UNIQUENESS OF LUMP SOLUTIONS OF KP-I EQUATION

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Abstract. The KP-I equation has family of solutions which decay to zero at space infinity. One of these solutions is the classical lump solution. This is a traveling wave, and the KP-I equation in this case reduces to the Boussinesq equation. In this paper we classify the lump type solutions of the Boussinesq equation. Using a robust inverse scattering transform developed by Bilman-Miller, we show that the lump type solutions are rational and their tau function has to be a polynomial of degree $k(k+1)$. In particular, this implies that the lump solution is the unique ground state of the KP-I equation (as conjectured by Klein and Saut in [27]). Our result generalizes a theorem by Airault-McKean-Moser on the classification of rational solutions for the KdV equation.

1. Introduction

The KP equation first appeared in the 1970 paper [25] by Kadomtsev and Petviashvili, where they studied the transverse stability of the line solitons of KdV equation. It can be written as

$$\partial_x \left(\partial_t U + \partial_x^3 U + 3\partial_x \left(U^2\right)\right) - \sigma \partial_y^2 U = 0. \tag{1}$$

If $\sigma = 1$, then it is called KP-I equation, while the case of $\sigma = -1$ is called KP-II.

KP equation is an integrable system and can be regarded as a two dimensional generalization of the classical KdV equation. It is an important PDE both in mathematics and physics. There are vast literature on KP equation. In the sequel, we shall briefly mention some results which are most closely related to our objective.

There are various different ways to study the KP equation. One of them is to use the inverse scattering transform. Manakov[36] studied the inverse scattering transform of KP equation on a formal level. Segur in [44] analyzed the direct scattering and rigorously obtained the solution for the direct problem under a small norm assumption. Lump solutions are not investigated in these works. Then Fokas-Ablowitz[19] obtained the lump solution in their inverse scattering setting. Their results are later extended to higher order rational solutions in [46]. Zhou[47] then studied the KP-I equation in a rigorous way. In that paper, the lump solutions correspond to poles of the associated eigenfunctions. Since the pole structure is actually still not well understood in the general case, therefore the lump solutions are not treated.

Observe that if U is a traveling wave of the form $u\,(x - t, y)$, then the KP-I equation reduces to the following Boussinesq equation:

$$\partial_x^2 \left(\partial_x^2 u + 3u^2 - u\right) - \partial_y^2 u = 0. \tag{2}$$

Due to the above mentioned difficulties, we would like to study the traveling wave solutions of the KP-I equation using the inverse scattering transform of the Boussinesq equation. This IST is first carried out in [13]. The first equation of the Lax pair turns out to be a third order ODE, in contrast with the second order ODE of the KdV equation. In this direction, the IST for first order ODE systems with generic potential(the poles are all simple) is studied in [5], [6] and the case of higher order ODE is treated in [4]. The case of general potentials have been studied in [14, 48] using the augmented contour approach.

Let us write the solution u of (2) in terms of the τ function: $u := 2\partial_x^2 \ln \tau$. Then it is well known that the Boussinesq equation in bilinear form is

$$\left(D_x^4 - D_x^2 - D_y^2\right) \tau \cdot \tau = 0, \tag{3}$$

where D is the bilinear derivative operator. One can check that the function $\tau(x, y) = x^2 + y^2 + 3$ is a solution of this bilinear equation. Note that this function is even in both x and y variables and corresponding to the lump solution. We point out that actually the lump solution is first obtained in [36, 43] using other methods. We have proved in [32] using the Backlund transformation that the lump solution is nondegenerated, in the sense that the linearized KP-I operator at this solution does not have nontrivial kernels. It is also known that KP equation is closely related to many other PDEs. For instance, in [7](see also the references therein), it is shown that KP equation is related to the GP equation. The nondegeneracy result for the lump can then be used to construct traveling wave solutions of the GP equation with subsonic speed, with a perturbation argument, see [33].

More general rational solutions of (3) with degree $k(k-1)$ have been found in [21], [40]. Then in [20] it is proved that around the higher energy lump type solutions, the KP-I equation has anomalous scattering with infinite phase shift. This indicates that the dynamics of the KP-I equation will be complicated than the KdV equation. Hence it is important to understand the structure of lump type solutions for the Boussinesq, as well as the KP-I equation. We also point out that KP-I is globally well-posed in the natural energy space, see, for instance [26],[38],[39] and the references therein for related results in this direction. A very fascinating and in-depth description of KP equation and related dynamical, variational, and other properties of its solutions, including lump, can be found in the book of Klein-Saut[28]. We refer to it and also its references for a detailed introduction on this subject.

It is worth mentioning that (2) is a special case of Boussinesq-type equation(its original form described by Boussinesq in 1870s)

$$\partial_x^2 \left(pu + u^2 + \partial_x^2 u\right) + \sigma^2 \partial_y^2 u = 0,$$

where $\sigma^2 = \pm 1$ and p is a constant. Rational solutions of this equation has been studied in [2], [3],[9], [21]. The special case of $(\sigma, p) = (1, 0)$ is considered in [12], using the theory of Painlevé equations.

In view of all these developments, it is desirable to have some classification on the solutions of the Boussinesq equation. In this paper, we would like to classify all the "lump type" solutions. The main result of this paper is the following

Theorem 1. *Suppose u is a real-valued smooth solution of the equation*

$$\partial_x^2 \left(\partial_x^2 u + 3u^2 - u\right) - \partial_y^2 u = 0 \text{ in } \mathbb{R}^2. \tag{4}$$

Assume

$$u(x,y) \to 0, \text{ as } x^2+y^2 \to +\infty.$$

Then $u = 2\partial_x^2 \ln \tau_k$, *where* τ_k *is a polynomial in* x, y *of degree* $k(k+1)$ *for some* $k \in \mathbb{N}$.

As in [15], problem (4) admits a variational structure which defines a ground state solution in suitable Sobolev space. As a corollary of this theorem and the algebraic decaying property of ground states (see Theorem 3.1 of [15]), we see that the classical lump solution is the unique ground state of the KP-I equation, due to the fact that the energy is determined by the degree of the tau function. This answers affirmatively the uniqueness question raised in Remark 18 and Remark 19 of [27]. As already pointed out there, while the uniqueness of ground state of the Schrodinger equation can be proved using ODE shooting method, the uniqueness of lump is more complicated since it is not radially symmetric. To our knowledge, our result seems to be the first classification result for solutions of semilinear elliptic equations without symmetry(also without any other assumptions like stable or finite Morse index).

Solutions satisfying the assumption of this theorem will be called lump type solutions. We remark that for each fixed k, there is a family of lump type solutions. We expect that all lump type solutions should be included in the family found in [21]. Those solutions will be recalled in the next section. However, a full classification of this type would need further detailed analysis, which will not be pursued in this paper. Such a full classification would presumably yields some information of the lump type solutions of the generalized KP equation.

Let us sketch the main ideas of our proof. We first use the robust inverse scattering transform developed by Bilman-Miller [8] to show that lump type solutions has to be rational. Then we use the technique of [3], appealing to the Boussinesq hierarchy, to show that the degree of the τ function has to be $k(k+1)$. This technique is used in [3] to prove that the τ function of the rational solution of the KdV equation necessarily is a polynomial of degree $k(k+1)/2$.

This paper is organized in the following way. In Section 2, we recall the construction of lump type solution appeared in various papers of Pelinovskii and his collaborators. We emphasize that actually there are many other constructions, using different methods, which we will not recall here. In Section 3, we use the robust inverse scattering transform to show that the solution is rational. In Section 4, we investigated the degree of the τ function, the main theorem then follows immediately. Some numerical computation for the even solution is carried out in the last section.

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2. Family of lump type solutions

For each $n \in \mathbb{N}$, real valued rational solutions of the Boussinesq equation are obtained in [40] by a limiting procedure. These solutions are even with respect to x

and y variable. Then in [21], families of rational solutions are derived using Wronskian representation of the solutions to the KP equation. For reader's convenience, Let us recall these results in this section. We adopt the notations used in [21].

Consider the KP-I equation

$$\left(-4u_{t_3}+\left(3u^2\right)t_1+u_{t_1t_1t_1}\right)_{t_1}-3u_{t_2}u_{t_2}=0.$$

This equation has solutions expressed in terms of the τ function:

$$u\left(t_1,t_2,t_3\right)=2\partial_{t_1}^2\ln\tau\left(t_1,t_2,t_3\right).$$

There are different forms for the τ functions. Let us explain it now.

Let $\Psi_n^{\pm}$ be solutions of the system of differential equations

$$\begin{cases}\pm i\partial_{t_2}\Psi_n^{\pm}=\partial_{t_1}^2\Psi_n^{\pm},\\ \partial_{t_3}\Psi_n^{\pm}=\partial_{t_1}^3\Psi_n^{\pm}.\end{cases}$$

We fix an integer N and define

$$\tau=\det M_N, \tag{5}$$

where M_N is the $N\times N$ matrix whose entries are given by $c_{nk}+I_{nk}, 1\leq n,k\leq N$. Here c_{nk} are arbitrary complex parameters and

$$I_{nk}=\int_{-\infty}^{t_1}\Psi_n^{+}\left(s,t_2,t_3\right)\Psi_k^{-}\left(s,t_2,t_3\right)ds.$$

The KP-I equation has another family of solutions, for which the τ function has the Wronskian form:

$$\tau=W\left(\Psi_1^{\pm},...,\Psi_N^{\pm}\right)=\det\left(J_{nk}^{\pm}\right), \tag{6}$$

where $J_{nk}^{\pm}=\partial_{t_1}^{k-1}\Psi_n^{\pm}$.

The two forms (5) and (6) are related to each other. If we choose in (5) the function

$$\Psi_k^{-}=\exp\left(p_kt_1-p_k^2t_2+p_k^3t_3\right)=\exp\left(\Phi_k^{-}\right),$$

then integration by parts yields

$$I_{nk}=\left(\frac{\Psi_n^{+}}{p_k}-\frac{\partial_{t_1}\Psi_n^{+}}{p_k^2}+\frac{\partial_{t_1}^2\Psi_n^{+}}{p_k^3}+...\right)\exp\left(\Phi_k^{-}\right).$$

Assuming $p_k>>1$, the leading terms of τ can be written as the product of a Vandermont determinant and the Wronskian $W\left(\Psi_1^{+},...,\Psi_N^{+}\right)$. Hence

$$\tau=\left[\frac{\prod\limits_{1\leq m<k\leq N}\left(p_k-p_m\right)}{\prod\limits_{k=1}^{N}p_k^N}W\left(\Psi_1^{+},...,\Psi_N^{+}\right)+O\left(p^{-\left(\frac{N(N+1)}{2}+1\right)}\right)\right]\exp\left(\sum_{k=1}^{N}\Phi_k^{-}\right).$$

Dividing the right hand side by $\exp\left(\sum_{k=1}^{N}\Phi_k^{-}\right)$ and the constant before the Wronskian W, letting $p\to 0$, we get (6).

Let $K\leq N$ be a fixed integer. If the above limiting procedure is only carried out for $\Phi_k^{-}, k=K+1,...,N$, then we obtain

$$\tau=\det\left(S_{nk}\right), \tag{7}$$

where

$$S_{nk} = \begin{cases} I_{nk}, \text{ for } k = 1, ..., K, \\ J^{+}_{n,k-K}, \text{ for } k = K+1, ..., N. \end{cases}$$

Let us now consider the function $\phi_m := \partial_p^m \exp\left(\Phi^+\left(t_1, t_2, t_3, p\right)\right)$, where

$$\Phi^+\left(t_1, t_2, t_3, p\right) = \sum_{j=1}^{\infty} \left(p^j t_j\right).$$

We have

$$\phi_m = P_m \exp\left(\Phi^+\left(t_1, t_2, t_3, p\right)\right).$$

Here P_m is a polynomial of the variables $\theta_1, ..., \theta_m$, given by $\theta_j = \frac{1}{m!}\partial_p^j \Phi^+$. In particular,

$$\begin{aligned} \theta_1 &= t_1 + 2pt_2 + 3p^2 t_3 + ..., \\ \theta_2 &= t_2 + 3pt_3 + ... \\ \theta_3 &= t_3 + ..., \end{aligned}$$

and θ_j only depends on $t_j, t_{j+1}, ...$ We have

$$P_1 = \theta_1, P_2 = 2\theta_2 + \theta_1^2.$$

To obtain a solution of the Boussinesq equation, let $v = -\frac{1}{3p}$ and define the vertex operator

$$\mathcal{S}(v) = \exp\left(-\sum_{m=1}^{+\infty} \frac{v^m}{m} \partial_{\theta_m}\right).$$

Then

$$\mathcal{S}(v) P_n = \left(1 - v\partial_{t_1}\right) P_n.$$

The tau function will be a solution of the Boussinesq equation if it depends on the variable $x = t_1 + 3p^2 t_3$ and t_2. This requires

$$\partial_{\theta_2}\tau = v\partial_{\theta_3}\tau.$$

The next step to construct solutions of the Boussinesq equation is choose p to be 1 and define

$$\begin{aligned} \Psi_n^+ &= S^{N-n}(v) P^+_{2n-1} \exp\left(\Phi^+\left(t_1, t_2, t_3, p\right)\right), \text{ for } 1 \leq n \leq N, \\ \Psi_k^- &= S^{K-k} P^-_{2k-1} \exp\left(\Phi^-\left(t_1, t_2, t_3, p\right)\right), 1 \leq k \leq K. \end{aligned}$$

Then the Tau function defined by (7) will correspond to a solution of the Boussinesq equation. In general, this solution is complex valued. But in the particular case of $K = N$, if we choose P_k^+, P_k^- such that $P_k^+ = \bar{P}_k^-$, then

$$\tau = \det\left(w^+\left(w^-\right)^T\right),$$

where

$$\begin{aligned} \left(w^+\right)_{nk} &= (-\mu)^{k-1} \partial_{t_1}^{k-1}\left[S^{-k}(\mu) S^{N-n}(v) P^+_{2n-1}\right], 1 \leq n \leq N, 1 \leq k \leq 2N-1, \\ w^-_{nk} &= \bar{w}^+_{nk}. \end{aligned}$$

Hence this determinant can be written as the sum of positive terms. Note that the condition $P_k^+ = \bar{P}_k^-$ requires t_{2k} is imaginary and t_{2k+1} is real. Hence there are in total $2N$ free (real)parameters, or N complex parameters.

We point out that in [50], an explicit family of rational solutions is obtained with different methods. It is not clear whether these two family of solutions are same. It is also worth mentioning that there are quite a vast literature on the construction of solutions to the KP equation, which we will not list them here.

3. The Lax pair and Beals-Coifman functions of the Boussinesq equation

We consider the Boussinesq equation in the following form

$$q_{yy} = 3q_{xxxx} - 12\left(q^2\right)_{xx} - 24q_{xx}. \tag{8}$$

This equation can be obtained from (2) by setting $q(x,y) = -6u\left(2\sqrt{2}x, 8\sqrt{3}y\right)$. We are interested in the solutions of this equation which decay to zero at infinity.

In this section, we will adopt the powerful method of robust inverse scattering transform, first developed in [8] for the Schrodinger equation to study the rogue wave solutions, to show that solution of the Boussinesq equation has to be rational.

The equation (8) can also be written into the following system of ODEs:

$$\begin{cases} q_y = -3p_x, \\ p_y = -q_{xxx} + 8qq_x + 8q_x. \end{cases}$$

This system has the following Lax pair(see [13]):

$$\frac{dL}{dy} = QL - LQ = [Q, L],$$

where

$$\begin{cases} L = i\frac{d^3}{dx^3} - i\left[\left(2(q+1)\frac{d}{dx} + q_x\right)\right] + p, \\ Q = i\left(3\frac{d^2}{dx^2} - 4(q+1)\right). \end{cases}$$

Let $k \in \mathbb{C}$ be a complex parameter. We consider the equation

$$Lf = \left(k^3 + 2k\right) f,$$

which can be recasted into the matrix form:

$$\frac{d}{dx}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q_x + pi - i\left(k^3 + 2k\right) & 2(q+1) & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{9}$$

The coefficient matrix of this system, denoted by A, will depend on the potential q. As $x \to \pm\infty$, A will tend to the constant matrix

$$T := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -i\left(k^3 + 2k\right) & 2 & 0 \end{pmatrix}.$$

The eigenvalues of T can be explicitly computed. They depend on the parameter k and are given by

$$\lambda_1 = ik, \quad \lambda_2 = \frac{-ik + \sqrt{3k^2+8}}{2}, \quad \lambda_3 = \frac{-ik - \sqrt{3k^2+8}}{2}.$$

It follows that T can be written as PMP^{-1}, where

$$P(k) = \begin{pmatrix} 1 & 1 & 1 \\ ik & \frac{-ik+\sqrt{3k^2+8}}{2} & \frac{-ik-\sqrt{3k^2+8}}{2} \\ -k^2 & \frac{k^2+4-ik\sqrt{3k^2+8}}{2} & \frac{k^2+4+ik\sqrt{3k^2+8}}{2} \end{pmatrix}, M(k) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Recall that for any constant matrix B, the matrix $e^{Tx}B$ is a solution of the equation $U' = TU$. We choose $B = P$ and get the following matrix solution

$$U_{bg}(k,x) := n(k)P(k)\,e^{M(k)x} := E(k)\,e^{M(k)x}.$$

Here $n(k)$ is chosen such that $\det(U_{bg}) = 1$. Explicitly,

$$n(k) = \left(\left(3k^2+2\right)\sqrt{3k^2+8}\right)^{-1}.$$

Let $k = s+ti$, with $x, y \in \mathbb{R}$. The condition $\operatorname{Re}(\lambda_2) = \operatorname{Re}(\lambda_3)$ implies $\operatorname{Re}\sqrt{3k^2+8} = 0$. That is,

$$s = 0, t^2 > \frac{8}{3}.$$

On the other hand, $\operatorname{Re}(\lambda_1) = \operatorname{Re}(\lambda_2)$ or $\operatorname{Re}(\lambda_3)$ requires

$$s^2 - 3t^2 + 2 = 0.$$

Let r be a fixed large constant and B_r be the ball of radius r centered at the origin. In the region $B_r^c := \mathbb{R}^2 \backslash B_r$, we consider the curve

$$\Sigma_1 := \left\{(s,t) \in B_r^c : s^2 - 3t^2 + 2 = 0\right\} \cup \left\{(s,t) \in B_r^c : s = 0 \text{ and } t^2 > \frac{8}{3}\right\}.$$

Let us define

$$\Omega_1 = B_r^c \backslash \Sigma_1.$$

Note that Ω_1 has six connected components, which we will denote them by $\Omega_{1,1}, ..., \Omega_{1,6}$.

In the ball B_r, we consider the curve

$$\Sigma_2 := \left\{(s,t) : s = 0, t^2 \leq \frac{8}{3}\right\}.$$

We also define $\Omega_2 := B_r \backslash \Sigma_2$.

Next we define the Beals-Coifman solution for (9). Note that if the matrix ϕ satisfies $\phi' = A\phi$, then $g := \phi e^{-Mx}$ will satisfy

$$\begin{aligned} g' &= \phi' e^{-Mx} + \phi(-M)\,e^{-Mx} \\ &= Ag - gM. \end{aligned}$$

For $k \in \mathbb{R}^2 \backslash B_r$, we choose to be the matrix solution such that $\|g(x)\|_{L^\infty(\mathbb{R})} < +\infty$, and

$$g(x) \to E(k), \text{ as } x \to -\infty.$$

We then define $U^{ou} = ge^{Mx}$. This solution is meromorphic in each $\Omega_{1,j}, j = 1, ..., 6$. The restriction of U^{ou} to $\Omega_{1,j}$ will be denoted by U_j^{ou}. On the common boundary of $\Omega_{1,j}$ and $\Omega_{1,j+1}$, U_{j+1}^{ou} and U_j^{ou} are related by the transfer matrix V_j. That is, for $j = 1, ..., 6$,

$$U_{j+1}^{ou} = U_j^{ou} V_j.$$

Here we set $U_7^{ou} = U_1^{ou}$.

Lemma 2. *The transfer matrix V_j is equal to I.*

Proof. The matrix V_j is independent of x. We would like to analyze the limit of V_j as y tends to infinity. We only like to estimate the difference between U_j^{ou} and $U_{bg} = E(k)\, e^{-M(k)x}$. Note that

$$e^{-Mx} = \begin{bmatrix} e^{-\lambda_1 x} & 0 & 0 \\ 0 & e^{-\lambda_2 x} & 0 \\ 0 & 0 & e^{-\lambda_3 x} \end{bmatrix}.$$

Let us denote the j-th column of $E(k)$ by ξ_j. For each $j = 1, 2, 3$, we would like to use an iteration scheme to construct a vector valued solution η satisfies the following condition: If $\mathrm{Re}(\lambda_j) \geq 0$, then

$$\eta_j e^{\lambda_j x} \to \xi_j \text{ as } x \to +\infty,$$

If $\mathrm{Re}(\lambda_j) \leq 0$, then

$$\eta_j e^{\lambda_j x} \to \xi_j \text{ as } x \to -\infty.$$

Using the same arguments as that of the proof Lemma 5.2 in [34], we find that

$$\left|\eta_j e^{\lambda_j x} - \xi_j\right| \leq \int_x^{\pm\infty} \frac{C}{s^2 + y^2} ds = \frac{C}{y} \arctan \frac{x}{y}.$$

Hence as $y \to \infty$, we find that uniformly

$$\eta_j e^{\lambda_j x} \to \xi_j.$$

This implies that the transfer matrix is identity. □

In Ω_2, we define U^{in}, matrix solution of (9), such that

$$U^{in}(0) = I.$$

Then one can show that U^{in} is holomorphic in Ω_2. This is a key property. In general, assuming the jump matrix from the interior to the outer solutions at the boundary circle ∂B_r has the form

$$G(k)\, E(k).$$

Then we have the following relation:

$$U^{ou} = U^{in} G(k)\, E(k). \tag{10}$$

Taking $(x, y) = (0, 0)$, we get

$$U^{ou}(0, 0) = G(k)\, E(k).$$

It turns our that the Beals-Coifman fundamental solution $U^{ou}(k; x, y)$ has the form

$$U^{ou}(k; x, y) = \left[I + \sum_{\lambda_j} \sum_{s=1}^{n_j} \left(\frac{A_{j,s}(x, y)}{(k - \lambda_j)^k} \right) \right] E(k)\, e^{M(k)x}.$$

Here $A_{j,s}$ are 3×3 matrices. We then obtain

$$\left[I + \sum_{\lambda_j} \sum_{s=1}^{n_j} \left(\frac{A_{j,s}(x, y)}{(k - \lambda_j)^k} \right) \right] E(k)\, e^{Mx} E(k)^{-1} G(k)^{-1} = U^{in}. \tag{11}$$

As we already mentioned, U^{in} is a holomorphic function in the radius r disk. This will yield a system of equations for the entries of $A_{j,s}$. Next we would like to show that the system has a unique solution.

Lemma 3. *For fixed G, the system* (10) *has a unique solution.*

Proof. We have

$$U^{ou}(k;x,y) = U^{in}(k;x,y)\,G(k).$$

Suppose there is another pair $\left(\tilde{U}^{ou}, \tilde{U}^{in}\right)$ such that

$$\tilde{U}^{ou}(k;x,y) = \tilde{U}^{in}(k;x,y)\,G(k).$$

We claim that $\tilde{U}^{ou} = U^{ou}$.

Indeed, since $\tilde{U}^{ou}$ is invertable, the matrix $U^{ou}\left(\tilde{U}^{ou}\right)^{-1}$ is holomorphic outside B_r, while $U^{in}\left(\tilde{U}^{in}\right)^{-1}$ is holomorphic inside B_r. Moreover, they are equal to each other on ∂B_r. Hence they patch up to an entire holomorphic function which is also bounded. Hence in view of their asymptotics at infinity, we obtain

$$U^{ou} = \tilde{U}^{ou}, \text{ and } U^{in} = \tilde{U}^{in}.$$

This finishes the proof. □

With this result at hand, next we show that the solution q has to be rational.

Theorem 4. *Suppose q is a solution of the Boussinesq equation satisfying the assumption of Theorem, then q is rational.*

Proof. The solution q, which appears as a potential in the Lax pair equation, is determined by the Beals-Coifman function U^{ou} and U^{in}. Hence we need to determine the matrices $A_{j,s}$ in (11).

Using the arguments of Lemma 2, one can show that the possible poles λ_j in (11) has to be $\lambda_{\pm}^* := \pm\sqrt{\frac{8}{3}}i$. The matrix $E(k)\,e^{Mx}E(k)^{-1}$ is holomorphic in k. As $k \to \lambda_{\pm}^*$, direct computation shows that it(as well as its derivatives in k) tends to a matrix whose entries are polynomial functions of x(multiplied by some exponential functions in x). Since the right hand side of (11) is holomorphic in k, we see that the matrices $A_{j,s}$ satisfy a system of linear equation whose entries are polynomial in x. Now by Lemma 3, the solution has to be unique. Hence q is rational in x. It then follows from the Krichever theorem(see [29, 30]) that the solution is also rational in y. □

4. The Boussinesq hierarchy and the structure of the rational solutions

In this section, we will use the technique developed in [3] for the KdV equation, to classify the degree of the tau function of the rational solution of the Boussinesq equation. Indeed, this problem is mentioned in P. 123-P. 124 of [3], for the hyperbolic case of Boussinesq equation.

In [35], Mckean found the Boussinesq hierarchy associated to the Boussinesq equation. The Boussinesq equation he studied has the following form

$$\partial_y^2 q = 3\partial_x^2\left(\partial_x^2 q + 4q^2\right). \tag{12}$$

We use D to denote the differentiation with respect to the x variable(not the bilinear derivative operator). Define the operator

$$\mathcal{D} = \begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix}.$$

Let

$$L_0 := D^5 + 5\left(qD^3 + D^3 q\right) \\ - 3\left(q''D + Dq''\right) + 16qDq,$$

and define

$$\mathcal{K}_0 = \begin{bmatrix} D^3 + qD + Dq & 3pD + 2p' \\ 3pD + p' & \frac{1}{3}L_0 \end{bmatrix}.$$

He then uses $\mathcal{K}_0$ to define recursively a sequence of vector fields, which generate the Boussinesq hierarchy.

In our case, we are actually considering those solutions of the Boussinesq equation (12) with nonzero boundary condition, say $\tilde{q} \to -\frac{1}{8}$ as $x^2 + y^2 \to +\infty$. Indeed, introducing new variable $\tilde{q}$ by $q = \tilde{q} - \frac{1}{8}$ in (12), we obtain

$$\partial_y^2 \tilde{q} = 3\partial_x^2 \left(\partial_x^2 \tilde{q} + 4\tilde{q}^2 - \tilde{q}\right). \tag{13}$$

If we set $\tilde{q}(x, y) = \frac{3}{4}u\left(x, \sqrt{3}y\right)$, then u satisfies the version of Boussinesq equation appeared in the first section, that is:

$$\partial_x^2\left(\partial_x^2 u + 3u^2 - u\right) - \partial_y^2 u = 0.$$

We are thus lead to consider the shifted operator L defined by

$$L := D^5 + 5\left[\left(q - \frac{1}{8}\right)D^3 + D^3\left(q - \frac{1}{8}\right)\right] \\ - 3\left(q''D + Dq''\right) + 16\left(q - \frac{1}{8}\right)D\left(q - \frac{1}{8}\right).$$

Note that

$$L = L_0 - \frac{5}{4}D^3 - 2\left(qD + Dq\right) + \frac{1}{4}D$$

We then define

$$\mathcal{K}_i = \begin{bmatrix} D^3 + \left(q - \frac{1}{8}\right)D + D\left(q - \frac{1}{8}\right) & 3\left(p + (-1)^i a\right)D + 2p' \\ 3\left(p + (-1)^i a\right)D + p' & \frac{1}{3}L \end{bmatrix}.$$

Here the constant a is chosen such that

$$(3a)^2 + \frac{1}{48} = 0.$$

Let $H_0 = \int \frac{3p}{2}$. Then a series of vector fields can be defined recursively by

$$X_{n+1} = \mathcal{K}_n \nabla H_n, \text{ and } \mathcal{D}\nabla H_n = X_n.$$

More precisely, once we obtained X_j, we can find ∇H_j by using the relation $\mathcal{D}\nabla H_j = X_j$. Then we can find X_{j+1} by $X_{j+1} = \mathcal{K}\nabla H_j$.

In particular,

$$X_0 = \mathcal{D}\nabla H_0 = 0, X_1 = \mathcal{K}\nabla H_0 = \left(3p', q''' + 8qq' - q'\right).$$

Hence X_2 is the Boussinesq flow. Here $'$ represents the derivative with respect to the x variable.

For real valued solutions, as we will see, the main order term of the τ function is $\left(x^2 + 3y^2\right)^n$.

Suppose u is a rational solution of the KP-I equation. Then from [29, 30], we know that u can be written in the form

$$u=-\frac{3}{2}\sum_{i=1}^{N}\frac{1}{\left(x-\xi_i\left(y,t\right)\right)^2}.$$

In this case, $u=\frac{3}{2}\partial_x^2\ln\tau$, where τ is a polynomial in the x variable.

For rational solutions q of the Boussinesq equation, we have

$$q=-\frac{3}{2}\sum_{i=1}^{N}\frac{1}{\left(x-\eta_i\left(y\right)\right)^2}. \tag{14}$$

inserting this into the equation (13) , we find that for each fixed index $i=1,...,n$, there holds

$$\partial_y^2\eta_i-\frac{72}{\left(\eta_i-\eta_j\right)^3}=0, \tag{15}$$

$$\eta_i'^2+36\Sigma_{j\neq i}\left(\eta_j-\eta_i\right)^{-2}+3=0.$$

Note that (15) is the Caloger-Moser system. More precisely, let $\partial_y\eta_i=\beta_i$. Then the CM flow is

$$\begin{cases}\partial_y\eta_i=\beta_i,\\ \partial_y\beta_i=\frac{72}{\left(\eta_i-\eta_j\right)^3}.\end{cases} \tag{16}$$

Indeed, one can show that the function (14) solves the Boussinesq equation if and only (η,β) satisfies the CM system (16) restricted to the set

$$M:=\left\{(\eta,\beta)\in\mathbb{C}^{2n}:\nabla\left(F_3+F_1\right)=0\right\},$$

where

$$F_1=3\sum_{j=1}^{n}\beta_j,$$

$$F_3=\frac{1}{3}\sum_{j=1}^{n}\beta_j^3+36\sum_{j=1}^{n}\sum_{k\neq j}\frac{\beta_j}{\left(\eta_j-\eta_k\right)^2}.$$

The proof follows from similar lines as that of [3], although the case of hyperbolic Boussinesq equation is treated there, instead of the elliptic case we are studying now. Therefore we omit the details.

Proposition 5. *The k-th Boussinesq flow $e\left(tX_k\right)$ induces a flow on M.*

Proof. The Boussinesq equation reads as

$$\begin{cases}q_y=3p',\\ p_y=q'''+8qq'-q'.\end{cases}$$

The rational solution q of the Boussinesq equation can be written as

$$q=-\frac{3}{2}\sum_{j=1}^{n}\frac{1}{\left(x-\eta_j\right)^2}.$$

Therefore,

$$p=\frac{1}{2}\sum_{j=1}^{n}\frac{\beta_j}{\left(x-\eta_j\right)^2}.$$

where $\beta_j = \partial_y \eta_j$.

We would like to write down the explicit form of each vector field X_k acting on (q,p). In view of the form of (q,p), it is natural to expect that

$$X_k(q,p) = \left(6\sum_{j=1}^{n}\frac{a_j}{(x-\eta_j)^3}, -\sum_{j=1}^{n}\left(\frac{2\beta_j a_j}{(x-\eta_j)^3} + \frac{b_j}{(x-\eta_j)^2}\right)\right). \tag{17}$$

Here $a_j = \frac{d\eta_j}{dy}, b_j = \frac{d\beta_j}{dy}$ lie in the tangent space of M. Once this is proved, we then deduce that X_k induces a flow on the locus.

We use the recursive formula of X_i defined through the operator $\mathcal{K}$. Let us set

$$m_j(x) := \frac{1}{(x-\eta_j)^2}, n_j(x) = \frac{1}{x-\eta_j}.$$

We also use $\Sigma_j = \sum_{j=1}^{n}$. Under these notations, we have

$$q = -\frac{3}{2}\Sigma_j m_j, p = \frac{1}{2}\Sigma_j(\beta_j m_j).$$

For this vector, we have

$$\begin{aligned}\nabla H_i &= \begin{bmatrix} 0 & D^{-1} \\ D^{-1} & 0\end{bmatrix}\left(6\Sigma_j\frac{a_j}{(x-\eta_j)^3}, -\Sigma_j\left(\frac{2\beta_j a_j}{(x-\eta_j)^3} + \frac{b_j}{(x-\eta_j)^2}\right)\right)^T \\ &= \left(\left(\Sigma_j(\beta_j a_j m_j + b_j n_j)\right), -3\Sigma_j(a_j m_j)\right).\end{aligned}$$

Let us denote the first component of $\mathcal{K}X_k$ by $(\mathcal{K}X_k)^{(1)}$. Then assuming $\mathcal{K}X_{k-1}$ has the form (17), we find that $(\mathcal{K}X_k)^{(1)}$ equals

$$\begin{aligned}&\left(D^3 + \left(q-\frac{1}{8}\right)D + D\left(q-\frac{1}{8}\right)\right)\left(\Sigma_j(\beta_j a_j m_j + b_j n_j)\right) \\ &+ \left(3(p+\alpha)D + 2p'\right)\left(-3\Sigma_j(a_j m_j)\right).\end{aligned}$$

The points $\eta_j, j = 1, ...,$ are possible poles. To analyze this function, we would like to expand it around each pole η_j.

Let us fix an index j. The coefficient before $\frac{1}{(x-\eta_j)^5}$ is

$$\begin{aligned}&\left(-24 + 2\left(-\frac{3}{2}\right)(-2) + \left(-\frac{3}{2}\right)(-2)\right)\beta_j a_j \\ &+ (-3)\left(3\left(\frac{1}{2}\right)(-2) + 2\left(\frac{1}{2}\right)(-2)\right)\beta_j a_j \\ &= 0.\end{aligned}$$

Therefore, $(\mathcal{K}\nabla H_k)^{(1)}$ does not have pole of order 5.

Next we consider the term $\frac{1}{(x-\eta_j)^4}$. We see that it only comes from

$$\left(D^3 + qD + Dq\right)(b_j n_j).$$

The coefficient vanishes, due to the fact that

$$b_j\left((-1)(-2)(-3) + \left(-\frac{3}{2}\right)(-1) + \left(-\frac{3}{2}\right)(-3)\right) = 0.$$

For the term $\frac{1}{(x-\eta_j)^3}$, we have

$$\left(D^3+\left(q-\frac{1}{8}\right)D+D\left(q-\frac{1}{8}\right)\right)\left(\sum_{j=1}^{N}\left(\beta_j a_j m_j+b_j n_j\right)\right)$$
$$+\left(3\left(p+\alpha\right)D+2p'\right)\left(-3\sum_{j=1}^{N}\left(a_j m_j\right)\right).$$

It comes from

$$\left(2q-\frac{1}{4}\right)\sum_{j=1}^{N}\left(\beta_j a_j m_j'\right)+q'\sum_{j=1}^{N}\left(\beta_j a_j m_j\right)$$
$$+q'\sum_{j=1}^{N}\left(b_j n_j\right)-9\left(p+\alpha\right)\sum_{j=1}^{N}\left(a_j m_j'\right)-6p'\sum_{j=1}^{N}\left(a_j m_j\right).$$

The coefficient I_3 equals

$$2\left(-\frac{3}{2}\right)\Sigma_k'\left(\left(-2\right)\beta_j a_j m_k\left(x_j\right)\right)-\frac{1}{4}\left(-2\right)\beta_j a_j$$
$$+\left(-\frac{3}{2}\right)\left(-2\right)\Sigma_k'\left(\beta_k a_k m_k\left(x_j\right)+b_k n_k\left(x_j\right)\right)$$
$$-9\left(\frac{1}{2}\right)\Sigma_k'\left(-2\beta_k a_j m_k\left(x_j\right)\right)-9\alpha\left(-2\right)a_j-6\left(\frac{1}{2}\right)\left(-2\right)\Sigma_k'\left(\beta_j a_k m_k\left(x_j\right)\right).$$

That is

$$I_3=6\Sigma_k'\left(\beta_j a_j m_k\left(x_j\right)\right)+\frac{1}{2}\beta_j a_j$$
$$+3\Sigma_k'\beta_k a_k m_k\left(x_j\right)+3\Sigma_k'\left(b_k n_k\left(x_j\right)\right)$$
$$+9\Sigma_k'\left(\beta_k a_j m_k\left(x_j\right)\right)+18\alpha a_j+6\Sigma_k'\left(\beta_j a_k m_k\left(x_j\right)\right).$$

For the $\frac{1}{(x-\eta_j)^2}$ term, it is related to

$$2q\sum_{j=1}^{N}\left(\beta_j a_j m_j'+b_j n_j'\right)+q'\sum_{j=1}^{N}\left(\beta_j a_j m_j+b_j n_j\right)-\frac{1}{4}\sum_{j=1}^{N}\left(b_j n_j'\right)$$
$$+3p\sum_{j=1}^{N}\left(-3a_j m_j'\right)+2p'\sum_{j=1}^{N}\left(-3a_j m_j\right).$$

Let us use Σ_k' to denote the summation over the index k which is not equal to j. Using the formula of q and p, we can compute its coefficient I_2 :

$$
\begin{aligned}
&2\left(-\frac{3}{2}\right)\left[(-2)\,\Sigma_k'\left[\beta_j a_j m_k'\left(x_j\right)\right]+\Sigma_k'\left[\beta_k a_k m_k'\left(x_j\right)\right]\right] \\
&+\left(-\frac{3}{2}\right)\left[\Sigma_k'\left[\beta_j a_j m_k'\left(x_j\right)\right]+(-2)\,\Sigma_k\left[\beta_k a_k m_k'\left(x_j\right)\right]\right] \\
&+2\left(-\frac{3}{2}\right)(-1)\,b_j\Sigma_k'\left(m_k\left(x_j\right)\right)+2\left(-\frac{3}{2}\right)\Sigma_k'\left[b_k n_k'\left(x_j\right)\right]+\left(-\frac{3}{2}\right)(-2)\,\Sigma_k'\left[b_k n_k'\left(x_j\right)\right]-\frac{1}{4}(-1)\,b_j \\
&-9\left(\frac{1}{2}\right)\left[(-2)\,\Sigma_k'\left[\beta_k a_j m_k'\left(x_j\right)\right]+\Sigma_k'\left[\beta_j a_k m_k'\left(x_j\right)\right]\right] \\
&-6\left(\frac{1}{2}\right)\left[\Sigma_k'\left[\beta_k a_j m_k'\left(x_j\right)\right]+(-2)\,\Sigma_k'\left[\beta_j a_k m_k'\left(x_j\right)\right]\right].
\end{aligned}
$$

Then I_2 equals

$$
\begin{aligned}
&\frac{9}{2}\Sigma_k'\left[\beta_j a_j m_k'\left(x_j\right)\right]+6\Sigma_k'\left[\beta_k a_j m_k'\left(x_j\right)\right]+\frac{3}{2}\Sigma_k'\left[\beta_j a_k m_k'\left(x_j\right)\right] \\
&+3b_j\Sigma_k'\left(m_k\left(x_j\right)\right)+\frac{1}{4}b_j.
\end{aligned}
$$

Note that

$$
\begin{aligned}
12\Sigma_k'\left(m_k\left(x_j\right)\right)+1&=\frac{1}{3}\beta_j^2, \\
\Sigma_k'\left[\beta_k m_k'\left(x_j\right)\right]&=-\Sigma_k'\left[\beta_j m_k'\left(x_j\right)\right].
\end{aligned}
$$

It follows that

$$
I_2=\frac{3}{2}\Sigma_k'\left[\beta_j\left(a_k-a_j\right)m_k'\left(x_j\right)\right]+\frac{1}{12}b_j\beta_j^2.
$$

Since $(a_1,...,a_N,b_1,...,b_N)$ is in the tangent space of M, we obtain $I_2=0$.

Next, we compute the $\frac{1}{(x-\eta_j)}$ term. It comes from

$$
2q\sum_{j=1}^{N}\left(\beta_j a_j m_j'+b_j n_j'\right)+q'\sum_{j=1}^{N}\left(\beta_j a_j m_j+b_j n_j\right)-9p\sum_{j=1}^{N}\left(a_j m_j'\right)-6p'\sum_{j=1}^{N}\left(a_j m_j\right).
$$

The corresponding coefficient I_1 is

$$
\begin{aligned}
&2\left(-\frac{3}{2}\right)\Sigma_k'\left(\beta_k a_k m_k''\left(x_j\right)+b_k n_k''\left(x_j\right)\right)+2\left(-\frac{3}{2}\right)\Sigma_k'\left((-2)\,\frac{1}{2}\beta_j a_j m_k''\left(x_j\right)\right)+2\left(-\frac{3}{2}\right)\Sigma_k'\left((-1)\,b_j m_k'\left(x_j\right)\right) \\
&+\left(-\frac{3}{2}\right)(-2)\,\Sigma_k'\left(\frac{1}{2}\beta_k a_k m_k''\left(x_j\right)+\frac{1}{2}b_k n_k''\left(x_j\right)\right)+\left(-\frac{3}{2}\right)\Sigma_k'\left(\beta_j a_j m_k''\left(x_j\right)\right)+\left(-\frac{3}{2}\right)\Sigma_k'\left(b_j m_k'\right) \\
&-9\left(\frac{1}{2}\right)\left(\Sigma_k'\left(\beta_j a_k m_k''\left(x_j\right)\right)+\Sigma_k'\left((-2)\,\frac{1}{2}\beta_k a_j m_k''\right)\right) \\
&-6\left(\frac{1}{2}\right)\left((-2)\,\Sigma_k'\left(\frac{1}{2}\beta_j a_k m_k''\left(x_j\right)\right)+\Sigma_k'\beta_k a_j m_k''\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
I_1=&\frac{3}{2}\Sigma_k'\left(\beta_j a_j m_k''\left(x_j\right)\right)+\frac{3}{2}\Sigma_k'\left(\beta_k a_j m_k''\left(x_j\right)\right)-\frac{3}{2}\Sigma_k'\left(\beta_j a_k m_k''\left(x_j\right)\right)-\frac{3}{2}\Sigma_k'\left(\beta_k a_k m_k''\left(x_j\right)\right) \\
&+\frac{3}{2}\Sigma_k'\left(b_j m_k'\left(x_j\right)\right)+\frac{3}{2}\Sigma_k'\left(b_k m_k'\left(x_j\right)\right).
\end{aligned}
$$

Note that on M, we have, for each fixed index j,

$$\Sigma_k' \left(\left(\beta_k + \beta_j \right) m_k' \left(x_j \right) \right) = 0.$$

As a consequence,

$$\Sigma_k' \left(\left(b_k + b_j \right) m_k' \left(x_j \right) \right) + \Sigma_k' \left(\left(\beta_k + \beta_j \right) \left(a_j - a_k \right) m_k'' \left(x_j \right) \right) = 0$$

This implies that $I_1 = 0$.

Now we consider the second component $\left(\mathcal{K} \nabla H_j \right)^{(2)}$ of the vector field $\mathcal{K} \nabla H_j$. We have

$$\left(\mathcal{K} \nabla H_j \right)^{(2)} = -L \Sigma_j \left(a_j m_j \right) + \left(3 \left(p - \alpha \right) D + p' \right) \Sigma_j \left(\beta_j a_j m_j + b_j n_j \right).$$

Similar(but more tedious, the most complicated term is $16qDq$) computation as above shows that the term $\frac{1}{(x-\eta_j)^k}$ vanishes for $k = 1, 4, 5, 6, 7$. Let us now compute the coefficient J_3 of $\frac{1}{(x-\eta_j)^3}$. Recall that equals

$$L = L_0 - \frac{5}{4} D^3 - 2 \left(qD + Dq \right) + \frac{1}{4} D$$

$$L_0 := D^5 + 5 \left(q D^3 + D^3 q \right) - 3 \left(q'' D + D q'' \right) + 16 q D q,$$

Observe that $D^3 \left(q \Sigma_j \left(a_j m_j \right) \right)$ does not contain $\frac{1}{(x-\eta_j)^3}$ term. Hence from the operator L_0, the contribution to the coefficient is:

$$\begin{aligned} & 5 \left(-\frac{3}{2} \right) (-24) \, \Sigma_k' \left(\frac{1}{2} a_j m_k'' \left(x_j \right) \right) \\ & - 3 \left(-\frac{3}{2} \right) \left(2 \, (6) \, \Sigma_k' a_k m_k'' \left(x_j \right) + 2 \, (-2) \, \Sigma_k' \left(a_j m_k'' \left(x_j \right) \right) + (-24) \, \Sigma_k' \left(\frac{1}{2} a_k m_k'' \left(x_j \right) \right) \right) \\ & + 16 \left(\frac{9}{4} \right) \left(\Sigma_k' \left((-2) \, a_j m_k'' \left(x_j \right) \right) + \Sigma_k' \left(m_k \left(x_j \right) \right) (-2) \, \Sigma_k' \left(\left(a_k + a_j \right) m_k \left(x_j \right) \right) \right). \end{aligned}$$

From the operator $-\frac{5}{4} D^3 - 2 \left(qD + Dq \right) + \frac{1}{4} D$, we get

$$-2 \left(-\frac{3}{2} \right) \left(2 \, (-2) \, \Sigma_k' \left(a_j m_k \left(x_j \right) \right) + (-2) \, \Sigma_k' \left(a_k m_k \left(x_j \right) \right) \right) + \frac{1}{4} \left(-2 a_j \right)$$

Finally, from

$$\left(3 \left(p + \alpha \right) D + p' \right) \Sigma_j \left(\beta_j a_j m_j + b_j n_j \right),$$

we obtain

$$\begin{aligned} & 3 \left(\frac{1}{2} \right) \Sigma_k' \left((-2) \, \beta_k \beta_j a_j m_k \left(x_j \right) \right) + 3 \alpha \, (-2) \, \beta_j a_j \\ & + \left(\frac{1}{2} \right) (-2) \, \Sigma_k' \left(\beta_j \beta_k a_k m_k \left(x_j \right) \right) + \frac{1}{2} \, (-2) \, \Sigma_k' \beta_j b_k n_k \left(x_j \right). \end{aligned}$$

Combining these, we obtain

$$\begin{aligned} J_3 = {} & 72 \Sigma_k' \left(m_k \left(x_j \right) \right) \Sigma_k' \left(\left(a_k + a_j \right) m_k \left(x_j \right) \right) \\ & + 12 \Sigma_k' \left(a_j m_k \left(x_j \right) \right) + 6 \Sigma_k' \left(a_k m_k \left(x_j \right) \right) + \frac{1}{2} a_j \\ & - 3 \Sigma_k' \left(\beta_k \beta_j a_j m_k \left(x_j \right) \right) - \Sigma_k' \left(\beta_j \beta_k a_k m_k \left(x_j \right) \right) \\ & - \Sigma_k' \left(\beta_j b_k n_k \left(x_j \right) \right) - 6 \alpha \beta_j a_j. \end{aligned}$$

Now using the identity

$$\beta_j^2 + 36 \Sigma_k' m_k \left(x_j \right) + 3 = 0,$$

we then see that $J_3 = \beta_j I_3$.

The coefficient of $\frac{1}{(x-\eta_j)^2}$ is also nonzero, and can computed in a similar way. However, to prove the proposition, it is not necessary to know its explicit formula.

In the sequel, for $j = 1, ..., n$, let us use I_j to denote the coefficient of degree -3 term for the pole η_j. We have now proved that $\mathcal{K}\nabla H_k$ has the form

$$\left(6\sum_{j=1}^{n}\frac{I_j}{(x-\eta_j)^3}, -\sum_{j=1}^{n}\left(\frac{2\beta_j I_j}{(x-\eta_j)^3}+\frac{B_j}{(x-\eta_j)^2}\right)\right) = \mathcal{D}\nabla H_{k+1},$$

Our next aim is to show that the vector

$$(I_1, ..., I_n, B_1, ..., B_n)$$

lies in the tangent space of M at the point $(\eta_1, ..., \eta_n, \beta_1, ..., \beta_n)$. To see this, it will be suffice to show that $\mathcal{K}\nabla H_{k+1}$ is residue free at each pole.

Let us write the operator $\mathcal{K}$ as

$$\mathcal{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}.$$

We also write $\nabla H_{k+1} = (\phi_1, \phi_2)^T$. That is,

$$(\phi_1, \phi_2) = (\Sigma_j (I_j \beta_j m_j + B_j n_j), -3\Sigma_j (I_j m_j)).$$

Introducing

$$\begin{aligned} \sigma &= \Sigma_j (a_j \beta_j m_j + b_j n_j), \\ \tau &= -3\Sigma_j (a_j m_j), \end{aligned}$$

we get

$$\begin{aligned} \phi_1' &= K_{21}\sigma + K_{22}\tau, \\ \phi_2' &= K_{11}\sigma + K_{12}\tau. \end{aligned} \tag{18}$$

Let l be a closed path around the pole η_j in the complex x plane. To see that the residue is zero(that is, does not have $\frac{1}{x-\eta_j}$ term in the Laurent expansion around η_j), we compute the integral

$$\begin{aligned} Q &:= \int_l (\mathcal{K}\nabla H_{k+1})^T dx \\ &= \int_l [K_{11}\phi_1 + K_{12}\phi_2, K_{21}\phi_1 + K_{22}\phi_2] dx. \end{aligned}$$

It is important to observe that each operator K_{11}, K_{22} is skew-symmetric, and moreover the adjoint of K_{12} is $-K_{21}$, that is,

$$\int (gK_{12}h) = -\int (hK_{21}g).$$

This is to say that the matrix operator $\mathcal{K}$ is skew-symmetric. Integrating by parts tells us that Q equals

$$-\int_l [\phi_1 K_{11}(1) + \phi_2 K_{21}(1), \phi_1 K_{12}(1) + \phi_2 K_{22}(1)] dx.$$

Let us define

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} := \mathcal{K}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} := \mathcal{K}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then for some functions w, s, we have

$$\mu = \mathcal{D}(w_1, w_2)^T \text{ and } v = \mathcal{D}(s_1, s_2)^T .$$

Explicitly,

$$\mu = (q', p')^T ,\ v = \left(2p', \frac{1}{3}(2q''' + 16qq' - 2q')\right)^T .$$

With these notations,

$$\begin{aligned} Q &= -\int_l [\phi_1\mu_1 + \phi_2\mu_2, \phi_1 v_1 + \phi_2 v_2]\, dx \\ &= -\int_l [\phi_1 w_2' + \phi_2 w_1', \phi_1 s_2' + \phi_2 s_1']\, dx \\ &= \int_l [\phi_1' w_2 + \phi_2' w_1, \phi_1' s_2 + \phi_2' s_1]\, dx. \end{aligned}$$

Using (18), we find that Q is equal to

$$\begin{aligned} &\int_l [(K_{21}\sigma + K_{22}\tau) w_2 + (K_{11}\sigma + K_{12}\tau) w_1, (K_{21}\sigma + K_{22}\tau) s_2 + (K_{11}\sigma + K_{12}\tau) s_1]\, dx \\ &= -\int_l [(K_{11}w_1 + K_{12}w_2)\sigma + (K_{21}w_1 + K_{22}w_2)\tau, (K_{11}s_1 + K_{12}s_2)\sigma + (K_{21}s_1 + K_{22}s_2)\tau]\, dx. \end{aligned}$$

Direct computation shows that

$$\begin{aligned} \mathcal{K}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= \left(-3\Sigma_j \left(s_j m_j'\right), \Sigma_j \left(s_j\beta_j m_j' + t_j n_j'\right)\right)^T , \\ \mathcal{K}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} &= \left(-3\Sigma_j \left(\bar{s}_j m_j'\right), \Sigma_j \left(\bar{s}_j\beta_j m_j' + \bar{t}_j n_j'\right)\right)^T , \end{aligned}$$

where $(s_1, ..., s_n, t_1, ..., t_n)$ and $(\bar{s}_1, ..., \bar{s}_n, \bar{t}_1, ..., \bar{t}_n)$ are in the tangent space of M. Then

$$\begin{aligned} &\int_l ((K_{11}w_1 + K_{12}w_2)\sigma + (K_{21}w_1 + K_{22}w_2)\tau)\, dx \\ &= -3\int_l \left[\Sigma_k (s_k m_k')\, \Sigma_j (a_j\beta_j m_j + b_j n_j) + \Sigma_j \left(s_j\beta_j m_j' + t_j n_j'\right) \Sigma_k (a_k m_k)\right] dx. \end{aligned}$$

Residue computation shows that this integral is zero. This finishes the proof. □

Next we show that the k the flow is trivial, if k is large.

Lemma 6. *Let n be fixed. Then for k large, at $y = 0$, $X_k^{(1)} = 0$.*

Proof. By our choice of the parameter a, if the index k is an odd number, then the main order term of X_k is $O\left(\frac{1}{x^{k+2}}\right)$. We define

$$\pi_k = \sum_{j=0}^{n} \eta_j^k, \Pi_k = \sum_{j=0}^{n} \left(\beta_j \eta_j^k\right).$$

Since

$$\frac{1}{(1-t)^2} = (k+1)\, t^k,$$

we can write

$$q = -\frac{3}{2}\sum_{k=0}^{\infty}\left(x^{-k-2}\left(\sum_{j}\eta_j^k\right)\right),$$

$$p = \frac{1}{2}\sum_{k=0}^{\infty}\left(x^{-k-2}\left(\sum_{j}\beta_j\eta_j^k\right)\right).$$

It follows that

$$X_k(q,p) = \sum_{j=0}^{\infty}\left(-\frac{3(j+1)X_k^{(1)}\pi_j}{2x^{j+2}}, \frac{(j+1)X_k^{(2)}\Pi_j}{2x^{j+2}}\right).$$

Since the main order of X_k is x^{-k-2}, we see that if $j<k$, then

$$X_k^{(1)}\pi_j = 0 \text{ and } X_k^{(2)}\Pi_j = 0.$$

However, if $k_0 \geq n$, then $\pi_1, ..., \pi_{k_0}$ form a basis of the locus. From this we deduce that if

$$n < k,$$

then the first component of the flow X_k is trivial. □

Lemma 7. *Suppose η is a complex-valued homogeneous polynomial in x, y of degree m and*

$$\left(D_x^2 + D_y^2\right)\eta\cdot\eta = 0.$$

Then

$$\eta(x,y) = a\left(x^2+y^2\right)^j (x+yi)^k,$$

where a is a constant and $2j+k=m$. In particular, if η is real-valued, then $\eta = a\left(x^2+y^2\right)^m$ for some real number a.

Proof. In the polar coordinate (r,θ), where $r=\sqrt{x^2+y^2}$, we can write $\eta = r^m g(\theta)$. Then

$$\begin{aligned}\left(D_x^2+D_y^2\right)\eta\cdot\eta &= 2\left(\eta\Delta\eta - |\nabla\eta|^2\right)\\ &= 2r^m g\left(m^2 r^{m-2} g + r^{m-2} g''\right) - 2\left(m^2 r^{2m-2} g^2 + r^{2m-2} g'^2\right).\end{aligned}$$

From this we obtain

$$gg'' - g'^2 = 0,$$

which implies $g(\theta) = ae^{b\theta}$ for some constants a and b. Since g has to be 2π-periodic in θ, we have $b = ki$ for some integer k. It follows that

$$\eta = ar^m\left(e^{i\theta}\right)^k = ar^{m-k}(x+yi)^k.$$

Setting $j=\frac{m-k}{2}$, we arrive at the desired result. □

Let τ be a polynomial solution of the bilinear equation

$$\left(D_x^4 - D_x^2 - D_y^2\right)\tau\cdot\tau = 0, \tag{19}$$

with $\deg(\tau) = m$. By Lemma 7, we can assume without loss of generality that the highest degree terms of τ are of the form

$$\left(x^2+y^2\right)^j (x+yi)^k = z^{j+k}\bar{z}^j := \tau_m,$$

where $z=x+yi$ and $\bar{z}=x-yi$ are complex variables. Let us denote those terms of τ with degree $m-1$ by τ_{m-1}. The previous lemma can also be proved using the $(z,\bar{z})$ coordinate. Indeed, we have the following

Lemma 8. *$\tau_{m-1}=a_1 z^{j+k-1}\bar{z}^j+a_2 z^{j+k}\bar{z}^{j-1}$ for some constants a_1,a_2. In particular, if τ is real-valued, then*

$$\tau_{m-1}=az^{j-1}\bar{z}^j+\bar{a}z^j\bar{z}^{j-1}.$$

Proof. The terms of degree $2m-3$ in the left hand side of (19) are of the form

$$-\left(D_x^2+D_y^2\right)\tau_m\cdot\tau_{m-1}.$$

Suppose $z^r\bar{z}^s$ is a term appearing in τ_{m-1}, then there holds

$$D_z D_{\bar{z}}\left(z^{j+k}\bar{z}^j\right)\cdot\left(z^r\bar{z}^s\right)=0.$$

Direct computation tells us that

$$\left[(j+k)\,j-(j+k)\,s-jr+rs\right]z^{j+k+r-1}\bar{z}^{j+s-1}=0.$$

In the case of $k=0$, we have

$$j^2-js-jr+rs=0.$$

That is, $r=j$ or $s=j$.

If the solution is real valued, then the degree $j+s$ term has to be

$$\begin{aligned}g\,(x,y)&:=cz^j\bar{z}^s+\bar{c}z^s\bar{z}^j\\&=(a+bi)\,(x+yi)^j\,(x-yi)^s+(a-bi)\,(x+yi)^s\,(x-yi)^j\,.\end{aligned}$$

Using $r+s=m-1$ and $2j+k=m$, we obtain

$$(j+k)\,j-(j+k)\,s-j\,(m-1-s)+(m-1-s)\,s=0.$$

That is,

$$-s^2+(2j-1)\,s-j\,(j-1)=0.$$

Therefore, $s=j$ or $j-1$.

If in addition τ is real-valued, then $k=0$ and $\tau_m=z^j\bar{z}^j$. Hence

$$\tau_{m-1}=az^{j-1}\bar{z}^j+\bar{a}z^j\bar{z}^{j-1}.$$

□

By this lemma, in the real-valued case, if we introduce new variables $Z=z+\frac{a}{j}$ and $\bar{Z}=\bar{z}+\frac{\bar{a}}{j}$, then we see that

$$z^j\bar{z}^j+az^{j-1}\bar{z}^j+\bar{a}z^j\bar{z}^{j-1}=Z^j\bar{Z}^j+P,$$

where P is a polynomial of $Z,\bar{Z}$ with degree less than $j-1$. This means that we can find real numbers b_1,b_2 such that in the new variables $\tilde{x}=x+b_1,\tilde{y}=y+b_2$, the highest degree term of τ is $\left(\tilde{x}^2+\tilde{y}^2\right)^j$ and τ does not have terms with degree $2j-1$.

Lemma 9. *Suppose $q=\frac{3}{2}\partial_x^2\ln\tau$ is a real valued rational solution of the Boussinesq equation* (13)*, where τ is a polynomial of degree $2n$. Let $p=\int_{-\infty}^x\partial_y q\,dx$. Then for x large, at $y=0$,*

$$q=-\frac{3n}{x^2}+O\left(x^{-3}\right),p=O\left(\frac{1}{x^5}\right).$$

Proof. Since τ is real valued, after a possible translation of the coordinate(and a scaling of the y variable), it has the form

$$\tau(x,y) = \left(x^2+y^2\right)^n + \sum_{j+k\leq 2n-2} \left(a_{jk}x^j y^k\right).$$

Therefore, $q = \frac{3n}{x^2} + O\left(x^{-3}\right)$. We also have

$$q = \frac{3n}{2}\partial_x^2 \ln\left(x^2+y^2\right) + 2\partial_x^2 \ln\left(\frac{1}{\left(x^2+y^2\right)^n}\sum_{j+k\leq 2n-2}\left(a_{jk}x^j y^k\right)\right).$$

Hence

$$\partial_y q = 2n\left(\frac{i}{(x+yi)^3} - \frac{i}{(x-yi)^3}\right) + O\left(x^{-5}\right).$$

The result then readily follows. □

Now we are at a position to prove the following

Theorem 10. *Assume that $\tilde{q}$ is a real valued rational solution of the Boussinesq equation* (13) *with*

$$\tilde{q}(x,y) \to 0, \text{ as } x^2+y^2 \to +\infty.$$

Then $\tilde{q} = \frac{3}{2}\partial_x^2 \ln\tau$, where τ is a polynomial in x, y with degree $k(k+1)$, $k \in \mathbb{N}$.

Proof. Since $(3a)^2 + \frac{1}{48} = 0$, we compute

$$\begin{bmatrix} -\frac{D}{4} & -3aD \\ -3aD & \frac{D}{12}\end{bmatrix}\begin{bmatrix} 0 & D^{-1} \\ D^{-1} & 0\end{bmatrix}\begin{bmatrix} -\frac{D}{4} & 3aD \\ 3aD & \frac{D}{12}\end{bmatrix} = 0.$$

This identity guarantees that if the main order term of the X_j is x^{-k}, then the main order term of X_{j+2} will be at the order x^{-k-2}.

For x large, the main order term of $\tilde{q}$ is $\frac{m}{x^2}$. Since the degree of the polynomial is expected to be $k(k+1)$, we expect m to be $-\frac{3}{2}k(k+1)$.

We compute

$$\begin{aligned}
&\left(D^3+qD+Dq\right)\left(\frac{1}{x^j}\right)\\
&= \left[-j(j+1)(j+2) - m(j+j+2)\right]\frac{1}{x^{j+3}}\\
&= -(j+1)(j(j+2)+2m)\frac{1}{x^{j+3}} := b_m(j)\frac{1}{x^{j+3}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left(-\frac{5}{4}D^3 - 2(qD+Dq)\right)\left(\frac{1}{x^j}\right)\\
&= \frac{5}{4}j(j+1)(j+2) + 2m(j+j+2)\frac{1}{x^{j+3}}\\
&= \frac{1}{8}(j+1)(10j(j+2)+32m)\frac{1}{x^{j+3}} =: B_m(j)\frac{1}{x^{j+3}}.
\end{aligned}$$

Vanishing of terms require

$$\frac{1}{4}b_m(j) - \frac{1}{4}B_m(j) = 0.$$

That is,

$$m = -\frac{3}{8} j (j+2).$$

Let $j = 2k$, we find that

$$m = -\frac{3}{2} k (k+1).$$

This completes the proof. □

5. Even solutions from numerical point of view

In this section, we would like to carry out some numerical computation to verify that the degree of the corresponding τ function of the even solution is $k(k+1)$.

Suppose q is an even solution, that is, $q(x,y) = q(x,-y) = q(-x,y)$. We assume that the degree of its tau function f is $2n$. From Lemma 7, we can assume that the sum of the degree $2n$ terms of f is $T_n = \left(x^2+y^2\right)^n$. We also denote the sum of the degree $2n-2j$ terms of f by $T_{n,j}$.

Let us define functions

$$g_j := \left(x^2+y^2\right)^{n-3j} x^{2j} y^{2j}, \quad \xi_{i,j} := \left(D_x^2 + D_y^2\right) g_i \cdot g_j.$$

Observe that actually $\xi_{i,j}$ can be divided by $\left(x^2+y^2\right)^{2n-3i-3j-1}$. We introduce the constants

$$d_{i,j} := \left.\frac{\left(D_x^2+D_y^2\right) g_i \cdot g_j}{\left(x^2+y^2\right)^{2n-3i-3j-1}}\right|_{(x^2=-1,y^2=1)}.$$

Lemma 11. $d_{i,j} = -12 (i-j)^2 (-1)^{i+j}$.

Proof. We compute

$$\left(D_x^2+D_y^2\right) g_i \cdot g_j = \Delta g_i g_j + g_i \Delta g_j - 2\nabla g_i \cdot \nabla g_j.$$

Note that

$$\begin{aligned}\Delta g_j &= (2n-6j)^2 r^{2n-6j-2} x^{2j} y^{2j} + 8j(2n-6j) r^{2n-6j-2} x^{2j} y^{2j} + r^{2n-6j} \Delta\left(x^{2j}y^{2j}\right) \\ &= (2n+2j)(2n-6j) r^{2n-6j-2} x^{2j} y^{2j} + r^{2n-6j} \Delta\left(x^{2j}y^{2j}\right).\end{aligned}$$

$$\begin{aligned}&\nabla g_i \cdot \nabla g_j \\ &= \left(x^2+y^2\right)^{2n-3i-3j-2} x^{2i+2j-2} y^{2i+2j} \left((2n-6i)x^2 + 2i\left(x^2+y^2\right)\right)\left((2n-6j)x^2 + 2j\left(x^2+y^2\right)\right) \\ &+ \left(x^2+y^2\right)^{2n-3i-3j-2} x^{2i+2j} y^{2i+2j-2} \left((2n-6i)y^2 + 2i\left(x^2+y^2\right)\right)\left((2n-6j)y^2 + 2j\left(x^2+y^2\right)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\left(D_x^2+D_y^2\right) g_i \cdot g_j}{\left(x^2+y^2\right)^{2n-3i-3j-1}}\Big|_{x^2=-1,y^2=1} &= (2n+2i)(2n-6i)(-1)^{i+j} \\ &+ (2n+2j)(2n-6j)(-1)^{i+j} \\ &- 2\left((2n-6i)(2n-6j) + 2(2n-6i)2j + 2(2n-6j)2i\right)(-1)^{i+j} \\ &= -12(i-j)^2(-1)^{i+j}.\end{aligned}$$

□

Let us now consider the function $D_x^4 \left(x^2+y^2\right)^{n-3i} \cdot \left(x^2+y^2\right)^{n-3j}$. Since we have taken the fourth order derivative, this function is dividable by $\left(x^2+y^2\right)^{2n-3i-3j-4}$. We also define

$$p_{i,j} = \left.\frac{D_x^4 g_i \cdot g_j}{\left(x^2+y^2\right)^{2n-3i-3j-4}}\right|_{(x^2=-1,y^2=1)}.$$

These constants depending on n, i, j. Explicitly, $(-)^{i+j} p_{i,j}$ is equal to

$$\begin{aligned}
&1296i^4 - 5184i^3 j + 7776i^2 j^2 - 5184ij^3 + 1296j^4 + 2592i^3 - 2592i^2 j \\
&- 1728i^2 n - 2592ij^2 + 3456ijn + 2592j^3 - 1728j^2 n + 1584i^2 - 1440ij \\
&- 576in + 1584j^2 - 576jn + 192n^2 + 288i + 288j - 192n.
\end{aligned}$$

In the special case $i = j$,

$$p_{i,j} = (-1)^{i+j}\, 192\,(3j-n+1)\,(3j-n)\,.$$

When $i = j-1$, we have $p_{i,j} = (-1)^{i+j}\, 192\,(3j-n+7)\,(3j-n)$. If $i = j-2$, then

$$p_{i,j} = (-1)^{i+j}\, 192\,(3j-n+30)\,(3j-n+1)\,.$$

Moreover, if $i = 0$, then

$$p_{i,j} = 1296j^4 + 2592j^3 - 1728j^2 n + 1584j^2 - 576jn + 192n^2 + 288j - 192n.$$

Now we would like to define a sequence of numbers $a_m, m = 0, 1, ...,$ depending on n, in the following way. Take $a_0 = 1$. Then a_m is determined by $a_1, ..., a_{m-1}$ through the following recursive relation:

$$\sum_{i,j\le m, i+j=m} a_i a_j d_{i,j} = \sum_{i,j\le m, i+j=m-1} a_i a_j p_{i,j}.$$

We regard a_j as a polynomial of the variable n. Now let us define the constant

$$J_n := \sum_{i,j\le[n/3], i+j=[n/3]+1} a_i a_j d_{i,j} - \sum_{i,j\le[n/3], i+j=[n/3]} a_i a_j p_{i,j}.$$

Proposition 12. *Let n be a fixed integer. If $J_n \neq 0$, then the Boussinesq equation has no rational even solution with degree $2n$.*

Proof. First of all, we claim that $T_{n,j}$ has the form

$$a_j \left(x^2+y^2\right)^{n-3j} x^{2j} y^{2j} + \left(x^2+y^2\right)^{n-3j+1} \Gamma(x,y)\,,$$

where Γ is a homogeneous polynomial in x, y with degree $4j-2$. Indeed,

Let us denote the function $\left(D_x^4 - D_x^2 - D_y^2\right) f \cdot f$ by K_f. Since we have chosen a_0 to be 1, K_f is a polynomial of degree at most $4n-4$. The terms with degree $4n-4$ are given by

$$D_x^4 T_{n,0} \cdot T_{n,0} - \left(D_x^2 + D_y^2\right) T_{n,0} \cdot T_{n,1}.$$

This function is dividable by $\left(x^2+y^2\right)^{2n-4}$. We write is as

$$b_1 \left(x^2+y^2\right)^{2n-4} x^2 y^2 + \left(x^2+y^2\right)^{2n-3} M(x,y)\,.$$

Inserting $x^2 = -1, y^2 = 1$ into this function, we find that necessary $b_1 = 0$. Therefore, we get

$$a_0^2 p_{0,0} - a_0 a_1 d_{0,1} = 0.$$

Similarly, consider the terms with degree $4n-6$, we get

$$D_x^4 T_{n,0}\cdot T_{n,1}-\left(D_x^2+D_y^2\right)T_{n,1}\cdot T_{n,1}-\left(D_x^2+D_y^2\right)T_{n,0}\cdot T_{n,2}=0.$$

Then

$$a_0a_1p_{0,1}-a_1^2d_{1,1}-a_0a_2d_{0,2}=0.$$

Similarly, for $m\leq[n/3]$,

$$\sum_{i,j\leq m,i+j=m}a_ia_jd_{i,j}=\sum_{i,j\leq m,i+j=m-1}a_ia_jp_{i,j}.$$

Since we require that the solution is a polynomial, the function

$$\sum_{i,j\leq[n/3],i+j=[n/3]+1}a_ia_j\left(D_x^2+D_y^2\right)T_{n,i}\cdot T_{n,j}-\sum_{i,j\leq[n/3],i+j=[n/3]}a_ia_jD_x^4T_{n,i}\cdot T_{n,j}$$

should dividable by $\left(x^2+y^2\right)^{n-1}$, this implies that $J_n=0$. □

We have computed the constants a_j and J_n, using mathematical software. It turns out that at least for $n\leq 300$, J_n is equal to zero if and only if $n=\frac{k(k+1)}{2}$ for some integer k.

References

[1] M. J. Ablowitz; S. Chakravarty; A. D. Trubatch; J. Villarroel, A novel class of solutions of the non-stationary Schrödinger and the Kadomtsev-Petviashvili I equations, *Phys. Lett. A* 267 (2000), no. 2-3, 132–146.

[2] Airault, H. Solutions of the Boussinesq equation. *Phys. D* 21 (1986), no. 1, 171–176.

[3] Airault, H.; McKean, H. P.; Moser, J. Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem. *Comm. Pure Appl. Math.* 30 (1977), no. 1, 95–148.

[4] Beals, Richard; Deift, Percy; Tomei, Carlos. Direct and inverse scattering on the line. Mathematical Surveys and Monographs, 28. American Mathematical Society, Providence, RI, 1988.

[5] Beals, R.; Coifman, R. R. Scattering and inverse scattering for first order systems. *Comm. Pure Appl. Math.* 37 (1984), no. 1, 39–90.

[6] Beals, Richard; Coifman, R. R. Scattering and inverse scattering for first-order systems. II. *Inverse Problems* 3 (1987), no. 4, 577–593.

[7] F. Bethuel, P. Gravejat, J.-C. Saut, On the KP I transonic limit of two-dimensional Gross-Pitaevskii travelling waves, *Dynamics of Partial Differential Equations.* 5 (2008) 241–280. https://doi.org/201308301911.

[8] Bilman, Deniz; Miller, Peter D. A robust inverse scattering transform for the focusing nonlinear Schrödinger equation. *Comm. Pure Appl. Math.* 72 (2019), no. 8, 1722–1805.

[9] Bogdanov, L. V.; Zakharov, V. E. The Boussinesq equation revisited. *Phys. D* 165 (2002), no. 3-4, 137–162.

[10] S. Chakravarty, M. Zowada, Classification of KPI lumps, *J. Phys. A: Math. Theor.* 55 (2022) 215701. https://doi.org/10.1088/1751-8121/ac624d.

[11] S. Chakravarty, M. Zowada, Multi-lump wave patterns of KPI via integer partitions, *Physica D: Nonlinear Phenomena.* 446 (2023) 133644. https://doi.org/10.1016/j.physd.2022.133644.

[12] Clarkson, Peter A. Rational solutions of the Boussinesq equation. *Anal. Appl. (Singap.)* 6 (2008), no. 4, 349–369.

[13] P. Deift, C. Tomoi, E. Trubowitz, Inverse scattering and the Boussinesq equation, *Comm. Pure Appl. Math.* 35, 567-628, (1982).

[14] Deift, Percy; Zhou, Xin. Direct and inverse scattering on the line with arbitrary singularities. *Comm. Pure Appl. Math.* 44 (1991), no. 5, 485–533.

[15] de Bouard, Anne; Saut, Jean-Claude Symmetries and decay of the generalized Kadomtsev-Petviashvili solitary waves. *SIAM J. Math. Anal.* 28 (1997), no. 5, 10641085.

[16] Dickson, Ronnie; Gesztesy, Fritz; Unterkofler, Karl. A new approach to the Boussinesq hierarchy. *Math. Nachr.* 198 (1999), 51–108.

[17] R. Dickson, F. Gesztesy, K. Unterkofler, A New Approach to the Boussinesq Hierarchy, *Math. Nachr.* 198 (1999) 51–108. https://doi.org/10.1002/mana.19991980105.
[18] P. G. Estevez; J. Prada, Lump solutions for PDE's: Algorithmic construction and Classification, *Journal of Nonlinear Mathematical Physics*, 15:sup3, 166–175.
[19] Fokas, A. S.; Ablowitz, M. J. On the inverse scattering of the time-dependent Schrödinger equation and the associated Kadomtsev-Petviashvili equation. *Stud. Appl. Math.* 69 (1983), no. 3, 211–228.
[20] K. A. Gorshkov; D. E. Pelinovskii, Yu. A. Stepanyants, Normal and anomalous scattering, formation and decay of bound states of two-dimensional solitons described by the Kadomtsev-Petviashvili equation, *JETP* 77(2), 1993, 237–245.
[21] V. M. Galkin; D. E. Pelinovsky; Yu. A. Stepanyants, The structure of the rational solutions to the Boussinesq equation. *Phys. D* 80 (1995), no. 3, 246–255.
[22] K. Hikami; M. Wadati, On additional symmetry: The many body problem related to the KP hierarchy, *Physics Letter A* 191(1994), 87–90.
[23] M. Ito, Symmetries and conservation laws of the classical Boussinesq equation, *Physics Letters A.* 104 (1984) 248–250. https://doi.org/10.1016/0375-9601(84)90061-6.
[24] V. G. Kac; J. W. van de Leur, Equivalence of formulations of the MKP hierarchy and its polynomial tau-functions, *Japan. J. Math.* 13, 235–271(2018).
[25] Kadomtsev, B.B., Petviashvili, V.I.: On the stability of solitary waves in weakly dispersive media. *Soviet Physics-Doklady* 15, 539541, 1970
[26] C. E. Kenig, On the local and global well-posedness theory for the KP-I equation, *Ann. Inst. H. Poincare Anal. Non-Linire.* 21 (2004) 827–838. https://doi.org/10.1016/j.anihpc.2003.12.002.
[27] C. Klein, J.-C. Saut, IST versus PDE: a comparative study, Fields Inst. Commun., 75 Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2015, 383-449.
[28] C. Klein, J.-C. Saut, Nonlinear Dispersive Equations: Inverse Scattering and PDE Methods, Springer International Publishing, Cham, 2021. https://doi.org/10.1007/978-3-030-91427-1.
[29] I. M. Krichever, Rational solutions of the Kadomtsev–Petviashvili equation and integrable systems of N particles on a line, *Functional Analysis and Its Applications* 1978, 12(1), 59–61.
[30] I. M. Krichever, Rational solutions of the Zakharov-Shabat equations and completely integrable systems of N particles on a line, *Journal of Soviet Mathematics* 21, 1983, 335–345.
[31] Q. M Liu; X. B. Hu; Y. Li, Rational solutions of the classical Boussinesq hierarchy, *J. Phy. A: Math. Gen.* 23 (1990), 585–591.
[32] Y. Liu, J. Wei, Nondegeneracy, Morse Index and Orbital Stability of the KP-I Lump Solution, *Arch Rational Mech Anal.* 234 (2019) 13351389. https://doi.org/10.1007/s00205-019-01413-5.
[33] Yong Liu, Zhengping Wang, Juncheng Wei, Wen Yang, From KP-I lump solution to travelling waves of Gross-Pitaevskii equation, arXiv:2110.15472 .
[34] Y. Liu, J. Wei, Classification of finite Morse index solutions to the elliptic sine-Gordon equation in the plane, *Rev. Mat. Iberoam.* 38 (2021) 355–432. https://doi.org/10.4171/RMI/1296.
[35] McKean, H. P., Boussinesq's equation as a Hamiltonian system. Topics in functional analysis (essays dedicated to M. G. Krein on the occasion of his 70th birthday), pp. 217–226, Adv. Math. Suppl. Stud., 3, Academic Press, New York-London, 1978.
[36] S. V. Manakov, The inverse scattering transform for the time-dependent Schrodinger equation and Kadomtsev-Petviashvili equation, *Physica D: Nonlinear Phenomena* 3(1981), 420–427.
[37] Manakov, S.V., Zakharov, V.E., Bordag, L.A., Its, A.R., Matveev, V.B.: Two dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction. *Physics letter A* 63, 205-206, 1977.
[38] L. Molinet, J.-C. Saut, N. Tzvetkov, Well-posedness and ill-posedness results for the Kadomtsev-Petviashvili-I equation, *Duke Math. J.* 115 (2002). https://doi.org/10.1215/S0012-7094-02-11525-7.
[39] L. Molinet, J.C. Saut, N. Tzvetkov, Global Well-Posedness for the KP-I Equation on the Background of a Non-Localized Solution, *Commun. Math. Phys.* 272 (2007) 775–810. https://doi.org/10.1007/s00220-007-0243-1.
[40] D. E. Pelinovskii; Yu. A. Stepanyants, New multisoliton solutions of the Kadomtsev-Petviashvili equation, *JETP Lett.* 57(1), 1993, 24–28.
[41] D. E. Pelinovsky, Rational solutions of the Kadomtsev-Petviashvili hierarchy and the dynamics of their poles. I. New form of a general rational solution. *J. Math. Phys.* 35 (1994), no. 11, 5820–5830.

[42] D. E. Pelinovsky, Rational solutions of the KP hierarchy and the dynamics of their poles. II. Construction of the degenerate polynomial solutions. *J. Math. Phys.* 39 (1998), no. 10, 5377–5395.
[43] Satsuma,J., Ablowitz, M.J.: Two-dimensional lumps in nonlinear dispersive systems. *J. Math. Phys.* 20(7), 1496-1503, 1979.
[44] Segur, Harvey. Comments on inverse scattering for the Kadomtsev-Petviashvili equation. Mathematical methods in hydrodynamics and integrability in dynamical systems (La Jolla, Calif., 1981), 211–228, AIP Conf. Proc., 88, Amer. Inst. Physics, New York, 1982.
[45] T. Shiota, Calogero-Moser hierarchy and KP hierarchy, *Journal of Mathematical Physics* 35, 5844 (1994).
[46] Villarroel, Javier; Ablowitz, Mark J. On the discrete spectrum of the nonstationary Schrodinger equation and multipole lumps of the Kadomtsev-Petviashvili I equation. *Comm. Math. Phys.* 207 (1999), no. 1, 1–42.
[47] X. Zhou, Inverse scattering transform for the time dependent Schrodinger equation with applications to the KPI equation, *Comm. Math. Phys.* 128 (1990), no. 3, 551–564.
[48] X. Zhou, Direct and inverse scattering transforms with arbitrary spectral singularities. *Comm. Pure Appl. Math.* 42 (1989), no. 7, 895–938.
[49] X. Zhou, The Riemann–Hilbert Problem and Inverse Scattering, *SIAM J. Math. Anal.* 20 (1989) 966–986. https://doi.org/10/d5s48h.
[50] B. Yang, J. Yang, General Rogue Waves in the Boussinesq Equation, *J. Phys. Soc. Jpn.* 89 (2020) 024003. https://doi.org/10.7566/JPSJ.89.024003.
[51] V.E. Zakharov, On stochastization of one-dimensional chains of nonlinear oscillators, *Soviet Physics JETP* Vol. 38, p.108, 1974.

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