THE NUMBER OF POSITIVE SOLUTIONS FOR n-COUPLED ELLIPTIC SYSTEMS

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ABSTRACT. We study the number of positive solutions to the n-coupled elliptic system

$$-\Delta u_i = \mu_i u_i^{2^*-1} + \sum_{j=1,\, j\neq i}^{n} \beta_{ij} u_i^{p_{ij}-1} u_j^{q_{ij}},\ u_i \in \mathscr{D}^{1,2}(\mathbb{R}^N),\ i = 1, 2, \cdots, n,$$

where $N \geq 3$, $n \geq 2$, $\mu_i > 0$, $\beta_{ij} > 0$, $p_{ij} < 2^*$, and $p_{ij} + q_{ij} = 2^*$ for $i \neq j \in \{1, 2, \cdots, n\}$. We prove new multiplicity and uniqueness results for positive solutions of the system, whether the system has a variational structure or not. In some cases we provide a rather complete characterization on the exact number of positive solutions. The results we obtain reveal that the positive solution set of this system has very different structures in the three cases $p_{ij} < 2$, $p_{ij} = 2$, and $2 < p_{ij} < 2^*$. Moreover, when $2 < p_{ij} < 2^*$, very different structures of the positive solution set can also be seen in the case where p_{ij} close to 2 and the case where p_{ij} close to 2^*. Similar results are given for elliptic systems with subcritical Sobolev exponents. These results substantially generalize and improve existing results in the literature. To show the effect of the uniqueness result, we apply it to prove existence of a positive solution to a 2-coupled nonlinear Schrödinger system with critical exponent and $L^{N/2}(\mathbb{R}^N)$ potentials.

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1. INTRODUCTION AND THE MAIN RESULTS

In this paper, we study the number of positive solutions to the system of n-coupled equations

$$-\Delta u_i + \lambda_i u_i = \sum_{j=1}^{n} \beta_{ij} |u_j|^{q_{ij}} |u_i|^{p_{ij}-2} u_i \quad \text{in } \Omega, \quad i = 1, 2, \cdots, n, \tag{1.1}$$

eq1.1

where $n \geq 2$, $\Omega \subset \mathbb{R}^N$ is a bounded domain or $\Omega = \mathbb{R}^N$ with $N \geq 1$, $\lambda_i \geq 0$ for $i = 1, 2, \cdots, n$, $\beta_{ij} > 0$, $q_{ij} > 0$, and $p_{ij} + q_{ij} = r$ for some $r \in (2, 2^*]$ and for $i, j = 1, 2, \cdots, n$.

Recall that $2^* = 2N/(N-2)$ for $N \geq 3$ and $2^* = +\infty$ for $N = 1, 2$. Most results of the paper are for the whole space $\mathbb{R}^N$ and the critical Sobolev exponent $r = 2^*$. Results for a bounded Ω and for $r < 2^*$, however, will be given.

A vector solution $(u_1, \cdots, u_n)$ is called *nontrivial* if each component u_i is nonzero. In the study of (1.1) it is a key issue to distinguish nontrivial solutions from *semitrivial* solutions which have at least one component being zero and at least one component being nonzero. Here we are interested in *positive* solutions, that is, nontrivial solutions $(u_1, u_2, \cdots, u_n)$ with $u_i(x) > 0$ for all $i = 1, 2, \cdots, n$ and all $x \in \mathbb{R}^N$.

This system admits a variational structure if and only if $p_{ij} = q_{ji}$ and $\frac{\beta_{ij}}{p_{ij}} = \frac{\beta_{ji}}{p_{ji}}$ for all $i \neq j$. In particular, if $p_{ij} = q_{ij} = \frac{r}{2}$ and $\beta_{ij} = \beta_{ji}$ then the associated energy functional is

$$J(u_1, \cdots, u_n) = \frac{1}{2} \sum_{i=1}^{n} \int_\Omega (|\nabla u_i|^2 + \lambda_i u_i^2) - \frac{1}{r} \sum_{i,j=1}^{n} \beta_{ij} \int_\Omega |u_i u_j|^{\frac{r}{2}}.$$

In this special case, any solution $(u_1, u_2, \cdots, u_n)$ of (1.1) corresponds to a solitary wave, i.e., a solution of the form $\Phi_i(x,t) = e^{\sqrt{-1}\lambda_i t} u_i(x)$ to the n-coupled Gross-Pitaevskii system

$$-\sqrt{-1}\frac{\partial \Phi_i}{\partial t} - \Delta \Phi_i = \sum_{j=1}^{n} \beta_{ij} |\Phi_j|^{\frac{r}{2}} |\Phi_i|^{\frac{r}{2}-2} \Phi_i \quad \text{in } \Omega \times \mathbb{R}^+, \quad i = 1, 2, \ldots, n. \tag{1.2}$$

eq1.2

System (1.2) arises in mathematical models describing various phenomena in physics. In the theory of Bose-Einstein condensates of n different hyperfine spin states, Φ_i is the wave function of the i-th condensate, and the constants β_{ii} and β_{ij} $(i \neq j)$ are the intraspecies and interspecies scattering lengths respectively; see [17, 36]. In nonlinear optics, (1.2) models n optical waves of different frequencies which co-propagate and interact nonlinearly in a medium, or n polarization components of a wave interacting nonlinearly at some central frequency, and Φ_i is the complex amplitude of the i-th electric field envelope, or the i-th polarization component; see [1, 29, 30]. The constants β_{ii} are parameters characteristic of self interaction, and β_{ij} $(i \neq j)$ are for the interaction between different waves or different polarization components. See [13, 18, 22, 44] for more physical background on (1.2).

When $p_{ij} = q_{ij} = \frac{r}{2}$ and $\beta_{ij} = \beta_{ji}$, system (1.1) has attracted tremendous attention and has been studied extensively over the last two decades both by mathematicians and by theoretical physicists. It turns out that (1.1) is much more complicated comparing to single equations. Moreover, the more equations that make up (1.1), the more complicated this system becomes. Up to now, many new properties of solutions, including existence and multiplicity of nontrivial solutions, segregation and synchronization of the components of solutions, and nodal properties of solutions, which a single equation does not possess, have been revealed; see, for example, [2, 4, 8, 9, 16, 23, 26, 27, 28, 31, 33, 37, 38, 43, 45, 46] and references therein for the subcritical case and [11, 12, 14, 15, 32, 34, 35, 41, 42, 51] for the critical case. We remark that systems studied in the above mentioned papers all have a variational structure and results are obtained by variational methods. In addition, most of the results are only proved for systems with two equations.

In the current paper, we consider system (1.1), which is much more general than those with variational structure studied extensively in the literature. We first focus on the critical Sobolev exponent case: $p_{ij} + q_{ij} = 2^*$, and we denote $\mu_i := \beta_{ii}$ and assume $\lambda_i = 0$, $\Omega = \mathbb{R}^N$ and $N \geq 3$ in (1.1). This leads us to study the system

$$\begin{cases} -\Delta u_i = \mu_i u_i^{2^*-1} + \sum\limits_{j=1,\, j\neq i}^{n} \beta_{ij} u_i^{p_{ij}-1} u_j^{q_{ij}} \quad \text{in } \mathbb{R}^N, \\ u_i > 0, \quad u_i \in \mathscr{D}^{1,2}(\mathbb{R}^N), \quad i = 1, 2, \cdots, n. \end{cases} \tag{1.3}$$

Uniqueness and multiplicity of solutions to (1.3) will be investigated. We remark that in this paper results on uniqueness, multiplicity and exact multiplicity of solutions of (1.3) are all up to translation and dilation. A solution $(u_1, u_2, \cdots, u_n)$ of (1.3) is called a *synchronized* solution if there exist positive numbers $k_1, k_2, \cdots, k_n$ such that $u_i = k_i U$ for $i = 1, 2, \cdots, n$, where U is a positive solution of the single equation

$$-\Delta u = u^{2^*-1} \quad \text{in } \mathbb{R}^N, \quad u \in \mathscr{D}^{1,2}(\mathbb{R}^N).$$

According to [3, 40], U is unique up to translation and dilation and has the expression

$$U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}.$$

Throughout the paper, we shall always assume that $\mu_i > 0$ which are arranged without loss of generality in the following order

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n.$$

For system (1.3), we shall assume in addition that

$$\beta_{ij} > 0, \quad p_{ij} < 2^*, \quad p_{ij} + q_{ij} = 2^* \quad \text{for } i \neq j \in \{1, 2, \cdots, n\}.$$

Various results will be obtained under this assumption and additional constraints. As a consequence of this assumption, we have $q_{ij} > 0$. However, we allow $p_{ij} \leq 0$ and accordingly $q_{ij} \geq 2^*$ in some of our results. For convenience, denote

$$B = (\beta_{ij})_{n\times n}.$$

Our first result deals with the case $p_{ij} < 2$ and gives existence of $2^n - 1$ synchronized positive solutions if β_{ij} $(i \neq j)$ are suitably small and existence of one synchronized positive solution if β_{ij} $(i \neq j)$ are suitably large and $p_{ij} = p$ for $i \neq j$. In either case, since we do not assume any symmetry condition the system is essentially more general than those having variational structure.

thm1.1 **Theorem 1.1.** *Assume $N \geq 3$, $\mu_i > 0$, $p_{ij} < 2$, and $q_{ij} = 2^* - p_{ij}$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. We have the following conclusions.*

(a) *If*

$$0 < \beta_{ij} < \beta_* := \frac{1}{2} \min_{1 \leq i \leq n} \Big[\sum_{j=1,\, j \neq i}^{n} \Big(\frac{1}{\mu_j}\Big)^{\frac{N-2}{4} q_{ij}} \Big(\frac{1}{2\mu_i}\Big)^{\frac{N-2}{4}(p_{ij}-2)} \Big]^{-1},$$

then (1.3) has at least $2^n - 1$ synchronized positive solutions.

(b) *Assume in addition that there exists $p < 2$ such that $p_{ij} = p$ for $i \neq j$. If the matrix $B = (\beta_{ij})_{n \times n}$ has an inverse $A = (a_{ij})_{n \times n}$ such that*

$$a_{ij} > 0 \text{ for } i \neq j \text{ and } \sum_{j=1}^{n} a_{ij} > 0 \text{ for } i = 1, 2, \cdots, n,$$

then (1.3) has at least one synchronized positive solution.

Remark 1.1. We remark that $p_{ij} \leq 0$ is allowed in Theorem 1.1, a case which has not been studied in the literature. In Theorem 1.1(a), β_{ij} $(i \neq j)$ are assumed to be suitably small. The proof of Theorem 1.1 in Section 2 shows that β_* can be replaced with

$$\beta_{**} = \max_{0 < \delta < 1} f(\delta)$$

where

$$f(\delta) = (1 - \delta) \min_{1 \leq i \leq n} \Big[\sum_{j=1,\, j \neq i}^{n} \Big(\frac{1}{\mu_j}\Big)^{\frac{N-2}{4} q_{ij}} \Big(\frac{\delta}{\mu_i}\Big)^{\frac{N-2}{4}(p_{ij}-2)} \Big]^{-1}.$$

Clearly, $\beta_{**} \geq f(1/2) = \beta_*$. We shall explain that the condition on the matrix B in Theorem 1.1(b) is satisfied if β_{ij} $(i \neq j)$ are close to a single number β with $\beta > \mu_n$ for all $i \neq j$ (see Proposition 2.1) or if β_{ij} $(i \neq j)$ are appropriately grouped with entries of each group close to a single number β properly large (see Proposition 2.2). Roughly speaking, Theorem 1.1(b) means that (1.3) has at least one synchronized positive solution if $p_{ij} = p$ for $i \neq j$ and β_{ij} $(i \neq j)$ are suitably large.

Now we consider a more specific case in which $p_{ij} = p$ and $\beta_{ij} = \beta$ for all $i, j \in \{1, 2, \cdots, n\}$ with $i \neq j$ and for some $p < 2$ and $\beta > 0$. For this case we have the following more refined theorem concerning existence, uniqueness and exact multiplicity of synchronized positive solutions, as well as uniqueness of positive solutions of (1.3).

thm1.2 **Theorem 1.2.** *Assume $N \geq 3$, $\mu_i > 0$, $p_{ij} = p < 2$, $q_{ij} = 2^* - p$ and $\beta_{ij} = \beta$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. We have the following conclusions.*

(a) *(1.3) has at least one synchronized positive solution for any $\beta > 0$.*

(b) *If $\beta \geq \mu_n$, or if $\beta \geq \mu_n - \delta_0$ in the case $\mu_{n-1} < \mu_n$ where δ_0 is some positive number, then (1.3) has exactly one synchronized positive solution.*

(c) *There exists $\beta_0 \in (0, \mu_1)$ such that (1.3) has exactly $2^n - 1$ synchronized positive solutions for $0 < \beta < \beta_0$.*

(d) *If $n = 2m$, $\mu_1 = \cdots = \mu_m =: \mu' \leq \mu_{m+1} = \cdots = \mu_{2m} =: \mu''$, and*

$$\beta > \frac{(m+1)\mu'' - (m-1)\mu' + \sqrt{(m+1)^2 \mu''^2 + (m-1)^2 \mu'^2 - 2(m^2+1)\mu'\mu''}}{2},$$

then (1.3) has exactly one positive solution.

Remark 1.2. Theorem 1.2(a)-(c) are all conclusions about positive solutions of the synchronized type. It would be interesting to prove that the number of synchronized positive solutions is decreasing with respect to $\beta > 0$, and right now we have a proof of this only for $n = 2$; see Theorem 1.5(c) below. It would also be interesting to prove a result concerning uniqueness of positive solutions more general than that stated in (d); cf. Theorem 1.5(b). In view of the discussions after each of the proofs of Lemmas 3.5, 3.7 and 3.8, δ_0 in (b) and β_0 in (c) can be given explicitly in terms of n, N and μ_i. If $n = 2$, that is, if the system consists of two equations, then the condition in Theorem 1.2(d) reduces to $\beta > \mu_2 + \sqrt{\mu_2^2 - \mu_1\mu_2}$. Again, we allow $p \leq 0$ in Theorem 1.2.

Remark 1.3. The conditions of Theorem 1.2 are restrictive, but compared with the conclusions in Theorem 1.1, those in Theorem 1.2 are much more delicate, which establish the existence of a synchronized positive solution for any $\beta > 0$ and give the exact number of synchronized positive solutions both for $\beta > 0$ large and $\beta > 0$ small. When the matrix B enjoys more structural conditions, Theorem 1.2 also determines uniqueness of the positive solution of (1.3) for $\beta > 0$ large.

Remark 1.4. Let us compare our results Theorems 1.1 and 1.2 with those in the literature. First of all, system (1.3) in Theorems 1.1 and 1.2 is much more general than the same type of systems studied via variational methods in the papers mentioned above, and variational methods can not be used to prove Theorems 1.1 and 1.2 since (1.3) may not have a variational structure. Especially, we allow $p_{ij} \leq 0$ and $q_{ij} \geq 2^*$ in Theorems 1.1 and 1.2, a case which, to our knowledge, has not been considered before. Note again (1.3) has a variational structure if and only if $p_{ij} = q_{ji}$ and $\frac{\beta_{ij}}{p_{ij}} = \frac{\beta_{ji}}{p_{ji}}$ for all $i \neq j$. Second, we find that the system has at least $2^n - 1$ synchronized positive solutions for β_{ij} small in Theorem 1.1(a), exactly $2^n - 1$ synchronized positive solutions for β small in Theorem 1.2(c) and exactly one synchronized positive solution for β large in Theorem 1.2(b); these results are completely new even for systems with variational structure, and they give a hint on the dependence of the bifurcation diagram of the solution set on n and β. Third, we give a uniqueness result in Theorem 1.2(d) on all positive solutions of the system, and this result is also completely new for this type of critical elliptic systems.

Remark 1.5. In the very special case where $n = 2$, $p_{12} = p_{21} = q_{12} = q_{21} = 2^*/2$, and $\beta := \beta_{12} = \beta_{21}$, (1.3) was studied in dimension $N \geq 4$ in [11, 12]. For $N \geq 5$, it was proved in [12, Theorem 1.6] that (1.3) has a least energy positive solution for any $\beta > 0$ and a second positive solution which is of the synchronized type for $\beta > 0$ suitably small. Later, it was showed in [51, Theorem 1.6] (see also [49, Theorem 1.2]) that the above least energy positive solution must also be of the synchronized type. In [12, Theorem 1.7] and [49, Theorem 1.2], it was established that the least energy positive solution is unique for $\beta > 0$ large. Our Theorems 1.1 and 1.2 considerably improve those existence, uniqueness and multiplicity results in several aspects: the system considered here is much more general, our results also cover the dimension $N = 3$, the multiplicity conclusions in Theorems 1.1 and 1.2 are much stronger, and the results on exact multiplicity of synchronized positive solutions and on the uniqueness of all positive solutions are completely new.

In both Theorem 1.1 and Theorem 1.2, we have studied the case $p_{ij} < 2$. Now we consider the case $p_{ij} = 2$. We have the following result about exact number of synchronized positive solutions, and nonexistence and uniqueness of positive solutions of (1.3).

thm1.3 **Theorem 1.3.** *Assume $N \geq 3$, $\mu_i > 0$, $p_{ij} = 2$, $q_{ij} = 2^* - 2$, and $\beta_{ij} = \beta$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. We have the following conclusions.*

- (a) *(1.3) has a synchronized positive solution if and only if $\beta > \mu_n$ or $0 < \beta < \mu_1$ or $\beta = \mu_1 = \mu_n$. Moreover, if $\beta > \mu_n$ or $0 < \beta < \mu_1$ then (1.3) has exactly one synchronized positive solution and if $\beta = \mu_1 = \mu_n$ then (1.3) has infinitely many synchronized positive solutions.*
- (b) *If $\mu_1 \leq \beta \leq \mu_n$ and $\mu_1 \neq \mu_n$ then (1.3) has no positive solution.*
- (c) *If $\beta > \mu_n$ then (1.3) has exactly one positive solution.*

Remark 1.6. Comparing Theorem 1.2 with Theorem 1.3, we can see two distinct structures of the solution set in the cases $p_{ij} = p < 2$ and $p_{ij} = 2$. For $p_{ij} = p < 2$ and for any $\beta > 0$, the system has a synchronized positive solution, while for $p_{ij} = 2$ and for $\mu_1 \leq \beta \leq \mu_n$ with $\mu_1 \neq \mu_n$, the system has no positive solution. For $p_{ij} = p < 2$ and for $\beta > 0$ small, the system has exactly $2^n - 1$ synchronized positive solutions, while for $p_{ij} = 2$ and for $\beta < \mu_1$, the system has exactly one synchronized positive solution. These differences give a hint on bifurcation phenomena of the solution set from the exponent $p_{ij} = 2$.

Remark 1.7. The assumptions in Theorem 1.3 make it impossible for the system to possess a variational structure, and no variational approach is applicable. In addition, Theorem 1.3 not only gives the exact number of synchronized positive solutions depending on the size of β, but also confirms the uniqueness of positive solutions for $\beta > \mu_n$. In the very special case where $n = 2$, $N = 4$, $\beta_{12} = \beta_{21} = \beta$ and $p_{12} = p_{21} = q_{12} = q_{21} = 2$, existence and nonexistence of positive solutions of (1.3) were investigated in [11] by variational methods.

The last case in which we deal with (1.3) is $p_{ij} \in (2, 2^*)$. The following theorem, as compared with Theorems 1.1-1.3, exhibits a completely new feature of the structure of the solution set in this case.

thm1.4 **Theorem 1.4.** *Assume $N \geq 3$, $\mu_i > 0$, $2 < p_{ij} < 2^*$, $q_{ij} = 2^* - p_{ij}$, and $\beta_{ij} > 0$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. Then*

(a) *(1.3) has a synchronized positive solution.*

Assume moreover $p_{ij} = p \in (2, 2^*)$, $q_{ij} = 2^* - p$, $\beta_{ij} = \beta$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. We have the following conclusions.

(b) *If $0 < \beta \leq \mu_1$, or if $0 < \beta \leq \mu_1 + \delta_0$ in the case $\mu_1 < \mu_2$ where δ_0 is some positive number, then (1.3) has exactly one synchronized positive solution.*
(c) *If $\beta > \mu_j$ then there exists $p_1 = p_1(\beta) \in (2, 2^*)$ such that for $p \in (2, p_1)$ (1.3) has at least $2^j - 1$ synchronized positive solutions. In particular, if $\beta > \mu_n$ then for p larger than and sufficiently close to 2, (1.3) has at least $2^n - 1$ synchronized positive solutions.*
(d) *If $\beta > \mu_1$ and*
$$\frac{\mu_1}{\beta} 2 + \Big(1 - \frac{\mu_1}{\beta}\Big) 2^* \leq p < 2^*,$$
then (1.3) has exactly one synchronized positive solution. In particular, this result together with (b) implies that for any $\beta > 0$, (1.3) has exactly one synchronized positive solution if p is less than and sufficiently close to 2^*.

Remark 1.8. Theorem 1.4 reveals that the structure of the solution set of (1.3) in the case $2 < p_{ij} < 2^*$ is incredibly different from what have been described in Theorems 1.1-1.3 in the cases $p_{ij} < 2$ and $p_{ij} = 2$. Let us examine more closely these differences only in the special case $p_{ij} = p$ and $\beta_{ij} = \beta$. When β is suitably small, (1.3) has exactly $2^n - 1$ synchronized positive solutions for $p < 2$ but exactly one synchronized positive solution for $2 \leq p < 2^*$. When $\beta > \mu_n$, (1.3) has exactly one synchronized positive solution for $p \leq 2$ but at least $2^n - 1$ synchronized positive solutions for p larger than and sufficiently close to 2. For any $\beta > 0$, (1.3) has a synchronized positive solution if $p \neq 2$ and $p < 2^*$, but for $\mu_1 \leq \beta \leq \mu_n$ with $\mu_1 \neq \mu_n$, (1.3) does not have any positive solution if $p = 2$. When $2 < p < 2^*$, a significant difference of the structure of the solution set can also be seen between p sufficiently close to 2^* and p sufficiently close to 2. From these results we see that the structure of the solution set of (1.3) is very complex and strongly depends on n, β_{ij} and p_{ij}.

Remark 1.9. The case $p_{ij} \in (2, 2^*)$ was rarely investigated in the literature and we are only aware of the paper [50] where a synchronized positive solution for any $\beta > 0$ was obtained for (1.3) in the case where $n = 2$, $N = 3$, $\beta_{12} = \beta_{21} = \beta$ and $p_{12} = p_{21} = q_{12} = q_{21} = 3$. Again, the system considered here is much more general and the conclusions of Theorem 1.4 are much stronger.

For (1.3) with two equations, that is, for $n = 2$, we now refine some of the results in Theorems 1.1-1.3 above, and we rewrite (1.3) as

$$\begin{cases} -\Delta u = \mu_1 u^{2^*-1} + \beta_1 u^{p_1-1} v^{q_1} & \text{in } \mathbb{R}^N, \\ -\Delta v = \mu_2 v^{2^*-1} + \beta_2 v^{p_2-1} u^{q_2} & \text{in } \mathbb{R}^N, \\ u > 0,\ v > 0,\ u, v \in \mathscr{D}^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.4}$$ eq1.4

Again, we may assume without loss of generality that $0 < \mu_1 \leq \mu_2$. In the following theorem, stronger results than those above on multiple synchronized positive solutions, exact multiplicity of synchronized positive solutions, and uniqueness of positive solutions will be exhibited for (1.4). We first formulate the following assumption.

(A) For $i, j \in \{1,2\}$ such that $i \neq j$,

$$\beta_i > \begin{cases} \text{either } \ 2\mu_j & \text{if } p_i = 2, \\ \text{or } \ \frac{2^{\frac{N-2}{4}q_i+2}}{N-2}\left(\frac{\mu_i}{2-p_i}\right)^{1-\frac{N-2}{4}q_i}\left(\frac{\mu_j}{q_i}\right)^{\frac{N-2}{4}q_i} & \text{if } 4-2^* < p_i < 2, \\ \text{or } \ \mu_i^{1-\frac{N-2}{4}q_i}\mu_j^{\frac{N-2}{4}q_i} & \text{if } p_i \le 4-2^*. \end{cases}$$

Note that this assumption totally consists of nine cases of combinations of p_1 and p_2 depending on either $p_1 = 2$, or $4-2^* < p_1 < 2$, or $p_1 < 4-2^*$ and either $p_2 = 2$, or $4-2^* < p_2 < 2$, or $p_2 < 4-2^*$. As examples, we only list the following two cases: $p_1 < 4-2^* < p_2 = 2$ and $4-2^* < p_1, p_2 < 2$.

thm1.5 **Theorem 1.5.** *Assume $N \ge 3$, $\mu_i > 0$, $\beta_i > 0$, $p_i \le 2$, and $q_i = 2^* - p_i$ for $i = 1, 2$.*

(a) *If for $i, j \in \{1,2\}$ and $i \neq j$, $p_i < 2$ and*

$$\beta_i < 2^{-\frac{N-2}{4}q_i}\mu_i^{1-\frac{N-2}{4}q_i}\mu_j^{\frac{N-2}{4}q_i},$$

then (1.4) has at least three synchronized positive solutions.

(b) *If (A) is satisfied, then (1.4) has a unique positive solution.*

(c) *Assume moreover $p := p_1 = p_2 < 2$ and $\beta := \beta_1 = \beta_2$. We have the following conclusions.*

(c_1) *If $\mu := \mu_1 = \mu_2$, then (1.4) has exactly three synchronized positive solutions for $0 < \beta < \beta_0$ and exactly one synchronized positive solution for $\beta \ge \beta_0$, where $\beta_0 = \frac{2\mu}{2(N-1)-(N-2)p}$.*

(c_2) *If $\mu_1 < \mu_2$, then there exists $\beta_0 \in (0, \mu_1)$ such that (1.4) has exactly three synchronized positive solutions for $0 < \beta < \beta_0$, exactly two synchronized positive solutions for $\beta = \beta_0$, and exactly one synchronized positive solution for $\beta > \beta_0$.*

Remark 1.10. Let us say some words about the relationship between Theorem 1.5 and Theorems 1.1-1.3. Theorem 1.5(a) improves Theorem 1.1(a) by allowing β_1 and β_2 from different ranges whose intersection is $(0, \beta_*)$, where β_* is the number defined in Theorem 1.1(a) and it has the following expression for $n = 2$:

$$\beta_* = \min_{i,j\in\{1,2\}, i\neq j} 2^{-\frac{N-2}{4}q_i}\mu_i^{1-\frac{N-2}{4}q_i}\mu_j^{\frac{N-2}{4}q_i}.$$

By allowing $\beta_1 \neq \beta_2$ and $p_1 \neq p_2$, Theorem 1.5(b) essentially generalizes Theorem 1.2(d) and Theorem 1.3(c) for $n = 2$. Clearly, Theorem 1.5(c) refines Theorem 1.2(a), (b) and (c) in the case $n = 2$. From the discussion after the proof of Theorem 1.5(c) in Section 6, we shall see that if $N = 8$, $p = \frac{4}{3}$ and $\mu_1 < \mu_2$ then β_0 is the unique positive root of the quartic polynomial

$$f(\beta) = -27\beta^4 + 18\mu_1\mu_2\beta^2 - 4(\mu_1^3 + \mu_2^3)\beta + \mu_1^2\mu_2^2.$$

For many problems in nonlinear analysis, existence of solutions of some elliptic equations can be derived from uniqueness of positive solutions of their limit equations. To show that this general principle also applies to the current situation and to illustrate the effect of the uniqueness result in Theorem 1.5(b), we shall apply it to prove existence of a solution of the nonlinear Schrödinger system

$$\begin{cases} -\Delta u + V_1(x)u = \mu_1 u^{2^*-1} + \frac{p\beta}{2^*}u^{p-1}v^q & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \mu_2 v^{2^*-1} + \frac{q\beta}{2^*}v^{q-1}u^p & \text{in } \mathbb{R}^N, \\ u > 0, \ v > 0, \ u, v \in \mathscr{D}^{1,2}(\mathbb{R}^N) \end{cases} \tag{1.5}$$

eq1.5

in the case where $N \ge 5$, $V_1, V_2 \in L^{N/2}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$, $0 < \mu_1 \le \mu_2$, $p, q \in (1, 2]$, $p + q = 2^*$, and $\beta > 0$. As $\|V_1\|_{L^{N/2}(\mathbb{R}^N)} + \|V_2\|_{L^{N/2}(\mathbb{R}^N)} \to 0$, the limit system of (1.5) is

$$\begin{cases} -\Delta u = \mu_1 u^{2^*-1} + \frac{p\beta}{2^*}u^{p-1}v^q & \text{in } \mathbb{R}^N, \\ -\Delta v = \mu_2 v^{2^*-1} + \frac{q\beta}{2^*}v^{q-1}u^p & \text{in } \mathbb{R}^N, \\ u > 0, \ v > 0, \ u, v \in \mathscr{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.4$'$}$$

which is exactly (1.4) when we choose there

$$p_1 = q_2 = p, \quad p_2 = q_1 = q, \quad \beta_1 = \frac{p\beta}{2^*}, \quad \beta_2 = \frac{q\beta}{2^*}.$$

Here we use the current symbols for convenience in the discussion concerning Theorem 1.6 below.

We introduce the notation

$$\tilde{\mu}_{12}=\frac{2^{\frac{N-2}{4}q+2}}{(N-2)p}\Big(\frac{\mu_1}{2-p}\Big)^{1-\frac{N-2}{4}q}\Big(\frac{\mu_2}{q}\Big)^{\frac{N-2}{4}q},\quad \tilde{\mu}_{21}=\frac{2^{\frac{N-2}{4}p+2}}{(N-2)q}\Big(\frac{\mu_2}{2-q}\Big)^{1-\frac{N-2}{4}p}\Big(\frac{\mu_1}{p}\Big)^{\frac{N-2}{4}p},$$

$$\hat{\mu}_{12}=\frac{1}{p}\mu_1^{1-\frac{N-2}{4}q}\mu_2^{\frac{N-2}{4}q},\quad \hat{\mu}_{21}=\frac{1}{q}\mu_2^{1-\frac{N-2}{4}p}\mu_1^{\frac{N-2}{4}p},$$

and define

$$\beta^0=\begin{cases} 2^*\mu_2, & \text{if } N=5,\ p=2,\ q=\frac{4}{3},\\ 2^*\max\{\mu_1,\ (\sqrt{2}\mu_2^{3/2})/(\sqrt{3}\mu_1^{1/2})\}, & \text{if } N=5,\ q=2,\ p=\frac{4}{3},\\ 2^*\max\{\tilde{\mu}_{12},\ \tilde{\mu}_{21}\}, & \text{if } 4-2^*<p,\,q<2,\\ 2^*\max\{\tilde{\mu}_{12},\ \hat{\mu}_{21}\}, & \text{if } 1<q\le 4-2^*<p<2,\\ 2^*\max\{\hat{\mu}_{12},\ \tilde{\mu}_{21}\}, & \text{if } 1<p\le 4-2^*<q<2,\\ 2^*\max\{\hat{\mu}_{12},\ \hat{\mu}_{21}\}, & \text{if } 1<p,\,q\le 4-2^*. \end{cases}$$

Since $p,q\in(1,2]$ and $p+q=2^*$, the dimension must be $N=5$ if $p=2$ or $q=2$. The dimension can be any $N\ge 5$ if $p,q<2$. For $(1.4')$ with $\beta>\beta^0$, it can be checked that the condition of Theorem 1.5(b) is satisfied, and then $(1.4')$ has exactly one positive solution, which is a synchronized positive solution according to the proof of Theorem 1.5(b) in Section 6. More precisely, by the proofs of Lemmas 6.1 and 6.2, this unique solution takes the form (k_1U,k_2U), where k_1,k_2 are positive numbers satisfying $0<k_i<(2\mu_i)^{-(N-2)/4}$ and are uniquely determined by μ_1,μ_2,β,p,q through the system of algebraic equations

$$\mu_1k_1^q-k_1^{2-p}+\frac{p\beta}{2^*}k_2^q=0,\quad \mu_2k_2^p-k_2^{2-q}+\frac{q\beta}{2^*}k_1^p=0.$$

Use $|\cdot|_r$ to represent the standard norm in $L^r(\mathbb{R}^N)$ and S the optimal constant of the Sobolev embedding $\mathscr{D}^{1,2}(\mathbb{R}^N)\hookrightarrow L^{2^*}(\mathbb{R}^N)$. Recall that

$$S=\inf_{u\in\mathscr{D}^{1,2}(\mathbb{R}^N),\ u\ne0}|\nabla u|_2^2/|u|_{2^*}^2,\quad |\nabla U|_2^2=|U|_{2^*}^{2^*}=S^{N/2}.$$

thm1.6 **Theorem 1.6.** *Assume $N\ge5$, $\mu_i>0$, $\beta>\beta^0$, $p,q\in(1,2]$ with $p+q=2^*$, $V_1,V_2\in L^{N/2}(\mathbb{R}^N)\cap L^\infty_{\text{loc}}(\mathbb{R}^N)$ are nonnegative functions, and*

$$\left(1+\frac{k_1^2|V_1|_{N/2}+k_2^2|V_2|_{N/2}}{(k_1^2+k_2^2)S}\right)^{N/2}<\min\left\{\frac{\mu_1^{-(N-2)/2}}{k_1^2+k_2^2},\ \frac{\mu_2^{-(N-2)/2}}{k_1^2+k_2^2},\ 2\right\}. \tag{1.6}$$

eq1.6

Then problem (1.5) admits a positive solution.

Remark 1.11. For $\beta>\beta^0$, we have $0<k_i<(2\mu_i)^{-(N-2)/4}$ as mentioned above, and then since $\mu_1\le\mu_2$ it is clear that (1.6) holds if

$$\left(1+\frac{|V_1|_{N/2}+|V_2|_{N/2}}{S}\right)^{\frac{N}{2}}<\min\left\{\frac{(2\mu_1)^{\frac{N-2}{2}}}{\mu_1^{\frac{N-2}{2}}+\mu_2^{\frac{N-2}{2}}},\ 2\right\}.$$

Note that this last inequality can only be satisfied when μ_2/μ_1 is not large enough. However, for any fixed N,p,q,μ_1,μ_2 and V_1,V_2 with $|V_1|_{N/2}+|V_2|_{N/2}$ small enough, (1.6) always holds if we enlarge β^0 adequately, regardless of the size of μ_2/μ_1. The reason for this is that $k_1^2+k_2^2$ can be very small if $\beta>\beta^0$ and β^0 is very large (see Remark 6.1).

Remark 1.12. A result of the same type as in Theorem 1.6 was first given in [6] for scalar field equations, which has been extended in [24] to system (1.5) with $p=q=2$ in dimension $N=4$. Theorem 1.6 extends the result of [24] to higher dimensions $N\ge5$. To prove the existence result Theorem 1.6, we need the uniqueness result Theorem 1.5(b) together with some new ideas incorporated.

Let us briefly sketch the ideas of the proofs of the above theorems. Most arguments of our proofs are new and are different from the methods used in the above mentioned papers. We shall prove Theorem 1.1 via the Brouwer degree theory. A synchronized positive solution $(k_1U, k_2U, \cdots, k_nU)$ of (1.3) corresponds to a positive solution of the algebraic system $f_i(k_1, k_2, \cdots, k_n) = 0$ $(i = 1, \cdots, n)$ defined in (2.1). For β_{ij} $(i \neq j)$ small enough, we shall find $2^n - 1$ mutually disjoint cuboids in $(0, +\infty)^n$ so that the Brouwer degree of $(f_1, \cdots, f_n)$ on each cuboid is nonzero, which shows that (1.3) has at least $2^n - 1$ synchronized positive solutions. However, for β_{ij} $(i \neq j)$ large enough, the algebraic system $f_i(k_1, k_2, \cdots, k_n) = 0$ may have no solution in any of those cuboids. The idea is that using the matrix A we convert the system in (2.1) to another algebraic system $g_i(k_1, k_2, \cdots, k_n) = 0$ $(i = 1, \cdots, n)$ so that a new cuboid can be constructed on which $(g_1, \cdots, g_n)$ has a nonzero Brouwer degree.

The idea of the proof of Theorem 1.2 is completely different. To prove Theorem 1.2(a), (b) and (c), we shall introduce a group of algebraic equations $G_s(\tau) = 0$ and show that the number of synchronized positive solutions of (1.3) is either equal to the number of positive solutions of a single algebraic equation (in the case $\beta \geq \mu_n$) or equal to the sum of the number of positive solutions of a group of at most 2^n algebraic equations (in the case $\beta < \mu_n$). We shall then study the number of positive solutions of these algebraic equations to achieve the conclusions. We prove Theorem 1.2(d) by showing that any positive solution must be of the synchronized type by comparing the components of the solution pairwise. Such an idea was first used in [47] and later generalized in [25], but still some essential new observations have to be incorporated into the proof.

The proof of Theorem 1.3 is relatively easy. Theorem 1.3(a) is proved by studying the associated algebraic system. Theorem 1.3(b) is obtained via an observation as in [5, Theorem 1.2]. Theorem 1.3(c) is verified using an idea from [47].

Again, Theorem 1.4(a) is proved via the Brouwer degree theory. The techniques for proving Theorem 1.4(b), (c) and (d) share some similarities with the proof of Theorem 1.2(a), (b) and (c), although the proofs are quite different in detail. Giving the functions G_s adjusted meaning, we also introduce a group of algebraic equations $G_s(\tau) = 0$, and show that the number of synchronized positive solutions of (1.3) is either equal to the number of positive solutions of a single algebraic equation (in the case $\beta \leq \mu_1$) or equal to the sum of the number of positive solutions of a group of at most 2^n algebraic equations (in the case $\beta > \mu_1$). Theorem 1.4(b) and (d) are verified by showing that there is only one equation $G_s(\tau) = 0$ which has exactly one solution. Theorem 1.4(c) is established by showing that there are at least $2^j - 1$ equations $G_s(\tau) = 0$ and any of them has at least one solution.

Once more, Theorem 1.5(a) and the existence part of Theorem 1.5(b) are proved using the Brouwer degree theory. In particular, under the conditions of Theorem 1.5(b) we shall prove that the algebraic system

$$\mu_1 k_1^{q_1} - k_1^{2-p_1} + \beta_1 k_2^{q_1} = 0, \quad \mu_2 k_2^{q_2} - k_2^{2-p_2} + \beta_2 k_1^{q_2} = 0$$

has a solution (k_1, k_2) in $Q_4 = (0, (2\mu_1)^{-(N-2)/4}) \times (0, (2\mu_2)^{-(N-2)/4})$. We shall show the following astonishing result: Existence of a solution of this algebraic system in Q_4 not only implies uniqueness of positive solutions of (1.4) but also implies uniqueness of solutions of the algebraic system itself in the whole $\mathbb{R}^+ \times \mathbb{R}^+$. In particular, this proves the uniqueness part of Theorem 1.5(b). We prove Theorem 1.5(c) by carefully analysing properties of the four algebraic equations $G_s(\tau) = 0$, $s = 1, 2, 3, 4$, depending on β.

With Theorem 1.5(b) in hand, we apply it to prove Theorem 1.6 using ideas from [6, 24]. Throughout the paper, only Theorem 1.6 is proved via variational methods and Theorems 1.1-1.5 cannot be obtained this way due to the lacking of variational structure.

The methods used to prove Theorems 1.1-1.4 are applicable to the following system with subcritical exponent

$$\begin{cases} -\Delta u_i + u_i = \mu_i u_i^{r-1} + \sum_{j=1,\, j\neq i}^{n} \beta_{ij} u_i^{p_{ij}-1} u_j^{q_{ij}} \quad \text{in } \mathbb{R}^N, \\ u_i > 0, \quad u_i \in H^1(\mathbb{R}^N), \quad i = 1, 2, \cdots, n, \end{cases} \tag{1.7}$$

eq1.7

where $N \geq 1$, $n \geq 2$, $2 < r < 2^*$, $2^* = +\infty$ if $N = 1, 2$ and $2^* = 2N/(N-2)$ if $N \geq 3$, $\mu_i > 0$, $\beta_{ij} > 0$, $p_{ij} < r$, and $p_{ij} + q_{ij} = r$ for $i \neq j \in \{1, 2, \cdots, n\}$. We shall give results on this system and shall discuss briefly applications of our methods to other related systems.

Before closing this section, we give a consequence of Theorems 1.2-1.5 on exact number of solutions of the problem

$$\begin{cases} -\Delta u_i = \mu_i u_i^{2^*-1} + \sum_{j=1,\, j\neq i}^{n} \beta_{ij} u_i^{p_{ij}-1} u_j^{q_{ij}} \quad \text{in } \mathbb{R}^N, \\ u_i > 0, \quad u_i \in W^{1,2}_{\text{loc}}(\mathbb{R}^N) \cap C^0(\mathbb{R}^N), \quad i = 1, 2, \cdots, n. \end{cases} \tag{1.8}$$

eq1.8

We first recall a result which is essentially due to [21].

lem1.1 **Lemma 1.1.** *Assume $N \geq 3$, $\mu_i > 0$, $\beta_{ij} > 0$, $p_{ij} + q_{ij} = 2^*$, and $q_{ij} > 0$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. Let $(u_1, u_2, \cdots, u_n)$ be a weak solution of (1.8). Then $(u_1, u_2, \cdots, u_n)$ is a synchronized solution of (1.3).*

Remark 1.13. Lemma 1.1 under the additional assumptions $p_{ij} \geq 1$ and $n = 2$ is proved in [21]; see [21, Theorems 3.1 and 3.2] and their proofs. The same methods yield the result in its present generality. By Lemma 1.1, Theorem 1.2(d) is a consequence of Theorem 1.2(b) and Theorem 1.3(c) is a consequence of Theorem 1.3(a). The reason we still keep Theorem 1.2(d) and Theorem 1.3(c) is that their proofs given in Sections 3 and 4 work for more general systems, say for subcritical systems like (1.7) (see the results in Section 7), while the methods in [21] for proving Lemma 1.1 is not applicable to subcritical systems.

As a direct consequence of Theorems 1.2-1.5 combined with Lemma 1.1, we have the following result on exact number of solutions of (1.8).

thm1.7 **Theorem 1.7.** *Assume $N \geq 3$, $\mu_i > 0$, $p_{ij} = p$, $q_{ij} = 2^* - p$ and $\beta_{ij} = \beta$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. We have the following conclusions.*

- (a) *Let $p < 2$. If either $\beta \geq \mu_n$ or $\beta \geq \mu_n - \delta_0$ in the case $\mu_{n-1} < \mu_n$ where δ_0 is some positive number, then (1.8) has exactly one solution.*
- (b) *Let $p < 2$. Then there exists $\beta_0 \in (0, \mu_1)$ such that (1.8) has exactly $2^n - 1$ solutions for $0 < \beta < \beta_0$.*
- (c) *Let $p = 2$. If $\beta > 0$ and $\beta \notin [\mu_1, \mu_n]$ then (1.8) has exactly one solution.*
- (d) *Let $2 < p < 2^*$. If either $0 < \beta \leq \mu_1$, or $0 < \beta \leq \mu_1 + \delta_0$ in the case $\mu_1 < \mu_2$ where δ_0 is some positive number, or $\beta > \mu_1$ and $\frac{\mu_1}{\beta} 2 + \left(1 - \frac{\mu_1}{\beta}\right) 2^* \leq p < 2^*$, then (1.8) has exactly one solution.*
- (e) *Let $n = 2$ and $p < 2$.*
  - (e_1) *If $\mu := \mu_1 = \mu_2$, then (1.8) has exactly three solutions for $0 < \beta < \beta_0$ and exactly one solution for $\beta \geq \beta_0$, where $\beta_0 = \frac{2\mu}{2(N-1)-(N-2)p}$.*
  - (e_2) *If $\mu_1 < \mu_2$, then there exists $\beta_0 \in (0, \mu_1)$ such that (1.8) has exactly three solutions for $0 < \beta < \beta_0$, exactly two solutions for $\beta = \beta_0$, and exactly one solution for $\beta > \beta_0$.*

Remark 1.14. The systems considered in this paper are in the cooperative case, that is $\beta_{ij} > 0$. Systems with critical exponent in the repulsive case, that is $\beta_{ij} < 0$, have also been studied; see, for example, [19, 20] for radial and nonradial bifurcation results.

The paper is organized as follows. We prove Theorems 1.1, 1.2, 1.3, 1.4 and 1.5 in Sections 2, 3, 4, 5 and 6, respectively. We shall give in Section 7 results on (1.7) similar as Theorems 1.1-1.4. In Section 8 we study (1.5) and prove Theorem 1.6.

2. Proof of Theorem 1.1

In this section, we assume $N \geq 3$, $\mu_i > 0$, $p_{ij} < 2$, and $q_{ij} = 2^* - p_{ij}$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. We prove Theorem 1.1 using the Brouwer degree theory. Elliptic system (1.3) has a synchronized positive solution of the form

$$(k_1 U, k_2 U, \cdots, k_n U)$$

if and only if $(k_1, k_2, \cdots, k_n)$ is a positive solution of the algebraic system

$$f_i(k_1, k_2, \cdots, k_n) := \mu_i k_i^{2^*-2} + \sum_{j=1,\, j\neq i}^{n} \beta_{ij} k_i^{p_{ij}-2} k_j^{q_{ij}} - 1 = 0, \quad i = 1, 2, \cdots, n. \tag{2.1}$$

eq2.1

For a vector solution $k = (k_1, k_2, \cdots, k_n)$ of (2.1), we call it a positive solution if $k_i > 0$ for all i. We denote $f = (f_1, f_2, \cdots, f_n)$. We shall first prove Theorem 1.1(a), showing that there exists $\beta_* > 0$ such that if $0 < \beta_{ij} < \beta_*$ then the above algebraic system has at least $2^n - 1$ positive solutions. The idea is to find $2^n - 1$ mutually disjoint n-dimensional cuboids in $(0, +\infty)^n$ so that the Brouwer degree of f on each cuboid is nonzero and thus the algebraic system (2.1) has a solution in each of these n-dimensional cuboids.

Proof of Theorem 1.1(a). First note that

$$f_i\Big(k_1, \cdots, k_{i-1}, \Big(\frac{1}{2\mu_i}\Big)^{\frac{N-2}{4}}, k_{i+1}, \cdots, k_n\Big) = -\frac{1}{2} + \sum_{j=1,\, j\neq i}^{n} \beta_{ij} k_j^{q_{ij}} \Big(\frac{1}{2\mu_i}\Big)^{\frac{N-2}{4}(p_{ij}-2)}$$

and

$$f_i\Big(k_1, \cdots, k_{i-1}, \Big(\frac{1}{\mu_i}\Big)^{\frac{N-2}{4}}, k_{i+1}, \cdots, k_n\Big) = \sum_{j=1,\, j\neq i}^{n} \beta_{ij} k_j^{q_{ij}} \Big(\frac{1}{\mu_i}\Big)^{\frac{N-2}{4}(p_{ij}-2)}.$$

Define

$$\beta_* = \frac{1}{2} \min_{1\le i\le n} \Big[\sum_{j=1,\, j\neq i}^{n} \Big(\frac{1}{\mu_j}\Big)^{\frac{N-2}{4}q_{ij}} \Big(\frac{1}{2\mu_i}\Big)^{\frac{N-2}{4}(p_{ij}-2)}\Big]^{-1}.$$

From the above observation we see that if $0 < \beta_{ij} < \beta_*$ then

$$f_i\Big(k_1, \cdots, \Big(\frac{1}{2\mu_i}\Big)^{\frac{N-2}{4}}, \cdots, k_n\Big) < 0 < f_i\Big(k_1, \cdots, \Big(\frac{1}{\mu_i}\Big)^{\frac{N-2}{4}}, \cdots, k_n\Big) \tag{2.2}$$

eq2.2

for all $k_j \in (0,\ (\frac{1}{\mu_j})^{\frac{N-2}{4}}]$ with $j \neq i$ and all $i = 1, 2, \cdots, n$. This implies the Brouwer degree

$$\deg(f, \Omega, 0) = 1,$$

where Ω is an n-dimensional cuboid defined as

$$\Omega := \prod_{i=1}^{n} \Big(\Big(\frac{1}{2\mu_i}\Big)^{\frac{N-2}{4}},\ \Big(\frac{1}{\mu_i}\Big)^{\frac{N-2}{4}}\Big).$$

Thus $f(k) = 0$ has a solution in Ω. In the following we assume that $0 < \beta_{ij} < \beta_*$ for $i \neq j$.

Let $1 \le \nu \le n-1$. There are $C_n^\nu = \frac{n!}{\nu!(n-\nu)!}$ different ways to decompose the index set $I := \{1, 2, \cdots, n\}$ into two disjoint nonempty subsets I_1 and I_2 so that the first subset I_1 has ν indices. For each of these decompositions, we prove that the algebraic system (2.1) has a solution $(k_1, k_2, \cdots, k_n)$ so that $k_i \in (0,\ (\frac{1}{2\mu_i})^{\frac{N-2}{4}})$ if $i \in I_1$ while $k_i \in ((\frac{1}{2\mu_i})^{\frac{N-2}{4}},\ (\frac{1}{\mu_i})^{\frac{N-2}{4}})$ if $i \in I_2$. If this is the case, then the algebraic system (2.1) has

$$1 + C_n^1 + C_n^2 + \cdots + C_n^{n-1} = 2^n - 1$$

positive solutions. Therefore system (1.3) has at least $2^n - 1$ synchronized positive solutions.

Without loss of generality, we assume that $I_1 = \{1, \cdots, \nu\}$ and $I_2 = \{\nu+1, \cdots, n\}$ and we prove that the system has a solution in the n-dimensional cuboid

$$\Omega_1 := \prod_{i=1}^{\nu} \Big(\epsilon,\ \Big(\frac{1}{2\mu_i}\Big)^{\frac{N-2}{4}}\Big) \times \prod_{i=\nu+1}^{n} \Big(\Big(\frac{1}{2\mu_i}\Big)^{\frac{N-2}{4}},\ \Big(\frac{1}{\mu_i}\Big)^{\frac{N-2}{4}}\Big),$$

for some $\epsilon > 0$ which will be specified next. Let $k_i \in [\epsilon, \ (\frac{1}{2\mu_i})^{\frac{N-2}{4}}]$ for $i = 1, 2, \cdots, \nu$ and $k_i \in [(\frac{1}{2\mu_i})^{\frac{N-2}{4}}, (\frac{1}{\mu_i})^{\frac{N-2}{4}}]$ for $i = \nu+1, \cdots, n$. If $1 \le i \le \nu$ then

$$f_i\Big(k_1, \cdots, k_{i-1}, \epsilon, k_{i+1}, \cdots, k_n\Big) = \mu_i \epsilon^{2^*-2} + \sum_{j=1,\, j\neq i}^{n} \beta_{ij} k_j^{q_{ij}} \epsilon^{p_{ij}-2} - 1$$
$$> \beta_{in}\Big(\frac{1}{2\mu_n}\Big)^{\frac{N-2}{4}q_{in}} \epsilon^{p_{in}-2} - 1 > 0 \tag{2.3}$$

eq2.3

for $\epsilon > 0$ sufficiently small, since $k_n \ge (\frac{1}{2\mu_n})^{\frac{N-2}{4}}$ and $p_{in} < 2$. Using (2.2) and (2.3), we see that

$$\deg(f, \Omega_1, 0) = (-1)^\nu.$$

This implies that $f(k) = 0$ has a solution in Ω_1. □

Now we turn to prove Theorem 1.1(b). In this case, it is not possible to find a positive solution of (2.1) in any of the n-dimensional cuboids constructed above. Indeed, it seems to be impossible to find an n-dimensional cuboid on which f itself has a nonzero degree. The idea to prove Theorem 1.1(b) is that we use the inverse matrix A of $B = (\beta_{ij})_{n\times n}$ to convert system (2.1) into a new system $g(k) = 0$ so that an n-dimensional cuboid on which g has a nonzero Brouwer degree can be constructed.

Proof of Theorem 1.1(b). Let $q = 2^* - p$. Then $q_{ij} = q$ for $i \neq j$. Using the inverse matrix $A = (a_{ij})_{n\times n}$ of $B = (\beta_{ij})_{n\times n}$, we write system (2.1) as

$$(k_1^q, k_2^q, \cdots, k_n^q)^T = A(k_1^{2-p}, k_2^{2-p}, \cdots, k_n^{2-p})^T,$$

where $(c_1, c_2, \cdots, c_n)^T$ means the transpose of a vector $(c_1, c_2, \cdots, c_n)$. Define

$$g_i(k_1, k_2, \cdots, k_n) := k_i^q - \sum_{j=1}^{n} a_{ij} k_j^{2-p}, \quad i = 1, 2, \cdots, n.$$

Then we convert (2.1) into

$$g_i(k_1, k_2, \cdots, k_n) = 0, \quad i = 1, 2, \cdots, n. \tag{2.4}$$

eq2.4

Since $q = 2^* - p > 2 - p$, we can choose $T > 0$ sufficiently large such that, for all $k_j \in (0, T]$ with $j \neq i$,

$$g_i(k_1, \cdots, k_{i-1}, T, k_{i+1}, \cdots, k_n) \ge T^q - \Big(\sum_{j=1}^{n} a_{ij}\Big) T^{2-p} > 0.$$

Let $\varepsilon \in (0, T)$. Since $p < 2$, $a_{ij} > 0$ for $i \neq j$, and $\sum_{j=1}^n a_{ij} > 0$ for $i = 1, 2, \cdots, n$, we have

$$g_i(k_1, \cdots, k_{i-1}, \varepsilon, k_{i+1}, \cdots, k_n) \le \varepsilon^q - \Big(\sum_{j=1}^{n} a_{ij}\Big) \varepsilon^{2-p} < 0,$$

for all $k_j \in [\varepsilon, T]$ with $j \neq i$, provided that ε is small enough. Set $g = (g_1, \cdots, g_n)$. Then

$$\deg(g,\ (\varepsilon, T)^n, 0) = 1.$$

Therefore, (2.4) has a solution in $(\varepsilon, T)^n$. □

In the following two propositions, we provide some examples to illustrate nature of the condition on the matrix B in Theorem 1.1(b). The first proposition is for any n but only for β_{ij} close to a single number, while the second proposition is for more flexible β_{ij} which may be close to different numbers but only for $n = 3, 4$. One may give more examples in this direction.

prop2.1 **Proposition 2.1.** *Let β be a number such that $\beta > \mu_n$. There exists $\delta_0 > 0$ such that if $|\beta_{ij} - \beta| < \delta_0$ for $i \neq j$ then the matrix $B = (\beta_{ij})_{n\times n}$ has an inverse $A = (a_{ij})_{n\times n}$ such that*

$$a_{ij} > 0 \text{ for } i \neq j \text{ and } \sum_{j=1}^{n} a_{ij} > 0 \text{ for } i = 1, 2, \cdots, n. \tag{2.5}$$

eq2.5

Proof. Replace β_{ij} $(i \neq j)$ in B with β and denote the new matrix by B^*. Since $\beta > \mu_n$, we have

$$\Delta_n(\mu_1, \mu_2, \cdots, \mu_n) := \det(B^*) = \prod_{j=1}^{n}(\mu_j - \beta)\Big(1 + \sum_{k=1}^{n} \frac{\beta}{\mu_k - \beta}\Big) \neq 0.$$

Then B^* has an inverse matrix $A^* = (a^*_{ij})_{n\times n}$ and a direct computation shows that

$$a^*_{ij} = \begin{cases} \dfrac{-\beta}{\Delta_n(\mu_1, \mu_2, \cdots, \mu_n)} \displaystyle\prod_{k=1,\, k\neq i,j}^{n} (\mu_k - \beta) & \text{if } i \neq j, \\ \dfrac{\Delta_{n-1}(\mu_1, \cdots, \mu_{i-1}, \mu_{i+1}, \cdots, \mu_n)}{\Delta_n(\mu_1, \mu_2, \cdots, \mu_n)} & \text{if } i = j. \end{cases}$$

Here, the product $\prod_{k=1,\, k\neq i,j}^{n}(\mu_k - \beta)$ is understood to be 1 if $n = 2$. By this expression, we have $a^*_{ij} > 0$ for $i \neq j$ and

$$\sum_{j=1}^{n} a^*_{ij} = \frac{\prod_{j=1, j\neq i}^{n}(\beta - \mu_j)}{|\Delta_n(\mu_1, \mu_2, \cdots, \mu_n)|} > 0,$$

for $i = 1, 2, \cdots, n$. The result follows by continuity. □

The proof of Proposition 2.1 also shows that if all β_{ij}'s $(i \neq j)$ are a single β and if $0 < \beta < \mu_1$ then the elements off the main diagonal of the inverse matrix A of B are all negative; this is in sharp contrast with the case $\beta > \mu_n$.

The formula for $\sum_{j=1}^{n} a^*_{ij}$ shows that $\sum_{j=1}^{n} a^*_{ij} \simeq \frac{1}{\beta}$ for β sufficiently large and for all $i = 1, 2, \cdots, n$. By the proof of Theorem 1.1(b), for any ε and T such that

$$0 < \varepsilon < \left(\min_{1\leq i\leq n} \sum_{j=1}^{n} a_{ij}\right)^{\frac{N-2}{4}} \leq \left(\max_{1\leq i\leq n} \sum_{j=1}^{n} a_{ij}\right)^{\frac{N-2}{4}} < T,$$

we have

$$g_i(k_1, \cdots, k_{i-1}, \varepsilon, k_{i+1}, \cdots, k_n) < 0 < g_i(k_1, \cdots, k_{i-1}, T, k_{i+1}, \cdots, k_n).$$

For β_{ij} close to β with β being sufficiently large, since $\sum_{j=1}^{n} a_{ij} \simeq \frac{1}{\beta}$ for all i, if $(k_1, k_2 \cdots, k_n)$ is any positive solution of the equation $g(k_1, k_2, \cdots, k_n) = 0$ then it must be that $k_i \simeq \beta^{-\frac{N-2}{4}}$ for all i. This justifies the statement before the proof of Theorem 1.1(b) that under the assumptions of Theorem 1.1(b), it is impossible to obtain a solution $(k_1, k_2 \cdots, k_n)$ of the equation $f(k_1, k_2, \cdots, k_n) = 0$ in any cuboids constructed in the proof of Theorem 1.1(a).

prop2.2 **Proposition 2.2.** *A matrix which is sufficiently close to any of the three matrices*

(a)
$$B = \begin{pmatrix} \mu_1 & \beta_1 & \beta_1 \\ \beta_1 & \mu_2 & \beta_2 \\ \beta_1 & \beta_2 & \mu_3 \end{pmatrix} \quad \textit{with} \begin{cases} \beta_1 > \max\{\mu_1, \beta_2\}, \\ \beta_2 \geq \mu_2 + \mu_3, \end{cases}$$

(b)
$$B = \begin{pmatrix} \mu_1 & \beta_1 & \beta_1 & \beta_1 \\ \beta_1 & \mu_2 & \beta_2 & \beta_2 \\ \beta_1 & \beta_2 & \mu_3 & \beta_2 \\ \beta_1 & \beta_2 & \beta_2 & \mu_4 \end{pmatrix} \quad \textit{with} \begin{cases} \beta_1 > \max\{\mu_1, \beta_2\}, \\ \beta_2 > \max\{\mu_2, \mu_3, \mu_4\}, \end{cases}$$

(c)
$$B = \begin{pmatrix} \mu_1 & \beta_2 & \beta_1 & \beta_1 \\ \beta_2 & \mu_2 & \beta_1 & \beta_1 \\ \beta_1 & \beta_1 & \mu_3 & \beta_3 \\ \beta_1 & \beta_1 & \beta_3 & \mu_4 \end{pmatrix} \quad \textit{with} \begin{cases} \beta_1 \geq \max\{\beta_2, \beta_3\}, \\ \beta_2 \geq \mu_1 + \mu_2, \\ \beta_3 \geq \mu_3 + \mu_4 \end{cases}$$

has an inverse $A = (a_{ij})_{n\times n}$ *such that*

$$a_{ij} > 0 \textit{ for } i \neq j \textit{ and } \sum_j a_{ij} > 0 \textit{ for all } i.$$

Proof. By elementary but tedious computations, it can be verified that a matrix B in any of the three forms (a), (b), and (c) has the required property. Then its small perturbation has the same property as continuity shows. □

From Proposition 2.1 we see that if all β_{ij}'s $(i \neq j)$ are a single β, then the inverse A of B exists and satisfies (2.5) for β suitably large. But if β_{ij}'s $(i \neq j)$ are not the same, except being suitably large they generally have to satisfy additional structural conditions to guarantee that the inverse A of B exists and satisfies (2.5); this is illustrated in Proposition 2.2.

Let the structural conditions for the three matrices B considered in Proposition 2.2 be satisfied. It can be proved that if β_1 is sufficiently large with all the other entries being fixed then $\sum_{j=1}^{n} a_{ij} \simeq \frac{1}{\beta_1}$ for all i. Therefore, for the matrices B considered in Proposition 2.2 with β_1 being sufficiently large, any solution of the equation $f(k_1, k_2, \cdots, k_n) = 0$ cannot be obtained in any cuboid constructed as in the proof of Theorem 1.1(a).

3. Proof of Theorem 1.2

In this section we always assume that $N \geq 3$, $\mu_i > 0$, $p_{ij} = p < 2$, $q_{ij} = 2^* - p$ and $\beta_{ij} = \beta$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. Denote $q = q_{ij} = 2^* - p$. Note that $(k_1U, k_2U, \cdots, k_nU)$ is a positive solution of (1.3) if and only if $k_i > 0$ and $(k_1, k_2, \cdots, k_n)$ is a solution of the system

$$\mu_i k_i^{2^*-1} + \beta k_i^{p-1} \sum_{j=1, j\neq i}^{n} k_j^q = k_i, \quad i = 1, 2, \cdots, n.$$

Letting $t_i = k_i^q$, in order to study the number of synchronized positive solutions of (1.3) we are led to study the number of solutions $(t_1, t_2, \cdots, t_n)$ with $t_i > 0$ of the nonlinear algebraic system

$$\mu_i t_i + \beta \sum_{j=1,\, j\neq i}^{n} t_j = t_i^{\frac{2-p}{q}}, \quad i = 1, 2, \cdots, n. \tag{3.1}$$ eq3.1

Indeed, the number of synchronized positive solutions of (1.3) is equal to the number of positive solutions $(t_1, t_2, \cdots, t_n)$ of (3.1).

To study the number of positive solutions of system (3.1), we convert it into another algebraic system with extended number of equations. For simplicity of notation, we denote

$$\alpha := \frac{2-p}{q} = \frac{2-p}{2^*-p} = \frac{q+2-2^*}{q}.$$

Then $\alpha \in (0, 1)$ and $(t_1, t_2, \cdots, t_n)$ is a positive solution of (3.1) if and only if, for some $\tau > 0$, $(t_1, t_2, \cdots, t_n, \tau)$ is a positive solution of the nonlinear expanded algebraic system with $n+1$ equations

$$\begin{cases} t_i^\alpha + (\beta - \mu_i)t_i = \tau, \quad i = 1, 2, \cdots, n, \\ \beta \sum_{i=1}^{n} t_i = \tau. \end{cases} \tag{3.2}$$ eq3.2

In view of the above reasoning, we have proved the following lemma.

lem3.1 **Lemma 3.1.** *The number of synchronized positive solutions of* (1.3) *is equal to the number of positive solutions of* (3.2).

We now show that the number of positive solutions of (3.2) is equal to either the number of positive solutions of a single equation (in the case $\beta \geq \mu_n$) or the number of positive solutions of any one of a group of at most 2^n equations (in the case $\beta < \mu_n$). To facilitate our further discussion, we introduce some notations. Denote

$$f_i(t) = t^\alpha + (\beta - \mu_i)t, \quad i = 1, 2, \cdots, n.$$

If $\beta \geq \mu_n$ then $\beta \geq \mu_i$ for all i since $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. This case is simple. For each i, f_i is strictly increasing from $(0, +\infty)$ onto $(0, +\infty)$ and f_i has an inverse function $h_i : (0, +\infty) \to (0, +\infty)$. Then $f_i(t_i) = t_i^\alpha + (\beta - \mu_i)t_i = \tau$ if and only if $h_i(\tau) = t_i$ and the number of positive solutions of system (3.2) is equal to the number of positive solutions of the single equation

$$\beta \sum_{i=1}^{n} h_i(\tau) = \tau, \quad \tau \in (0, +\infty). \tag{3.3}$$ eq3.3

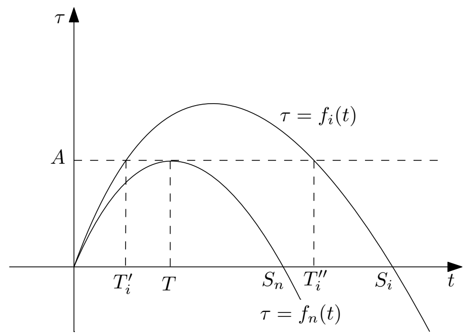


FIGURE 1. The case $p < 2$ and $0 < \beta < \mu_i$

The case $\beta < \mu_n$ is far more complex. In this case f_n achieves its maximum

$$A = A(\beta) := \frac{\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}}{(\mu_n - \beta)^{\frac{\alpha}{1-\alpha}}} = \frac{[((2-p)/q)^{\frac{(N-2)(2-p)}{4}} - ((2-p)/q)^{\frac{(N-2)q}{4}}]}{(\mu_n - \beta)^{\frac{(N-2)(2-p)}{4}}}$$

at

$$T = T(\beta) := \Big(\frac{\alpha}{\mu_n - \beta}\Big)^{\frac{1}{1-\alpha}} = \Big(\frac{2-p}{q(\mu_n - \beta)}\Big)^{\frac{(N-2)q}{4}},$$

f_n is strictly increasing in $(0, T]$ and strictly decreasing in $[T, +\infty)$. For each i, there exists uniquely a number T'_i such that

$$0 < T'_i \le T, \quad f_i(T'_i) = A,$$

and that $f_i|_{(0,T'_i]}$ is strictly increasing from $(0, T'_i]$ onto $(0, A]$. We denote the inverse function of $f_i|_{(0,T'_i]}$ by $h_i : (0, A] \to (0, T'_i]$. Note that we have given h_i different meanings, which shall incur no confusion. We shall use other symbols like this. If $\beta < \mu_i$ for some i then there exists uniquely a second number T''_i such that

$$T \le T''_i, \quad f_i(T''_i) = A,$$

and that $f_i|_{[T''_i,S_i)}$ is strictly decreasing from $[T''_i, S_i)$ onto $(0, A]$, where

$$S_i := \Big(\frac{1}{\mu_i - \beta}\Big)^{\frac{1}{1-\alpha}} = \Big(\frac{1}{\mu_i - \beta}\Big)^{\frac{(N-2)q}{4}}.$$

In this case we denote the inverse function of $f_i|_{[T''_i,S_i)}$ by $k_i : (0, A] \to [T''_i, S_i)$. Clearly, $T'_n = T''_n = T$, each h_i $(i = 1, 2, \cdots, n)$ is well defined, but k_i is well defined if and only if $\beta < \mu_i$. For $0 < \beta < \mu_i$, the graphs of f_n and f_i are illustrated in Figure 1.

Let $j \ge 0$ be the smallest integer such that $\beta < \mu_{j+1}$ and k the smallest integer such that $\mu_{k+1} = \cdots = \mu_n$ with $k \ge j$. Let $(t_1, t_2, \cdots, t_n, \tau)$ be a positive solution of (3.2). Then

$$0 < \tau = f_n(t_n) \le \max_{t \ge 0} f_n(t) = A,$$

$t_i = h_i(\tau) \in (0, T'_i]$ for $i = 1, 2, \cdots, j$, and either $t_i = h_i(\tau) \in (0, T'_i]$ or $t_i = k_i(\tau) \in [T''_i, S_i)$ for $i = j+1, j+2, \cdots, n$. Note that for $j + 1 \le i \le k$ and $\tau \in (0, A)$

$$h_i(\tau) < h_i(A) = T'_i < T < T''_i = k_i(A) < k_i(\tau),$$

and for $k + 1 \le i \le n$ and $\tau \in (0, A)$

$$h_i(\tau) < h_i(A) = T'_i = T = T''_i = k_i(A) < k_i(\tau).$$

The index set $\{j + 1, j + 2, \cdots, k\}$ has 2^{k-j} subsets which we mark as $J_1, J_2, \cdots, J_{2^{k-j}}$. We understand that this index set is the empty set if $k = j$. Let σ^* be the number of J_s for which the equality

$$\beta \sum_{i \in I \setminus J_s} h_i(A) + \beta \sum_{i \in J_s} k_i(A) = A \tag{3.4}$$

eq3.4

holds. Recall that $I = \{1, 2, \cdots, n\}$. Then σ^* is the number of positive solutions $(t_1, \cdots, t_n, \tau)$ of system (3.2) with $\tau = A$. We label with $I_1, I_2, \cdots, I_{2^{n-j}}$ the 2^{n-j} subsets of the index set $\{j+1, j+2, \cdots, n\}$. For $s = 1, 2, \cdots, 2^{n-j}$, let σ_s be the number of solutions of the equation

$$\beta \sum_{i \in I \setminus I_s} h_i(\tau) + \beta \sum_{i \in I_s} k_i(\tau) = \tau, \quad \tau \in (0, A). \tag{3.5}$$

eq3.5

Set $\sigma^{**} = \sum_{s=1}^{2^{n-j}} \sigma_s$. The number of positive solutions $(t_1, \cdots, t_n, \tau)$ of system (3.2) with $\tau \in (0, A)$ equals σ^{**}.

We summarize the above conclusions in the next lemma.

lem3.2 **Lemma 3.2.** (a) *If $\beta \geq \mu_n$ then the number of positive solutions of system (3.2) is equal to the number of solutions of the single equation (3.3).*

(b) *If $\beta < \mu_n$ and $j \geq 0$ is the smallest integer such that $\beta < \mu_{j+1}$ then the number of positive solutions of system (3.2) equals $\sigma^* + \sigma^{**}$, with σ^* and σ^{**} being defined as above via equations (3.4) and (3.5).*

We have seen that to study the number of positive solutions of system (3.2) is equivalent to study the number of positive solutions of (3.3) (if $\beta \geq \mu_n$) or to study the number of positive solutions of (3.5) plus the number of J_s satisfying (3.4) (if $\beta < \mu_n$). We shall fulfil this task with a series of lemmas in what follows and prove at appropriate stages Theorem 1.2(a), (b), (c), and (d), respectively.

lem3.3 **Lemma 3.3.** *If $\beta \geq \mu_n$ then (3.2) has a unique positive solution.*

Proof. Since $\beta \geq \mu_n$, by Lemma 3.2(a), the number of positive solutions of system (3.2) equals the number of solutions of the single equation (3.3). Since $0 < \alpha < 1$ and $\lim_{\tau \to 0^+} \frac{h_i(\tau)}{\tau^{1/\alpha}} = 1$, we have $\beta \sum_{i=1}^n h_i(\tau) < \tau$ for $\tau > 0$ sufficiently small. If $\beta > \mu_n$ then $\lim_{\tau \to +\infty} \frac{h_n(\tau)}{\tau} = \frac{1}{\beta - \mu_n}$, and if $\beta = \mu_n$ then $\lim_{\tau \to +\infty} \frac{h_n(\tau)}{\tau} = \lim_{\tau \to +\infty} \tau^{\frac{1}{\alpha} - 1} = +\infty$. In either case we have $\beta \sum_{i=1}^n h_i(\tau) > \tau$ for $\tau > 0$ sufficiently large. Then we see that (3.3) has a solution. Set $G_1(\tau) := \beta \sum_{i=1}^n h_i(\tau) - \tau$. Since for $\tau \in (0, +\infty)$,

$$G_1'(\tau) = \beta \sum_{i=1}^n h_i'(\tau) - 1 = \beta \sum_{i=1}^n \frac{1}{\alpha h_i^{\alpha-1}(\tau) + \beta - \mu_i} - 1$$

and $h_i^\alpha(\tau) + (\beta - \mu_i) h_i(\tau) = \tau$, we see that

$$G_1'(\tau) > \beta \sum_{i=1}^n \frac{1}{h_i^{\alpha-1}(\tau) + \beta - \mu_i} - 1 = \frac{\beta}{\tau} \sum_{i=1}^n h_i(\tau) - 1 = \frac{1}{\tau} G_1(\tau).$$

This implies $G_1'(\tau) > 0$ when $G_1(\tau) = 0$. Therefore (3.3) has a unique solution and (3.2) has a unique positive solution. □

lem3.4 **Lemma 3.4.** *For any $\beta > 0$, (3.2) has a positive solution.*

Proof. If $\beta \geq \mu_n$ then the result follows from Lemma 3.3. Now we assume $0 < \beta < \mu_n$. Then all the functions $h_i : (0, A] \to (0, T_i']$, $i = 1, 2, \cdots, n$, are defined. Set $G_1(\tau) := \beta \sum_{i=1}^n h_i(\tau) - \tau$ for $\tau \in (0, A]$. If $G_1(A) = 0$ then we are done since $\sigma^* \geq 1$. If $G_1(A) > 0$ then the equation $G_1(\tau) = 0$ has a solution in $(0, A)$ since $G_1(\tau) < 0$ for $\tau > 0$ sufficiently small as seen in the proof of Lemma 3.3. In the remaining case $G_1(A) < 0$, we set $G_2(\tau) := \beta \sum_{i=1}^{n-1} h_i(\tau) + \beta k_n(\tau) - \tau$ for $\tau \in (0, A]$. Since

$$G_2(A) = \beta \sum_{i=1}^{n-1} h_i(A) + \beta k_n(A) - A = \beta \sum_{i=1}^n h_i(A) - A = G_1(A) < 0$$

and since

$$\lim_{\tau \to 0^+} G_2(\tau) = \beta S_n > 0,$$

the equation $G_2(\tau) = 0$ has a solution in $(0, A)$. This shows that if $G_1(A) \neq 0$ then for at least one s, (3.5) has a solution. In summary, (3.2) has a positive solution for any $\beta > 0$. □

Proof of Theorem 1.2(a). The result follows from Lemmas 3.1 and 3.4. □

The next lemma shows that the region of β in which (3.2) has a unique positive solution as concluded in Lemma 3.3 can be enlarged if $\mu_{n-1} < \mu_n$.

lem3.5 **Lemma 3.5.** *Assume* $\mu_{n-1} < \beta < \mu_n$ *and*

$$\frac{1}{\beta} + \frac{(N-2)q}{4(\mu_n - \beta)} \geq \sum_{i=1}^{n-1} \frac{1}{\mu_n - \mu_i}. \tag{3.6}$$ eq3.6

Then (3.2) *has a unique positive solution.*

Proof. Let $G_1(\tau)$ and $G_2(\tau)$ be defined as in Lemma 3.4. As in the proof of Lemma 3.3, we have

$$G_1'(\tau) > \frac{1}{\tau} G_1(\tau), \quad \tau \in (0, A). \tag{3.7}$$ eq3.7

Note that

$$G_2'(\tau) = \sum_{i=1}^{n-1} \frac{\beta}{\alpha h_i^{\alpha-1}(\tau) + \beta - \mu_i} + \frac{\beta}{\alpha k_n^{\alpha-1}(\tau) + \beta - \mu_n} - 1, \quad \tau \in (0, A),$$

and that $\alpha h_i^{\alpha-1}(\tau) + \beta - \mu_i > 0$ and $\alpha k_n^{\alpha-1}(\tau) + \beta - \mu_n < 0$. For $\tau \in (0, A)$, since $h_i(\tau) < h_n(\tau) < T$ for $1 \leq i \leq n-1$ and $k_n^{\alpha-1}(\tau) > \mu_n - \beta$, and since $\alpha T^{\alpha-1} + \beta = \mu_n$, using the assumption we see that

$$G_2'(\tau) < \sum_{i=1}^{n-1} \frac{\beta}{\mu_n - \mu_i} - \frac{\beta}{(1-\alpha)(\mu_n - \beta)} - 1 \leq 0. \tag{3.8}$$ eq3.8

Since $\mu_{n-1} < \beta < \mu_n$ there are only two equations in the form of (3.5): $G_1(\tau) = 0$ and $G_2(\tau) = 0$, $\tau \in (0, A)$. Clearly $G_1(A) = G_2(A)$. If $G_1(A) = G_2(A) = 0$ then $\sigma^* = 1$ and by (3.7) and (3.8) the two equations $G_1(\tau) = 0$ and $G_2(\tau) = 0$ have no solution in $(0, A)$. If $G_1(A) = G_2(A) > 0$ then $\sigma^* = 0$ and by (3.7) and (3.8) again the equation $G_1(\tau) = 0$ has a unique solution in $(0, A)$ while $G_2(\tau) = 0$ has no solution in $(0, A)$. If $G_1(A) = G_2(A) < 0$ then $\sigma^* = 0$ and by (3.7) and (3.8) once more the equation $G_1(\tau) = 0$ has no solution in $(0, A)$ while $G_2(\tau) = 0$ has a unique solution in $(0, A)$. This implies, by Lemma 3.2(b), (3.2) has a unique positive solution. □

Note that assumption (3.6) holds if $\beta < \mu_n$ and β is sufficiently close to μ_n, and in particular if

$$\mu_n - \frac{(N-2)q}{4} \left(\sum_{i=1}^{n-1} (\mu_n - \mu_i)^{-1} \right)^{-1} \leq \beta < \mu_n.$$

Proof of Theorem 1.2(b). The result follows from Lemmas 3.1, 3.3 and 3.5. □

The next lemma gives a multiplicity result on positive solutions of (3.2).

lem3.6 **Lemma 3.6.** *If* $\beta < \mu_1$ *and*

$$\beta(\mu_n - \beta)^{\frac{(N-2)(2-p)}{4}} \sum_{i=1}^{n} (\mu_i - \beta)^{-\frac{(N-2)q}{4}} \leq \left(\frac{2-p}{q}\right)^{\frac{(N-2)(2-p)}{4}} - \left(\frac{2-p}{q}\right)^{\frac{(N-2)q}{4}}, \tag{3.9}$$ eq3.9

then (3.2) *has at least* $2^n - 1$ *positive solutions.*

Proof. For the 2^n subsets $I_1, I_2, \cdots, I_{2^n}$ of the index set $I = \{1, 2, \cdots, n\}$, we assume $I_1 = \emptyset$ so that $I_s \neq \emptyset$ for $s \neq 1$. For $s = 1, 2, \cdots, 2^n$, define

$$G_s(\tau) = \beta \sum_{i \in I \setminus I_s} h_i(\tau) + \beta \sum_{i \in I_s} k_i(\tau) - \tau, \quad \tau \in (0, A].$$

Since $\beta < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, for $\tau \in (0, A]$ we have $h_1(\tau) \leq h_2(\tau) \leq \cdots \leq h_n(\tau) \leq T$ and $k_1(\tau) \geq k_2(\tau) \geq \cdots \geq k_n(\tau) \geq T$. Then for any s, by (3.9),

$$G_s(A) \leq \beta \sum_{i=1}^{n} k_i(A) - A < \beta \sum_{i=1}^{n} S_i - A \leq 0.$$

For $2 \leq s \leq 2^n$, since $I_s \neq \emptyset$ we have

$$\lim_{\tau \to 0^+} G_s(\tau) = \beta \sum_{i \in I_s} S_i > 0.$$

Therefore for $s = 2, 3, \cdots, 2^n$, $G_s(\tau) = 0$ has a solution in $(0, A)$. That is, for $s = 2, 3, \cdots, 2^n$, (3.5) has a solution and $\sigma_s \geq 1$. This implies that (3.2) has at least $2^n - 1$ positive solutions, since for $s_1 \neq s_2$, the two equations $G_{s_1}(\tau) = 0$ and $G_{s_2}(\tau) = 0$ give different solutions of (3.2). □

Assumption (3.9) holds for $\beta > 0$ small enough. In the following two lemmas, we give two different types of conditions both of which guarantee that (3.2) has exactly $2^n - 1$ positive solutions.

lem3.7 **Lemma 3.7.** *If $\beta < \mu_1$ and there exists $\nu \in (0, 1)$ such that*

$$\beta(\mu_n - \beta)^{\frac{(N-2)q}{4}-1} \sum_{i=1}^{n} (\mu_i - \beta)^{-\frac{(N-2)q}{4}} \leq \nu\Big(\Big(\frac{2-p}{q}\Big)^{\frac{(N-2)q}{4}-1} - \Big(\frac{2-p}{q}\Big)^{\frac{(N-2)q}{4}}\Big) \tag{3.10}$$ eq3.10

and

$$\beta\Big(\frac{n-1}{\nu^{-4/[(N-2)(2-p)]} - 1} - \frac{(N-2)(2-p)}{4}\Big) \leq \mu_n, \tag{3.11}$$ eq3.11

then (3.2) has exactly $2^n - 1$ positive solutions.

Proof. Let G_s be as in the proof of Lemma 3.6. For $\tau \in [\nu A, A]$, by assumption (3.10),

$$\beta \sum_{i=1}^{n} k_i(\tau) < \beta \sum_{i=1}^{n} S_i \leq \nu A \leq \tau.$$

This implies that $G_s(\tau) = 0$ has no solution in $[\nu A, A]$ for any $s = 1, 2, \cdots, 2^n$ and that $G_1(\tau) = \beta \sum_{i=1}^{n} h_i(\tau) - \tau = 0$ has no solution in $(0, A]$ since $G_1(\tau) < 0$ for $\tau > 0$ small, $G_1(A) < 0$, and $G_1'(\tau) > \frac{1}{\tau} G_1(\tau)$ for $\tau \in (0, A)$.

Now assume $\tau \in (0, \nu A)$. For $s = 2, 3, \cdots, 2^n$, $I_s \neq \emptyset$ and

$$G_s'(\tau) = \beta \sum_{i \in I \setminus I_s} \frac{1}{\alpha h_i^{\alpha-1}(\tau) + \beta - \mu_i} + \beta \sum_{i \in I_s} \frac{1}{\alpha k_i^{\alpha-1}(\tau) + \beta - \mu_i} - 1.$$

We have $\alpha h_i^{\alpha-1}(\tau) + \beta - \mu_i \geq \alpha h_n^{\alpha-1}(\tau) + \beta - \mu_n > 0$ for $i \in I \setminus I_s$ and $0 > \alpha k_i^{\alpha-1}(\tau) + \beta - \mu_i > -(1-\alpha)(\mu_n - \beta)$ for $i \in I_s$. Inserting these estimates into the formula for $G_s'(\tau)$ yields

$$G_s'(\tau) < \frac{(n-1)\beta}{\alpha h_n^{\alpha-1}(\tau) + \beta - \mu_n} - \frac{\beta}{(1-\alpha)(\mu_n - \beta)} - 1,$$

since the first summation in the expression of $G_s'(\tau)$ has at most $n-1$ terms. Use $\alpha h_n^{\alpha-1}(\tau) + \beta - \mu_n > 0$ and $h_n^{\alpha}(\tau) + (\beta - \mu_n) h_n(\tau) = \tau$ to deduce that, for $\tau \in (0, \nu A)$, $h_n(\tau) < (\frac{\tau}{1-\alpha})^{1/\alpha} < (\frac{\nu A}{1-\alpha})^{1/\alpha}$. Then, using the expression of A, we have $\alpha h_n^{\alpha-1}(\tau) + \beta - \mu_n > (\nu^{\frac{\alpha-1}{\alpha}} - 1)(\mu_n - \beta) > 0$. By assumption (3.11), we then see that for $\tau \in (0, \nu A)$ and $s = 2, 3, \cdots, 2^n$,

$$G_s'(\tau) < \frac{(n-1)\beta}{(\nu^{\frac{\alpha-1}{\alpha}} - 1)(\mu_n - \beta)} - \frac{\beta}{(1-\alpha)(\mu_n - \beta)} - 1 < 0.$$

Since $G_s(\nu A) < 0$ and $\lim_{\tau \to 0^+} G_s(\tau) > 0$, each equation $G_s(\tau) = 0$ has exactly one solution in $(0, \nu A)$. This implies that (3.2) has exactly $2^n - 1$ positive solution. □

The assumptions of Lemma 3.7 are satisfied for $\beta > 0$ sufficiently small, and in particular, for $\beta > 0$ such that $\beta \leq \frac{1}{2}\mu_1$, $\beta \leq \frac{2^{4/[(N-2)(2-p)]} - 1}{n-1} \mu_n$, and

$$\beta \leq \frac{2}{(N-2)(2-p)} \Big(\frac{2-p}{q}\Big)^{\frac{(N-2)q}{4}} \mu_n^{1-\frac{(N-2)q}{4}} \left(\sum_{i=1}^{n} \Big(\mu_i - \frac{1}{2}\mu_1\Big)^{-\frac{(N-2)q}{4}}\right)^{-1}.$$

lem3.8 **Lemma 3.8.** *If $\beta < \mu_1$,*

$$\beta(\mu_n - \beta)^{\frac{(N-2)q}{4}-1} \sum_{i=1}^{n} (\mu_i - \beta)^{-\frac{(N-2)q}{4}} < \Big(\frac{2-p}{q}\Big)^{\frac{(N-2)q}{4}-1} - \Big(\frac{2-p}{q}\Big)^{\frac{(N-2)q}{4}}, \tag{3.12}$$ eq3.12

and

$$\sum_{i=2}^{n} \frac{\beta}{\chi_i(\beta)} - \frac{(N-2)q\beta}{4(\mu_n - \beta)} \leq 1, \tag{3.13}$$ eq3.13

where

$$\chi_i(\beta) := \frac{2-p}{q}\Big(\frac{4}{(N-2)q}\Big)^{\frac{4}{(N-2)(2-p)}}\Big[\beta\sum_{j=1}^{n}(\mu_j-\beta)^{-\frac{(N-2)q}{4}}\Big]^{-\frac{4}{(N-2)(2-p)}} - (\mu_i-\beta),$$

then (3.2) *has exactly* 2^n-1 *positive solutions.*

Proof. Let G_s be as in Lemma 3.6. If $i \le j$ and $\tau \in (0, A)$ then $\alpha h_i^{\alpha-1}(\tau)+\beta-\mu_i \ge \alpha h_j^{\alpha-1}(\tau)+\beta-\mu_j > 0$ and $-(1-\alpha)(\mu_n-\beta) < \alpha k_i^{\alpha-1}(\tau)+\beta-\mu_i < 0$. This implies, for $s = 2, 3, \cdots, 2^n$ and $\tau \in (0, A)$,

$$G_s'(\tau) < \beta\sum_{i=2}^{n}\frac{1}{\alpha h_i^{\alpha-1}(\tau)+\beta-\mu_i} - \frac{\beta}{(1-\alpha)(\mu_n-\beta)} - 1.$$

For any s, by assumption (3.12),

$$G_s(A) \le \beta\sum_{i=1}^{n}k_i(A) - A = \beta\sum_{i=1}^{n}T_i'' - A < \beta\sum_{i=1}^{n}S_i - A < 0.$$

This implies in particular that $G_1(\tau) = 0$ has no solution in $(0, A]$ and for $s = 2, 3, \cdots, 2^n$, $G_s(\tau) = 0$ has a solution in $(0, A)$. For $s = 2, 3, \cdots, 2^n$, if $\tau \in (0, A)$ is a solution of $G_s(\tau) = 0$ then

$$\tau = \beta\sum_{i\in I\setminus I_s}h_i(\tau) + \beta\sum_{i\in I_s}k_i(\tau) < \beta\sum_{i=1}^{n}S_i = \beta\sum_{i=1}^{n}(\mu_i-\beta)^{-\frac{1}{1-\alpha}}.$$

Since $\alpha h_i^{\alpha-1}(\tau) - (\mu_i-\beta) > 0$ and $h_i^{\alpha}(\tau) - (\mu_i-\beta)h_i(\tau) = \tau$, we have

$$h_i(\tau) < \Big(\frac{\tau}{1-\alpha}\Big)^{1/\alpha} < (1-\alpha)^{-(1/\alpha)}\Big(\beta\sum_{j=1}^{n}(\mu_j-\beta)^{-\frac{1}{1-\alpha}}\Big)^{1/\alpha}.$$

This estimate implies $\alpha h_i^{\alpha-1}(\tau) - (\mu_i-\beta) > \chi_i(\beta)$. By assumption (3.12) again,

$$\chi_i(\beta) \ge \alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}\Big[\beta\sum_{j=1}^{n}(\mu_j-\beta)^{-\frac{1}{1-\alpha}}\Big]^{-\frac{1-\alpha}{\alpha}} - (\mu_n-\beta) > 0.$$

Then we have $\alpha h_i^{\alpha-1}(\tau) - (\mu_i-\beta) > \chi_i(\beta) > 0$. Now, for $s = 2, 3, \cdots, 2^n$ and $\tau \in (0, A)$, if $G_s(\tau) = 0$ then using (3.13)

$$G_s'(\tau) < \sum_{i=2}^{n}\frac{\beta}{\chi_i(\beta)} - \frac{\beta}{(1-\alpha)(\mu_n-\beta)} - 1 \le 0.$$

This implies, for $s = 2, 3, \cdots, 2^n$, there exists exactly one $\tau \in (0, A)$ such that $G_s(\tau) = 0$ and (3.2) has exactly $2^n - 1$ positive solutions. □

All the conditions of Lemma 3.8 are met in particular for $\beta > 0$ such that $\beta < \frac{1}{2}\mu_1$, $\beta \le \frac{1}{n-1}\mu_n$, and

$$\beta \le \frac{4}{(N-2)(2-p)}\Big(\frac{2-p}{q}\Big)^{\frac{(N-2)q}{4}}(2\mu_n)^{1-\frac{(N-2)q}{4}}\left(\sum_{i=1}^{n}\Big(\mu_i-\frac{1}{2}\mu_1\Big)^{-\frac{(N-2)q}{4}}\right)^{-1}.$$

since, according to the last inequality, it can be checked that $\chi_i(\beta) \ge \mu_n$ for all i.

Under the assumptions of Lemmas 3.7 and 3.8, we have proved that for $s = 2, 3, \cdots, 2^n$ and $\tau \in (0, A)$, $G_s'(\tau) < 0$ if $G_s(\tau) = 0$. The uniqueness of solution of the equation $G_s(\tau) = 0$ is then a consequence of that fact. We remark that it is generally impossible to have $G_s'(\tau) < 0$ for all $\tau \in (0, A)$. In fact, if $n \in I\setminus I_s$ and $\mu_{n-1} < \mu_n$ then $\lim_{\tau\to A^-} G_s'(\tau) = +\infty$, since as $\tau \to A^-$, the term $\beta h_n'(\tau)$ in the expression of $G_s'(\tau)$ tends to $+\infty$ and all the other terms tend to finite numbers.

Proof of Theorem 1.2(c). The result follows from Lemmas 3.1, 3.6, 3.7 and 3.8. □

Proof of Theorem 1.2(d). Now we prove the uniqueness result. We have $\beta > \mu'' \geq \mu'$ by the assumption. Then according to Lemma 3.3, (3.1) has a unique positive solution which we denote by $(t_1, t_2, \cdots, t_n)$. Let $(u_1, u_2, \cdots, u_n)$ be any positive solution of (1.3) and set $k_j = t_j^{1/q}$ and $U_j = k_j^{-1} u_j$. Existence of $(u_1, u_2, \cdots, u_n)$ is guaranteed by Lemma 3.3. Since $\mu_1 = \cdots = \mu_m = \mu'$ and $\mu_{m+1} = \cdots = \mu_{2m} = \mu''$, it is easy to see that $t_1 = \cdots = t_m$ and $t_{m+1} = \cdots = t_{2m}$. Therefore, $k_1 = \cdots = k_m$ and $k_{m+1} = \cdots = k_{2m}$. In order to see that $(u_1, u_2, \cdots, u_n)$ is the unique positive solution of (1.3) it suffices to prove that $U_1 = U_2 = \cdots = U_n = U$.

We first prove that $U_1 = \cdots = U_m$ and $U_{m+1} = \cdots = U_{2m}$, and for this we only prove $U_1 = U_2$ since the proof of other equalities is the same. Suppose $U_1 \neq U_2$ and assume $\Omega = \{x \in \mathbb{R}^N \mid U_1(x) > U_2(x)\} \neq \emptyset$. For convenience, we denote $\mu := \mu'$ and $t := t_1 = t_2$. Then U_1 and U_2 satisfies

$$\begin{cases} -\Delta U_1 = t^{-\alpha}\Big(\mu t U_1^{2^*-1} + \beta t U_1^{p-1} U_2^q + \beta \sum\limits_{j=3}^{n} t_j U_1^{p-1} U_j^q\Big), \\ -\Delta U_2 = t^{-\alpha}\Big(\mu t U_2^{2^*-1} + \beta t U_1^{q} U_2^{p-1} + \beta \sum\limits_{j=3}^{n} t_j U_2^{p-1} U_j^q\Big). \end{cases}$$

Multiplying the first equation with U_2 and the second equation with U_1 and taking integral on Ω yields

$$\begin{aligned} \int_\Omega \big[(-\Delta U_1)U_2 + (\Delta U_2)U_1\big] = {} & t^{-\alpha} \int_\Omega \Big(\mu t U_1^{2^*-1} U_2 + \beta t U_1^{p-1} U_2^{q+1} - \mu t U_2^{2^*-1} U_1 - \beta t U_1^{q+1} U_2^{p-1}\Big) \\ & + t^{-\alpha} \beta \sum_{j=3}^{n} t_j \int_\Omega U_j^q U_1 U_2 \Big(U_1^{p-2} - U_2^{p-2}\Big) \triangleq I_1 + I_2. \end{aligned}$$

For the left hand side, we have

$$\text{LHS} = \int_{\partial\Omega} \Big(-\frac{\partial U_1}{\partial n} + \frac{\partial U_2}{\partial n}\Big) U_1 \geq 0.$$

There are two terms on the right hand side. The second term $I_2 < 0$, since $p < 2$ and $U_1 > U_2$ imply the integrand of I_2 is negative on Ω. To estimate the integral in I_1 we split and recombine the four terms of its integrand as

$$\begin{aligned} & \mu t U_1^{2^*-1} U_2 + \beta t U_1^{p-1} U_2^{q+1} - \mu t U_2^{2^*-1} U_1 - \beta t U_1^{q+1} U_2^{p-1} \\ & \quad = \mu t U_1^{p-1} U_2 \Big(U_1^q - U_2^q\Big) + (\beta+\mu) t U_1^{p-1} U_2^{q+1} + \mu t U_1 U_2^{p-1} \Big(U_1^q - U_2^q\Big) - (\beta+\mu) t U_1^{q+1} U_2^{p-1}. \end{aligned}$$

Then, since $U_1 > U_2$ on Ω and $\mu < \frac{\beta+\mu}{2}$, rearrange the terms on the right hand side and factorize to see that

$$\begin{aligned} & \mu t U_1^{2^*-1} U_2 + \beta t U_1^{p-1} U_2^{q+1} - \mu t U_2^{2^*-1} U_1 - \beta t U_1^{q+1} U_2^{p-1} \\ & \qquad < \frac{\beta+\mu}{2} t \Big(U_1^{q+1} U_2 + U_1 U_2^{q+1}\Big)\Big(U_1^{p-2} - U_2^{p-2}\Big) < 0. \end{aligned}$$

Thus $I_1 < 0$. Then we arrive at a contradiction $0 \leq I_1 + I_2 < 0$. Therefore $U_1 = \cdots = U_m$ and $U_{m+1} = \cdots = U_{2m}$.

Now we prove $U_1 = U_{m+1}$. Since we have proved $U_1 = \cdots = U_m$ and $U_{m+1} = \cdots = U_{2m}$, U_1 and U_{m+1} satisfy

$$\begin{cases} -\Delta U_1 = t_1^{1-\alpha}\big(\mu' + (m-1)\beta\big) U_1^{2^*-1} + m\beta t_1^{-\alpha} t_{m+1} U_1^{p-1} U_{m+1}^q, \\ -\Delta U_{m+1} = t_{m+1}^{1-\alpha}\big(\mu'' + (m-1)\beta\big) U_{m+1}^{2^*-1} + m\beta t_1 t_{m+1}^{-\alpha} U_1^{q} U_{m+1}^{p-1}. \end{cases}$$

If $U_1 \neq U_{m+1}$, we may assume $\Omega = \{x \in \mathbb{R}^N \mid U_1(x) > U_{m+1}(x)\} \neq \emptyset$. Then taking integral on Ω we have

$$\begin{aligned} \int_\Omega \big[(-\Delta U_1)U_{m+1} + (\Delta U_{m+1})U_1\big] = {} & \int_\Omega \Big(t_1^{1-\alpha}\big(\mu' + (m-1)\beta\big) U_1^{2^*-1} U_{m+1} + m\beta t_1^{-\alpha} t_{m+1} U_1^{p-1} U_{m+1}^{q+1} \\ & - t_{m+1}^{1-\alpha}\big(\mu'' + (m-1)\beta\big) U_1 U_{m+1}^{2^*-1} - m\beta t_1 t_{m+1}^{-\alpha} U_1^{q+1} U_{m+1}^{p-1}\Big). \end{aligned}$$

For the left hand side,

$$\text{LHS} = \int_{\partial\Omega} \Big(-\frac{\partial U_1}{\partial n} + \frac{\partial U_{m+1}}{\partial n} \Big) U_1 \geq 0.$$

Denote by G the integrand on the right hand side and split the four terms and recombine them as

$$\begin{aligned} G = & t_1^{1-\alpha}\big(\mu' + (m-1)\beta\big)U_1^{p-1}U_{m+1}\Big(U_1^q - U_{m+1}^q\Big) + U_1^{p-1}U_{m+1}^{q+1} \\ & + t_{m+1}^{1-\alpha}\big(\mu'' + (m-1)\beta\big)U_1 U_{m+1}^{p-1}\Big(U_1^q - U_{m+1}^q\Big) - U_1^{q+1}U_{m+1}^{p-1}, \end{aligned}$$

since by (3.1) t_1 and t_m satisfy the equations

$$\big(\mu' + (m-1)\beta\big)t_1^{1-\alpha} + m\beta t_1^{-\alpha}t_{m+1} = 1 = \big(\mu'' + (m-1)\beta\big)t_{m+1}^{1-\alpha} + m\beta t_1 t_{m+1}^{-\alpha}. \tag{3.14}$$

eq3.14

From the facts that $\mu' \leq \mu'' < \beta$ and $(\beta - \mu')t_1 + t_1^\alpha = (\beta - \mu'')t_{m+1} + t_{m+1}^\alpha$, it can be deduced that

$$t_1 \leq t_{m+1} \quad \text{and} \quad (\beta - \mu'')t_{m+1} \leq (\beta - \mu')t_1. \tag{3.15}$$

eq3.15

Now using the assumption in Theorem 1.2(d) we see that $\beta^2 - \big((m+1)\mu'' - (m-1)\mu'\big)\beta + \mu'\mu'' > 0$, that is

$$\frac{\mu'' + (m-1)\beta}{m\beta} < \frac{\beta - \mu''}{\beta - \mu'}. \tag{3.16}$$

eq3.16

Combining the second inequality in (3.15) and (3.16) we have

$$\big(\mu'' + (m-1)\beta\big)t_{m+1} < m\beta t_1. \tag{3.17}$$

eq3.17

This together with the second equality in (3.14) implies $\big(\mu'' + (m-1)\beta\big)t_{m+1}^{1-\alpha} < \frac{1}{2}$. Since $\mu' \leq \mu''$ and $t_1 \leq t_m$, by (3.17) we have $\big(\mu' + (m-1)\beta\big)t_1 < m\beta t_{m+1}$, which combined with the first equality in (3.14) leads to $\big(\mu' + (m-1)\beta\big)t_1^{1-\alpha} < \frac{1}{2}$. Inserting these inequalities into the expression of G, we have on Ω

$$G < \frac{1}{2}\Big(U_1^{q+1}U_{m+1} + U_1 U_{m+1}^{q+1}\Big)\Big(U_1^{p-2} - U_{m+1}^{p-2}\Big) < 0.$$

But this gives a contradiction $0 \leq \int_\Omega G < 0$. Therefore, $U_1 = U_2 = \cdots = U_n$. Then $U_1 = U_2 = \cdots = U_n = U$ and uniqueness of positive solutions of (1.3) follows. □

4. Proof of Theorem 1.3

In this section we prove Theorem 1.3 and we always assume $N \geq 3$, $\mu_i > 0$, $p_{ij} = 2$, $q_{ij} = 2^* - 2$, and $\beta_{ij} = \beta$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. Recall that $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. Note $(k_1 U, k_2 U, \cdots, k_n U)$ is a synchronized positive solution of (1.3) if and only if $(k_1, k_2, \cdots, k_n)$ is a positive solution of the algebraic system

$$\mu_i k_i^{2^*-2} + \beta \sum_{j=1, j\neq i}^{n} k_j^{2^*-2} = 1, \quad i = 1, 2, \cdots, n.$$

Proof of Theorem 1.3. (a) We write the above system in the following form

$$1 + (\beta - \mu_1)k_1^{2^*-2} = 1 + (\beta - \mu_2)k_2^{2^*-2} = \cdots = 1 + (\beta - \mu_n)k_n^{2^*-2} = \beta \sum_{j=1}^{n} k_j^{2^*-2}. \tag{4.1}$$

eq4.1

If (4.1) has a positive solution then it has to be that $\beta > \mu_n$ or $0 < \beta < \mu_1$ or $\beta = \mu_1 = \mu_n$. Conversely, if $\beta > \mu_n$ or $0 < \beta < \mu_1$, then (4.1) has a unique positive solution $(k_1, k_2, \cdots, k_n)$ and

$$k_i = \Big((\beta - \mu_i)\Big(\sum_{j=1}^{n} \frac{\beta}{\beta - \mu_j} - 1 \Big) \Big)^{\frac{-(N-2)}{4}},$$

and if $\beta = \mu_1 = \mu_n$ then any $(k_1, k_2, \cdots, k_n)$ with positive components satisfying $\beta \sum_{j=1}^n k_j^{2^*-2} = 1$ is a solution of (4.1).

(b) Since $\mu_1 \leq \beta \leq \mu_n$ and $\mu_1 \neq \mu_n$, we may assume for some $i \in \{1, 2, \cdots, n-1\}$, $\mu_i \leq \beta \leq \mu_{i+1}$ and either $\mu_i < \beta$ or $\beta < \mu_{i+1}$. If (1.3) has a positive solution $(u_1, u_2, \cdots, u_n)$, subtracting the

$(i+1)$-th equation multiplied with u_i from the i-th equation multiplied with u_{i+1} and taking integral, we have a contradiction

$$0 = \int_{\mathbb{R}^N} \Big((\mu_i - \beta) u_i^{2^*-1} u_{i+1} + (\beta - \mu_{i+1}) u_i u_{i+1}^{2^*-1} \Big) < 0.$$

(c) Let $(u_1, u_2, \cdots, u_n)$ be any positive solution of (1.3) and $(k_1, k_2, \cdots, k_n)$ be the unique positive solution of (4.1). Set $U_i = \frac{1}{k_i} u_i$ and $t_i = k_i^{2^*-2}$. It suffices to prove that

$$U_1 = U_2 = \cdots = U_n = U.$$

If not, we may assume $\Omega = \{x \in \mathbb{R}^N \,|\, U_1(x) > U_2(x)\} \neq \emptyset$. From the first two equations of (1.3)

$$\begin{cases} -\Delta U_1 = \mu_1 t_1 U_1^{2^*-1} + \beta t_2 U_1 U_2^{2^*-2} + \beta \sum_{j=3}^n t_j U_1 U_j^{2^*-2}, \\ -\Delta U_2 = \mu_2 t_2 U_2^{2^*-1} + \beta t_1 U_1^{2^*-2} U_2 + \beta \sum_{j=3}^n t_j U_2 U_j^{2^*-2}, \end{cases}$$

we have

$$\int_\Omega ((-\Delta U_1) U_2 + (\Delta U_2) U_1) = \int_\Omega \Big(\mu_1 t_1 U_1^{2^*-1} U_2 + \beta t_2 U_1 U_2^{2^*-1} - \mu_2 t_2 U_1 U_2^{2^*-1} - \beta t_1 U_1^{2^*-1} U_2 \Big).$$

Since $\beta > \mu_1$, this leads to a contradiction

$$0 \leq \int_{\partial\Omega} \Big(-\frac{\partial U_1}{\partial n} + \frac{\partial U_2}{\partial n} \Big) U_1 = \int_\Omega t_1 (\mu_1 - \beta) U_1 U_2 \big(U_1^{2^*-2} - U_2^{2^*-2} \big) < 0.$$

The proof is finished. □

5. Proof of Theorem 1.4

In this section we assume $N \geq 3$, $\mu_i > 0$, $2 < p_{ij} < 2^*$, $p_{ij} + q_{ij} = 2^*$, and $\beta_{ij} > 0$.

Proof of Theorem 1.4(a). Consider the algebraic system in (2.1):

$$f_i(k_1, k_2, \cdots, k_n) := \mu_i k_i^{2^*-2} + \sum_{j=1,\, j\neq i}^n \beta_{ij} k_i^{p_{ij}-2} k_j^{q_{ij}} - 1 = 0, \quad i = 1, 2, \cdots, n.$$

Since $2 < p_{ij} < 2^*$ and $p_{ij} + q_{ij} = 2^*$, if $\varepsilon > 0$ is small enough then

$$f_i(k_1, \cdots, \varepsilon, \cdots, k_n) < 0 < f_i(k_1, \cdots, \mu_i^{-(N-2)/4}, \cdots, k_n)$$

for all $k_j \in [\varepsilon,\ \mu_j^{-(N-2)/4}]$ with $j \neq i$ and all $i = 1, 2, \cdots, n$. This implies the Brouwer degree

$$\deg(f, \Omega, 0) = 1,$$

where Ω is an n-dimensional cuboid defined as $\Omega := \prod_{i=1}^n \big(\varepsilon,\ \mu_i^{-(N-2)/4}\big)$. This yields a synchronized positive solution of (1.3). □

For the rest of this section, we assume in addition that $p_{ij} = p \in (2, 2^*)$, $q_{ij} = q = 2^* - p$, $\beta_{ij} = \beta$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. We shall use the same symbols as in Section 3, but with adjusted meanings.

Recall that $\alpha = \frac{2-p}{q}$. Unlike the number $\alpha \in (0,1)$ in Section 3, here α ranges from $-\infty$ to 0. As we have seen in Section 3, the number of synchronized positive solutions of (1.3) is equal to the number of positive solutions $(t_1, \cdots, t_n, \tau)$ of (3.2). As in Section 3, define $f_i(t)$ for $t \in (0, +\infty)$ and $i = 1, 2, \cdots, n$.

Recall that $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. If $\beta \leq \mu_1$ then, for all i, f_i is strictly decreasing from $(0, S_i)$ onto $(0, +\infty)$ and $f_i|_{(0,S_i)}$ has an inverse decreasing function $h_i : (0, +\infty) \to (0, S_i)$, where $S_i = (\mu_i - \beta)^{-\frac{1}{1-\alpha}} = (\mu_i - \beta)^{-\frac{(N-2)q}{4}}$ if $\beta < \mu_i$ and $S_i = +\infty$ if $\beta = \mu_i$. In this case the number of synchronized positive solutions of (1.3) is equal to the number of positive solutions of the single algebraic equation

$$G_1(\tau) := \beta \sum_{i=1}^n h_i(\tau) - \tau = 0, \quad \tau \in (0, +\infty). \tag{5.1}$$

eq5.1

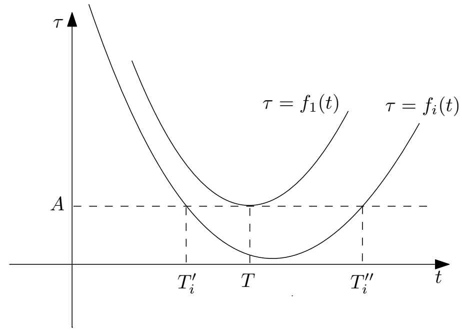


FIGURE 2. The case $p > 2$ and $\beta > \mu_i$

Since G_1 is strictly decreasing and since

$$\lim_{\tau\to 0^+} G_1(\tau) = \beta\sum_{i=1}^{n} S_i > 0, \quad \lim_{\tau\to+\infty} G_1(\tau) = -\infty,$$

(5.1) has a unique solution. Thus we have the following lemma.

lem5.1 **Lemma 5.1.** *If $\beta \le \mu_1$ then (1.3) has exactly one synchronized positive solution.*

Now we turn to consider the case $\beta > \mu_1$. In this case, f_1 achieves its minimum

$$A := \min_{0<t<+\infty} f_1(t) = \frac{(-\alpha)^{\frac{\alpha}{1-\alpha}} + (-\alpha)^{\frac{1}{1-\alpha}}}{(\beta-\mu_1)^{\frac{\alpha}{1-\alpha}}} = \frac{((p-2)/q)^{(N-2)(2-p)/4} + ((p-2)/q)^{(N-2)q/4}}{(\beta-\mu_1)^{(N-2)(2-p)/4}}$$

at

$$T := \Big(\frac{-\alpha}{\beta-\mu_1}\Big)^{\frac{1}{1-\alpha}} = \Big(\frac{p-2}{q(\beta-\mu_1)}\Big)^{(N-2)q/4}.$$

For each i, there exists uniquely a number T_i' such that

$$0 < T_i' \le T, \quad f_i(T_i') = A,$$

and that $f_i|_{(0,T_i']}$ is strictly decreasing from $(0, T_i']$ onto $[A, +\infty)$, and we denote the inverse decreasing function of $f_i|_{(0,T_i']}$ by $h_i : [A, +\infty) \to (0, T_i']$. For some i, if $\beta > \mu_i$ then there exists uniquely a second number T_i'' such that

$$T \le T_i'', \quad f_i(T_i'') = A,$$

and that $f_i|_{[T_i'',+\infty)}$ is strictly increasing from $[T_i'', +\infty)$ onto $[A, +\infty)$, and we denote the inverse increasing function of $f_i|_{[T_i'',+\infty)}$ by $k_i : [A, +\infty) \to [T_i'', +\infty)$. Note that $T_1' = T_1'' = T$, all h_i $(i = 1, 2, \cdots, n)$ are defined, and k_i is well defined if and only if $\beta > \mu_i$. In addition, for $\tau \in [A, +\infty)$,

$$h_n(\tau) \le \cdots \le h_2(\tau) \le h_1(\tau) \le h_1(A) = T$$

and, if $\beta > \mu_i$,

$$k_i(\tau) \ge \cdots \ge k_2(\tau) \ge k_1(\tau) \ge k_1(A) = T.$$

For $\beta > \mu_i$, the graphs of f_1 and f_i are illustrated in Figure 2.

We define σ^* as follows. We denote $I = \{1, 2, \cdots, n\}$ as in Section 2. Let j be the largest subscript such that $\mu_j < \beta$, k the largest subscript such that $\mu_1 = \mu_2 = \cdots = \mu_k$ with $k \le j$, and $J_1, J_2, \cdots, J_{2^{j-k}}$ the subsets of the index set $\{k+1, k+2, \cdots, j\}$. Note that the index set is empty if $k = j$ and we only have J_1 in this case which is the empty set. Define σ^* to be the number of index sets J_s for which the equality

$$\beta\sum_{i\in I\setminus J_s} h_i(A) + \beta\sum_{i\in J_s} k_i(A) = A \tag{5.2}$$ eq5.2

holds. Note that σ^* is the number of positive solutions $(t_1, \cdots, t_n, \tau)$ of (3.2) with $\tau = A$. The index set $\{1, 2, \cdots, j\}$ has 2^j subsets which we label with $I_1, I_2, \cdots, I_{2^j}$. For the sake of later discussion, we assume $I_1 = \emptyset$. For $s = 1, 2, \cdots, 2^j$, let σ_s be the number of solutions of the equation

$$\beta \sum_{i \in I \setminus I_s} h_i(\tau) + \beta \sum_{i \in I_s} k_i(\tau) = \tau, \quad \tau \in (A, +\infty). \tag{5.3}$$ eq5.3

Denote $\sigma^{**} = \sum_{s=1}^{2^j} \sigma_s$. Then σ^{**} is the number of positive solutions $(t_1, \cdots, t_n, \tau)$ of (3.2) with $\tau > A$.

At this point, we have proved the next lemma.

lem5.2 **Lemma 5.2.** *Let $\beta > \mu_1$. Then the number of synchronized positive solutions of (1.3) equals $\sigma^* + \sigma^{**}$.*

We now prove Theorem 1.4(b).

Proof of Theorem 1.4(b). The result follows from Lemma 5.1 if $\beta \le \mu_1$. Now we assume $\mu_1 < \beta < \mu_2$. For $\tau \in [A, +\infty)$, define

$$G_1(\tau) := \beta \sum_{i=1}^{n} h_i(\tau) - \tau, \quad G_2(\tau) := \beta \sum_{i=2}^{n} h_i(\tau) + \beta k_1(\tau) - \tau.$$

According to Lemma 5.2, the number of synchronized positive solutions of (1.3) equals the number of solutions of $G_1(\tau) = 0$ in $[A, +\infty)$ plus the number of solutions of $G_2(\tau) = 0$ in $(A, +\infty)$. Note that $G_1(A) = G_2(A)$, $G_1(\tau)$ is strictly decreasing in $[A, +\infty)$ and $\lim_{\tau \to +\infty} G_1(\tau) = -\infty$. Since for τ large enough

$$G_2(\tau) \simeq \frac{\beta}{\beta - \mu_1}\tau - \tau = \frac{\mu_1}{\beta - \mu_1}\tau,$$

we see that $\lim_{\tau \to +\infty} G_2(\tau) = +\infty$. We claim the existence of a $\delta_0 \in (0, \mu_2 - \mu_1)$ such that G_2 is strictly increasing in $[A, +\infty)$ if $\mu_1 < \beta < \mu_1 + \delta_0$. If this is the case, then $G_1(\tau) = 0$ has a unique solution in $[A, +\infty)$ and $G_2(\tau) = 0$ has no solution in $(A, +\infty)$ when $G_1(A) \ge 0$, and vice versa, $G_1(\tau) = 0$ has no solution in $[A, +\infty)$ and $G_2(\tau) = 0$ has a unique solution in $(A, +\infty)$ when $G_1(A) < 0$. Therefore (1.3) has exactly one synchronized positive solution for $\mu_1 < \beta < \mu_1 + \delta_0$. This together with Lemma 5.1 concludes Theorem 1.4(b).

Indeed, for $\tau \in (A, +\infty)$ and $i \ge 2$, since $\alpha < 0$ and $h_i(\tau) < h_i(A) < T$, we have, $\alpha h_i^{\alpha-1}(\tau) + \beta - \mu_i < \alpha T^{\alpha-1} + \beta - \mu_i < 0$. We also have $0 < \alpha k_1^{\alpha-1}(\tau) + \beta - \mu_1 < \beta - \mu_1$. Then, for $\tau \in (A, +\infty)$,

$$G_2'(\tau) = \sum_{i=2}^{n} \frac{\beta}{\alpha h_i^{\alpha-1}(\tau) + \beta - \mu_i} + \frac{\beta}{\alpha k_1^{\alpha-1}(\tau) + \beta - \mu_1} - 1 > -\sum_{i=2}^{n} \frac{\beta}{\mu_i - \mu_1} + \frac{\beta}{\beta - \mu_1} - 1.$$

From this estimate, it is easy to find a positive number δ_0 such that if $\mu_1 < \beta \le \mu_1 + \delta_0$ then for $\tau \in (A, +\infty)$, $G_2'(\tau) > 0$ and thus $G_2(\tau)$ is strictly increasing in $[A, +\infty)$. □

Now we prove Theorem 1.4(c). We assume $\beta > \mu_1$ and let j be the largest integer such that $\beta > \mu_j$. For $s = 2, \cdots, 2^j$, we define

$$G_s(\tau) := \beta \sum_{i \in I \setminus I_s} h_i(\tau) + \beta \sum_{i \in I_s} k_i(\tau) - \tau, \quad \tau \in [A, +\infty),$$

and we consider the equation $G_s(\tau) = 0$ for $\tau \in (A, +\infty)$. Since $\emptyset \ne I_s \subset \{1, \cdots, j\}$ and for τ large enough $G_s(\tau) \simeq \sum_{i \in I_s} \frac{\beta\tau}{\beta - \mu_i} - \tau$, we have $\lim_{\tau \to +\infty} G_s(\tau) = +\infty$. To achieve the conclusion of Theorem 1.4(c), we prove that $G_s(A) < 0$ if $p > 2$ and p is sufficiently close to 2. We have

$$G_s(A) = \beta \sum_{i \in I \setminus I_s} T_i' + \beta \sum_{i \in I_s} T_i'' - A \le n\beta T_j'' - A.$$

We need to estimate A and T_j'' and we first have the next lemma.

lem5.3 **Lemma 5.3.** *Assume $2 < p < 1 + \frac{2^*}{2}$. Then*

$$\min\{1, \sqrt{\beta - \mu_1}\} \le A \le \left(e^{e^{-1}} + 1\right) \max\{1, \sqrt{\beta - \mu_1}\}.$$

Proof. Since $2 < p < 1 + \frac{2^*}{2}$, we have $-1 < \alpha = \frac{2-p}{2^*-p} < 0$. It is easy to see that $1 \le (-\alpha)^{\frac{\alpha}{1-\alpha}} \le e^{e^{-1}}$, $0 < (-\alpha)^{\frac{1}{1-\alpha}} < 1$, and

$$\min\{1, \sqrt{\beta - \mu_1}\} \le (\beta - \mu_1)^{-\frac{\alpha}{1-\alpha}} \le \max\{1, \sqrt{\beta - \mu_1}\}.$$

Then the result follows from the definition of A. □

The estimate of T_j'' is a little more delicate. Since T_j'' depends on p implicitly we write $T_j'' = T_j''(p)$ and we have the following lemma.

lem5.4 **Lemma 5.4.** *For any $\nu \in (0, 1)$ we have*

$$T_j''(p) = O((p-2)^\nu) \quad \text{as } p \to 2^+.$$

Proof. Assume $2 < p < 1 + \frac{2^*}{2}$. Then $(-\alpha)^\nu \in (0, 1)$. Since $A = (T_j'')^\alpha + (\beta - \mu_j)T_j'' \ge (\beta - \mu_j)T_j''$, we have $T_j'' \le \frac{1}{\beta - \mu_j} A$. Using Lemma 5.3, we obtain an upper bound for T_j'':

$$T_j'' \le M := \frac{1}{\beta - \mu_j}\big(e^{e^{-1}} + 1\big) \max\{1, \sqrt{\beta - \mu_1}\}.$$

By the Young inequality, we have

$$A = (T_j'')^\alpha + (\beta - \mu_j)T_j'' \ge \big(r(\beta - \mu_j)T_j''\big)^{\frac{1}{r}} \big(s(T_j'')^\alpha\big)^{\frac{1}{s}},$$

where $r = \frac{1}{(-\alpha)^\nu}$ and $s = \frac{1}{1-(-\alpha)^\nu}$. Since $\alpha < 0$, using the upper bound of T_j'' yields

$$A \ge r^{\frac{1}{r}} s^{\frac{1}{s}} (\beta - \mu_j)^{\frac{1}{r}} M^{\frac{\alpha}{s}} (T_j'')^{\frac{1}{r}}.$$

Then we have $T_j'' \le r^{-1} s^{-\frac{r}{s}} (\beta - \mu_j)^{-1} M^{-\frac{r\alpha}{s}} A^r$, that is,

$$T_j'' \le (\beta - \mu_j)^{-1} (-\alpha)^\nu (1 - (-\alpha)^\nu)^{(-\alpha)^{-\nu} - 1} M^{(-\alpha)^{1-\nu} + \alpha} A^{(-\alpha)^{-\nu}}.$$

In view of the expression of A, we see that

$$A^{(-\alpha)^{-\nu}} = (-\alpha)^{-(-\alpha)^{1-\nu}/(1-\alpha)} (1-\alpha)^{(-\alpha)^{-\nu}} (\beta - \mu_1)^{(-\alpha)^{1-\nu}/(1-\alpha)},$$

from which a simple computation shows that $\lim_{\alpha \to 0^-} A^{(-\alpha)^{-\nu}} = 1$. Moreover,

$$\lim_{\alpha \to 0^-} (1 - (-\alpha)^\nu)^{(-\alpha)^{-\nu} - 1} M^{(-\alpha)^{1-\nu} + \alpha} = e^{-1}.$$

Since $p \to 2^+$ is equivalent to $\alpha \to 0^-$, we have, as $p \to 2^+$,

$$T_j''(p) = O((-\alpha)^\nu) = O((p-2)^\nu).$$

The proof is complete. □

Proof of Theorem 1.4(c). Fix $\nu \in (0, 1)$. By Lemmas 5.3 and 5.4, there exist $C = C(\beta) > 0$ and $p_0 = p_0(\beta) > 2$ such that for $p \in (2, p_0)$ and $s = 2, \cdots, 2^j$,

$$G_s(A) \le n\beta T_j'' - A \le C(p-2)^\nu - \min\{1, \sqrt{\beta - \mu_1}\}.$$

Then there exists $p_1 = p_1(\beta) \in (2, p_0)$ such that for $p \in (2, p_1)$ and for $s = 2, \cdots, 2^j$, $G_s(A) < 0$. This implies that, for $s = 2, \cdots, 2^j$, the equation $G_s(\tau) = 0$ has a solution in $(A, +\infty)$ as $G_s(\tau) > 0$ for τ sufficiently large. Then by Lemma 5.2, (1.3) has at least $2^j - 1$ synchronized positive solutions. □

Proof of Theorem 1.4(d). For $s = 2, 3, \cdots, 2^j$, since $I_s \ne \emptyset$ and $k_i(\tau) \ge k_1(\tau)$, we have

$$G_s(\tau) = \beta \sum_{i \in I \setminus I_s} h_i(\tau) + \beta \sum_{i \in I_s} k_i(\tau) - \tau > \beta k_1(\tau) - \tau, \quad \tau \in [A, +\infty).$$

Since $k_1(\tau) \ge k_1(A) = T$ and $\alpha < 0$, we see that $k_1^{\alpha-1}(\tau) \le T^{\alpha-1} = \frac{\mu_1 - \beta}{\alpha}$. Using the assumption $p \ge \frac{\mu_1}{\beta} 2 + \left(1 - \frac{\mu_1}{\beta}\right) 2^*$ yields $\frac{\mu_1 - \beta}{\alpha} \le \mu_1$. Thus we have, for $s = 2, \cdots, 2^j$ and $\tau \in [A, +\infty)$,

$$G_s(\tau) > \beta k_1(\tau) - [k_1^\alpha(\tau) + (\beta - \mu_1)k_1(\tau)] = k_1(\tau)(\mu_1 - k_1^{\alpha-1}(\tau)) \ge 0.$$

This implies for $s = 2, \cdots, 2^j$, $G_s(\tau) = 0$ has no solution in $[A, +\infty)$. Since $G_1(\tau) = \beta \sum_{i=1}^n h_i(\tau) - \tau$ is strictly decreasing and $\lim_{\tau \to +\infty} G_1(\tau) = -\infty$ and since

$$G_1(A) > \beta h_1(A) - A = \beta k_1(A) - A \geq 0,$$

$G_1(\tau) = 0$ has exactly one solution in $[A, +\infty)$. Therefore, (1.3) has a unique synchronized positive solution. □

6. Proof of Theorem 1.5

In this section we assume $N \geq 3$, $\mu_i > 0$, $\beta_i > 0$, $p_i \leq 2$, and $q_i = 2^* - p_i$ for $i = 1, 2$ and prove Theorem 1.5. Let $k_1 > 0$ and $k_2 > 0$. Then $(k_1 U, k_2 U)$ is a synchronized positive solution of (1.4) if and only if (k_1, k_2) is a positive solution of the system

$$\begin{cases} f_1(k_1, k_2) := \mu_1 k_1^{q_1} - k_1^{2-p_1} + \beta_1 k_2^{q_1} = 0, \\ f_2(k_1, k_2) := \mu_2 k_2^{q_2} - k_2^{2-p_2} + \beta_2 k_1^{q_2} = 0. \end{cases} \tag{6.1}$$ eq6.1

Proof of Theorem 1.5(a). We assume $p_i < 2$. Set for $j = 1, 2$, $\alpha_j = (2\mu_j)^{-\frac{N-2}{4}}$ and $\tau_j = \mu_j^{-\frac{N-2}{4}}$. For $k_2 \in [0, \tau_2]$, since $p_1 + q_1 = 2^*$ and $\beta_1 < 2^{-\frac{N-2}{4}q_1} \mu_1^{1-\frac{N-2}{4}q_1} \mu_2^{\frac{N-2}{4}q_1}$, we see that

$$f_1(\alpha_1, k_2) \leq -\mu_1 (2\mu_1)^{-\frac{N-2}{4}q_1} + \beta_1 \mu_2^{-\frac{N-2}{4}q_1} < 0.$$

In addition, for $k_2 \in (0, \tau_2]$,

$$f_1(\tau_1, k_2) > \mu_1^{1-\frac{N-2}{4}q_1} - \mu_1^{-\frac{N-2}{4}(2-p_1)} = 0.$$

We have showed that $f_1(\alpha_1, k_2) < 0 < f_1(\tau_1, k_2)$ for $k_2 \in (0, \tau_2]$. In the same way, we have $f_2(k_1, \alpha_2) < 0 < f_2(k_1, \tau_2)$ for $k_1 \in (0, \tau_1]$. This implies

$$\deg((f_1, f_2), Q_1, (0, 0)) = 1 \quad \text{ for } Q_1 := (\alpha_1, \tau_1) \times (\alpha_2, \tau_2).$$

Then (6.1) has a solution in Q_1. Since $p_1 < 2$ and $q_1 > 0$, for $k_2 \in [\alpha_2, \tau_2]$ and $\epsilon > 0$ sufficiently small,

$$f_1(\epsilon, k_2) = \mu_1 \epsilon^{q_1} - \epsilon^{2-p_1} + \beta_1 k_2^{q_1} > 0.$$

Fix such an $\epsilon > 0$. Then we have $f_1(\alpha_1, k_2) < 0 < f_1(\epsilon, k_2)$ for $k_2 \in [\alpha_2, \tau_2]$ and $f_2(k_1, \alpha_2) < 0 < f_2(k_1, \tau_2)$ for $k_1 \in [\epsilon, \alpha_1]$. Therefore,

$$\deg((f_1, f_2), Q_2, (0, 0)) = -1 \quad \text{ for } Q_2 := (\epsilon, \alpha_1) \times (\alpha_2, \tau_2),$$

and (6.1) has a solution in Q_2. Similarly,

$$\deg((f_1, f_2), Q_3, (0, 0)) = -1 \quad \text{ for } Q_3 := (\alpha_1, \tau_1) \times (\epsilon, \alpha_2),$$

and (6.1) has a solution in Q_3. Then (1.4) has three synchronized positive solutions. □

We now study existence of solutions of (6.1) in $Q_4 := (0, \alpha_1) \times (0, \alpha_2)$. As an astonishing fact we shall see that existence of a solution of (6.1) in Q_4 not only implies that (6.1) has a unique solution in $(0, +\infty) \times (0, +\infty)$ but also implies that (1.4) has exactly one positive solution.

Set $I_i = [0, \alpha_i]$ and $g_i(t) = \mu_i t^{q_i} - t^{2-p_i}$. It is easy to see that

$$\inf_{t \in I_i} g_i(t) = -1 \qquad \text{if } p_i = 2, \tag{6.2}$$ eq6.2

$$\inf_{t \in I_i} g_i(t) = -\frac{4}{N-2} q_i^{-\frac{N-2}{4}q_i} \left(\frac{\mu_i}{2 - p_i} \right)^{1-\frac{N-2}{4}q_i} \qquad \text{if } 4 - 2^* < p_i < 2, \tag{6.3}$$ eq6.3

and

$$\inf_{t \in I_i} g_i(t) = -\mu_i (2\mu_i)^{-\frac{N-2}{4}q_i} \qquad \text{if } p_i \leq 4 - 2^*. \tag{6.4}$$ eq6.4

lem6.1 **Lemma 6.1.** *Under assumption (A), (6.1) has a solution in Q_4.*

Proof. Let $k_1 \in [0, \alpha_1]$. We have

$$f_1(k_1, \alpha_2) \geq \inf_{t \in I_1} g_1(t) + \beta_1 (2\mu_2)^{-\frac{N-2}{4} q_1}.$$

Using assumption (A), we see that $f_1(k_1, \alpha_2) > 0$ by (6.2) if $p_1 = 2$, by (6.3) if $4 - 2^* < p_1 < 2$, and by (6.4) if $p_1 \leq 4 - 2^*$. Now we show that $f_1(k_1, \epsilon) < 0$ for $k_1 \in [\epsilon, \alpha_1]$ and for $\epsilon > 0$ small. We first consider the case $p_1 < 2$. Then $g_1(t) < 0$ for $t \in (0, \alpha_1]$, $g_1(0) = 0$, and $g_1'(t) = \mu_1 q_1 t^{q_1 - 1} - (2 - p_1) t^{1 - p_1} < 0$ for $t > 0$ small as $p_1 + q_1 = 2^* > 2$. Then for $\epsilon > 0$ small and for $k_1 \in [\epsilon, \alpha_1]$,

$$f_1(k_1, \epsilon) \leq \max_{t \in [\epsilon, \alpha_1]} g_1(t) + \beta_1 \epsilon^{q_1} = g_1(\epsilon) + \beta_1 \epsilon^{q_1} = (\mu_1 + \beta_1) \epsilon^{q_1} - \epsilon^{2 - p_1} < 0.$$

If $p_1 = 2$ then $q_1 = \frac{4}{N-2}$. For $\epsilon > 0$ small and for $k_1 \in [\epsilon, \alpha_1]$,

$$f_1(k_1, \epsilon) \leq \max_{t \in [\epsilon, \alpha_1]} g_1(t) + \beta_1 \epsilon^{q_1} = g_1(\alpha_1) + \beta_1 \epsilon^{q_1} = -\frac{1}{2} + \beta_1 \epsilon^{q_1} < 0.$$

The above discussion shows that there exists $\epsilon \in (0, \min\{\alpha_1, \alpha_2\})$ such that $f_1(k_1, \epsilon) < 0 < f_1(k_1, \alpha_2)$ for $k_1 \in [\epsilon, \alpha_1]$. In exactly the same way, we have $f_2(\epsilon, k_2) < 0 < f_2(\alpha_1, k_2)$ for $k_2 \in [\epsilon, \alpha_2]$. Denote $Q_4^\epsilon := (\epsilon, \alpha_1) \times (\epsilon, \alpha_2)$. Then

$$\deg((f_1, f_2), Q_4^\epsilon, (0, 0)) = -1.$$

Therefore, (6.1) has a solution in $Q_4^\epsilon \subset Q_4$. □

lem6.2 **Lemma 6.2.** *If (6.1) has a solution in Q_4 then (1.4) has a unique positive solution.*

Proof. Let us fix a solution $(k_1, k_2) \in Q_4$ of (6.1). Then $(k_1 U, k_2 U)$ is a positive solution of (1.4). Now let (u, v) be any positive solution of (1.4). Set $u = k_1 \varphi$ and $v = k_2 \psi$. Then (φ, ψ) is a positive solution of

$$\begin{cases} -\Delta \varphi = \mu_1 k_1^{2^*-2} \varphi^{2^*-1} + \beta_1 k_1^{p_1 - 2} k_2^{q_1} \varphi^{p_1 - 1} \psi^{q_1} & \text{in } \mathbb{R}^N, \\ -\Delta \psi = \mu_2 k_2^{2^*-2} \psi^{2^*-1} + \beta_2 k_1^{q_2} k_2^{p_2 - 2} \varphi^{q_2} \psi^{p_2 - 1} & \text{in } \mathbb{R}^N, \\ \varphi, \psi \in \mathscr{D}^{1,2}(\mathbb{R}^N). \end{cases}$$

We prove that $\varphi = \psi$. If not we may assume that $\Omega = \{x \in \mathbb{R}^N \mid \varphi(x) > \psi(x)\} \neq \emptyset$. Then

$$\int_\Omega [(-\Delta \varphi)\psi - (-\Delta \psi)\varphi] = \int_\Omega \Big[\mu_1 k_1^{2^*-2} \varphi^{2^*-1} \psi + \beta_1 k_1^{p_1-2} k_2^{q_1} \varphi^{p_1-1} \psi^{q_1+1} - \mu_2 k_2^{2^*-2} \varphi \psi^{2^*-1} - \beta_2 k_1^{q_2} k_2^{p_2-2} \varphi^{q_2+1} \psi^{p_2-1} \Big].$$

The left hand side is nonnegative. Denote the integrand of the right hand side by G. Since (k_1, k_2) is a solution of (6.1) and since $p_i + q_i = 2^*$, we have

$$G = \varphi\psi \big[\mu_1 k_1^{2^*-2} \varphi^{p_1-2} (\varphi^{q_1} - \psi^{q_1}) + \varphi^{p_1-2} \psi^{q_1} + \mu_2 k_2^{2^*-2} \psi^{p_2-2} (\varphi^{q_2} - \psi^{q_2}) - \varphi^{q_2} \psi^{p_2-2} \big].$$

Since $(k_1, k_2) \in Q_4$ implies $\mu_1 k_1^{2^*-2} < \frac{1}{2}$ and $\mu_2 k_2^{2^*-2} < \frac{1}{2}$ and since $q_i > 0$, $p_i \leq 2$, we have on Ω

$$G < \frac{1}{2} \varphi\psi \big[\varphi^{q_2} (\varphi^{p_2-2} - \psi^{p_2-2}) + \psi^{q_1} (\varphi^{p_1-2} - \psi^{p_1-2}) \big] \leq 0.$$

We arrive at a contradiction. Then $\varphi = \psi \in \mathscr{D}^{1,2}(\mathbb{R}^N)$ is a positive solution of the equation $-\Delta u = u^{2^*-1}$. Therefore, $\varphi = \psi = U$. □

Proof of Theorem 1.5(b). Combining Lemma 6.1 and Lemma 6.2 we obtain the conclusion of Theorem 1.5(b). □

As a byproduct of the above discussions, we have the following astonishing result asserting that if (6.1) has a solution in Q_4 then not only (6.1) has exactly one solution in Q_4 but it also has exactly one solution in the whole $(0, +\infty) \times (0, +\infty)$.

prop6.1 **Proposition 6.1.** *If (6.1) has a solution in Q_4 then (6.1) has a unique solution in $(0, +\infty) \times (0, +\infty)$.*

Proof. If (6.1) has two solutions in $(0, +\infty) \times (0, +\infty)$, then (1.4) has at least two positive solutions. But this is a contradiction with Lemma 6.2. □

rem6.1 **Remark 6.1.** For $m \geq 2$, let $\alpha_i(m) = (m\mu_i)^{-\frac{N-2}{4}}$, $I_i(m) = (0, \alpha_i(m))$, and $Q_4(m) = I_1(m) \times I_2(m)$. Then

$$\inf_{t\in I_i(m)} g_i(t) = \begin{cases} -1 & \text{if } p_i = 2, \\ -\frac{4}{N-2} q_i^{-\frac{N-2}{4}q_i} \big(\frac{\mu_i}{2-p_i}\big)^{1-\frac{N-2}{4}q_i} & \text{if } \frac{2m-2^*}{m-1} < p_i < 2, \\ -(m-1)\mu_i (m\mu_i)^{-\frac{N-2}{4}q_i} & \text{if } p_i \leq \frac{2m-2^*}{m-1}. \end{cases}$$

Now we assume, for $i, j \in \{1, 2\}$ such that $i \neq j$,

$$\beta_i > \begin{cases} \text{either } \ m\mu_j & \text{if } p_i = 2, \\ \text{or } \ \frac{4}{N-2}\big(\frac{\mu_i}{2-p_i}\big)^{1-\frac{N-2}{4}q_i} \big(\frac{m\mu_j}{q_i}\big)^{\frac{N-2}{4}q_i} & \text{if } \frac{2m-2^*}{m-1} < p_i < 2, \\ \text{or } \ (m-1)\mu_i^{1-\frac{N-2}{4}q_i} \mu_j^{\frac{N-2}{4}q_i} & \text{if } p_i \leq \frac{2m-2^*}{m-1}. \end{cases}$$

Then the proof of Lemma 6.1 shows that (6.1) has a solution in $Q_4(m)$. By Proposition 6.1, for all $m \geq 2$, this solution obtained in $Q_4(m)$ must be the unique solution (k_1, k_2) obtained in Q_4. This means that, under the above assumption, $0 < k_i < \alpha_i(m)$. Therefore, k_i $(i = 1, 2)$ are sufficiently small if β_i $(i = 1, 2)$ are sufficiently large, since $\lim_{m\to+\infty} \alpha_i(m) = 0$.

Now we prove Theorem 1.5(c) and in this particular case we rewrite (1.4) as

$$\begin{cases} -\Delta u = \mu_1 u^{2^*-1} + \beta u^{p-1}v^q & \text{in } \mathbb{R}^N, \\ -\Delta v = \mu_2 v^{2^*-1} + \beta u^q v^{p-1} & \text{in } \mathbb{R}^N, \\ u > 0, \ v > 0, \ u, v \in \mathscr{D}^{1,2}(\mathbb{R}^N). \end{cases} \tag{6.5}$$ eq6.5

Recall that we assume $N \geq 3$, $\mu_2 \geq \mu_1 > 0$, $\beta > 0$, $p < 2$, and $q = 2^* - p$ in Theorem 1.5(c).

We first consider the case $\mu := \mu_1 = \mu_2$ and prove the first part of Theorem 1.5(c). By Theorem 1.2(b), (6.5) has exactly one synchronized positive solution if $\beta \geq \mu$. Now we assume $0 < \beta < \mu$. Since $\mu_1 = \mu_2$, we have

$$h(\tau) := h_1(\tau) = h_2(\tau) \ \text{ and } \ k(\tau) := k_1(\tau) = k_2(\tau) \quad \text{for } \ 0 \leq \tau \leq A.$$

Recall that $\beta_0 = \frac{2\mu}{2(N-1)-(N-2)p}$, $\alpha = \frac{2-p}{q} = \frac{2-p}{2^*-p}$, and

$$A = \big(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\big)(\mu-\beta)^{-\frac{\alpha}{1-\alpha}}, \qquad T = h(A) = k(A) = \alpha^{\frac{1}{1-\alpha}}(\mu-\beta)^{-\frac{1}{1-\alpha}}.$$

By the discussions in Section 3, to study the number of synchronized positive solutions of (6.5) we need to consider for $\tau \in (0, A]$ the number of zeros of the three functions

$$G_1(\tau) = 2\beta h(\tau) - \tau, \quad G_2(\tau) = \beta(h(\tau) + k(\tau)) - \tau, \quad G_3(\tau) = 2\beta k(\tau) - \tau.$$

We have $G_1(A) = G_2(A) = G_3(A) = 2\beta T - A$. Using the formulas for β_0, T, and A, it is easy to see that $0 < \beta_0 < \mu$ and

$$G_1(A) = G_2(A) = G_3(A) \begin{cases} < 0, & \text{if } 0 < \beta < \beta_0, \\ = 0, & \text{if } \beta = \beta_0, \\ > 0, & \text{if } \beta_0 < \beta < \mu. \end{cases}$$

We first show that G_2 is strictly decreasing in $(0, A]$. This is not obvious since the term $\beta h(\tau)$ in $G_2(\tau)$ is strictly increasing in $(0, A]$ and especially $\lim_{\tau\to A^-} h'(\tau) = +\infty$.

lem6.3 **Lemma 6.3.** *$G_2'(\tau) < 0$ for $\tau \in (0, A)$.*

Proof. For $\tau \in (0, A)$, $G_2'(\tau) = \frac{\beta}{H(\tau)} + \frac{\beta}{K(\tau)} - 1$, where $H(\tau) = \alpha h^{\alpha-1}(\tau) + \beta - \mu$ and $K(\tau) = \alpha k^{\alpha-1}(\tau) + \beta - \mu$. We have

$$\frac{d}{d\tau} H^2(\tau) = 2H(\tau)H'(\tau) = 2H(\tau)\alpha(\alpha-1)h^{\alpha-2}(\tau)h'(\tau) = 2\alpha(\alpha-1)h^{\alpha-2}(\tau).$$

For $\tau \in (0, A)$, since $h(\tau) < h(A) = T$ and $\alpha \in (0, 1)$, we have $\frac{d}{d\tau}H^2(\tau) < 2\alpha(\alpha-1)T^{\alpha-2}$. Taking integral from τ to A and in view of the fact that $H(A) = \alpha h^{\alpha-1}(A) + \beta - \mu = 0$, we see that $-H^2(\tau) < 2\alpha(\alpha-1)T^{\alpha-2}(A-\tau)$. Then

$$H(\tau) > \sqrt{2\alpha(1-\alpha)T^{\alpha-2}(A-\tau)} \ \text{ for } \ 0 < \tau < A.$$

In the same way, using the facts $k(\tau) > k(A) = T$ and $K(\tau) < K(A) = 0$, we have

$$K(\tau) > -\sqrt{2\alpha(1-\alpha)T^{\alpha-2}(A-\tau)} \text{ for } 0 < \tau < A.$$

Therefore, $H(\tau) + K(\tau) > 0$ and

$$G_2'(\tau) = \frac{\beta(H(\tau) + K(\tau))}{H(\tau)K(\tau)} - 1 < 0 \text{ for } 0 < \tau < A.$$

The proof is complete. □

Proof of Theorem 1.5(c) in the case $\mu_1 = \mu_2$. By Lemma 6.3, G_2 is strictly decreasing in $(0, A]$. From the discussion in Section 3, we know that $G_1(0) = 0$, $G_1(\tau) < 0$ for $\tau > 0$ small, $G_1'(\tau) > 0$ when $\tau \in (0, A)$ is a solution of $G_1(\tau) = 0$, and

$$\lim_{\tau \to A^-} G_1'(\tau) = \lim_{\tau \to A^-} \frac{2\beta}{H(\tau)} - 1 = +\infty.$$

Moreover, $G_2(0) > 0$, $G_3(0) > 0$, and G_3 is also strictly decreasing in $(0, A]$.

If $\beta_0 < \beta < \mu$ then $G_1(A) = G_2(A) = G_3(A) > 0$. In this case neither $G_2(\tau) = 0$ nor $G_3(\tau) = 0$ has any solution in $(0, A)$, and $G_1(\tau) = 0$ has in $(0, A)$ exactly one solution denoted by τ_1. Then (6.5) has exactly one synchronized positive solution $(h^{1/q}(\tau_1)U, h^{1/q}(\tau_1)U)$.

If $\beta = \beta_0$ then $G_1(A) = G_2(A) = G_3(A) = 0$. In this case no equation $G_i(\tau) = 0$ $(i = 1, 2, 3)$ has any solution in $(0, A)$. Then $(T^{1/q}U, T^{1/q}U)$ is the unique synchronized positive solution of (6.5).

If $0 < \beta < \beta_0$ then $G_1(A) = G_2(A) = G_3(A) < 0$. In this case $G_1(\tau) = 0$ has no solution in $(0, A)$ and both $G_2(\tau) = 0$ and $G_3(\tau) = 0$ have a unique solution in $(0, A)$, denoted by τ_2 and τ_3 respectively. We then conclude that (6.5) has exactly three synchronized positive solutions $(h^{1/q}(\tau_2)U, k^{1/q}(\tau_2)U)$, $(k^{1/q}(\tau_2)U, h^{1/q}(\tau_2)U)$, and $(k^{1/q}(\tau_3)U, k^{1/q}(\tau_3)U)$. □

Now we turn to consider the case $\mu_1 < \mu_2$. By Theorem 1.2(b), (6.5) has exactly one synchronized positive solution if $\beta \geq \mu_2$. In what follows, we assume $0 < \beta < \mu_2$ and we write

$$A = A(\beta) = \big(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\big)(\mu_2 - \beta)^{-\frac{\alpha}{1-\alpha}} \text{ and } T = T(\beta) = \alpha^{\frac{1}{1-\alpha}}(\mu_2 - \beta)^{-\frac{1}{1-\alpha}}$$

to stress the dependence on β. Set, for $\tau \in [0, A(\beta)]$,

$$G_1(\tau) = G_1(\beta, \tau) = \beta(h_1(\beta, \tau) + h_2(\beta, \tau)) - \tau, \quad G_2(\tau) = G_2(\beta, \tau) = \beta(h_1(\beta, \tau) + k_2(\beta, \tau)) - \tau.$$

If $\beta < \mu_1$, we also set, for $\tau \in [0, A(\beta)]$,

$$G_3(\tau) = G_3(\beta, \tau) = \beta(k_1(\beta, \tau) + k_2(\beta, \tau)) - \tau, \quad G_4(\tau) = G_4(\beta, \tau) = \beta(k_1(\beta, \tau) + h_2(\beta, \tau)) - \tau.$$

Again we add β as an independent variable and write $h_i(\tau) = h_i(\beta, \tau)$ and $k_i(\tau) = k_i(\beta, \tau)$ in the above expressions in order to emphasize dependence on β of the functions h_i, k_i and G_i. Define for $0 < \beta < \mu_2$

$$\eta_1(\beta) = G_1(\beta, A(\beta)) = G_2(\beta, A(\beta)) = \beta(h_1(\beta, A(\beta)) + h_2(\beta, A(\beta))) - A(\beta),$$

and for $0 < \beta < \mu_1$

$$\eta_2(\beta) = G_3(\beta, A(\beta)) = G_4(\beta, A(\beta)) = \beta(k_1(\beta, A(\beta)) + h_2(\beta, A(\beta))) - A(\beta).$$

We show that both η_1 and η_2 are strictly increasing functions.

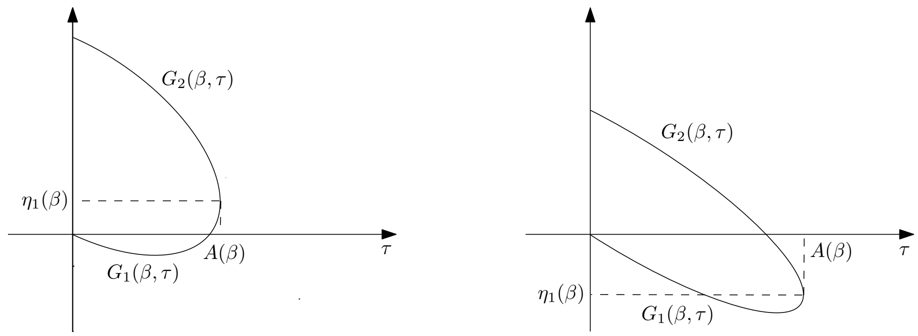
lem6.4 **Lemma 6.4.** *$\eta_1'(\beta) > 0$ for $\beta \in (0, \mu_2)$ and $\eta_2'(\beta) > 0$ for $\beta \in (0, \mu_1)$.*

Proof. For $0 < \beta < \mu_2$, since $h_2(\beta, A(\beta)) = T(\beta)$ we have $\eta_1(\beta) = \beta h_1(\beta, A(\beta)) + \beta T(\beta) - A(\beta)$. Then, since $A'(\beta) = T(\beta)$, we see that

$$\eta_1'(\beta) = h_1(\beta, A(\beta)) + \beta\Big[\frac{\partial h_1}{\partial \beta}(\beta, A(\beta)) + \frac{\partial h_1}{\partial \tau}(\beta, A(\beta))\frac{dA}{d\beta}\Big] + \beta T'(\beta).$$

For later discussions, we denote, for $\tau \in (0, A(\beta))$,

$$H_i(\beta, \tau) = \alpha h_i^{\alpha-1}(\beta, \tau) + \beta - \mu_i, \quad K_i(\beta, \tau) = \alpha k_i^{\alpha-1}(\beta, \tau) + \beta - \mu_i.$$



Case : $\tilde\beta < \beta < \mu_2,\ \eta_1(\beta) > 0$ Case : $0 < \beta < \tilde\beta,\ \eta_1(\beta) < 0$

FIGURE 3. The graphs of G_1 and G_2

From the equation $h_1^\alpha(\beta, A(\beta)) + (\beta - \mu_1)h_1(\beta, A(\beta)) = A(\beta)$, we have

$$\frac{\partial h_1}{\partial \beta}(\beta, A(\beta)) + \frac{\partial h_1}{\partial \tau}(\beta, A(\beta))\frac{dA}{d\beta} = \frac{T(\beta) - h_1(\beta, A(\beta))}{H_1(\beta, A(\beta))} > 0$$

since $T(\beta) > h_1(\beta, A(\beta))$ and $H_1(\beta, A(\beta)) > 0$. This implies $\eta_1'(\beta) > 0$ for $0 < \beta < \mu_2$ since we also have $T'(\beta) > 0$. Similarly, we have

$$\eta_2'(\beta) = k_1(\beta, A(\beta)) + \beta\Big[\frac{\partial k_1}{\partial \beta}(\beta, A(\beta)) + \frac{\partial k_1}{\partial \tau}(\beta, A(\beta))\frac{dA}{d\beta}\Big] + \beta T'(\beta)$$

and

$$\frac{\partial k_1}{\partial \beta}(\beta, A(\beta)) + \frac{\partial k_1}{\partial \tau}(\beta, A(\beta))\frac{dA}{d\beta} = \frac{T(\beta) - k_1(\beta, A(\beta))}{K_1(\beta, A(\beta))}.$$

Since $T(\beta) < k_1(\beta, A(\beta))$ and $K_1(\beta, A(\beta)) < 0$, $\eta_2'(\beta) > 0$ for $\beta \in (0, \mu_1)$ and we arrive at the conclusion. □

In the next lemma, we give properties of the two functions $G_1(\beta,\tau)$ and $G_2(\beta,\tau)$ as well as $\eta_1(\beta) = G_1(\beta, A(\beta)) = G_2(\beta, A(\beta))$, which will be used to study the number of zeros $\tau \in (0, A(\beta)]$ of the two equations $G_1(\beta,\tau) = 0$ and $G_2(\beta,\tau)$=0 for $0 < \beta < \mu_2$. We illustrate the graphs of G_1 and G_2 in Figure 3. It can be seen that these graphs and part of the vertical axis form a loop.

lem6.5 **Lemma 6.5.** *Let $0 < \beta < \mu_2$. We have the following properties for G_1, G_2 and η_1:*

(a) *$G_1(\beta, 0) = 0$, $G_1(\beta,\tau) < 0$ for $\tau > 0$ small, $\frac{\partial}{\partial\tau}G_1(\beta,\tau) > 0$ if $G_1(\beta,\tau) = 0$, and $\frac{\partial}{\partial\tau}G_1(\beta,\tau) \to +\infty$ as $\tau \to A(\beta)^-$;*

(b) *$G_2(\beta, 0) > 0$, $G_2(\beta,\tau)$ is strictly decreasing in $\tau \in [0, A(\beta)]$, and $\frac{\partial}{\partial\tau}G_2(\beta,\tau) \to -\infty$ as $\tau \to A(\beta)^-$;*

(c) *there exists $\tilde\beta \in (0, \mu_2)$ such that $\eta_1(\beta) < 0$ if $0 < \beta < \tilde\beta$, $\eta_1(\beta) = 0$ if $\beta = \tilde\beta$, and $\eta_1(\beta) > 0$ if $\tilde\beta < \beta < \mu_2$.*

Proof. (a) In account of the discussions in Section 3, we see that $G_1(\beta, 0) = 0$, $G_1(\beta,\tau) < 0$ for $\tau > 0$ small, and $\frac{\partial}{\partial\tau}G_1(\beta,\tau) > 0$ if $G_1(\beta,\tau) = 0$. For $\tau \in (0, A(\beta))$, we have

$$\frac{\partial}{\partial\tau}G_1(\beta,\tau) = \frac{\beta}{H_1(\beta,\tau)} + \frac{\beta}{H_2(\beta,\tau)} - 1.$$

By the fact that $H_i(\beta,\tau) > 0$ and $\lim_{\tau\to A(\beta)^-} H_2(\beta,\tau) = \alpha T^{\alpha-1}(\beta) + \beta - \mu_2 = 0$, we arrive at $\lim_{\tau\to A(\beta)^-} \frac{\partial}{\partial\tau}G_1(\beta,\tau) = +\infty$.

(b) We have $G_2(\beta, 0) = \beta k_2(\beta, 0) = \beta(\mu_2 - \beta)^{-\frac{1}{1-\alpha}} > 0$. As in the proof of Lemma 6.3, we have

$$H_1^2(\beta, A(\beta)) - H_1^2(\beta,\tau) < 2\alpha(\alpha-1)T^{\alpha-2}(\beta)(A(\beta) - \tau) < K_2^2(\beta, A(\beta)) - K_2^2(\beta,\tau).$$

Since

$$K_2(\beta,\tau) < K_2(\beta, A(\beta)) = 0 < H_1(\beta, A(\beta)) < H_1(\beta,\tau), \tag{6.6}$$ eq6.6

it can be deduced that for $0<\tau<A(\beta)$

$$H_1(\beta,\tau) > \sqrt{2\alpha(1-\alpha)T^{\alpha-2}(\beta)(A(\beta)-\tau)} > -K_2(\beta,\tau) > 0.$$

Therefore,

$$\frac{\partial}{\partial\tau}G_2(\beta,\tau) = \frac{\beta(H_1(\beta,\tau)+K_2(\beta,\tau))}{H_1(\beta,\tau)K_2(\beta,\tau)} - 1 < 0 \quad \text{for} \quad 0<\tau<A(\beta),$$

which implies $G_2(\beta,\tau)$ is strictly decreasing in $\tau\in[0,A(\beta)]$. In account of (6.6) again, we have $\lim_{\tau\to A(\beta)^-}\frac{\partial}{\partial\tau}G_2(\beta,\tau) = -\infty$.

(c) We estimate $\eta_1(\beta)$ as

$$\eta_1(\beta) \le 2\beta T(\beta) - A(\beta) = 2\beta\alpha^{\frac{1}{1-\alpha}}(\mu_2-\beta)^{-\frac{1}{1-\alpha}} - \big(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\big)(\mu_2-\beta)^{-\frac{\alpha}{1-\alpha}}$$

and

$$\eta_1(\beta) \ge \beta T(\beta) - A(\beta) = \beta\alpha^{\frac{1}{1-\alpha}}(\mu_2-\beta)^{-\frac{1}{1-\alpha}} - \big(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\big)(\mu_2-\beta)^{-\frac{\alpha}{1-\alpha}},$$

from which we see that $\lim_{\beta\to0^+}\eta_1(\beta) = -\big(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\big)\mu_2^{-\frac{\alpha}{1-\alpha}} < 0$ and $\lim_{\beta\to\mu_2^-}\eta_1(\beta) = +\infty$. Then using Lemma 6.4 we claim that there exists $\tilde\beta\in(0,\mu_2)$ such that $\eta_1(\beta)<0$ if $0<\beta<\tilde\beta$, $\eta_1(\beta)=0$ if $\beta=\tilde\beta$, and $\eta_1(\beta)>0$ if $\tilde\beta<\beta<\mu_2$. □

Let us now assume $0<\beta<\mu_1$ and study properties of $G_4(\beta,\tau)$ which is the most complex function among all the G_i $(i=1,2,3,4)$. We set

$$\gamma(\beta) := \min_{0\le\tau\le A(\beta)} G_4(\beta,\tau).$$

lem6.6 **Lemma 6.6.** *There exists $\tau=\tau(\beta)\in(0,A(\beta))$ such that $\frac{\partial}{\partial\tau}G_4(\beta,\tau)<0$ for $\tau\in(0,\tau(\beta))$ and $\frac{\partial}{\partial\tau}G_4(\beta,\tau)>0$ for $\tau\in(\tau(\beta),A(\beta))$. As a consequence,*

$$\gamma(\beta) = \min_{0\le\tau\le A(\beta)} G_4(\beta,\tau) = G_4(\beta,\tau(\beta)).$$

Proof. For $\tau\in(0,A(\beta))$,

$$\frac{\partial}{\partial\tau}G_4(\beta,\tau) = \frac{\beta}{K_1(\beta,\tau)} + \frac{\beta}{H_2(\beta,\tau)} - 1.$$

Since $\lim_{\tau\to0^+}K_1(\beta,\tau) = -(1-\alpha)(\mu_1-\beta)$ and $\lim_{\tau\to0^+}H_2(\beta,\tau)=+\infty$, we see that

$$\lim_{\tau\to0^+}\frac{\partial}{\partial\tau}G_4(\beta,\tau) = -\frac{\beta}{(1-\alpha)(\mu_1-\beta)} - 1 < 0.$$

Since $K_1(\beta,\tau) < \alpha k_1^{\alpha-1}(\beta,A(\beta)) + \beta - \mu_1 < 0$, $H_2(\beta,\tau)>0$, and $\lim_{\tau\to A(\beta)^-}H_2(\beta,\tau)=0$, we have

$$\lim_{\tau\to A(\beta)^-}\frac{\partial}{\partial\tau}G_4(\beta,\tau) = +\infty.$$

Then the minimum $\gamma(\beta)$ can only be attained at some τ in the open interval $(0,A(\beta))$, where $\frac{\partial}{\partial\tau}G_4(\beta,\tau)=0$. We show that there is only one such τ. Let $\tau\in(0,A(\beta))$ be such that $\frac{\partial}{\partial\tau}G_4(\beta,\tau)=0$. Then

$$\frac{1}{H_2^3(\beta,\tau)} = \frac{1}{\beta^3} - \frac{1}{K_1^3(\beta,\tau)} - \frac{3}{\beta K_1(\beta,\tau)H_2(\beta,\tau)}.$$

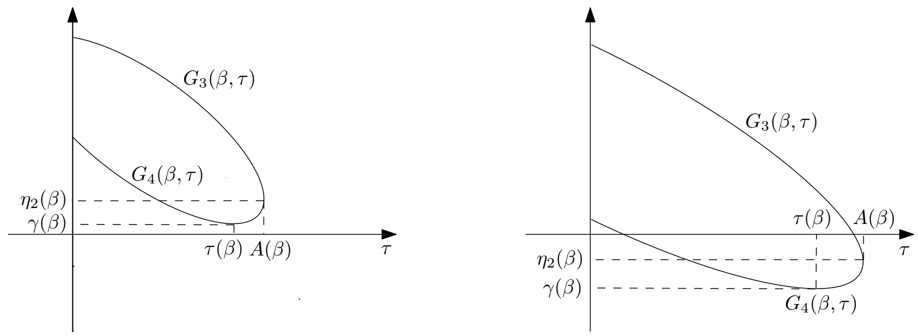
Using this equation, we compute the second partial derivative of G_4 with respect to τ:

$$\frac{\partial^2}{\partial\tau^2}G_4(\beta,\tau) = \beta\alpha(1-\alpha)\left(\frac{k_1^{\alpha-2}(\beta,\tau)-h_2^{\alpha-2}(\beta,\tau)}{K_1^3(\beta,\tau)} + h_2^{\alpha-2}(\beta,\tau)\Big(\frac{1}{\beta^3} - \frac{3}{\beta K_1(\beta,\tau)H_2(\beta,\tau)}\Big)\right).$$

Since $\alpha\in(0,1)$, $k_1(\beta,\tau)>h_2(\beta,\tau)>0$ and $K_1(\beta,\tau)<0<H_2(\beta,\tau)$, we have $\frac{\partial^2}{\partial\tau^2}G_4(\beta,\tau)>0$. This implies that there is exactly one $\tau=\tau(\beta)\in(0,A(\beta))$ such that $\frac{\partial}{\partial\tau}G_4(\beta,\tau(\beta))=0$ and the result follows. □

The next lemma shows that $\gamma(\beta)$ is strictly increasing in $\beta\in(0,\mu_1)$.

lem6.7 **Lemma 6.7.** *$\gamma'(\beta)>0$ for $\beta\in(0,\mu_1)$.*



Case : $\beta_0 < \beta < \mu_1,\ \gamma(\beta) > 0$ Case : $0 < \beta < \beta_0,\ \gamma(\beta) < 0$

FIGURE 4. The graphs of G_3 and G_4

Proof. Indeed, for $\beta \in (0, \mu_1)$,
$$\frac{d\gamma(\beta)}{d\beta} = \frac{\partial}{\partial\beta} G_4(\beta, \tau(\beta)) + \frac{\partial}{\partial\tau} G_4(\beta, \tau(\beta)) \frac{d\tau(\beta)}{d\beta} = \frac{\partial}{\partial\beta} G_4(\beta, \tau(\beta)).$$
Taking partial derivative in the equation $k_1^\alpha(\beta,\tau) + (\beta - \mu_1)k_1(\beta,\tau) = \tau$ with respect to β, we obtain $\frac{\partial}{\partial\beta} k_1(\beta,\tau) = -\frac{k_1(\beta,\tau)}{K_1(\beta,\tau)}$. Similarly, $\frac{\partial}{\partial\beta} h_2(\beta,\tau) = -\frac{h_2(\beta,\tau)}{H_2(\beta,\tau)}$. Then
$$\frac{d\gamma(\beta)}{d\beta} = k_1(\beta,\tau(\beta)) + h_2(\beta,\tau(\beta)) + \beta\Big(- \frac{k_1(\beta,\tau(\beta))}{K_1(\beta,\tau(\beta))} - \frac{h_2(\beta,\tau(\beta))}{H_2(\beta,\tau(\beta))}\Big).$$
Using again the fact that $K_1^{-1} + H_2^{-1} = \beta^{-1}$ at $\tau = \tau(\beta)$, we arrive at
$$\frac{d\gamma(\beta)}{d\beta} = k_1(\beta,\tau(\beta)) + \frac{\beta}{K_1(\beta,\tau(\beta))}(h_2(\beta,\tau(\beta)) - k_1(\beta,\tau(\beta))).$$
Since $k_1(\beta,\tau(\beta)) > h_2(\beta,\tau(\beta)) > 0$ and $K_1(\beta,\tau(\beta)) < 0$, we obtain $\gamma'(\beta) > 0$, as required. □

lem6.8 **Lemma 6.8.** *We have*
$$\lim_{\beta\to\mu_1^-} \eta_2(\beta) = \lim_{\beta\to\mu_1^-} \gamma(\beta) = +\infty$$
and
$$\lim_{\beta\to 0^+} \eta_2(\beta) = \lim_{\beta\to 0^+} \gamma(\beta) = -\big(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\big)\mu_2^{-\frac{\alpha}{1-\alpha}} < 0.$$

Proof. For $\beta \in (0,\mu_1)$, since $k_1(\beta,\tau) > \Big(\frac{\alpha}{\mu_1-\beta}\Big)^{\frac{1}{1-\alpha}}$, we see that
$$\eta_2(\beta) > \gamma(\beta) \geq \beta\alpha^{\frac{1}{1-\alpha}}(\mu_1-\beta)^{-\frac{1}{1-\alpha}} - \big(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\big)(\mu_2-\beta)^{-\frac{\alpha}{1-\alpha}}.$$
This implies $\lim_{\beta\to\mu_1^-}\eta_2(\beta) = \lim_{\beta\to\mu_1^-}\gamma(\beta) = +\infty$. Since $0 < k_1(\beta, A(\beta)) + h_2(\beta, A(\beta)) < 2(\mu_1 - \beta)^{-\frac{1}{1-\alpha}}$, we have, for $\beta \in (0,\mu_1)$,
$$-A(\beta) < \gamma(\beta) < \eta_2(\beta) < 2\beta(\mu_1-\beta)^{-\frac{1}{1-\alpha}} - A(\beta),$$
and thus
$$\lim_{\beta\to 0^+}\eta_2(\beta) = \lim_{\beta\to 0^+}\gamma(\beta) = -\lim_{\beta\to 0^+} A(\beta) = -\big(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\big)\mu_2^{-\frac{\alpha}{1-\alpha}} < 0.$$
The proof is complete. □

In the next lemma, we list properties of G_3, G_4, η_2 and γ which will be used to study the number of zeros $\tau \in (0, A(\beta)]$ of the two equations $G_3(\beta,\tau) = 0$ and $G_4(\beta,\tau)=0$ for $\beta \in (0,\mu_1)$. We illustrate the graphs of G_3 and G_4 in Figure 4. These graphs and part of the vertical axis make up a loop.

lem6.9 **Lemma 6.9.** *G_3, G_4, η_2 and γ have the following properties:*

(a) *$G_3(\beta,\tau) > G_4(\beta,\tau)$ for $0 \leq \tau < A(\beta)$;*
(b) *$G_3(\beta,\tau)$ is strictly decreasing in $\tau \in [0, A(\beta)]$;*
(c) *$G_4(\beta,\tau)$ is strictly decreasing in $\tau \in [0,\tau(\beta)]$ and strictly increasing in $\tau \in [\tau(\beta), A(\beta)]$;*
(d) *$\eta_2(\beta)$ and $\gamma(\beta)$ are strictly increasing in $\beta \in (0,\mu_1)$;*

(e) *there exist* $\hat\beta \in (0, \mu_1)$ *and* $\beta_0 \in (\hat\beta, \mu_1)$ *such that* $\eta_2(\beta) < 0$ *if* $0 < \beta < \hat\beta$, $\eta_2(\beta) = 0$ *if* $\beta = \hat\beta$, *and* $\eta_2(\beta) > 0$ *if* $\hat\beta < \beta < \mu_1$, *and that* $\gamma(\beta) < 0$ *if* $0 < \beta < \beta_0$, $\gamma(\beta) = 0$ *if* $\beta = \beta_0$, *and* $\gamma(\beta) > 0$ *if* $\beta_0 < \beta < \mu_1$.

Proof. (a) This is directly from the definitions of G_3 and G_4.

(b) This is true since both $k_1(\beta, \tau)$ and $k_2(\beta, \tau)$ are strictly decreasing in $\tau \in [0, A(\beta)]$.

(c) This is deduced from Lemma 6.6.

(d) This follows from Lemmas 6.4 and 6.7.

(e) This is a consequence of (d) combined with Lemma 6.8. □

Proof of Theorem 1.5(c) in the case $\mu_1 < \mu_2$. As we have seen before, to find the exact number of synchronized positive solutions of (6.5) it is sufficient to study the exact number of solutions of the four equations $G_i(\tau) = G_i(\beta, \tau) = 0$ in $\tau \in (0, A(\beta)]$ for $i = 1, 2, 3, 4$.

Let us first consider the two equations $G_1(\beta, \tau) = 0$ and $G_2(\beta, \tau) = 0$ for $\beta \in (0, \mu_2)$. By Lemma 6.5, if $0 < \beta < \tilde\beta$ then the equation $G_1(\beta, \tau) = 0$ has no solution and the equation $G_2(\beta, \tau) = 0$ has exactly one solution τ_1 which lies in $(0, A(\beta))$; if $\beta = \tilde\beta$ then the two equations have the same solution $\tau = A(\beta)$; and if $\tilde\beta < \beta < \mu_2$ then the first equation has exactly one solution τ_2 which lies in $(0, A(\beta))$ and the second equation has no solution. In each of these three cases, exactly one synchronized positive solution, which is $(h_1^{1/q}(\tau_1)U, k_2^{1/q}(\tau_1)U)$ if $0 < \beta < \tilde\beta$, $(h_1^{1/q}(A)U, h_2^{1/q}(A)U)$ if $\beta = \tilde\beta$, or $(h_1^{1/q}(\tau_2)U, h_2^{1/q}(\tau_2)U)$ if $\tilde\beta < \beta < \mu_2$, of equation (6.5) with $0 < \beta < \mu_2$ can be deduced from the two equations $G_1(\beta, \tau) = 0$ and $G_2(\beta, \tau) = 0$.

Second, we consider the two equations $G_3(\beta, \tau) = 0$ and $G_4(\beta, \tau) = 0$ for $\beta \in (0, \mu_1)$. By Lemma 6.9, if $0 < \beta < \hat\beta$ then each of the two equations $G_3(\beta, \tau) = 0$ and $G_4(\beta, \tau) = 0$ has exactly one solution, denoted τ_3 and τ_4 respectively, which lies in $(0, A(\beta))$; if $\beta = \hat\beta$ then the equation $G_3(\beta, \tau) = 0$ has no solution in $(0, A(\beta))$, the equation $G_4(\beta, \tau) = 0$ has exactly one solution τ_5 which is in $(0, A(\beta))$, and both the equations have the same solution $\tau = A(\beta)$; if $\hat\beta < \beta < \beta_0$ then the equation $G_3(\beta, \tau) = 0$ has no solution and the equation $G_4(\beta, \tau) = 0$ has exactly two solutions τ_6 and τ_7 which lie in $(0, A(\beta))$; if $\beta = \beta_0$ then the equation $G_3(\beta, \tau) = 0$ has no solution and the equation $G_4(\beta, \tau) = 0$ has exactly one solution τ_8 in $(0, A(\beta))$; if $\beta_0 < \beta < \mu_1$ then both $G_3(\beta, \tau) = 0$ and $G_4(\beta, \tau) = 0$ have no solution. This implies that the two equations $G_3(\beta, \tau) = 0$ and $G_4(\beta, \tau) = 0$ produce exactly two synchronized positive solutions for (6.5) with $0 < \beta < \beta_0$, which are $(k_1^{1/q}(\tau_3)U, k_2^{1/q}(\tau_3)U)$ and $(k_1^{1/q}(\tau_4)U, h_2^{1/q}(\tau_4)U)$ for $0 < \beta < \hat\beta$, $(k_1^{1/q}(\tau_5)U, h_2^{1/q}(\tau_5)U)$ and $(k_1^{1/q}(A)U, T^{1/q}U)$ for $\beta = \hat\beta$, and $(k_1^{1/q}(\tau_6)U, h_2^{1/q}(\tau_6)U)$ and $(k_1^{1/q}(\tau_7)U, h_2^{1/q}(\tau_7)U)$ for $\hat\beta < \beta < \beta_0$; they yield exactly one synchronized positive solution for (6.5) with $\beta = \beta_0$ which is $(k_1^{1/q}(\tau_8)U, h_2^{1/q}(\tau_8)U)$; and they result in no synchronized positive solution for (6.5) with $\beta > \beta_0$.

We remark that the synchronized positive solution obtained via $G_1(\beta, \tau) = 0$ and $G_2(\beta, \tau) = 0$ is different from those obtained via $G_3(\beta, \tau) = 0$ and $G_4(\beta, \tau) = 0$. Combining the results obtained via the equations $G_i(\beta, \tau) = 0$, $i = 1, 2, 3, 4$, we see that (6.5) has exactly three synchronized positive solutions if $0 < \beta < \beta_0$, exactly two synchronized positive solutions if $\beta = \beta_0$, and exactly one synchronized positive solution if $\beta > \beta_0$. □

In the special case where $N = 8$ and $p = q = \frac{4}{3}$, we present an elementary proof of Theorem 1.5(c) and give a more specific expression for β_0. In the present case, in account of the discussions at the beginning of Section 3, the number of synchronized positive solutions of (6.5) is equal to the the number of positive solutions of the system

$$\mu_1 t_1 + \beta t_2 = t_1^{\frac{1}{2}}, \quad \beta t_1 + \mu_2 t_2 = t_2^{\frac{1}{2}}.$$

Set $x = \sqrt{t_1}$ and $y = \sqrt{t_2}$. Then we rewrite the above system as

$$\mu_1 x^2 + \beta y^2 = x, \quad \beta x^2 + \mu_2 y^2 = y. \tag{6.7}$$

eq6.7

We solve the first equation for y and obtain

$$y = \sqrt{\frac{x - \mu_1 x^2}{\beta}}. \tag{6.8}$$

eq6.8

Inserting this expression into the second equation, after simplifying it we have an equation for x:

$$(\beta^2-\mu_1\mu_2)^2x^3+2\mu_2(\beta^2-\mu_1\mu_2)x^2+(\beta\mu_1+\mu_2^2)x-\beta=0. \tag{6.9}$$ eq6.9

Here we use the fact that we are searching for positive solutions and thus $x\neq 0$. If $\beta=\sqrt{\mu_1\mu_2}$ then (6.9) is a linear equation and (6.7) has a unique positive solution $\big(\frac{\mu_1^{1/2}}{\mu_1^{3/2}+\mu_2^{3/2}},\frac{\mu_2^{1/2}}{\mu_1^{3/2}+\mu_2^{3/2}}\big)$. In what follows we assume $\beta\neq\sqrt{\mu_1\mu_2}$. Then (6.9) is a cubic equation. Recall that the general cubic ax^3+bx^2+cx+d has three distinct real roots if and only if its discriminant

$$\Delta:=18abcd-4b^3d+b^2c^2-4ac^3-27a^2d^2>0$$

and has only one real root if $\Delta<0$. The cubic may have only one real root or two distinct real roots if $\Delta=0$. For (6.9) the discriminant is

$$\Delta=\beta^2(\beta^2-\mu_1\mu_2)^2\big(-27\beta^4+18\beta^2\mu_1\mu_2-4\beta(\mu_1^3+\mu_2^3)+\mu_1^2\mu_2^2\big).$$

Set

$$f(\beta)=-27\beta^4+18\beta^2\mu_1\mu_2-4\beta(\mu_1^3+\mu_2^3)+\mu_1^2\mu_2^2. \tag{6.10}$$ eq6.10

The elementary inequality $\beta\mu_1\mu_2\le 3\beta^3+\frac19\mu_1^3+\frac19\mu_2^3$ implies that $f'(\beta)=-108\beta^3+36\beta\mu_1\mu_2-4(\mu_1^3+\mu_2^3)\le 0$ and $f'(\beta)=0$ if and only if $\mu_1=\mu_2=3\beta$. Thus $f'(\beta)<0$ except for the case $\mu_1=\mu_2=3\beta$. If $\mu_1=\mu_2=3\beta$ then $f(\beta)=f'(\beta)=f''(\beta)=0$ and $f'''(\beta)=-648\beta<0$. In any case, the quartic polynomial $f(\beta)$ in (6.10) has a unique positive zero, denoted β_0, so that $f(\beta_0)=0$, $f(\beta)>0$ for $0<\beta<\beta_0$, and $f(\beta)<0$ for $\beta>\beta_0$. Since $\mu_1^2\mu_2\le\frac{4\sqrt2\mu_1^3+\mu_2^3}{6}$ and $\mu_1\mu_2^2\le\frac{\mu_1^3+2\mu_2^3}{3}$, we have

$$f(\mu_1)=\mu_1(-31\mu_1^3-4\mu_2^3+18\mu_1^2\mu_2+\mu_1\mu_2^2)\le-\frac{\mu_1}{3}\big((92-36\sqrt2)\mu_1^3+\mu_2^3\big)<0.$$

This shows that $\beta_0\in(0,\mu_1)$. The above discussion shows that (6.9) has exactly one real solution for $\beta>\beta_0$ and exactly three real solutions for $0<\beta<\beta_0$. Writing (6.9) in the form

$$x\big[(\beta^2-\mu_1\mu_2)x+\mu_2\big]^2+\beta(\mu_1x-1)=0,$$

we see that its real solutions must lie in the range $(0,\mu_1^{-1})$. Thus any real solution of (6.9) uniquely determines a positive y through (6.8). Therefore, (6.7) has exactly one positive solution for $\beta>\beta_0$ and exactly three positive solutions for $0<\beta<\beta_0$, which implies that (1.3) has exactly one synchronized positive solution for $\beta>\beta_0$ and exactly three synchronized positive solutions for $0<\beta<\beta_0$.

Now we determine the number of positive solutions of (6.7) for $\beta=\beta_0$. We first consider the case $\mu_1=\mu_2$ and we observe that $\beta_0=\frac13\mu_1$ through the above discussion. Let $\beta=\beta_0$ in this case. Then equation (6.9) is reduced to

$$64\beta^4x^3-48\beta^3x^2+12\beta^2x-\beta=0.$$

From this equation and (6.8) we see that (6.7) has exactly one positive solution $x=y=\frac{1}{4\beta}$. Now we assume $\mu_1\neq\mu_2$ and show that (6.9) has exactly two real solutions. If not, then it has only one real solution x_0 which must satisfy

$$x_0=\frac{2\mu_2}{3(\mu_1\mu_2-\beta^2)},\quad x_0^2=\frac{\beta\mu_1+\mu_2^2}{3(\mu_1\mu_2-\beta^2)^2},\quad x_0^3=\frac{\beta}{(\mu_1\mu_2-\beta^2)^2}.$$

But these expressions imply $\mu_1=\mu_2=3\beta$, a contradiction.

7. Results on systems with subcritical exponents

Let $N\ge1$ and $n\ge2$. The arguments used above are applicable to system (1.7) with subcritical exponents:

$$\begin{cases}-\Delta u_i+u_i=\mu_iu_i^{r-1}+\sum\limits_{j=1,\,j\neq i}^{n}\beta_{ij}u_i^{p_{ij}-1}u_j^{q_{ij}} & \text{in } \mathbb{R}^N,\\ u_i>0,\quad u_i\in H^1(\mathbb{R}^N),\quad i=1,2,\cdots,n,\end{cases}$$

where $0<\mu_1\le\cdots\le\mu_n$, $\beta_{ij}>0$, $2<r<2^*$, $p_{ij}<r$, and $p_{ij}+q_{ij}=r$ for $i\neq j\in\{1,2,\cdots,n\}$. For this system synchronized positive solutions are of the form $(k_1U,\cdots,k_nU)$ with k_i positive and U being the unique positive radial solution of

$$-\Delta u+u=u^r\quad\text{in } \mathbb{R}^N,\quad u\in H^1(\mathbb{R}^N).$$

We have the following four theorems, which are stated without proofs since their proofs are easy adaptations of those of Theorems 1.1-1.4. We remark that in this section results on uniqueness, multiplicity and exact multiplicity of solutions are all up to translation.

thm7.1 **Theorem 7.1.** *Assume $N \geq 1$, $\mu_i > 0$, $2 < r < 2^*$, $p_{ij} < 2$, and $p_{ij}+q_{ij} = r$ for $i \neq j \in \{1, 2, \cdots, n\}$. We have the following conclusions.*

(a) *There exists $\beta_* > 0$ such that if $0 < \beta_{ij} < \beta_*$ then (1.7) has at least $2^n - 1$ synchronized positive solutions.*
(b) *Assume in addition that there exists $p < 2$ such that $p_{ij} = p$ for $i \neq j$. If the matrix B has an inverse $A = (a_{ij})_{n\times n}$ such that*

$$a_{ij} > 0 \text{ for } i \neq j \text{ and } \sum_{j=1}^{n} a_{ij} > 0 \text{ for } i = 1, 2, \cdots, n,$$

then (1.7) has at least one synchronized positive solution.

Remark 7.1. As for the critical case an explicit β_* can be given in terms of μ_i, r, and p_{ij}.

thm7.2 **Theorem 7.2.** *Assume $N \geq 1$, $\mu_i > 0$, $2 < r < 2^*$, $p_{ij} = p < 2$, $q_{ij} = r - p$ and $\beta_{ij} = \beta$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. We have the following conclusions.*

(a) *(1.7) has at least one synchronized positive solution for any $\beta > 0$.*
(b) *If $\beta \geq \mu_n$, or if $\mu_{n-1} < \mu_n$ and $\beta \geq \mu_n - \delta_0$ where δ_0 is some positive number, then (1.7) has exactly one synchronized positive solution.*
(c) *There exists $\beta_0 \in (0, \mu_1)$ such that (1.7) has exactly $2^n - 1$ synchronized positive solutions for $0 < \beta < \beta_0$.*
(d) *Assume moreover $n = 2m$, $\mu_1 = \cdots = \mu_m =: \mu' \leq \mu_{m+1} = \cdots = \mu_{2m} =: \mu''$, and*

$$\beta > \frac{(m+1)\mu'' - (m-1)\mu' + \sqrt{(m+1)^2\mu''^2 + (m-1)^2\mu'^2 - 2(m^2+1)\mu'\mu''}}{2}.$$

Then (1.7) has exactly one positive solution.

thm7.3 **Theorem 7.3.** *Assume $N \geq 1$, $\mu_i > 0$, $2 < r < 2^*$, $p_{ij} = 2$, $q_{ij} = r - 2$, and $\beta_{ij} = \beta$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. We have the following conclusions.*

(a) *(1.7) has a synchronized positive solution if and only if $\beta > \mu_n$ or $0 < \beta < \mu_1$ or $\beta = \mu_1 = \mu_n$. Moreover, if $\beta > \mu_n$ or $0 < \beta < \mu_1$ then (1.7) has exactly one synchronized positive solution and if $\beta = \mu_1 = \mu_n$ then (1.7) has infinitely many synchronized positive solutions.*
(b) *If $\mu_1 \leq \beta \leq \mu_n$ and $\mu_1 \neq \mu_n$ then (1.7) has no positive solution.*
(c) *If $\beta > \mu_n$ then (1.7) has exactly one positive solution.*

thm7.4 **Theorem 7.4.** *Assume $N \geq 1$, $\mu_i > 0$, $2 < r < 2^*$, $2 < p_{ij} < r$, $p_{ij} + q_{ij} = r$, and $\beta_{ij} > 0$. Then*

(a) *(1.7) has a synchronized positive solution.*

Assume moreover $p_{ij} = p \in (2, r)$, $q_{ij} = r - p$, $\beta_{ij} = \beta$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. Then we have the following conclusions.

(b) *If $0 < \beta \leq \mu_1$, or if $\mu_1 < \mu_2$ and $\beta < \mu_1 + \delta_0$ where δ_0 is some positive number, then (1.7) has exactly one synchronized positive solution.*
(c) *If $\beta > \mu_j$ then there exists $p_1 = p_1(\beta) \in (2, r)$ such that for $p \in (2, p_1)$ (1.7) has at least $2^j - 1$ synchronized positive solutions. In particular, if $\beta > \mu_n$ then for p larger than and sufficiently close to 2, (1.7) has at least $2^n - 1$ synchronized positive solutions.*
(d) *If $\beta > \mu_1$ and*

$$\frac{\mu_1}{\beta}2 + \Big(1 - \frac{\mu_1}{\beta}\Big)r \leq p < r,$$

then (1.7) has exactly one synchronized positive solution. In particular, for any $\beta > 0$, (1.7) has exactly one synchronized positive solution if $p < r$ and p is sufficiently close to r.

The methods used in this paper are also applicable to other systems of elliptic equations, for example, to the m-Laplacian system

$$-\Delta_m u_i + \lambda_i |u_i|^{m-2} u_i = \sum_{j=1}^{n} \beta_{ij} |u_j|^{q_{ij}} |u_i|^{p_{ij}-2} u_i \quad \text{in } \Omega, \quad i = 1, 2, \cdots, n,$$

where $\Delta_m u = \operatorname{div}(|\nabla u|^{m-2}\nabla u)$, $1 < m < N$, $\lambda_i \ge 0$, $\beta_{ij} > 0$, $p_{ij} + q_{ij} = r$ for some $r \le m^* = \frac{mN}{N-m}$, and Ω is either a bounded domain in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$.

8. Proof of Theorem 1.6

In this section we always assume that $N \ge 5$, $\mu_i > 0$, $p, q \in (1, 2]$, $p + q = 2^*$, $\beta > \beta^0$, and $V_1, V_2 \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ are nonnegative functions. By the maximum principle, nontrivial solutions of the system

$$\begin{cases} -\Delta u + V_1(x) u = \mu_1 (u^+)^{2^*-1} + \frac{p\beta}{2^*} (u^+)^{p-1} (v^+)^q & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x) v = \mu_2 (v^+)^{2^*-1} + \frac{q\beta}{2^*} (u^+)^{p} (v^+)^{q-1} & \text{in } \mathbb{R}^N, \\ u, v \in \mathscr{D}^{1,2}(\mathbb{R}^N) \end{cases} \tag{8.1}$$ eq8.1

are positive solutions of (1.5), where $u^+ = \max\{u, 0\}$ and $v^+ = \max\{v, 0\}$. Note that (8.1) has a variational structure. We shall prove as in [24] the existence of a nontrivial solution of (8.1) by a variational approach.

We first introduce some notation and recall some known facts. Let $\|\cdot\|$ and $|\cdot|_s$ be the usual norms in $\mathscr{D}^{1,2}(\mathbb{R}^N)$ and $L^s(\mathbb{R}^N)$ respectively. Set $\mathcal{H} = \mathscr{D}^{1,2}(\mathbb{R}^N) \times \mathscr{D}^{1,2}(\mathbb{R}^N)$ equipped with the norm $\|(u, v)\| = (\|u\|^2 + \|v\|^2)^{\frac{1}{2}}$. Clearly, critical points of the energy functional

$$I(u, v) = \frac{1}{2}\|(u, v)\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} \left(V_1(x)u^2 + V_2(x)v^2\right) - \frac{1}{2^*}\int_{\mathbb{R}^N} \left(\mu_1 (u^+)^{2^*} + \beta (u^+)^p (v^+)^q + \mu_2 (v^+)^{2^*}\right)$$

defined on $\mathcal{H}$ are solutions of system (8.1). In order to avoid semitrivial critical points, we need to also consider for $j = 1, 2$ the scalar field equation $-\Delta u + V_j(x) u = \mu_j (u^+)^{2^*-1}$, $u \in \mathscr{D}^{1,2}(\mathbb{R}^N)$, and its associated functional ϕ_j. Define the infimum

$$c = \inf_{(u,v)\in\mathcal{N}} I(u, v) > 0$$

on the Nehari manifold

$$\mathcal{N} = \{(u, v) \in \mathcal{H} \mid (u, v) \ne (0, 0),\ \langle I'(u, v), (u, v)\rangle = 0\}.$$

For $j = 1, 2$, let γ_j be the infimum of the functional ϕ_j on its Nehari manifold. If $V_1 = V_2 = 0$, we denote I, $\mathcal{N}$, c and γ_j by I_∞, $\mathcal{N}_\infty$, c_∞ and $\gamma_{j\infty}$, respectively. Since $V_j \ge 0$ and $V_j \in L^{\frac{N}{2}}(\mathbb{R}^N)$, the argument in [6, Proposition 2.2] implies that

$$\gamma_j = \gamma_{j\infty} = \frac{1}{N}\mu_j^{(2-N)/2} S^{N/2}. \tag{8.2}$$ eq8.2

It is also well known that $\gamma_{j\infty}$ is attained by the functions $W_{\delta,z,j}(x) = \mu_j^{-\frac{N-2}{4}} U_{\delta,z}(x)$, where $U_{\delta,z}(x) = \delta^{(2-N)/2} U((x - z)/\delta)$ for $\delta > 0$ and $z \in \mathbb{R}^N$.

By Theorem 1.5(b), (1.4′) with $\beta > \beta^0$ has exactly one positive solution. Moreover, by the proofs of Lemmas 6.1 and 6.2, this solution takes the form $(k_1 U, k_2 U)$, where (k_1, k_2) is a positive solution, which is unique according to Proposition 6.1, of the algebraic system

$$\mu_1 k_1^q - k_1^{2-p} + \frac{p\beta}{2^*} k_2^q = 0, \quad \mu_2 k_2^p - k_2^{2-q} + \frac{q\beta}{2^*} k_1^p = 0. \tag{8.3}$$ eq8.3

lem8.1 **Lemma 8.1.** *There holds* $c_\infty = \frac{1}{N}(k_1^2 + k_2^2) S^{\frac{N}{2}}$.

Proof. Let $\epsilon > 0$ be such that $|u^+|_{2^*}^{2^*} + |v^+|_{2^*}^{2^*} \ge 2\epsilon$ for any $(u, v) \in \mathcal{N}_\infty$. Let $(u, v) \in \mathcal{N}_\infty$ and assume without loss of generality that $|v^+|_{2^*}^{2^*} \ge \epsilon$. Setting $t = |u^+|_{2^*}^2 / |v^+|_{2^*}^2 \ge 0$ and using the Sobolev inequality, we have

$$\|(u, v)\|^2 \ge S\big(|u^+|_{2^*}^2 + |v^+|_{2^*}^2\big) = S(1 + t)|v^+|_{2^*}^2. \tag{8.4}$$ eq8.4

By the Hölder inequality,

$$\|(u,v)\|^2 \le \mu_1|u^+|_{2^*}^{2^*} + \beta|u^+|_{2^*}^p|v^+|_{2^*}^q + \mu_2|v^+|_{2^*}^{2^*} = (\mu_1 t^{\frac{2^*}{2}} + \beta t^{\frac{p}{2}} + \mu_2)|v^+|_{2^*}^{2^*},$$

which combined with (8.4) implies that

$$\|(u,v)\|^2 \ge g(t) := \frac{S^{\frac{N}{2}}(1+t)^{\frac{N}{2}}}{\big(\mu_1 t^{\frac{2^*}{2}} + \beta t^{\frac{p}{2}} + \mu_2\big)^{\frac{N-2}{2}}}.$$

This implies

$$c_\infty \ge \frac{1}{N}\inf_{t\ge 0} g(t). \tag{8.5}$$

eq8.5

The derivative of g has the form

$$g'(t) = \frac{NS^{\frac{N}{2}}(1+t)^{\frac{N-2}{2}}h(t)}{2\big(\mu_1 t^{\frac{2^*}{2}} + \beta t^{\frac{p}{2}} + \mu_2\big)^{\frac{N}{2}}},$$

where $h(t) = \frac{q\beta}{2^*}t^{\frac{p}{2}} - \frac{p\beta}{2^*}t^{\frac{p-2}{2}} - \mu_1 t^{\frac{2}{N-2}} + \mu_2$. Since $p,q\in(1,2]$ and $p+q=2^*$ and thus $\frac{p}{2} > \frac{2}{N-2}$, we see that $h(t)<0$ for $t>0$ small and $h(t)>0$ for $t>0$ large. Then there exists $t_1>0$ such that $g(t_1) = \inf_{t\ge 0} g(t)$. By (8.5), we have

$$c_\infty \ge \frac{1}{N}g(t_1). \tag{8.6}$$

eq8.6

Let $(u_0,v_0) = (\sqrt{t_1}bU_{1,0}, bU_{1,0})$ where $b = \frac{(1+t_1)^{(N-2)/4}}{(\mu_1 t_1^{2^*/2} + \beta t_1^{p/2} + \mu_2)^{(N-2)/4}}$. Then we have $(u_0,v_0)\in\mathcal{N}_\infty$ and $I_\infty(u_0,v_0) = \frac{1}{N}g(t_1)$. Hence $c_\infty \le \frac{1}{N}g(t_1)$, which combined with (8.6) implies $c_\infty = I_\infty(u_0,v_0) = \frac{1}{N}g(t_1)$. As a minimizer of $I_\infty|_{\mathcal{N}_\infty}$, it is easy to see that (u_0,v_0) is a synchronized positive solution of (1.4). By Theorem 1.5(b) and the definition of (k_1,k_2), we have $k_1 = \sqrt{t_1}b$ and $k_2 = b$. Then $t_1 = k_1^2/k_2^2$. Since $\mu_1 k_1^{2^*} + \beta k_1^p k_2^q + \mu_2 k_2^{2^*} = k_1^2 + k_2^2$, we finally obtain

$$c_\infty = \frac{1}{N}g(t_1) = \frac{1}{N}(k_1^2+k_2^2)S^{\frac{N}{2}}.$$

The proof is complete. □

As a byproduct of the proof of Lemma 8.1, using (8.2) we have

$$c_\infty < \frac{1}{N}\min\left\{\lim_{t\to+\infty} g(t),\ g(0)\right\} = \min\{\gamma_{1\infty},\ \gamma_{2\infty}\}.$$

Using an argument similar to the proof of [24, Lemma 2.6] which is for $N=4$, we can prove the following lemma. We omit the details.

lem8.2

Lemma 8.2. *If $V_j \in L^{\frac{N}{2}}(\mathbb{R}^N)$, $V_j \ge 0$, and $|V_1|_{N/2} + |V_2|_{N/2} > 0$, then $c = c_\infty$ and c is not achieved.*

The following lemma provides decomposition of Palais-Smale sequences of I and will be used to derive compactness of Palais-Smale sequences at certain levels. Similar results are well known for scalar field equations ([6, 10, 39]), which have been extended to coupled systems (see [24, 32]). The proof of the lemma can be conducted in the same spirit as in [24, 32] and is thus dropped.

lem8.3

Lemma 8.3. *Assume that $\{(u_n,v_n)\}\subset\mathcal{H}$ is a Palais-Smale sequence for the functional I at level d. Then there exist a nonnegative integer l, a solution (u^0,v^0) of (8.1), l nonzero solutions (u^j,v^j) $(1\le j\le l)$ of (8.1) with $V_1=V_2=0$, l sequences of positive numbers $\{\sigma_n^j\}$ $(1\le j\le l)$ and l sequences of points $\{z_n^j\}\subset\mathbb{R}^N$ $(1\le j\le l)$ such that, up to a subsequence and as $n\to\infty$,*

$$\Big\|(u_n,v_n) - (u^0,v^0) - \sum_{j=1}^{l}\Big((\sigma_n^j)^{-\frac{N-2}{2}}u^j\Big(\frac{\cdot - z_n^j}{\sigma_n^j}\Big), (\sigma_n^j)^{-\frac{N-2}{2}}v^j\Big(\frac{\cdot - z_n^j}{\sigma_n^j}\Big)\Big)\Big\| \to 0$$

and

$$I(u^0,v^0) + \sum_{j=1}^{l} I_\infty(u^j,v^j) = d.$$

Using Theorem 1.5(b), Lemmas 8.2 and 8.3, we prove the next lemma.

lem8.4 **Lemma 8.4.** *Let $d \in (c_\infty,\, \min\{\gamma_{1\infty}, \gamma_{2\infty}, 2c_\infty\})$ and assume $\{(u_n, v_n)\} \subset \mathcal{N}$ satisfies*

$$I(u_n, v_n) \to d, \quad (I|_{\mathcal{N}})'(u_n, v_n) \to 0, \quad \text{as } n \to \infty.$$

Then $\{(u_n, v_n)\}$ contains a subsequence converging in $\mathcal{H}$.

Proof. It is easy to check that $I'(u_n, v_n) \to 0$ in $\mathcal{H}^{-1}$. Let (u^0, v^0), (u^j, v^j), $\{\sigma_n^j\}$ and $\{z_n^j\}$, $j = 1, 2, \cdots, l$, be as in Lemma 8.3. Since $d < 2c_\infty$, either $l = 0$ or $l = 1$. To conclude, it suffices to prove that $l = 0$. Otherwise, $d = I(u^0, v^0) + I_\infty(u^1, v^1)$. If $(u^0, v^0) \neq (0, 0)$ then by Lemma 8.2 we have $d \geq 2c_\infty$, which is a contradiction. Thus $(u^0, v^0) = (0, 0)$ and $d = I_\infty(u^1, v^1)$. Since $u^1 \geq 0$ and $v^1 \geq 0$, by Theorem 1.5(b) and the uniqueness of positive solutions of the scalar equation $-\Delta u = \mu_j u^{2^*-1}$ in $\mathbb{R}^N$, (u^1, v^1) must be either $(k_1 U, k_2 U)$, or $(\mu_1^{(2-N)/4} U, 0)$, or $(0, \mu_2^{(2-N)/4} U)$, up to translation and dilation. Then either $d = c_\infty$, or $d = \gamma_{1\infty}$, or $d = \gamma_{2\infty}$, yielding a contradiction with the condition $d \in (c_\infty,\, \min\{\gamma_{1\infty}, \gamma_{2\infty}, 2c_\infty\})$. $\square$

By Lemma 8.4, to obtain a nontrivial solution of (8.1) it suffices to construct a Palais-Smale sequence of $I|_{\mathcal{N}}$ at some level $d \in (c_\infty,\, \min\{\gamma_{1\infty}, \gamma_{2\infty}, 2c_\infty\})$. This will be fulfilled by constructing a linking structure as follows.

For $(u, v) \in \mathcal{H}$ with $(u^+, v^+) \neq (0, 0)$, we define

$$\xi(u, v) = \frac{\int_{\mathbb{R}^N} \frac{x}{1+|x|} \big(\mu_1 (u^+)^{2^*} + \beta (u^+)^p (v^+)^q + \mu_2 (v^+)^{2^*}\big)}{\int_{\mathbb{R}^N} \big(\mu_1 (u^+)^{2^*} + \beta (u^+)^p (v^+)^q + \mu_2 (v^+)^{2^*}\big)}$$

and

$$\eta(u, v) = \frac{\int_{\mathbb{R}^N} \big|\frac{x}{1+|x|} - \xi(u, v)\big| \big(\mu_1 (u^+)^{2^*} + \beta (u^+)^p (v^+)^q + \mu_2 (v^+)^{2^*}\big)}{\int_{\mathbb{R}^N} \big(\mu_1 (u^+)^{2^*} + \beta (u^+)^p (v^+)^q + \mu_2 (v^+)^{2^*}\big)}.$$

Fix a number κ with $0 < \kappa < 1$ and set

$$\widehat{\mathcal{N}} = \{(u, v) \in \mathcal{N} \mid \xi(u, v) = 0,\ \eta(u, v) = \kappa\}.$$

We define

$$\hat{c} = \inf_{(u,v) \in \widehat{\mathcal{N}}} I(u, v).$$

From Theorem 1.5(b), Lemmas 8.1, 8.2 and 8.3, a similar argument as in [24, Lemma 4.1] leads to the following result.

lem8.5 **Lemma 8.5.** *If $V_j \in L^{\frac{N}{2}}(\mathbb{R}^N)$, $V_j \geq 0$, and $|V_1|_{N/2} + |V_2|_{N/2} > 0$, then $\hat{c} > c_\infty$.*

Now we construct a family of subsets of $\mathcal{N}$ which link with $\widehat{\mathcal{N}}$. By Lemmas 8.1 and 8.5 we have $\frac{1}{N}(k_1 + k_2) S^{N/2} < \hat{c}$. By condition (1.6), we also have

$$\frac{1}{N}(k_1^2 + k_2^2) S^{\frac{N}{2}} \Big(1 + \frac{k_1^2 |V_1|_{N/2} + k_2^2 |V_2|_{N/2}}{(k_1^2 + k_2^2) S}\Big)^{\frac{N}{2}} < \min\{\gamma_{1\infty},\ \gamma_{2\infty},\ 2c_\infty\}.$$

Then as in [7] and [48, P. 35], we can construct a nonnegative function $w \in C_0^\infty(B_1(0))$ such that $\int_{\mathbb{R}^N} w^{2^*} = \|w\|^2 > S^{\frac{N}{2}}$,

$$\frac{1}{N}(k_1^2 + k_2^2) \|w\|^2 < \hat{c}, \tag{8.7}$$ eq8.7

and

$$\frac{1}{N}(k_1^2 + k_2^2) \|w\|^2 \Big(1 + \frac{k_1^2 |V_1|_{N/2} + k_2^2 |V_2|_{N/2}}{(k_1^2 + k_2^2) S}\Big)^{\frac{N}{2}} < \min\{\gamma_{1\infty},\ \gamma_{2\infty},\ 2c_\infty\}. \tag{8.8}$$ eq8.8

For $\delta > 0$ and $z \in \mathbb{R}^N$, let $w_{\delta,z} = \delta^{(2-N)/2} w((\cdot - z)/\delta)$. We define $g_0 : \mathbb{R}^+ \times \mathbb{R}^N \to \mathcal{N}$ by $g_0(\delta, z) = (k_1 t_{\delta,z} w_{\delta,z}, k_2 t_{\delta,z} w_{\delta,z})$, where

$$t_{\delta,z} = \left(\frac{\|(k_1 w_{\delta,z}, k_2 w_{\delta,z})\|^2 + \int_{\mathbb{R}^N} \big(V_1(x)(k_1 w_{\delta,z})^2 + V_2(x)(k_2 w_{\delta,z})^2\big)}{\int_{\mathbb{R}^N} \big(\mu_1 (k_1 w_{\delta,z})^{2^*} + \beta (k_1 w_{\delta,z})^p (k_2 w_{\delta,z})^q + \mu_2 (k_2 w_{\delta,z})^{2^*}\big)}\right)^{\frac{N-2}{4}}.$$

Since (k_1, k_2) is a solution of (8.3) and $\int_{\mathbb{R}^N} w^{2^*} = \|w\|^2$, we have, for $\delta > 0$ and $z \in \mathbb{R}^N$,

$$I(g_0(\delta, z)) = \frac{\left((k_1^2 + k_2^2)\|w\|^2 + k_1^2 \int_{\mathbb{R}^N} V_1(x) w_{\delta,z}^2 + k_2^2 \int_{\mathbb{R}^N} V_2(x) w_{\delta,z}^2\right)^{\frac{N}{2}}}{N(k_1^2 + k_2^2)^{\frac{N-2}{2}} \|w\|^{N-2}}. \tag{8.9}$$ eq8.9

We need the next lemma to show the linking structure.

lem8.6 **Lemma 8.6.** *There exists $R_0 > 0$ and δ_1, δ_2 with $0 < \delta_1 < \kappa < \delta_2 < +\infty$ such that*

(a) *$I(g_0(\delta, z)) < \hat{c}$ if either $\delta = \delta_1$ or $\delta = \delta_2$ or $|z| = R_0$;*

(b) *$\langle \xi(g_0(\delta, z)), z\rangle_{\mathbb{R}^N} > 0$ if $|z| = R_0$, $\eta(g_0(\delta_1, z)) < \kappa$, and $\eta(g_0(\delta_2, z)) > \kappa$.*

Proof. Since the proof is similar to that in [24], it is only sketchy.

(a) As in [24, Lemma 4.2], $\int_{\mathbb{R}^N} V_j(x) w_{\delta,z}^2 \to 0$ uniformly in $z \in \mathbb{R}^N$ if either $\delta \to 0$ or $\delta \to +\infty$ and uniformly in $\delta > 0$ if $|z| \to +\infty$. Then the expression of $I(g_0(\delta, z))$ together with (8.7) implies that any δ_1 sufficiently small and R_0, δ_2 sufficiently large are eligible candidates.

(b) This is proved in the same way as in [24, Lemma 4.3], decreasing δ_1 and enlarging δ_2 from (a) if necessary. □

Set $D = \{(\delta, z) \mid \delta_1 < \delta < \delta_2,\ z \in \mathbb{R}^N,\ |z| < R_0\}$ and define $\Gamma = \{g \in C(\overline{D}, \mathcal{N}) \mid g|_{\partial D} = g_0|_{\partial D}\}$. From Lemma 8.6(b), using the argument of [24, Lemma 4.5], it is easy to prove the following result.

lem8.7 **Lemma 8.7.** $\deg((\eta \circ g, \xi \circ g),\ D,\ (\kappa, 0)) = 1$ *for any $g \in \Gamma$.*

For $g \in \Gamma$, by Lemma 8.6(a) we have $\widehat{\mathcal{N}} \cap g(\partial D) = \emptyset$, while by Lemma 8.7 we obtain $\widehat{\mathcal{N}} \cap g(D) \neq \emptyset$. In other words, the family $\{g(D) \mid g \in \Gamma\}$ and $\widehat{\mathcal{N}}$ link. Define

$$d := \inf_{g \in \Gamma} \max_{(\delta, z) \in \overline{D}} I(g(\delta, z)).$$

Since $\widehat{\mathcal{N}} \cap g(D) \neq \emptyset$, we have $d \geq \hat{c}$. Then using Lemma 8.5, we arrive at $d \geq \hat{c} > c_\infty$.

On the other hand, we have $d \leq \max_{(\delta,z)\in\overline{D}} I(g_0(\delta, z))$. From (8.9), using the Hölder and the Sobolev inequalities, we obtain

$$d \leq \frac{\left((k_1^2 + k_2^2)\|w\|^2 + k_1^2 |V_1|_{N/2} S^{-1} \|w\|^2 + k_2^2 |V_2|_{N/2} S^{-1} \|w\|^2\right)^{\frac{N}{2}}}{N(k_1^2 + k_2^2)^{\frac{N-2}{2}} \|w\|^{N-2}}.$$

The last inequality together with (8.8) implies that $d < \min\{\gamma_{1\infty},\ \gamma_{2\infty},\ 2c_\infty\}$.

Proof of Theorem 1.6. For any $g \in \Gamma$ and $(\delta, z) \in \partial D$, by Lemma 8.6(a) we have $I(g(\delta, z)) = I(g_0(\delta, z)) < \hat{c}$. Since $\hat{c} \leq d$, a standard deformation argument implies the existence of a Palais-Smale sequence $\{(u_n, v_n)\} \subset \mathcal{N}$ of $I|_{\mathcal{N}}$ at the level d. Then by Lemma 8.4, since $c_\infty < \hat{c} \leq d < \min\{\gamma_{1\infty},\ \gamma_{2\infty},\ 2c_\infty\}$, $\{(u_n, v_n)\}$ has a convergent subsequence in $\mathcal{H}$ and therefore I has a critical point (u, v) with $I(u, v) = d$, which is a solution of (8.1). This solution must be nontrivial since $d < \min\{\gamma_{1\infty},\ \gamma_{2\infty}\}$. By the maximum principle, (u, v) is a positive solution of (1.5). □

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