0 and α_0 so that setting $\mathbf{p}_0=(\lambda_0,\alpha_0,0)$ we have

$$\int_{\mathbb{R}^2}u_1(\mathbf{p}_0)dx = 8\pi, \tag{5.22}$$

$$\int_{\mathbb{R}^2}S(u_1(\mathbf{p}_0))|x-\xi|^2dx = O\Big(\frac{1}{t^{\frac32+\sigma}}\Big), \tag{5.23}$$

for some $\sigma>0$. The reason for allowing in (5.23) an error is that it is difficult to solve with right hand side equal to 0 and a remainder of size $O(t^{-\frac32-\sigma})$ with $\sigma>0$ is sufficiently small to proceed with the rest of the construction.

Assuming that (5.22) holds, we get

$$\int_{\mathbb{R}^2} S(u_1)|x-\xi|^2 dx = 4\int_{\mathbb{R}^2} \varphi_\lambda dx - \alpha \int_{\mathbb{R}^2} \tilde{E}(x-\xi,t;\lambda)|x-\xi|^2 dx$$
$$+ \int_{\mathbb{R}^2} \nabla\varphi_\lambda \, dx \cdot \dot{\xi} - \frac{\dot{\alpha}}{\lambda^2}\int_{\mathbb{R}^2} U\chi|x-\xi|^2 dx - (1-\alpha)\int_{\mathbb{R}^2} E(x-\xi,t;\lambda)|x-\xi|^2 dx.$$

It turns out that the main terms in the expression for $\int_{\mathbb{R}^2} S(u_1)|x-\xi|^2 dx$ are the first two. So the equation

$$\int_{\mathbb{R}^2} S(u_1(\mathbf{p_0}))|x-\xi|^2 dx = 0$$

is at main order given by

$$4\int_{\mathbb{R}^2} \varphi_\lambda dx - \int_{\mathbb{R}^2} \tilde{E}|x-\xi|^2 dx = 0.$$

It will be shown later that

$$\int_{\mathbb{R}^2} \tilde{E}|x-\xi|^2 dx = -64\pi\Upsilon\frac{\lambda^2}{t} + O\Big(\frac{\lambda^4}{t^2}\Big), \tag{5.24}$$

see Lemma 7.5, where Υ is given in (2.7), so that the equation we want to solve becomes at main order,

$$\int_{\mathbb{R}^2} \varphi_\lambda dx + 16\pi\Upsilon\frac{\lambda^2}{t} = 0.$$

In §7 we will show that

$$\int_{\mathbb{R}^2} \varphi_\lambda dx = -4\pi\int_{t/2}^{t-\lambda^2} \frac{\lambda\dot{\lambda}}{t-s} ds - 2\pi\frac{\lambda^2}{t} - 16\pi\Upsilon\frac{\lambda^2}{t} + O\Big(\frac{\lambda^4 \log\log t}{t}\Big) \tag{5.25}$$

see Corollary 7.1. Using (5.25) we see that

$$\int_{\mathbb{R}^2} \varphi_\lambda dx + 16\pi\Upsilon\frac{\lambda^2}{t} = -4\pi\Big[\int_{t/2}^{t-\lambda^2} \frac{\lambda\dot{\lambda}}{t-s} ds + \frac{\lambda^2}{2t}\Big] + O\Big(\frac{\lambda^4 \log\log t}{t}\Big) \tag{5.26}$$

so that the equation for λ is at main order

$$\int_{t/2}^{t-\lambda^2} \frac{\lambda\dot{\lambda}}{t-s} ds + \frac{\lambda^2}{2t} = 0.$$

One can check that $\lambda^*(t) = \frac{c_0}{\sqrt{\log t}}$, where $c_0 > 0$ is an arbitrary constant, is an approximate solution. Indeed

$$\begin{aligned}\int_{t/2}^{t-(\lambda^*)^2} \frac{\lambda^*(s)\dot{\lambda}^*(s)}{t-s} ds + \frac{(\lambda^*)^2}{2t} &\approx \lambda^*(t)\dot{\lambda}^*(t)\int_{t/2}^{t-(\lambda^*)^2} \frac{ds}{t-s} + \frac{\lambda^*(t)^2}{2t}\\ &\approx \lambda^*(t)\dot{\lambda}^*(t)\log t + \frac{\lambda^*(t)^2}{2t}\\ &= \frac{1}{2}\frac{d}{dt}\big[\lambda^*(t)^2\log t\big] = 0.\end{aligned}$$

The error left out in the approximation (5.26) is too big. We give next a result that shows that for an appropriate modification of λ^* we can achieve a smaller error. Let us write $\tilde{E}(\lambda)$ the expression defined in (3.11) with the explicit dependence on λ.

Proposition 5.1. *Let $c_0 > 0$ be fixed. For $t_0 > 0$ sufficiently large there exists $\lambda_0 : [\frac{t_0}{2}, \infty) \to (0,\infty)$ such that*

$$\int_{\mathbb{R}^2} \varphi_{\lambda_0} dx - \frac{1}{4}\int_{\mathbb{R}^2} \tilde{E}(\lambda_0)|x-\xi|^2 dx = O\Big(\frac{1}{t^{\frac{3}{2}+\sigma}}\Big), \quad t > t_0, \tag{5.27}$$

for some $\sigma > 0$. Moreover, for arbitrarily $\varepsilon > 0$ small, λ_0 has the expansion

$$\begin{aligned}\lambda_0(t) &= \frac{c_0}{\sqrt{\log t}} + O\Big(\frac{1}{(\log t)^{\frac{3}{2}-\varepsilon}}\Big),\\ \dot{\lambda}_0(t) &= -\frac{c_0}{2t(\log t)^{3/2}} + O\Big(\frac{1}{t(\log t)^{\frac{5}{2}-\varepsilon}}\Big),\\ |\ddot{\lambda}_0(t)| &\le \frac{C}{t^2(\log t)^{3/2}},\end{aligned}$$

as $t \to \infty$.

We will prove this result in §7.1.

Once λ_0 is constructed in Proposition 5.1 we choose α_0 so that (5.22) holds, by imposing

$$\alpha_0(t) \int_{\mathbb{R}^2} U(y)\chi_0\Big(\frac{\lambda_0(t)y}{\sqrt{t}}\Big)\,dy + \int_{\mathbb{R}^2} \varphi_{\lambda_0}(x,t)\,dx = 8\pi, \quad t > t_0. \tag{5.28}$$

We note that by (2.6), (5.27) and (5.24) we get

$$\alpha_0(t) = 1 + O\Big(\frac{1}{t^{\frac32+\sigma}}\Big)$$

as $t \to \infty$. A byproduct of the proof of Proposition 5.1 is that

$$\Big|\frac{d}{dt}\int_{\mathbb{R}^2} \varphi_{\lambda_0} dx\Big| \le \frac{C}{t^2}, \tag{5.29}$$

and from this and (5.28) we get

$$|\dot\alpha_0(t)| \le \frac{C}{t^2}. \tag{5.30}$$

As a corollary of Proposition 5.1 we get:

Corollary 5.1. *Let* $\boldsymbol{p}_0 = (\lambda_0, \alpha_0, 0)$ *with* α_0 *defined by* (5.22) *and* λ_0 *be given by Proposition 5.1. Then*

$$\int_{\mathbb{R}^2} S(u_1(\boldsymbol{p}_0))|x-\xi|^2 dx = O\Big(\frac{1}{t^{\frac32+\sigma}}\Big),$$

for some $\sigma > 0$.

Proof. Using Lemma 5.2 we have

$$\begin{aligned}\int_{\mathbb{R}^2} S(u_1)|x-\xi|^2 dx &= 4\int_{\mathbb{R}^2} \varphi_{\lambda_0} dx - \int_{\mathbb{R}^2} \tilde E(x-\xi,t;\lambda_0)|x-\xi|^2 dx \\ &\quad - \frac{\dot\alpha_0}{\lambda^2}\int_{\mathbb{R}^2} U\chi|x-\xi|^2 dx - (1-\alpha_0)\int_{\mathbb{R}^2} E(x-\xi,t;\lambda_0)|x-\xi|^2 dx \\ &= O\Big(\frac{1}{t^{\frac32+\sigma}}\Big),\end{aligned}$$

for some $\sigma > 0$, since $\dot\alpha_0(t) = O(\frac{1}{t^2\log t})$ and

$$\int_{\mathbb{R}^2} E(x-\xi,t;\lambda_0)|x-\xi|^2 dx = O\Big(\frac{\lambda_0^2}{t}\Big)$$

by (5.24) and a direct estimate for the remaining terms in E (c.f. (3.10)). ☐

5.2. **A further improvement of the approximation.** We introduce a correction $\phi_0^i(y)$, $y = \frac{x-\xi}{\lambda}$ in the inner approximation to eliminate the radial part of $S(u_1(\mathbf{p}))$ (defined in (4.2)), which we define as

$$\begin{aligned}S_0(u_1(\mathbf{p})) &= -\frac{\dot\alpha}{\lambda^2}U(y)\chi + (\alpha-1)\frac{\dot\lambda}{\lambda^3}Z_0\chi + \frac{(\alpha-1)}{2t}\frac{1}{\lambda^2}U\nabla_z\chi_0\cdot\frac{x-\xi}{\sqrt{t}} \\ &\quad + \frac{2(\alpha-1)}{\lambda^3 t^{1/2}}\nabla_z\chi_0\cdot\nabla_y U + \frac{(\alpha-1)}{t}\Delta\chi_0\frac{1}{\lambda^2}U - \frac{\alpha^2-1}{\lambda^3\sqrt{t}}U\nabla_z\chi_0\cdot\nabla_y\Gamma_0 - \frac{\alpha}{\lambda^3\sqrt{t}}U\nabla_z\chi_0\cdot\nabla_y\mathcal{R} \\ &\quad - \frac{\alpha(\alpha-1)\chi}{\lambda^4}\nabla_y\cdot(U\nabla_y\Gamma_0) + \frac{\alpha^2\chi(1-\chi)}{\lambda^4}U^2 - \frac{\alpha\chi}{\lambda^4}\nabla_y U\cdot\nabla_y\mathcal{R}. \\ &\quad - \frac{4}{r}\partial_r\varphi_\lambda - \nabla\cdot(\varphi_\lambda\nabla v_0) - \nabla\cdot(u_0\nabla\psi_\lambda) - \nabla\cdot(\varphi_\lambda\nabla\psi_\lambda).\end{aligned} \tag{5.31}$$

With this definition

$$S(u_1) = S_0(u_1) + \frac{\alpha}{\lambda^3}\dot\xi\cdot\nabla_y U(y)\,\chi + \frac{\alpha}{\lambda^2\sqrt{t}}U(y)\dot\xi\cdot\nabla\chi_0,$$

and the terms not in $S(u_1)$ correspond to $\frac{\alpha}{\lambda^3}\dot\xi\cdot\nabla_y U(y)\,\chi + \frac{\alpha}{\lambda^2\sqrt{t}}U(y)$ which are in mode 1.

Then we want ϕ_0^i to be an appropriate solution to the equation

$$L[\phi_0^i] + \lambda^4 S_0(u_1(\mathbf{p}_0))(x,t) = c_0(t)W_2 \quad \text{in } \mathbb{R}^2, \quad x = \xi + \lambda y, \tag{5.32}$$

where L is the linear operator (5.4), $t > t_0$ is regarded as a parameter, $W_2(y)$ is a fixed smooth radial function with compact support, and

$$\int_{\mathbb{R}^2} W_2(y)dy = 0, \quad \int_{\mathbb{R}^2} W_2(y)|y|^2 dy = 1. \tag{5.33}$$

By Lemma 5.2 and Proposition 5.1, the choice $\mathbf{p} = \mathbf{p}_0$ is so that (5.22), (5.23) hold. Since the difference between $S(u_1)$ and $S_0(u_1)$ contains terms in mode 1 only, we get from Corollary 5.1

$$\int_{\mathbb{R}^2} \lambda^4 S_0(u_1(\mathbf{p}_0))|y|^2 dy = O\Big(\frac{1}{t^{\frac{3}{2}+\sigma}}\Big). \tag{5.34}$$

In (5.32) we select $c_0(t)$ such that

$$\int_{\mathbb{R}^2} [\lambda^4 S_0(u_1(\mathbf{p}_0)) + c_0(t)W_2]|y|^2 dy = 0, \quad t > t_0$$

and thanks to (5.34) we have

$$|c_0(t)| \le \frac{C}{t^{\frac{3}{2}+\sigma}}, \quad t > t_0. \tag{5.35}$$

Note that we have

$$\int_{\mathbb{R}^2} S_0(u_1(\mathbf{p}_0))dx = 0,$$

which follows from the constant mass in time of $u_1(\mathbf{p}_0)$ in (5.22) and the form of the operator S_0 (5.31).

We let ϕ_0^i be the solution to (5.32) constructed in Lemma 5.1. By (5.15) and (4.5)

$$|\phi_0^i(y,t)| \le \frac{C}{t}\frac{\log(1+|y|)}{1+|y|^4}, \tag{5.36}$$

and

$$\int_{\mathbb{R}^2} \phi_0^i(y,t)dy = 0, \quad t > t_0.$$

5.3. **Reformulation of the system.** In the outer problem (5.9) we would like to separate the effect of the initial condition from the coupling $G(\phi^i, \varphi^o, \mathbf{p})$.

We take the initial condition in (5.9) to be

$$\varphi^o(\cdot, t_0) = \varphi_0^*,$$

and let $\varphi^*(x,t)$ denote the solution of

$$\begin{cases} \partial_t \varphi^* = \Delta\varphi^* - \nabla_x \Gamma_0\Big(\dfrac{x-\xi}{\lambda}\Big)\cdot\nabla\varphi^* & \text{in } \mathbb{R}^2\times(t_0,\infty) \\ \varphi^*(\cdot,t_0) = \varphi_0^* \quad \text{in } \mathbb{R}^2. \end{cases} \tag{5.37}$$

The initial condition $\varphi_0^*(x)$ will be later used to prove the stability claimed in Theorem 1.1. The topology for φ_0^* will be specified later on.

Note that $\nabla_x\Gamma_0(\frac{x-\xi}{\lambda}) = -4\frac{x-\xi}{|x-\xi|^2+\lambda^2}$ so that φ^* is a function of the parameters λ, ξ. Therefore we will write $\varphi^*(x,t;\mathbf{p})$ when convenient.

We decompose

$$\begin{cases} \phi^i = \phi_0^i + \phi \\ \varphi^o = \varphi^* + \varphi \\ \mathbf{p} = \mathbf{p}_0 + \mathbf{p}_1 \end{cases} \tag{5.38}$$

where

$$\mathbf{p}_0 = (\lambda_0, \alpha_0, 0), \quad \mathbf{p}_1 = (\lambda_1, \alpha_1, \xi_1),$$

with λ_0 the function constructed in Proposition 5.1 and α_0 chosen so that (5.22) holds.

We substitute the expressions for ϕ^i, φ^o and $\mathbf{p}$ in (5.38) into the equations (5.5), (5.9), and are led to the following problem for ϕ, φ

$$\begin{cases} \lambda^2\partial_t\phi = L[\phi] + B_0[\phi] + E_2\tilde\chi_2 + F_2(\phi,\varphi,\mathbf{p}_1,\varphi_0^*)\tilde\chi & \text{in } \mathbb{R}^2\times(t_0,\infty) \\ \phi(\cdot,t_0) = \phi_0 \quad \text{in } \mathbb{R}^2 \end{cases} \tag{5.39}$$

$$\begin{cases} \partial_t \varphi = \Delta\varphi - \nabla_x \Gamma_0(\frac{x-\xi}{\lambda}) \cdot \nabla\varphi + G_2(\phi,\varphi,\mathbf{p}_1,\varphi_0^*) \quad \text{in } \mathbb{R}^2\times(t_0,\infty) \\ \varphi(\cdot,t_0) = 0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{5.40}$$

where $\tilde\chi$ is defined in (5.8),

$$E_2 = -\partial_t \phi_0^i + B_0[\phi_0^i] + c_0(t) W_2$$

$$\begin{aligned} F_2(\phi,\varphi,\mathbf{p}_1,\varphi_0^*) &= F(\phi_0^i+\phi,\varphi^*+\varphi,\mathbf{p}_0+\mathbf{p}_1) + \lambda^4[S_0(u_1(\mathbf{p}_0+\mathbf{p}_1)) - S_0(u_1(\mathbf{p}_0))] \\ &\quad + \lambda\alpha\dot\xi_1\cdot\nabla_y U(y)\,\chi + \frac{\alpha\lambda^2}{\sqrt{t}} U(y)\dot\xi_1\cdot\nabla\chi_0 \end{aligned} \tag{5.41}$$

$$G_2(\phi,\varphi,\mathbf{p}_1,\varphi_0^*) = G(\phi_0^i+\phi,\varphi^*+\varphi,\mathbf{p}_0+\mathbf{p}_1) + \lambda^{-4}E_2(1-\tilde\chi_2)\chi \tag{5.42}$$

$$\tilde\chi_2(x,t) = \chi_0\Big(\frac{x-\xi}{t^{\frac12-\delta}}\Big),$$

$\delta>0$ is a small constant to be fixed later on, and χ_0 is as in (2.5). We recall that F and G are defined in (5.6) and (5.10). The expressions for F_2 and G_2 depend on the initial condition φ_0^* through φ^* (5.37) and ϕ_0. The role of ϕ_0 will be clarified later on.

By the estimate for $\ddot\lambda_0$ in Proposition 5.1 and (5.35) we get

$$|E_2(y,t)| \le \frac{C}{t^2(\log t)^2}\frac{\log(1+|y|)}{1+|y|^4} + \frac{C}{t^{\frac32+\sigma}}|W_2(y)|, \quad |y|\le C\sqrt{t\log t}. \tag{5.43}$$

The reason that we introduce the cut-off $\tilde\chi_2$ is to achieve

$$|E_2\tilde\chi_2(y,t)| \le \frac{C}{t^\nu(1+|y|)^{6+\sigma}},$$

if $\nu<1+2\delta-\frac{\sigma}{2}$. We will choose δ and σ positive small numbers such that $2\delta-\frac{\sigma}{2}>0$ so that we can find $1<\nu<1+2\delta-\frac{\sigma}{2}$.

5.4. **Splitting the inner solution ϕ.** We perform one more change in the formulation (5.39), (5.40), which consists in decomposing

$$\phi = \phi_1 + \phi_2.$$

The function ϕ_1 will solve an equation with part of the right hand side of (5.39), which will be projected so that it satisfies the zero second moment condition.

For any $h(y,t)$ with sufficient spatial decay we define

$$m_0[h](t) = \int_{\mathbb{R}^2} h(y,t)dy, \quad m_2[h](t) = \int_{\mathbb{R}^2} h(y,t)|y|^2dy, \tag{5.44}$$

and

$$m_{1,j}[h](t) = \int_{\mathbb{R}^2} h(y,t)y_j dy, \quad j=1,2,$$

which denote the mass, second moment and center of mass of h.

Let $W_0\in C^\infty(\mathbb{R}^2)$ be radial with compact support such that

$$\int_{\mathbb{R}^2} W_0 dy = 1, \quad \int_{\mathbb{R}^2} W_0|y|^2dy = 0.$$

Let $W_{1,j}$, $j=1,2$ be a smooth functions with compact support and with the form $W_{1,j}(y)=\tilde W(|y|)y_j$ so that

$$\int_{\mathbb{R}^2} W_{1,j}(y)y_j = 1. \tag{5.45}$$

We recall that W_2 defined in (5.33).

Then, $h-m_0[h]W_0$ has zero mass, $h-m_2[h]W_2$ has zero second moment, and $h-m_{1,1}[h]W_{1,1}-m_{1,2}[h]W_{1,2}$ has zero center of mass.

We modify of the operator B_0 appearing in (5.39), and defined in (5.7). The idea is to work with a variant of it, which coincides with it for radial functions, but for functions without radial part it is cutoff outside the region $|y| \lesssim \frac{\sqrt{t}}{\lambda}$. More precisely, we decompose ϕ in a radial part $[\phi]_{rad}$ defined by

$$[\phi]_{rad}(\rho, t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\rho e^{i\theta}, t) d\theta \tag{5.46}$$

and a term with no radial mode $\phi_1 = \phi - [\phi]_{rad}$. We note that the other linear terms in the equation behave well with this decomposition. Then we define

$$B[\phi] = \lambda\dot\lambda(2[\phi]_{rad} + y\cdot\nabla[\phi]_{rad}) + \lambda\dot\lambda(2\phi_1 + y\cdot\nabla\phi_1)\chi_0\Big(\frac{\lambda y}{5\sqrt{t}}\Big) \tag{5.47}$$

where χ_0 is a smooth cut-off in $\mathbb{R}$ with $\chi_0(s) = 1$ for $s \le 1$ and $\chi_0(s) = 1$ for $s \ge 2$.

With these definitions we introduce the following system for ϕ_1, ϕ_2, φ, $\mathbf{p}_1$,

$$\begin{cases} \lambda^2\partial_t\phi_1 = L[\phi_1] + B[\phi_1] + F_3(\phi_1+\phi_2, \varphi, \mathbf{p}_1, \varphi_0^*) \\ \qquad\qquad - m_0[F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)]W_0 - m_2[F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)]W_2 \\ \qquad\qquad + \displaystyle\sum_{j=1}^2 \mu_j W_{1,j} \qquad \text{in } \mathbb{R}^2\times(t_0,\infty) \\ \phi_1(\cdot,t_0) = 0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{5.48}$$

$$\begin{cases} \lambda^2\partial_t\phi_2 = L[\phi_2] + B[\phi_2] + m_2[F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)]W_2 \quad \text{in } \mathbb{R}^2\times(t_0,\infty) \\ \phi_2(\cdot,t_0) = \phi_0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{5.49}$$

$$\begin{cases} \quad \partial_t\varphi = \Delta\varphi - \nabla\Gamma_0\cdot\nabla\varphi + G_2(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*) \quad \text{in } \mathbb{R}^2\times(t_0,\infty) \\ \varphi(\cdot,t_0) = 0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{5.50}$$

where

$$F_3(\phi,\varphi,\mathbf{p}_1,\varphi_0^*) = E_2\tilde\chi_2 + F_2(\phi,\varphi,\mathbf{p}_1,\varphi_0^*)\tilde\chi, \tag{5.51}$$

In (5.48) $\mu_j(t)$ are functions so that the right hand side has center of mass equal to zero. A solution ϕ_1, ϕ_2, φ to (5.48), (5.49) and (5.50) gives a solution to the system (5.39), (5.40) provided $\mathbf{p}_1$ is such that the following equations are satisfied

$$\begin{cases} 0 = m_0[F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)](t), \quad \forall t > t_0, \\ 0 = \mu_j(t), \quad \forall t>t_0,\ j = 1,2. \end{cases} \tag{5.52}$$

5.5. **Mass and second moment.** In this section we derive some formulas for the mass and second moment appearing in the right hand side of (5.48).

In the computation of $m_0[F_3(\phi,\varphi,\mathbf{p}_1,\varphi_0^*)]$ and $m_2[F_3(\phi,\varphi,\mathbf{p}_1,\varphi_0^*)]$, the following formulas will be useful.

Lemma 5.3. *We have*

$$\begin{aligned}\int_{\mathbb{R}^2} S(u_1(\mathbf{p}))dx &= -\partial_t\int_{\mathbb{R}^2} u_0 dx - \partial_t\int_{\mathbb{R}^2}\varphi_\lambda dx \\ &= -\partial_t\Big\{8\pi\alpha\Big[1 + 2\Upsilon\frac{\lambda^2}{t}\Big] + \alpha e_1\Big(\frac{\lambda^2}{t}\Big) + \int_{\mathbb{R}^2}\varphi_\lambda dx\Big\}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^2}(S(u_1(\mathbf{p})) - S(u_1(\mathbf{p}_0)))dx = -\partial_t\Big\{&\alpha_1\Big[8\pi\Big(1+2\Upsilon\frac{\lambda^2}{t}\Big) + e_1\Big(\frac{\lambda^2}{t}\Big)\Big] + 16\pi\alpha_0\Upsilon\frac{\lambda^2-\lambda_0^2}{t} \\ &+ \alpha_0\Big(e_1\Big(\frac{\lambda^2}{t}\Big) - e_1\Big(\frac{\lambda_0^2}{t}\Big)\Big) + \int_{\mathbb{R}^2}(\varphi_\lambda - \varphi_{\lambda_0})dx\Big\},\end{aligned}$$

where $e_1(s)$ is defined by

$$\int_{\mathbb{R}^2} u_0 dx = 8\pi\alpha\Big[1+2\Upsilon\frac{\lambda^2}{t}\Big] + \alpha e_1\Big(\frac{\lambda^2}{t}\Big). \tag{5.53}$$

Recall that Υ is given in (2.7) and note that

$$e_1(s) = O(s^2), \quad \text{as } s \to 0.$$

Proof. For this we recall that (c.f. (2.8))

$$S(u_1(\mathbf{p})) = -\partial_t u_0 - \partial_t \varphi_\lambda + \mathcal{E}(u_0 + \varphi_\lambda),$$

so

$$\begin{aligned}\int_{\mathbb{R}^2} S(u_1(\mathbf{p}))dx &= -\partial_t \int_{\mathbb{R}^2} u_0 dx - \partial_t \int_{\mathbb{R}^2} \varphi_\lambda dx \\ &= -\partial_t \Big\{ 8\pi\alpha\Big[1 + 2\Upsilon\frac{\lambda^2}{t}\Big] + \alpha e_1\Big(\frac{\lambda^2}{t}\Big) + \int_{\mathbb{R}^2} \varphi_\lambda dx \Big\}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^2} (S(u_1(\mathbf{p})) - S(u_1(\mathbf{p}_0)))dx = -\partial_t \Big\{ \alpha_1\Big[8\pi\Big(1 + 2\Upsilon\frac{\lambda^2}{t}\Big) + e_1\Big(\frac{\lambda^2}{t}\Big)\Big] + 16\pi\alpha_0\Upsilon\frac{\lambda^2 - \lambda_0^2}{t} \\ + \alpha_0\Big(e_1\Big(\frac{\lambda^2}{t}\Big) - e_1\Big(\frac{\lambda_0^2}{t}\Big)\Big) + \int_{\mathbb{R}^2} (\varphi_\lambda - \varphi_{\lambda_0})dx \Big\}.\end{aligned}$$

□

Lemma 5.4. *We have*

$$\begin{aligned}&\lambda^4 m_2[S_0(u_1(\boldsymbol{p}_0 + \boldsymbol{p}_1)) - S_0(u_1(\boldsymbol{p}_0))] \\ &\quad = -32\pi\alpha_1 - \frac{\dot\alpha}{\lambda^2}\int_{\mathbb{R}^2} U(\frac{x-\xi}{\lambda})\chi_0(\frac{x-\xi}{\lambda})|x-\xi|^2\,dx + \frac{\dot\alpha_0}{\lambda_0^2}\int_{\mathbb{R}^2} U(\frac{x}{\lambda_0})\chi(\frac{x}{\lambda_0})|x|^2\,dx \\ &\quad - 4\Big[\alpha e_1\Big(\frac{\lambda^2}{t}\Big) - \alpha_0 e_1\Big(\frac{\lambda_0^2}{t}\Big)\Big] - \Big[\alpha e_2\Big(\frac{\lambda^2}{t}\Big) - \alpha_0 e_2\Big(\frac{\lambda_0^2}{t}\Big)\Big] \\ &\quad - \Big(\int_{\mathbb{R}^2} (\varphi_\lambda - \varphi_{\lambda_0})\,dx\Big)^2 \\ &\quad - (1-\alpha)\int_{\mathbb{R}^2} E(x-\xi,t,\lambda)|x-\xi|^2\,dx + (1-\alpha_0)\int_{\mathbb{R}^2} E(x,t,\lambda_0)|x|^2\,dx \\ &\quad - |\xi|^2\int_{\mathbb{R}^2} S(u_1(\mathbf{p}_0))dx.\end{aligned}$$

Proof. We have defined the second moment m_2 (5.44) integrating with respect to y. Note that

$$\lambda^4 \int_{\mathbb{R}^2} f(y)|y|^2 dy = \int_{\mathbb{R}^2} f\Big(\frac{x-\xi}{\lambda}\Big)|x-\xi|^2 dx,$$

and therefore

$$\begin{aligned}\lambda^4 m_2[S_0(u_1(\mathbf{p}_0 + \mathbf{p}_1)) - S_0(u_1(\mathbf{p}_0))] &= \lambda^4 \int_{\mathbb{R}^2} S_0(u_1(\mathbf{p}_0 + \mathbf{p}_1))(\xi + \lambda y)|y|^2 dy \\ &\quad - \lambda^4 \int_{\mathbb{R}^2} S_0(u_1(\mathbf{p}_0))(\xi + \lambda y)|y|^2 dy \\ &= \int_{\mathbb{R}^2} S_0(u_1(\mathbf{p}_0 + \mathbf{p}_1))(x)|x-\xi|^2 dy \\ &\quad - \int_{\mathbb{R}^2} S_0(u_1(\mathbf{p}_0))(x)|x-\xi|^2 dy.\end{aligned}$$

We have by Lemma 5.2,

$$\begin{aligned}\int_{\mathbb{R}^2} S(u_1)|x-\xi|^2 dx &= 4\int_{\mathbb{R}^2} \varphi_\lambda dx - \alpha \int_{\mathbb{R}^2} \tilde E(x-\xi,t;\lambda)|x-\xi|^2 dx \\ &\quad + \int_{\mathbb{R}^2} \nabla\varphi_\lambda\,dx \cdot \dot\xi - \frac{\dot\alpha}{\lambda^2}\int_{\mathbb{R}^2} U\chi|x-\xi|^2 dx \\ &\quad - (1-\alpha)\int_{\mathbb{R}^2} E(x-\xi,t;\lambda)|x-\xi|^2 dx \\ &\quad + 4\Big(\int_{\mathbb{R}^2} u_0 + \int_{\mathbb{R}^2} \varphi_\lambda\Big)\Big(1 - \frac{1}{8\pi}\int_{\mathbb{R}^2} u_0 - \frac{1}{8\pi}\int_{\mathbb{R}^2} \varphi_\lambda\Big).\end{aligned} \tag{5.54}$$

where E, $\tilde{E}$ are defined in (3.10), (3.11). Let

$$m = \int_{\mathbb{R}^2} (u_0 + \varphi_\lambda) dx, \quad \delta m = m - 8\pi.$$

Since

$$\int_{\mathbb{R}^2} (u_0 + \varphi_{\lambda_0}) dx = 8\pi,$$

by (5.22), we have

$$\delta m = \int_{\mathbb{R}^2} (\varphi_\lambda - \varphi_{\lambda_0})\, dx.$$

Replacing m in (5.54) we get

$$\begin{aligned}\int_{\mathbb{R}^2} S(u_1(\mathbf{p}))|x-\xi|^2 dx = 32\pi - 4\int_{\mathbb{R}^2} u_0 dx - \frac{1}{2\pi}(\delta m)^2 - \alpha \int_{\mathbb{R}^2} \tilde{E}(x-\xi, t; \lambda)|x-\xi|^2 dx \\ + \int_{\mathbb{R}^2} \nabla\varphi_\lambda\, dx \cdot \dot{\xi} - \frac{\dot{\alpha}}{\lambda^2} \int_{\mathbb{R}^2} U\chi |x-\xi|^2 dx \\ - (1-\alpha)\int_{\mathbb{R}^2} E(x-\xi, t; \lambda)|x-\xi|^2 dx. \end{aligned} \tag{5.55}$$

Also under (4.1) we have by (5.24):

$$\int_{\mathbb{R}^2} \tilde{E}|x-\xi|^2 dx = -64\pi\Upsilon \frac{\lambda^2}{t} + e_2\Big(\frac{\lambda^2}{t}\Big), \tag{5.56}$$

where

$$e_2(s) = O(s^2), \quad \text{as } s \to 0.$$

Combining (5.55), (5.53) and (5.56) we get

$$\begin{aligned}\int_{\mathbb{R}^2} S(u_1(\mathbf{p}))|x-\xi|^2 dx = 32\pi(1-\alpha) - \frac{1}{2\pi}(\delta m)^2 - \frac{\dot{\alpha}}{\lambda^2}\int_{\mathbb{R}^2} U\chi|x-\xi|^2 dx \\ + \int_{\mathbb{R}^2} \nabla\varphi_\lambda\, dx \cdot \dot{\xi} - (1-\alpha)\int_{\mathbb{R}^2} E(x-\xi, t;\lambda)|x-\xi|^2 dx \\ - 4\alpha e_1\Big(\frac{\lambda^2}{t}\Big) - \alpha e_2\Big(\frac{\lambda^2}{t}\Big).\end{aligned}$$

We can apply this formula to $\mathbf{p} = \mathbf{p}_0$ and get

$$\begin{aligned}\int_{\mathbb{R}^2} S(u_1(\mathbf{p}_0))|x|^2 dx = 32\pi(1-\alpha_0) - \frac{\dot{\alpha}_0}{\lambda_0^2}\int_{\mathbb{R}^2} U(\frac{x}{\lambda_0})\chi|x|^2 dx \\ - (1-\alpha_0)\int_{\mathbb{R}^2} E(x, t; \lambda_0)|x|^2 dx \\ - 4\alpha_0 e_1\Big(\frac{\lambda_0^2}{t}\Big) - \alpha_0 e_2\Big(\frac{\lambda_0^2}{t}\Big).\end{aligned}$$

Note that

$$\begin{aligned}\int_{\mathbb{R}^2} S_0(u_1(\mathbf{p}_0))|x-\xi|^2 dx &= \int_{\mathbb{R}^2} S_0(u_1(\mathbf{p}_0))|x|^2 dx + |\xi|^2 \int_{\mathbb{R}^2} S_0(u_1(\mathbf{p}_0))dx \\ &= \int_{\mathbb{R}^2} S_0(u_1(\mathbf{p}_0)|x|^2 dx + |\xi|^2 \int_{\mathbb{R}^2} S(u_1(\mathbf{p}_0)dx\end{aligned}$$

because

$$\int_{\mathbb{R}^2} S_0(u_1(\mathbf{p}_0)x_j dx = 0.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^2} [S(u_1(\mathbf{p})) - S(u_1(\mathbf{p}_0))]|x-\xi|^2 dx \\
&= -32\pi\alpha_1 - \frac{\dot\alpha}{\lambda^2}\int_{\mathbb{R}^2} U(\frac{x-\xi}{\lambda})\chi_0(\frac{x-\xi}{\lambda})|x-\xi|^2\,dx + \frac{\dot\alpha_0}{\lambda_0^2}\int_{\mathbb{R}^2} U(\frac{x}{\lambda_0})\chi(\frac{x}{\lambda_0})|x|^2\,dx \\
&\quad - 4\Big[\alpha e_1\Big(\frac{\lambda^2}{t}\Big) - \alpha_0 e_1\Big(\frac{\lambda_0^2}{t}\Big)\Big] - \Big[\alpha e_2\Big(\frac{\lambda^2}{t}\Big) - \alpha_0 e_2\Big(\frac{\lambda_0^2}{t}\Big)\Big] \\
&\quad - \Big(\int_{\mathbb{R}^2} (\varphi_\lambda - \varphi_{\lambda_0})\,dx\Big)^2 \\
&\quad - (1-\alpha)\int_{\mathbb{R}^2} E(x-\xi,t,\lambda)|x-\xi|^2\,dx + (1-\alpha_0)\int_{\mathbb{R}^2} E(x,t,\lambda_0)|x|^2\,dx \\
&\quad - |\xi|^2\int_{\mathbb{R}^2} S(u_1(\mathbf{p}_0)dx.
\end{aligned}$$

□

6. Proof of Theorem 1.1

Next we define norms, which are suitably adapted to the terms in the inner linear problems (5.48), (5.49). Let us write the linearized versions of these problems as

$$\begin{cases} \lambda^2\partial_t\phi = L[\phi] + B[\phi] + h(y,t) & \text{in } \mathbb{R}^2\times(t_0,\infty), \\ \phi(\cdot,t_0) = 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{6.1}$$

Given positive numbers ν, p, ϵ and $m\in\mathbb{R}$, we let

$$\|h\|_{0,\nu,m,p,\epsilon} = \inf K \quad \text{such that} \tag{6.2}$$

$$|h(y,t)| \le \frac{K}{t^\nu(\log t)^m}\frac{1}{(1+|y|)^p}\begin{cases} 1 & |y|\le\sqrt{t\log t}, \\ \dfrac{(t\log t)^{\epsilon/2}}{|y|^\epsilon} & |y|\ge\sqrt{t\log t}. \end{cases}$$

We also defie

$$\|\phi\|_{1,\nu,m,p,\epsilon} = \inf K \quad \text{such that}$$

$$|\phi(y,t)| + (1+|y|)|\nabla_y\phi(y,t)| \le \frac{K}{t^\nu(\log t)^m}\frac{1}{(1+|y|)^p}\begin{cases} 1 & |y|\le\sqrt{t\log t}, \\ \dfrac{(t\log t)^{\epsilon/2}}{|y|^\epsilon} & |y|\ge\sqrt{t\log t}. \end{cases}$$

We develop a solvability theory of problem (6.1) that involves uniform space-time bounds in terms of the above norms. We will establish two results: one in which the solution “loses” one power of t on bounded sets with respect to the time-decay of h, under radial symmetry and the condition of spatial average 0 at all times. Our second result states that for a general h this loss is only $t^{\frac12}$ if in addition the center of mass and second-moment of h are zero at all times.

For the first result we introduce a parameter in the problem in order to get a fast decay of the solution:

$$\begin{cases} \lambda^2\partial_t\phi = L[\phi] + B[\phi] + h(y,t) & \text{in } \mathbb{R}^2\times(t_0,\infty), \\ \phi(\cdot,t_0) = c_1\tilde Z_0 & \text{in } \mathbb{R}^2, \end{cases} \tag{6.3}$$

where $\tilde Z_0$ is defined as

$$\tilde Z_0(\rho) = (Z_0(\rho) - m_{Z_0}U)\chi_0\Big(\frac{\rho}{3\lambda(t_0)\sqrt{t_0}}\Big), \tag{6.4}$$

where m_{Z_0} is such that

$$\int_{\mathbb{R}^2}\tilde Z_0 = 0.$$

Proposition 6.1. *Assume* (4.1). *Let* $\sigma > 0$, $\epsilon > 0$ *with* $\sigma+\epsilon < 2$ *and* $1 < \nu < \frac{7}{4}$. *Let* $0 < q < 1$. *Then there exists a number* $C > 0$ *such that for* t_0 *sufficiently large and all radially symmetric* $h = h(|y|, t)$ *with* $\|h\|_{0,\nu,m,6+\sigma,\epsilon} < \infty$ *and*

$$\int_{\mathbb{R}^2} h(y,t)dy = 0, \quad \text{for all } t > t_0,$$

there exists $c_1 \in \mathbb{R}$ *and solution* $\phi(y,t) = \mathcal{T}_{\boldsymbol{p}}^{i,2}[h]$ *of problem* (6.3) *that defines a linear operator of* h *and satisfies the estimate*

$$\|\phi\|_{1,\nu-1,m+q-1,4,2+\sigma+\epsilon} \le \frac{C}{(\log t_0)^{1-q}} \|h\|_{0,\nu,m,6+\sigma,\epsilon}.$$

Moreover c_1 *is a linear operator of* h *and*

$$|c_1| \le C \frac{1}{t_0^{\nu-1}(\log t_0)^m} \|h\|_{0,\nu,m,6+\sigma,\epsilon}.$$

We also consider the problem

$$\begin{cases} \lambda^2 \partial_t \phi = L[\phi] + B[\phi] + h(y,t) + \displaystyle\sum_{j=1}^{2} \mu_j(t) W_{1,j} & \text{in } \mathbb{R}^2 \times (t_0, \infty), \\ \phi(\cdot, t_0) = 0 \quad \text{in } \mathbb{R}^2. \end{cases} \tag{6.5}$$

where the function $W_{1,j}$ have been defined in (5.45).

Proposition 6.2. *Assume* (4.1). *Let* $0 < \sigma < 1$, $\epsilon > 0$ *with* $\sigma+\epsilon < \frac{3}{2}$ *and* $1 < \nu < \min(1+\frac{\epsilon}{2}, 3-\frac{\sigma}{2}, \frac{5}{4})$. *Let* $0 < q < 1$. *Then there is* C *such that for* t_0 *large the following holds. Suppose that* h *satisfies* $\|h\|_{0,\nu,m,6+\sigma,\epsilon} < \infty$ *and*

$$\int_{\mathbb{R}^2} h(y,t)dy = 0, \quad \int_{\mathbb{R}^2} h(y,t)|y|^2 dy = 0, \quad \text{for all } t > t_0.$$

Then there exists a solution $\phi(y,t)$, $\mu_j(t)$ *of problem* (6.5) *that defines a linear operator of* h *and satisfies*

$$\|\phi\|_{1,\nu-\frac{1}{2},m+\frac{q-1}{2},4,2+\sigma+\epsilon} \le C\|h\|_{0,\nu,m,6+\sigma,\epsilon}.$$

The parameters μ_j *satisfy*

$$\mu_j(t) = -\int_{\mathbb{R}^2} h(y,t) y_j dy + \tilde{\mu}_j[h](t)$$

where $\tilde{\mu}_j$ *are linear functions of* h *with*

$$|\tilde{\mu}_j[h]| \le \frac{C}{t^{\nu+1}(\log t)^{\nu+m+2}} \|h\|_{\nu,m,5+\sigma,\epsilon}.$$

We denote this solution by $\phi = \mathcal{T}_{\boldsymbol{p}}^{i,1}[h]$.

The proof of the Propositions 6.1 and 6.2 is divided into different steps and presented in sections 8–12.

Next we consider the linear outer problem:

$$\begin{cases} \partial_t \phi^o = L^o[\phi^o] + g(x,t), & \text{in } \mathbb{R}^2 \times (t_0, \infty) \\ \phi^o(\cdot, t_0) = \phi_0^o, & \text{in } \mathbb{R}^2. \end{cases} \tag{6.6}$$

where

$$L^o[\varphi] := \Delta_x \varphi - \nabla_x \Big[\Gamma_0 \Big(\frac{x - \xi(t)}{\lambda(t)} \Big) \Big] \cdot \nabla_x \varphi.$$

For a given function $g(x,t)$ we consider the norm $\|g\|_{**,o}$ defined as the least $K \ge 0$ such that for all $(x,t) \in \mathbb{R}^2 \times (t_0, \infty)$

$$|g(x,t)| \le K \frac{1}{t^a (\log t)^\beta} \frac{1}{1+|\zeta|^b}, \quad \zeta = \frac{x-\xi(t)}{\sqrt{t}}. \tag{6.7}$$

Accordingly, we consider for a function $\phi^o(x,t)$ the norm $\|\phi\|_{*,o}$ defined as the least $K \ge 0$ such that

$$|\phi^o(x,t)| + (\lambda + |x-\xi|)|\nabla_x \phi^o(x,t)| \le K \frac{1}{t^{a-1}(\log t)^\beta} \frac{1}{1+|\zeta|^b}, \quad \zeta = \frac{x-\xi}{\sqrt{t}} \tag{6.8}$$

for all $(x,t) \in \mathbb{R}^2 \times (t_0, \infty)$.

We assume that the parameters a, b, β satisfy the constraints

$$1 < a < 4, \quad 2 < b < 6, \quad a < 1 + \frac{b}{2}, \qquad \beta \in \mathbb{R}. \tag{6.9}$$

Proposition 6.3. *Assume that the parameter functions $\boldsymbol{p} = (\lambda, \alpha, \xi)$ satisfy conditions (4.1) and the numbers a, b, β satisfy (6.9). Then there is a constant C so that for t_0 sufficiently large and for $\|g\|_{**,o} < \infty$, there exists a solution $\phi^o = \mathcal{T}^o_{\boldsymbol{p}}[g]$ of (6.6) with $\phi^o_0 = 0$, which defines a linear operator of g and satisfies*

$$\|\phi^o\|_{*,o} \le C\|g\|_{**,o}.$$

For the initial condition ϕ^o_0 in (6.6) we consider the norm $\|\varphi^o_0\|_{*,b}$ defined as

$$\|\phi^o_0\|_{*,b} = \inf K \quad \text{such that}$$
$$|\phi^o_0(x)| + (\lambda(t_0) + |x|)|\nabla_x \phi^o(x)| \le \frac{K}{(1 + \frac{|x|}{\sqrt{t_0}})^b}. \tag{6.10}$$

We have an estimate for the solution of (6.6) with $g = 0$ and $\|\phi^o_0\|_{*,b} < \infty$.

Proposition 6.4. *Assume that the parameter functions $\boldsymbol{p} = (\lambda, \alpha, \xi)$ satisfy conditions (4.1) and the numbers a, b, β satisfy (6.9). Then there is a constant C so that for t_0 sufficiently large and for $\|\phi^o_0\|_{*,b} < \infty$ there exists a solution ϕ^o of (6.6), which defines a linear operator of ϕ^o_0 and satisfies*

$$\|\phi^o\|_{*,o} \le C t_0^{a-1} (\log t_0)^\beta \|\phi^o_0\|_{*,b}.$$

The proofs of Propositions 6.3 and 6.4 are contained in Section 13.

In what follows we work with $\mathbf{p}_1$ of the form

$$\mathbf{p}_1 = (0, \alpha_1, \xi_1),$$

that is, we take $\lambda = \lambda_0$, $\alpha = \alpha_0 + \alpha_1$, $\xi = \xi_1$, where λ_0 and α_0 have been fixed in Section 5.1, and we write

$$\mathbf{p} = \mathbf{p}_0 + \mathbf{p}_1.$$

Next we define suitable operators that allow us to formulate the system of equations (5.48), (5.49), (5.50), and (5.52) as a fixed point problem. We let

$$\begin{aligned}
\mathcal{A}_{i1}[\phi_1, \phi_2, \varphi, \mathbf{p}_1] = \mathcal{T}^{i,1}_{\mathbf{p}} \Big[& F_3(\phi_1 + \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0) \\
& - m_0[F_3(\phi_1 + \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0)] W_0 - m_2[F_3(\phi_1 + \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0)] W_2 \\
& - m_{1,1}[F_3(\phi_1 + \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0)] W_{1,1} - m_{1,2}[F_3(\phi_1 + \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0)] W_{1,2} \Big]
\end{aligned}$$

$$\mathcal{A}_{i2}[\phi_1, \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0] = \mathcal{T}^{i,2}_{\mathbf{p}} \left[m_2[F_3(\phi_1 + \phi_2, \varphi, \mathbf{p}_1), \varphi^*_0] W_2 \right]$$

$$A_o[\phi_1, \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0] = \mathcal{T}^o_{\mathbf{p}}[G_2(\phi_1 + \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0)].$$

Then the equations (5.48), (5.49),(5.50) can be written as

$$\begin{aligned}
\phi_1 &= \mathcal{A}_{i1}[\phi_1, \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0] \\
\phi_2 &= \mathcal{A}_{i2}[\phi_1, \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0] \\
\varphi &= \mathcal{A}_o[\phi_1, \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0.]
\end{aligned}$$

Next we consider the equations (5.52), that is, $m_0[F_3(\phi_1 + \phi_2, \varphi, \mathbf{p}_1, \varphi^*_0)](t) \equiv 0$ and $\mu_j(t) \equiv 0$. By (5.51) and (5.41)

$$\begin{aligned}
m_0[F_3(\phi, \varphi, \mathbf{p}_1, \varphi^*_0)\tilde\chi] = {} & \lambda_0^4 m_0[S_0(u_1(\mathbf{p}_0 + \mathbf{p}_1)) - S_0(u_1(\mathbf{p}_0))] + m_0[E_2 \tilde\chi_2] \\
& + m_0[F(\phi^i_0 + \phi, \varphi^* + \varphi, \mathbf{p}_0 + \mathbf{p}_1)\tilde\chi] \\
& + \lambda_0^4 m_0[(S_0(u_1(\mathbf{p}_0 + \mathbf{p}_1)) - S_0(u_1(\mathbf{p}_0)))(\tilde\chi - 1)],
\end{aligned}$$

and using Lemma 5.3,

$$\begin{aligned}
m_0[F_3(\phi, \varphi, \mathbf{p}_1, \varphi^*_0)\tilde\chi] = {} & -\lambda_0^2 \partial_t \Big\{ \alpha_1 \Big[8\pi \Big(1 + 2\Upsilon \frac{\lambda_0^2}{t}\Big) + e_1\Big(\frac{\lambda_0^2}{t}\Big) \Big] \Big\} + m_0[E_2 \tilde\chi_2] \\
& + m_0[F(\phi^i_0 + \phi, \varphi^* + \varphi, \mathbf{p}_0 + \mathbf{p}_1)\tilde\chi] \\
& + \lambda_0^4 m_0[(S_0(u_1(\mathbf{p}_0 + \mathbf{p}_1)) - S_0(u_1(\mathbf{p}_0)))(\tilde\chi - 1)].
\end{aligned}$$

This motivates the definition

$$\begin{aligned} &\mathcal{A}_{\alpha_1}[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*] \\ &= -\frac{1}{8\pi(1+2\Upsilon\frac{\lambda_0^2}{t})+e_1(\frac{\lambda_0^2}{t})}\int_t^\infty \frac{1}{\lambda_0^2}\Big\{m_0[E_2\tilde\chi_2](s)+m_0[F(\phi_0^i+\phi,\varphi^*+\varphi,\mathbf{p}_0+\mathbf{p}_1)\tilde\chi](s) \\ &\qquad + \lambda_0^4 m_0[(S_0(u_1(\mathbf{p}_0+\mathbf{p}_1))-S_0(u_1(\mathbf{p}_0)))(\tilde\chi-1)](s)\Big\}ds \quad (6.11)\end{aligned}$$

Similarly, by (5.51) and (5.41), asking that $\mu_j\equiv 0$ in (5.48) is equivalent to

$$\begin{aligned} 0 &= \lambda_0\alpha\dot\xi_{1,j}\int_{\mathbb{R}^2}\partial_{y_j}U(y)y_j\tilde\chi dy+\frac{\alpha\lambda_0^2}{\sqrt t}\dot\xi_{1,j}\int_{\mathbb{R}^2}U(y)\partial_{z_j}\chi_0(\frac{\lambda y}{\sqrt t})y_jdy \\ &\quad + m_{1,j}[E_2\tilde\chi_2]+m_{1,j}[F(\phi_0^i+\phi,\varphi^*+\varphi,\mathbf{p}_0+\mathbf{p}_1)\tilde\chi]+m_{1,j}[B[\phi_1]].\end{aligned}$$

This motivates the definition

$$\begin{aligned} &\mathcal{A}_{\xi_1}[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*] \\ &= \int_t^\infty \frac{1}{\lambda_0\alpha\int_{\mathbb{R}^2}\partial_{y_j}U(y)y_j\tilde\chi dy}\Big\{\frac{\alpha\lambda_0^2}{\sqrt t}\dot\xi_{1,j}\int_{\mathbb{R}^2}U(y)\partial_{z_j}\chi_0(\frac{\lambda y}{\sqrt t})y_jdy \\ &\quad + m_{1,j}[E_2\tilde\chi_2](s)+m_{1,j}[F(\phi_0^i+\phi,\varphi^*+\varphi,\mathbf{p}_0+\mathbf{p}_1)\tilde\chi](s)+m_{1,j}[B[\phi_1]](s)\Big\}ds \qquad (6.12)\end{aligned}$$

Then we define $\mathcal{A}_p$ by

$$\mathcal{A}_p[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*]=(0,\mathcal{A}_{\alpha_1}[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*],\mathcal{A}_{\xi_1}[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*]). \qquad (6.13)$$

Then

$$\mathbf{p}_1=\mathcal{A}_p[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*]$$

is equivalent to the equations (5.52).

We write

$$\vec\phi=(\phi_1,\phi_2,\varphi,\mathbf{p}_1),$$

and

$$\mathcal{A}[\vec\phi]=(\mathcal{A}_{i1}[\vec\phi,\varphi_0^*],\mathcal{A}_{i2}[\vec\phi,\varphi_0^*],\mathcal{A}_o[\vec\phi,\varphi_0^*],\mathcal{A}_p[\vec\phi,\varphi_0^*]),$$

and the objective is to find $\vec\phi$ such that

$$\vec\phi=\mathcal{A}[\vec\phi].$$

The operator $\mathcal{A}$ depends on the initial condition φ_0^* appearing in the parabolic problem (5.37), and we will stress its dependence later on when proving the stability assertion in Theorem 1.1.

We define the spaces on which we will consider the operator $\mathcal{A}$ to set up the fixed point problem. For certain choices of constants ν, q, σ, ϵ, a, b, β, γ, Θ that we will make precise later, we let

$$\begin{aligned} X_i = \Big\{ &\phi\in L^\infty(\mathbb{R}^2\times(t_0,\infty)) \mid \nabla_y\phi\in L^\infty(\mathbb{R}^2\times(t_0,\infty)),\ \|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}<\infty, \\ &\int_{\mathbb{R}^2}\phi(y,t)dy=0,\ \int_{\mathbb{R}^2}\phi(y,t)ydy=0,\quad t>t_0\Big\},\end{aligned}$$

$$X_o=\{\varphi\in L^\infty(\mathbb{R}^2\times(t_0,\infty)) \mid \nabla_y\phi\in L^\infty(\mathbb{R}^2\times(t_0,\infty)),\ \|\varphi\|_{*,o}<\infty\},$$

$$X_p=\{(0,\alpha_1,\xi_1)\in C^1([t_0,\infty)) \mid \|\alpha_1\|_{C^1,\nu+\frac12,\Theta}<\infty,\ \|\xi_1\|_{C^1,\gamma,0}<\infty\}$$

where the norms $\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}$ and $\|\varphi\|_{*,o}$ are defined in (6.2), (6.7) and $\|\xi_1\|_{C^1,\mu,m}$ is defined by

$$\|g\|_{C^0,\mu,m}=\sup_{t\ge t_0}t^\mu(\log t)^m|g(t)|.$$

$$\|g\|_{C^1,\mu,m}=\|g\|_{C^0,\mu,m}+\|\dot g\|_{C^0,\mu+1,m}.$$

for a function $g\in C^1([t_0,\infty))$.

We choose in the definition of the outer norm (6.8)

$$a=\nu+\frac52,\quad 2\nu+3<b<6,\quad \beta<\frac{1+q}{2}. \qquad (6.14)$$

With these choices we see that (6.9) are satisfied. Also ν will be in the range $1<\nu<\frac32$ so the interval for b is not empty in (6.14).

We use the following notation: for $\mathbf{p}_1 = (0, \alpha_1, \xi_1)$

$$\|\mathbf{p}_1\|_{X_p} = \|\alpha_1\|_{C^1,\nu+\frac{1}{2},\Theta} + \|\xi_1\|_{C^1,1+\gamma,0},$$

and for $\vec{\phi} = (\phi_1, \phi_2, \varphi, \mathbf{p}_1)$

$$\|\vec{\phi}\|_X = \|\phi_1\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\epsilon} + \|\phi_2\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\epsilon} + \|\varphi\|_{*,o} + \|\mathbf{p}_1\|_{X_p}. \tag{6.15}$$

With the above notation, given φ_0^* with $\|\varphi_0^*\|_{*,b}$ sufficiently small, we consider the fixed point problem

$$\vec{\phi} = \mathcal{A}[\vec{\phi}], \tag{6.16}$$

with $\vec{\phi}$ in a suitable close ball of X. A solution of this fixed point problem yields a solution of the system of equations (5.48), (5.49), (5.50), (5.52), which in turn gives a solution to (3.1).

We claim that for some constant C independent of $t_0 \gg 1$, if $t_0^{a-1}(\log t_0)^\beta \|\varphi_0^*\|_{*,b} \le 1$, and $\|\vec{\phi}\|_X \le 1$, then

$$\|\mathcal{A}_{i1}[\phi_1, \phi_2, \varphi, \mathbf{p}_1, \varphi_0^*]\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\epsilon} \le \frac{C}{t_0^\vartheta} + C(\log t_0)^{\frac{\sigma}{2}} t_0^{\nu+1+\frac{\sigma}{2}} \|\varphi_0^*\|_{*,b}, \tag{6.17}$$

for some $\vartheta > 0$ small, a constant C independent of t_0, and t_0 sufficiently large.

Indeed, by Proposition 6.2 we have

$$\|\mathcal{A}_{i1}[\phi_1, \phi_2, \varphi, \mathbf{p}_1, \varphi_0^*]\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\epsilon} \le C\|F_3(\phi_1+\phi_2, \varphi, \mathbf{p}_1, \varphi_0^*)\|_{0,\nu,6+\sigma,\epsilon}.$$

We recall the expansion of F_3 in (5.51). To estimate $E_2\tilde{\chi}_2$ we use (5.43) to get

$$\|E_2\tilde{\chi}\|_{0,\nu,0,6+\sigma,\epsilon} \le \frac{C}{t_0^{1+2\delta-\frac{\sigma}{2}-\nu}(\log t_0)^2}$$

where δ, σ are positive small constants and are assumed to satisfy $2\delta - \frac{\sigma}{2} > 0$. Then we take ν in the range

$$1 < \nu < 1 + 2\delta - \frac{\sigma}{2}, \tag{6.18}$$

with ν close to 1.

Let us consider the term $\lambda^4[S_0(\mathbf{p}_0+\mathbf{p}_1) - S_0(\mathbf{p}_0)]$ in $F_3(\phi_1+\phi_2, \varphi, \mathbf{p}_1, \varphi_0^*)$ (c.f. (5.51)). The formula $\lambda^4[S_0(\mathbf{p}_0+\mathbf{p}_1) - S_0(\mathbf{p}_0)]$ (c.f. (5.31)) contains for example the term, evaluated at $y = \frac{x-\xi_1}{\lambda_0}$,

$$\begin{aligned}
&-\lambda_0^2\dot{\alpha}U(y)\chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big) + \lambda_0^2\dot{\alpha}_0 U\Big(\frac{\xi_1+\lambda_0 y}{\lambda_0}\Big)\chi_0\Big(\frac{\xi_1+\lambda_0 y}{\sqrt{t}}\Big)\\
&= -\lambda_0^2\dot{\alpha}_1 U(y)\chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big) - \lambda_0^2\dot{\alpha}_0\Big[U(y) - U\Big(\frac{\xi_1+\lambda_0 y}{\lambda_0}\Big)\Big]\chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big)\\
&\quad - \lambda_0^2\dot{\alpha}_0 U\Big(\frac{\xi_1+\lambda_0 y}{\lambda_0}\Big)\Big[\chi_0\Big(\frac{\xi_1+\lambda_0 y}{\sqrt{t}}\Big) - \chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big)\Big]
\end{aligned} \tag{6.19}$$

But

$$\Big|-\lambda_0^2\dot{\alpha}_1 U(y)\chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big)\Big| \le C\frac{1}{t^{\nu+\frac{1-\sigma}{2}}(\log t)^{\Theta-\frac{\sigma}{2}}}\frac{1}{(1+|y|)^{6+\sigma}}\chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big)\|\alpha_1\|_{C^1,\nu+\frac{1}{2},\Theta},$$

so

$$\Big\|-\lambda_0^2\dot{\alpha}_1 U(y)\chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big)\Big\|_{0,\nu,6+\sigma,\epsilon} \le \frac{C}{t_0^\vartheta}\|\alpha_1\|_{C^1,\nu+\frac{1}{2},\Theta},$$

for some $\vartheta > 0$.

Similarly,

$$\begin{aligned}
\Big|-\lambda_0^2\dot{\alpha}_0\Big[U(y) - U\Big(\frac{\xi_1+\lambda_0 y}{\lambda_0}\Big)\Big]\chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big)\Big| &\le C\frac{1}{\log t}\frac{1}{t^2\log t}\frac{1}{(1+|y|)^5}\frac{|\xi_1|}{\lambda_0}\chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big)\\
&\le C\frac{1}{t^{2+\gamma}(\log t)^{\frac{3}{2}}}\frac{(t\log t)^{\frac{1+\sigma}{2}}}{(1+|y|)^{6+\sigma}}\chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big)\|\xi_1\|_{C^1,\gamma,0}\\
&\le C\frac{1}{t^{\frac{3-\sigma}{2}+\gamma}(\log t)^{1-\frac{\sigma}{2}}}\frac{1}{(1+|y|)^{6+\sigma}}\chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big)\|\xi_1\|_{C^1,\gamma,0}
\end{aligned}$$

so

$$\Big\| -\lambda_0^2\dot\alpha_0\Big[U(y)-U\Big(\frac{\xi_1+\lambda_0 y}{\lambda_0}\Big)\Big]\chi_0\Big(\frac{\lambda_0 y}{\sqrt{t}}\Big)\Big\|_{0,\nu,6+\sigma,\epsilon} \le \frac{C}{t_0^{\vartheta}}\|\alpha_1\|_{C^1,\nu+\frac12,\Theta},$$

for some $\vartheta>0$. The last term in the expression (6.19) is similar.

The terms in $\lambda^4[S_0(\mathbf{p}_0+\mathbf{p}_1)-S_0(\mathbf{p}_0)]$ that contain the function φ_{λ_0} are

$$\begin{aligned}&{\lambda_0}^4\Big[-\frac{4}{r}\partial_r\varphi_{\lambda_0}-\nabla\cdot(\varphi_{\lambda_0}\nabla v_0)-\nabla\cdot(u_0\nabla\psi_{\lambda_0})-\nabla\cdot(\varphi_{\lambda_0}\nabla\psi_{\lambda_0})\Big]\\ &=4\frac{{\lambda_0}^2}{\rho(\rho^2+1)}\partial_\rho\varphi_{\lambda_0}-(\alpha-1){\lambda_0}^2\nabla_y\varphi_{\lambda_0}\cdot\nabla_y\Gamma_0+{\lambda_0}^2\nabla_y\varphi_{\lambda_0}\cdot\nabla_y\mathcal{R}+2{\lambda_0}^2U\chi\varphi_{\lambda_0}\\ &\quad-\alpha\nabla_y(U\chi)\cdot\nabla_y\psi_{\lambda_0}-{\lambda_0}^2\nabla_y(\varphi_{\lambda_0}\nabla_y\psi_{\lambda_0}).\end{aligned}$$

In ${\lambda_0}^4[S_0(\mathbf{p}_0+\mathbf{p}_1)-S_0(\mathbf{p}_0)]$ these terms appear evaluated at y and then at $\frac{\xi_1}{\lambda_0}+y$. Using estimates for the the second derivative of φ_{λ_0} similar to Lemma 4.1 and assuming

$$\sigma<1,\quad \nu<1+\gamma, \tag{6.20}$$

we get

$$\|\lambda^4[S_0(\mathbf{p}_0+\mathbf{p}_1)-S_0(\mathbf{p}_0)]\|_{0,\nu,6+\sigma,\epsilon}\le C\frac{1}{t_0^{\vartheta}}\|\vec\phi\|_X.$$

The main term in $F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)$ that depends on the outer solution is $\lambda^2U\varphi^o$ with $\varphi^o=\varphi^*+\varphi$ defined in (5.38). Then we have

$$\begin{aligned}|\lambda^2U\varphi(y,t)\tilde\chi| &\le \frac{\lambda^2}{t^{a-1}(\log t)^\beta}\frac{1}{(1+|y|)^4}\tilde\chi\|\varphi\|_{*,o}\\ &\le C\frac{1}{t^{\nu+\frac32}(\log t)^{\beta+1}}\frac{1}{(1+|y|)^4}\tilde\chi\|\varphi\|_{*,o}\\ &\le C\frac{(t\log t)^{1+\frac{\sigma}{2}}}{t^{\nu+\frac32}(\log t)^{\beta+1}}\frac{1}{(1+|y|)^{6+\sigma}}\tilde\chi\|\varphi\|_{*,o}\\ &\le C\frac{1}{t^{\nu+\frac{1-\sigma}{2}}(\log t)^{\beta-\frac{\sigma}{2}}}\frac{1}{(1+|y|)^{6+\sigma}}\tilde\chi\|\varphi\|_{*,o}.\end{aligned}$$

Therefore

$$\|\lambda^2U\varphi\tilde\chi\|_{0,\nu,6+\sigma,\epsilon}\le C\frac{1}{t_0^{\frac{1-\sigma}{2}}(\log t_0)^{\beta-\frac{\sigma}{2}}}\|\varphi\|_{*,o}.$$

Regarding the function φ^* (c.f. (5.37)) we note that it has the estimate

$$|\varphi^*(x,t)|\le t_0^{a-1}(\log t_0)^\beta\|\varphi_0^*\|_{*,b}\frac{1}{t^{a-1}(\log t)^\beta}\frac{1}{1+|\zeta|^b},\quad \zeta=\frac{x-\xi}{\sqrt{t}} \tag{6.21}$$

by Proposition 6.4, provided (6.9) holds, and therefore

$$\|\lambda^2U\varphi^*\tilde\chi\|_{0,\nu,6+\sigma,\epsilon}\le Ct_0^{\nu+1+\frac{\sigma}{2}}(\log t_0)^{\frac{\sigma}{2}}\|\varphi_0^*\|_{*,b}.$$

Let us analyze some of the terms in $F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)$ that depend on the inner solutions ϕ_1 and ϕ_2. For instance

$$(\alpha-1)\nabla_y\cdot(\phi_j\nabla_y\Gamma_0)=(\alpha-1)\nabla_y\phi_j\cdot\nabla_y\Gamma_0-(\alpha-1)\phi_jU.$$

We have the estimate

$$\begin{aligned}|(\alpha-1)\nabla_y\phi_j\cdot\nabla_y\Gamma_0\tilde\chi| &\le \frac{C}{t\log t}\frac{1}{t^{\nu-\frac12}(\log t)^{\frac{q-1}{2}}}\frac{1}{(1+|y|)^6}\|\phi_j\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}\\ &\le C\frac{1}{t^{\nu+\frac12-\frac{\sigma}{2}}(\log t)^{1+\frac{q-1}{2}-\frac{\sigma}{2}}}\frac{1}{(1+|y|)^{6+\sigma}}\|\phi_j\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon},\end{aligned}$$

and we get

$$\|(\alpha-1)\nabla_y\phi_j\cdot\nabla_y\Gamma_0\tilde\chi\|_{0,\nu,6+\sigma,\epsilon}\le\frac{C}{t_0^{\vartheta}}\|\phi_j\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon},$$

for some $\vartheta>0$.

We also have, writing $\phi = \phi_1 + \phi_2$,

$$\|\lambda \dot\xi_1 \nabla \phi \tilde\chi\|_{0,\nu,6+\sigma,\epsilon} \le \frac{C}{t_0^{\vartheta}} \|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}$$

for some $\vartheta > 0$, if

$$\gamma > \frac{\sigma}{2}. \tag{6.22}$$

Let us estimate the term $\nabla_y \cdot (U \nabla_y(\hat\psi - \psi))\tilde\chi$ appearing in (5.6), where $\hat\psi = (-\Delta)^{-1}(\lambda^{-2}\phi^i\chi)$, $\psi = (-\Delta)^{-1}(\lambda^{-2}\phi^i)$. We recall that $\phi^i = \phi_0^i + \phi$, c.f. (5.38), and therefore we can decompose $\hat\psi = \hat\psi_0^i + \hat\psi_1$ where $\hat\psi_0^i = (-\Delta)^{-1}(\lambda^{-2}\phi_0^i\chi)$ and $\hat\psi_1 = (-\Delta)^{-1}(\lambda^{-2}\phi\chi)$. Similarly, we can decompose $\psi = \psi_0^i + \psi_1$ where $\psi_0^i = (-\Delta)^{-1}(\lambda^{-2}\phi_0^i)$ and $\psi_1 = (-\Delta)^{-2}(\lambda^{-1}\phi)$. By linearity we need to estimate separately $\nabla_y \cdot (U\nabla_y(\hat\psi_0^i - \psi_0^i))$ and $\nabla_y \cdot (U\nabla_y(\hat\psi_1 - \psi_1))$. Let us consider the latter one. Note that

$$\hat\psi_1 - \psi_1 = (-\Delta)^{-1}[\lambda^{-2}\phi(1-\chi)].$$

From the definition of the norm $\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}$

$$|\phi(y,t)| \le \|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon} \frac{1}{t^{\nu-\frac12}(\log t)^{\frac{q-1}{2}}} \frac{1}{(1+|y|)^4} \tag{6.23}$$

and so

$$|\nabla_y(\hat\psi_1 - \psi_1)(y,t)| \le C\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon} \frac{1}{t^{\nu-\frac12}(\log t)^{\frac{q-1}{2}}} \frac{1}{(t\log t)^{\frac32}}, \quad \text{for } |y| \le 2\frac{\sqrt t}{\lambda}.$$

Then

$$|\nabla_y U \cdot \nabla_y(\hat\psi_1 - \psi_1))(y,t)| \le C\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon} \frac{1}{t^{\nu+\frac{1-\sigma}{2}}(\log t)^{\frac{q+1-\sigma}{2}}} \frac{1}{(1+|y|)^{6+\sigma}}, \quad \text{for } |y| \le 2\frac{\sqrt t}{\lambda}.$$

This and a similar estimate for $U\phi(1-\chi)$ give

$$\|\nabla_y \cdot (U\nabla_y(\hat\psi_1 - \psi_1))\tilde\chi\|_{0,\nu,6+\sigma,\epsilon} \le \frac{C}{t_0^{\vartheta}}\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}$$

for some $\vartheta > 0$. A similar estimate is obtained for $\|\nabla_y \cdot (U\nabla_y(\hat\psi_0^i - \psi_0^i))\tilde\chi\|_{0,\nu,6+\sigma,\epsilon}$ using (5.36).

Let us estimate next the term $\lambda^2\nabla_y \cdot (\varphi_\lambda \nabla_y \psi)\tilde\chi$, where we recall, $\psi = (-\Delta)^{-1}(\lambda^{-2}\phi)$. To do this we use that $\phi = \phi_1 + \phi_2$ has zero mass and center of mass, that is,

$$\int_{\mathbb{R}^2} \phi(y,t)\,dy = \int_{\mathbb{R}^2} \phi(y,t) y_j\,dy = 0, \quad t > t_0.$$

This and the estimate (6.23) imply

$$|\psi(y,t)| + (1+|y|)|\nabla_y\psi(y,t)| \le C\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon} \frac{1}{t^{\nu-\frac12}(\log t)^{\frac{q-1}{2}}} \frac{\log(2+|y|)}{(1+|y|)^2},$$

by an argument similar to Remark 9.1. On the other hand, from (4.3)

$$|\nabla_y\varphi_\lambda(y,t)| \le \frac{C}{t\log t}\frac{1}{(1+|y|)^3}, \quad |y| \le 2\frac{\sqrt t}{\lambda}.$$

Therefore

$$\begin{aligned} |\lambda^2(\nabla_y\varphi_\lambda \cdot \nabla_y\psi)(y,t)| &\le C\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon} \frac{\lambda^2}{t^{\nu+\frac12}(\log t)^{\frac{q+1}{2}}} \frac{\log(2+|y|)}{(1+|y|)^6} \\ &\le C\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon} \frac{1}{t^{\nu+\frac{1-\sigma}{2}}(\log t)^{\frac{q+1-\sigma}{2}}} \frac{1}{(1+|y|)^{6+\sigma}}, \quad |y| \le 2\frac{\sqrt t}{\lambda}. \end{aligned}$$

From this coupled with a similar estimate for $\lambda^2\varphi_\lambda\phi$ we get

$$\|\lambda^2\nabla_y \cdot (\varphi_\lambda\nabla_y\psi)\tilde\chi\|_{0,\nu,6+\sigma,\epsilon} \le \frac{C}{t_0^{\vartheta}}\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}$$

for some $\vartheta > 0$.

The remaining terms in $F_3(\phi_1+\phi_2, \varphi, \mathbf{p}_1, \varphi_0^*)$ are estimated in a similar way and we get the validity (6.17).

Proceeding in the same way we get a Lipschitz bound. Assuming $t_0^{a-1}(\log t_0)^\beta\|\varphi_0^*\|_{*,b} \le 1$, for $\|\vec\phi_1\|_X \le 1$ and $\|\vec\phi_2\|_X \le 1$ we have

$$\|\mathcal{A}_{i1}[\vec\phi_1,\varphi_0^*] - \mathcal{A}_{i1}[\vec\phi_2,\varphi_0^*]\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon} \le \frac{C}{t_0^\vartheta}\|\vec\phi_1-\vec\phi_2\|_X,$$

for some $\vartheta>0$ small, a constant C independent of t_0, and t_0 sufficiently large. Indeed, the Lipschitz estimate with respect to ϕ_1, ϕ_2, and φ is direct from the explicit dependence of $F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)$ on these variables, which is either linear or quadratic. The Lipschitz dependence on ξ_1 (where $\mathbf{p}_1=(\alpha_1,\xi_1)$) is also direct from the explicit form of $F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)$. The Lipschitz condition with respect to α_1 appears as an explicit dependence on this variable in $F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)$.

Let us estimate the operator $\mathcal{A}_{i2}$. We claim that if $t_0^{a-1}(\log t_0)^\beta\|\varphi_0^*\|_{*,b}\le 1$ and $\|\vec\phi\|_X\le 1$, then

$$\|\mathcal{A}_{i2}[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*]\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon} \le C(\log t_0)^{-\frac{1-q}{2}-\Theta} + Ct_0^{a-1}(\log t_0)^{\frac{1-q}{2}}\|\varphi_0^*\|_{*,b}. \tag{6.24}$$

Indeed, we apply Proposition 6.1 to get

$$\|\mathcal{A}_{i2}[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*]\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon} \le \frac{C}{(\log t_0)^{1-q}}\|m_2[F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)]W_2\|_{0,\nu+\frac12,\frac{1-q}{2},6+\sigma,\epsilon}$$

and since W_2 has compact support,

$$\begin{aligned}&\|\mathcal{A}_{i2}[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*]\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}\\ &\le \frac{C}{(\log t_0)^{1-q}}\sup_{t>t_0} t^{\nu+\frac12}(\log t)^{\frac{1-q}{2}}|m_2[F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)](t)|.\end{aligned}$$

Using the definition of F_3 (5.51)

$$m_2[F_3(\phi,\varphi,\mathbf{p}_1,\varphi_0^*)] = m_2[E_2\tilde\chi_2] + m_2[F_2(\phi,\varphi,\mathbf{p}_1,\varphi_0^*)\tilde\chi]$$

We have by (5.43) (assuming $\sigma<\frac12$),

$$|m_2[E_2\tilde\chi_2](t)| \le \frac{C}{t^{\frac{3+\sigma}{2}}}.$$

Therefore, asking that

$$\nu+\frac12 < \frac{3+\sigma}{2} \Leftrightarrow \nu < 1+\frac{\sigma}{2} \tag{6.25}$$

we get

$$\sup_{t>t_0} t^{\nu+\frac12}(\log t)^{\frac{1-q}{2}}|m_2[E_2\tilde\chi_2](t)| \le \frac{C}{t_0^\vartheta},$$

for some $\vartheta>0$.

By (5.41)

$$\begin{aligned} m_2[F_2(\phi,\varphi,\mathbf{p}_1,\varphi_0^*)\tilde\chi] &= \lambda^4 m_2[S_0(u_1(\mathbf{p}_0+\mathbf{p}_1)) - S_0(u_1(\mathbf{p}_0))] + m_2[F(\phi_0^i+\phi,\varphi^*+\varphi,\mathbf{p}_0+\mathbf{p}_1)\tilde\chi]\\ &\quad + \lambda^4 m_2[(S_0(u_1(\mathbf{p}_0+\mathbf{p}_1)) - S_0(u_1(\mathbf{p}_0)))(\tilde\chi-1)].\end{aligned}$$

Of these terms, the largest is the first one. By Lemma 5.4, and since $\lambda=\lambda_0$, we get

$$\begin{aligned}&\lambda^4 m_2[S_0(u_1(\mathbf{p}_0+\mathbf{p}_1)) - S_0(u_1(\mathbf{p}_0))]\\ &= -32\pi\alpha_1 - \frac{\dot\alpha_1}{\lambda_0^2}\int_{\mathbb{R}^2} U(\frac{x-\xi}{\lambda_0})\chi_0(\frac{x-\xi}{\lambda_0})|x-\xi|^2\,dx\\ &\quad + \alpha_1\int_{\mathbb{R}^2} E(x-\xi,t,\lambda_0)|x-\xi|^2\,dx - |\xi|^2\int_{\mathbb{R}^2} S(u_1(\mathbf{p}_0))dx. \end{aligned}\tag{6.26}$$

But

$$\sup_{t>t_0} t^{\nu+\frac12}(\log t)^{\frac{1-q}{2}}|\alpha_1(t)| \le C(\log t_0)^{\frac{1-q}{2}-\Theta}\|\alpha_1\|_{C^1,\nu+\frac12,\Theta}, \tag{6.27}$$

under the assumption

$$\Theta > \frac{1-q}{2}. \tag{6.28}$$

The second term in (6.26) is much smaller. For the last term in (6.26) we use Lemma 5.3 and (5.29), (5.30) to get

$$\left|\int_{\mathbb{R}^2} S(u_1(\mathbf{p}_0))dx\right| \le \frac{C}{t^2} \tag{6.29}$$

and therefore

$$|\xi(t)|^2 \left| \int_{\mathbb{R}^2} S(u_1(\mathbf{p}_0))dx \right| \le \frac{C}{t^{2+2\gamma}} \|\xi_1\|^2_{C^1,\gamma,0}.$$

Combining (6.26), (6.27) and (6.29) we get

$$\frac{C}{(\log t_0)^{1-q}} \sup_{t>t_0} t^{\nu+\frac{1}{2}} (\log t)^{\frac{1-q}{2}} \lambda^4 |m_2[S_0(u_1(\mathbf{p}_0+\mathbf{p}_1)) - S_0(u_1(\mathbf{p}_0))](t)| \le C(\log t_0)^{-\frac{1-q}{2}-\Theta} \|\mathbf{p}_1\|_{X_p}.$$

Let's estimate the remaining terms in $m_2[F_3(\phi,\varphi,\mathbf{p}_1,\varphi_0^*)]$. Consider

$$A(t) := \int_{\mathbb{R}^2} \nabla_y \cdot (\lambda^2 \varphi_\lambda \nabla_y (-\Delta_y)^{-1}\phi)\tilde{\chi}|y|^2 dy + \int_{\mathbb{R}^2} \nabla_y \cdot (\phi \nabla_y \psi_\lambda)\tilde{\chi}|y|^2 dy$$

which appears in the definition of F, where $\phi = \phi_1 + \phi_2$. Let us recall that $\psi_\lambda = (-\Delta_x)^{-1}\varphi_\lambda$ and let's write

$$\psi = (-\Delta_y)^{-1}\phi$$

Integrating by parts

$$A(t) = \int_{\mathbb{R}^2} (\Delta_y \psi_\lambda \nabla_y \psi + \Delta_y \psi \nabla_y \psi_\lambda) \cdot y(2\tilde{\chi} + y \cdot \nabla_y \tilde{\chi})dy.$$

Using the following Pohozaev type identity

$$\Delta_y \psi_\lambda (\nabla_y \psi \cdot y) + \Delta_y \psi (\nabla_y \psi_\lambda \cdot y) = \nabla_y \cdot [\nabla_y \psi_\lambda (\nabla_y \psi \cdot y) + \nabla_y \psi (\nabla_y \psi_\lambda \cdot y) - y \nabla_y \psi_\lambda \cdot \nabla_y \psi]$$

and integrating by parts we get

$$A(t) = -\int_{\mathbb{R}^2} [\nabla_y \psi_\lambda (\nabla_y \psi \cdot y) + \nabla_y \psi (\nabla_y \psi_\lambda \cdot y) - y \nabla_y \psi_\lambda \cdot \nabla_y \psi] \cdot [2\nabla_y \tilde{\chi} + y \Delta_y \tilde{\chi}]dy.$$

Therefore

$$|A(t)| \le C \int_{2\sqrt{t}/\lambda \le |y| \le 4\sqrt{t}/\lambda} |\nabla_y \psi|\, \nabla \psi| dy.$$

Using that $\psi = (-\Delta)^{-1}\phi$, and

$$\int_{\mathbb{R}^2} \phi(y,t)dy = 0, \quad \int_{\mathbb{R}^2} \phi(y,t)y dy = 0,$$

we have (see Remark 9.1) for any $\varrho > 0$ small,

$$|\nabla \psi(y,t)| \le \frac{C}{1+|y|^{2-\varrho}} \frac{1}{t^{\nu-\frac{1}{2}}(\log t)^{\frac{q-1}{2}}} \|\phi\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\epsilon}.$$

Using that φ_λ, and ψ_λ are radial and

$$\left| \int_{\mathbb{R}^2} \varphi_\lambda dx \right| \le \frac{C}{t \log t},$$

by Lemma 4.1 we have

$$|\psi_\lambda(y,t)| \le \frac{C}{t\sqrt{\log t}} \frac{1}{1+|y|}.$$

Then

$$|A(t)| \le C \frac{1}{t^{\nu+1-\frac{\varrho}{2}}(\log t)^{\frac{q}{2}-1-\frac{\varrho}{2}}} \|\phi\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\epsilon}.$$

Let us consider the contribution of the term $\lambda^2 U \varphi^*$. Thanks to (6.21)

$$\|m_2[\lambda^2 U \varphi^* \tilde{\chi}] W_2\|_{0,\nu+\frac{1}{2},\frac{1-q}{2},6+\sigma,\epsilon} \le C t_0^{a-1} (\log t_0)^{\frac{1-q}{2}} \|\varphi_0^*\|_{*,b},$$

under the condition

$$\beta > \frac{1-q}{2}. \tag{6.30}$$

The other terms in m_2 are estimated in a similar way and we get (6.24).

Similarly we get that if $t_0^{a-1}(\log t_0)^\beta \|\varphi_0^*\|_{*,b} \le 1$, then for $\|\vec{\phi}_1\|_X \le 1$ and $\|\vec{\phi}_2\|_X \le 1$ we have

$$\|\mathcal{A}_{i2}[\vec{\phi}_1] - \mathcal{A}_{i2}[\vec{\phi}_2]\|_{0,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\epsilon} \le C(\log t_0)^{-\frac{1-q}{2}-\Theta} \|\vec{\phi}_1 - \vec{\phi}_2\|_X,$$

for a constant C independent of t_0, where t_0 sufficiently large.

Let us estimate the operator $\mathcal{A}_o[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*]$. We claim that if $t_0^{a-1}(\log t_0)^\beta\|\varphi_0^*\|_{*,b}\le 1$, then for $\|\vec\phi\|_X\le 1$,

$$\|\mathcal{A}_o[\vec\phi,\varphi_0^*]\|_{*,o}\le \frac{C}{(\log t_0)^{\frac{q+1}{2}-\beta}}+Ct_0^{a-2}(\log t_0)^{\beta-1}\|\varphi_0^*\|_{*,b}, \tag{6.31}$$

and for $\|\vec\phi_1\|_X\le 1$, $\|\vec\phi_2\|_X\le 1$ and $t_0^{a-1}(\log t_0)^\beta\|\varphi_0^*\|_{*,b}\le 1$,

$$\|\mathcal{A}_o[\vec\phi_1,\varphi_0^*]-\mathcal{A}_o[\vec\phi_2,\varphi_0^*]\|_{*,o}\le \frac{C}{(\log t_0)^{\frac{q+1}{2}-\beta}}\|\vec\phi_1-\vec\phi_2\|_X.$$

Note that $\frac{q+1}{2}-\beta>0$ by (6.14).

Indeed, by Proposition 6.3

$$\|\mathcal{A}_o[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*]\|_{*,o}\le C\|G_2(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)\|_{**,o},$$

where we recall G_2 defined in (5.42).

We start with the term $\lambda^{-4}E_2(1-\tilde\chi_2)\chi$. Using the estimate (5.43) we get

$$\|\lambda^{-4}E_2(1-\tilde\chi_2)\chi\|_{**,o}\le\frac{C}{t_0^\vartheta}$$

for some $\vartheta>0$ provided

$$a<4(1-\delta).$$

We also directly get from (4.6)

$$\|S(u_1)(1-\chi)\|_{**,o}\le\frac{C}{t_0^\vartheta}$$

for some $\vartheta>0$ if $a<4$.

Regarding the terms in G (c.f. (5.10)) that the depend linearly on $\phi^i=\phi_0^i+\phi$, we have for $\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}<\infty$

$$\begin{aligned}\Big|\frac{1}{\lambda^2}\phi\Delta\chi\Big|(x,t)&\le\frac{C}{\lambda^2}\frac{1}{t^{\nu-\frac12}(\log t)^{\frac{q-1}{2}}}\frac{1}{(|x-\xi|/\lambda|)^4}\frac1t\Big|\Delta_z\chi_0(\frac{x-\xi}{\sqrt t})\Big|\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}\\&\le C\frac{1}{t^{\nu+\frac52}(\log t)^{\frac{q+1}{2}}}\frac{1}{(1+|x-\xi|/\sqrt t)^b}\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}\end{aligned}\tag{6.32}$$

which implies

$$\Big\|\frac{1}{\lambda^2}\phi\Delta\chi\Big\|_{**,o}\le\frac{C}{(\log t_0)^{\frac{q+1}{2}-\beta}}\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon},$$

since $\beta<\frac{q+1}{2}$, which is one of the conditions in (6.14).

We also have, using (5.36),

$$\Big\|\frac{1}{\lambda^2}\phi_0^i\Delta\chi\Big\|_{**,o}\le\frac{C}{t_0^\vartheta}$$

for some $\vartheta>0$ if

$$a<4.$$

A similar estimate holds for the other terms depending on ϕ^i.

Some of the terms in G that depend on $\varphi^o=\varphi^*+\varphi$ are

$$\begin{aligned}\Big|\frac{1}{\lambda^2}U\varphi^o(1-\chi)\Big|&\le C\frac{\lambda^2}{|x-\xi|^4}\frac{1}{t^{a-1}(\log t)^\beta}\frac{1}{(1+|x-\xi|/\sqrt t)^b}(1-\chi)\|\varphi^o\|_{*,o}\\&\le\frac{C}{t_0\log t_0}\frac{1}{t^a(\log t)^\beta}\frac{1}{(1+|x-\xi|/\sqrt t)^b}(\|\varphi^*\|_{*,o}+\|\varphi\|_{*,o})\end{aligned}$$

which implies that

$$\Big\|\frac{1}{\lambda^2}U\varphi^o(1-\chi)\Big\|_{**,o}\le\frac{C}{t_0\log t_0}\|\varphi\|_{*,o}+Ct_0^{a-2}(\log t_0)^{\beta-1}\|\varphi_0^*\|_{*,b},$$

by Proposition 6.4. Other terms are estimated in a similar way.

Let us estimate the operator $\mathcal{A}_p$, which is defined by the equations (6.13). We claim that if

$$(0,\tilde\alpha_1,\tilde\xi_1)=\mathcal{A}_p[\phi_1,\phi_2,\varphi,\mathbf{p}_1]$$

and $t_0^{a-1}(\log t_0)^\beta\|\varphi_0^*\|_{*,b}\le 1$, $\|\vec\phi\|_X\le 1$, $\vec\phi=(\phi_1,\phi_2,\varphi,\mathbf{p}_1)$, then

$$\begin{aligned}\|\tilde\alpha_1\|_{C^1,\nu+\frac12,\Theta}&\le C(\log t_0)^{\Theta-\beta}+Ct_0^{a-1}(\log t_0)^{\Theta}\|\varphi_0^*\|_{*,b}\\ \|\tilde\xi_1\|_{*,\gamma,0}&\le \frac{C}{t_0^\vartheta}+Ct_0^{1+\gamma}(\log t_0)^{\frac12}\|\varphi_0^*\|_{*,b},\end{aligned}\tag{6.33}$$

for some $\vartheta>0$. Similarly, we have the following Lipschitz estimate. If $t_0^{a-1}(\log t_0)^\beta\|\varphi_0^*\|_{*,b}\le 1$, then for some $\vartheta>0$, and for $\|\vec\phi_1\|_X\le 1$, $\|\vec\phi_2\|_X\le 1$,

$$\|\mathcal{A}_p[\vec\phi_1,\varphi_0^*]-\mathcal{A}_p[\vec\phi_2,\varphi_0^*]\|_{X_p}\le C(\log t_0)^{\Theta-\beta}\|\vec\phi_1-\vec\phi_2\|_X,\tag{6.34}$$

for some $\vartheta>0$.

Indeed, by (6.11)

$$|\mathcal{A}_{\alpha_1}[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*](t)|\le|I_1(t)|+|I_2(t)|+|I_3(t)|$$

where

$$\begin{aligned}I_1(t)&=\int_t^\infty\frac{1}{\lambda_0^2}m_0[E_2\tilde\chi_2](s)ds\\ I_2(t)&=\int_t^\infty\frac{1}{\lambda_0^2}m_0[F(\phi_0^i+\phi,\varphi^*+\varphi,\mathbf{p}_0+\mathbf{p}_1)\tilde\chi](s)ds\\ I_3(t)&=\int_t^\infty\lambda_0^2m_0[(S_0(u_1(\mathbf{p}_0+\mathbf{p}_1))-S_0(u_1(\mathbf{p}_0)))(\tilde\chi-1)](s)ds.\end{aligned}$$

Using (5.43) and $\int_{\mathbb{R}^2}E_2dy=0$ we get

$$\left|\frac{1}{\lambda_0^2}m_0[E_2\tilde\chi_2](t)\right|\le C\frac{1}{t^{3-2\delta}}.$$

This gives

$$\|I_1\|_{C^1,\nu+\frac12,\Theta}\le Ct_0^{\nu-\frac32+2\delta},\tag{6.35}$$

under the assumption

$$\nu<\frac32-2\delta.$$

The largest contribution in I_2 comes from the term $\lambda^2U\varphi^o$ in $F(\phi_0^i+\phi,\varphi^*+\varphi,\mathbf{p}_0+\mathbf{p}_1)$ (c.f. (5.6)). The estimate of this term is

$$\left|\frac{1}{\lambda_0^2(t)}\int_{\mathbb{R}^2}\lambda_0(t)^2U(y)\varphi^o(y,t)dy\right|\le C\frac{1}{t^{\nu+\frac32}(\log t)^\beta}\|\varphi^o\|_{*,o}\tag{6.36}$$

and so

$$\left\|\int_{\mathbb{R}^2}U(y)\varphi^o(y,t)dy\right\|_{C^1,\nu+\frac12,\Theta}\le C(\log t_0)^{\Theta-\beta}\|\varphi^o\|_{*,o},$$

under the assumption

$$\Theta<\beta.\tag{6.37}$$

Similar estimates for the remaining terms give

$$\|I_2\|_{C^1,\nu+\frac12,\Theta}\le C(\log t_0)^{\Theta-\beta}\|\vec\phi\|_X+Ct_0^{a-1}(\log t_0)^\Theta\|\varphi_0^*\|_{*,b}.\tag{6.38}$$

Regarding I_3, using (4.5) we have

$$\lambda_0^2m_0[S_0(u_1(\mathbf{p}))(\tilde\chi-1)]\le\frac{C}{t^3\log t}.\tag{6.39}$$

Putting together (6.35), (6.38), and (6.39) we get

$$\|\mathcal{A}_{\alpha_1}[\phi_1,\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*]\|_{C^1,\nu+\frac12,\Theta}\le C(\log t_0)^{\Theta-\beta}\|\vec\phi\|_X+Ct_0^{a-1}(\log t_0)^\Theta\|\varphi_0^*\|_{*,b}$$

assuming also that

$$\nu<\frac32.$$

The computations leading to (6.33) are very similar, under the assumption

$$\gamma<\nu-\frac12.\tag{6.40}$$

This restriction arises when considering the largest term in the expression (6.12), namely comes from estimating the term $\lambda_0^2 m_{1,j}[\varphi_{\lambda_0}\phi\tilde\chi]$ ($\lambda_0^2\varphi_{\lambda_0}\phi$ is one of the terms in (5.6))

$$\begin{aligned}\frac{1}{\lambda_0}\lambda_0^2|m_{1,j}[\varphi_{\lambda_0}\phi\tilde\chi](t)| &\le C\lambda_0\int_{\mathbb{R}^2}|\varphi_{\lambda_0}\phi y_j|dy\\ &\le C\lambda_0\frac{1}{t(\log t)^2}\frac{1}{t^{\nu-\frac12}(\log t)^{\frac{q-1}{2}}}\|\phi\|_{1,\nu-\frac12,\frac{q-1}{2},4,2+\sigma+\epsilon}\end{aligned}$$

Let us summarize the restrictions on the parameters. We let $0<q<1$ be fixed. We take

$$0<\delta<\sigma<\min(1,4\delta),$$

and

$$1<\nu<\min\Big(1+2\delta-\frac{\sigma}{2},\frac32,1+\gamma,1+\frac{\sigma}{2}\Big).$$

because of (6.18), (6.20), (6.25). We also need

$$\frac{1-q}{2}<\Theta<\beta<\frac{1+q}{2}$$

by (6.28), (6.37) and by (6.30) and (6.14). We take

$$\frac{\sigma}{2}<\gamma<\nu-\frac12$$

by (6.22) and (6.40).

Together with the above inequalities we want also the relations $\sigma+\epsilon<2$, $\nu+\frac12<\frac74$ for Proposition 6.1 and $\sigma+\epsilon<\frac32$, $\nu<\min(1+\frac{\epsilon}{2},3-\frac{\sigma}{2},\frac54)$ for Proposition 6.2. The condition (6.9) for Propositions 6.3 and 6.4 hold by (6.14). We see that all these restrictions are satisfied by choosing first δ, $\sigma>0$ small so that $2\delta-\frac{\sigma}{2}>0$. Then we take $\nu>1$ close to 1, then let $a=\nu+\frac52$ and b satisfying (6.14). Then Θ, β and γ can be selected. Note that with the above procedure we are getting the restriction $b>5$.

We already have all elements to solve the fixed point problem (6.16), which we recall

$$\vec\phi=\mathcal{A}[\vec\phi],\quad \vec\phi\in\mathcal{B},$$

where $\mathcal{B}$ is the closed unit ball in the Banach space of functions $\vec\phi$ with $\|\vec\phi\|_X<+\infty$ and the norm defined in (6.15). Thus

$$\mathcal{B}=\{\,\vec\phi\in X\,|\,\|\vec\phi\|_X\le 1\,\}.$$

Let φ_0^* be such that $t_0^{a-1}(\log t_0)^\beta\|\varphi_0^*\|_{*,b}\le 1$. Estimates (6.17), (6.24), (6.31) and (6.34), imply that, enlarging the parameter t_0 if necessary, $\mathcal{A}$ maps $\mathcal{B}$ into itself. We also get that $\mathcal{A}$ is a contraction mapping on $\mathcal{B}$. The contraction mapping principle yields the existence of a unique fixed point in $\mathcal{B}$, which then yields the required existence result.

6.1. **Stability.** Theorem 1.1 gives that if φ_0^* has mass zero and is small so that $t_0^{\nu+\frac32}(\log t_0)^\beta\|\varphi_0^*\|_{*,b}\le 1$, then the function

$$\begin{aligned}u(x,t)=\frac{\alpha(t)}{\lambda_0(t)^2}&\Big[U\Big(\frac{x-\xi(t)}{\lambda_0(t)}\Big)+\phi_0^i\Big(\frac{x-\xi(t)}{\lambda_0(t)},t\Big)+\phi\Big(\frac{x-\xi(t)}{\lambda_0(t)},t\Big)\Big]\chi(x,t)\\ &+\tilde\varphi_{\lambda_0}(x-\xi(t),t)+\varphi(x,t)+\varphi^*(x,t),\end{aligned}\tag{6.41}$$

solves (3.1) and blows-up in the way described in Theorem 1.1. This follows from the form of the ansatz (3.2), (3.13), (5.1), (5.38), where $\phi=\phi_1+\phi_2$, and ϕ_1, ϕ_2, φ, φ^* satisfy respectively the equations (5.48), (5.49), (5.50) and (5.37). The initial value of u is

$$\begin{aligned}u^*(x;\varphi_0^*)=\frac{\alpha(t_0;\varphi_0^*)}{\lambda_0(t_0)^2}&\Big[U\Big(\frac{x-\xi(t_0;\varphi_0^*)}{\lambda_0(t_0)}\Big)+\phi_0^i\Big(\frac{x-\xi(t_0;\varphi_0^*)}{\lambda_0(t_0)}\Big)+c_1(\varphi_0^*)\tilde Z_0\Big(\frac{x-\xi(t_0)}{\lambda_0(t_0)}\Big)\Big]\\ &\cdot\chi_0\Big(\frac{x-\xi(t_0;\varphi_0^*)}{\sqrt{t_0}}\Big)+\tilde\varphi_{\lambda_0}(x-\xi(t_0;\varphi_0^*),t_0)+\varphi_0^*(x).\end{aligned}$$

We recall that $\tilde\varphi_\lambda$ is defined in (3.9). The function $\tilde\varphi$ doesn't depend on ξ and is radial about the origin.

We let $u_0^*(x)=u^*(x;0)$. Note that u_0^* is radial and so its center of mass is zero.

To prove stability, we would like to prove the following intermediate step: given v defined on $\mathbb{R}^2$ small, with mass zero and under some additional assumptions to be defined later on, we would like to find φ_0^* with mass zero such that

$$u^*(\varphi_0^*) = u_0^* + v. \tag{6.42}$$

The equation (6.42) for φ_0^* has the form

$$\begin{aligned}
&\frac{\alpha(t_0;\varphi_0^*)}{\lambda_0(t_0)^2}\Big[U\Big(\frac{x-\xi(t_0;\varphi_0^*)}{\lambda_0(t_0)}\Big) + \phi_0^i\Big(\frac{x-\xi(t_0;\varphi_0^*)}{\lambda_0(t_0)}\Big) + c_1(\varphi_0^*)\tilde Z_0\Big(\frac{x-\xi(t_0)}{\lambda_0(t_0)}\Big)\Big]\cdot \chi_0\Big(\frac{x-\xi(t_0;\varphi_0^*)}{\sqrt{t_0}}\Big)\\
&\quad + \tilde\varphi_{\lambda_0}(x-\xi(t_0;\varphi_0^*),t_0) + \varphi_0^*(x)\\
&= \frac{\alpha(t_0;0)}{\lambda_0(t_0)^2}\Big[U\Big(\frac{x}{\lambda_0(t_0)}\Big) + \phi_0^i\Big(\frac{x}{\lambda_0(t_0)}\Big) + c_1(0)\tilde Z_0\Big(\frac{x}{\lambda_0(t_0)}\Big)\Big]\cdot\chi_0\Big(\frac{x}{\sqrt{t_0}}\Big)\\
&\quad + \tilde\varphi_{\lambda_0}(x,t_0) + v.
\end{aligned} \tag{6.43}$$

Computing the mass we find that $\alpha(t_0;\varphi_0^*) = \alpha(t_0;0)$. Note that $\lim_{t\to\infty}\xi(t) = 0$ by (6.12). Then the center of mass of $u(\cdot,t)$ satisfies

$$\lim_{t\to\infty}\int_{\mathbb{R}^2} u(x,t)x\,dx = 0.$$

Since the center of mass is preserved

$$\int_{\mathbb{R}^2} u(x,t_0)x\,dx = 0.$$

Let's assume that the center of mass of v and φ_0^* are both zero. Then, computing the center of mass we find that

$$\xi(t_0;\varphi_0^*) = 0. \tag{6.44}$$

Then the equation (6.43) reduces to

$$(c_1(\varphi_0^*) - c_1(0))\frac{\alpha(t_0;0)}{\lambda_0(t_0)^2}\tilde Z_0\Big(\frac{x}{\lambda_0(t_0)}\Big) + \varphi_0^*(x) = v. \tag{6.45}$$

We will prove at the end of this section the following.

Proposition 6.5. *There is $\delta>0$ so that if $t_0^{\nu+\frac32}(\log t_0)^\beta\|v\|_{*,b}\le\delta$, v has mass and center of mass equal to zero, then*

$$\int_{\mathbb{R}^2} v(x)|x|^2dx = 0,$$

is equivalent to

$$c_1(v) - c_1(0) = 0.$$

To prove stability we first observe that if $v:\mathbb{R}^2\to\mathbb{R}$ satisfies $t_0^{\nu+\frac32}(\log t_0)^\beta\|v\|_{*,b}\le\delta$, has mass zero, and

$$\int_{\mathbb{R}^2} v(x)x_j\,dx = 0, \quad \int_{\mathbb{R}^2} v(x)|x|^2dx = 0,$$

then $u_0^*+v = u^*(\varphi_0^*)$ for $\varphi_0^* = v$, by Proposition 6.5.

Now consider a general v with $t_0^{\nu+\frac32}(\log t_0)^\beta\|v\|_{*,b}\le\delta$ (for a possibly smaller $\delta>0$), and mass zero. We want to show that the initial condition u_0^*+v produces a solution to (3.1) with infinite time blow as described in Theorem 1.1. Consider

$$u_{\Lambda,p}(x) = \frac{1}{\Lambda^2}\Big[u_0^*\Big(\frac{x-p}{\Lambda}\Big) + v\Big(\frac{x-p}{\Lambda}\Big)\Big],$$

where $p\in\mathbb{R}^2$ and $\Lambda>0$. Note that $u_{\Lambda,p}$ has mass 8π. Then we select Λ and p such that

$$\int_{\mathbb{R}^2} u_{\Lambda,p}(x)x_j\,dx = 0, \quad \int_{\mathbb{R}^2} u_{\Lambda,p}(x)|x|^2dx = \int_{\mathbb{R}^2} u_0^*(x)|x|^2dx.$$

Note that $|\Lambda^2-1|\le Ct_0^2\|v\|_{*,b}\ll 1$ and $|p|\le Ct_0\|v\|_{*,b}\ll 1$. Then we expand

$$u_{\Lambda,p}(x) = u_0^* + w$$

and w satisfies $t_0^{\nu+\frac32}(\log t_0)^\beta\|w\|_{*,b}\le C\delta$, has mass zero, center of mass zero and second moment equal to 0. By the previous claim, the initial condition $u_{\Lambda,p}(x) = u_0^*+w$ is such that the solution to (3.1) blows up as in Theorem 1.1. Then the same is true for the initial condition $u_0^* + v$ after a scaling and translation in space.

6.2. **Proof of Proposition 6.5.**

Lemma 6.1. *Assume that $t_0^{\nu+\frac{3}{2}}(\log t_0)^\beta \|v\|_{*,b} \leq 1$, that v has mass and center of mass equal to zero, and that*

$$c_1(v) - c_1(0) = 0. \tag{6.46}$$

Then

$$\int_{\mathbb{R}^2} v(x)|x|^2 dx = 0.$$

Proof. From (6.46), $\varphi_0^* = v$ solves (6.45), and therefore $u_0^* + v$ is an initial condition for (3.1) for which the solution blows up in infinite time. The solution u to (3.1) preserves the second moment:

$$\int_{\mathbb{R}^2} u(x,t)|x|^2 dx = \text{const}.$$

We compute the expansion of $\int_{\mathbb{R}^2} u(x,t)|x|^2 dx$ as $t \to \infty$, based on the expression (6.41).

Note that $\lim_{t\to\infty} \xi(t) = 0$ by (6.12). Then

$$\frac{\alpha(t)}{\lambda_0(t)^2} \int_{\mathbb{R}^2} U\Big(\frac{x-\xi(t)}{\lambda_0(t)}\Big)\chi(x,t)|x|^2 dx = \frac{\alpha(t)}{\lambda_0(t)^2} \int_{\mathbb{R}^2} U\Big(\frac{x-\xi(t)}{\lambda_0(t)}\Big)\chi(x,t)|x-\xi(t)|^2 dx + o(1),$$

as $t \to \infty$. By explicit computation

$$\frac{1}{\lambda_0(t)^2} \int_{\mathbb{R}^2} U\Big(\frac{x-\xi(t)}{\lambda_0(t)}\Big)\chi(x,t)|x-\xi(t)|^2 dx = 8\pi\lambda_0^2 \log\Big(\frac{t}{\lambda_0^2}\Big) + O(\lambda_0^2) \tag{6.47}$$

as $t \to \infty$.

Using Lemma 4.1

$$\int_{\mathbb{R}^2} \varphi_{\lambda_0}(x,t)|x-\xi(t)|^2 dx \leq \frac{C}{\log(t)}.$$

Using also Lemma 4.1 to estimate the mass and first moment of φ_{λ_0} we get

$$\int_{\mathbb{R}^2} \varphi_{\lambda_0}(x,t)|x|^2 dx \leq \frac{C}{\log(t)}. \tag{6.48}$$

Using (6.47), (6.48) and the estimates for ϕ_0^i (5.36), $\phi = \phi_1 + \phi_2$ that arise from $\|\phi\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\epsilon} < \infty$, and φ, φ^* which arise from $\|\varphi\|_{*,o} < \infty$, $\|\varphi^*\|_{*,o} < \infty$, we get that

$$\int_{\mathbb{R}^2} u(x,t)|x|^2 dx = 8\pi\lambda_0^2 \log\Big(\frac{t}{\lambda_0^2}\Big) + O(\lambda_0^2)$$

as $t \to \infty$. But λ_0 was constructed in Proposition 5.1 with the expansion

$$\lambda_0(t) = \frac{c_0}{\sqrt{\log t}} + O\Big(\frac{1}{(\log t)^{\frac{3}{2}-\varepsilon}}\Big),$$

as $t \to \infty$, where $c_0 > 0$ is a constant. Therefore

$$\int_{\mathbb{R}^2} u(x,t)|x|^2 dx = 8\pi c_0^2$$

and evaluating at $t = t_0$ we obtain

$$\int_{\mathbb{R}^2} u_0^*(x)|x|^2 dx + \int_{\mathbb{R}^2} v(x)|x|^2 dx = 8\pi c_0^2.$$

We can apply the previous calculation to $v = 0$ and arrive at

$$\int_{\mathbb{R}^2} u_0^*(x)|x|^2 dx = 8\pi c_0^2.$$

This shows that

$$\int_{\mathbb{R}^2} v(x)|x|^2 dx = 0.$$

□

We need an expansion for $c_1(\varphi_0^*) - c_1(0)$.

Lemma 6.2. *Assume that* $t_0^{\nu+\frac{3}{2}}(\log t_0)^\beta \|\varphi_0^*\|_{*,b} \le 1$ *and that* φ_0^* *has mass and center of mass equal to zero. Then*

$$c_1(\varphi_0^*) - c_1(0) = a_0 \int_{\mathbb{R}^2} \varphi_0^*(x)|x|^2 dx + R_0(\varphi_0^*),$$

where $a_0 \neq 0$ *and* R_0 *satisfies*

$$|R_0(\varphi_0^*)| \le C t_0 \|\varphi_0^*\|_{*,b}. \tag{6.49}$$

Proof. In the following calculations $\lambda = \lambda_0$.

First we need to estimate the Lipschitz constant of the solutions ϕ_1, ϕ_2, and φ with respect to φ_0^*. We claim that

$$\begin{aligned} &\|\phi_1(\varphi_0^*) - \phi_1(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\varepsilon} + \|\phi_2(\varphi_0^*) - \phi_2(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\varepsilon} \\ &\qquad \le C \frac{1}{\log t_0} t_0^{\nu+\frac{1}{2}} (\log t_0)^{\beta-1} \|\varphi_0^*\|_{*,b} \end{aligned} \tag{6.50}$$

$$|\varphi(x,t;\varphi_0^*) - \varphi(x,t;0)| \le C \frac{1}{t^{\nu+\frac{3}{2}} (\log t)^{\frac{q+1}{2}}} \frac{1}{(1+|x-\xi|/\sqrt{t})^b} t_0^{\nu+\frac{1}{2}} (\log t_0)^{\beta-1} \|\varphi_0^*\|_{*,b} \tag{6.51}$$

$$|\alpha_1[\varphi_0^*](t) - \alpha_1[0](t)| \le C \frac{1}{t^{\nu+\frac{1}{2}} (\log t)^{\frac{q+1}{2}}} t_0^{\nu+\frac{1}{2}} (\log t_0)^{\beta-1} \|\varphi_0^*\|_{*,b}. \tag{6.52}$$

We discuss briefly the proof of these estimates. One of the main terms in the right hand side of (5.50), written for the difference $\varphi(\varphi_0^*) - \varphi(0)$ is

$$\begin{aligned} &\Big| \frac{1}{\lambda^2} [\phi(\varphi_0^*) - \phi(0)] \Delta\chi \Big| (x,t) \\ &\le \frac{C}{\lambda^2} \frac{1}{t^{\nu-\frac{1}{2}} (\log t)^{\frac{q-1}{2}}} \frac{1}{(|x-\xi|/\lambda|)^4} \frac{1}{t} \Big| \Delta_z \chi_0 (\frac{x-\xi}{\sqrt{t}}) \Big| \|\phi(\varphi_0^*) - \phi(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\epsilon} \\ &\le C \frac{1}{t^{\nu+\frac{5}{2}} (\log t)^{\frac{q+1}{2}}} \frac{1}{(1+|x-\xi|/\sqrt{t})^b} \|\phi(\varphi_0^*) - \phi(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\epsilon}, \end{aligned}$$

which implies

$$\Big\| \frac{1}{\lambda^2} [\phi(\varphi_0^*) - \phi(0)] \Delta\chi \Big\|_{**,o} \le \frac{C}{(\log t_0)^{\frac{q+1}{2}-\beta}} \|\phi(\varphi_0^*) - \phi(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\epsilon},$$

since $\beta < \frac{q+1}{2}$, which is one of the conditions in (6.14). (Here χ depends on φ_0^*. There is another ther in the difference that depends on $\chi(\varphi_0^*) - \chi(0)$ and is estimated similarly.) Then

$$\begin{aligned} &|\varphi(x,t;\varphi_0^*) - \varphi(x,t;0)| \\ &\le C \frac{1}{t^{\nu+\frac{3}{2}} (\log t)^{\frac{q+1}{2}}} \frac{1}{(1+|x-\xi|/\sqrt{t})^b} [\|\phi(\varphi_0^*) - \phi(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\varepsilon} + t_0^{\nu+\frac{1}{2}} (\log t_0)^{\beta-1} \|\varphi_0^*\|_{*,b}]. \end{aligned} \tag{6.53}$$

Considering φ as an operator of ϕ we examine the effect of the therm $\lambda^2 U \varphi$. This term appears in the right hand side of (5.48), where the effect is less important, and in the computation of α_1. Estimating the right hand side of (5.50) as in (6.32), using Proposition 6.3 gives that

$$\begin{aligned} |\alpha_1[\phi(\varphi_0^*](t) - \alpha_1[\phi(0)](t)| &\le C \int_t^\infty \int_{\mathbb{R}^2} U(y) |\varphi(\xi+\lambda y, t, ; \varphi_0^*) - \varphi(\xi+\lambda y, t, ; 0)| dy \\ &\le C \frac{1}{t^{\nu+\frac{1}{2}} (\log t)^{\frac{q+1}{2}}} [\|\phi(\varphi_0^*) - \phi(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\varepsilon} + t_0^{\nu+\frac{1}{2}} (\log t_0)^{\beta-1} \|\varphi_0^*\|_{*,b}]. \end{aligned}$$

We consider now the effect of $|\alpha_1[\phi](t)|$ in the right hand side of (5.49), where thanks to Lemma 5.4 appears mainly as $\alpha_1(t) W_2(y)$, where W_2 is radial with compact support. Then Proposition 6.1 gives

$$\begin{aligned} |\phi_2(y,t;\varphi_0^*) - \phi_2(y,t;0)| &\le C \frac{1}{(\log t_0)^{1-q}} \frac{1}{t^{\nu-\frac{1}{2}} (\log t)^{\frac{q+1}{2}+q-1}} \frac{1}{(1+|y|)^4} \min\Big(1, \frac{(t\log t)^{1/2}}{|y|}\Big)^{2+\sigma+\epsilon} \\ &\quad \cdot [\|\phi(\varphi_0^*) - \phi(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\varepsilon} + t_0^{\nu+\frac{1}{2}} (\log t_0)^{\beta-1} \|\varphi_0^*\|_{*,b}]. \end{aligned}$$

Then

$$\|\phi_2(\varphi_0^*) - \phi_2(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\varepsilon} \le C \frac{1}{\log t_0} [\|\phi(\varphi_0^*) - \phi(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\varepsilon} + t_0^{\nu+\frac{1}{2}} (\log t_0)^{\beta-1} \|\varphi_0^*\|_{*,b}].$$

The estimate for ϕ_1 is actually better, and therefore

$$\begin{aligned}
&\|\phi_1(\varphi_0^*) - \phi_1(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\varepsilon} + \|\phi_2(\varphi_0^*) - \phi_2(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\varepsilon} \\
&\qquad \le C\frac{1}{\log t_0}[\|\phi_1(\varphi_0^*) - \phi_1(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\varepsilon} + \|\phi_2(\varphi_0^*) - \phi_2(0)\|_{1,\nu-\frac{1}{2},\frac{q-1}{2},4,2+\sigma+\varepsilon} \\
&\qquad\quad + t_0^{\nu+\frac{1}{2}}(\log t_0)^{\beta-1}\|\varphi_0^*\|_{*,b}].
\end{aligned}$$

This implies (6.50). Replacing this in (6.53) we obtain (6.51), and similarly we get (6.52).

The parameter c_1 appears in the second inner equation in (5.49), which we write as

$$\begin{cases} \lambda^2\partial_t\phi_2 = L[\phi_2] + B[\phi_2] + h(t)W_2 \quad \text{in } \mathbb{R}^2\times(t_0,\infty) \\ \phi_2(\cdot,t_0) = c_1\tilde{Z}_0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{6.54}$$

where

$$h(t,\varphi_0^*) = m_2[F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)](t).$$

Note that ϕ_2 in (6.54) is radial, so the operator B defined (5.47) reduces to $B[\phi] = \lambda\dot{\lambda}(2\phi + y\cdot\nabla\phi) = \lambda\dot{\lambda}\nabla\cdot(y\phi)$. Multiplying by $|y|^2$ and integrating on $\mathbb{R}^2$ gives

$$\lambda^2\partial_t\int_{\mathbb{R}^2}\phi|y|^2dy + 2\lambda\dot{\lambda}\int_{\mathbb{R}^2}\phi|y|^2dy = h(t).$$

Then

$$\lambda^2\int_{\mathbb{R}^2}\phi(y,t)|y|^2dy = -\int_t^\infty h(s)ds.$$

But $\phi_2(y,t_0) = c_1\tilde{Z}_0(y)$ so

$$c_1(\varphi_0^*) = -\frac{1}{\lambda(t_0)^2\int_{\mathbb{R}^2}\tilde{Z}_0(y)|y|^2dy}\int_{t_0}^\infty h(s,\varphi_0^*)ds.$$

In particular

$$c_1(\varphi_0^*) - c_1(0) = -\frac{1}{\lambda(t_0)^2\int_{\mathbb{R}^2}\tilde{Z}_0(y)|y|^2dy}\int_{t_0}^\infty [h(s,\varphi_0^*) - h(s,0)]ds. \tag{6.55}$$

The function $h(t,\varphi_0^*) = m_2[F_3(\phi_1+\phi_2,\varphi,\mathbf{p}_1,\varphi_0^*)](t)$ is analyzed near (6.25). We follow the same steps. Using the definition of F_3 (5.51)

$$m_2[F_3(\phi,\varphi,\mathbf{p}_1,\varphi_0^*)] = m_2[E_2\tilde{\chi}_2(\varphi_0^*)] + m_2[F_2(\phi,\varphi,\mathbf{p}_1,\varphi_0^*)\tilde{\chi}(\varphi_0^*)].$$

We note that $\tilde{\chi}$, $\tilde{\chi}_2$ also depend on φ_0^* because ξ depends on φ_0^*. By (5.41)

$$m_2[F_2(\phi,\varphi,\mathbf{p}_1,\varphi_0^*)\tilde{\chi}(\varphi_0^*)] = I(\varphi_0^*) + II(\varphi_0^*) + III(\varphi_0^*)$$

where

$$\begin{aligned}
I(t,\varphi_0^*) &= \lambda^4 m_2[S_0(u_1(\mathbf{p}_0+\mathbf{p}_1)) - S_0(u_1(\mathbf{p}_0))] \\
II(t,\varphi_0^*) &= m_2[F(\phi_0^i+\phi,\varphi^*+\varphi,\mathbf{p}_0+\mathbf{p}_1)\tilde{\chi}(\varphi_0^*)] \\
III(t,\varphi_0^*) &= \lambda^4 m_2[(S_0(u_1(\mathbf{p}_0+\mathbf{p}_1)) - S_0(u_1(\mathbf{p}_0)))(\tilde{\chi}(\varphi_0^*)-1)].
\end{aligned}$$

The main term is $I(\varphi_0^*)$ and the others are treated as perturbations.

By Lemma 5.4, since $\lambda = \lambda_0$, we get

$$I(t,\varphi_0^*) = -32\pi\alpha_1(\varphi_0^*) + I_0(\varphi_0^*), \tag{6.56}$$

where

$$\begin{aligned}
I_0(t,\varphi_0^*) = &-\frac{\dot{\alpha}_1(\varphi_0^*)}{\lambda_0^2}\int_{\mathbb{R}^2}U(\frac{x}{\lambda_0})\chi_0(\frac{x}{\lambda_0})|x|^2\,dx + \alpha_1(\varphi_0^*)\int_{\mathbb{R}^2}E(x,t,\lambda_0)|x|^2\,dx \\
&- |\xi(\varphi_0^*)|^2\int_{\mathbb{R}^2}S(u_1(\mathbf{p}_0))dx.
\end{aligned}$$

By (6.11)

$$\alpha_1(t,\varphi_0^*) - \alpha_1(t,0) = A_1(t,\varphi_0^*) + A_2(t,\varphi_0^*) \tag{6.57}$$

where

$$A_1(t,\varphi_0^*) = -\frac{1}{8\pi(1+2\Upsilon\frac{\lambda_0^2}{t})+e_1(\frac{\lambda_0^2}{t})}\int_t^\infty \frac{1}{\lambda_0^2}\Big\{m_0[F(\phi_0^i+\phi(\varphi_0^*),\varphi^*+\varphi(\varphi_0^*),\mathbf{p}_0+\mathbf{p}_1(\varphi_0^*))\tilde\chi(\varphi_0^*)](s)$$
$$- m_0[F(\phi_0^i+\phi(0),\varphi(0),\mathbf{p}_0+\mathbf{p}_1(0))\tilde\chi(0)](s)\Big\}ds$$

$$A_2(t,\varphi_0^*) = -\frac{1}{8\pi(1+2\Upsilon\frac{\lambda_0^2}{t})+e_1(\frac{\lambda_0^2}{t})}\int_t^\infty \lambda_0^2\Big\{m_0[(S_0(u_1(\mathbf{p}_0+\mathbf{p}_1(\varphi_0^*)))-S_0(u_1(\mathbf{p}_0)))(\tilde\chi(\varphi_0^*)-1)](s)$$
$$- m_0[(S_0(u_1(\mathbf{p}_0+\mathbf{p}_1(0)))-S_0(u_1(\mathbf{p}_0)))(\tilde\chi(0)-1)](s)\Big\}ds$$

Let

$$\tilde m_0(t,\varphi_0^*) = \frac{1}{\lambda_0^2}m_0[F(\phi_0^i+\phi(\varphi_0^*),\varphi^*+\varphi(\varphi_0^*),\mathbf{p}_0+\mathbf{p}_1(\varphi_0^*))\tilde\chi(\varphi_0^*)](t)$$
$$-\int_{\mathbb{R}^2} U(y)\varphi^o(\xi(t,\varphi_0^*)+\lambda y,t,\varphi_0^*)dy$$

so that

$$m_0[F(\phi_0^i+\phi(\varphi_0^*),\varphi^*+\varphi(\varphi_0^*),\mathbf{p}_0+\mathbf{p}_1(\varphi_0^*))\tilde\chi(\varphi_0^*)](t) = \lambda_0^2(t)\int_{\mathbb{R}^2} U(y)\varphi^o(\xi(t,\varphi_0^*)+\lambda y,t,\varphi_0^*)dy$$
$$+\lambda_0^2(t)\tilde m_0(t,\varphi_0^*).$$

By (5.38), $\varphi^o = \varphi^*+\varphi$, where $\varphi=\varphi(\varphi_0^*)$ solves (5.50) and φ^* solves (5.37). Therefore

$$A_1(t,\varphi_0^*) = -\frac{1}{8\pi(1+2\Upsilon\frac{\lambda_0^2}{t})+e_1(\frac{\lambda_0^2}{t})}\int_t^\infty\int_{\mathbb{R}^2} U(y)\varphi^*(\xi(s,\varphi_0^*)+\lambda y,s,\varphi_0^*)dyds$$
$$+\tilde A_1(t,\varphi_0^*)$$

where

$$\tilde A_1(t,\varphi_0^*) = -\frac{1}{8\pi(1+2\Upsilon\frac{\lambda_0^2}{t})+e_1(\frac{\lambda_0^2}{t})}\int_t^\infty\int_{\mathbb{R}^2} U(y)[\varphi(\xi(s,\varphi_0^*)+\lambda y,s,\varphi_0^*)-\varphi(\xi(s,0)+\lambda y,s,0)]dyds$$
$$-\frac{1}{8\pi(1+2\Upsilon\frac{\lambda_0^2}{t})+e_1(\frac{\lambda_0^2}{t})}\int_t^\infty[\tilde m_0(s,\varphi_0^*)-\tilde m_0(s,0)]ds.$$

Integrating (5.37) on $\mathbb{R}^2$ we find that

$$\partial_t\int_{\mathbb{R}^2}\varphi^*(x,t)dx = \lambda(t)^{-2}\int_{\mathbb{R}^2} U\Big(\frac{x-\xi}{\lambda}\Big)\varphi^*(x,t)\,dx = \int_{\mathbb{R}^2} U(y)\varphi^*(\xi+\lambda y,t)\,dy,$$

and therefore

$$A_1(t,\varphi_0^*) = \frac{1}{8\pi(1+2\Upsilon\frac{\lambda_0^2}{t})+e_1(\frac{\lambda_0^2}{t})}\int_{\mathbb{R}^2}\varphi^*(x,t,\varphi_0^*)dx+\tilde A_1(t,\varphi_0^*).$$

Then from (6.57)

$$\alpha_1(t,\varphi_0^*)-\alpha_1(t,0) = \frac{1}{8\pi(1+2\Upsilon\frac{\lambda_0^2}{t})+e_1(\frac{\lambda_0^2}{t})}\int_{\mathbb{R}^2}\varphi^*(x,t,\varphi_0^*)dx+\tilde A_1(t,\varphi_0^*)+A_2(t,\varphi_0^*).$$

Using this and (6.56) we get

$$I(t,\varphi_0^*)-I(t,0) = -\frac{4}{1+2\Upsilon\frac{\lambda_0^2}{t}+\frac{1}{8\pi}e_1(\frac{\lambda_0^2}{t})}\int_{\mathbb{R}^2}\varphi^*(x,t,\varphi_0^*)dx-32\pi\tilde A_1(t,\varphi_0^*)-32\pi A_2(t,\varphi_0^*)$$
$$+I_0(t,\varphi_0^*)-I_0(t,0).$$

Hence

$$h(t,\varphi_0^*)-h(t,0) = -4\int_{\mathbb{R}^2}\varphi^*(x,t,\varphi_0^*)dx+\tilde h(t,\varphi_0^*)$$

where

$$\tilde h(\varphi_0^*,t) = 4\frac{2\Upsilon\frac{\lambda_0^2}{t}+\frac{1}{8\pi}e_1(\frac{\lambda_0^2}{t})}{1+2\Upsilon\frac{\lambda_0^2}{t}+\frac{1}{8\pi}e_1(\frac{\lambda_0^2}{t})}\int_{\mathbb{R}^2}\varphi^*(x,t,\varphi_0^*)dx-32\pi\tilde A_1(t,\varphi_0^*)-32\pi A_2(t,\varphi_0^*)$$
$$+I_0(t,\varphi_0^*)-I_0(t,0)+II(t,\varphi_0^*)-II(t,0)+III(t,\varphi_0^*)-III(t,0).$$

From (6.55) it follows that

$$c_1(\varphi_0^*) - c_1(0) = \frac{4}{\lambda(t_0)^2 \int_{\mathbb{R}^2} \tilde{Z}_0(y)|y|^2 dy} \int_{t_0}^{\infty} \int_{\mathbb{R}^2} \varphi^*(x,s,\varphi_0^*) dx ds + \tilde{c}_1(\varphi_0^*) \tag{6.58}$$

where

$$\tilde{c}_1(\varphi_0^*) = -\frac{1}{\lambda(t_0)^2 \int_{\mathbb{R}^2} \tilde{Z}_0(y)|y|^2 dy} \int_{t_0}^{\infty} \tilde{h}(s,\varphi_0^*) ds.$$

We can relate the integral $\int_{t_0}^{\infty} \int_{\mathbb{R}^2} \varphi^*(x,s,\varphi_0^*) dx ds$ with the second moment of φ_0^* as follows. We multiply the equation of φ^* (5.37) by $|x-\xi(t)|^2$ and integrate on $\mathbb{R}^2$ to get

$$\begin{aligned}\partial_t \int_{\mathbb{R}^2} \varphi^*(x,t)|x-\xi(t)|^2 dx = \int_{\mathbb{R}^2} \Delta\varphi^*(x,t)|x-\xi(t)|^2 dx - \int_{\mathbb{R}^2} \nabla_x \Gamma_0\Big(\frac{x-\xi(t)}{\lambda}\Big) \cdot \nabla\varphi^*(x,t)|x-\xi(t)|^2 dx \\ - 2\dot{\xi}(t) \cdot \int_{\mathbb{R}^2} \varphi^*(x,t)(x-\xi(t)) dx.\end{aligned}$$

But

$$\int_{\mathbb{R}^2} \Delta\varphi^* |x-\xi(t)|^2 dx = 4 \int_{\mathbb{R}^2} \varphi^* dx$$

and

$$\int_{\mathbb{R}^2} \nabla_x \Gamma_0\Big(\frac{x-\xi(t)}{\lambda}\Big) \cdot \nabla\varphi^*(x,t)|x-\xi(t)|^2 dx = -\int_{\mathbb{R}^2} \varphi^*(x+\xi(t)) \Big[\Delta_x \Gamma_0\Big(\frac{x}{\lambda}\Big)|x|^2 + 2\nabla_x \Gamma_0\Big(\frac{x}{\lambda}\Big) \cdot x\Big] dx.$$

Using the explicit expressions for U and Γ_0 and writing $y = \frac{x}{\lambda}$, $\rho = |y|$, we get

$$\begin{aligned}\Delta_x \Gamma_0\Big(\frac{x}{\lambda}\Big)|x|^2 + 2\nabla_x \Gamma_0\Big(\frac{x}{\lambda}\Big) \cdot x &= -\frac{1}{\lambda^2} U\Big(\frac{x}{\lambda}\Big)|x|^2 + 2\nabla_y \Gamma_0\Big(\frac{x}{\lambda}\Big) \cdot \frac{x}{\lambda} \\ &= -\frac{8\rho^2}{(1+\rho^2)^2} - \frac{8\rho^2}{1+\rho^2} \\ &= -\Big[\frac{8\rho^2}{(1+\rho^2)^2} + 8 - \frac{8}{1+\rho^2}\Big] \\ &= -8 + \frac{8}{(1+\rho^2)^2}.\end{aligned}$$

So

$$\int_{\mathbb{R}^2} \varphi^*(x+\xi(t)) \Big[\Delta_x \Gamma_0\Big(\frac{x}{\lambda}\Big)|x|^2 + 2\nabla_x \Gamma_0\Big(\frac{x}{\lambda}\Big) \cdot x\Big] dx = -8 \int_{\mathbb{R}^2} \varphi^*(x,t) dx + \int_{\mathbb{R}^2} U\Big(\frac{x-\xi(t)}{\lambda}\Big) \varphi^*(x,t) dx$$

and we find that

$$\begin{aligned}\partial_t \int_{\mathbb{R}^2} \varphi^*(x,t)|x-\xi(t)|^2 dx = -4 \int_{\mathbb{R}^2} \varphi^*(x,t) dx + \int_{\mathbb{R}^2} U\Big(\frac{x-\xi(t)}{\lambda}\Big) \varphi^*(x,t) dx \\ - 2\dot{\xi}(t) \int_{\mathbb{R}^2} \varphi^*(x,t)(x-\xi(t)) dx.\end{aligned}$$

Integrating and using (6.58) we find that

$$c_1(\varphi_0^*) - c_1(0) = \frac{1}{\lambda(t_0)^2 \int_{\mathbb{R}^2} \tilde{Z}_0(y)|y|^2 dy} \int_{\mathbb{R}^2} \varphi^*(x,t_0)|x|^2 dx + R_0(\varphi_0^*),$$

by (6.44), where

$$\begin{aligned}R_0(\varphi_0^*) = \frac{1}{\lambda(t_0)^2 \int_{\mathbb{R}^2} \tilde{Z}_0(y)|y|^2 dy} \Big[\int_{t_0}^{\infty} \int_{\mathbb{R}^2} U\Big(\frac{x-\xi(s)}{\lambda}\Big) \varphi^*(x,s) dx ds \\ - 2 \int_{t_0}^{\infty} \dot{\xi}(s) \int_{\mathbb{R}^2} \varphi^*(x,s)(x-\xi(s)) dx ds\Big] + \tilde{c}_1(\varphi_0^*).\end{aligned}$$

We claim that $R_0(\varphi_0^*)$ satisfies (6.49). Indeed, let us look at

$$\int_{t_0}^{\infty} [I_0(s,\varphi_0^*) - I_0(s,0)] ds.$$

Similarly (6.36), we have

$$\left| \frac{\dot{\alpha}_1(t,\varphi_0^*) - \dot{\alpha}_1(t,0)}{\lambda_0^2} \int_{\mathbb{R}^2} U(\frac{x}{\lambda_0}) \chi_0(\frac{x}{\lambda_0}) |x|^2 \, dx \right| \le C \frac{t_0^{\nu+\frac{3}{2}} (\log t_0)^{\beta}}{t^{\nu+\frac{3}{2}} (\log t)^{\beta}} \lambda_0^2 \|\varphi_0^*\|_{*,b}$$

Similar computations for the other terms of I_0 give

$$|I_0(t,\varphi_0^*) - I_0(t,0)| \le C \frac{t_0^{\nu+\frac32}(\log t_0)^\beta}{t^{\nu+\frac32}(\log t)^\beta}\lambda_0^2\|\varphi_0^*\|_{*,b}.$$

It follows that

$$\left|\int_{t_0}^\infty [I_0(s,\varphi_0^*) - I_0(s,0)]ds\right| \le C\frac{t_0}{\log(t_0)}\|\varphi_0^*\|_{*,b}.$$

The other terms in R_0 are estimated similarly. □

Proof of Proposition 6.5. If $t_0^{\nu+\frac32}(\log t_0)^\beta\|v\|_{*,b} \le 1$ and $c_1(v) - c_1(0) = 0$, then Lemma 6.1 implies that $\int_{\mathbb{R}^2} v(x)|x|^2 dx = 0$.

To prove the converse, let

$$v_1(x) = \frac{1}{(1+\frac{|x|}{\sqrt{t_0}})^b}$$

so that $\|v_1\|_{*,b} = 1$ (norm defined in (6.10)). Assuming $\mu t_0^{\nu+\frac32}(\log t_0)^\beta \le \delta$ and $\delta > 0$ small, we have by Lemma 6.2

$$c_1(v+\mu v_1) - c_1(0) = c\mu t_0^2 + R_0(v+\mu v_1),$$

for some constant $c \ne 0$. Note that is $c_1(\varphi_0^*)$ continuous function of φ_0^*, and so is $R_0(\varphi_0^*)$. By the intermediate value theorem, there is $\mu = O(t_0\|v\|_{*,b})$ such that $c_1(v+\mu v_1) - c_1(0) = 0$. By Lemma 6.1 $\int_{\mathbb{R}^2}()(v(x)+\mu v_1(x))|x|^2 dx = 0$, which implies that $\mu = 0$. But then $c_1(v) - c_1(0) = 0$. □

7. The mass of φ_λ

We devote this section to prove Proposition 5.1. To that purpose, a basic step is to derive a formula for the mass of φ_λ defined in (3.12).

Let us write

$$\varphi_\lambda = \varphi_\lambda^{(1)} + \varphi_\lambda^{(2)} \tag{7.1}$$

where $\varphi_\lambda^{(1)}$ and $\varphi_\lambda^{(2)}$ are the solutions, given by Duhamel's formula, of the following problems

$$\begin{cases} \partial_t\varphi_\lambda^{(1)} = \Delta_6\varphi_\lambda^{(1)} + \dfrac{\dot\lambda}{\lambda^3}Z_0(\dfrac{x}{\lambda})\chi_0(z) & \text{in } \mathbb{R}^2\times(\frac{t_0}{2},\infty) \\ \varphi_\lambda^{(1)}(\cdot,\frac{t_0}{2}) = 0 \end{cases} \tag{7.2}$$

$$\begin{cases} \partial_t\varphi_\lambda^{(2)} = \Delta_6\varphi_\lambda^{(2)} + \dfrac{1}{2\lambda^2 t}U\nabla_z\chi_0(z)\cdot z + \tilde E, & \text{in } \mathbb{R}^2\times(\frac{t_0}{2},\infty), \quad z = \dfrac{x}{\sqrt t}, \\ \varphi_\lambda^{(2)}(\cdot,\frac{t_0}{2}) = 0 \end{cases} \tag{7.3}$$

where the operator Δ_6 is defined in (3.8) and $\tilde E$ in (3.11). We let $\varphi[p,\lambda](r,t)$ be the solution of the problem

$$\begin{cases} \partial_t\varphi[p,\lambda] = \Delta_6\varphi[p,\lambda] + \dfrac{p}{\lambda^4}Z_0\Big(\dfrac{r}{\lambda}\Big)\chi\Big(\dfrac{r}{\sqrt t}\Big) & \text{in } \mathbb{R}^2\times(\frac{t_0}{2},\infty), \\ \varphi[p,\lambda](\cdot,\frac{t_0}{2}) = 0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{7.4}$$

given by Duhamel's formula. By definition, we have

$$\varphi_\lambda^{(1)} = \varphi[\lambda\dot\lambda,\lambda].$$

In definitions (7.2), (7.3), (7.4), the parameter function $\lambda(t)$ is assumed to be defined for $t > \frac{t_0}{2}$. In the rest of this section we also assume the validity of the condition stated for λ in (4.1), namely

$$|\lambda(t)| + t\log(t)|\dot\lambda(t)| \le \frac{C}{\sqrt{\log(t)}}, \quad t > \frac{t_0}{2}, \tag{7.5}$$

for some fixed constant C. Let us define

$$\|p\|_{\gamma,m} = \sup_{t\ge t_0/2} t^\gamma(\log t)^m|p(t)|. \tag{7.6}$$

In what follows we shall only deal with radial functions on $\mathbb{R}^2$ and sometimes we will consider them as radial functions on $\mathbb{R}^6$. For a fixed constant $c_0 > 0$ we let

$$\lambda^*(t) = \frac{c_0}{\sqrt{\log t}}. \tag{7.7}$$

The following expansion holds.

Lemma 7.1. *Assume that λ satisfies (7.5). Let $0 < \gamma < 2$, $m \in \mathbb{R}$ and suppose that $\|p\|_{\gamma,m} < \infty$. Then*

$$\int_{\mathbb{R}^2} \varphi[p,\lambda](x)dx = -4\pi \int_{t/2}^{t-\lambda(t)^2} \frac{p(s)}{t-s}ds + R[p,\lambda]$$

where $R[p,\lambda]$ satisfies

$$\|R[p,\lambda]\|_{\gamma,m} \le C\|p\|_{\gamma,m}.$$

If λ_1, λ_2 satisfy

$$\Big\|\frac{\lambda_j}{\lambda^*}\Big\|_{L^\infty(t_0/2,\infty)} < \frac{1}{2}, \quad j = 1,2,$$

then we also have

$$\|R[p,\lambda^*+\lambda_1] - R[p,\lambda^*+\lambda_2]\|_{\gamma,m} \le C\|p\|_{\gamma,m} \Big\|\frac{\lambda_1-\lambda_2}{\lambda^*}\Big\|_{L^\infty(t_0/2,\infty)}. \tag{7.8}$$

For the proof of the above result we will need the following calculation.

Lemma 7.2. *Let*

$$f(w) = \frac{1}{(4\pi)^3}\int_{\mathbb{R}^6} e^{-\frac{|z|^2}{4}} \frac{1}{|w-z|^4}dz, \quad w \in \mathbb{R}^6.$$

Then

$$f(w) = \frac{1}{|w|^4}\Big[1 - e^{\frac{-|w|^2}{4}}\Big(1 + \frac{|w|^2}{4}\Big)\Big]. \tag{7.9}$$

Proof. Let φ_0 be given by

$$\varphi_0(x,t) = \frac{1}{(4\pi)^3}\frac{1}{t^3}\int_{\mathbb{R}^6} e^{-\frac{|y|^2}{4t}}\frac{1}{|x-y|^4}dy, \quad x \in \mathbb{R}^6,\ t > 0,$$

which solves

$$\partial_t \varphi_0 = \Delta_{\mathbb{R}^6}\varphi_0 \quad \text{in } \mathbb{R}^6 \times (0,\infty)$$
$$\varphi_0(x,0) = \frac{1}{|x|^4}.$$

Then

$$f(w) = \varphi_0(w,1).$$

Write

$$\varphi_0(x,t) = \frac{1}{t^2}q\Big(\frac{|x|}{\sqrt{t}}\Big).$$

Then

$$q''(s) + \frac{5}{s}q'(s) + \frac{s}{2}q'(s) + 2q(s) = 0$$

and we want $q(s)$ bounded for $s \to 0$, $q(s) = s^{-4}(1+o(1))$ as $s \to \infty$. A calculation using the explicit element in the kernel of the linear operator, s^{-4}, gives

$$q(s) = \frac{1}{s^4}\Big[1 - e^{-\frac{s^2}{4}}\Big(1 + \frac{s^2}{4}\Big)\Big], \quad s > 0,$$

and then (7.9) follows.

□

Proof of Lemma 7.1. The solution $\varphi[p,\lambda]$ of (7.4) has the formula

$$\varphi[p,\lambda](x,t) = \frac{1}{(4\pi)^3}\int_{t_0/2}^t \frac{p(s)}{\lambda^4(s)}\frac{1}{(t-s)^3}\int_{\mathbb{R}^6} e^{-\frac{|x-y|^2}{4(t-s)}} Z_0\Big(\frac{y}{\lambda(s)}\Big)\chi\Big(\frac{y}{\sqrt{s}}\Big)dyds, \quad x \in \mathbb{R}^6.$$

Writing

$$\varphi = \varphi[p,\lambda]$$

we have

$$\begin{aligned}\int_{\mathbb{R}^2}\varphi(x,t)\,dx &= \frac{2}{\pi^2}\int_{\mathbb{R}^6}\varphi(x,t)|x|^{-4}dx\\ &= \frac{2}{\pi^2}\frac{1}{(4\pi)^3}\int_{t_0/2}^{t}\frac{p(s)}{\lambda^4(s)}\frac{1}{(t-s)^3}\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}e^{-\frac{|x-y|^2}{4(t-s)}}|x|^{-4}dxZ_0\Big(\frac{y}{\lambda(s)}\Big)\chi\Big(\frac{y}{\sqrt{s}}\Big)dyds\\ &= \frac{2}{\pi^2}\frac{1}{(4\pi)^3}\int_{t_0/2}^{t}\frac{p(s)}{\lambda(s)^4}\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}e^{-\frac{|z|^2}{4}}\frac{1}{|y-\sqrt{t-s}z|^4}dzZ_0\Big(\frac{y}{\lambda(s)}\Big)\chi\Big(\frac{y}{\sqrt{s}}\Big)dyds\end{aligned}$$

Using (7.9) we have

$$\begin{aligned}\int_{\mathbb{R}^2}\varphi(x,t)dx &= \frac{2}{\pi^2}\int_{t_0/2}^{t}\frac{p(s)}{\lambda(s)^4}\int_{\mathbb{R}^6}\frac{1}{(t-s)^2}f((t-s)^{-1/2}|y|)Z_0\Big(\frac{y}{\lambda(s)}\Big)\chi\Big(\frac{y}{\sqrt{s}}\Big)dyds\\ &= 2\pi\int_{t_0/2}^{t}\frac{p(s)}{\lambda(s)^4}\int_0^\infty\Big[1-e^{-\frac{r^2}{4(t-s)}}\Big(1+\frac{r^2}{4(t-s)}\Big)\Big]Z_0\Big(\frac{r}{\lambda(s)}\Big)\chi\Big(\frac{r}{\sqrt{s}}\Big)rdrds.\end{aligned}$$

Let us notice that

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbb{R}^2}\varphi(x,t)dx\\ &= \int_{t_0/2}^{t}\frac{p(s)(t-s)}{\lambda(s)^4}\int_0^\infty\Big[1-e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big]Z_0\Big(\frac{z\sqrt{t-s}}{\lambda(s)}\Big)\chi\Big(\frac{z\sqrt{t-s}}{\sqrt{s}}\Big)zdzds.\end{aligned}$$

We decompose

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\varphi(x,t)dx = I_1+I_2+I_3$$

where

$$\begin{aligned}I_1 &= \int_{t_0/2}^{t/2}\dots\\ I_2 &= \int_{t/2}^{t-\lambda(t)^2}\dots\\ I_3 &= \int_{t-\lambda(t)^2}^{t}\dots\end{aligned}$$

and separately estimate each term. To estimate I_1 we note that for $s\le t/2$ we have $\frac{s}{t-s}\le 1$. Assuming that $\chi(x)=0$ for $x\ge 2$ we obtain

$$\int_0^\infty\Big[1-e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big]Z_0\Big(\frac{z\sqrt{t-s}}{\lambda(s)}\Big)\chi\Big(\frac{z\sqrt{t-s}}{\sqrt{s}}\Big)zdz = \int_0^{2\frac{\sqrt{s}}{\sqrt{t-s}}}\dots$$

We estimate for $s\le t/2$,

$$\begin{aligned}&\Big|\int_0^{2\frac{\sqrt{s}}{\sqrt{t-s}}}\Big[1-e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big]Z_0\Big(\frac{z\sqrt{t-s}}{\lambda(s)}\Big)\chi\Big(\frac{z\sqrt{t-s}}{\sqrt{s}}\Big)zdz\Big|\\ &\le C\int_0^{2\frac{\sqrt{s}}{\sqrt{t-s}}}z^4\frac{\lambda(s)^4}{(t-s)^2z^4}zdz\\ &\le C\frac{\lambda(s)^4 s}{(t-s)^3},\end{aligned}$$

where we have used that $Z_0(\rho)\le C/\rho^4$ and $1-e^{-\frac{z^2}{4}}(1+\frac{z^2}{4})\le Cz^4$. Therefore

$$|I_1|\le\int_{t_0/2}^{t/2}\frac{|p(s)|s}{(t-s)^2}ds\le\|p\|_{\gamma,m}\int_0^{t/2}\frac{s^{1-\gamma}}{(t-s)^2(\log s)^m}ds\le\frac{C}{t^\gamma(\log t)^m}\|p\|_{\gamma,m}.$$

Let us analyze I_2. We write

$$I_2 = I_{2,*}+I_{2,a}+I_{2,b}+I_{2,c}+I_{2,d}$$

where

$$I_{2,*} = -16\int_{t/2}^{t-\lambda(t)^2}\frac{p(s)(t-s)}{\lambda(s)^4}\int_0^\infty\Big[1-e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big]\frac{\lambda(s)^4}{(t-s)^2z^4}zdzds$$

and

$$I_{2,a} = \int_{t/2}^{t-\lambda(t)^2} \frac{p(s)(t-s)}{\lambda(s)^4} \int_0^{\frac{\lambda(s)}{\sqrt{t-s}}} \Big[1 - e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big] Z_0\Big(\frac{z\sqrt{t-s}}{\lambda(s)}\Big)\chi\Big(\frac{z\sqrt{t-s}}{\sqrt{s}}\Big) z dz ds$$

$$I_{2,b} = 16\int_{t/2}^{t-\lambda(t)^2} \frac{p(s)(t-s)}{\lambda(s)^4} \int_0^{\frac{\lambda(s)}{\sqrt{t-s}}} \Big[1 - e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big] \frac{\lambda(s)^4}{(t-s)^2z^4}\chi\Big(\frac{z\sqrt{t-s}}{\sqrt{s}}\Big) z dz ds$$

$$I_{2,c} = \int_{t/2}^{t-\lambda(t)^2} \frac{p(s)(t-s)}{\lambda(s)^4} \int_{\frac{\lambda(s)}{\sqrt{t-s}}}^{\infty} \Big[1 - e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big] \Big[Z_0\Big(\frac{z\sqrt{t-s}}{\lambda(s)}\Big) + 16\frac{\lambda(s)^4}{(t-s)z^4}\Big] z dz ds$$

$$I_{2,d} = \int_{t/2}^{t-\lambda(t)^2} \frac{p(s)(t-s)}{\lambda(s)^4} \int_0^{\infty} \Big[1 - e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big] Z_0\Big(\frac{z\sqrt{t-s}}{\lambda(s)}\Big)\Big[\chi\Big(\frac{z\sqrt{t-s}}{\sqrt{s}}\Big) - 1\Big] z dz ds$$

A calculation gives that

$$I_{2,*} = -2\int_{t/2}^{t-\lambda(t)^2} \frac{p(s)}{t-s} ds. \tag{7.10}$$

Next we find a bound for $I_{2,a}$. Using that Z_0 is a bounded function and $|1-e^{-\frac{z^2}{4}}(1+\frac{z^2}{4})| \le Cz^4$, we get

$$\begin{aligned} &\Big|\int_0^{\frac{\lambda(s)}{\sqrt{t-s}}} \Big[1 - e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big] Z_0\Big(\frac{z\sqrt{t-s}}{\lambda(s)}\Big)\chi\Big(\frac{z\sqrt{t-s}}{\sqrt{s}}\Big) z dz\Big| \\ &\quad \le C\int_0^{\frac{\lambda(s)}{\sqrt{t-s}}} z^5 dz \le C\frac{\lambda(s)^6}{(t-s)^3}. \end{aligned}$$

It follows that

$$\begin{aligned} |I_{2,a}| &\le C\int_{t/2}^{t-\lambda(t)^2} \frac{|p(s)|\lambda(s)^2}{(t-s)^2} ds \le \frac{C}{t^\gamma(\log t)^m}\|p\|_{\gamma,m}\int_{t/2}^{t-\lambda(t)^2}\frac{\lambda(s)^2}{(t-s)^2}ds \\ &\le \frac{C}{t^\gamma(\log t)^m}\|p\|_{\gamma,m}. \end{aligned}$$

Using that $|1-e^{-\frac{z^2}{4}}(1+\frac{z^2}{4})| \le Cz^4$, we get

$$\begin{aligned} \Big|\int_0^{\frac{\lambda(s)}{\sqrt{t-s}}} \Big[1 - e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big] \frac{\lambda(s)^4}{(t-s)^2z^4}\chi\Big(\frac{z\sqrt{t-s}}{\sqrt{s}}\Big) z dz\Big| C &\le \frac{\lambda(s)^4}{(t-s)^2}\int_0^{\frac{\lambda(s)}{\sqrt{t-s}}} z dz \\ &\le C\frac{\lambda(s)^6}{(t-s)^3}, \end{aligned}$$

and similarly as before,

$$|I_{2,b}| \le \frac{C}{t^\gamma(\log t)^m}\|p\|_{\gamma,m}.$$

Using that

$$Z_0\Big(\frac{z\sqrt{t-s}}{\lambda(s)}\Big) = -16\frac{\lambda(s)^4}{(t-s)^2z^4} + O\Big(\frac{\lambda(s)^6}{(t-s)^3z^6}\Big), \quad \frac{z\sqrt{t-s}}{\lambda(s)} \ge 1,$$

we get

$$\begin{aligned} |I_{2,c}| &\le C\int_{t/2}^{t-\lambda(t)^2} \frac{p(s)(t-s)}{\lambda(s)^4} \int_{\frac{\lambda(s)}{\sqrt{t-s}}}^{\infty} \Big[1 - e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big] \frac{\lambda(s)^6}{(t-s)^3z^6}\chi\Big(\frac{z\sqrt{t-s}}{\sqrt{s}}\Big) z dz \\ &\le C\int_{t/2}^{t-\lambda(t)^2} \frac{p(s)\lambda(s)^2}{(t-s)^2} \int_{\frac{\lambda(s)}{\sqrt{t-s}}}^{\infty} \Big[1 - e^{-\frac{z^2}{4}}\Big(1+\frac{z^2}{4}\Big)\Big] \frac{1}{z^5} dz. \end{aligned}$$

But $\frac{\lambda(s)}{\sqrt{t-s}} \le 2$ in the considered range of s, and then

$$
\begin{aligned}
|I_{2,c}| &\le C \int_{t/2}^{t-\lambda(t)^2} \frac{|p(s)|\lambda(s)^2}{(t-s)^2} \log\Big(\frac{\lambda(s)^2}{t-s}\Big) ds \\
&\le \frac{C}{t^\gamma (\log t)^m} \|p\|_{\gamma,m} \int_{t/2}^{t-\lambda(t)^2} \frac{\lambda(s)^2}{(t-s)^2} \log\Big(\frac{\lambda(s)^2}{t-s}\Big) ds \\
&\le \frac{C}{t^\gamma (\log t)^m} \|p\|_{\gamma,m}.
\end{aligned}
$$

Finally, for $I_{2,d}$,

$$
\begin{aligned}
&\Big| \int_0^\infty \Big[1 - e^{-\frac{z^2}{4}} \Big(1 + \frac{z^2}{4}\Big)\Big] Z_0\Big(\frac{z\sqrt{t-s}}{\lambda(s)}\Big) \Big[\chi\Big(\frac{z\sqrt{t-s}}{\sqrt{s}}\Big) - 1\Big] z dz \Big| \\
&\le \int_{2\sqrt{s}/\sqrt{t-s}}^\infty \Big[1 - e^{-\frac{z^2}{4}} \Big(1 + \frac{z^2}{4}\Big)\Big] \frac{\lambda(s)^4}{(t-s)^2 z^4} z dz \\
&\le \frac{\lambda(s)^4}{(t-s)^2} \int_{2\sqrt{s}/\sqrt{t-s}}^\infty z^{-3} dz \\
&\le C \frac{\lambda(s)^4}{(t-s)s}.
\end{aligned}
$$

Then

$$
|I_{2,d}| \le C \int_{t/2}^{t-\lambda(t)^2} \frac{|p(s)|(t-s)}{\lambda(s)^4} \frac{\lambda(s)^4}{(t-s)s} ds \le \frac{C}{t^\gamma (\log t)^m} \|p\|_{\gamma,m}.
$$

Finally we estimate

$$
\begin{aligned}
|I_3| &= \Big| \int_{t-\lambda(t)^2}^t \frac{p(s)}{\lambda(s)^2} \int_0^\infty \Big[1 - e^{-\frac{\rho^2\lambda^2}{4(t-s)}} \Big(1 + \frac{\rho^2 \lambda^2}{4(t-s)}\Big)\Big] Z_0(\rho) \chi\Big(\frac{\lambda\rho}{\sqrt{s}}\Big) \rho d\rho ds \Big| \\
&\le C \int_{t-\lambda(t)^2}^t \frac{|p(s)|}{\lambda(s)^2} ds \\
&\le \frac{C}{t^\gamma (\log t)^m} \|p\|_{\gamma,m}.
\end{aligned}
$$

In summary, by (7.10) we have written

$$
\frac{1}{2\pi} \int_{\mathbb{R}^2} \varphi(x,t) dx = -2 \int_{t/2}^{t-\lambda(t)^2} \frac{p(s)}{t-s} ds + I_1 + I_{2,a} + I_{2,b} + I_{2,c} + I_{2,d} + I_3,
$$

and each of the expressions I_1, $I_{2,a}$, $I_{2,b}$, $I_{2,c}$, $I_{2,d}$, I_3 are linear operators of p with the estimate

$$
\|I_j[p]\|_{\gamma,m} \le C \|p\|_{\gamma,m}.
$$

The proof of (7.8) follows from the explicit expressions for the terms I_j in R, and similar estimates as before.

□

Lemma 7.3. *Suppose that λ satisfies (7.5) and $\varphi_\lambda^{(2)}$ be given by (7.3). Then*

$$
\varphi_\lambda^{(2)}(0,t;\lambda) = -\frac{\lambda(t)^2}{4t^2} + O\Big(\frac{1}{t^2(\log t)^2}\Big), \tag{7.11}
$$

as $t \to \infty$, where $O(\frac{1}{t^2(\log t)^2})$ is uniform in t_0. With λ^* given by (7.7), if λ_1, λ_2 satisfy

$$
\Big\| \frac{\lambda_j}{\lambda^*} \Big\|_{L^\infty(t_0/2,\infty)} < \frac{1}{2}, \quad j = 1, 2,
$$

then we also have

$$
|\varphi^{(2)}_{\lambda^*+\lambda_1}(0,t) - \varphi^{(2)}_{\lambda^*+\lambda_2}(0,t)| \le \frac{C}{t^2 \log t} \Big\| \frac{\lambda_1 - \lambda_2}{\lambda^*} \Big\|_{L^\infty(t_0/2,\infty)}. \tag{7.12}
$$

Proof. For simplicity of notation let us write $\varphi(x,t;\lambda) = \varphi_\lambda^{(2)}(x,t)$. Let us write the right hand side of equation (7.3) in the following form

$$E_2(x,t;\lambda) = -\frac{1}{2\lambda^2 t}U(y)\nabla_z z_0(z)\cdot z + \frac{2}{\lambda^3 t^{1/2}}\nabla_z\chi_0(z)\cdot\nabla_y U(y) + \frac{1}{\lambda^2 t}\Delta_z\chi_0(z)U(y) - \frac{1}{\lambda^3 t^{1/2}}U(y)\nabla_z\chi_0(z)\cdot\nabla_y\Gamma_0(y), \quad y = \frac{x}{\lambda},\ z = \frac{x}{\sqrt{t}}.$$

To compute $\varphi(0,t;\lambda)$ let us define the following approximation of it

$$\hat\varphi(r,t) = \lambda^2\tilde\varphi(r,t),$$

where $\tilde\varphi(r,t)$ solves the radial heat equation in dimension 6:

$$\begin{cases} \partial_t\tilde\varphi = \partial_r^2\tilde\varphi + \frac{5}{r}\partial_r\tilde\varphi + \frac{1}{t^3}h\Big(\frac{r}{\sqrt{t}}\Big), \\ \tilde\varphi(r,0) = 0, \end{cases} \tag{7.13}$$

and

$$h(\zeta) = \frac{8}{\zeta^4}\left[\chi_0'' - \frac{3}{\zeta}\chi_0'(\zeta) + \frac{\zeta}{2}\chi_0'(\zeta)\right].$$

The solution $\tilde\varphi(r,t)$ to problem (7.13) can be expressed in self-similar form as

$$\tilde\varphi(r,t) = \frac{1}{t^2}g(\zeta), \quad \zeta = \frac{r}{\sqrt{t}}.$$

We find for g the equation

$$g'' + \frac{5}{\zeta}g' + \frac{\zeta}{2}g' + 2g + h(\zeta) = 0, \quad \zeta\in(0,\infty). \tag{7.14}$$

Using that the function $\frac{1}{\zeta^4}$ is in the kernel of the homogeneous equation, we find the explicit solution of (7.14),

$$g_0(\zeta) = -\frac{1}{\zeta^4}\int_0^\zeta x^3 e^{-\frac14 x^2}\int_0^x h(y)e^{\frac14 y^2}y\,dydx.$$

To find the solution $\tilde\varphi$ with suitable decay at infinity we let

$$g(\zeta) = g_0(\zeta) + \frac18\bar z(\zeta)I, \tag{7.15}$$

where

$$\bar z(\zeta) = \frac{1}{\zeta^4}\int_0^\zeta x^3 e^{-\frac14 x^2}\,dx$$

is a second solution of the homogeneous equation, linearly independent of $\frac{1}{\zeta^4}$ and

$$I = \int_0^\infty x^3 e^{-\frac14 x^2}\int_0^x h(y)e^{\frac14 y^2}y\,dydx.$$

We observe that

$$g(\zeta) = O(e^{-\frac14\zeta^2}) \quad \text{as } \zeta\to+\infty,$$

which makes the solution (7.15) the only one with decay faster than $O(\zeta^{-4})$ as $\zeta\to+\infty$. An explicit calculation gives that $I=-8$, and therefore

$$\hat\varphi(0,t) = -\frac{\lambda(t)^2}{4t^2}. \tag{7.16}$$

Then, using a barrier for the equation satisfied by $\varphi(x,t;\lambda) - \hat\varphi(x,t)$ we get

$$|\varphi(x,t;\lambda) - \hat\varphi(x,t)| \le C\frac{1}{t^2(\log t)^2}e^{-c\frac{|x|^2}{t}}, \tag{7.17}$$

for $t\ge 2$, where $0<c<\frac14$. From (7.16) and (7.17) we obtain (7.11).

The proof of (7.12) is similar. □

Lemma 7.4. *Suppose that λ satisfies (7.5) and $\varphi_\lambda^{(2)}$ be given by (7.3). Then*

$$\int_{\mathbb{R}^2}\varphi_\lambda^{(2)} = -2\pi\frac{\lambda^2}{t} - 16\pi\Upsilon\frac{\lambda^2}{t} + O\Big(\frac{1}{t^2(\log t)^2}\Big). \tag{7.18}$$

where Υ is defined in (2.7), that, is, $\Upsilon = \int_0^\infty(\chi_0(s)-1)s^{-3}ds$.

Proof. Integrating (7.3)

$$\frac{d}{dt}\int_{\mathbb{R}^2}\varphi_\lambda^{(2)} = -4\varphi_\lambda^{(2)}(0,t) - \frac{1}{2\lambda^2 t}\int_{\mathbb{R}^2} U(y)\nabla_z\chi_0(z)\cdot z dx + \int_{\mathbb{R}^2}\tilde{E}dx.$$

From (7.11)

$$\varphi_\lambda^{(2)}(0,t) = -\frac{\lambda(t)^2}{4t^2} + O\Big(\frac{1}{t^2(\log t)^2}\Big)$$

and we compute

$$\begin{aligned}
&-\frac{1}{2\lambda^2 t}U(y)\nabla_z\chi_0(z)\cdot z + \tilde{E}\\
&= -\frac{1}{\lambda^2}U(y)\nabla_z\chi_0(z)\cdot z + \frac{2}{\lambda^2}\nabla_x\chi\cdot\nabla_x U + \frac{1}{\lambda^2}\Delta_x\chi U - \frac{1}{\lambda^2}U\nabla\chi\cdot\nabla\Gamma_0\\
&= \Big[4\frac{\lambda^2}{t^3}\chi_0'(s)\frac{1}{s^3} - 64\frac{\lambda^2}{t^3}\chi_0'(s)\frac{1}{s^5} + 8\frac{\lambda^2}{t^3}(\chi_0''(s) + \frac{1}{s}\chi_0'(s))\frac{1}{s^4}\\
&\quad + 32\frac{\lambda^2}{t^3}\chi_0'(s)\frac{1}{s^5}\Big] + O\Big(\frac{\lambda^4}{t^4}\Big)\chi_{\{1\le s\le 2\}}\\
&= 8\frac{\lambda^2}{t^3}\frac{1}{s^4}\Big[\frac{s}{2}\chi_0'(s) - \frac{3}{s}\chi_0'(s) + \chi_0''(s)\Big] + O\Big(\frac{\lambda^4}{t^4}\Big)\chi_{\{1\le s\le 2\}}
\end{aligned}$$

where $s = \frac{r}{\sqrt{t}}$. Then

$$\begin{aligned}
&-\frac{1}{2\lambda^2 t}\int_{\mathbb{R}^2}U(y)\nabla_z\chi_0(z)\cdot z dx + \int_{\mathbb{R}^2}\tilde{E}dx\\
&= 2\pi\frac{8\lambda^2}{t^2}\int_0^\infty \frac{1}{s^4}\Big[\frac{s}{2}\chi_0'(s) - \frac{3}{s}\chi_0'(s) + \chi_0''(s)\Big]s ds + O\Big(\frac{\lambda^4}{t^3}\Big)\\
&= 16\pi\frac{\lambda^2}{t^2}\Big[\int_0^\infty(\chi_0(s)-1)s^{-3}ds + \int_0^\infty (s^{-3}\chi_0')'ds\Big] + O\Big(\frac{\lambda^4}{t^3}\Big)\\
&= 16\pi\frac{\lambda^2}{t^2}\Upsilon + O\Big(\frac{\lambda^4}{t^3}\Big).
\end{aligned}$$

Therefore

$$\frac{d}{dt}\int_{\mathbb{R}^2}\varphi_\lambda^{(2)} = 2\pi\frac{\lambda(t)^2}{t^2} + 16\pi\Upsilon\frac{\lambda^2}{t^2} + O\Big(\frac{1}{t^3(\log t)^2}\Big)$$

and integrating we get

$$\int_{\mathbb{R}^2}\varphi_\lambda^{(2)} = -2\pi\frac{\lambda^2}{t} - 16\pi\Upsilon\frac{\lambda^2}{t} + O\Big(\frac{1}{t^2(\log t)^2}\Big).$$

This is the desired expansion (7.18). □

As a corollary from Lemma 7.1 and Lemma 7.4 we get:

Corollary 7.1. *Assume λ satisfies (7.5). Then*

$$\int_{\mathbb{R}^2}\varphi_\lambda dx = -4\pi\int_{t/2}^{t-\lambda(t)^2}\frac{\lambda\dot{\lambda}(s)}{t-s}ds - 2\pi\frac{\lambda^2(t)}{t} - 16\pi\Upsilon\frac{\lambda^2(t)}{t} + O\Big(\frac{1}{t^2(\log t)^2}\Big) + R[\lambda\dot{\lambda},\lambda],$$

where R is as in Lemma 7.1.

Lemma 7.5. *Let $\tilde{E}$ be defined by (3.11). Assume that λ satisfies (7.5). Then*

$$\int_{\mathbb{R}^2}\tilde{E}|x|^2dx = -64\pi\Upsilon\frac{\lambda^2}{t} + O\Big(\frac{1}{t^2(\log t)^2}\Big). \tag{7.19}$$

Proof. Similarly to the proof of Lemma 7.4 we have

$$\tilde{E} = 8\frac{\lambda^2}{t^3}\frac{1}{s^4}\Big[-\frac{3}{s}\chi_0'(s) + \chi_0''(s)\Big] + O\Big(\frac{\lambda^4}{t^4}\Big)\chi_{\{1\le s\le 2\}},$$

where $r = |x|$, $s = \frac{r}{\sqrt{t}}$, and so

$$\begin{aligned}
\int_{\mathbb{R}^2}\tilde{E}|x|^2dx &= 16\pi\frac{\lambda^2}{t}\int_0^\infty\frac{1}{s^4}\Big[-\frac{3}{s}\chi_0'(s) + \chi_0''(s)\Big]s^3ds + O\Big(\frac{\lambda^4}{t^2}\Big)\\
&= -64\pi\frac{\lambda^2}{t}\Upsilon + O\Big(\frac{\lambda^4}{t^2}\Big).
\end{aligned}$$

This is (7.19). □

Lemma 7.6. *Let E be defined by (3.10). Assume that λ satisfies (7.5). Then*

$$\left| \int_{\mathbb{R}^2} E|x|^2 dx \right| \leq \frac{C}{t\log(t)}.$$

Proof. We have from (3.10)

$$E(\zeta,t;\lambda) = \frac{\dot{\lambda}}{\lambda^3} Z_0\Big(\frac{\zeta}{\lambda}\Big)\chi_0\Big(\frac{\zeta}{\sqrt{t}}\Big) + \frac{1}{2\lambda^2 t} U\Big(\frac{\zeta}{\lambda}\Big)\nabla_z\chi_0(z)\cdot z + \tilde{E}(x,t),$$

and we have already computed $\int_{\mathbb{R}^2} \tilde{E}|x|^2 dx$ in (7.19). We have

$$\begin{aligned}\int_{\mathbb{R}^2} Z_0\Big(\frac{\zeta}{\lambda}\Big)\chi_0\Big(\frac{\zeta}{\sqrt{t}}\Big)|\zeta|^2\, d\zeta &= 2\pi\lambda^4 \int_0^\infty Z_0(\rho)\chi_0\Big(\frac{\lambda\rho}{\sqrt{t}}\Big)\rho^3\, d\rho \\ &= O(\lambda^4\log(t)),\end{aligned}$$

and so

$$\left| \frac{\dot{\lambda}}{\lambda^3}\int_{\mathbb{R}^2} Z_0\Big(\frac{\zeta}{\lambda}\Big)\chi_0\Big(\frac{\zeta}{\sqrt{t}}\Big)|\zeta|^2\, d\zeta \right| \leq \frac{C}{t\log t}.$$

□

7.1. **Proof of Proposition 5.1.** Let

$$I[\lambda] = 4\int_{\mathbb{R}^2} \varphi_\lambda dx - \int_{\mathbb{R}^2} \tilde{E}(\lambda)|x|^2 dx.$$

For the proof we proceed by linearization, that is we look for a function λ_0 satisfying

$$|I[\lambda_0](t)| \leq C\frac{1}{t^{\frac{3}{2}+\sigma}}, \quad t > t_0$$

with the expansion

$$\lambda_0(t) = \lambda^*(t) + \tilde{\lambda}_0(t)$$

where λ^* was defined in (7.7), that is, $\lambda^*(t) = \frac{c_0}{\sqrt{\log t}}$ and $\tilde{\lambda}_0(t)$, $t > \frac{t_0}{2}$, is a correction. Here $c_0 > 0$ is a fixed constant.

We claim that

$$|I[\lambda^*](t)| \leq C\frac{\log(\log t)}{t(\log t)^2}, \quad t > \frac{t_0}{2}, \tag{7.20}$$

with C independent of t_0. In the rest of the proof C will be a constant independent of t_0 (for t_0 large).

Indeed, using the decomposition (7.1) and the notation (7.4) we have

$$\int_{\mathbb{R}^2} \varphi_{\lambda^*} dx = \int_{\mathbb{R}^2} \varphi^{(1)}_{\lambda^*} dx + \int_{\mathbb{R}^2} \varphi^{(2)}_{\lambda^*} dx$$

and

$$\int_{\mathbb{R}^2} \varphi^{(1)}_{\lambda^*} dx = \int_{\mathbb{R}^2} \varphi[p^*,\lambda^*] dx, \quad p^* = \lambda^*\dot{\lambda}^*.$$

By Lemma 7.1 we have

$$\left| \int_{\mathbb{R}^2} \varphi[p^*,\lambda^*]dx + 4\pi\int_{t/2}^{t-\lambda^*(t)^2} \frac{p^*(s)}{t-s} ds \right| \leq C\frac{1}{t(\log t)^2}, \quad t > \frac{t_0}{2}.$$

Therefore

$$\left| \int_{\mathbb{R}^2} \varphi[p^*,\lambda^*]dx + 4\pi\log(t)p^*(t) \right| \leq C\frac{\log(\log t)}{t(\log t)^2}, \quad t > \frac{t_0}{2}.$$

On the other hand, by Lemma 7.4 we have

$$\int_{\mathbb{R}^2} \varphi^{(2)}_{\lambda^*} dx = -2\pi\frac{\lambda^*(t)^2}{t} - 16\pi\Upsilon\frac{\lambda^*(t)^2}{t} + O\Big(\frac{1}{t(\log t)^2}\Big),$$

and by Lemma 7.5

$$\int_{\mathbb{R}^2} \tilde{E}(\lambda^*)|x|^2 dx = -64\pi\Upsilon\frac{\lambda^*(t)^2}{t} + O\Big(\frac{1}{t^2(\log t)^2}\Big).$$

Using the explicit form of λ^* and the previous formulas we deduce (7.20).

Next let us rewrite slightly the operator $I[\lambda]$ as follows. We have

$$I[\lambda] = 4\int_{\mathbb{R}^2} \varphi[\lambda\dot\lambda,\lambda]dx + 4\int_{\mathbb{R}^2} \varphi^{(2)}_\lambda dx - \int_{\mathbb{R}^2} \tilde E(\lambda)|x|^2 dx.$$

Let us define

$$R[p,\lambda] = \int_{\mathbb{R}^2} \varphi[p,\lambda]dx + 4\pi \int_{t/2}^{t-\lambda^*(t)^2} \frac{p(s)}{t-s} ds.$$

This is similar to the decomposition given in Lemma 7.1, but we have changed the interval of integration to $[\frac{t}{2}, t-\lambda^*(t)^2]$. We decompose the integral

$$\begin{aligned}\int_{t/2}^{t-\lambda^*(t)^2} \frac{p(s)}{t-s} ds &= \int_{t/2}^{t-t^{1-\vartheta}} \frac{p(s)}{t-s} ds + \int_{t-t^{1-\vartheta}}^{t-\lambda^*(t)^2} \frac{p(s)}{t-s} ds \\ &= \int_{t/2}^{t-t^{1-\vartheta}} \frac{p(s)}{t-s} ds + p(t)\int_{t-t^{1-\vartheta}}^{t-\lambda^*(t)^2} \frac{1}{t-s} ds \\ &\quad - \int_{t-t^{1-\vartheta}}^{t-\lambda^*(t)^2} \frac{p(t)-p(s)}{t-s} ds\end{aligned}$$

where $0<\vartheta<\frac12$ is a fixed constant.

We change variables $\mu=\lambda^2$, so that

$$\begin{aligned}I[\lambda] &= -8\pi\dot\mu(t)((1-\vartheta)\log(t) - 2\log(\lambda^*(t))) - 8\pi \int_{t/2}^{t-t^{1-\vartheta}} \frac{\dot\mu(s)}{t-s} ds \\ &\quad + 4\int_{\mathbb{R}^2} \varphi^{(2)}_{\sqrt{\mu}} dx + 2R[\dot\mu,\sqrt{\mu}] - \int_{\mathbb{R}^2} \tilde E(\sqrt{\mu})|x|^2 dx \\ &\quad + 8\pi \int_{t-t^{1-\vartheta}}^{t-\lambda^*(t)^2} \frac{\dot\mu(t)-\dot\mu(s)}{t-s} ds.\end{aligned}$$

Let η be a smooth cut-off such that $\eta(t)=0$ for $t<\frac34 t_0$, $\eta(t)=1$ for $t>t_0$. We define

$$\begin{aligned}\tilde I[\mu] &= -8\pi\dot\mu(t)((1-\vartheta)\log(t) - 2\log(\lambda^*(t))) - 8\pi\eta(t) \int_{t/2}^{t-t^{1-\vartheta}} \frac{\dot\mu(s)}{t-s} ds \\ &\quad + 4\eta(t)\int_{\mathbb{R}^2} \varphi^{(2)}_{\sqrt{\mu}} dx + 2\eta(t)R[\dot\mu,\sqrt{\mu}] - \eta(t)\int_{\mathbb{R}^2} \tilde E(\sqrt{\mu})|x|^2 dx \\ &\quad + 8\pi\eta(t) \int_{t-t^{1-\vartheta}}^{t-\lambda^*(t)^2} \frac{\dot\mu(t)-\dot\mu(s)}{t-s} ds\end{aligned}$$

which we write

$$\tilde I[\mu] = \ell[\mu] + N[\mu] + R[\mu],$$

where

$$\begin{aligned}\ell[\mu](t) &= -8\pi\dot\mu(t)((1-\vartheta)\log(t) - 2\log(\lambda^*(t))) - 8\pi\eta(t) \int_{t/2}^{t-t^{1-\vartheta}} \frac{\dot\mu(s)}{t-s} ds \\ N[\mu](t) &= 4\eta(t)\int_{\mathbb{R}^2} \varphi^{(2)}_{\sqrt{\mu}} dx + 2\eta(t)R[\dot\mu,\sqrt{\mu}] - \eta(t)\int_{\mathbb{R}^2} \tilde E(\sqrt{\mu})|x|^2 dx \\ R[\mu](t) &= 8\pi\eta(t) \int_{t-t^{1-\vartheta}}^{t-\lambda^*(t)^2} \frac{\dot\mu(t)-\dot\mu(s)}{t-s} ds.\end{aligned}$$

Note that $I[\lambda](t) = \tilde I[\lambda^2](t)$ for $t\ge t_0$.

Instead of finding λ such that $I[\lambda]=0$ for $t>t_0$ we are going to construct μ such that

$$|\tilde I[\mu](t)| \le \frac{C}{t^{\frac32+\sigma}}, \quad t>\frac{t_0}{2},$$

for some $\sigma>0$.

Let $\mu^* = (\lambda^*)^2$ where λ^* is defined in (7.7). In a first step we will find μ_1 so that

$$\ell[\mu^*+\mu_1] + N[\mu^*+\mu_1] + R[\mu^*] = 0, \quad t>\frac{t_0}{2}. \tag{7.21}$$

We will look for μ_1 with $\|\mu_1\|_{*,\gamma,m} < \infty$ where, for a function $\mu_1 \in C^1([\frac{t_0}{2}, \infty))$ with $\lim_{t\to\infty} \mu_1(t) = 0$ we define

$$\|\mu_1\|_{*,\gamma,m} = \sup_{t\geq t_0/2} t^\gamma (\log t)^m |\dot\mu_1(t)| = \|\dot\mu_1\|_{\gamma,m}.$$

Equation (7.21) takes the form

$$\begin{aligned} 0 = -8\pi\dot\mu_1((1-\vartheta)\log(t) - 2\log(\lambda^*(t))) - 8\pi\eta(t)\int_{t/2}^{t-t^{1-\vartheta}} \frac{\dot\mu_1(s)}{t-s} ds \\ + \eta(t)e_1(t) + \eta(t)F_1[\mu_1](t), \quad t > \frac{t_0}{2}, \end{aligned} \tag{7.22}$$

where

$$e_1(t) = \tilde I[\mu^*]$$

and F_1 is an operator with the following properties:

$$\|F_1[\tilde\mu_1]\|_{\gamma,m} \leq C\|\tilde\mu_1\|_{*,\gamma,m}, \tag{7.23}$$

$$\|F_1[\tilde\mu_1] - F_1[\tilde\mu_2]\|_{\gamma,m} \leq C\|\tilde\mu_1 - \tilde\mu_2\|_{*,\gamma,m}, \tag{7.24}$$

for $\tilde\mu_j$ satisfying $\|\tilde\mu_j\|_{*,\gamma,m} \leq 1$, with $0<\gamma<2$, $m \in \mathbb{R}$, where $\|\ \|_{\gamma,m}$ is defined in (7.6). From (7.20) we find

$$|e_1(t)| \leq C\frac{\log(\log t)}{t(\log t)^2}, \quad t > \frac{t_0}{2}.$$

Now we apply the contraction mapping principle to the equation (7.22) written in the form

$$\dot\mu_1 = -\eta(t)I_r[\dot\mu_1] + \frac{1}{8\pi((1-\vartheta)\log(t) - 2\log(\lambda^*(t)))}\eta(t)\big[e_1(t) + F[\mu_1](t)\big], \ t > \frac{t_0}{2}, \tag{7.25}$$

where

$$I_r[\dot\mu_1] = \frac{1}{((1-\vartheta)\log(t) - 2\log(\lambda^*(t))}\int_{t/2}^{t-t^{1-\vartheta}} \frac{\dot\mu_1(s)}{t-s} ds.$$

We directly check that

$$\|I_r[\dot\mu]\|_{\gamma,m} \leq \frac{\vartheta}{1-\vartheta}\|\dot\mu_1\|_{\gamma,m}.$$

Let X be the space $X = \{\mu_1 \in C^1([\frac{t_0}{2},\infty)) \,|\, \lim_{t\to\infty}\mu_1(t) = 0\}$ with the norm $\|\mu_1\|_X = \|\mu_1\|_{*,1,3-\varepsilon}$, where $0<\varepsilon<1$. It follows that if $\vartheta < \frac12$ the equation (7.25) has a unique solution μ_1 in the ball $\overline{B}_1(0)$ of X.

Therefore we have found μ_1 with $\|\mu_1\|_{*,1,3-\varepsilon} \leq 1$ so that $\mu = \mu^* + \mu_1$ satisfies

$$\tilde I[\mu] = -8\pi\eta(t)\int_{t-t^{1-\vartheta}}^{t-\lambda^*(t)^2} \frac{\dot\mu_1(t) - \dot\mu_1(s)}{t-s} ds. \tag{7.26}$$

To estimate this remainder we then need a bound for $\ddot\mu$. Differentiating with respect to t in the decompositions used in Lemmas 7.1, 7.4, 7.5 we obtain

$$|\dot e_1(t)| \leq C\frac{\log(\log t)}{t^2(\log t)^2}, \quad t > \frac{t_0}{2}.$$

Differentiating in t equation (7.25) and using the contraction mapping principle we get that for any $\varepsilon > 0$ small

$$|\ddot\mu_1(t)| \leq \frac{C}{t^{2-\varepsilon}}.$$

Using this we find that the remainder (7.26) has the estimate

$$\left| \int_{t-t^{1-\vartheta}}^{t-\lambda^*(t)^2} \frac{\dot\mu(t) - \dot\mu(s)}{t-s} ds \right| \leq \frac{C}{t^{1+\vartheta-\varepsilon}}, \quad t > \frac{t_0}{2},$$

where $\mu = \mu^* + \mu_1$.

Next we introduce another correction μ_2 to improve the decay of the remainder. We consider $\mu = \mu^* + \mu_1 + \mu_2$ and we consider the following equation for μ_2:

$$\ell[\mu^* + \mu_1 + \mu_2] + N[\mu^* + \mu_1 + \mu_2] + R[\mu^* + \mu_1] = 0, \quad t > \frac{t_0}{2}.$$

Similarly as before, this equation can be written as

$$0 = -8\pi\dot\mu_2((1-\vartheta)\log(t) - 2\log(\lambda^*(t))) - 8\pi\eta(t)\int_{t/2}^{t-t^{1-\vartheta}} \frac{\dot\mu_2(s)}{t-s}ds$$
$$+ \eta(t)e_2(t) + \eta(t)F_2[\mu_2](t), \quad t > \frac{t_0}{2}, \tag{7.27}$$

where F_2 satisfies the same estimate (7.23) (7.24), and e_2 has the estimate

$$|e_2(t)| \le \frac{C}{t^{1+\vartheta-\varepsilon}}, \quad t > \frac{t_0}{2}.$$

Using again the contraction mapping principle we find a solution μ_2 of (7.27) with $\|\mu_2\|_{*,1+\vartheta-\varepsilon,1} \le 1$. Then for $\mu = \mu^* + \mu_1 + \mu_2$

$$\tilde I[\mu](t) = -8\pi\eta(t)\int_{t-t^{1-\vartheta}}^{t-\lambda^*(t)^2} \frac{\dot\mu_2(t)-\dot\mu_2(s)}{t-s}ds.$$

To estimate this remainder we need the following bound for $\ddot\mu_2$

$$|\ddot\mu_2(t)| \le \frac{C}{t^{2+\vartheta-\varepsilon}} \tag{7.28}$$

which is obtained from an estimate for $\dot e_2$, differentiating with respect to t equation (7.27). The estimate for $\dot e_2$ is obtained from an analogous estimate for $\frac{d^3\mu_1}{dt^3}$.

From (7.28) we find

$$|\tilde I[\mu](t)| \le \frac{C}{t^{1+2\vartheta-\varepsilon}} \quad t > \frac{t_0}{2},$$

where we recall that $0 < \vartheta < \frac12$ is arbitrary.

Thus letting $\lambda_0 = \sqrt{\mu}$, $\mu = \mu^* + \mu_1 + \mu_2$ we obtain

$$|I[\lambda_0]| \le \frac{C}{t^{1+2\vartheta-\varepsilon}} \quad t > t_0.$$

Choosing $\vartheta > \frac14$ and $\varepsilon > 0$ small, we obtain the properties stated in Proposition 5.1.

□

8. Inner linear theory

In this section we consider the problem

$$\begin{cases} \lambda^2\partial_t\phi = L[\phi] + B[\phi] + h(y,t) + \sum_{j=1}^{2}\mu_j(t)W_{1,j} & \text{in } \mathbb{R}^2\times(t_0,\infty) \\ \phi(\cdot,t_0) = 0 \quad \text{in } \mathbb{R}^2. \end{cases} \tag{8.1}$$

that appears in the inner equations (5.48) and (5.49), where, we recall

$$L[\phi] = \nabla\cdot\Big[U\nabla\Big(\frac{\phi}{U} - (-\Delta)^{-1}\phi\Big)\Big],$$
$$(-\Delta)^{-1}\phi(y,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\Big(\frac{1}{|y-z|}\Big)\phi(z,t)dz. \tag{8.2}$$

Slightly more general than the operator B defined in (5.47) we will consider

$$B[\phi] = \zeta_1(t)[\phi]_{rad} + \zeta_2(t)y\cdot\nabla[\phi]_{rad} + (\zeta_1(t)\phi_1 + \zeta_2(t)y\cdot\nabla\phi_1)\chi_0\Big(\frac{\lambda y}{5\sqrt t}\Big)$$

where $[\phi]_{rad}$ is the radial part of ϕ (defined in (5.46)) and $\phi_1 = \phi - [\phi]_{rad}$, and where χ_0 is the smooth cut-off function defined in (2.5). In the sequel we will keep the same notation for B.

In what follows we will analyze the linear initial value problem (8.1) where we assume that the functions $\lambda(t)$, $\zeta_i(t)$ are continuous, $t_0 > 1$ and that for some positive numbers c, C we have

$$\frac{c}{\sqrt{\log t}} \le \lambda(t) \le \frac{C}{\sqrt{\log t}} \quad \text{for all } t > t_0,$$

$$|\zeta_i(t)| \le \frac{C}{t\log^2 t} \quad \text{for all } t > t_0.$$

We change the time variable into

$$\tau = \tau_0 + \int_{t_0}^{t} \frac{1}{\lambda(s)^2} ds,$$

where $\tau_0 = t_0 \log t_0$. Then

$$\tilde c_1 t \log t \ \le\ \tau \ \le\ \tilde c_2 t \log t$$

for some $\tilde c_1, \tilde c_2 > 0$. Identifying $\phi(y,t)$ and $h(y,t)$ with $\phi(y,\tau)$ and $h(y,\tau)$ we rewrite (8.1) as

$$\begin{cases} \partial_\tau \phi = L[\phi] + B[\phi] + h + \sum_{j=1}^{2} \mu_j(\tau) W_{1,j} \quad \text{in } \mathbb{R}^2 \times (\tau_0, \infty) \\ \phi(\cdot, \tau_0) = 0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{8.3}$$

We consider problem (8.3) for functions $h(y,\tau)$ that have fast decay in space. More precisely, we assume that for all $T > 0$ there is C_T such that

$$|h(y,\tau)| \le \frac{C_T}{1+|y|^6} \quad \text{for all } (y,\tau) \in \mathbb{R}^2 \times (\tau_0, T).$$

In this case, by a solution $\phi(y,\tau)$ of (8.3) we understand a continuous function $\phi(y,\tau)$, of class C^1 in y, such that for any $T > \tau_0$ there exists a $C_T > 0$ with

$$|\phi(y,\tau)| + (1+|y|)|\nabla_y \phi(y,\tau)| \ \le\ \frac{C_T}{1+|y|^6} \quad \text{for all } (y,\tau) \in \mathbb{R}^2 \times (\tau_0, T), \tag{8.4}$$

and satisfies the integral equation

$$\begin{aligned} \phi(y,\tau) = \int_{\tau_0}^{\tau} \int_{\mathbb{R}^2} G(y-z, \tau - s)\, [-\nabla\phi \nabla \Gamma_0 - \nabla U \nabla (-\Delta)^{-1}\phi \\ + 2U\phi + B[\phi] + h + \sum_{j=1}^{2} \mu_j(s) W_{1,j}](z,s)\, dz ds, \end{aligned} \tag{8.5}$$

where $(-\Delta)^{-1}\phi$ is defined in (8.2) and $G(y,\tau)$ is the two-dimensional heat kernel,

$$G(y,\tau) = \frac{1}{4\pi\tau} e^{-\frac{|y|^2}{4\tau}}.$$

From the formula

$$\nabla(-\Delta)^{-1} h(y) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y-z}{|y-z|^2} h(z) dz$$

we see that if $|\phi(y)| \le \frac{C}{1+|y|^6}$ then

$$|\nabla(-\Delta)^{-1}\phi(y)| \le \frac{C}{1+|y|} \|(1+|y|^6)\phi\|_{L^\infty(\mathbb{R}^2)}.$$

Using this estimate, existence and uniqueness of a solution of (8.5) satisfying (8.4) are standard. For a short time $T > \tau_0$ this is established by a contraction mapping argument in an appropriate L^∞-weighted space. Then a direct linear continuation procedure applies.

A first natural condition to impose on h in (8.3) is that

$$\int_{\mathbb{R}^2} h(y,\tau) dy = 0 \quad \text{for all } \tau > \tau_0,$$

in order to achieve that the solution has also zero mass at all times.

We want to find solutions to (8.3) that have fast decay in space and time. For this we need to assume fast space-time decay of the right hand side, which we do by working with the following class of norms.

Given positive numbers ν, p, ϵ and $m \in \mathbb{R}$, we let $\|h\|_{\nu,m,p,\epsilon}$ denote the least $K \ge 0$ such that for all $\tau > \tau_0$ and for all $y \in \mathbb{R}^2$

$$|h(y,\tau)| \ \le\ \frac{K}{\tau^\nu (\log\tau)^m} \frac{1}{(1+|y|)^p} \begin{cases} 1 & |y| \le \sqrt{\tau}, \\ \frac{\tau^{\epsilon/2}}{|y|^\epsilon} & |y| \ge \sqrt{\tau}. \end{cases} \tag{8.6}$$

This is similar to the norm introduced in (6.2) but defined using τ instead of t. We will give the results in Sections 9–12 using the norm (8.6).

Still, fast decay of the right hand side doesn't imply fast decay of the solution. For example, consider equation (8.1) without the operator B and without the μ_j, that is,

$$\begin{cases} \partial_\tau \phi = L[\phi] + h(y,t) \quad \text{in } \mathbb{R}^2 \times (\tau_0, \infty) \\ \phi(\cdot, \tau_0) = 0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{8.7}$$

and suppose that h has compact support in space and time, and that ϕ has sufficient space-time decay. Then, multiplying (8.7) by $|y|^2$ and integrating in $\mathbb{R}^2 \times (\tau_0, \infty)$ gives

$$\int_{\tau_0}^\infty \int_{\mathbb{R}^2} h(y,\tau)|y|^2 dy d\tau = 0,$$

because if ϕ is a regular function with fast decay, then

$$\int_{\mathbb{R}^2} L[\phi]|y|^2 dy = 0,$$

see Remark 9.2 below. It is then necessary to impose a condition on h, or to adjust a parameter in the problem in order to get a fast decay of the solution. We develop here the theory by adjusting the parameter c_1 in the equation below

$$\begin{cases} \partial_\tau \phi = L[\phi] + B[\phi] + h(y,t) \quad \text{in } \mathbb{R}^2 \times (\tau_0, \infty), \\ \phi(\cdot, t_0) = c_1 \tilde{Z}_0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{8.8}$$

where $\tilde{Z}_0$ is defined in (6.4).

Proposition 8.1. *Let $\sigma > 0$, $\epsilon > 0$ with $\sigma + \epsilon < 2$ and $1 < \nu < \frac{7}{4}$. Let $0 < q < 1$. Then there exists a number $C > 0$ such that for t_0 sufficiently large and all radially symmetric $h = h(|y|, \tau)$ with $\|h\|_{\nu,m,6+\sigma,\epsilon} < \infty$ and*

$$\int_{\mathbb{R}^2} h(y,\tau) dy = 0, \quad \text{for all } \tau > \tau_0,$$

there exists $c_1 \in \mathbb{R}$ and solution $\phi(y,\tau) = \mathcal{T}_{\boldsymbol{p}}^{i,2}[h]$ of problem (8.8) that defines a linear operator of h and satisfies the estimate

$$\|\phi\|_{\nu-1,m+q,4,2+\sigma+\epsilon} \le \frac{C}{(\log \tau_0)^{1-q}} \|h\|_{\nu,m,6+\sigma,\epsilon}.$$

Moreover c_1 is a linear operator of h and

$$|c_1| \le C \frac{1}{\tau_0^{\nu-1} (\log \tau_0)^{m+1}} \|h\|_{\nu,m,6+\sigma,\epsilon}.$$

We have stated this result only in the radial setting, because this is what is needed, but there is a version of it in the non-radial case.

The next result is for the problem

$$\begin{cases} \partial_\tau \phi = L[\phi] + B[\phi] + h(y,\tau) + \sum_{j=1}^2 \mu_j W_{1,j} \quad \text{in } \mathbb{R}^2 \times (\tau_0, \infty), \\ \phi(\cdot, \tau_0) = 0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{8.9}$$

and holds without the radial symmetry assumption.

Proposition 8.2. *Let $0 < \sigma < 1$, $\epsilon > 0$ with $\sigma + \epsilon < \frac{3}{2}$ and $1 < \nu < \min(1 + \frac{\epsilon}{2}, 3 - \frac{\sigma}{2}, \frac{5}{4})$. Let $0 < q < 1$. Then there is C such that for τ_0 large the following holds. Suppose that h satisfies $\|h\|_{\nu,m,6+\sigma,\epsilon} < \infty$ and*

$$\int_{\mathbb{R}^2} h(y,\tau) dy = 0, \quad \int_{\mathbb{R}^2} h(y,\tau)|y|^2 dy = 0, \quad \text{for all } \tau > \tau_0.$$

Then there exists a solution $\phi(y,\tau)$, μ_j of problem (8.9) that defines a linear operator of h and satisfies

$$\|\phi\|_{\nu-\frac{1}{2},m+\frac{q}{2},4,2+\sigma+\epsilon} \le C \|h\|_{\nu,m,6+\sigma,\epsilon}.$$

The parameters μ_j satisfy

$$\mu_j(\tau) = -\int_{\mathbb{R}^2} h(y,\tau) y_j dy + \tilde{\mu}_j[h](\tau)$$

where $\tilde{\mu}_j$ are linear functions of h with

$$|\tilde{\mu}_j[h](\tau)| \le C \frac{1}{\tau^{\nu+1} (\log \tau)^{m+1}} \|h\|_{\nu,m,6+\sigma,\epsilon}.$$

We denote this solution by $\phi = \mathcal{T}_p^{i,1}[h]$.

Propositions 6.1 and 6.2 given in Section 6 are direct corollaries of Propositions 8.1 and 8.2. The only changes are due to the change in the time variable, because $\tau \sim t\log t$, and the fact that the norms for the solutions in Propositions 6.1 and 6.2 include a gradient term. The estimate for the gradient follows from the weighted L^∞ estimate, scaling and standard parabolic estimates.

The proofs of Propositions 8.1 and 8.2 are contained in Sections 9–12. They are based on an energy inequality obtained by multiplying the equation by a suitable test function, and using an inequality for a quadratic form. Section 9 contains some preliminaries on this quadratic form.

In Proposition 10.1, we obtain an additive decomposition of the solution $\phi(y,\tau)$ of (8.8) into a part with a relatively slow space decay that loses $\tau^{1/2}$ with respect to the time decay of the right hand side, and a term along $Z_0(y)$ that loses an entire power of τ. This is the key element for the proof of Proposition 8.1 in Section 10 (p.80).

Then the proof of Proposition 8.2 in the radial case uses Proposition 10.1 after formally applying the operator L^{-1} to the original equation and performing a *concentration procedure* that improves the space decay of the resulting error. This is done on Section 11, and we give there a proof of Proposition 8.2 in the case of radial functions.

The proof of Proposition 8.2 in the general case is in Section 12 (p.96).. The idea is that the decomposition obtained in Proposition 10.1 for solutions with no radial mode does not contain the term along Z_0, which allows us to obtain a much better estimate.

9. Preliminaries for the linear theory

A central ingredient in obtaining good estimates for the linearized parabolic operator associated to the inner problem is the analysis of the quadratic form

$$\phi \mapsto \int_{\mathbb{R}^2} g\phi, \quad g = \frac{\phi}{U} - (-\Delta)^{-1}\phi. \tag{9.1}$$

This quadratic form arises when considering the linearized Keller-Segel problem (8.1). Indeed, $L[\phi] = \nabla\cdot(U\nabla g)$ and it is natural to test the equation (8.1) with g, since

$$\int_{\mathbb{R}^2} L[\phi]g = \int_{\mathbb{R}^2} \nabla\cdot(U\nabla g)g = -\int_{\mathbb{R}^2} U|\nabla g|^2.$$

But from the time derivative we get $\lambda^2 \int_{\mathbb{R}^2} \partial_t\phi g$, which leads to (9.1).

We observe that g has degeneracy directions. Indeed, if $\psi = (-\Delta)^{-1}\phi$ then

$$\Delta\psi + U(y)\psi = -Ug \quad \text{in } \mathbb{R}^2.$$

The operator $\Delta\psi + U(y)\psi$ is classical. It corresponds to linearizing the Liouville equation

$$\Delta v + e^v = 0 \quad \text{in } \mathbb{R}^2$$

around the solution $\Gamma_0 = \log U$. It is well known that the bounded kernel of this linearization is spanned by the generators of rigid motions, namely dilation and translations of the equation, which are precisely the functions z_0, z_1, z_2 defined by

$$\begin{cases} z_0(y) = \nabla\Gamma_0(y)\cdot y + 2 \\ z_j(y) = \partial_{y_j}\Gamma_0(y), \quad j = 1,2. \end{cases} \tag{9.2}$$

Note that g is precisely annihilated at the linear combinations of these functions. In the rest of this section we will state and prove several estimates that take into account this issue, which will be crucial later on.

The quadratic form (9.1) can be naturally transformed into a similar one in S^2 by stereographic projection $\Pi : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$

$$\Pi(y_1,y_2,y_3) = \Big(\frac{y_1}{1-y_3}, \frac{y_2}{1-y_3}\Big).$$

For $\varphi : \mathbb{R}^2 \to \mathbb{R}$ we write

$$\tilde\varphi = \varphi\circ\Pi, \quad \tilde\varphi : S^2\setminus\{(0,0,1)\} \to \mathbb{R}.$$

Then we have the following formulas

$$\int_{S^2} \tilde\varphi = \frac{1}{2}\int_{\mathbb{R}^2} \varphi U$$

$$\int_{S^2} \tilde U |\nabla_{S^2}\tilde\varphi|^2 = \int_{\mathbb{R}^2} U|\nabla_{\mathbb{R}^2}\varphi|^2$$

$$\frac{1}{2}\tilde U \Delta_{S^2}\tilde\varphi = (\Delta_{\mathbb{R}^2}\varphi)\circ\Pi.$$

9.1. **The Liouville equation.** Here we consider the linearized Liouville equation

$$\Delta\psi + U\psi + h = 0 \quad \text{in } \mathbb{R}^2. \tag{9.3}$$

The stereographic projection transforms the linearized Liouville equation (9.3) into

$$\Delta_{S^2}\tilde\psi + 2\tilde\psi + 2\tilde h = 0 \tag{9.4}$$

in $S^2 \setminus \{P\}$, $P = (0,0,1)$, where $\tilde\psi = \psi\circ\Pi$, $\tilde h = (U^{-1}h)\circ\Pi$.

The functions in (9.2) are transformed through the stereographic projection into constant multiples of the coordinate functions

$$\tilde z_j(\omega) = c_j\omega_j, \quad j = 1,2, \quad \tilde z_0(\omega) = c_0\omega_3, \quad \omega = (\omega_1,\omega_2,\omega_3)\in S^2.$$

By standard elliptic theory, if $\tilde h \in L^p(S^2)$, $p > 2$, then exists a solution $\tilde\psi_0 \in W^{2,p}(S^2)$ to (9.4) in S^2 if and only if $\tilde h$ satisfies

$$\int_{S^2} \tilde h \tilde z_j = 0, \quad j = 1,2,3.$$

This solution is unique if we normalize it such that

$$\int_{S^2} \tilde\psi_0 \tilde z_j = 0, \quad j = 1,2,3,$$

and then satisfies the estimate

$$\|\tilde\psi_0\|_{C^{1,\alpha}(S^2)} \le C\|\tilde h\|_{L^p(S^2)}$$

where $\alpha = 1 - \frac{2}{p}$. By subtracting off a suitable linear combination of the functions $\tilde z_j$, $j = 0,1,2$ we obtain the unique solution $\tilde\psi_1$ to (9.4) in S^2 satisfying

$$\tilde\psi_1(P) = 0, \quad \nabla_{S^2}\tilde\psi_1(P) = 0. \tag{9.5}$$

For this solution we also have the estimate

$$\|\tilde\psi_1\|_{C^{1,\alpha}(S^2)} \le C\|\tilde h\|_{L^p(S^2)}. \tag{9.6}$$

Lemma 9.1. *Let $0 < \sigma < 1$. Then there is C such that if ψ satisfies (9.3) and $\psi(y) \to 0$ as $|y| \to \infty$ with h satisfying $\|(1+|y|)^{3+\sigma}h\|_{L^\infty(\mathbb{R}^2)} < +\infty$ and*

$$\int_{\mathbb{R}^2} (U\psi + h(y))dy = 0, \quad \int_{\mathbb{R}^2} (U\psi + h(y))y_j\,dy = 0, \quad j = 1,2, \tag{9.7}$$

then

$$\|(1+|y|)^{1+\sigma}\psi\|_{L^\infty(\mathbb{R}^2)} \le C\|(1+|y|)^{3+\sigma}h\|_{L^\infty(\mathbb{R}^2)}.$$

Remark 9.1. *Let $h : \mathbb{R}^2 \to \mathbb{R}$ satisfy $\|(1+|y|)^{2+\sigma}h\|_{L^\infty(\mathbb{R}^2)} < +\infty$ where $0 < \sigma < 1$. If*

$$\int_{\mathbb{R}^2} h(y)dy = 0$$

then

$$|(-\Delta)^{-1}h(y)| \le \frac{C}{(1+|y|)^\sigma}\|(1+|y|)^{2+\sigma}h\|_{L^\infty(\mathbb{R}^2)}.$$

If $h : \mathbb{R}^2 \to \mathbb{R}$ satisfy $\|(1+|y|)^{3+\sigma}h\|_{L^\infty(\mathbb{R}^2)} < +\infty$ where $0 < \sigma < 1$ and in addition to mass zero we have

$$\int_{\mathbb{R}^2} h(y)y_j dy = 0, \quad j = 1,2,$$

then

$$|(-\Delta)^{-1}h(y)| \le \frac{C}{(1+|y|)^{1+\sigma}}\|(1+|y|)^{3+\sigma}h\|_{L^\infty(\mathbb{R}^2)}.$$

The first claim is standard. For the second, write

$$(-\Delta)^{-1}h(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} (\log|x| - \log|x-y| + \frac{y\cdot x}{|x|^2})h(y)dy$$

and estimate the integral after splitting it into the regions $|y| < \frac{|x|}{2}$ *and its complement.*

Proof of Lemma 9.1. We claim that $\psi = (-\Delta)^{-1}(U\psi + h)$. Indeed the function $\psi - (-\Delta)^{-1}(U\psi + h)$ is harmonic in $\mathbb{R}^2$ and decays to 0 at infinity, and therefore it is equal to 0. The assumptions (9.7) and Remark 9.1 imply that

$$\|(1+|y|)^{1+\sigma}\psi\|_{L^\infty(\mathbb{R}^2)} \le C\|(1+|y|)^{3+\sigma}h\|_{L^\infty(\mathbb{R}^2)} + C\|\psi\|_{L^\infty(\mathbb{R}^2)}. \tag{9.8}$$

Let $\tilde\psi = \psi\circ\Pi$, so that it satisfies (9.4) in $S^2\setminus\{P\}$ with $\tilde h = (U^{-1}h)\circ\Pi$. Note that $\tilde h \in L^p(S^2)$ for some $p > 2$. More precisely

$$\|\tilde h\|_{L^p(S^2)} \le C\|(1+|y|)^{3+\sigma}h\|_{L^\infty(\mathbb{R}^2)}, \tag{9.9}$$

with $p < \frac{2}{1-\sigma}$. The singularity at P is removable and thus $\tilde\psi$ satisfies (9.4) in S^2. By elliptic regularity $\tilde\psi \in C^{1,\alpha}(S^2)$ for some $\alpha > 0$. Since ψ decays at infinity, $\tilde\psi(P) = 0$. By (9.8) we have also $\nabla_{S^2}\tilde\psi(P) = 0$.

We let $\tilde\psi_1$ denote the solution to (9.4) satisfying (9.5). The solution to (9.4) in S^2 satisfying (9.5) is unique, so that we have $\tilde\psi = \tilde\psi_1$ and by estimate (9.6), (9.9) and (9.8) we obtain

$$\|(1+|y|)^{1+\sigma}\psi\|_{L^\infty(\mathbb{R}^2)} \le C\|(1+|y|)^{3+\sigma}h\|_{L^\infty(\mathbb{R}^2)}.$$

□

9.2. **A quadratic form.** Here we discuss properties of the quadratic form (9.1). For this we consider a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ with sufficient decay, in the form,

$$|\phi(y)| \le \frac{1}{(1+|y|)^{2+\sigma}}, \tag{9.10}$$

with $0 < \sigma < 1$, and zero mass:

$$\int_{\mathbb{R}^2} \phi\, dy = 0. \tag{9.11}$$

We recall g defined in (9.1) $g = \frac{\phi}{U} - (-\Delta)^{-1}\phi$, and use the notation

$$\psi = (-\Delta)^{-1}\phi$$

so that

$$-\Delta\psi - U\psi = Ug \quad \text{in } \mathbb{R}^2.$$

We next introduce a normalized version of g, namely $g^\perp$ defined by

$$g^\perp = g + a,$$

where $a \in \mathbb{R}$ is chosen so that

$$\int_{\mathbb{R}^2} g^\perp U dy = 0.$$

As shown in Lemma 9.3 below, the quadratic form $\int_{\mathbb{R}^2}\phi g$ is equivalent to $\int_{\mathbb{R}^2} U(g^\perp)^2$.

It will be convenient to work with functions $\phi^\perp$, $\psi^\perp$, which are analogues of ϕ, ψ but associated to $g^\perp$. In particular, we want a choice of $\psi^\perp$ such that

$$-\Delta\psi^\perp - U\psi^\perp = Ug^\perp, \quad \psi^\perp(y) \to 0 \quad \text{as } |y| \to \infty. \tag{9.12}$$

Let $\psi_0 = 1 + \frac{1}{2}z_0$, where z_0 is defined in (9.2), and observe that

$$-\Delta\psi_0 - U\psi_0 = -U, \quad \psi_0(y) \to 0 \quad \text{as } |y| \to \infty.$$

Then $\psi^\perp$ defined by

$$\psi^\perp = \psi - a\Big(1 + \frac{1}{2}z_0\Big) = \psi - a\psi_0,$$

indeed satisfies (9.12).

Define

$$\phi^\perp = U(g^\perp + \psi^\perp),$$

and obtain the relations

$$\phi = \phi^\perp + \frac{a}{2} U z_0, \quad -\Delta \psi^\perp = \phi^\perp, \quad \int_{\mathbb{R}^2} \phi^\perp = 0.$$

We note that $\phi - \phi^\perp = \frac{a}{2} U z_0$ is a constant times $Z_0 = U z_0$, which is in the kernel of the operator L.

Lemma 9.2. *If $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfies (9.10) and (9.11), then*

$$\int_{\mathbb{R}^2} g U z_j = \int_{\mathbb{R}^2} g^\perp U z_j = 0, \quad j = 0, 1, 2,$$

where z_j are the functions defined in (9.2).

Proof. By the definition of ψ and from (9.10), (9.11) we have

$$|\psi(y)| + (1 + |y|) |\nabla \psi(y)| \le \frac{C}{(1 + |y|)^\sigma},$$

and hence also

$$|\psi^\perp(y)| + (1 + |y|) |\nabla \psi^\perp(y)| \le \frac{C}{(1 + |y|)^\sigma}. \tag{9.13}$$

We multiply (9.12) by z_j, integrate in the ball $B_R(0)$ and let $R \to \infty$. Since z_j is in the kernel of $\Delta + U$ we just have to check that

$$\int_{\partial B_R} \Big(\frac{\partial \psi^\perp}{\partial \nu} z_j - \psi^\perp \frac{\partial z_j}{\partial \nu} \Big) \to 0, \quad \text{as } R \to \infty,$$

where ν is the exterior normal vector to ∂B_R. This follows from (9.13), and the explicit bounds

$$|z_0(y)| \le C, \quad |z_j(y)| \le \frac{C}{(1 + |y|)}, \quad j = 1, 2,$$

$$|\nabla z_j(y)| \le \frac{C}{(1 + |y|)^2}.$$

□

A consequence of the previous lemma is the following.

Remark 9.2. *Suppose that $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfies (9.10) and (9.11). Then*

$$\int_{\mathbb{R}^2} L[\phi] |y|^2 dy = 0.$$

Indeed, integrating on B_R, with the notation $g = \frac{\phi}{U} - (-\Delta)^{-1} \phi$,

$$\begin{aligned} \int_{B_R} L[\phi] |y|^2 dy &= \int_{B_R} \nabla \cdot (U \nabla g) |y|^2 dy \\ &= -2 \int_{B_R} U \nabla g \cdot y dy + R^2 \int_{\partial B_R} U \nabla g \cdot \nu dS(y) \\ &= 2 \int_{B_R} g Z_0 dy - 2 \int_{\partial B_R} U g y \cdot \nu dy + R^2 \int_{\partial B_R} U \nabla g \cdot \nu dS(y). \end{aligned}$$

By (9.10) and (9.11), $g(y) = O(|y|^{2-\sigma})$, $\nabla g(y) = O(|y|^{1-\sigma})$ as $|y| \to \infty$. Therefore the boundary terms tend to 0 as $R \to \infty$, and we get

$$\int_{\mathbb{R}^2} L[\phi] |y|^2 dy = 2 \int_{\mathbb{R}^2} g Z_0 dy = 0,$$

by Lemma 9.2.

Lemma 9.3. *There are constants $c_1 > 0$, $c_2 > 0$ such that if $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfies*

$$|\phi(y)| \le \frac{1}{(1 + |y|)^{3+\sigma}}, \quad 0 < \sigma < 1$$

and (9.11), then

$$c_1 \int_{\mathbb{R}^2} U (g^\perp)^2 \le \int_{\mathbb{R}^2} \phi g^\perp \le c_2 \int_{\mathbb{R}^2} U (g^\perp)^2. \tag{9.14}$$

Proof. By Lemma 9.2

$$\int_{\mathbb{R}^2} \phi g = \int_{\mathbb{R}^2} (\phi^\perp + \frac{a}{2} U z_0) g = \int_{\mathbb{R}^2} \phi^\perp (g^\perp + a) = \int_{\mathbb{R}^2} \phi^\perp g^\perp$$
$$= \int_{\mathbb{R}^2} U(g^\perp + \psi^\perp) g^\perp.$$

Let $\tilde g^\perp = g^\perp \circ \Pi$, $\tilde\psi^\perp = \psi^\perp \circ \Pi$ and write (9.12) as

$$-\Delta_{S^2} \tilde\psi^\perp - 2\tilde\psi^\perp = 2\tilde g^\perp, \quad \text{in } S^2. \tag{9.15}$$

We also get

$$\frac{1}{2}\int_{\mathbb{R}^2} \phi g = \int_{S^2} [(\tilde g^\perp)^2 + \tilde\psi^\perp \tilde g^\perp].$$

Multiplying (9.15) by $\tilde\psi^\perp$ we find that

$$\int_{S^2} \tilde g^\perp \tilde\psi^\perp = \frac{1}{2}\int_{S^2} |\nabla_{S^2} \tilde\psi^\perp|^2 - \int_{S^2} (\tilde\psi^\perp)^2$$

and hence

$$\frac{1}{2}\int_{\mathbb{R}^2} \phi g = \int_{S^2} (\tilde g^\perp)^2 + \frac{1}{2}\int_{S^2} |\nabla_{S^2} \tilde\psi^\perp|^2 - \int_{S^2} (\tilde\psi^\perp)^2.$$

We recall that the eigenvalues of $-\Delta$ on S^2 are given by $\{k(k+1) \mid k \geq 0\}$. The eigenvalue 0 has a constant eigenfunction and the eigenvalue 2 has eigenspace spanned by the coordinate functions $\pi_i(x_1, x_2, x_3) = x_i$, for $(x_1, x_2, x_3) \in S^2$ and $i = 1, 2, 3$. Let $(\lambda_j)_{j\geq 0}$ denote all eigenvalues, repeated according to multiplicity, with $\lambda_0 = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 2$, and let $(e_j)_{j\geq 0}$ denote the corresponding eigenfunctions so that they form an orthonormal system in $L^2(S^2)$, and e_1, e_2, e_3 are multiples of the coordinate functions π_1, π_2, π_3. We decompose $\tilde\psi$ and $\tilde g$:

$$\tilde\psi^\perp = \sum_{j=0}^\infty \tilde\psi_j^\perp e_j, \quad \tilde g^\perp = \sum_{j=0}^\infty \tilde g_j^\perp e_j, \tag{9.16}$$

where

$$\tilde\psi_j^\perp = \langle \tilde\psi^\perp, e_j \rangle_{L^2(S^2)}, \quad \tilde g_j^\perp = \langle \tilde g^\perp, e_j \rangle_{L^2(S^2)}.$$

Then

$$\frac{1}{2}\int_{\mathbb{R}^2} \phi g = \sum_{j=0}^\infty (\tilde g_j^\perp)^2 + \frac{1}{2}\sum_{j=0}^\infty (\lambda_j - 2)(\tilde\psi_j^\perp)^2$$
$$= \sum_{j=0}^\infty (\tilde g_j^\perp)^2 - (\tilde\psi_0^\perp)^2 + \frac{1}{2}\sum_{j=4}^\infty (\lambda_j - 2)(\tilde\psi_j^\perp)^2.$$

Equation (9.15) gives us that

$$(\lambda_j - 2)\tilde\psi_j^\perp = 2\tilde g_j^\perp, \tag{9.17}$$

and then

$$\frac{1}{2}\int_{\mathbb{R}^2} \phi g = \sum_{j=1}^\infty (\tilde g_j^\perp)^2 + \sum_{j=4}^\infty \frac{2}{\lambda_j - 2}(\tilde g_j^\perp)^2.$$

By Lemma 9.2 $\tilde g_1^\perp = \tilde g_2^\perp = \tilde g_3^\perp = 0$. Therefore

$$\frac{1}{2}\int_{\mathbb{R}^2} \phi g = \sum_{j=4}^\infty \frac{\lambda_j}{\lambda_j - 2}(\tilde g_j^\perp)^2 \tag{9.18}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^2} (g^\perp)^2 U = \sum_{j=4}^\infty (\tilde g_j^\perp)^2.$$

This proves (9.14).

∎

Lemma 9.4. *There exist positive constants c_1, c_2 such that if $\phi : \mathbb{R}^2 \to \mathbb{R}$ is radially symmetric and satisfies $(1+|y|)^{3+\sigma}\phi \in L^\infty(\mathbb{R}^2)$ with $0 < \sigma < 1$, and*

$$\int_{\mathbb{R}^2} \phi(y)dy = 0,$$

then

$$c_1 \int_{\mathbb{R}^2} U(g^\perp)^2 \le \int_{\mathbb{R}^2} U^{-1}(\phi^\perp)^2 \le c_2 \int_{\mathbb{R}^2} U(g^\perp)^2, \tag{9.19}$$

$$\int_{\mathbb{R}^2} U(\psi^\perp)^2 \le c_2 \int_{\mathbb{R}^2} U(g^\perp)^2. \tag{9.20}$$

Proof. Using the same notation as in the proof of Lemma 9.3, we have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^2} U^{-1}(\phi^\perp)^2 &= \frac{1}{2}\int_{\mathbb{R}^2} U[(\psi^\perp)^2 + 2\psi^\perp g^\perp + (g^\perp)^2] \\ &= \int_{S^2} [(\tilde\psi^\perp)^2 + 2\tilde\psi^\perp \tilde g^\perp + (\tilde g^\perp)^2] \\ &= \sum_{j=0}^\infty [(\tilde\psi_j^\perp)^2 + 2\tilde\psi_j^\perp \tilde g_j^\perp + (\tilde g_j^\perp)^2].\end{aligned}$$

As in the previous proof, $\tilde g_j^\perp = 0$ for $j = 0, 1, 2, 3$. Using (9.17) we get

$$\frac{1}{2}\int_{\mathbb{R}^2} U^{-1}(\phi^\perp)^2 = \sum_{j=0}^3 (\tilde\psi_j^\perp)^2 + \sum_{j=4}^\infty \frac{\lambda_j^2}{(\lambda_j - 2)^2} (\tilde g_j^\perp)^2.$$

This formula already gives

$$\int_{\mathbb{R}^2} U(g^\perp)^2 \le C \int_{\mathbb{R}^2} U^{-1}(\phi^\perp)^2.$$

We observe that $\tilde\psi_1^\perp = \tilde\psi_2^\perp = 0$ by radial symmetry. We also have $\tilde\psi_0^\perp = 0$, by (9.17). Let

$$\hat\psi = \sum_{j=4}^\infty \tilde\psi_j^\perp e_j$$

and note that it satisfies

$$-\Delta_{S^2}\hat\psi - 2\hat\psi = 2\tilde g^\perp \quad \text{in } S^2.$$

By (9.17),

$$\|\hat\psi\|_{L^2(S^2)} \le C\|\tilde g^\perp\|_{L^2(S^2)},$$

and from elliptic estimates

$$\|\hat\psi\|_{C^\alpha(S^2)} \le C\|\tilde g^\perp\|_{L^2(S^2)}, \tag{9.21}$$

for any $0 < \alpha < 1$. Since $(1+|y|)^{3+\sigma}\phi \in L^\infty(\mathbb{R}^2)$ and ϕ has total mass 0, we have $(1+|y|)^{1+\sigma}\psi \in L^\infty(\mathbb{R}^2)$ (here the functions are radial) and also $(1+|y|)^{1+\sigma}\psi^\perp \in L^\infty(\mathbb{R}^2)$. It follows that $\tilde\psi^\perp(P) = 0$ where $P = (0,0,1)$. Since $\tilde\psi^\perp$ and $\hat\psi$ differ by a constant times π_3 we have

$$\tilde\psi^\perp = \hat\psi - \frac{\hat\psi(P)}{\pi_3(P)}\pi_3,$$

where $\pi_3(x_1,x_2,x_3) = x_3$. This implies, by (9.21),

$$\|\tilde\psi^\perp\|_{L^2(S^2)} \le C\|\hat\psi\|_{L^2(S^2)} + C|\hat\psi(P)| \le C\|\tilde g^\perp\|_{L^2(S^2)}.$$

This proves the other inequality in (9.19) and (9.20).

□

Lemma 9.5. *Suppose that $\phi = \phi(y,t)$, $y \in \mathbb{R}^2$, $t > 0$ is a function satisfying*

$$|\phi(y,t)| \le \frac{1}{(1+|y|)^{2+\sigma}},$$

with $0 < \sigma < 1$,

$$\int_{\mathbb{R}^2} \phi(y,t)\,dy = 0, \quad \forall t > 0,$$

and that ϕ is differentiable with respect to t and ϕ_t satisfies also

$$|\phi_t(y,t)| \le \frac{1}{(1+|y|)^{2+\sigma}}.$$

Then

$$\int_{\mathbb{R}^2} \phi_t g = \frac{1}{2}\partial_t \int_{\mathbb{R}^2} \phi g$$

where for each t, $g(y,t)$ is defined as

$$g = \frac{\phi}{U} - (-\Delta^{-1})\phi + c(t)$$

and $c(t) \in \mathbb{R}$ is chosen so that

$$\int_{\mathbb{R}^2} g(y,t)U(y)\,dy = 0.$$

Proof. Using the notation of the previous lemma, we have

$$\int_{\mathbb{R}^2} \phi_t g = \int_{\mathbb{R}^2} U(g_t + \psi_t)g = 2\int_{S^2} (\tilde g_t \tilde g + \tilde\psi_t \tilde g).$$

We have

$$-\Delta_{S^2}\tilde\psi - 2\tilde\psi = 2\tilde g, \quad \text{in } S^2.$$

And differentiating in t we get

$$-\Delta_{S^2}\tilde\psi_t - 2\tilde\psi_t = 2\tilde g_t, \quad \text{in } S^2. \tag{9.22}$$

Multiplying by $\tilde g$ and integrating we find that

$$\int_{S^2} \tilde\psi_t \tilde g = -\frac{1}{2}\int_{S^2} \Delta\tilde\psi_t \tilde g - \int_{S^2} \tilde g_t \tilde g.$$

Thus

$$\int_{\mathbb{R}^2} \phi_t g = -\int_{S^2} \Delta\tilde\psi_t \tilde g$$

Decompose as in (9.16) and find that

$$\int_{\mathbb{R}^2} \phi_t g = \sum_{j=0}^{\infty} \lambda_j (\tilde\psi_j)_t \tilde g_j$$

But from (9.22)

$$(\lambda_j - 2)(\tilde\psi_j)_t = 2(\tilde g_j)_t.$$

We note that $\tilde g_j = 0$ for $j = 0,1,2,3$. Indeed, this is true for $j=0$ by the assumption $\int_{\mathbb{R}^2} gU = 0$. By Lemma 9.2 this is true also for $j = 1,2,3$. Then

$$\frac{1}{2}\int_{\mathbb{R}^2} \phi_t g = \sum_{j=4}^{\infty} \frac{\lambda_j}{\lambda_j - 2}(\tilde g_j)_t \tilde g_j$$

and the desired conclusion follows from (9.18). □

9.3. A Poincaré inequality.

Lemma 9.6. *Let $B_R(0) \subset \mathbb{R}^2$ be the open ball centered at 0 of radius R. There exists $C > 0$ such that, for any $R > 0$ large and any $g \in H^1(B_R)$ with $\int_{B_R} g\,U\,dx = 0$ we have*

$$\frac{C}{R^2}\int_{B_R} g^2 U \le \int_{B_R} |\nabla g|^2 U.$$

Proof. Using a Fourier decomposition we only need to consider the radial case, that is, we claim that if $g(r)$ satisfies

$$\int_0^R g(r)\frac{r}{(1+r^2)^2}\,dr = 0, \tag{9.23}$$

then there is C such that for all R large

$$\int_0^R g(r)^2 \frac{r}{(1+r^2)^2}\,dr \le CR^2 \int_0^R g'(r)^2 \frac{r}{(1+r^2)^2}\,dr.$$

Let $0<\delta<1$ to be fixed later on. From (9.23) we have

$$\int_\delta^R g(r)\frac{r}{(1+r^2)^2}dr = -\int_0^\delta g(r)\frac{r}{(1+r^2)^2}dr.$$

But

$$\begin{aligned}\int_\delta^R g(r)\frac{r}{(1+r^2)^2}dr &= -\frac12\int_\delta^R g(r)\frac{d}{dr}\Big(\frac{1}{1+r^2}\Big)dr\\ &= -\frac12\frac{g(R)}{1+R^2}+\frac12\frac{g(\delta)}{1+\delta^2}+\frac12\int_\delta^R g'(r)\frac{1}{1+r^2}dr.\end{aligned}$$

Therefore

$$\frac12\frac{|g(\delta)|}{1+\delta^2}\le\frac12\frac{|g(R)|}{1+R^2}+\frac12\int_\delta^R|g'(r)|\frac{1}{1+r^2}dr+\int_0^\delta|g(r)|\frac{r}{(1+r^2)^2}dr.$$

By the Cauchy-Schwarz inequality

$$\int_\delta^R|g'(r)|\frac{1}{1+r^2}dr\le\Big(\int_\delta^R g'(r)^2\frac{r}{(1+r^2)^2}dr\Big)^{1/2}(\log R-\log\delta)^{1/2}$$

$$\int_0^\delta|g(r)|\frac{r}{(1+r^2)^2}dr\le\delta\Big(\int_0^\delta g(r)^2\frac{r}{(1+r^2)^2}dr\Big)^{1/2}.$$

Hence

$$g(\delta)^2\le 2\frac{g(R)^2}{R^4}+2(\log R-\log\delta)\int_\delta^R g'(r)^2\frac{r}{(1+r^2)^2}dr+4\delta^2\int_0^\delta g(r)^2\frac{r}{(1+r^2)^2}dr. \tag{9.24}$$

We compute now

$$\begin{aligned}\int_\delta^R g(r)^2\frac{r}{(1+r^2)^2}dr &= -\frac12\int_\delta^R g(r)^2\frac{d}{dr}\Big(\frac{1}{1+r^2}\Big)dr\\ &= -\frac12\frac{g(R)^2}{1+R^2}+\frac12\frac{g(\delta)^2}{1+\delta^2}+\int_\delta^R g(r)g'(r)\frac{1}{1+r^2}dr.\end{aligned}$$

Using (9.24) and the Cauchy-Schwartz inequality we get

$$\begin{aligned}\int_\delta^R g(r)^2\frac{r}{(1+r^2)^2}dr\le&-\frac12\frac{g(R)^2}{1+R^2}+\frac{g(R)^2}{R^4}+(\log R-\log\delta)\int_\delta^R g'(r)^2\frac{r}{(1+r^2)^2}dr\\ &+2\delta^2\int_0^\delta g(r)^2\frac{r}{(1+r^2)^2}dr+AR^2\int_\delta^R g'(r)^2\frac{r}{(1+r^2)^2}dr\\ &+\frac{1}{AR^2}\int_\delta^R g(r)^2\frac1r dr.\end{aligned}$$

But $\frac{1}{AR^2r}\le\frac12\frac{r}{(1+r^2)^2}$ for $r\in[\delta,R]$ if $A=4(1+\frac{1}{\delta^2})$ and $R\ge1$. Choosing $A=4(1+\frac{1}{\delta^2})$ and $R\ge2$ we have

$$\begin{aligned}\int_\delta^R g(r)^2\frac{r}{(1+r^2)^2}dr\le&[2AR^2+2(\log R-\log\delta)]\int_\delta^R g'(r)^2\frac{r}{(1+r^2)^2}dr\\ &+4\delta^2\int_0^\delta g(r)^2\frac{r}{(1+r^2)^2}dr\end{aligned} \tag{9.25}$$

With $\delta>0$ still to be chosen we get from (9.24) for $0<x<\delta$

$$g(x)^2\le 2\frac{g(R)^2}{R^4}+2(\log R-\log x)\int_0^R g'(r)^2\frac{r}{(1+r^2)^2}dr+4x^2\int_0^\delta g(r)^2\frac{r}{(1+r^2)^2}dr.$$

Integrating we get

$$\int_0^\delta g(r)^2\frac{r}{(1+r^2)^2}dr\le\delta^2\frac{g(R)^2}{R^4}+2\log R\int_0^R g'(r)^2\frac{r}{(1+r^2)^2}dr+\delta^4\int_0^\delta g(r)^2\frac{r}{(1+r^2)^2}dr. \tag{9.26}$$

Using the condition (9.23) we obtain

$$\int_0^R g(r)\frac{r}{(1+r^2)^2}dr = \frac{1}{2}\int_0^R g(r)\frac{d}{dr}\Big(\frac{r^2}{1+r^2}\Big)dr$$
$$= \frac{1}{2}g(R)\frac{R^2}{1+R^2} - \frac{1}{2}\int_0^R g'(r)\frac{r^2}{1+r^2}dr.$$

Then

$$g(R)^2 \le 4R^4\int_0^R g'(r)^2\frac{r}{(1+r^2)^2}dr.$$

Using this combined with (9.26) we get

$$\int_0^\delta g(r)^2\frac{r}{(1+r^2)^2}dr \le \delta^2 4\int_0^R g'(r)^2\frac{r}{(1+r^2)^2}dr + 2\log R\int_0^R g'(r)^2\frac{r}{(1+r^2)^2}dr$$
$$+ \delta^4\int_0^\delta g(r)^2\frac{r}{(1+r^2)^2}dr.$$

Taking $\delta = \frac{1}{2}$ (this fixes A) gives

$$\int_0^\delta g(r)^2\frac{r}{(1+r^2)^2}dr \le 4(\log R + 1)\int_0^R g'(r)^2\frac{r}{(1+r^2)^2}dr.$$

Combining this with (9.25) we get

$$\int_0^R g(r)^2\frac{r}{(1+r^2)^2}dr \le CR^2\int_0^R g'(r)^2\frac{r}{(1+r^2)^2}dr.$$

□

10. Linear theory: a decomposition

Here we consider

$$\begin{cases} \partial_\tau\phi = L[\phi] + B[\phi] + h, & \text{in } \mathbb{R}^2\times(\tau_0,\infty), \\ \phi(\cdot,\tau_0) = \phi_0 & \text{in } \mathbb{R}^2. \end{cases} \tag{10.1}$$

The results of this section are going to be used later only in the case of radial functions, so we make this assumption here. We write in the rest of this section $\phi = \phi(y,\tau) = \phi(\rho,\tau)$, where $y\in\mathbb{R}^2$, $\rho = |y|$.

The operator B is assumed to be one of the following two:

$$B[\phi] = \zeta(\tau)(2\phi + y\cdot\nabla\phi) = \zeta(\tau)\nabla\cdot(y\phi), \tag{10.2}$$

or

$$B[\phi] = \zeta(\tau)y\cdot\nabla\phi, \tag{10.3}$$

where

$$\zeta(\tau) = -\frac{\zeta_0}{\tau\log\tau} + O\Big(\frac{1}{\tau(\log\tau)^{1+\sigma_0}}\Big), \quad \text{as } \tau\to\infty,$$

for some constants $\zeta_0 > 0$, $0 < \sigma_0 < 1$.

We assume that $\|h\|_{**} < \infty$ where

$$\|h\|_{**} = \inf K, \quad \text{such that}$$
$$|h(y,\tau)| \le K\frac{1}{\tau^\nu(\log\tau)^m}\frac{1}{(1+|y|)^{6+\sigma}}\min\Big(1,\frac{\tau^{\epsilon/2}}{|y|^\epsilon}\Big), \quad \tau>\tau_0,\ y\in\mathbb{R}^2,$$

where $\nu > 1$, $\epsilon > 0$, $\sigma > 0$, $m\in\mathbb{R}$. This is the same norm as in (8.6).

We also assume that h has zero mass

$$\int_{\mathbb{R}^2} h(y,\tau)dy = 0 \quad \text{for all } \tau > \tau_0, \tag{10.4}$$

and the same for the initial condition

$$\int_{\mathbb{R}^2}\phi_0 dy = 0. \tag{10.5}$$

It follows from the equation (10.1), (10.4), and (10.5) that the solution ϕ to (10.1) defined in §8 satisfies

$$\int_{\mathbb{R}^2} \phi(y,\tau)dy = 0 \quad \text{for all } \tau > \tau_0.$$

We recall the decomposition of ϕ introduced in §9.2. Given $\phi : \mathbb{R}^2 \to \mathbb{R}$ with sufficient decay and mass zero, we let $g = \frac{\phi}{U} - (-\Delta^{-1})\phi$, and define a so that $\int_{\mathbb{R}^2}(g+a)U dy = 0$. Then define $g^\perp = g + a$, $\psi^\perp = \psi - a(1 + \frac{1}{2}z_0)$, and

$$\phi^\perp = \phi - \frac{a}{2} Z_0. \tag{10.6}$$

Actually a is directly computed by

$$a = -\frac{1}{8\pi}\int_{\mathbb{R}^2} Ug = \frac{1}{8\pi}\int_{\mathbb{R}^2} U(-\Delta)^{-1}\phi = \frac{1}{8\pi}\int_{\mathbb{R}^2}\Gamma_0\phi. \tag{10.7}$$

In the time dependent situation $a = a(\tau)$ and all functions depend on $y \in \mathbb{R}^2$ and τ.

A difficulty to obtain estimates is the presence of a kernel in the linear operator if $B = 0$, since Z_0 satisfies $L[Z_0] = 0$. It can be proved that the solution ϕ of (10.1) with zero initial condition and $\|h\|_{**} < \infty$ has the bound

$$\sup_y |\phi(y,\tau)| \le C\Big(\frac{\log\tau_0}{\log\tau}\Big)^{2\zeta_0 - \sigma_0}\|h\|_{**},$$

and probably this estimate cannot be improved much. Also ϕ has a some decay at spatial infinity and in particular it has finite second moment

$$\int_{\mathbb{R}^2} |\phi(y,\tau)|\,|y|^2\,dy < \infty, \quad \tau > \tau_0.$$

Therefore Z_0 doesn't describe well the class of solution we want to consider, even for the case $B = 0$, in which $\zeta(\tau) \equiv 0$.

A better candidate to describe the solutions ϕ of (10.1) with zero initial condition and $\|h\|_{**} < \infty$ is obtained by considering the initial value problem

$$\begin{cases} \partial_\tau Z_B = L[Z_B] + B[Z_B], & \text{in } \mathbb{R}^2 \times (\tau_0,\infty), \\ Z_B(\cdot,\tau_0) = \tilde Z_0 & \text{in } \mathbb{R}^2. \end{cases} \tag{10.8}$$

where $\tilde Z_0$ is defined in (6.4). Note that since Z_0 has mass zero and decays like $1/\rho^4$ we have $m_{Z_0} = O(\frac{1}{\tau_0})$.

We will then consider the problem

$$\begin{cases} \partial_\tau \phi = L[\phi] + B[\phi] + h, & \text{in } \mathbb{R}^2 \times (\tau_0,\infty), \\ \phi(\cdot,\tau_0) = c_1\tilde Z_0 & \text{in } \mathbb{R}^2, \end{cases} \tag{10.9}$$

for radial functions ϕ, h, ϕ_0, where $c_1 \in \mathbb{R}$ is a parameter. We assume that $\|h\|_{**} < \infty$.

Proposition 10.1. *Let us assume that $1 < \nu < \frac{7}{4}$. Then there is $C > 0$ such that for any τ_0 sufficiently large the following holds. Suppose that $\|h\|_{**} < \infty$ is radially symmetric and satisfies the zero mass condition (10.4). Then there exists c_1 such that the solution $\phi = \phi^\perp + \frac{a}{2}Z_0$ of (10.9) satisfies*

$$|a(\tau)| \le C\frac{f(\tau)R(\tau)^2}{(\log\tau_0)^{1-q}}\|h\|_{**},$$

$$|\phi^\perp(\rho,\tau)| \le Cf(\tau)R(\tau)\frac{1}{1+|y|^2}\|h\|_{**}. \tag{10.10}$$

where $R(\tau) > 0$ is defined by

$$R(\tau)^2 = \frac{\tau}{(\log\tau)^q}, \tag{10.11}$$

where $0 < q < 1$, and

$$f(\tau) = \frac{1}{\tau^\nu(\log\tau)^m}. \tag{10.12}$$

Moreover c_1 is a linear function of h and satisfies

$$|c_1| \le C \frac{f(\tau_0)R(\tau_0)^2}{(\log \tau_0)^{1-q}} \|h\|_{**}.$$

We always decompose ϕ as in (10.6):

$$\phi = \phi^\perp + \frac{a(\tau)}{2} Z_0$$

and write

$$g = \frac{\phi}{U} - (-\Delta)^{-1}\phi, \quad g^\perp = \frac{\phi^\perp}{U} - (-\Delta)^{-1}\phi^\perp.$$

Let us denote

$$\omega(\tau) = \Big(\int_{\mathbb{R}^2 \setminus B_{R(\tau)}(0)} U g(\tau)^2 \Big)^{1/2}. \tag{10.13}$$

The strategy for the proof of Proposition 10.1 is contained in the following lemmas. The first one is an a-priori estimate for the solution, assuming that $a(T_2) = 0$ for some T_2.

Lemma 10.1. *There is C such that for τ_0 large the following holds. Suppose that $\|h\|_{**} < \infty$ is radially symmetric and satisfies the zero mass condition (10.4) and consider (10.9). Let $\phi^\perp$, a be the decomposition (10.6). Suppose that for some $c_1 \in \mathbb{R}$ there is $T_2 > \tau_0$ is such that*

$$a(T_2) = 0.$$

Then

$$|a(\tau)| \le C \frac{f(\tau)R(\tau)^2}{(\log \tau_0)^{1-q}} \|h\|_{**}, \quad \tau \in [\tau_0, T_2] \tag{10.14}$$

$$|\omega(\tau)| \le C \frac{f(\tau)R(\tau)}{(\log \tau_0)^{1-q}} \|h\|_{**}, \quad \tau \in [\tau_0, T_2] \tag{10.15}$$

$$|c_1| \le C \frac{f(\tau_0)R(\tau_0)^2}{(\log \tau_0)^{1-q}} \|h\|_{**}. \tag{10.16}$$

The constant C is independent of T_2 and c_1.

There is a variant of the previous lemma, where the hypothesis $a(T_2) = 0$ is replaced by an assumption about its time decay.

Lemma 10.2. *There is C such that for τ_0 large the following holds. Suppose that $\|h\|_{**} < \infty$ is radially symmetric and satisfies the zero mass condition (10.4) and consider (10.9). Let $\phi^\perp$, a be the decomposition (10.6). Suppose that for some $c_1 \in \mathbb{R}$,*

$$\frac{a}{fR^2} \in L^\infty(\tau_0, \infty).$$

Then

$$|a(\tau)| \le C \frac{f(\tau)R(\tau)^2}{(\log \tau_0)^{1-q}} \|h\|_{**}, \quad \tau > \tau_0 \tag{10.17}$$

$$|\omega(\tau)| \le C \frac{f(\tau)R(\tau)}{(\log \tau_0)^{1-q}} \|h\|_{**}, \quad \tau > \tau_0, \tag{10.18}$$

$$|c_1| \le C \frac{f(\tau_0)R(\tau_0)^2}{(\log \tau_0)^{1-q}} \|h\|_{**}. \tag{10.19}$$

Lemma 10.3. *Let Z_B be the solution to (10.8) and write it as $Z_B = Z_B^\perp + \frac{a_Z}{2} Z_0$ according to the decomposition (10.6). Then $a_Z(\tau) \neq 0$ for all $\tau \ge \tau_0$.*

Lemma 10.4. *There is C such that for τ_0 large the following holds. Suppose that $\|h\|_{**} < \infty$ is radially symmetric and satisfies the zero mass condition (10.4). Then there is a unique $c_1 \in \mathbb{R}$ such that the solution $\phi = \phi^\perp + \frac{a}{2} Z_0$ of (10.9) (as in (10.6)) satisfies (10.17), (10.18) and (10.19).*

In the first results we do some computations and obtain some estimates, which are used as technical steps in the main argument.

The next lemma is a calculation to help us deal with the term B when we multiply the equation by a suitable test function. It holds for operators more general than B as in (10.2) and (10.3). Let

$$\tilde{B}[\phi] = \zeta_1(\tau)\phi + \zeta_2(\tau) y \cdot \nabla \phi,$$

with $\zeta_1(\tau)$, $\zeta_2(\tau)$ satisfying

$$|\zeta_i(\tau)| \le \frac{C}{\tau \log \tau} \quad \text{for all } \tau > \tau_0. \tag{10.20}$$

Lemma 10.5. *We have*

$$\left| \int_{\mathbb{R}^2} \tilde{B}[\phi] g^\perp \right| \le \frac{C}{\tau \log \tau} \int_{\mathbb{R}^2} U (g^\perp)^2 dy + C \frac{|a(\tau)|}{\tau \log \tau} \| \nabla g^\perp U^{\frac12} \|_{L^2}. \tag{10.21}$$

Proof. We have

$$\int_{\mathbb{R}^2} \tilde{B}[\phi] g^\perp dy = \int_{\mathbb{R}^2} [\zeta_1(\tau)\phi + \zeta_2(\tau) y \cdot \nabla \phi] g^\perp dy.$$

By Lemma 9.3 and the hypothesis (10.20) we have

$$\left| \zeta_1(\tau) \int_{\mathbb{R}^2} \phi g^\perp dy \right| \le \frac{C}{\tau \log \tau} \int_{\mathbb{R}^2} U (g^\perp)^2 dy. \tag{10.22}$$

Let us write

$$\int_{\mathbb{R}^2} y \cdot \nabla \phi(y) g^\perp(y) dy = \int_{\mathbb{R}^2} y \cdot \nabla \phi^\perp(y) g^\perp(y) dy + \frac{a(\tau)}{2} \int_{\mathbb{R}^2} y \cdot \nabla Z_0(y) g^\perp(y) dy.$$

We claim that

$$\left| \int_{\mathbb{R}^2} y \cdot \nabla \phi^\perp(y) g^\perp(y) dy \right| \le C \int_{\mathbb{R}^2} (g^\perp)^2 U dy. \tag{10.23}$$

Indeed, we write

$$\int_{\mathbb{R}^2} y \cdot \nabla \phi^\perp(y) g^\perp(y) dy = \int_{\mathbb{R}^2} y \cdot \nabla (U g^\perp) g^\perp(y) dy + \int_{\mathbb{R}^2} y \cdot \nabla (U \psi^\perp) g^\perp(y) dy. \tag{10.24}$$

But

$$\begin{aligned} \int_{\mathbb{R}^2} y \cdot \nabla (U g^\perp) g^\perp(y) dy &= \int_{\mathbb{R}^2} y \cdot \nabla U (g^\perp)^2(y) dy + \int_{\mathbb{R}^2} U y \cdot \nabla g^\perp g^\perp(y) dy \\ &= \int_{\mathbb{R}^2} y \cdot \nabla U (g^\perp)^2(y) dy + \frac12 \int_{\mathbb{R}^2} U y \cdot \nabla [(g^\perp)^2](y) dy \\ &= \frac12 \int_{\mathbb{R}^2} y \cdot \nabla U (g^\perp)^2(y) dy - \int_{\mathbb{R}^2} U (g^\perp)^2(y) dy, \end{aligned}$$

and so

$$\left| \int_{\mathbb{R}^2} y \cdot \nabla (U g^\perp) g^\perp(y) dy \right| \le C \int_{\mathbb{R}^2} (g^\perp)^2 U dy. \tag{10.25}$$

The second term in (10.24) is:

$$\int_{\mathbb{R}^2} y \cdot \nabla (U \psi^\perp) g^\perp(y) dy = \int_{\mathbb{R}^2} (y \cdot \nabla U) \psi^\perp g^\perp(y) dy + \int_{\mathbb{R}^2} U (y \cdot \nabla \psi^\perp) g^\perp(y) dy.$$

We estimate the first term above

$$\begin{aligned} \left| \int_{\mathbb{R}^2} (y \cdot \nabla U) \psi^\perp g^\perp(y) dy \right| &\le C \Big(\int_{\mathbb{R}^2} (\psi^\perp)^2 U dy \Big)^{1/2} \Big(\int_{\mathbb{R}^2} (g^\perp)^2 U dy \Big)^{1/2} \\ &\le C \int_{\mathbb{R}^2} (g^\perp)^2 U dy, \end{aligned} \tag{10.26}$$

by (9.20). To estimate $\int_{\mathbb{R}^2} U (y \cdot \nabla \psi^\perp) g^\perp(y) dy$ we write it using radial symmetry:

$$\int_{\mathbb{R}^2} U (y \cdot \nabla \psi^\perp) g^\perp(y) dy = 2\pi \int_0^\infty U(\rho) (\psi^\perp)'(\rho) g^\perp(\rho) \rho^2 d\rho.$$

We use that $\psi^\perp$ satisfies

$$-\Delta \psi^\perp - U \psi^\perp = U g^\perp \quad \text{in } \mathbb{R}^2, \quad \psi^\perp(\rho, \tau) \to 0 \quad \text{as } \rho \to \infty.$$

Then, by the variations of parameters formula, since that $\int_{\mathbb{R}^2} U g^\perp z_0 dy = 0$, we have

$$(\psi^\perp)'(\rho) = z_0'(\rho) \int_\rho^\infty U(r) g^\perp(r) \bar z_0(r) r\, dr + \bar z_0'(\rho) \int_0^\rho U(r) g^\perp(r) z_0(r) r\, dr,$$

where $\bar z_0$ is a second linear independent function in the kernel of $\Delta + U$ satisfying

$$|\bar z_0(\rho)| \le C(|\log \rho| + 1).$$

We then compute

$$\int_0^\infty U(\rho)(\psi^\perp)'(\rho) g^\perp(\rho)\rho^2 d\rho = I_1 + I_2$$

where

$$I_1 = \int_0^\infty \int_\rho^\infty U(\rho)U(r) z_0'(\rho) \bar z_0(r) g^\perp(r) g^\perp(\rho)\rho^2 r dr d\rho$$

$$I_2 = -\int_0^\infty \int_\rho^\infty U(\rho)U(r) \bar z_0'(\rho) z_0(r) g^\perp(r) g^\perp(\rho)\rho^2 r dr d\rho.$$

We directly check that

$$|I_1| + |I_2| \le C \int_{\mathbb{R}^2} (g^\perp)^2 U dy.$$

From this we get that

$$\left| \int_{\mathbb{R}^2} U(y\cdot \nabla \psi^\perp) g^\perp(y) dy \right| \le C \int_{\mathbb{R}^2} (g^\perp)^2 U dy. \tag{10.27}$$

Combining (10.24), (10.25), (10.26), (10.27) we obtain (10.23).

Next we claim that

$$\left| \int_{\mathbb{R}^2} y\cdot \nabla Z_0(y) g^\perp(y) dy \right| \le C \|\nabla g^\perp U^{\frac12}\|_{L^2}. \tag{10.28}$$

Indeed, write

$$y\cdot\nabla Z_0 = \nabla\cdot(yZ_0) - 2Z_0 = \nabla\cdot(yZ_0 - 2\nabla z_0) - 4Z_0$$

where z_0 is defined in (9.2) and satisfies the linearized Liouville equation $\Delta z_0 + U z_0 = 0$. We have used here that $Z_0 = U z_0$. So

$$\int_{\mathbb{R}^2} y\cdot\nabla Z_0(y) g^\perp(y) dy = -\int_{\mathbb{R}^2} (yZ_0 - 2\nabla z_0)\nabla g^\perp dy - 4\int_{\mathbb{R}^2} g^\perp Z_0 dy.$$

But $\int_{\mathbb{R}^2} Z_0 g^\perp dy = \int_{\mathbb{R}^2} U z_0 g^\perp dy = 0$ by Lemma 9.2, and $|yZ_0 - 2\nabla z_0| \le \frac{C}{|y|^4}$, so

$$\left| \int_{\mathbb{R}^2} y\cdot\nabla Z_0(y) g^\perp(y) dy \right| \le C\Big(\int_{\mathbb{R}^2} \frac{1}{(1+|y|)^4} |\nabla g^\perp|^2 dy \Big)^{\frac12} \le C \|\nabla g^\perp U^{\frac12}\|_{L^2}.$$

This proves (10.28).

From (10.22), (10.23) and (10.28) we conclude the validity of (10.21).

□

In the next lemma we get an estimate for $\int_{\mathbb{R}^2} \phi g^\perp$, but with right hand side that depends on the solution.

Lemma 10.6. *We make the same assumptions of Proposition 10.1. Let f be given by (10.12), ω be defined in (10.13) and let $R : [\tau_0, \infty) \to (0,\infty)$ be continuous. There is $c > 0$, $\varepsilon > 0$ and $C > 0$ such that for τ_0 sufficiently large, if*

$$\sup_{\tau\ge\tau_0} \frac{R^2(\tau)}{\tau\log\tau} \le \varepsilon \tag{10.29}$$

then

$$\partial_\tau \int_{\mathbb{R}^2} \phi g^\perp + \frac{c}{R^2}\int_{\mathbb{R}^2} \phi g^\perp \le C f(\tau)^2 \|h\|_{**}^2 + C\frac{a(\tau)^2}{R^4} + C\frac{\omega(\tau)^2}{R^2},$$

for some constant $c > 0$.

Proof. Equation (10.9) can be written in the form

$$\partial_\tau \phi = \nabla \cdot (U \nabla g^\perp) + B[\phi] + h, \quad \text{in } \mathbb{R}^2 \times (\tau_0, \infty).$$

We multiply this equation by $g^\perp$ and integrate on $\mathbb{R}^2$, using Lemma 9.5:

$$\frac{1}{2}\partial_\tau \int_{\mathbb{R}^2} \phi g^\perp + \int_{\mathbb{R}^2} U|\nabla g^\perp|^2 = \int_{\mathbb{R}^2} B[\phi] g^\perp + \int_{\mathbb{R}^2} h g^\perp. \tag{10.30}$$

Let $H = (-\Delta)^{-1} h$, and observe that, since h is radial and $\int_{\mathbb{R}^2} h dy = 0$,

$$|\nabla H(\rho,\tau)| = \left| \frac{1}{\rho} \int_\rho^\infty h(s,\tau) s ds \right| \le C f(\tau) \|h\|_{**} \frac{1}{(1+\rho)^{5+\sigma}}.$$

It follows that

$$\begin{aligned} \left| \int_{\mathbb{R}^2} h g^\perp \right| &= \left| \int_{\mathbb{R}^2} \nabla \cdot \nabla H g^\perp \right| = \left| \int_{\mathbb{R}^2} \nabla H \cdot \nabla g^\perp \right| \\ &\le \frac{1}{2} \int_{\mathbb{R}^2} U |\nabla g^\perp|^2 + C \int_{\mathbb{R}^2} |\nabla H|^2 U^{-1} \\ &\le \frac{1}{2} \int_{\mathbb{R}^2} U |\nabla g^\perp|^2 + C f(\tau)^2 \|h\|_{**}^2. \end{aligned}$$

This combined with (10.30) gives

$$\frac{1}{2}\partial_\tau \int_{\mathbb{R}^2} \phi g^\perp + \frac{1}{2} \int_{\mathbb{R}^2} U|\nabla g^\perp|^2 \le \left| \int_{\mathbb{R}^2} B[\phi] g^\perp \right| + C f(\tau)^2 \|h\|_{**}^2. \tag{10.31}$$

We use the inequality in Lemma 9.6 to get

$$\frac{c}{R^2} \int_{B_R} (g^\perp - \bar{g}_R^\perp)^2 U \le \int_{\mathbb{R}^2} U |\nabla g^\perp|^2, \tag{10.32}$$

for some $c > 0$, where

$$\bar{g}_R^\perp = \frac{1}{\int_{B_R} U} \int_{B_R} g^\perp U.$$

From

$$\int_{B_R} (g^\perp)^2 U = \int_{B_R} (g^\perp - \bar{g}_R^\perp)^2 U + 2 \int_{B_R} g^\perp \bar{g}_R^\perp U - \int_{B_R} (\bar{g}_R^\perp)^2 U$$

we get

$$\int_{B_R} (g^\perp)^2 U \le 2 \int_{B_R} (g^\perp - \bar{g}_R^\perp)^2 U + C (\bar{g}_R^\perp)^2.$$

so, using (10.32),

$$\frac{c}{R^2} \int_{B_R} (g^\perp)^2 U \le \int_{\mathbb{R}^2} U |\nabla g^\perp|^2 + C \frac{1}{R^2} (\bar{g}_R^\perp)^2,$$

for a new $c > 0$. This implies

$$\frac{c}{R^2} \int_{\mathbb{R}^2} (g^\perp)^2 U \le \int_{\mathbb{R}^2} U |\nabla g^\perp|^2 + C \frac{1}{R^2} (\bar{g}_R^\perp)^2 + C \frac{1}{R^2} \int_{\mathbb{R}^2 \setminus B_R} U (g^\perp)^2.$$

Using that $g^\perp = g + a$ we get

$$\frac{c}{R^2} \int_{\mathbb{R}^2} (g^\perp)^2 U \le \int_{\mathbb{R}^2} U |\nabla g^\perp|^2 + C \frac{1}{R^2} (\bar{g}_R^\perp)^2 + C \frac{\omega^2}{R^2} + C \frac{a^2}{R^4}. \tag{10.33}$$

But

$$\int_{\mathbb{R}^2} g^\perp U dy = 0$$

and this implies

$$\bar{g}_R^\perp = -\frac{1}{\int_{B_R} U} \int_{\mathbb{R}^2 \setminus B_R} g^\perp U$$

so

$$(\bar{g}_R^\perp)^2 \le \frac{C}{R^2} \int_{\mathbb{R}^2 \setminus B_R} (g^\perp)^2 U \le \frac{C a^2}{R^4} + \frac{C}{R^2} \int_{\mathbb{R}^2 \setminus B_R} g^2 U.$$

This combined with (10.33) gives

$$\frac{c}{R^2}\int_{\mathbb{R}^2}(g^\perp)^2U \le \int_{\mathbb{R}^2} U|\nabla g^\perp|^2 + C\frac{a^2}{R^4} + C\frac{\omega^2}{R^2}.$$

We use this together with (10.31) to obtain (for a new $c>0$)

$$\begin{aligned}\frac{1}{2}\partial_\tau\int_{\mathbb{R}^2}\phi g^\perp + \frac{c}{R^2}\int_{\mathbb{R}^2}(g^\perp)^2U + \frac{1}{4}\int_{\mathbb{R}^2}U|\nabla g^\perp|^2&\\ \le \Big|\int_{\mathbb{R}^2}B[\phi]g^\perp\Big| + Cf(\tau)^2\|h\|_{**}^2 + C\frac{a^2}{R^4} + C\frac{\omega^2}{R^2}.&\end{aligned} \tag{10.34}$$

We obtain from Lemma 10.5 and the assumption (10.29) that

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}B[\phi]g^\perp\Big| &\le \frac{C}{\tau\log\tau}\int_{\mathbb{R}^2}(g^\perp)^2U dy + C\frac{|a(\tau)|}{\tau\log\tau}\|\nabla g^\perp U^{\frac12}\|_{L^2}\\ &\le \frac{C}{\tau\log\tau}\int_{\mathbb{R}^2}(g^\perp)^2U dy + \frac{|a(\tau)|^2}{R^4} + C\frac{R^4}{\tau^2(\log\tau)^2}\|\nabla g^\perp U^{\frac12}\|_{L^2}^2.\\ &= C\frac{\varepsilon}{R^2}\int_{\mathbb{R}^2}(g^\perp)^2U dy + \frac{|a(\tau)|^2}{R^4} + C\varepsilon^2\|\nabla g^\perp U^{\frac12}\|_{L^2}^2.\end{aligned} \tag{10.35}$$

Taking $\varepsilon>0$ small, and combining (10.34) and (10.35) we get

$$\partial_\tau\int_{\mathbb{R}^2}\phi g^\perp + \frac{1}{R^2}\int_{\mathbb{R}^2}(g^\perp)^2U \le Cf(\tau)^2\|h\|_{**}^2 + \frac{Ca^2}{R^4} + C\frac{\omega^2}{R^2}.$$

By Lemma 9.3 we obtain

$$\partial_\tau\int_{\mathbb{R}^2}\phi g^\perp + \frac{c}{R^2}\int_{\mathbb{R}^2}\phi g^\perp \le Cf(\tau)^2\|h\|_{**}^2 + C\frac{a^2}{R^4} + C\frac{\omega^2}{R^2},$$

for some constant $c>0$, which is the desired conclusion. □

The next lemma provides a pointwise estimate for $g=\frac{\phi}{U}-(-\Delta^{-1})\phi$ assuming a certain bound for $\|U^{1/2}g\|_{L^2}$.

Lemma 10.7. *Assume $\nu>0$. Let ϕ be the solution to (10.9) as in §8. Suppose that $\tau_1\ge\tau_0$ and*

$$\|g(\tau)U^{\frac12}\|_{L^2(\mathbb{R}^2)} \le K_1 f_1(\tau), \quad \tau\in[\tau_0,\tau_1], \tag{10.36}$$

where $K_1\ge 0$ and

$$f_1(\tau) = \frac{(\log\tau)^\mu}{\tau^{\nu-1}},$$

where $\mu\in\mathbb{R}$. Then

$$|U(y)g(y,\tau)| \le C\Big(K_1 + \frac{\|h\|_{**}}{R(\tau_0)} + \frac{|c_1|}{f_1(\tau_0)}\Big)f_1(\tau)\frac{1}{(1+|y|)^2}, \quad \tau\in[\tau_0,\tau_1].$$

Proof. We define

$$g_0 = Ug,$$

and obtain from (10.1) the equation

$$\begin{aligned}\partial_\tau g_0 = U\partial_\tau g &= \partial_\tau\phi - U(-\Delta^{-1})\partial_\tau\phi\\ &= \nabla\cdot\Big[U\nabla\Big(\frac{g_0}{U}\Big)\Big] - U(-\Delta)^{-1}\left[\nabla\cdot(U\nabla g)\right] + h - U(-\Delta)^{-1}h\\ &\quad + B[g_0] + B[U\psi[g_0]] - U(-\Delta)^{-1}(B[g_0+U\psi[g_0]]),\end{aligned} \tag{10.37}$$

where we regard $\psi[g_0]$ as the operator that maps g_0 to the unique radial solution to

$$-\Delta\psi - U\psi = g_0 \quad \text{in } \mathbb{R}^2, \quad \psi(\rho,\tau)\to 0 \quad \text{as } \rho\to\infty. \tag{10.38}$$

We note that this problem has indeed a solution since $\int_{\mathbb{R}^2} g_0 z_0 dy = 0$ by Lemma 9.2, which is unique by imposing $\psi(\rho,\tau)\to 0$ as $\rho\to\infty$ in the radial setting. This solution is given by the variations of parameters formula

$$\psi(\rho,\tau) = z_0(\rho)\int_\rho^\infty g_0(r,\tau)\bar z_0(r)r\,dr + \bar z_0(\rho)\int_0^\rho g_0(r,\tau)z_0(r)r\,dr,$$

where $\bar z_0$ is a second linear independent function in the kernel of $\Delta+U$ satisfying $|\bar z_0(\rho)|\le C(|\log\rho|+1)$.

We compute

$$\nabla\cdot(U\nabla g) = \Delta g U + \nabla U\cdot\nabla g = \Delta(gU) - \nabla U\cdot\nabla g - g\Delta U,$$

and hence

$$\begin{aligned}(-\Delta)^{-1}[\nabla\cdot(U\nabla g)] &= -gU - (-\Delta)^{-1}\left[\nabla U\cdot\nabla g + g\Delta U\right]\\ &= -gU - v\end{aligned}$$

where

$$v := (-\Delta)^{-1}(\nabla\cdot(g_0\nabla\Gamma_0)). \tag{10.39}$$

We write (10.37) as

$$\partial_\tau g_0 = \Delta g_0 - \nabla g_0\cdot\nabla\Gamma_0 + 2Ug_0 + B[g_0] + \tilde h \tag{10.40}$$

where

$$\tilde h = Uv + B[U\psi[g_0]] - U(-\Delta)^{-1}(B[g_0 + U\psi[g_0]]) + h - U(-\Delta)^{-1}h. \tag{10.41}$$

Note that since we are working with radial functions, we can integrate (10.39) explicitly and obtain

$$v(\rho,\tau) = \int_\rho^\infty g_0(s,\tau)\Gamma_0'(s)ds. \tag{10.42}$$

We claim that for any $y\in\mathbb{R}^2$:

$$\|\tilde h\|_{L^p(B_1(y))} \le C\Big(K_1 + \frac{\|h\|_{**}}{R(\tau_0)}\Big) f_1(\tau)\frac{1}{(1+|y|)^{4-\frac4p}}, \quad \tau\in[\tau_0,\tau_1]. \tag{10.43}$$

Indeed, let us start with

$$\begin{aligned}\int_0^\infty |v(\rho)|^p U(\rho)\rho d\rho &\le \int_0^\infty \Big(\int_\rho^\infty U(s)g(s)^2 sds\Big)^{p/2}\Big(\int_\rho^\infty U(s)\frac{\Gamma_0'(s)^2}{s}ds\Big)^{p/2} U(\rho)\rho d\rho\\ &\le C\|gU^{\frac12}\|_{L^2(\mathbb{R}^2)}^p,\end{aligned} \tag{10.44}$$

which follows from (10.42) and Hölder's inequality

Let us write $\psi=\psi[g_0]$ and $\tilde\psi = \psi\circ\Pi$, where Π is the stereographic projection. Writing (10.38) in S^2 and using standard L^p theory we find that for any $p>2$

$$\|\tilde\psi\|_{L^\infty(S^2)} + \|\nabla_{S^2}\tilde\psi\|_{L^p(S^2)} \le C\|gU^{\frac12}\|_{L^2(\mathbb{R}^2)},$$

which implies

$$\|\psi\|_{L^\infty(\mathbb{R}^2)} + \left(\int_{\mathbb{R}^2}|\nabla\psi|^p U^{1-\frac p2}\right)^{\frac1p} \le C\|gU^{\frac12}\|_{L^2(\mathbb{R}^2)}. \tag{10.45}$$

Let $y\in\mathbb{R}^2$. From (10.36) we see that

$$\|g_0(\cdot,\tau)\|_{L^2(B_1(y))} \le CK_1 f_1(\tau)\frac{1}{(1+|y|)^2}, \quad \tau\in[\tau_0,\tau_1],$$

and from (10.36) and (10.44) we have

$$\|Uv(\cdot,\tau)\|_{L^p(B_1(y))} \le CK_1 f_1(\tau)\frac{1}{(1+|y|)^{4-\frac4p}}, \quad \tau\in[\tau_0,\tau_1]. \tag{10.46}$$

Similarly, inequalities (10.45) and (10.36) imply

$$\|B[U\psi[g_0]]\|_{L^p(B_1(y))} \le CK_1 f_1(\tau)\frac{1}{\tau\log\tau}\frac{1}{(1+|y|)^4}, \quad \tau\in[\tau_0,\tau_1]. \tag{10.47}$$

Let's estimate

$$(-\Delta)^{-1}(B[g_0+U\psi[g_0]]) = \zeta_1(\tau)(-\Delta)^{-1}(y\cdot\nabla(g_0+U\psi[g_0])) + \zeta_2(\tau)(-\Delta)^{-1}(g_0+U\psi[g_0]).$$

Note that $\psi = (-\Delta)^{-1}\phi = (-\Delta)^{-1}(g_0+U\psi)$. But we can estimate ψ from

$$\psi(\rho) = z_0(\rho)\int_\rho^\infty \frac{1}{z_0(r)^2 r}\int_r^\infty g_0(s)z_0(s)sds, \quad \rho>1. \tag{10.48}$$

Then (10.36) yields

$$|\psi(\rho,\tau)| \le \frac{C}{1+\rho}\|gU^{\frac12}\|_{L^2(\mathbb{R}^2)} \le CK_1 f_1(\tau)\frac{1}{1+\rho}, \quad \tau\in[\tau_0,\tau_1], \tag{10.49}$$

and so

$$|U(-\Delta)^{-1}(g_0 + U\psi[g_0])| \le CK_1 f_1(\tau)\frac{1}{1+|y|^5}, \quad \tau \in [\tau_0, \tau_1]. \tag{10.50}$$

Concerning the term $(-\Delta)^{-1}(y\cdot\nabla(g_0+U\psi))$, we notice that if we let $w = g_0+U\psi$, then $\int_{\mathbb{R}^2} y\cdot\nabla w = 0$, and

$$(-\Delta)^{-1}(y\cdot\nabla w)(\rho) = \int_\rho^\infty rw(r,\tau)dr - 2\psi(\rho,\tau).$$

Using (10.36) and (10.49) we get

$$\left|((-\Delta)^{-1}(y\cdot\nabla(g_0 + U\psi)))(\rho,\tau)\right| \le CK_1 f_1(\tau)\frac{1}{1+\rho}, \quad \tau \in [\tau_0, \tau_1].$$

From this and (10.50) we find that

$$|U(-\Delta)^{-1}(B[g_0 + U\psi[g_0]])(y,\tau)| \le CK_1 f_1(\tau)\frac{1}{\tau\log\tau(1+|y|)^5}, \quad \tau \in [\tau_0, \tau_1]. \tag{10.51}$$

Finally the estimates

$$\|h\|_{L^p(B_1(y))} + \|U(-\Delta)^{-1}h\|_{L^p(B_1(y))} \le C\frac{\|h\|_{**}}{R(\tau_0)} f_1(\tau)\frac{1}{(1+|y|)^{4-\frac{4}{p}}} \tag{10.52}$$

are directly obtained.

Combining (10.46), (10.47), (10.51), and (10.52) we deduce (10.43).

From equation (10.40), the estimate (10.43), standard parabolic L^p estimates restricted to $B_1(y)\times(\max(\tau-1,\tau_0),\tau)$ and embedding into Hölder spaces, we deduce that

$$|g_0(y,\tau)| \le C\Big(K_1 + \frac{\|h\|_{**}}{R(\tau_0)} + \frac{|c_1|}{f_1(\tau_0)}\Big) f_1(\tau)\frac{1}{(1+|y|)^2}, \quad \tau \in [\tau_0, \tau_1]. \tag{10.53}$$

This is the desired conclusion. We also get from (10.53):

$$|v(y,\tau)| \le C\Big(K_1 + \frac{\|h\|_{**}}{R(\tau_0)} + \frac{|c_1|}{f_1(\tau_0)}\Big) f_1(\tau)\frac{1}{(1+|y|)^2}, \quad \tau \in [\tau_0, \tau_1]. \tag{10.54}$$

□

In some of the proofs below the following barrier will be useful. Consider the equation

$$\begin{aligned} &\partial_\tau\phi = \Delta_{\mathbb{R}^6}\phi + h \quad \text{in } (\tau_0,\infty)\times\mathbb{R}^6 \\ &\phi(\tau_0,\cdot) = 0 \end{aligned} \tag{10.55}$$

where $\Delta_{\mathbb{R}^6}$ is the laplacian in $\mathbb{R}^6$. Suppose that h has the estimate

$$|h(y,\tau)| \le \frac{1}{\tau^{\gamma+1}}\frac{1}{(1+|y|/\sqrt{\tau})^b}$$

for some $\gamma, b \in \mathbb{R}$.

If $\gamma < 3$ and $\gamma < \frac{b}{2}$ then there is a barrier satisfying

$$C_1\frac{1}{\tau^\gamma}\frac{1}{(1+|y|/\sqrt{\tau})^b} \le \phi(y,\tau) \le C_2\frac{1}{\tau^\gamma}\frac{1}{(1+|y|/\sqrt{\tau})^b}.$$

Indeed, we can consider all functions to be radial and write $\rho = |y|$, $y \in \mathbb{R}^6$. Let

$$\bar\phi(\rho,\tau) = \frac{1}{\tau^\gamma}g\Big(\frac{\rho}{\sqrt{\tau}}\Big), \quad \zeta = \frac{\rho}{\sqrt{\tau}}. \tag{10.56}$$

Then

$$\partial_\tau\bar\phi - \Big(\partial_{\rho\rho} + \frac{5}{\rho}\partial_\rho\Big)\bar\phi = -\frac{1}{\tau^{\gamma+1}}\Big[g''(\zeta) + \frac{5}{\zeta}g'(\zeta) + \frac{\zeta}{2}g'(\zeta) + \gamma g(\zeta)\Big].$$

Let $g_1(\zeta) = \frac{1}{(1+\zeta^2)^{b/2}}$. Since $\gamma < \frac{b}{2}$ we have

$$-\Big[g_1''(\zeta) + \frac{5}{\zeta}g_1'(\zeta) + \frac{\zeta}{2}g_1'(\zeta) + \gamma g_1(\zeta)\Big] \ge \frac{c}{\zeta^b}, \quad \zeta \ge M,$$

for some $c, M > 0$. Let $g_0(\zeta) = e^{-\frac{\zeta^2}{4}}$ be the Gaussian kernel, which satisfies

$$g_0''(\zeta) + \frac{5}{\zeta}g_0'(\zeta) + \frac{\zeta}{2}g_0'(\zeta) + 3g_0(\zeta) = 0.$$

Let $g = C_1 g_0 + g_1$. Since $\gamma < 3$, we can find C_1 large so that

$$-\Big[g''(\zeta) + \frac{5}{\zeta}g'(\zeta) + \frac{\zeta}{2}g'(\zeta) + \gamma g(\zeta)\Big] \geq \frac{c}{1+\zeta^b}, \quad \zeta > 0.$$

Then $\bar\phi$ defined by (10.56) with $g = C_1 g_0 + g_1$ is a supersolution to (10.55).

In the next lemma we improve the spatial decay of $g = \frac{\phi}{U} - (-\Delta^{-1})\phi$.

Lemma 10.8. *Assume $1 < \nu < \frac{7}{4}$. Let ϕ be the solution to (10.9) as in §8. Suppose that $\tau_1 \geq \tau_0$ and*

$$\|g(\tau)U^{\frac12}\|_{L^2(\mathbb{R}^2)} \leq K_1 f_1(\tau), \quad \tau \in [\tau_0, \tau_1],$$

where $K_1 \geq 0$ and

$$f_1(\tau) = \frac{(\log\tau)^\mu}{\tau^{\nu-1}},$$

where $\mu \in \mathbb{R}$. Then

$$|U(\rho)g(\rho,\tau)| \leq C\Big(K_1 + \frac{\|h\|_{**}}{R(\tau_0)} + \frac{|c_1|}{f_1(\tau_0)}\Big) f_1(\tau)\frac{1}{(1+\rho)^4}, \quad \tau \in [\tau_0, \tau_1]. \tag{10.57}$$

Proof. We us the same notation as in Lemma 10.7 and consider (10.40) for $g_0 = Ug$ with $\tilde h$ defined in (10.41). We are going to use barriers to estimate g_0.

We claim that $\tilde h$ satisfies

$$|\tilde h(y,\tau)| \leq C\Big(K_1 + \frac{\|h\|_{**}}{R(\tau_0)}\Big) f_1(\tau)\Big(\frac{1}{(1+|y|)^6} + \frac{1}{\tau\log\tau(1+|y|)^5}\Big), \quad \tau \in [\tau_0,\tau_1]. \tag{10.58}$$

Indeed, from (10.53) and (10.54) we find that

$$|-Uv + h - U(-\Delta)^{-1}h| \leq C\Big(K_1 + \frac{\|h\|_{**}}{R(\tau_0)}\Big) f_1(\tau)\frac{1}{(1+|y|)^6}, \quad \tau \in [\tau_0,\tau_1]. \tag{10.59}$$

To estimate $B[U\psi[g_0]]$ we use (10.49) a similar estimate for $\partial_\rho\psi$, and the assumptions on ζ_1, ζ_2 in (10.20), to obtain

$$|B[U\psi[g_0]| \leq CK_1 f_1(\tau)\frac{1}{\tau\log\tau}\frac{1}{1+|y|^5}, \quad \tau \in [\tau_0,\tau_1].$$

This, (10.59) and (10.51) prove (10.58).

To get better spatial decay we construct a barrier and apply the maximum principle to equation (10.40) in $(\mathbb{R}^2 \setminus B_{R_0}(0)) \times (\tau_0, \tau_1)$, where R_0 is a fixed large constant. Several of constants C below depend on R_0 but we will not keep track of the explicit dependence.

The linear operator for g_0 in (10.40), acting on radial functions with $\rho = |y|$, is given by:

$$\begin{aligned}\partial_\tau g_0 - [\Delta g_0 - \nabla g_0\cdot\nabla\Gamma_0 + B[g_0] + 2Ug_0] &= \partial_\tau g_0 - \partial_{\rho\rho}g_0 - \frac{1}{\rho}\partial_\rho g_0 - \frac{4\rho}{1+\rho^2}\partial_\rho g_0 + O\Big(\frac{1}{1+\rho^4}\Big)g_0 \\ &\quad + O\Big(\frac{1}{\tau\log\tau}\Big)g_0 + O\Big(\frac{1}{\tau\log\tau}\Big)\rho\partial_\rho g_0.\end{aligned}$$

The main part outside of a ball $B_{R_0}(0)$ with R_0 big is given by $\partial_\tau - \partial_{\rho\rho} - \frac{5}{\rho}\partial_\rho$.

By (10.58) we need to construct $\bar g_1$ such that

$$\partial_\tau \bar g_1 - [\Delta\bar g_1 - \nabla\bar g_1\cdot\nabla\Gamma_0 + B[\bar g_1] + 2U\bar g_1] \geq h_1$$

where

$$h_1(\rho,\tau) = f_1(\tau)\Big(\frac{1}{(1+\rho)^6} + \frac{1}{\tau\log\tau(1+\rho)^5}\Big).$$

To construct $\bar g_1$, let $0 < \vartheta < 1$, and let $\tilde g_1(\rho)$ be radial and solve

$$-\Delta_6 \tilde g_1 = \frac{1}{1+\rho^{6-\vartheta}} \quad \text{in } \mathbb{R}^6,$$

such that $\tilde g_1(\rho)(1+\rho^{4-\vartheta})$ is bounded below and above by positive constants. Let

$$\bar g_1(\rho,\tau) = f_1(\tau)\tilde g_1(\rho)\chi_0\Big(\frac{\rho}{\delta\sqrt\tau}\Big) + C_1\frac{f_1(\tau)}{\tau^{2-\vartheta/2}(1+\rho/\sqrt\tau)^5} + C_2\frac{f_1(\tau)}{\tau^{2-\vartheta/2}}e^{-\frac{\rho^2}{4\tau}},$$

For appropriate $\delta > 0$, C_1, and C_2, the function $\bar g_1(\rho,\tau)$ is a supersolution in $(\mathbb{R}^2 \setminus B_{R_0}(0)) \times (\tau_0,\tau_1)$ for the right hand side h_1. More precisely, writing $M = R_0^{2-\vartheta}(K_1 + \frac{\|h\|_{**}}{R(\tau_0)} + \frac{1}{f_1(\tau_0)}|c_1|)$, we have

$$(\partial_\tau - [\Delta - \nabla(\cdot)\cdot\nabla\Gamma_0 + B])\, M\bar g_1 \geq |\tilde h|, \quad \text{in } (\mathbb{R}^2 \setminus B_{R_0}(0)) \times (\tau_0,\tau_1),$$

$$M\bar g_1 \geq |g_0|, \quad \text{on } \rho = R_0,\ \tau \in (\tau_0,\tau_1),$$

because of Lemma 10.7, and

$$M\bar g_1(\tau_0) \geq \left|c_1 U g_{\tilde Z_0}\right|, \quad \text{in } \mathbb{R}^2,$$

where

$$g_{\tilde Z_0} = \frac{\tilde Z_0}{U} - (-\Delta)^{-1}\tilde Z_0,$$

is the function g associated to $\tilde Z_0$ defined in (6.4). We note that $|U g_{\tilde Z_0}(\rho)| \leq C\frac{1}{1+\rho^4}$ and is supported on $\rho \leq 2\sqrt{\tau_0}$. Here we are using that $\nu < \frac{3}{2} + \frac{\vartheta}{2}$. notation

Using the maximum principle we get

$$|g_0(y,\tau)| \leq C\Big(K_1 + \frac{\|h\|_{**}}{R(\tau_0)} + \frac{|c_1|}{f_1(\tau_0)}\Big) f_1(\tau)\frac{1}{(1+\rho)^{4-\vartheta}}, \quad \tau \in [\tau_0,\tau_1].$$

The constant C here depends on R_0, but R_0 is fixed and we will not keep track of the dependence of C on R_0.

By (10.42) and (10.48) we have

$$|\tilde h(y,\tau)| \leq C\Big(K_1 + \frac{\|h\|_{**}}{R(\tau_0)} + \frac{|c_1|}{f_1(\tau_0)}\Big) f_1(\tau)\Big(\frac{1}{(1+\rho)^{6+\sigma}} + \frac{1}{\tau\log\tau(1+\rho)^{6-\vartheta}}\Big).$$

We can now repeat the argument with a new barrier. Consider $\tilde g_2(\rho)$ the radial solution to

$$-\Delta_6 \tilde g_2 = \frac{1}{1+\rho^{6+\sigma}} \quad \text{in } \mathbb{R}^6, \quad c_1\frac{1}{1+\rho^4} \leq \tilde g_2(\rho) \leq c_2\frac{1}{1+\rho^4}, \tag{10.60}$$

where c_1, $c_2 > 0$. Let

$$\bar g_2(\rho,\tau) = f_1(\tau)\tilde g_2(\rho)\chi_0\Big(\frac{\rho}{\delta\sqrt{\tau}}\Big) + C_1\frac{f_1(\tau)}{\tau^2(1+\rho/\sqrt{\tau})^{6-\vartheta}} + C_2\frac{f_1(\tau)}{\tau^2}e^{-\frac{\rho^2}{4\tau}}.$$

For appropriate constants δ, C_1, C_2, and assuming that $\nu < 2 - \frac{\vartheta}{2}$ we get a suitable supersolution and we obtain

$$|g_0(y,\tau)| \leq C\Big(K_1 + \frac{\|h\|_{**}}{R(\tau_0)} + \frac{|c_1|}{f_1(\tau_0)}\Big) f_1(\tau)\frac{1}{(1+\rho)^4}.$$

This proves (10.57).

The restriction on ν were $\nu < \frac{3}{2} + \frac{\vartheta}{2}$ and $\nu < 2 - \frac{\vartheta}{2}$. Choosing $\vartheta = \frac{1}{4}$ we find that for $\nu < \frac{7}{4}$ both barriers work. □

The next result is a technical step used in several places.

Lemma 10.9. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be radial such that $\int_{\mathbb{R}^2}\phi = 0$ and $|\phi(y)| \leq \frac{C}{(1+|y|)^{2+\sigma}}$ for some $\sigma > 0$. Let $g = \frac{\phi}{U} - (-\Delta)^{-1}\phi$ and assume that $\|g\|_{L^\infty} < \infty$. Then*

$$|\phi(y)| \leq C\frac{\|g\|_{L^\infty}}{(1+|y|)^4}. \tag{10.61}$$

Proof. Let $\psi = (-\Delta)^{-1}\phi$. Since ψ satisfies

$$-\Delta\psi - U\psi = Ug \quad \text{in } \mathbb{R}^2, \quad \psi(\rho) \to 0 \quad \text{as } \rho \to \infty,$$

we have necessarily

$$\int_{\mathbb{R}^2} U g z_0 dy = 0.$$

We have the variations of parameters formula

$$\psi(\rho) = z_0(\rho)\int_\rho^\infty \frac{1}{z_0(r)^2 r}\int_r^\infty Ug(s,\tau)z_0(s)s\,ds\,dr, \quad \rho > 1. \tag{10.62}$$

From (10.62) we find

$$|\psi(\rho,\tau)| \leq C\|g\|_{L^\infty}.$$

This and the formula $\phi = Ug + U\psi$ gives (10.61). □

Next we give a proof of Proof of Lemma 10.1, but first we point some estimates of $\tilde Z_0$ defined in (6.4). Using the general decomposition (10.6), we write

$$\tilde Z_0 = \tilde Z_0^\perp + \frac{\tilde a_0}{2} Z_0.$$

By (10.7)

$$\tilde a_0 = \frac{1}{8\pi}\int_{\mathbb{R}^2} \Gamma_0 \tilde Z_0 = 2 + O(\frac{\log \tau_0}{\tau_0}).$$

Hence $\tilde Z_0^\perp$ satisfies

$$\begin{aligned}\tilde Z_0^\perp(\rho) &= \tilde Z_0(\rho) - \frac{\tilde a_0}{2} Z_0(\rho)\\ &= (Z_0(\rho) - m_{Z_0} U(\rho))\chi_3 - \Big(1 + O(\frac{\log\tau_0}{\tau_0})\Big) Z_0(\rho)\\ &= Z_0(\rho)(\chi_3 - 1) + O\Big(\frac{\log\tau_0}{\tau_0}\frac{1}{1+\rho^4}\Big)\end{aligned}$$

where

$$\chi_3(\rho) = \chi_0\Big(\frac{\rho}{3\sqrt{\tau_0}}\Big).$$

Let $\tilde g_0 = \frac{\tilde Z_0}{U} - (-\Delta_y)^{-1}\tilde Z_0$ and $\tilde g_0^\perp = \tilde g_0 + \tilde a_0$. Note that since Z_0 has mass zero and decays like $1/\rho^4$ we have $m_{Z_0} = O(\frac{1}{\tau_0})$. We claim that

$$\Big|\int_{\mathbb{R}^2} \tilde Z_0 g_0^\perp dy\Big| \le C\frac{\log\tau_0}{\tau^2}. \tag{10.63}$$

Indeed, let us use the notation

$$\tilde\psi_0 = (-\Delta_y)^{-1}\tilde Z_0$$

so that

$$\tilde g_0 = \frac{\tilde Z_0}{U} - \tilde\psi_0.$$

Let us write

$$\tilde Z_0 = Z_0 + h,$$

where

$$h = Z_0(\chi_3 - 1) - m_{Z_0} U\chi_3.$$

Since $\Delta z_0 + U z_0 = 0$ and $\lim_{\rho\to\infty} z_0(\rho) = -2$ we have $(-\Delta)^{-1} Z_0 = z_0 + 2$. Therefore

$$\tilde\psi_0 = (-\Delta)^{-1}\tilde Z_0 = z_0 + 2 + (-\Delta)^{-1} h.$$

Since the mass of $\tilde Z_0$ is zero

$$\begin{aligned}\int_{\mathbb{R}^2} \tilde Z_0 \tilde g_0^\perp &= \int_{\mathbb{R}^2} \tilde Z_0 \tilde g_0\\ &= \int_{\mathbb{R}^2} (Z_0 + h)\Big(\frac{Z_0 + h}{U} - \tilde\psi_0\Big) dy\\ &= \int_{\mathbb{R}^2} (Z_0 + h)\Big(\frac{Z_0 + h}{U} - z_0 - 2 - (-\Delta)^{-1} h\Big) dy\\ &= \int_{\mathbb{R}^2} \frac{Z_0^2 + 2Z_0 h + h^2}{U} dy - \int_{\mathbb{R}^2} Z_0(z_0 + 2 + (-\Delta)^{-1} h) dy\\ &\quad - \int_{\mathbb{R}^2} h(z_0 + 2 + (-\Delta)^{-1} h) dy\end{aligned}$$

But $Z_0 = U z_0$ and the mass of h is zero, so

$$\begin{aligned}\int_{\mathbb{R}^2} \tilde Z_0 g^\perp &= \int_{\mathbb{R}^2} z_0 h dy + \int_{\mathbb{R}^2} \frac{h^2}{U} dy - \int_{\mathbb{R}^2} Z_0 (-\Delta)^{-1} h dy - \int_{\mathbb{R}^2} h(-\Delta)^{-1} h dy\\ &= \int_{\mathbb{R}^2} \frac{h^2}{U} dy - \int_{\mathbb{R}^2} h(-\Delta)^{-1} h dy,\end{aligned}$$

because, integrating by parts,

$$\int_{\mathbb{R}^2} Z_0(-\Delta)^{-1}h dy = \int_{\mathbb{R}^2}(-\Delta z_0)(-\Delta)^{-1}h dy = \int_{\mathbb{R}^2} z_0 h.$$

By direct computation

$$|(-\Delta_y)^{-1}h(\rho)| \le C\log(\tau_0)\begin{cases}\frac{1}{\rho^2} & \rho \ge \sqrt{\tau_0}\\ \frac{1}{\tau_0} & \rho \le \sqrt{\tau_0}.\end{cases}$$

With this inequality we estimate

$$\Big|\int_{\mathbb{R}^2}\frac{h^2}{U}dy\Big| \le \frac{C}{\tau^2}$$

and

$$\Big|\int_{\mathbb{R}^2} h(-\Delta)^{-1}h dy\Big| \le C\frac{\log\tau_0}{\tau^2}.$$

This proves (10.63).

Proof of Lemma 10.1. We let R be defined by (10.11). We multiply equation (10.9) by $g^\perp$ and integrate in $\mathbb{R}^2$. Using Lemmas 10.6 and 9.3 we get

$$\partial_\tau\int_{\mathbb{R}^2}\phi g^\perp + \frac{c}{R^2}\int_{\mathbb{R}^2}\phi g^\perp \le Cf(\tau)^2\|h\|_{**}^2 + \frac{Ca^2}{R^4} + \frac{C}{R^2}\omega(\tau)^2, \tag{10.64}$$

for some $c>0$, where

$$\omega(\tau) = \Big(\int_{\mathbb{R}^2\setminus B_{R(\tau)}} g^2 U\Big)^{1/2}.$$

Let us write

$$\|\varphi\|_{\infty,T_2} = \|\varphi\|_{L^\infty(\tau_0,T_2)},$$

and note that

$$\Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2}^2 < \infty, \quad \Big\|\frac{\omega}{Rf}\Big\|_{\infty,T_2} < \infty.$$

The following inequalities are valid for $\tau_0<\tau<T_2$. From (10.64) we get

$$\partial_\tau\int_{\mathbb{R}^2}\phi g^\perp + \frac{c}{R^2}\int_{\mathbb{R}^2}\phi g^\perp \le Cf(\tau)^2\Big(\|h\|_{**}^2 + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2}^2 + \Big\|\frac{\omega}{Rf}\Big\|_{\infty,T_2}^2\Big).$$

By Gronwall's inequality and Lemma 9.3 we get

$$\int_{\mathbb{R}^2}(g^\perp)^2 U \le Cf(\tau)^2R(\tau)^2\Big(\|h\|_{**}^2 + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2}^2 + \Big\|\frac{\omega}{Rf}\Big\|_{\infty,T_2}^2 + c_1^2 D(\tau_0)^2\Big) \tag{10.65}$$

where

$$D(\tau_0) = \frac{1}{f(\tau_0)R(\tau_0)}\frac{\sqrt{\log\tau_0}}{\tau_0},$$

and we have used (10.63).

From (10.65) we find

$$\int_{\mathbb{R}^2} g^2 U \le Cf(\tau)^2R(\tau)^4\Big(\frac{1}{R(\tau_0)^2}\|h\|_{**}^2 + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2}^2 + \frac{1}{R(\tau_0)^2}\Big\|\frac{\omega}{Rf}\Big\|_{\infty,T_2}^2 + c_1^2\frac{D(\tau_0)^2}{R(\tau_0)^2}\Big) \tag{10.66}$$

Using Lemma 10.8 we get

$$|Ug| \le Cf(\tau)R(\tau)^2\Big(\frac{1}{R(\tau_0)}\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} + \frac{1}{R(\tau_0)}\Big\|\frac{\omega}{Rf}\Big\|_{\infty,T_2} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big)\frac{1}{(1+\rho)^4}, \tag{10.67}$$

where we have used that for τ_0 large, $\frac{D(\tau_0)}{R(\tau_0)} < \frac{1}{f(\tau_0)R(\tau_0)^2}$.

We use this to estimate

$$\int_{\mathbb{R}^2\setminus B_R} g^2 U \le Cf(\tau)^2R(\tau)^2\Big(\frac{1}{R(\tau_0)}\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} + \frac{1}{R(\tau_0)}\Big\|\frac{\omega}{Rf}\Big\|_{\infty,T_2} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big)^2,$$

which implies

$$\frac{\omega(\tau)}{R(\tau)f(\tau)} \le C\Big(\frac{1}{R(\tau_0)}\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} + \frac{1}{R(\tau_0)}\Big\|\frac{\omega}{Rf}\Big\|_{\infty,T_2} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big).$$

We deduce that

$$\Big\|\frac{\omega}{Rf}\Big\|_{\infty,T_2} \le C\Big(\frac{1}{R(\tau_0)}\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big). \tag{10.68}$$

Combining this inequality with (10.66) we obtain

$$\int_{\mathbb{R}^2} g^2 U \le C f(\tau)^2 R(\tau)^4 \Big(\frac{1}{R(\tau_0)}\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big)^2, \tag{10.69}$$

and with (10.65) we get

$$\int_{\mathbb{R}^2} (g^\perp)^2 U \le C f(\tau)^2 R(\tau)^2 \Big(\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big)^2. \tag{10.70}$$

Going back to (10.67) we find

$$|Ug(\rho,\tau)| \le C f(\tau) R(\tau)^2 \Big(\frac{1}{R(\tau_0)}\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big)\frac{1}{1+\rho^4}. \tag{10.71}$$

Using Lemma 10.9 we also obtain

$$|\phi(\rho,\tau)| \le C f(\tau) R(\tau)^2 \Big(\frac{1}{R(\tau_0)}\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big)\frac{1}{1+\rho^4}. \tag{10.72}$$

We multiply the equation satisfied by ϕ (10.9) by $|y|^2\chi_0(\frac{y}{R})$, and integrate on $\mathbb{R}^2$

$$\begin{aligned}\partial_\tau \int_{\mathbb{R}^2} \phi |y|^2 \chi_0\Big(\frac{y}{R}\Big)dy = \int_{\mathbb{R}^2} (L[\phi]+h)|y|^2\chi_0(\frac{y}{R})dy + \int_{\mathbb{R}^2} B[\phi]|y|^2\chi_0(\frac{y}{R})dy \\ - \frac{R'(\tau)}{R}\int_{\mathbb{R}^2} \phi|y|^2\nabla\chi_0(\frac{y}{R})\cdot\frac{y}{R}dy,\end{aligned} \tag{10.73}$$

where $R' = \frac{dR}{d\tau}$.

We integrate (10.73) from τ to T_2, use the decomposition (10.6) and that $a(T_2)=0$ to get

$$\begin{aligned}|a(\tau)|\log\tau \le& \Big|\int_\tau^{T_2}\int_{\mathbb{R}^2}(L[\phi(s)]+h)|y|^2\chi_0(\frac{y}{R(s)})dyds\Big| + \Big|\int_\tau^{T_2}\int_{\mathbb{R}^2} B[\phi(s)]|y|^2\chi_0(\frac{y}{R(s)})dyds\Big| \\ &+ \Big|\int_\tau^{T_2}\frac{R'(s)}{R(s)}\int_{\mathbb{R}^2}\phi(s)|y|^2\nabla\chi_0(\frac{y}{R(s)})\cdot\frac{y}{R(s)}dyds\Big| \\ &+ \Big|\int_{\mathbb{R}^2}\phi^\perp(T_2)|y|^2\chi_0\Big(\frac{y}{R(T_2)}\Big)dy\Big| + \Big|\int_{\mathbb{R}^2}\phi^\perp(\tau)|y|^2\chi_0\Big(\frac{y}{R(\tau)}\Big)dy\Big|.\end{aligned} \tag{10.74}$$

By Lemma 9.4 and (10.70)

$$\begin{aligned}\int_{B_{2R(\tau)}} |\phi(\tau)^\perp||y|^2dy &\le CR(\tau)\Big(\int_{\mathbb{R}^2}(\phi^\perp(\tau))^2U^{-1}\Big)^{1/2} \\ &\le CR(\tau)\Big(\int_{\mathbb{R}^2}(g^\perp(\tau))^2U\Big)^{1/2} \\ &\le Cf(\tau)R(\tau)^2\Big(\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big).\end{aligned} \tag{10.75}$$

Analogously,

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\phi^\perp(T_2)|y|^2\chi_0\Big(\frac{y}{R(T_2)}\Big)dy\Big| &\le Cf(T_2)R(T_2)^2\Big(\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big) \\ &\le Cf(\tau)R(\tau)^2\Big(\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big).\end{aligned} \tag{10.76}$$

Integrating by parts

$$\begin{aligned}\Big|\int_\tau^{T_2}\int_{\mathbb{R}^2} B[\phi(s)]|y|^2\chi_0(\frac{y}{R(s)})dyds\Big| \le& C\int_\tau^{T_2}\frac{1}{s\log s}\int_{\mathbb{R}^2}|\phi(y,s)||y|^2\chi_0(\frac{y}{R(s)})dyds \\ &+ C\int_\tau^{T_2}\frac{1}{s\log s}\int_{\mathbb{R}^2}|\phi(y,s)||y|^2|\nabla\chi_0(\frac{y}{R(s)})|dyds.\end{aligned} \tag{10.77}$$

Let's estimate, using (10.72)

$$\begin{aligned}&\int_\tau^{T_2} \frac{1}{s\log s}\int_{\mathbb{R}^2} |\phi(y,s)||y|^2\chi_0(\frac{y}{R(s)})dyds\\&\le C\int_\tau^{T_2}\frac{1}{s}f(s)R(s)^2ds\Big(\frac{1}{R(\tau_0)}\|h\|_{**}+\Big\|\frac{a}{R^2f}\Big\|_{\infty,T_2}+|c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big)\\&\le Cf(\tau)R(\tau)^2\Big(\frac{1}{R(\tau_0)}\|h\|_{**}+\Big\|\frac{a}{R^2f}\Big\|_{\infty,T_2}+|c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big).\end{aligned}$$

The second term in (10.77) is even smaller, and we deduce that

$$\begin{aligned}\Big|\int_\tau^{T_2}\int_{\mathbb{R}^2} B[\phi(s)]|y|^2\chi_0(\frac{y}{R(s)})dyds\Big| &\le Cf(\tau)R(\tau)^2\Big(\frac{1}{R(\tau_0)}\|h\|_{**}+\Big\|\frac{a}{R^2f}\Big\|_{\infty,T_2}\\&\quad+|c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big).\end{aligned}\tag{10.78}$$

From (10.72) we also get

$$\begin{aligned}&\Big|\int_\tau^{T_2}\frac{R'(s)}{R(s)}\int_{\mathbb{R}^2}\phi(s)|y|^2\nabla\chi_0(\frac{y}{R(s)})\cdot\frac{y}{R(s)}dyds\Big|\\&\le C\int_\tau^{T_2}\frac{R'(s)}{R(s)}f(s)R(s)^2ds\Big(\frac{1}{R(\tau_0)}\|h\|_{**}+\Big\|\frac{a}{R^2f}\Big\|_{\infty,T_2}+|c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big)\\&\le Cf(\tau)R(\tau)^2\Big(\frac{1}{R(\tau_0)}\|h\|_{**}+\Big\|\frac{a}{R^2f}\Big\|_{\infty,T_2}+|c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big).\end{aligned}\tag{10.79}$$

Next we look at

$$\begin{aligned}\int_{\mathbb{R}^2}L[\phi]|y|^2\chi_0(\frac{y}{R})dy &= -2\int_{\mathbb{R}^2}U\nabla g\cdot y\chi_0(\frac{y}{R})dy-\frac{1}{R}\int_{\mathbb{R}^2}U|y|^2\nabla g\cdot\nabla\chi_0(\frac{y}{R})dy\\&=2\int_{\mathbb{R}^2}gZ_0\chi_0(\frac{y}{R})dy+\frac{4}{R}\int_{\mathbb{R}^2}gUy\cdot\nabla\chi_0(\frac{y}{R})dy\\&\quad+\frac{1}{R}\int_{\mathbb{R}^2}g|y|^2\nabla U\cdot\nabla\chi_0(\frac{y}{R})dy+\frac{1}{R^2}\int_{\mathbb{R}^2}g|y|^2U\Delta\chi_0(\frac{y}{R})dy.\end{aligned}\tag{10.80}$$

We have $\int_{\mathbb{R}^2}gZ_0=0$ by Lemma 9.2 and therefore, using (10.71), we find that

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}gZ_0\chi_0(\frac{y}{R(\tau)})dy\Big| &\le \Big|\int_{\mathbb{R}^2\setminus B_{R(\tau)}(0)}gZ_0dy\Big|\\&\le f(\tau)\Big(\frac{1}{R(\tau_0)}\|h\|_{**}+\Big\|\frac{a}{R^2f}\Big\|_{\infty}+|c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big).\end{aligned}$$

The remaining terms in (10.80) are estimated using (10.69) or (10.71) and we get

$$\Big|\int_{\mathbb{R}^2}L[\phi]|y|^2\chi_0(\frac{y}{R})dy\Big|\le Cf(\tau)\Big(\frac{1}{R(\tau_0)}\|h\|_{**}+\Big\|\frac{a}{R^2f}\Big\|_{\infty,T_2}+|c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big).$$

Therefore

$$\begin{aligned}\Big|\int_\tau^{T_2}\int_{\mathbb{R}^2}L[\phi(s)]|y|^2\chi_0(\frac{y}{R(s)})dyds\Big| &\le Cf(\tau)R(\tau)^2(\log\tau)^q\Big(\frac{1}{R(\tau_0)}\|h\|_{**}+\Big\|\frac{a}{R^2f}\Big\|_{\infty,T_2}\\&\quad+|c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big).\end{aligned}\tag{10.81}$$

Finally

$$\Big|\int_\tau^{T_2}\int_{\mathbb{R}^2}h|y|^2\chi_0(\frac{y}{R(s)})dyds\Big|\le Cf(\tau)R(\tau)^2(\log\tau)^q\|h\|_{**}.\tag{10.82}$$

From (10.74), (10.75), (10.76), (10.78), (10.79), (10.81), and (10.82) we get

$$|a(\tau)|\log\tau\le Cf(\tau)R(\tau)^2(\log\tau)^q\Big(\|h\|_{**}+\Big\|\frac{a}{R^2f}\Big\|_{\infty,T_2}+|c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big).\tag{10.83}$$

Assuming τ_0 large, we deduce that

$$\Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} \leq \frac{C}{(\log\tau_0)^{1-q}}\Big(\|h\|_{**} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big). \tag{10.84}$$

Note that $a(\tau_0)$ and c_1 are related. Indeed, the initial condition is $\phi_0 = c_1\tilde Z_0 = \phi_0^\perp + \frac{a(\tau_0)}{2}Z_0$ with

$$a(\tau_0) = \frac{c_1}{8\pi}\int_{\mathbb{R}^2}\tilde Z_0\Gamma_0,$$

by (10.7). We note that $\int_{\mathbb{R}^2}\tilde Z_0\Gamma_0 = 16\pi + O(\frac{\log\tau_0}{\tau_0})$. So by (10.84)

$$|c_1| \leq C|a(\tau_0)| \leq C\frac{f(\tau_0)R(\tau_0)^2}{(\log\tau_0)^{1-q}}\|h\|_{**} + C\frac{1}{(\log\tau_0)^{1-q}}|c_1|.$$

For τ_0 large, we deduce that

$$|c_1| \leq C|a(\tau_0)| \leq C\frac{f(\tau_0)R(\tau_0)^2}{(\log\tau_0)^{1-q}}\|h\|_{**}. \tag{10.85}$$

This proves (10.16). Replacing this in (10.84) we get

$$\Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} \leq \frac{C}{(\log\tau_0)^{1-q}}\|h\|_{**}, \tag{10.86}$$

which proves (10.14). Combining (10.68), (10.85) and (10.86) we obtain (10.15).

Finally, we also obtain from (10.72)

$$|\phi(\rho,\tau)| \leq C\frac{f(\tau)R(\tau)^2}{(\log\tau_0)^{1-q}}\frac{1}{1+\rho^4}\|h\|_{**}. \tag{10.87}$$

□

Proof of Lemma 10.2. The proof is a slight modification of the one of Lemma 10.1. Using the same notation as in that proof, integrating (10.73) from τ to $T_2 > \tau$ yields

$$\begin{aligned}\int_{\mathbb{R}^2}\phi(T_2)|y|^2\chi_0\Big(\frac{y}{R(T_2)}\Big)dy - \int_{\mathbb{R}^2}\phi(\tau)|y|^2\chi_0\Big(\frac{y}{R(\tau)}\Big)dy \\ = \int_\tau^{T_2}\int_{\mathbb{R}^2}(L[\phi(s)] + h)|y|^2\chi_0(\frac{y}{R(s)})dyds \\ + \int_\tau^{T_2}\int_{\mathbb{R}^2}B[\phi(s)]|y|^2\chi_0(\frac{y}{R})dyds \\ - \int_\tau^{T_2}\frac{R'(s)}{R(s)}\int_{\mathbb{R}^2}\phi(s)|y|^2\nabla\chi_0(\frac{y}{R(s)})\cdot\frac{y}{R(s)}dyds,\end{aligned}$$

Similarly to (10.83) we obtain

$$\begin{aligned}|a(\tau)|\log\tau \leq Cf(\tau)R(\tau)^2(\log\tau)^q\Big(\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big) \\ + C|a(T_2)|\log(T_2).\end{aligned} \tag{10.88}$$

The assumption $\frac{a}{fR^2} \in L^\infty(\tau_0,\infty)$ implies that

$$\lim_{\tau\to\infty} a(\tau)\log\tau = 0.$$

Letting $T_2 \to \infty$ in (10.88) we obtain

$$|a(\tau)|\log\tau \leq Cf(\tau)R(\tau)^2(\log\tau)^q\Big(\|h\|_{**} + \Big\|\frac{a}{R^2 f}\Big\|_{L^\infty(\tau_0,\infty)} + |c_1|\frac{1}{f(\tau_0)R(\tau_0)^2}\Big).$$

Then the same argument as in Lemma 10.1 gives the estimates for a, ω and c_1. □

Proof of Lemma 10.3. Here Z_B is the solution to (10.8). Assume to the contrary that there is some $T_2 > \tau_0$ such that

$$a_Z(T_2) = 0.$$

We follow the same computations as in the proof of Lemma 10.1 with $h = 0$ and $c_1 = 1$. By the inequality (10.84) in the proof of Lemma 10.1

$$\Big\|\frac{a}{R^2 f}\Big\|_{\infty,T_2} \leq \frac{C}{(\log\tau_0)^{1-q}}\frac{1}{f(\tau_0)R(\tau_0)^2}$$

which implies

$$|a(\tau_0)| \le \frac{C}{(\log \tau_0)^{1-q}}. \tag{10.89}$$

But by (10.7)

$$a(\tau_0) = \frac{1}{8\pi}\int_{\mathbb{R}^2} \Gamma_0 \tilde Z_0 = 2 + O(\frac{\log \tau_0}{\tau_0})$$

which contradicts (10.89). □

Proof of Lemma 10.4. We let T_n be a sequence such that $T_n \to \infty$ as $n \to \infty$. Let $\bar\phi$ be the solution to (10.1) with initial condition equal to 0. This solution exists but for the moment we don't have any control of its asymptotic behavior as $\tau \to \infty$. Let $\bar\phi^\perp$, $\bar a(\tau)$ be the decomposition (10.6) of $\bar\phi$. Let $Z_B^\perp$, $a_Z(\tau)$ be the decomposition (10.6) of Z_B. Using Lemma 10.3 there is $c_n \in \mathbb{R}$ such that

$$\bar a(T_n) + c_n a_Z(T_n) = 0.$$

Let us define

$$\phi_n = \bar\phi + c_n Z_B,$$

and let

$$\phi_n = \phi_n^\perp + \frac{a_n}{2} Z_0$$

be the decomposition (10.6) of ϕ_n. Then by Lemma 10.1 we have

$$|a_n(\tau)| \le C\frac{f(\tau)R(\tau)^2}{(\log\tau_0)^{1-q}}\|h\|_{**}, \quad \tau\in[\tau_0,T_n]$$

$$|\omega_n(\tau)| \le C\frac{f(\tau)R(\tau)}{(\log\tau_0)^{1-q}}\|h\|_{**}, \quad \tau\in[\tau_0,T_n]$$

$$|c_n| \le C\frac{f(\tau_0)R(\tau_0)^2}{(\log\tau_0)^{1-q}}\|h\|_{**}.$$

Moreover, we also have the uniform estimate

$$|\phi_n(\rho,\tau)| \le C\frac{f(\tau)R(\tau)^2}{(\log\tau_0)^{1-q}}\frac{1}{1+\rho^4}\|h\|_{**}$$

for $\tau\in[\tau_0,T_n]$ from (10.87).

By using standard parabolic estimates, passing to a subsequence we may assume that $c_n \to c_1$ and $\phi_n \to \phi$ locally uniformly in space-time, and that ϕ is a solution of (10.9) for some c_1 such that

$$|c_1| \le C\frac{f(\tau_0)R(\tau_0)^2}{(\log\tau_0)^{1-q}}\|h\|_{**}.$$

Moreover ϕ satisfies

$$|\phi(\rho,\tau)| \le C\frac{f(\tau)R(\tau)^2}{(\log\tau_0)^{1-q}}\frac{1}{1+\rho^4}\|h\|_{**}$$

and writing the decomposition (10.6) as $\phi = \phi^\perp + \frac{a}{2}Z_0$ we have

$$|a(\tau)| \le C\frac{f(\tau)R(\tau)^2}{(\log\tau_0)^{1-q}}\|h\|_{**}.$$

We also get

$$|\omega_n(\tau)| \le C\frac{f(\tau)R(\tau)}{(\log\tau_0)^{1-q}}\|h\|_{**},$$

where ω is defined in (10.13).

The uniqueness of c_1 is a consequence of Lemma 10.2. □

Proof of Proposition 10.1. We have already constructed ϕ and c_1 in Lemma 10.4, we have the uniqueness of ϕ and the estimates for a and c_1 in Lemma 10.2.

We only need to prove the estimate for $\phi^\perp$ stated in (10.10). By the construction of ϕ in Lemma 10.4 and (10.70), (10.85) and (10.86), we get

$$\int_{\mathbb{R}^2}(g^\perp)^2 U \le Cf(\tau)^2R(\tau)^2\|h\|_{**}^2, \quad \tau>\tau_0. \tag{10.90}$$

We claim that from this inequality we have

$$U|g^{\perp}(y,\tau)| \le Cf(\tau)R(\tau)\frac{1}{(1+|y|)^2}\|h\|_{**}, \quad \tau > \tau_0.$$

The proof of this estimate is similar to that of (10.57) in Lemma 10.8.

Indeed, we define

$$g_0^{\perp} = Ug^{\perp}$$

and obtain the equation

$$\begin{aligned}\partial_\tau g_0^{\perp} &= \nabla\cdot\Big(U\nabla\Big(\frac{g_0^{\perp}}{U}\Big)\Big) - U(-\Delta)^{-1}\nabla\cdot\Big(U\nabla\Big(\frac{g_0^{\perp}}{U}\Big)\Big)\\ &\quad + h - U(-\Delta)^{-1}h\\ &\quad + B[g_0^{\perp}] - U(-\Delta)^{-1}B[g_0^{\perp}] + B[U\psi[g_0^{\perp}] - U(-\Delta)^{-1}B[U\psi[g_0^{\perp}]\\ &\quad + a'(\tau)U + \frac{a}{2}B[Z_0] - \frac{a}{2}U(-\Delta)^{-1}B[Z_0].\end{aligned} \tag{10.91}$$

Here the notation $\psi[g_0^{\perp}]$ is the one introduced in the proof of Lemma 10.7 in (10.38).

To get an estimate for the solution we need an estimate for $a'(\tau)$. Since $g^{\perp} = g + a$ and $\int_{\mathbb{R}^2} Ug^{\perp} = 0$ we have

$$a(\tau) = -\frac{1}{8\pi}\int_{\mathbb{R}^2} Ug(\tau)dy = -\frac{1}{8\pi}\int_{\mathbb{R}^2} g_0(\tau)dy.$$

But integrating (10.37) we find

$$\begin{aligned}\partial_\tau\int_{\mathbb{R}^2} g_0(\tau)dy &= -\int_{\mathbb{R}^2} U(-\Delta)^{-1}\Big(\nabla\cdot(U\nabla\frac{g_0}{U})\Big)dy - \int_{\mathbb{R}^2} U(-\Delta)^{-1}hdy\\ &\quad - \int_{\mathbb{R}^2} U(-\Delta)^{-1}(B[g_0 + U\psi[g_0]])dy,\end{aligned}$$

which gives the expression

$$\begin{aligned}a'(\tau) &= \frac{1}{8\pi}\int_{\mathbb{R}^2} U(-\Delta)^{-1}\Big(\nabla\cdot(U\nabla\frac{g_0}{U})\Big)dy + \frac{1}{8\pi}\int_{\mathbb{R}^2} U(-\Delta)^{-1}hdy\\ &\quad + \frac{1}{8\pi}\int_{\mathbb{R}^2} U(-\Delta)^{-1}(B[g_0 + U\psi[g_0]])dy.\end{aligned}$$

We claim that

$$|a'(\tau)| \le Cf(\tau)R(\tau)\|h\|_{**}. \tag{10.92}$$

Indeed, we have

$$\begin{aligned}\int_{\mathbb{R}^2} U(-\Delta)^{-1}\Big(\nabla\cdot(U\nabla\frac{g_0}{U})\Big)dy &= \int_{\mathbb{R}^2}\Gamma_0\nabla\cdot(U\nabla g^{\perp})dy = -\int_{\mathbb{R}^2}\nabla U\cdot\nabla g^{\perp}dy\\ &= \int_{\mathbb{R}^2}\Delta U g^{\perp}.\end{aligned}$$

Then, by (10.90)

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} U(-\Delta)^{-1}\Big(\nabla\cdot(U\nabla\frac{g_0}{U})\Big)dy\Big| &\le C\Big(\int_{\mathbb{R}^2}(g^{\perp})^2U\Big)^{1/2}\\ &\le Cf(\tau)R(\tau)\|h\|_{**}.\end{aligned}$$

We also have, for the case of the operator (10.2),

$$\begin{aligned}\int_{\mathbb{R}^2} U(-\Delta)^{-1}(B[g_0])\,dy &= \int_{\mathbb{R}^2}\Gamma_0 B[g_0] = \zeta(\tau)\int_{\mathbb{R}^2}\Gamma_0\nabla\cdot(yg_0)dy\\ &= -\zeta(\tau)\int_{\mathbb{R}^2}\nabla\Gamma_0\cdot yUgdy\end{aligned}$$

But by construction and (10.69), (10.85) and (10.86), we get

$$\Big(\int_{\mathbb{R}^2} g^2U\Big)^{1/2} \le \frac{C}{(\log\tau_0)^{1-q}}f(\tau)R(\tau)^2\|h\|_{**}. \tag{10.93}$$

so, using (10.93)

$$\begin{aligned}\left|\int_{\mathbb{R}^2} U(-\Delta)^{-1}(B[g_0])\,dy\right| &\le \frac{C}{\tau\log\tau}\Big(\int_{\mathbb{R}^2} Ug^2\Big)^{1/2}\\ &\le \frac{C}{\tau\log\tau}\frac{1}{(\log\tau_0)^{1-q}}f(\tau)R(\tau)^2\|h\|_{**}\\ &\le Cf(\tau)\|h\|_{**}\\ &\le Cf(\tau)R(\tau)\|h\|_{**}.\end{aligned}$$

The last term is estimated similarly and we get (10.92).

Repeating the argument in of Lemma 10.7 we obtain from (10.90)

$$|g_0^\perp(y,\tau)| \le Cf(\tau)R(\tau)\|h\|_{**}\frac{1}{(1+|y|)^2}.$$

An argument similar to Lemma 10.9 gives

$$|\phi^\perp(\rho,\tau)| \le Cf(\tau)R(\tau)\frac{1}{(1+|y|)^2}\|h\|_{**}.$$

□

We have an estimate for $\phi^\perp$ stronger than (10.10) under a stricter assumption on ν.

Lemma 10.10. *Let us assume that $1<\nu<\frac{3}{2}$. Under the same assumption of Proposition 10.1 let $\phi=\phi^\perp+\frac{a}{2}Z_0$ be the solution of (10.9). Then*

$$|\phi^\perp(y,\tau)| \le CR(\tau)f(\tau)\|h\|_{**}\begin{cases}\frac{1}{(1+|y|)^2} & |y|\le\sqrt{\tau}\\ \frac{\tau}{|y|^4} & |y|\ge\sqrt{\tau},\end{cases}$$

Proof. We write (10.91) as

$$\partial_\tau g_0^\perp = \Delta g_0^\perp - \nabla g_0^\perp\cdot\nabla\Gamma_0 + 2Ug_0^\perp + B[g_0^\perp] + \tilde h_1 \tag{10.94}$$

where

$$\begin{aligned}\tilde h_1 &= -U(-\Delta)^{-1}(\nabla\cdot(g_0^\perp\nabla\Gamma_0))\\ &\quad - U(-\Delta)^{-1}B[g_0^\perp] + B[U\psi[g_0^\perp]] - U(-\Delta)^{-1}B[U\psi[g_0^\perp]]\\ &\quad + a'(\tau)U + \frac{a}{2}B[Z_0] - \frac{a}{2}U(-\Delta)^{-1}B[Z_0]\\ &\quad + h - U(-\Delta)^{-1}h.\end{aligned}$$

Then, similarly to (10.58), we have

$$|\tilde h_1(y,\tau)| \le Cf(\tau)R(\tau)\|h\|_{**}\frac{1}{(1+|y|)^4}.$$

Let

$$\bar g^\perp(\rho,\tau) = f(\tau)R(\tau)\tilde g_3(\rho)\chi_0\Big(\frac{\rho}{\delta\sqrt{\tau}}\Big) + A_1\frac{f(\tau)R(\tau)}{\tau}\frac{1}{(1+\rho/\sqrt{\tau})^4} + A_2\frac{f(\tau)R(\tau)}{\tau}e^{-\frac{\rho^2}{4\tau}}$$

where $-\Delta_6\tilde g_3=\frac{1}{1+\rho^4}$ with $\tilde g_2(\rho)\to0$ as $\rho\to\infty$. If $\nu<\frac{3}{2}$, for appropriate positive constants δ, A_1, A_2, and C, the function $C\|h\|_{**}\bar g^\perp$ is supersolution to (10.94) in $\{(y,\tau)|\tau>\tau_0,\ |y|>R_0\}$. We deduce that

$$|g_0^\perp(y,\tau)| \le Cf(\tau)R(\tau)\|h\|_{**}\frac{\min(1,\frac{\tau}{|y|^2})}{(1+|y|)^2}.$$

An argument similar to Lemma 10.9 gives

$$|\phi^\perp(\rho,\tau)| \le Cf(\tau)R(\tau)\|h\|_{**}\begin{cases}\frac{1}{1+|y|^2} & |y|\le\sqrt{\tau}\\ \frac{\tau}{|y|^4} & |y|\ge\sqrt{\tau}.\end{cases}$$

□

Proof of Proposition 8.1. By Proposition 10.1 there is c_1 such that the solution ϕ to (10.9) has the properties stated in Proposition 10.1. We recall that by (10.87) ϕ satisfies

$$|\phi(\rho,\tau)| \le C\frac{f(\tau)R(\tau)^2}{(\log\tau_0)^{1-q}}\frac{1}{1+\rho^4}\|h\|_{**}. \tag{10.95}$$

We will construct a barrier to estimate ϕ for $|y|\ge R_0$, where R_0 is a large constant. We consider the equation (10.9) in $\mathbb{R}^2\setminus B_{R_0}(0)$ written in the form

$$\partial_\tau\phi = \Delta\phi - 4\nabla\Gamma_0\nabla\phi + 2U\phi + B[\phi] + \bar h, \tag{10.96}$$

where

$$\bar h = -\nabla U\nabla\psi + h.$$

Since $\psi = (-\Delta)^{-1}\phi$, from (10.95) we get

$$|\nabla\psi(\rho,\tau)| \le C\frac{f(\tau)R(\tau)^2}{(\log\tau_0)^{1-q}}\frac{1}{1+\rho^3}\|h\|_{**}.$$

This gives

$$|\nabla U\cdot\nabla\psi| \le C\frac{f(\tau)R(\tau)^2}{(\log\tau_0)^{1-q}}\frac{1}{1+\rho^8}\|h\|_{**}. \tag{10.97}$$

By (10.97) and the definition of the norm $\|h\|_{**}$,

$$|\bar h(y,\tau)| \le C\frac{f(\tau)R(\tau)^2}{(\log\tau_0)^{1-q}}\frac{1}{(1+\rho)^{6+\sigma}}\min\Big(1,\frac{\tau^{\epsilon/2}}{\rho^\epsilon}\Big)\|h\|_{**},$$

where we have used that $\sigma+\epsilon<2$. Let $\tilde g_2$ be defined by (10.60) and let

$$\bar\phi(\rho,\tau) = f(\tau)R(\tau)^2\tilde g_2(\rho)\chi_0\Big(\frac{\rho}{\delta\sqrt{\tau}}\Big) + A_1\frac{f(\tau)R(\tau)^2}{\tau^2}\frac{1}{(1+\rho/\sqrt{\tau})^{6+\sigma+\epsilon}} + A_2\frac{f(\tau)R(\tau)^2}{\tau^2}e^{-\frac{\rho^2}{4\tau}}.$$

Then for suitable positive constants δ, A_1, A_2, and C, the function $C(\log\tau_0)^{q-1}\|h\|_{**}\bar\phi$ is a super-solution to (10.96) in $\{(y,\tau)|\tau>\tau_0,\ |y|>R_0\}$. For this we need $\nu<2$. Moreover $|\phi(\rho,\tau)|\le C\bar\phi(\rho,\tau)(\log\tau_0)^{q-1}\|h\|_{**}$ at $\rho=R_0$ by (10.95). By the maximum principle

$$|\phi(y,\tau)| \le C\bar\phi(y,\tau)(\log\tau_0)^{q-1}\|h\|_{**},\quad |y|>R_0.$$

This gives the explicit bound

$$|\phi(\rho,\tau)| \le C\frac{f(\tau)R(\tau)^2}{(\log\tau_0)^{1-q}}\frac{1}{(1+\rho^4)}\min\Big(1,\frac{\tau^{1/2}}{\rho}\Big)^{2+\sigma+\epsilon}\|h\|_{**}$$

□

We include here some results that will be useful later. Let

$$\hat Z_0 = L[\tilde Z_0].$$

Lemma 10.11. *The function $\hat Z_0$ satisfies*

$$|\hat Z_0(\rho)| \le C\frac{1}{\tau_0(1+\rho)^4} \tag{10.98}$$

and is supported on $\rho\le 2\tau_0$.

Proof. Let $\psi = (-\Delta)^{-1}\tilde Z_0$ and $g = \frac{\tilde Z_0}{U} - \psi$. By (6.4) and using that $Z_0 = Uz_0$, z_0 defined in (9.2),

$$g = \frac{(Z_0 - m_{Z_0}U)\chi}{U} - \psi = z_0\chi - m_{Z_0}\chi - \psi,$$

where $\chi(\rho) = \chi_0(\frac{\rho}{\sqrt{\tau_0}})$. Note that $\tilde Z_0$ has mass zero and support in $B_{2\sqrt{\tau_0}}$. It follows that ψ has also support contained in $B_{2\sqrt{\tau_0}}$ and then g has support contained in $B_{2\sqrt{\tau_0}}$. Therefore $\hat Z_0 = L[\tilde Z_0] = \nabla\cdot(U\nabla g)$ has also support contained in $B_{2\sqrt{\tau_0}}$.

To get an estimate for $\hat Z_0$ let us write

$$\psi = (-\Delta)^{-1}(Z_0 - m_{Z_0}U)\chi) = (-\Delta)^{-1}Z_0 + \psi_1,$$

where

$$\psi_1 = (-\Delta)^{-1}(Z_0(\chi-1) - m_{Z_0}U\chi).$$

Since $\Delta z_0 + U z_0 = 0$ and $\lim_{\rho\to\infty} z_0(\rho) = -2$ we have $(-\Delta)^{-1} Z_0 = z_0 + 2$. So

$$\psi = z_0 + 2 + \psi_1$$

Hence

$$g = z_0(\chi - 1) - 2 - m_{Z_0}\chi - \psi_1$$

and so

$$\begin{aligned}\hat{Z}_0 = L[\tilde{Z}_0] &= \nabla\cdot(U\nabla g)\\ &= \nabla\cdot(U(\nabla z_0(\chi-1) + z_0\nabla\chi - m_{Z_0}\nabla\chi - \nabla\psi_1)).\end{aligned} \tag{10.99}$$

Using radial symmetry and $m_{Z_0} = O(\frac{1}{\tau_0})$ we get

$$|\nabla\psi_1(\rho)| \le C\frac{1}{\tau_0(1+\rho)}.$$

From this and (10.99) we get (10.98). □

Consider the initial value problem

$$\begin{cases}\partial_\tau\phi_1 = L[\phi_1] + B[\phi_1] & \text{in } \mathbb{R}^2\times(\tau_0,\infty),\\ \phi_1(\cdot,\tau_0) = \hat{Z}_0 & \text{in } \mathbb{R}^2.\end{cases} \tag{10.100}$$

Lemma 10.12. *Let $0<\gamma<2$. Let $1<\nu_0<\frac{7}{4}$*

$$f_0(\tau) = \frac{1}{\tau^{\nu_0}}.$$

and let $R(\tau)$ be as in (10.11). Then the solution ϕ_1 of (10.100) satisfies

$$|\phi_1(\rho,\tau)| \le C\frac{f_0(\tau)R(\tau)^2}{\tau_0 f_0(\tau_0)R(\tau_0)^2}\frac{1}{(1+\rho^4)}\min\Big(1,\frac{\tau^{1/2}}{\rho}\Big)^{2+\gamma}.$$

Proof. A suitable modification in the proof of Proposition 10.1 gives the following result. Consider

$$\begin{cases}\partial_\tau\phi = L[\phi] + B[\phi] & \text{in } \mathbb{R}^2\times(\tau_0,\infty),\\ \phi(\cdot,t_0) = \phi_0 + c_1\tilde{Z}_0 & \text{in } \mathbb{R}^2,\end{cases} \tag{10.101}$$

Then there is $C>0$ such that for any τ_0 sufficiently large the following holds. Suppose that ϕ_0 is a radial function with zero mass in $\mathbb{R}^2$, supported in $B_{2\sqrt{\tau_0}}(0)$, and such that

$$|\phi_0(\rho)| \le M\frac{1}{1+\rho^4}.$$

Then there exists c_1 such that the solution ϕ of (10.101) satisfies

$$|\phi(\rho,\tau)| \le CM\frac{f_0(\tau)R(\tau)^2}{f_0(\tau_0)R(\tau_0)^2}\frac{1}{(1+\rho^4)}\min\Big(1,\frac{\tau^{1/2}}{\rho}\Big)^{2+\gamma}.$$

Moreover c_1 is a linear function of ϕ_0 and satisfies

$$|c_1| \le CM\frac{1}{(\log\tau_0)^{1-q}}.$$

Let us apply this statement to $\phi_0 = L[\tilde{Z}_0]$, which is radial, with mass zero, support in $B_{2\sqrt{\tau_0}}(0)$, and satisfies

$$|\phi_0(\rho)| \le \frac{1}{\tau_0}\frac{1}{1+\rho^4},$$

by Lemma 10.11. Then there exists c_1 such that the solution $\tilde{\phi}$ to (10.101) with $\phi_0 = L[\tilde{Z}_0]$ satisfies

$$|\tilde{\phi}(\rho,\tau)| \le C\frac{f_0(\tau)R(\tau)^2}{\tau_0 f_0(\tau_0)R(\tau_0)^2}\frac{1}{(1+\rho^4)}\min\Big(1,\frac{\tau^{1/2}}{\rho}\Big)^{2+\gamma}. \tag{10.102}$$

We claim that $c_1 = 0$. To prove this, we multiply (10.101) by $|y|^2$ and integrate on $\mathbb{R}^2\times(\tau_0,\infty)$. Let's work with

$$B[\phi] = \zeta(\tau)\nabla\cdot(y\phi).$$

The case of the operator (10.3) is similar. Then we get

$$\partial_\tau\int_{\mathbb{R}^2}\tilde{\phi}(y,\tau)|y|^2dy = -2\zeta(\tau)\int_{\mathbb{R}^2}\tilde{\phi}(y,\tau)|y|^2dy,$$

because $\int_{\mathbb{R}^2} L[\phi]|y|^2 dy = 0$, see Remark 9.2. Integrating

$$\int_{\mathbb{R}^2} \tilde\phi(y,\tau)|y|^2 dy = e^{-2\int_{\tau_0}^{\tau}\zeta} \int_{\mathbb{R}^2} \tilde\phi(y,\tau_0)|y|^2 dy = c_1 e^{-2\int_{\tau_0}^{\tau}\zeta} \int_{\mathbb{R}^2} \tilde Z_0(y)|y|^2 dy,$$

because $\int_{\mathbb{R}^2} L[\tilde Z_0]|y|^2 dy = 0$. Using the asymptotic expansion of ζ one gets

$$e^{-2\int_{\tau_0}^{\tau}\zeta} \to \infty, \quad \text{as } \tau \to \infty.$$

But the bound (10.102) implies that

$$\lim_{\tau\to\infty} \int_{\mathbb{R}^2} \tilde\phi(y,\tau)|y|^2 dy = 0.$$

This only can happen if $c_1 = 0$.

We deduce that ϕ_1 defined in (10.100) coincides with $\tilde\phi$, and then (10.102) holds for ϕ_1. □

11. Linear estimate with second moment (radial)

We will prove in this section Proposition 8.2 in the radial case $h(\rho,\tau)$. In this case $\mu_j \equiv 0$.

Proposition 11.1. *Let $0<\sigma<1$, $\epsilon>0$ with $\sigma+\epsilon<2$ and $1<\nu<\min(1+\frac{\epsilon}{2}, 3-\frac{\sigma}{2}, \frac{5}{4})$. Let $0<q<1$. Then there is C such that for τ_0 large the following holds. Suppose that h satisfies $\|h\|_{\nu,m,6+\sigma,\epsilon}<\infty$ and*

$$\int_{\mathbb{R}^2} h(y,\tau)dy = 0, \quad \int_{\mathbb{R}^2} h(y,\tau)|y|^2 dy = 0.$$

Then the solution $\phi(y,\tau)$ of problem (8.9) satisfies

$$\|\phi\|_{\nu-\frac12, m+\frac{q}{2}, 4, 2+\sigma+\epsilon} \le C\|h\|_{\nu,m,6+\sigma,\epsilon}.$$

To describe the idea of the proof more easily let us consider for a moment the equation (8.9) without B:

$$\begin{cases} \partial_\tau\phi = L[\phi] + h(y,t) & \text{in } \mathbb{R}^2\times(\tau_0,\infty) \\ \phi(\cdot,\tau_0) = 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{11.1}$$

The idea is to formally apply a suitable left inverse L^{-1} of L to (11.1) (to be defined later on in Lemma 11.1). If we call $\Phi = L^{-1}\phi$, $H = L^{-1}h$, then we would like to solve

$$\begin{cases} \partial_\tau\Phi = L[\Phi] + H(y,t) & \text{in } \mathbb{R}^2\times(\tau_0,\infty) \\ \Phi(\cdot,\tau_0) = 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{11.2}$$

In order to get good properties of H, in this step we have already used that h satisfies the second moment condition. At this point we would like to apply Proposition 10.1, which gives a decomposition

$$\Phi = \Phi^\perp + \frac{a(\tau)}{2} Z_0.$$

Note that $\Phi^\perp$ decays in time like $1/\tau^{\nu-1/2}$ and so $\phi = L\Phi$ also decays in time like $1/\tau^{\nu-1/2}$, which is better than the estimate provided by Proposition 8.1. It turns out that H decays in space like $1/\rho^{4+\sigma}$ so we can't apply directly Proposition 10.1 to (11.2). What we do is *concentrate* H by solving first a nicer problem. We write $\Phi = \Phi_1 + \Phi_2$ where Φ_1 is asked to solve

$$\begin{cases} \partial_\tau\Phi_1 = L_0[\Phi_1] + H(y,t) & \text{in } \mathbb{R}^2\times(\tau_0,\infty) \\ \Phi_1(\cdot,\tau_0) = 0 & \text{in } \mathbb{R}^2. \end{cases}$$

where

$$L_0[\phi] = \nabla\cdot\Big(U\nabla\Big(\frac{\phi}{U}\Big)\Big) = \Delta\phi - \nabla\phi\cdot\nabla\Gamma_0 + U\phi. \tag{11.3}$$

Lemma 11.2 below deals with Φ_1. Then the problem for Φ_2 becomes

$$\begin{cases} \partial_\tau\Phi_2 = L_0[\Phi_2] + L[\Phi_1] - L_0[\Phi_1] & \text{in } \mathbb{R}^2\times(\tau_0,\infty) \\ \Phi_1(\cdot,\tau_0) = 0 & \text{in } \mathbb{R}^2. \end{cases}$$

It turns out that the right hand side in this equation has better spatial decay and we can apply Proposition 10.1.

In the next lemmas we give some preliminary results, and the proof of Proposition 11.1 is given at the end of this section.

We define the inverse of L that we use. For $h:\mathbb{R}^2\to\mathbb{R}$ define $\|h\|_{\tau,6+\sigma,\epsilon}$ as the smallest K such that

$$|h(y)|\le\frac{K}{(1+|y|)^{6+\sigma}}\begin{cases}1 & |y|\le\sqrt{\tau}\\ \frac{\tau^{\epsilon/2}}{|y|^{\epsilon}} & |y|\ge\sqrt{\tau}.\end{cases}$$

which depends on τ, treated as parameter here, σ, and ε.

Lemma 11.1. *Let $\sigma,\epsilon>0$. Let $h=h(\rho)$ be radial and satisfy $\|h\|_{\tau,6+\sigma,\epsilon}<\infty$ and*

$$\int_{\mathbb{R}^2}h\,dy=\int_{\mathbb{R}^2}h|y|^2dy=0.$$

Then there exists H radially symmetric such that $L[H]=h$ in $\mathbb{R}^2$ and satisfies

$$\|H\|_{\tau,4+\sigma,\epsilon}\le C\|h\|_{\tau,6+\sigma,\epsilon} \tag{11.4}$$

Moreover, H defines a linear operator of h and satisfies

$$\int_{\mathbb{R}^2}H\,dy=0. \tag{11.5}$$

Proof. Write the equation $L[H]=h$ as

$$\nabla\cdot(U\nabla g)=h$$

where $g=\frac{H}{U}-(-\Delta)^{-1}H$. We choose g as

$$g(\rho)=-\int_\rho^\infty\frac{1}{rU(r)}\int_0^r h(s)s\,ds\,dr.$$

Using that $\int_{\mathbb{R}^2}h=0$ we check that

$$|g(\rho)|\le C\|h\|_{\tau,6+\sigma,\epsilon}\begin{cases}\frac{1}{(1+\rho)^{\sigma}} & \rho\le\sqrt{\tau}\\ \frac{\tau^{\epsilon/2}}{\rho^{\sigma+\epsilon}} & \rho\ge\sqrt{\tau}\end{cases}$$

Now we solve Liouville's equation

$$-\Delta\psi-U\psi=Ug\quad\text{in }\mathbb{R}^2,\quad\psi(\rho)\to0\quad\text{as }\rho\to\infty,$$

Since $\int_{\mathbb{R}^2}h|y|^2dy=0$ we check that

$$\int_{\mathbb{R}^2}gZ_0\,dy=0.$$

Then we can use the variations of parameter formula, and get

$$|\psi(\rho)|\le C\|h\|_{\tau,6+\sigma,\epsilon}\begin{cases}\frac{1}{(1+\rho)^{2+\sigma}} & \rho\le\sqrt{\tau}\\ \frac{\tau^{\epsilon/2}}{\rho^{2+\sigma+\varepsilon}} & \rho\ge\sqrt{\tau}\end{cases}$$

Then define $H=U(g+\psi)$, which is the desired solution, and note that it satisfies (11.4). Property (11.5) follows from $H=-\Delta\psi$ and the decay of ψ. □

To take into account the operator B we define

$$\Lambda[\phi]=y\cdot\nabla\phi,$$

and compute

$$\Lambda\circ L[\Phi]-L\circ\Lambda[\Phi]=\nabla\cdot(\Phi Uy)-2L[\Phi]-\nabla\cdot((y\cdot\nabla U+2U)\nabla(-\Delta)^{-1}\Phi). \tag{11.6}$$

Indeed, write $\Psi=(-\Delta)^{-1}\Phi$. Then

$$L\Phi=\Delta\Phi-\nabla\Gamma_0\cdot\nabla\Phi-\nabla U\cdot\nabla\Psi+2U\Phi. \tag{11.7}$$

By direct computation

$$\Lambda\Delta\Phi=\Delta\Lambda\Phi-2\Delta\Phi \tag{11.8}$$

$$\Lambda(\nabla\Gamma_0\cdot\nabla\Phi)=\nabla(\Lambda\Gamma_0)\cdot\nabla\Phi+\nabla\Gamma_0\cdot\nabla(\Lambda\Phi)-2\nabla\Gamma_0\cdot\nabla\Phi \tag{11.9}$$

$$\Lambda(\nabla U\cdot\nabla\Psi)=\nabla(\Lambda U)\cdot\nabla\Psi+\nabla U\cdot\nabla(\Lambda\Psi)-2\nabla U\cdot\nabla\Psi. \tag{11.10}$$

But $-\Delta\Psi=\Phi$ and therefore

$$-\Delta(\Lambda\Psi)+2\Delta\Psi=\Lambda\Phi.$$

Applying $(-\Delta)^{-1}$ gives

$$\Lambda\Psi = (-\Delta)^{-1}(\Lambda\Phi) + 2\Psi.$$

Substituting this into (11.10) we obtain

$$\begin{aligned}\Lambda(\nabla U\cdot\nabla\Psi) &= \nabla(\Lambda U)\cdot\nabla\Psi + \nabla U\cdot\nabla[(-\Delta)^{-1}(\Lambda\Phi)+2\Psi] - 2\nabla U\cdot\nabla\Psi\\ &= \nabla(\Lambda U)\cdot\nabla\Psi + \nabla U\cdot\nabla[(-\Delta)^{-1}(\Lambda\Phi)].\end{aligned} \tag{11.11}$$

Combining (11.7), (11.8), (11.9), (11.11) we find that

$$\Lambda L\Phi = L\Lambda\Phi - 2L\Phi + 4U\Phi - 2\nabla U\cdot\nabla\Psi - \nabla(\Lambda\Gamma_0)\cdot\nabla\Phi - \nabla(\Lambda U)\cdot\nabla\Psi + 2\Lambda(U)\Phi.$$

But

$$\begin{aligned}-2\nabla U\cdot\nabla\Psi - \nabla(\Lambda U)\cdot\nabla\Psi &= -\nabla Z_0\cdot\nabla\Psi\\ &= -\nabla\cdot(Z_0\nabla\Psi) - Z_0\Phi,\end{aligned}$$

so that

$$\Lambda L\Phi = L\Lambda\Phi - 2L\Phi + 4U\Phi - \nabla(\Lambda\Gamma_0)\cdot\nabla\Phi + 2\Lambda(U)\Phi - \nabla\cdot(Z_0\nabla\Psi) - Z_0\Phi.$$

Using that

$$2\Lambda(U)\Phi - Z_0\Phi = -2U\Phi + \Lambda(U)\Phi$$

we then obtain

$$\Lambda L\Phi = L\Lambda\Phi - 2L\Phi + 2U\Phi - \nabla(\Lambda\Gamma_0)\cdot\nabla\Phi + \Lambda(U)\Phi - \nabla\cdot(Z_0\nabla\Psi)$$

Let's consider the terms $2U\Phi - \nabla(\Lambda\Gamma_0)\cdot\nabla\Phi + \Lambda(U)\Phi$. Noting that $\nabla(\Lambda\Gamma_0) = \nabla(y\cdot\nabla\Gamma + 2) = \nabla z_0$ and that $Z_0 = 2U + \Lambda(U)$, we can write

$$\begin{aligned}2U\Phi - \nabla(\Lambda\Gamma_0)\cdot\nabla\Phi + \Lambda(U)\Phi &= 2U\Phi - \nabla z_0\cdot\nabla\Phi + \Lambda(U)\Phi\\ &= Z_0\Phi - \nabla\cdot(\nabla z_0\Phi) + \Delta z_0\Phi.\end{aligned}$$

But $\Delta z_0 + Z_0 = 0$, so

$$\Lambda L\Phi = L\Lambda\Phi - 2L\Phi - \nabla\cdot(\nabla z_0\Phi) - \nabla\cdot(Z_0\nabla\Psi).$$

We can again write $\nabla z_0 = \nabla(y\cdot\nabla\Gamma_0)$ and using the radial symmetry of the functions Γ_0, z_0 and the notation $\rho = |y|$

$$\nabla z_0 = \frac{y}{\rho}\partial_\rho z_0 = \frac{y}{\rho}\partial_\rho(\rho\partial_\rho\Gamma_0) = y\Delta\Gamma_0 = -yU.$$

Then

$$\Lambda L\Phi = L\Lambda\Phi - 2L\Phi + \nabla\cdot(yU\Phi) - \nabla\cdot(Z_0\nabla\Psi).$$

This proves (11.6).

Formula (11.6) leads us to consider the following equation for $\Phi = L^{-1}[\phi]$:

$$\begin{cases}\partial_\tau\Phi = L[\Phi] + \tilde B[\Phi] + \zeta_1(\tau)A[\Phi] + H & \text{in } \mathbb{R}^2\times(\tau_0,\infty)\\ \Phi(\cdot,\tau_0) = 0\end{cases} \tag{11.12}$$

where

$$A[\Phi] = L^{-1}[\nabla\cdot(\Phi U y) - \nabla\cdot(Z_0\nabla(-\Delta)^{-1}\Phi)],$$

$Z_0(y) = 2U(y) + y\cdot\nabla_y U(y)$, and $\tilde B$ has the same form as B:

$$\tilde B[\Phi] = \tilde\zeta_1(\tau)y\cdot\nabla\Phi + \tilde\zeta_2(\tau)\Phi$$

with $\tilde\zeta_1(\tau)$, $\tilde\zeta_2(\tau)$ satisfying

$$|\tilde\zeta_i(\tau)| \le \frac{C}{\tau\log\tau} \quad \text{for all } \tau > \tau_0. \tag{11.13}$$

and ζ_1 satisfies the same restriction, that is, (10.20).

The next lemma allows us to reduce to an equation like (11.12) but with a right hand side with more spatial decay.

Lemma 11.2. *Let $\sigma > 0$, $\epsilon > 0$ and $1 < \nu < \min(1+\frac{\epsilon}{2}, 3-\frac{\sigma}{2})$. Let $H(y,\tau)$ be radial in y and satisfy*

$$\int_{\mathbb{R}^2} H(\cdot,\tau) = 0 \tag{11.14}$$

and $\|H\|_{\nu,m,4+\sigma,\epsilon} < \infty$. Then there exists H_1 and Φ_1 such that

$$\begin{cases} \partial_\tau \Phi_1 = L[\Phi_1] + \tilde{B}[\Phi_1] + H - H_1, \quad \text{in } \mathbb{R}^2 \times (\tau_0, \infty) \\ \Phi_1(\cdot,\tau_0) = 0 \quad \text{in } \mathbb{R}^2. \end{cases}$$

Moreover Φ_1 and H_1 are linear operators of H and satisfy

$$|\Phi_1(\rho,\tau)| \le \frac{C}{\tau^\nu (\log\tau)^m} \|H\|_{\nu,m,4+\sigma,\epsilon} \begin{cases} \frac{1}{(1+\rho)^{2+\sigma}} & \rho \le \sqrt{\tau} \\ \frac{\tau^{1+\epsilon/2}}{(1+\rho)^{4+\sigma+\epsilon}} & \rho \ge \sqrt{\tau}. \end{cases} \tag{11.15}$$

$$|H_1(\rho,\tau)| \le \frac{C}{\tau^\nu (\log\tau)^m} \|H\|_{\nu,m,4+\sigma,\epsilon} \begin{cases} \frac{1}{(1+\rho)^{6+\sigma}} & \rho \le \sqrt{\tau} \\ \frac{\tau^{\epsilon/2}}{(1+\rho)^{6+\sigma+\epsilon}} & \rho \ge \sqrt{\tau}. \end{cases}, \tag{11.16}$$

$$\int_{\mathbb{R}^2} \Phi_1 dy = 0 \tag{11.17}$$

$$\int_{\mathbb{R}^2} H_1(\cdot,\tau) = 0. \tag{11.18}$$

Proof. Write the operator L as

$$L[\phi] = L_0[\phi] - \nabla\cdot(U\nabla(-\Delta)^{-1}\phi)$$

where L_0 is defined in (11.3). Consider the problem

$$\begin{cases} \partial_\tau \Phi_1 = L_0[\Phi_1] + \tilde{B}[\Phi_1] + H, \quad \text{in } \mathbb{R}^2 \times (\tau_0,\infty), \\ \Phi_1(\cdot,\tau_0) = 0 \quad \text{in } \mathbb{R}^2. \end{cases}$$

The idea is to formally apply L_0^{-1} to this equation. Similarly to the proof of (11.6) we compute

$$\Lambda \circ L_0[\Phi] - L_0 \circ \Lambda[\Phi] = \nabla\cdot(\Phi U y) - 2L_0[\Phi].$$

This leads us to consider the problem

$$\begin{cases} \partial_\tau \tilde{\Phi} = L_0[\tilde{\Phi}] + B_1[\tilde{\Phi}] + \tilde{H}, \quad \text{in } \mathbb{R}^2 \times (\tau_0,\infty), \\ \tilde{\Phi}(\cdot,\tau_0) = 0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{11.19}$$

where $\tilde{H}$ is a radial function satisfying

$$L_0[\tilde{H}] = H \quad \text{in } \mathbb{R}^2$$

and

$$B_1[\tilde{\Phi}] = \hat{\zeta}_1(\tau) y\cdot\nabla\tilde{\Phi} + \hat{\zeta}_2(\tau)\tilde{\Phi}$$

with

$$\hat{\zeta}_1(\tau) = \tilde{\zeta}_1(\tau) = O\Big(\frac{1}{\tau\log\tau}\Big), \quad \hat{\zeta}_2(\tau) = \tilde{\zeta}_2(\tau) - 2\tilde{\zeta}_1(\tau) = O\Big(\frac{1}{\tau\log\tau}\Big), \tag{11.20}$$

by (11.13).

We claim that there is a choice of $\tilde{H}$, which defines a linear operator of H, and satisfies

$$|\tilde{H}| + (1+\rho)|\nabla\tilde{H}| \le C\frac{1}{\tau^\nu(\log\tau)^m}\|H\|_{\nu,m,4+\sigma,\epsilon} \begin{cases} \frac{1}{(1+\rho)^{2+\sigma}} & \rho \le \sqrt{\tau} \\ \frac{\tau^{\epsilon/2}}{(1+\rho)^{2+\sigma+\epsilon}} & \rho \ge \sqrt{\tau}. \end{cases} \tag{11.21}$$

Indeed, the equation $L_0[\tilde{H}] = H$ for radial functions has the form

$$\partial_\rho\Big(\rho U \partial_\rho\Big(\frac{\tilde{H}}{U}\Big)\Big) = \rho H.$$

We select the solution

$$\tilde{H}(\rho,\tau) = U(\rho)\int_0^\rho \frac{1}{rU(r)}\int_0^r H(s,\tau) s\,ds\,dr.$$

Using (11.14) we get (11.21).

Instead of (11.19) we consider

$$\begin{cases} \partial_\tau \tilde\Phi_1 = \Delta_{\mathbb{R}^2}\tilde\Phi_1 - \nabla\Gamma_0\cdot\nabla\tilde\Phi_1 + B_1[\tilde\Phi_1] + \tilde H, & \text{in } \mathbb{R}^2\times(\tau_0,\infty),\\ \tilde\Phi_1(\cdot,\tau_0) = 0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{11.22}$$

We then have the following estimate for $\tilde\Phi_1$:

$$|\tilde\Phi_1| \le \frac{C}{\tau^\nu(\log\tau)^m}\|H\|_{\nu,m,4+\sigma,\epsilon}\begin{cases}\frac{1}{(1+\rho)^\sigma} & \rho\le\sqrt\tau\\ \frac{\tau^{1+\epsilon/2}}{(1+\rho)^{2+\sigma+\epsilon}} & \rho\ge\sqrt\tau.\end{cases} \tag{11.23}$$

For the proof of this we construct a barrier. First we find a solution to

$$\Delta_{\mathbb{R}^2}\phi_1 - \nabla\Gamma_0\cdot\nabla\phi_1 + \frac{1}{(1+\rho)^{2+\sigma}} = 0 \quad \text{in } \mathbb{R}^2,$$
$$\phi_1(\rho)\to 0 \quad \text{as } \rho\to\infty.$$

The equation may be integrated explicitly, noting that

$$\Delta_{\mathbb{R}^2}\phi - \nabla\Gamma_0\cdot\nabla\phi = \phi_{\rho\rho} + \Big(\frac{1}{\rho}+\frac{4\rho}{1+\rho^2}\Big)\phi_\rho$$

and that the constants are in the kernel of this operator. We then have

$$\phi_1(\rho) = \int_\rho^\infty \frac{1}{r(1+r^2)^2}\int_0^r \frac{1}{(1+s)^{2+\sigma}}s(1+s^2)^2\,ds\,dr$$

and this implies

$$|\phi_1(\rho)| + (1+\rho)|\phi_1'(\rho)| \le \frac{C}{(1+\rho)^\sigma}$$

Let

$$\chi(\rho,\tau) = \chi_0\Big(\frac{\rho}{\delta\sqrt\tau}\Big),$$

where $\chi_0\in C^\infty(\mathbb{R})$, $\chi_0(s)=1$ for $s\le 1$ and $\chi_0(s)=0$ for $s\ge 2$. Define $\tilde\phi_1 = \frac{1}{\tau^\nu(\log\tau)^m}\phi_1\chi$. We have

$$\begin{aligned}&(\partial_\tau - \Delta_{\mathbb{R}^2} + \nabla\Gamma_0\cdot\nabla)\tilde\phi_1\\ &\ge \frac{1}{\tau^\nu(\log\tau)^m(1+\rho)^{2+\sigma}}\chi - \frac{C_1}{\tau^{\nu+\sigma/2+1}(\log\tau)^m}\chi_{\{\delta\sqrt\tau\le\rho\le 2\delta\sqrt\tau\}},\end{aligned}$$

for some $C_1>0$, $\delta>0$ (assuming τ_0 large). Now consider

$$\phi_2(\rho,\tau) = \frac{1}{\tau^{\nu+\sigma/2}(\log\tau)^m}\frac{1}{(1+\rho/\sqrt\tau)^{2+\sigma+\epsilon}}, \qquad \phi_3(\rho,\tau) = \frac{1}{\tau^{\nu+\sigma/2}(\log\tau)^m}e^{-\frac{\rho^2}{4\tau}}.$$

A computation, using (11.20), shows that

$$\bar\phi = A_1\tilde\phi_1 + A_2\phi_2 + A_3\phi_3$$

satisfies

$$(\partial_\tau - \Delta_{\mathbb{R}^2} + \nabla\Gamma_0\cdot\nabla + B_1)\bar\phi \ge \frac{c}{\tau^\nu(\log\tau)^m}\begin{cases}\frac{1}{(1+\rho)^{2+\sigma}} & \rho\le\sqrt\tau\\ \frac{\tau^{\epsilon/2}}{(1+\rho)^{2+\sigma+\epsilon}} & \rho\ge\sqrt\tau.\end{cases}$$

for some $c>0$. This step needs $\nu-1<\frac{\epsilon}{2}$ and $\nu+\frac{\sigma}{2}<3$. By comparison, we find that $\tilde\Phi_1$ satisfies (11.23).

The solution $\tilde\Phi_1$ of (11.22) satisfies

$$\partial_\tau\tilde\Phi_1 = L_0[\tilde\Phi_1] - U\tilde\Phi_1 + B_1[\tilde\Phi_1] + \tilde H$$

Applying L_0 to this equation we find that

$$\Phi_1 = L_0[\tilde\Phi_1]$$

satisfies

$$\partial_\tau\Phi_1 = L[\Phi_1] + \tilde B[\Phi_1] + H - H_1$$

with

$$H_1 = -\nabla\cdot(U\nabla\Psi_1) + L_0[U\tilde\Phi_1] + \tilde\zeta_1\nabla\cdot(\tilde\Phi_1 Uy), \quad \Psi_1 = (-\Delta)^{-1}\Phi_1. \tag{11.24}$$

Let us verify that Φ_1 and H_1 satisfy the conditions stated in (11.15), (11.16), (11.18). Indeed, from standard parabolic estimates and (11.23) we have

$$|\nabla\tilde\Phi_1| \le \frac{C}{\tau^\nu(\log\tau)^m}\|H\|_{\nu,m,4+\sigma,\epsilon}\begin{cases}\frac{1}{(1+\rho)^{1+\sigma}} & \rho\le\sqrt{\tau}\\ \frac{\tau^{1+\epsilon/2}}{(1+\rho)^{3+\sigma+\epsilon}} & \rho\ge\sqrt{\tau}.\end{cases} \tag{11.25}$$

Differentiating in y_j, $j=1,2$ the equation (11.22) and using standard parabolic estimates, together with (11.21), (11.25) we obtain

$$|D^2\tilde\Phi_1| \le \frac{C}{\tau^\nu(\log\tau)^m}\|H\|_{\nu,m,4+\sigma,\epsilon}\begin{cases}\frac{1}{(1+\rho)^{2+\sigma}} & \rho\le\sqrt{\tau}\\ \frac{\tau^{1+\epsilon/2}}{(1+\rho)^{4+\sigma+\epsilon}} & \rho\ge\sqrt{\tau}.\end{cases} \tag{11.26}$$

The definition $\Phi_1 = L_0[\tilde\Phi_1]$ and the estimates (11.23), (11.25), (11.26) give the estimate (11.15).

We compute

$$H_1 = -\nabla U\cdot\nabla\Psi_1 + U\Phi_1 + \nabla U\cdot\nabla\tilde\Phi_1 + U\Delta\tilde\Phi_1 + \tilde\zeta_1\nabla\cdot(\tilde\Phi_1 U y).$$

Note that $\int_{\mathbb{R}^2}\Phi_1(\cdot,\tau)=0$. So, by a direct radial computation of $\Psi_1=(-\Delta)^{-1}\Phi_1$ and (11.15) we obtain

$$|\nabla\Psi_1(\rho,\tau)| \le \frac{C}{\tau^\nu(\log\tau)^m}\|H\|_{\nu,m,4+\sigma,\epsilon}\begin{cases}\frac{1}{(1+\rho)^{1+\sigma}} & \rho\le\sqrt{\tau}\\ \frac{\tau^{1+\epsilon/2}}{(1+\rho)^{3+\sigma+\epsilon}} & \rho\ge\sqrt{\tau}.\end{cases}$$

This estimate and the ones already obtained for $\tilde\Phi_1$ (11.25), (11.26) and for Φ_1 (11.15) yield

$$|H_1(\rho,\tau)| \le \frac{C}{\tau^\nu(\log\tau)^m}\|H\|_{\nu,m,4+\sigma,\epsilon}\begin{cases}\frac{1}{(1+\rho)^{6+\sigma}} & \rho\le\sqrt{\tau}\\ \frac{\tau^{\epsilon/2}}{(1+\rho)^{6+\sigma+\epsilon}} & \rho\ge\sqrt{\tau}.\end{cases},$$

which is the desired estimate (11.16).

Finally, the zero mass condition (11.18) follows from the form of H_1 (11.24) and its decay. The mass condition for Φ_1 (11.17) follows from $\Phi_1=L_0[\tilde\Phi_1]$ and the decay of $\tilde\Phi_1$ (11.23) and (11.25). ☐

Next we would like to obtain a result similar to Proposition 10.1 for the problem (11.12). In order to simplify this step, we will modify this equation by allowing a parameter in the initial condition. This technical obstruction will be removed in the proof of Proposition 11.1. Thus we consider

$$\begin{cases}\partial_\tau\Phi = L[\Phi]+\tilde B[\Phi]+\zeta_1(\tau)A[\Phi]+H & \text{in } \mathbb{R}^2\times(\tau_0,\infty)\\ \Phi(\cdot,\tau_0)=c_1\tilde Z_0,\end{cases} \tag{11.27}$$

where $\tilde Z_0$ is defined in (6.4).

The next result allows us to say that if in equation (11.27) the right hand side has fast decay, then we can decompose the solution similarly as in Proposition 10.1. This result is an extension of that proposition to an equation that has the extra operator A in it, which is treated as a perturbation.

Lemma 11.3. *Let $0<\sigma<1$, $\epsilon>0$, $\sigma+\epsilon<2$, $1<\nu<\min(1+\frac{\epsilon}{2},3-\frac{\sigma}{2},\frac{3}{2})$. Let $0<q<1$. Then there is $C>0$ such that for τ_0 sufficiently large and for H radially symmetric with $\|H\|_{\nu,m,4+\sigma,\epsilon}<\infty$ and*

$$\int_{\mathbb{R}^2} H(y,\tau)dy = 0 \quad \text{for all } \tau>\tau_0$$

the solution Φ to (11.27) can be decomposed as $\Phi=\Phi_0+\frac{a(\tau)}{2}Z_0$ with the estimates

$$|\Phi_0(\rho,\tau)| \le C\|H\|_{\nu,m,4+\sigma,\epsilon}\frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac{q}{2}}}\min\Big(\frac{1}{(1+|y|)^2},\frac{\tau}{|y|^4}\Big)$$

$$|a(\tau)| \le C\|H\|_{\nu,m,4+\sigma,\epsilon}\frac{1}{\tau^{\nu-1}(\log\tau)^{m+q}}.$$

Moreover Φ_0 and a are linear operators of H.

Proof of Lemma 11.3. We will treat the operator A as a perturbation and therefore consider

$$\begin{cases}\partial_\tau\Phi = L[\Phi]+\tilde B[\Phi]+H & \text{in } \mathbb{R}^2\times(\tau_0,\infty)\\ \Phi(\cdot,\tau_0)=c_1\tilde Z_0.\end{cases} \tag{11.28}$$

Let Φ_1, H_1 be the functions constructed in Lemma 11.2. Setting $\Phi = \Phi_1 + \Phi_2$, (11.28) is equivalent to the following equation for Φ_2

$$\begin{cases} \partial_\tau \Phi_2 = L[\Phi_2] + \tilde{B}[\Phi_2] + H_1, & \text{in } \mathbb{R}^2 \times (\tau_0, \infty), \\ \Phi_2(\cdot, \tau_0) = c_1 \tilde{Z}_0 & \text{in } \mathbb{R}^2. \end{cases} \tag{11.29}$$

We now apply Proposition 10.1 to (11.29). We have that $\|H_1\|_{\nu,m,6+\sigma,\epsilon} < \infty$ by (11.16), H_1 is radial and satisfies the zero mass condition (11.18). By Proposition 10.1 and Lemma 10.10 there exists c_1 such that the solution Φ_2 of (11.29) satisfies

$$\Phi_2(y,\tau) = \Phi_2^\perp(y,\tau) + \frac{a(\tau)}{2} Z_0(y),$$

with the estimates

$$|\Phi_2^\perp(y,\tau)| \le C \frac{\|H_1\|_{\nu,m,6+\sigma,\epsilon}}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac{q}{2}}} \begin{cases} \dfrac{1}{(1+|y|)^2} & |y| \le \sqrt{\tau} \\ \dfrac{\tau}{|y|^4} & |y| \ge \sqrt{\tau}, \end{cases} \tag{11.30}$$

$$|a(\tau)| \le C \frac{\|H_1\|_{\nu,m,6+\sigma,\epsilon}}{\tau^{\nu-1}(\log\tau)^{m+q}}. \tag{11.31}$$

(We are ignoring the factor $\frac{1}{(\log\tau_0)^{1-q}}$ in the estimate of $a(\tau)$.) We also know that c_1 is a linear function of H_1 and satisfies

$$|c_1| \le C \frac{\|H_1\|_{\nu,m,6+\sigma,\epsilon}}{\tau_0^{\nu-1}(\log\tau_0)^{m+1}}.$$

Combining (11.15) and (11.30) we conclude that Φ, the solution to (11.28), can be decomposed as

$$\Phi = \Phi_0 + \frac{a(\tau)}{2} Z_0$$

where $\Phi_0(y,\tau) = \Phi_1 + \Phi_2^\perp$ is radial and satisfies

$$|\Phi_0(y,\tau)| \le C \frac{\|H\|_{\nu,m,4+\sigma,\epsilon}}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac{q}{2}}} \min\Big(\frac{1}{(1+|y|)^2}, \frac{\tau}{|y|^4}\Big)$$

and $a(\tau)$ satisfies, combining (11.16) and (11.31),

$$|a(\tau)| \le C \frac{1}{\tau^{\nu-1}(\log\tau)^{m+q}} \|H\|_{\nu,m,4+\sigma,\epsilon}.$$

We summarize the previous finding as follows. Given H radial satisfying $\int_{\mathbb{R}^2} H(\cdot,\tau) = 0$ for $\tau > \tau_0$ and $\|H\|_{\nu,m,4+\sigma,\epsilon} < \infty$, let us denote $T_0(H) = \Phi_0 = \Phi_1 + \Phi_2^\perp$ and $T_a(H) = a(\tau)$ so that the solution Φ of (11.28), is $\Phi = \Phi_0 + \frac{a(\tau)}{2} Z_0 = T_0[H] + \frac12 T_a[H] Z_0$. Then T_0, T_a are linear and have the estimates

$$\|T_0[H]\|_0 \le C \|H\|_{\nu,m,4+\sigma,\epsilon} \tag{11.32}$$

$$\|T_a[H]\|_a \le C \|H\|_{\nu,m,4+\sigma,\epsilon}, \tag{11.33}$$

where

$$\|\Phi_0\|_0 = \sup_{\tau>\tau_0,\, y\in\mathbb{R}^2} \tau^{\nu-\frac12}(\log\tau)^{m+\frac{q}{2}} \frac{1}{\min\Big(\frac{1}{(1+|y|)^2}, \frac{\tau}{|y|^4}\Big)} |\Phi_0(y,\tau)|$$

$$\|a\|_a = \sup_{\tau>\tau_0} \tau^{\nu-1}(\log\tau)^{m+q}|a(\tau)|.$$

Moreover c_1 is a linear function of H and satisfies

$$|c_1| \le C \frac{\|H\|_{\nu,m,4+\sigma,\epsilon}}{\tau_0^{\nu-1}(\log\tau_0)^{m+1}}.$$

We will apply these estimates to treat problem (11.27), which can be written as the fixed point problem

$$\Phi_0 = T_0[H + \zeta_1 A[\Phi_0 + a Z_0]]$$

$$a = T_a[H + \zeta_1 A[\Phi_0 + a Z_0]]$$

By (11.32) and (11.33)

$$\|T_0[\zeta_1 A[\Phi_0 + aZ_0]]\|_0 + \|T_a[\zeta_1 A[\Phi_0 + aZ_0]]\|_a \le C \|\zeta_1 A[\Phi_0 + aZ_0]\|_{\nu,m,4+\sigma,\epsilon}.$$

We claim that

$$\|\zeta_1 A[\Phi_0]\|_{\nu,m,4+\sigma,\epsilon} \le C\tau_0^{-\vartheta}\|\Phi_0\|_0, \tag{11.34}$$

for some $\vartheta > 0$, where C is independent of τ_0, and

$$\|\zeta_1 A[aZ_0]\|_{\nu,m,4+\sigma,\epsilon} \le \frac{C}{(\log\tau_0)^{1+q}}\|a\|_a. \tag{11.35}$$

Assume for the moment that (11.34), (11.35) hold. The we see that

$$\|\Phi_0\|_0 + \|a\|_a \le \frac{C}{(\log\tau_0)^{1+q}}(\|\Phi_0\|_0 + \|a\|_a) + C\|H\|_{\nu,m,4+\sigma,\epsilon}.$$

For τ_0 large this gives

$$\|\Phi_0\|_0 + \|a\|_a \le C\|H\|_{\nu,m,4+\sigma,\epsilon},$$

which is the desired result.

For the proof of estimates (11.34), (11.35) we will need the following property. If Φ satisfies $|\Phi(y)| \le \frac{1}{(1+|y|)^{2+\kappa}}$ for some $\kappa > 0$ and $\int_{\mathbb{R}^2}\Phi dy = 0$, then

$$\int_{\mathbb{R}^2}\nabla\cdot[\Phi U y - Z_0\nabla\Psi]|y|^2 dy = 0, \quad \Psi = (-\Delta)^{-1}\Phi. \tag{11.36}$$

Indeed,

$$\begin{aligned}\int_{\mathbb{R}^2}\nabla\cdot(\Phi U y)|y|^2 dy &= -2\int_{\mathbb{R}^2}\Phi U|y|^2 dy = 2\int_{\mathbb{R}^2}\Delta\Psi U|y|^2 dy\\ &= -2\int_{\mathbb{R}^2}\nabla\Psi\cdot\nabla(U|y|^2)dy\\ &= -2\int_{\mathbb{R}^2}\nabla\Psi\cdot yZ_0 dy\end{aligned}$$

and

$$\int_{\mathbb{R}^2}\nabla\cdot(Z_0\nabla\Psi)|y|^2 dy = -2\int_{\mathbb{R}^2}Z_0\nabla\Psi\cdot y dy.$$

To prove (11.34), let us write $\Psi_0 = (-\Delta)^{-1}\Phi_0$. Then

$$A[\Phi_0] = L^{-1}[\nabla\cdot(\Phi_0 U y - Z_0\nabla\Psi_0)].$$

Using the definition of L^{-1} given in Lemma 11.1 we have that

$$L^{-1}[\nabla\cdot(\Phi U y) - \nabla\cdot(Z_0\nabla(-\Delta)^{-1}\Phi)] = Ug + U\psi$$

where

$$g(\rho,\tau) = -\int_\rho^\infty\Big[\Phi_0(s,\tau)s - \frac{Z_0(s)}{U(s)}\partial_\rho\Psi_0(s,\tau)\Big]ds, \tag{11.37}$$

and ψ is the decaying solution to the Liouville equation

$$-\Delta\psi - U\psi = Ug.$$

From the definition $\Psi_0 = (-\Delta)^{-1}\Phi_0$ and using that $\int_{\mathbb{R}^2}\Phi_0 dy = 0$ we have

$$\partial_\rho\Psi_0(\rho,\tau) = \frac{1}{\rho}\int_\rho^\infty\Phi_0(s,\tau)s ds$$

which gives the estimate

$$|\partial_\rho\Psi_0(\rho,\tau)| \le C\|\Phi_0\|_0\frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac q2}}\begin{cases}\frac{\log(\frac{2\sqrt\tau}{1+\rho})}{1+\rho} & \rho\le\sqrt\tau,\\ \frac{\tau}{\rho^3} & \rho\ge\sqrt\tau.\end{cases}$$

Then formula (11.37) gives

$$\begin{aligned}|g(\rho,\tau)| &\le C\|\Phi_0\|_0\frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac q2}}\begin{cases}\log^2(\frac{2\sqrt\tau}{1+\rho}) & \rho\le\sqrt\tau,\\ \frac{\tau}{\rho^2} & \rho\ge\sqrt\tau.\end{cases}\\ &\le C\|\Phi_0\|_0\frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac q2-2}}\min\Big(1,\frac{\tau}{\rho^2}\Big).\end{aligned}$$

We note that by (11.36) we have $\int_{\mathbb{R}^2} Ugz_0 dy = 0$. Then, ψ has the estimate

$$|\psi(\rho,\tau)| \le C\|\Phi_0\|_0 \frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac{q}{2}-2}} \frac{1}{(1+\rho)^2} \min\Big(1, \frac{\tau}{\rho^2}\Big).$$

It follows that $A[\Phi_0] = Ug + U\psi$ satisfies

$$|A[\Phi_0](\rho,\tau)| \le C\|\Phi_0\|_0 \frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac{q}{2}-2}} \frac{1}{(1+\rho)^4} \min\Big(1, \frac{\tau}{\rho^2}\Big).$$

From this inequality we obtain (11.34).

The proof of (11.35) is similar. This time $A[aZ_0] = Ug_1 + U\psi_1$ where

$$g_1(\rho,\tau) = -a(\tau)\int_\rho^\infty \Big[Z_0(s)s - \frac{Z_0(s)}{U(s)} z_0'(s)\Big] ds,$$

and ψ_1 is the radial decaying solution to

$$-\Delta\psi_1 - U\psi_1 = Ug_1.$$

We then obtain that

$$|A[aZ_0](\rho,\tau)| \le C\|a\|_a \frac{1}{\tau^{\nu-1}(\log\tau)^{m+q}} \frac{1}{(1+\rho)^6}.$$

From this estimate we deduce (11.35). ☐

Before proving Proposition 11.1 as stated, we obtain a version of it for the problem

$$\begin{cases} \partial_\tau\phi = L[\phi] + B[\phi] + h(y,\tau) \quad \text{in } \mathbb{R}^2\times(\tau_0,\infty), \\ \phi(\cdot,\tau_0) = c_1\hat{Z}_0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{11.38}$$

where

$$\hat{Z}_0 = L[\tilde{Z}_0].$$

Lemma 11.4. *Let $0<\sigma<1$, $\epsilon>0$, $\sigma+\epsilon<2$ and $1<\nu<\min(1+\frac{\epsilon}{2}, 3-\frac{\sigma}{2}, \frac32)$. Let $0<q<1$. Then there is C such that for τ_0 large the following holds. Suppose that h is radially symmetric, satisfies $\|h\|_{\nu,m,6+\sigma,\epsilon}<\infty$ and*

$$\int_{\mathbb{R}^2} h(y,\tau)dy = 0, \quad \int_{\mathbb{R}^2} h(y,\tau)|y|^2 dy = 0, \quad \tau>\tau_0.$$

Then there exist $c_1\in\mathbb{R}$ and a solution $\phi(y,\tau)$ of problem (11.38) that define linear operators of h and satisfy

$$\|\phi\|_{\nu-\frac12, m+\frac{q-1}{2}, 4, 2+\sigma+\epsilon} \le C\|h\|_{\nu,m,6+\sigma,\epsilon}.$$

$$|c_1| \le C\frac{1}{\tau_0^{\nu-1}(\log\tau_0)^{m+1}}\|h\|_{\nu,m,6+\sigma,\epsilon}.$$

Proof. Consider equation (11.27), where H is the function constructed in Lemma 11.1. By Lemma 11.3, there is c_1 such that the solution Φ of (11.27) can be decomposed as $\Phi = \Phi_0 + \frac{a(\tau)}{2}Z_0$, where Φ_0 and a satisfy the estimates stated in that proposition. In combination with (11.4) we find

$$|\Phi_0(\rho,\tau)| \le C\|h\|_{\nu,m,6+\sigma,\epsilon} \frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac{q}{2}}} \min\Big(\frac{1}{(1+|y|)^2}, \frac{\tau}{|y|^4}\Big) \tag{11.39}$$

$$|a(\tau)| \le C\|h\|_{\nu,m,6+\sigma,\epsilon} \frac{1}{\tau^{\nu-1}(\log\tau)^{m+q}}.$$

$$|c_1| \le C\frac{1}{\tau_0^{\nu-1}(\log\tau_0)^{m+1}}\|h\|_{\nu,m,6+\sigma,\epsilon}. \tag{11.40}$$

Moreover Φ_0, a, c_1 are linear operators of H.

From standard parabolic estimates and (11.39) we obtain

$$|\nabla\Phi_0(\rho,\tau)| \le C\|h\|_{\nu,m,6+\sigma,\epsilon} \frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac{q}{2}}} \min\Big(\frac{1}{(1+|y|)^3}, \frac{\tau}{|y|^5}\Big). \tag{11.41}$$

We consider the equation for $\Phi_0 = \Phi - \frac{a(\tau)}{2} Z_0$, obtained from (11.27), and differentiate with respect to y_j, $j = 1, 2$. Using standard parabolic estimates, together with (11.39), (11.41), and the bound for $a'(\tau)$ in (10.92), we obtain

$$|D^2\Phi_0(\rho,\tau)| \le C\|h\|_{\nu,m,6+\sigma,\epsilon} \frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac q2}} \min\Big(\frac{1}{(1+|y|)^4}, \frac{\tau}{|y|^6}\Big). \tag{11.42}$$

Let us define $\phi = L[\Phi]$. Then ϕ satisfies (11.38) because $L[Z_0] = 0$ and thanks to (11.39), (11.41), (11.42) we find

$$|\phi(\rho,\tau)| \le C\|h\|_{\nu,m,6+\sigma,\epsilon} \frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac q2}} \min\Big(\frac{1}{(1+|y|)^4}, \frac{\tau}{|y|^6}\Big). \tag{11.43}$$

In the rest of the proof we show that

$$|\phi(\rho,\tau)| \le C\|h\|_{\nu,m,6+\sigma,\epsilon} \frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac q2}} \frac{1}{(1+\rho)^4} \begin{cases} 1 & \rho \le \sqrt{\tau} \\ \frac{\tau^{1+\sigma/2+\epsilon/2}}{\rho^{2+\sigma+\epsilon}} & \rho \ge \sqrt{\tau}. \end{cases}$$

For this we consider the equation (11.38) written in the form

$$\partial_\tau \phi = \Delta\phi - \nabla\Gamma_0\nabla\phi + 2U\phi + B[\phi] + \bar h, \tag{11.44}$$

where

$$\bar h = -\nabla U \nabla \psi + h.$$

Using (11.43) and the radial formula for $\psi = (-\Delta)^{-1}\phi$, we get

$$|\nabla\psi(y,\tau)| \le C\|h\|_{\nu,m,6+\sigma,\epsilon} \frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac q2}} \begin{cases} \frac{1}{(1+\rho)^3} & \rho \le \sqrt{\tau} \\ \frac{\tau}{\rho^5} & \rho \ge \sqrt{\tau}. \end{cases}$$

This estimate and the definition of the norm $\|h\|_{\nu,m,6+\sigma,\epsilon}$, give

$$|\bar h(y,\tau)| \le C \frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac q2}} \|h\|_{\nu,m,6+\sigma,\epsilon} \begin{cases} \frac{1}{(1+|y|)^{6+\sigma}} & |y| \le \sqrt{\tau} \\ \frac{1}{\tau^{3+\sigma/2}(\frac{|y|}{\sqrt{\tau}})^{6+\sigma+\epsilon}} & |y| \ge \sqrt{\tau}. \end{cases}$$

We now construct a barrier very similar to the proof of Proposition 8.1

$$\begin{aligned}\bar\phi(\rho,\tau) = {} & A_1 \frac{1}{\tau^{\nu-\frac12}(\log\tau)^{m+\frac q2}} \tilde g_2(\rho)\chi_0\Big(\frac{\rho}{\sqrt{\tau}}\Big) + A_2 \frac{1}{\tau^{\nu+\frac32}(\log\tau)^{m+\frac q2}} \frac{1}{(1+\rho/\sqrt{\tau})^{6+\sigma+\epsilon}} \\ & + A_3 \frac{1}{\tau^{\nu+\frac32}(\log\tau)^{m+\frac q2}} e^{-\frac{\rho^2}{4\tau}},\end{aligned}$$

where $\tilde g_2$ is the function (10.60). We consider (11.44) in $\{\,(y,\tau) \mid \tau > \tau_0,\ |y| > R_0\,\}$ where $R_0 > 0$ is a large constant. For suitable constants A_1, A_2, A_3, C the function $C\|h\|_{\nu,m,6+\sigma,\epsilon}\bar\phi$ is a supersolution. This computation requires $\nu < \frac32$.

Moreover $\phi(y,\tau) \le C\|h\|_{\nu,m,6+\sigma,\epsilon}\bar\phi(y,\tau)$ at $|y| = R_0$. The initial conditions also compare well. Indeed, by Lemma 10.11 and (11.40)

$$|\phi(\rho,\tau_0)| = c_1|\hat Z_0(\rho)| \le C \frac{1}{\tau_0^{\nu-1}(\log\tau_0)^{m+1}} \|h\|_{\nu,m,6+\sigma,\epsilon} \frac{1}{\tau_0} \frac{1}{1+\rho^6},$$

and this is supported on $\rho \le 2\sqrt{\tau_0}$, so

$$|\phi(\rho,\tau_0)| \le C\|h\|_{\nu,m,6+\sigma,\epsilon}\bar\phi(y,\tau).$$

By the maximum principle

$$|\phi(y,\tau)| \le C\bar\phi(y,\tau)\|h\|_{\nu,m,6+\sigma,\epsilon}, \quad |y| > R_0.$$

This finishes the proof. ∎

Proof of Proposition 11.1. Let $\hat\phi$, c_1 be the solution to (11.38) constructed in Lemma 11.4. Let ϕ_1 be the solution to (10.100). By Lemma 10.12 ϕ_1 satisfies

$$|\phi_1(\rho,\tau)| \le C \frac{\tau_0^{\nu_0-1}R(\tau)^2}{\tau^{\nu_0}R(\tau_0)^2} \frac{1}{(1+\rho^4)} \min\Big(1, \frac{\tau^{1/2}}{\rho}\Big)^{2+\sigma+\epsilon}, \tag{11.45}$$

where $1 < \nu_0 < \frac74$. Then the solution ϕ to (8.9) that we construct is given by

$$\phi = \hat\phi - c_1\phi_1.$$

To get the desired estimate on ϕ we need to estimate $|c_1\phi_1|$. Let f be given by (10.12). By (11.40) and (11.45)

$$
\begin{aligned}
|c_1\phi_1(\rho,\tau)| &\le C\frac{1}{\tau_0^{\nu-1}(\log\tau_0)^{m+1}}\frac{\tau_0^{\nu_0-1}R(\tau)^2}{\tau^{\nu_0}R(\tau_0)^2}\frac{1}{(1+\rho^4)}\min\Big(1,\frac{\tau^{1/2}}{\rho}\Big)^{2+\sigma+\epsilon}\|h\|_{\nu,m,6+\sigma,\epsilon}\\
&\le C\frac{1}{\log\tau_0 R(\tau_0)}f(\tau)R(\tau)\frac{1}{(1+\rho^4)}\min\Big(1,\frac{\tau^{1/2}}{\rho}\Big)^{2+\sigma+\epsilon}\|h\|_{\nu,m,6+\sigma,\epsilon}\\
&\le Cf(\tau)R(\tau)\frac{1}{(1+\rho^4)}\min\Big(1,\frac{\tau^{1/2}}{\rho}\Big)^{2+\sigma+\epsilon}\|h\|_{\nu,m,6+\sigma,\epsilon}
\end{aligned}
$$

provided $\frac{1}{2}+\nu-\nu_0<0$. But ν_0 can be taken close to $\frac{7}{4}$, so we obtain the result by assuming $\nu<\frac{5}{4}$ in addition to the other constraints needed in Lemma 11.4, namely $1<\nu<\min(1+\frac{\epsilon}{2},3-\frac{\sigma}{2},\frac{3}{2})$. $\square$

12. Linear estimate with second moment (general)

A convenient property of problem (8.3) is that it can be split into Fourier modes. If we decompose

$$h(y,\tau)=h_0(|y|,\tau)+h_1(y,\tau),\quad h_0(\rho,\tau)=\frac{1}{2\pi}\int_0^{2\pi}h(\rho e^{i\theta},\tau)d\theta \tag{12.1}$$

$$\phi(y,\tau)=\phi_0(|y|,\tau)+\phi_1(y,\tau),\quad \phi_0(\rho,\tau)=\frac{1}{2\pi}\int_0^{2\pi}\phi(\rho e^{i\theta},\tau)d\theta, \tag{12.2}$$

then ϕ solves (8.3) if and only if ϕ_i solves (8.3) where h is replaced with h_i, for $i=0,1$. If $h=h_1$ we say that *h has no radial mode.*

For the proof Proposition 8.2 in the general case we will consider in a first step the equation (8.3) but without the operator B, namely,

$$\begin{cases}\partial_\tau\phi=L[\phi]+h, & \text{in } \mathbb{R}^2\times(\tau_0,\infty),\\ \phi(\cdot,\tau_0)=0 & \text{in } \mathbb{R}^2,\end{cases} \tag{12.3}$$

for functions with no radial mode, as explained at the beginning of Section 11. Later on, we will consider equation (8.3) for functions with no radial mode, where we will treat the operator $B[\phi]$ as a perturbation term that can be assimilated to the right hand side.

The main step in the proof is the following estimate, valid when the functions involved have no radial mode.

Proposition 12.1. *Let $0<\sigma<1$, $0<\epsilon<2$, $0<\nu<\min(1+\frac{\epsilon}{2},\frac{3}{2}-\frac{\sigma}{2})$, $m\in\mathbb{R}$. Then there is a $C>0$ such that for any τ_0 sufficiently large the following holds. Suppose that $h(y,\tau)$ has no radial mode and satisfies $\|h\|_{\nu,m,5+\sigma,\epsilon}<\infty$,*

$$\int_{\mathbb{R}^2}h(y,\tau)y_j\,dy=0\quad \text{for all } \tau>\tau_0,\quad j=1,2. \tag{12.4}$$

Then the solution $\phi(y,\tau)$ of (12.3) satisfies

$$|\phi(y,\tau)|\le C\frac{\|h\|_{\nu,m,5+\sigma,\epsilon}}{\tau^\nu(\log\tau)^m}\begin{cases}\frac{1}{(1+|y|)^{3+\sigma}}, & |y|\le\sqrt{\tau}.\\ \frac{\tau^{1+\frac{\epsilon}{2}}}{|y|^{5+\sigma+\epsilon}}, & |y|\ge\sqrt{\tau}.\end{cases} \tag{12.5}$$

Proof. Since $h(y,\tau)$ has no radial mode, all functions involved in the proof have also this property. We use the notation from §9.2, particular $g=\frac{\phi}{U}-(-\Delta)^{-1}\phi$, $g^\perp=g-a$ with $a(\tau)\in\mathbb{R}$ such that

$$\int_{\mathbb{R}^2}g^\perp(y,\tau)U\,dy=0.$$

But

$$\int_{\mathbb{R}^2}g(y,\tau)U\,dy=0$$

because g has no radial mode, so that $a(\tau)=0$, $g^\perp=g$, $\phi^\perp=\phi$. Then the proof proceeds as the proof of Proposition 10.1 with some simplifications, since there is no need to estimate a.

We write (12.3) as

$$\partial_\tau\phi=\nabla\cdot(U\nabla g^\perp)+h,\quad \text{in } \mathbb{R}^2\times(\tau_0,\infty).$$

We multiply this equation by g and integrate in $\mathbb{R}^2$.

Let $R > 0$ be a large fixed constant and let

$$f(\tau) = \frac{1}{\tau^\nu (\log \tau)^m}.$$

Let $T_2 > \tau_0$ and let

$$\|\varphi\|_{\infty, T_2} = \sup_{\tau \in [\tau_0, T_2]} |\varphi(\tau)|.$$

The following estimates are valid for $\tau \in [\tau_0, T_2]$. As in the proof of Proposition 10.1 we get

$$\int_{\mathbb{R}^2} g^2 U \le C f(\tau)^2 R^2 \Big(\|h\|_{\nu,m,5+\sigma,\epsilon}^2 + \Big\| \frac{\omega}{fR} \Big\|_{\infty, T_2}^2 \Big), \tag{12.6}$$

where

$$\omega(\tau) = \Big(\int_{\mathbb{R}^2 \setminus B_R} g(\tau)^2 U \Big)^{1/2}.$$

Similarly as in Lemma 10.8, from (12.6) we get

$$|U g(y,\tau)| \le C f(\tau) R \Big(\|h\|_{\nu,m,5+\sigma,\epsilon} + \Big\| \frac{\omega}{fR} \Big\|_{\infty, T_2} \Big) \frac{1}{(1+|y|)^{3+\sigma}}. \tag{12.7}$$

The proof is presented below. We use this to estimate

$$\omega(\tau) = \Big(\int_{\mathbb{R}^2 \setminus B_R} g^2 U \Big)^{1/2} \le C f(\tau) R^{1-\sigma} \Big(\|h\|_{\nu,m,5+\sigma,\epsilon} + \Big\| \frac{\omega}{fR} \Big\|_{\infty, T_2} \Big),$$

which implies

$$\frac{\omega(\tau)}{f(\tau) R} \le C R^{-\sigma} \|h\|_{\nu,m,5+\sigma,\epsilon} + C R^{-\sigma} \Big\| \frac{\omega}{fR} \Big\|_{\infty, T_2}.$$

We deduce that

$$\Big\| \frac{\omega}{fR} \Big\|_{\infty, T_2} \le C R^{-\sigma} \|h\|_{\nu,m,5+\sigma,\epsilon},$$

by choosing R as a large constant.

Now we let $T_2 \to \infty$ and find

$$\omega(\tau) \le C f(\tau) R \|h\|_{\nu,m,5+\sigma,\epsilon}^2, \quad \tau > \tau_0. \tag{12.8}$$

The inequalities that follow hold for $\tau > \tau_0$.

Combining (12.8) with (12.6) we obtain

$$\int_{\mathbb{R}^2} g^2 U \le C f(\tau)^2 R^2 \|h\|_{\nu,m,5+\sigma,\epsilon}^2, \quad \tau > \tau_0.$$

and using (12.7) we also get

$$|U g(y,\tau)| \le C f(\tau) R \|h\|_{\nu,m,5+\sigma,\epsilon} \frac{1}{(1+|y|)^{3+\sigma}}.$$

Let $\psi = (-\Delta)^{-1} \phi$ so that $\phi = Ug + U\psi$. Using Lemma 9.1 and the previous estimate we obtain

$$|\psi(y,\tau)| + (1+|y|) |\nabla \psi(y,\tau)| \le C \frac{R}{\tau^\nu (\log \tau)^m} \frac{1}{(1+|y|)^{1+\sigma}} \|h\|_{\nu,m,5+\sigma,\epsilon}. \tag{12.9}$$

We consider the equation (12.3) in $\mathbb{R}^2 \setminus B_R(0)$ written in the form

$$\partial_\tau \phi = \Delta \phi - \nabla \Gamma_0 \nabla \phi + 2U\phi + \bar{h},$$

where

$$\bar{h} = -\nabla U \nabla \psi + h.$$

By (12.9) and the definition of the norm $\|h\|_{\nu,m,5+\sigma,\epsilon}$,

$$|\bar{h}(y,\tau)| \le C \|h\|_{\nu,m,5+\sigma,\epsilon} \frac{1}{\tau^\nu (\log \tau)^m} \frac{1}{(1+|y|)^{5+\sigma}} \begin{cases} 1 & |y| \le \sqrt{\tau} \\ \frac{\tau^{\epsilon/2}}{|y|^\epsilon} & |y| \ge \sqrt{\tau}. \end{cases}$$

Here we are using $\epsilon < 2$. Using barriers as in the proof of Lemma 10.8 we get

$$|\phi(y,\tau)| \le C \|h\|_{\nu,m,5+\sigma,\epsilon} \frac{1}{\tau^\nu (\log \tau)^m} \frac{1}{(1+|y|)^{3+\sigma}} \begin{cases} 1 & |y| \le \sqrt{\tau} \\ \frac{\tau^{1+\epsilon/2}}{|y|^{2+\epsilon}} & |y| \ge \sqrt{\tau}. \end{cases}$$

(For this we need $\nu < 1 + \frac{\epsilon}{2}$, $\nu + \frac{\sigma}{2} < \frac{3}{2}$.) This proves (12.5).

□

Proof of (12.7). We define

$$g_0 = Ug,$$

which satisfies the equation

$$\partial_\tau g_0 = \Delta g_0 - \nabla g_0 \cdot \nabla \Gamma_0 + 2U g_0 + \tilde h \tag{12.10}$$

where

$$\tilde h = Uv + h - U(-\Delta)^{-1} h$$

and

$$v := (-\Delta)^{-1}(\nabla \cdot (g_0 \nabla \Gamma_0)).$$

As in the proof of Lemma 10.7 we obtain

$$|g_0(y,\tau)| \le C \frac{R}{\tau^\nu (\log\tau)^m (1+|y|)^2} K, \tag{12.11}$$

where

$$K = \|h\|_{\nu,m,5+\sigma,\epsilon} + \Big\| \frac{\omega}{fR} \Big\|_{\infty,T_2}.$$

Applying parabolic estimates to (12.10) and a scaling argument we find

$$|\nabla g_0(y,\tau)| \le C \frac{RK}{\tau^\nu (\log\tau)^m (1+|y|)^3}. \tag{12.12}$$

Using (12.11), (12.12) and $g_0 = gU$ we get that

$$|\nabla U \cdot \nabla g + g \Delta U| \le C \frac{RK}{\tau^\nu (\log\tau)^m (1+|y|)^4}.$$

We observe that for $i = 1, 2$

$$\int_{\mathbb{R}^2} \nabla (U \nabla g) y_i \, dy = 0. \tag{12.13}$$

Indeed,

$$\int_{\mathbb{R}^2} \nabla (U \nabla g) y_i \, dy = -\int_{\mathbb{R}^2} U \nabla g e_i = \int_{\mathbb{R}^2} g \nabla U e_i.$$

But from $g = \frac{\phi}{U} - \psi$, $\psi = (-\Delta)^{-1}\phi$ we have

$$-\Delta \psi - U\psi = Ug = g_0.$$

Multiplying this equation by $z_i = \nabla \Gamma_0 e_i$ defined in (9.2) and integrating we get

$$\int_{\mathbb{R}^2} g U \nabla \Gamma_0 e_i = 0,$$

which is the desired claim (12.13). We note that

$$\begin{cases} -\Delta v = \nabla U \cdot \nabla g + g \Delta U = \nabla \cdot (g \nabla U) & \text{in } \mathbb{R}^2. \\ v(y) \to 0 \quad \text{as } |y| \to \infty. \end{cases}$$

Now we can apply Remark 9.1 and deduce that for any $\vartheta \in (0,1)$ there is C such that

$$|v(y,\tau)| \le C \frac{RK}{\tau^\nu (\log\tau)^m (1+|y|)^{2-\vartheta}}. \tag{12.14}$$

We next estimate $\tilde h$. From Remark 9.1 and the assumptions on h, in particular (12.4), we have

$$|((-\Delta)^{-1} h)(y,\tau)| \le C \frac{\|h\|}{\tau^\nu (\log\tau)^m (1+|y|)^{2-\vartheta}}, \tag{12.15}$$

for any $\vartheta \in (0,1)$. Also from (12.11) we have

$$|U g_0(y,\tau)| \le C \frac{R}{\tau^\nu (\log\tau)^m (1+|y|)^6} K.$$

Therefore, from (12.15), (12.11), (12.14) we find that for any $\vartheta > 0$

$$|\tilde h(y,\tau)| \le C \frac{RK}{\tau^\nu (\log\tau)^m} \Big[\frac{1}{(1+|y|)^{5+\sigma}} \min\Big(1, \frac{\tau^{\epsilon/2}}{\rho^\epsilon}\Big) + \frac{1}{(1+|y|)^{6-\vartheta}} \Big].$$

We now use a barrier as in the proof of Lemma 10.8, in a domain of the form $(\mathbb{R}^2 \setminus B_{R_0}) \times (\tau_0, \infty)$ where R_0 is a large constant. We let $\tilde g(y)$ be the radial decaying solution to $-\Delta_6 \tilde g = \frac{1}{(1+|y|)^{5+\sigma}}$ and

$$\bar g(y,\tau) = \frac{1}{\tau^\nu (\log\tau)^m} \tilde g(y) \chi_0\Big(\frac{y}{\delta\sqrt{\tau}}\Big) + C_1 \frac{1}{\tau^{\nu+\frac32+\frac\sigma2}(\log\tau)^m}\Big[\frac{1}{(1+|y|/\sqrt{\tau})^\mu} + C_2 e^{-\frac{|y|^2}{4\tau}}\Big]$$

where

$$\mu = \min(5+\sigma+\epsilon, 6-\vartheta).$$

We assume that $\nu < \frac32 - \frac\sigma2 - \frac\vartheta2$, $\nu < 1 + \frac\epsilon2$, $\nu + \frac\sigma2 < \frac32$, and $\sigma + \vartheta < 1$. Since $\vartheta > 0$ is arbitrary we only need $\nu < \frac32 - \frac\sigma2$, $\nu < 1+\frac\epsilon2$ and $\sigma < 1$. Then, for an appropriate choice of C_1, C_2, the function $RK\bar g(y,\tau)$ is a supersolution. By the maximum principle

$$|g_0(y,\tau)| \le CRK\bar g(y,\tau).$$

This proves the desired estimate (12.7).

□

Next we consider equation (8.3), which we recall,

$$\begin{cases} \partial_\tau \phi = L[\phi] + B[\phi] + h + \displaystyle\sum_{j=1}^2 \mu_j(\tau) W_{1,j} \quad \text{in } \mathbb{R}^2 \times (\tau_0,\infty) \\ \phi(\cdot,\tau_0) = 0 \quad \text{in } \mathbb{R}^2. \end{cases} \tag{12.16}$$

For ϕ with no radial mode we can write

$$B[\phi] = (\zeta_1(t)\phi + \zeta_2(t) y\cdot\nabla\phi)\chi_0\Big(\frac{\lambda y}{5\sqrt{t}}\Big).$$

Corollary 12.1. *Let $0<\sigma<1$, $0<\epsilon<2$, $1<\nu<\min(1+\frac\epsilon2, \frac32-\frac\sigma2)$, $m\in\mathbb{R}$. Then there is a $C>0$ such that for any τ_0 sufficiently large the following holds. Suppose that $h(y,\tau)$ has no radial mode and satisfies $\|h\|_{\nu,m,5+\sigma,\epsilon}<\infty$. Then there is a solution $\phi(y,\tau)$, μ_j of (12.16) that is a linear operator of h and satisfies*

$$\|\phi\|_{\nu,m,3+\sigma,2+\epsilon} \le C\|h\|_{\nu,m,5+\sigma,\epsilon} \tag{12.17}$$

$$\mu_j(\tau) = -\int_{\mathbb{R}^2} h(y,\tau) y_j\,dy + \tilde\mu_j[h](\tau) \tag{12.18}$$

$$|\tilde\mu_j[h]| \le \frac{C}{\tau^{\nu+1+\sigma}(\log\tau)^{m+1}} \|h\|_{\nu,m,5+\sigma,\epsilon}.$$

Proof. Using Proposition 12.1, there is a linear operator T so that given h with $\|h\|_{\nu,m,5+\sigma,\epsilon}<\infty$, with no radial mode, and satisfying the condition (12.4) associates the solution ϕ of (12.3). Then the solution ϕ of (12.16) can be written as

$$\phi = T\Big[B[\phi] + h + \sum_{j=1}^2 \mu_j(\tau) W_{1,j}\Big],$$

where μ_j is chosen so that

$$\int_{\mathbb{R}^2} (B[\phi]+h) y_j\,dy + \mu_j(\tau) = 0, \quad \forall \tau > \tau_0. \tag{12.19}$$

The estimate (12.5) implies

$$\|\phi\|_{\nu,m,3+\sigma,2+\epsilon} \le \|B[\phi]+h\|_{\nu,m,5+\sigma,\epsilon} + \sup_{\tau>\tau_0} \tau^\nu (\log\tau)^m \sum_{j=1}^2 |\mu_j(\tau)|.$$

Using standard parabolic estimates we also get

$$\||y|\nabla\phi\|_{\nu,m,3+\sigma,2+\epsilon} \le \|B[\phi]+h\|_{\nu,m,5+\sigma,\epsilon} + \sup_{\tau>\tau_0} \tau^\nu (\log\tau)^m \sum_{j=1}^2 |\mu_j(\tau)|.$$

To estimate μ_j note that multiplying (12.16) by y_j and integrating we get that

$$\int_{\mathbb{R}^2} \phi y_j\,dy = 0, \quad \forall \tau > \tau_0.$$

Therefore

$$\left|\int_{\mathbb{R}^2} B(\phi)dy\right| \le \frac{C}{\tau^{\nu+1+\sigma}(\log\tau)^{m+1}}\|\phi\|_{\nu,m,3+\sigma,2+\epsilon},$$

and from the definition (12.19)

$$\sup_{\tau>\tau_0}\tau^\nu(\log\tau)^m\sum_{j=1}^2|\mu_j(\tau)| \le C\|h\|_{\nu,m,5+\sigma,\epsilon} + \frac{C}{\tau_0^{1+\sigma}\log\tau_0}\|\phi\|_{\nu,m,3+\sigma,2+\epsilon}.$$

We also have that

$$\|B[\phi]\|_{\nu,m,5+\sigma,\epsilon} \le \frac{C}{\log\tau_0}\||\phi|+|y||\nabla\phi|\|_{\nu,m,3+\sigma,2+\epsilon}.$$

Then for τ_0 large we deduce the estimate (12.17).

Finally, from (12.19) we get (12.18) with $\tilde\mu_j$ a linear operator of h satisfying

$$\begin{aligned}|\tilde\mu_j[h]| = \left|\int_{\mathbb{R}^2} B(\phi)dy\right| &\le \frac{C}{\tau^{\nu+1+\sigma}(\log\tau)^{m+1}}\|\phi\|_{\nu,m,3+\sigma,2+\epsilon}\\ &\le \frac{C}{\tau^{\nu+1+\sigma}(\log\tau)^{m+1}}\|h\|_{\nu,m,5+\sigma,\epsilon}.\end{aligned}$$

□

We are now in a position to prove Proposition 8.2 in the general case.

Proof of Proposition 8.2. We decompose $h = h_0 + h_1$ and $\phi = \phi_0 + \phi_1$ as in (12.1), (12.2). We apply Proposition 11.1 to get

$$\|\phi_0\|_{\nu-\frac12,m+\frac q2,4,2+\sigma+\epsilon} \le C\|h\|_{\nu,m,6+\sigma,\epsilon}.$$

To estimate ϕ_1 we use Corollary 12.1. First we select $0<\vartheta<\frac12$. Then note that

$$\|h_1\|_{\nu,m,6-\vartheta,\sigma+\epsilon+\vartheta} \le C\|h\|_{\nu,m,6+\sigma,\epsilon}.$$

Then Corollary 12.1 gives a solution ϕ_1 of (12.16) such that

$$\|\phi_1\|_{\nu,m,3+\bar\sigma,2+\bar\epsilon} \le C\|h_1\|_{\nu,m,5+\bar\sigma,\bar\epsilon}.$$

We take $\bar\sigma = 1-\vartheta$ and $\bar\varepsilon = \varepsilon+\sigma+\vartheta$ and get

$$\|\phi_1\|_{\nu,m,4-\vartheta,2+\sigma+\epsilon+\vartheta} \le C\|h_1\|_{\nu,m,6-\vartheta,\sigma+\epsilon+\vartheta}$$

and

$$\mu_j(\tau) = -\int_{\mathbb{R}^2} h(y,\tau)y_j dy + \tilde\mu_j[h](\tau)$$

because h_0 is radial, with

$$|\tilde\mu_j[h]| \le \frac{C}{\tau^{\nu+2-\vartheta}(\log\tau)^{m+1}}\|h_1\|_{\nu,m,6-\vartheta,\sigma+\epsilon+\vartheta}.$$

But

$$\|\phi_1\|_{\nu-\frac12,m+\frac q2,4,2+\sigma+\epsilon} \le \|\phi_1\|_{\nu,m+\frac q2,4-\vartheta,2+\sigma+\epsilon+\vartheta}$$

and hence

$$\|\phi_1\|_{\nu-\frac12,m+\frac q2,4,2+\sigma+\epsilon} \le C\|h\|_{\nu,m,6+\sigma,\epsilon}.$$

To apply Corollary 12.1 we need $1<\nu<1+\frac{\epsilon}{2}$ and $\nu<1+\frac{\vartheta}{2}$. Given $1<\nu<\min(1+\frac{\epsilon}{2}, 3-\frac{\sigma}{2}, \frac54)$ we can select $\vartheta\in(0,\frac12)$ such that $\nu<1+\frac{\vartheta}{2}$ and then proceed. This concludes the proof.

□

13. The outer problem

We consider the linear outer problem:

$$\begin{cases} \partial_t \phi^o = L^o[\phi^o] + g(x,t), & \text{in } \mathbb{R}^2 \times (t_0, \infty) \\ \phi^o(\cdot, t_0) = 0, & \text{in } \mathbb{R}^2. \end{cases} \tag{13.1}$$

where

$$L^o[\varphi] := \Delta_x \varphi - \nabla_x \Big[\Gamma_0 \Big(\frac{x - \xi(t)}{\lambda(t)} \Big) \Big] \cdot \nabla_x \varphi = \Delta_x \varphi + 4 \frac{(x-\xi)}{|x-\xi|^2 + \lambda^2} \cdot \nabla_x \varphi.$$

For $g : \mathbb{R}^2 \times (t_0, \infty) \to \mathbb{R}$ we consider the norm $\|g\|_{**,o}$ defined as the least K such that for all $(x,t) \in \mathbb{R}^2 \times (t_0, \infty)$

$$|g(x,t)| \le K \frac{1}{(t - t_0 + A)^a (\log t)^\beta} \frac{1}{1 + |\zeta|^b}, \quad \zeta = \frac{x - \xi(t)}{\sqrt{t - t_0 + A}},$$

where $A > 0$ is a constant.

We also define the norm $\|\phi\|_{*,o}$ as the least K such that

$$|\phi^o(x,t)| + (\lambda + |x - \xi|) |\nabla_x \phi^o(x,t)| \le K \frac{1}{(t - t_0 + A)^{a-1} (\log t)^\beta} \frac{1}{1 + |\zeta|^b}, \quad \zeta = \frac{x - \xi}{\sqrt{t - t_0 + A}}$$

for all $(x,t) \in \mathbb{R}^2 \times (t_0, \infty)$.

We assume that the parameters a, b satisfy the constraints

$$1 < a < 4, \quad 2 < b < 6, \quad a < 1 + \frac{b}{2}. \tag{13.2}$$

There is no restriction on β.

We recall from (4.1) that we are assuming that

$$|\dot\lambda(t)| \le \frac{C}{t (\log t)^{3/2}}, \quad t > t_0, \tag{13.3}$$

and

$$|\dot\xi(t)| \le \frac{C}{t^{\frac{3}{2} + \sigma}}, \quad t > t_0, \tag{13.4}$$

where $0 < \sigma < \frac{1}{2}$.

Proposition 13.1. *Assume that a, b satisfy (13.2), $\frac{A}{\lambda(t_0)^2}$ is sufficiently large, and λ, ξ satisfy (13.3), (13.4). Then there is a constant C so that for t_0 sufficiently large and for $\|g\|_{**,o} < \infty$ there exists a solution $\phi^o = \mathcal{T}_p^o[g]$ of (13.1), which defines a linear operator of g and satisfies*

$$\|\phi^o\|_{*,o} \le C \|g\|_{**,o}.$$

Proposition 6.3 in Section 6 follows from Proposition 13.1 with $A = t_0$.

Lemma 13.1. *Let $2 < \beta < 6$ and $h(r)$ satisfy*

$$|h(r)| \le \frac{\lambda^{-2}}{(r/\lambda + 1)^\beta} = \frac{\lambda^{\beta - 2}}{(r + \lambda)^\beta}, \tag{13.5}$$

where $\lambda > 0$. Then there is a unique bounded radial function $\varphi(r)$ satisfying

$$L^o[\varphi] + h = 0 \quad \textit{in } \mathbb{R}^2.$$

Moreover φ satisfies

$$|\varphi(r)| + (\lambda + r) |\partial_r \varphi(r)| \le \frac{C}{(1 + r/\lambda)^{\beta - 2}} = C \frac{\lambda^{\beta - 2}}{(r + \lambda)^{\beta - 2}} \tag{13.6}$$

Proof. The equation for φ is given by

$$\partial_{rr} \varphi(r) + \Big(\frac{1}{r} + \frac{4r}{\lambda^2 + r^2} \Big) \partial_r \varphi(r) + h(r) = 0, \quad r > 0.$$

We change variables $\rho = \frac{r}{\lambda}$ and let $\varphi(r) = \bar\varphi(\frac{r}{\lambda})$. Then we need to solve

$$\partial_{\rho\rho} \bar\varphi + \Big(\frac{1}{\rho} + \frac{4\rho}{1 + \rho^2} \Big) \partial_\rho \bar\varphi + \bar h(\rho) = 0, \quad \rho > 0,$$

where

$$\bar h(\rho) = \lambda^2 h(\lambda \rho).$$

By (13.5)

$$|\bar{h}(\rho)| \le \frac{1}{(1+\rho)^\beta}.$$

The bounded solution is given by

$$\bar{\varphi}(\rho) = \int_\rho^\infty \frac{1}{v(1+v^2)^2} \int_0^v \bar{h}(s)s(1+s^2)^2\, ds\, dv.$$

By direct computation we get

$$|\bar{\varphi}(\rho)| + (1+\rho)|\partial_\rho \bar{\varphi}(\rho)| \le \frac{C}{(1+\rho)^{\beta-2}},$$

and this implies (13.6).

□

Proof of Proposition 13.1. To find a pointwise estimate for the solution ϕ^o we construct a barrier.

Using polar coordinates $x - \xi(t) = re^{i\theta}$, L^o can be written as:

$$L^o[\varphi] = \partial_{rr}\varphi + \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\varphi + \frac{1}{r^2}\partial_{\theta\theta}\varphi.$$

First we construct a function $\tilde{\psi}(r,t)$ such that

$$\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\tilde{\psi} \ge \frac{1}{(t-t_0+A)^a(\log t)^\beta}\frac{1}{(1+r/\sqrt{t-t_0+A})^b}.$$

Let

$$\psi_1(r,t) = \frac{1}{(t-t_0+A)^{a-1}(\log t)^\beta}\Big[\frac{1}{(1+\frac{r^2}{t-t_0+A})^{b/2}} + C_1 e^{-\frac{r^2}{4(t-t_0+A)}}\Big].$$

Choosing a large constant C_1, ψ_1 satisfies

$$\partial_t\psi_1 - \partial_{rr}\psi_1 - \frac{5}{r}\partial_r\psi_1 \ge c\frac{1}{(t-t_0+A)^a(\log t)^\beta}\frac{1}{(1+\frac{r}{\sqrt{t-t_0+A}})^b}, \quad \text{for } r>0,\ t>t_0,$$

where $c>0$. Here we require $a<4$ and $a<1+\frac{b}{2}$, which are part of the conditions (13.2). Then

$$\begin{aligned}\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\psi_1 &= \Big[\partial_t - \partial_{rr} - \frac{5}{r}\partial_r\Big]\psi_1 + 4\frac{\lambda^2}{r(r^2+\lambda^2)}\partial_r\psi_1\\ &\ge c\frac{1}{(t-t_0+A)^a(\log t)^\beta}\frac{1}{(1+\frac{r}{\sqrt{t-t_0+A}})^b} - 4\frac{\lambda^2}{r(r^2+\lambda^2)}|\partial_r\psi_1|.\end{aligned} \tag{13.7}$$

But

$$\partial_r\psi_1 = \frac{r}{(t-t_0+A)^a(\log t)^\beta}\Big[-\frac{b}{(1+\frac{r^2}{t-t_0+A})^{b/2+1}} - \frac{C_1}{2}e^{-\frac{r^2}{4(t-t_0+A)}}\Big]$$

and so

$$\frac{\lambda^2}{r(r^2+\lambda^2)}|\partial_r\psi_1| \le C\frac{\lambda^2}{r^2+\lambda^2}\frac{1}{(t-t_0+A)^a(\log t)^\beta}\frac{1}{(1+\frac{r^2}{t-t_0+A})^{b/2+1}}. \tag{13.8}$$

We note that for $r \le \sqrt{t-t_0+A}$ we have

$$\frac{\lambda^2}{r(r^2+\lambda^2)}|\partial_r\psi_1| \le \frac{\lambda^2}{r^2+\lambda^2}\frac{1}{(t-t_0+A)^a(\log t)^\beta} \le C\frac{\lambda^2}{(r^2+\lambda^2)^2}\frac{1}{(t-t_0+A)^{a-1}(\log t)^\beta}, \tag{13.9}$$

where we have used that $A \ge \lambda(t)^2$.

Let $\tilde{\psi}_2(r;\lambda)$ be the bounded solution of

$$-\Big[\partial_{rr} + \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\tilde{\psi}_2 = \frac{\lambda^2}{(r^2+\lambda^2)^2}, \quad r>0,$$

given by Lemma 13.1. Then $\tilde{\psi}_2$ can be written as

$$\tilde{\psi}_2(r;\lambda) = \bar{\psi}_2\Big(\frac{r}{\lambda}\Big),$$

for a function $\bar{\psi}_2$ satisfying

$$|\bar{\psi}_2(\rho)| + (1+\rho)|\bar{\psi}_2'(\rho)| \le \frac{C}{1+\rho^2}. \tag{13.10}$$

Let

$$\psi_2(r,t) = \frac{1}{(t-t_0+A)t^{a-1}(\log t)^\beta}\tilde\psi_2(r;\lambda(t)).$$

Then, using (13.10) and (13.3), we get

$$\begin{aligned}
&\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\psi_2 \\
&= \frac{1}{(t-t_0+A)^{a-1}(\log t)^\beta}\frac{\lambda^2}{(r^2+\lambda^2)^2}\Big[1 - \Big(\frac{a-1}{t-t_0+A} + \frac{\beta}{t\log t}\Big)\tilde\psi_2(r)\frac{(r^2+\lambda^2)^2}{\lambda^2} \\
&\qquad - \frac{\dot\lambda}{\lambda}\bar\psi_2'\Big(\frac{r}{\lambda}\Big)\frac{r}{\lambda}\frac{(r^2+\lambda^2)^2}{\lambda^2}\Big] \\
&\ge \frac{1}{(t-t_0+A)^{a-1}(\log t)^\beta}\frac{\lambda^2}{(r^2+\lambda^2)^2}\Big[1 - C\frac{r^2+\lambda^2}{t-t_0+A}\Big].
\end{aligned}$$

Therefore there is $\delta > 0$ (fixed independent of t_0) such that for all t_0 large,

$$\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\psi_2 \ge \frac{1}{2}\frac{1}{(t-t_0+A)^{a-1}(\log t)^\beta}\frac{\lambda^2}{(r^2+\lambda^2)^2}, \quad \text{for } r \le 2\delta\sqrt{t}. \tag{13.11}$$

Let $\chi_0 \in C^\infty(\mathbb{R})$ be such that $\chi_0(s) = 1$ if $s \le 1$ and $\chi_0(s) = 0$ if $s \ge 2$ and define

$$\chi_\delta(r,t) = \chi_0\Big(\frac{r}{\delta\sqrt{t-t_0+A}}\Big).$$

We consider

$$\tilde\psi = \psi_1 + M\psi_2\chi_\delta,$$

where $M > 0$ is a constant to be fixed later. We compute, using (13.7)

$$\begin{aligned}
\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\tilde\psi &\ge c\frac{1}{(t-t_0+A)^a(\log t)^\beta(1+\frac{r}{\sqrt{t-t_0+A}})^b} - 4\frac{\lambda^2}{r(r^2+\lambda^2)}|\partial_r\psi_1| \\
&\quad + M\chi_\delta\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\tilde\psi_2 + R(r,t),
\end{aligned} \tag{13.12}$$

where

$$R = M\Big[\psi_2\partial_t\chi_\delta - 2\partial_r\tilde\psi_2\partial_r\chi_\delta - \tilde\psi_2\Big(\partial_{rr}\chi_\delta + \frac{1}{r}\partial_r\chi_\delta + 4\frac{r}{r^2+\lambda^2}\partial_r\chi_\delta\Big)\Big].$$

We have, by (13.10),

$$|R(r,t)| \le C_2M\lambda^2\frac{1}{(t-t_0+A)^{a+1}(\log t)^\beta}, \tag{13.13}$$

where C_2 is independent of M (although it depends on δ), and is supported on $\delta\sqrt{t-t_0+A} \le r \le 2\delta\sqrt{t-t_0+A}$.

We claim that there is $M > 0$ and $\tilde c > 0$ so that for all t_0 sufficiently large

$$\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\tilde\psi \ge \tilde c\frac{1}{t^a(\log t)^\beta(1+r/\sqrt{t})^b}, \tag{13.14}$$

for all $r > 0$, $t > t_0$.

Indeed, if $r \le \delta\sqrt{t-t_0+A}$, then from (13.12), (13.7), (13.11) and (13.9) we get

$$\begin{aligned}
\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\tilde\psi &\ge c\frac{1}{(t-t_0+A)^a(\log t)^\beta(1+\frac{r}{\sqrt{t-t_0+A}})^b} \\
&\quad - C\frac{\lambda^2}{(r^2+\lambda^2)}\frac{1}{(t-t_0+A)^{a-1}(\log t)^\beta} \\
&\quad + M\chi_\delta\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\tilde\psi_2 \\
&\ge c\frac{1}{(t-t_0+A)^a(\log t)^\beta(1+\frac{r}{\sqrt{t-t_0+A}})^b},
\end{aligned} \tag{13.15}$$

if $M \ge C$. Here we fix $M = C$.

If $\delta\sqrt{t-t_0+A} \le r \le 2\delta\sqrt{t-t_0+A}$, then by (13.12), (13.7), (13.9) and (13.13) we get

$$\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\tilde\psi \ge c\frac{1}{(t-t_0+A)^a(\log t)^\beta(1+\frac{r}{\sqrt{t-t_0+A}})^b} - C_2M\lambda^2\frac{1}{(t-t_0+A)^{a+1}(\log t)^\beta}$$
$$= \frac{1}{(t-t_0+A)^a(\log t)^\beta}\Big(\frac{c}{3^b} - \frac{C_2M\lambda^2}{t-t_0+A}\Big)$$

By taking $\frac{A}{\lambda(t_0)^2}$ large, we get

$$\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\tilde\psi \ge \frac{c}{2}\frac{1}{(t-t_0+A)^a(\log t)^\beta(1+\frac{r}{\sqrt{t-t_0+A}})^b}, \tag{13.16}$$

for $\delta\sqrt{t-t_0+A} \le r \le 2\delta\sqrt{t-t_0+A}$.

If $r \ge 2\delta\sqrt{t-t_0+A}$, by (13.12) and (13.8)

$$\Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\psi_1 \ge c\frac{1}{(t-t_0+A)^a(\log t)^\beta(1+\frac{r}{\sqrt{t-t_0+A}})^b}$$
$$- C\frac{\lambda^2}{r^2+\lambda^2}\frac{1}{(t-t_0+A)^a(\log t)^\beta}\frac{1}{(1+\frac{r^2}{t-t_0+A})^{b/2+1}}$$
$$\ge \frac{1}{(t-t_0+A)^a(\log t)^\beta(1+\frac{r}{\sqrt{t-t_0+A}})^b}\Big[c - C\frac{\lambda^2}{t-t_0+A}\Big]$$
$$\ge \frac{c}{2}\frac{1}{(t-t_0+A)^a(\log t)^\beta(1+\frac{r}{\sqrt{t-t_0+A}})^b} \tag{13.17}$$

if $\frac{A}{\lambda(t_0)^2}$ is sufficiently large.

Combining (13.15), (13.16) and (13.17) we deduce the estimate (13.14).

Let

$$\psi(x,t) = \tilde\psi(|x-\xi|,t).$$

Then by (13.14)

$$(\partial_t - L^o)[\psi] = \Big[\partial_t - \partial_{rr} - \Big(\frac{1}{r} + \frac{4r}{\lambda^2+r^2}\Big)\partial_r\Big]\tilde\psi - \partial_r\tilde\psi\frac{(x-\xi)\cdot\dot\xi}{|x-\xi|}$$
$$\ge \tilde c\frac{1}{(t-t_0+A)^a(\log t)^\beta(1+\frac{r}{\sqrt{t-t_0+A}})^b} - |\dot\xi|\,|\partial_r\tilde\psi|.$$

But

$$|\partial_r\psi| \le C\frac{1}{(t-t_0+A)^{a-1/2}(\log t)^\beta}\frac{1}{(1+\frac{r}{\sqrt{t-t_0+A}})^{b+1}} + C\frac{1}{(t-t_0+A)^{a-1}(\log t)^\beta}\frac{1}{\lambda}\frac{1}{(1+r/\lambda)^3}\chi_\delta(r,t)$$
$$+ C\frac{1}{\delta(t-t_0+A)^{a-1/2}(\log t)^\beta}\frac{1}{(1+r/\lambda)^2}\chi_0'\Big(\frac{r}{\delta\sqrt{t-t_0+A}}\Big).$$

Using (13.4) we see that if t_0 is sufficiently large,

$$(\partial_t - L^o)[\psi] \ge \frac{\tilde c}{2}\frac{1}{(t-t_0+A)^a(\log t)^\beta(1+\frac{r}{\sqrt{t-t_0+A}})^b}.$$

□

A direct consequence of the proof of Proposition 13.1 (using the same barriers) is the following, for the initial value problem

$$\begin{cases} \partial_t\phi^o = L^o[\phi^o], & \text{in } \mathbb{R}^2\times(t_0,\infty) \\ \phi^o(\cdot,t_0) = \phi^o_0, & \text{in } \mathbb{R}^2. \end{cases} \tag{13.18}$$

Consider the norm

$$\|\phi^o_0\|_{*,b} = \inf K \quad \text{such that}$$
$$|\phi^o_0(x)| \le \frac{K}{(1+\frac{|x-\xi(0)|}{\sqrt{t-t_0+A}})^b}$$

where $b\in(2,6)$, $A>0$.

Proposition 13.2. *Assume that* a, b *satisfy* (13.2), $\frac{A}{\lambda(t_0)^2}$ *is sufficiently large, and* λ, ξ *satisfy* (13.3), (13.4). *Then there is a constant* C *so that for* t_0 *sufficiently large and for* $\|\phi_0^o\|_{*,b} < \infty$ *there exists a solution* ϕ^o *of* (13.18)*, which defines a linear operator of* ϕ_0^o *and satisfies*

$$\|\phi^o\|_{*,o} \leq CA^{a-1}(\log t_0)^{\beta}\|\phi_0^o\|_{*,b}.$$

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