STABLE AND UNSTABLE PERIODIC SPIKY SOLUTIONS FOR THE GRAY-SCOTT SYSTEM AND THE SCHNAKENBERG SYSTEM

DANIEL GOMEZ, LINFENG MEI, AND JUNCHENG WEI

Abstract. The Hopf bifurcations for the classical Gray-Scott system and the Schnakenberg system in an one-dimensional interval are considered. For each system, the existence of time-periodic solutions near the Hopf bifurcation parameter for a boundary spike is rigorously proved by the classical Crandall-Rabinowitz theory. The criteria for the stability of limit cycles are determined and it is shown that the Hopf bifurcation is *supercritical* for the Schnakenberg system, and hence the bifurcating periodic solutions are linearly stable. For the Gray-Scott system, there is a *critical feeding rate*, when the feeding rate of the system is greater than this critical feeding rate, the Hopf bifurcation is *supercritical* which implies the bifurcating periodic solutions are *linearly stable*, while when the feeding rate is smaller than this critical feeding rate, the Hopf bifurcation is *subcritical*, implying the bifurcating periodic solutions are *linearly unstable*.

1. Introduction

The study of localised patterns in the so-called Turing's diffusion-driven-instability reaction-diffusion systems has been a very active field of research for the last couple of decades ([19]). The canonical model systems such as the Gierer-Meinhardt system ([10], [20]), the Gray-Scott system ([11, 23]) and the Schankenberg system ([25]) have been intensively studied in many papers. For the existence and stability of steady spiky patterns in a bounded interval or the whole space, we refer to [8], [5], [29], [13], [22], [39] and the book [41] for the Gierer-Meihardt system, [17, 36, 38, 37, 6, 9] for the Gray-Scott system, and [14, 33] for the Schnakenberg system. The dynamics of spiky patterns for these systems have been studied in [7, 26, 2, 1] and [40]. For Hopf bifurcations out of spiky patterns for one-dimensional systems, we refer to [31, 32, 30] and [28].

Spatially nonhomogeneous periodic patterns are observed to happen in these pattern formatting reaction-diffusion systems. For the classic activator-inhibitor system Gierer-Meinhardt model, the time periodic patterns are believed to exist but unstable. However, in the activator-substrate systems such as the Gray-Scott model and the Schnakenberg model, nonhomogeneous time periodic patterns were observed to exist. In this paper, we give a rigorous mathematical proof of the existence of stable periodic spike solutions for the Gray-Scott system and the Schnakenberg system. We would like here to mention a paper by F. Veerman [30], where a schematic procedure to deal with the nonhomogeneous Hopf bifurcation problem by singular perturbation and centre manifold reduction has been presented. However, to use the centre manifold reduction method, one needs to verify very carefully the spectral properties, such as the transversality condition, to the associated linearized eigenvalue problem around the steady state solution. This verification task is often not easy, especially in our context that the underlying solution of the linearized problem has singular patterns. And it is unclear if the same

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technique works for bounded intervals. Our proof is more PDE-oriented. By using the nonlocal eigenvalue problem we are able to treat these difficulties nicely. We believe that the techniques and computations presented in this paper can be used for the study of sub-criticality or sup-criticality of Hopf bifurcations of spiky patterns in many other Turing systems.

After proper rescaling ([38, 37]), the Gray-Scott system can be written in the form

$$\begin{cases} u_t = \varepsilon^2 u_{xx} - u + Avu^2, & 0 < x < 1,\ t > 0, \\ v_t = \dfrac{1}{\tau_\varepsilon}\left\{D_\varepsilon v_{xx} + (1-v) - \varepsilon^{-1}vu^2\right\}, & 0 < x < 1,\ t > 0, \\ u_x(0,t) = u_x(1,t) = 0, & t > 0, \\ v_x(0,t) = v_x(1,t) = 0, & t > 0, \end{cases} \tag{1.1}$$

where $(u,v)^T$ are the unknowns, the parameters $A, \varepsilon, \tau_\varepsilon, D_\varepsilon$ satisfy $A > \sqrt{12}$, $0 < \varepsilon \ll 4A^{-2}$, $\tau_\varepsilon > 0$ and $D_\varepsilon > 0$. We emphasize that A is the so called feeding rate of the system.

After proper rescaling ([16, 14, 40])the Schnakenberg system can be written in the form

$$\begin{cases} u_t = \varepsilon^2 u_{xx} - u + vu^2, & 0 < x < 1,\ t > 0, \\ v_t = \dfrac{1}{\tau_\varepsilon}\left\{D_\varepsilon v_{xx} + 1 - \varepsilon^{-1}vu^2\right\}, & 0 < x < 1,\ t > 0, \\ u_x(0,t) = u_x(1,t) = 0, & t > 0, \\ v_x(0,t) = v_x(1,t) = 0, & t > 0, \end{cases} \tag{1.2}$$

where $(u,v)^T$ are the unknowns, the parameters $\varepsilon, \tau_\varepsilon, D_\varepsilon$ satisfy $0 < \varepsilon \ll 1$, $\tau_\varepsilon > 0$ and $D_\varepsilon > 0$.

Using the reduction techniques of [34] ([14, 40]), one can easily show that the stationary system of (1.1) ((1.9)) has solutions with a single boundary spike at $x = 0$, as $\epsilon \to 0$ and $D = D(\epsilon) \to \infty$ at a suitable speed. (See also early work [27] for the Gierer-Meinhardt system.)

To prove the existence, uniqueness, and stability of the Hopf bifurcation of (1.1) and (1.2) we use the classical Crandall-Rabinowitz bifurcation theory ([3]). More precisely we use a more concise formulation given in Theorem I.8.2 of [15]. The linear stability of the bifurcating periodic solutions is obtained using Corollary I.12.3 in [15]. Specifically, stability is determined by the sign of certain Floquet multipliers relative to a transversality condition. To apply these results we need to write (1.1) and (1.2) in the form of an evolution equation

$$\Phi_t = \mathcal{F}_\epsilon(\Phi) \equiv \mathcal{L}_\epsilon \Phi + R_\varepsilon(\tau_\varepsilon, \Phi), \tag{1.3}$$

where

$$\Phi = \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} u - u_\varepsilon^S \\ v - v_\varepsilon^S \end{pmatrix}$$

and $\mathcal{L}_\varepsilon = D_\Phi \mathcal{F}_\varepsilon$ denote the perturbation and linearization about the stationary single-spike solution $(u_\varepsilon^S, v_\varepsilon^S)^T$ respectively, and $R_\varepsilon(\tau_\varepsilon, \Phi)$ indicates the remaining higher order term. For the Gray-Scott system (1.1),

$$\mathcal{L}_\varepsilon = \begin{pmatrix} \varepsilon^2 \frac{d^2}{dx^2} + 2Au_\varepsilon^S v_\varepsilon^S - 1 & A(u_\varepsilon^S)^2 \\ -2\tau_\varepsilon^{-1}\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S & \tau_\varepsilon^{-1}[D_\varepsilon \frac{d^2}{dx^2} - \varepsilon^{-1}(u_\varepsilon^S)^2 - 1] \end{pmatrix}, \tag{1.4}$$

and

$$R_\varepsilon(\tau_\varepsilon, \Phi) = \begin{pmatrix} Av_\varepsilon^S \phi^2 + 2Au_\varepsilon^S \phi\psi + A\phi^2\psi \\ -\tau_\varepsilon^{-1}\varepsilon^{-1}[v_\varepsilon^S \phi^2 + 2u_\varepsilon^S \phi\psi + \phi^2\psi] \end{pmatrix}. \tag{1.5}$$

For the Schankenberg system (1.2),

$$\mathcal{L}_\varepsilon = \begin{pmatrix} \varepsilon^2 \frac{d^2}{dx^2} + 2u_\varepsilon^S v_\varepsilon^S - 1 & (u_\varepsilon^S)^2 \\ -2\tau_\varepsilon^{-1}\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S & \tau_\varepsilon^{-1}[D_\varepsilon \frac{d^2}{dx^2} - \varepsilon^{-1}(u_\varepsilon^S)^2] \end{pmatrix}. \tag{1.6}$$

and

$$R_\varepsilon(\tau_\varepsilon, \Phi) = \begin{pmatrix} v_\varepsilon^S \phi^2 + 2u_\varepsilon^S \phi\psi + \phi^2\psi \\ -\tau_\varepsilon^{-1}\varepsilon^{-1}[v_\varepsilon^S \phi^2 + 2u_\varepsilon^S \phi\psi + \phi^2\psi] \end{pmatrix}. \tag{1.7}$$

To consider positive solutions of (1.1) and (1.2) with a single spike at the boundary $x = 0$, it is convenient to consider, by even extension, the positive solutions of the following two systems

$$\begin{cases} u_t = \varepsilon^2 u_{xx} - u + Avu^2, & -1 < x < 1,\ t > 0, \\ v_t = \dfrac{1}{\tau_\varepsilon}\left\{D_\varepsilon v_{xx} + (1-v) - \varepsilon^{-1}vu^2\right\}, & -1 < x < 1,\ t > 0, \\ u_x(-1,t) = u_x(1,t) = 0, & t > 0, \\ v_x(-1,t) = v_x(1,t) = 0, & t > 0, \end{cases} \tag{1.8}$$

and

$$\begin{cases} u_t = \varepsilon^2 u_{xx} - u + vu^2, & -1 < x < 1,\ t > 0, \\ v_t = \dfrac{1}{\tau_\varepsilon}\left\{D_\varepsilon v_{xx} + 1 - \varepsilon^{-1}vu^2\right\}, & -1 < x < 1,\ t > 0, \\ u_x(-1,t) = u_x(1,t) = 0, & t > 0, \\ v_x(-1,t) = v_x(1,t) = 0, & t > 0, \end{cases} \tag{1.9}$$

with a single symmetrical spike at the center of the interval $(-1, 1)$, respectively.

To develop our theory, we collect some preliminaries. Let $X = L^2([-1, 1])$ be the usual Hilbert space endowed with the inner product

$$\langle \phi_1, \phi_2 \rangle_X = \int_{-1}^{1} \phi\, \overline{\psi}\, dx, \qquad \phi_1, \phi_2 \in X, \tag{1.10}$$

where the over-bar denotes the complex conjugate. Let $Z = X \times X$ be endowed with the inner product

$$\langle \Phi_1, \Phi_2 \rangle_Z = \langle \phi_1, \phi_2 \rangle_X + \langle \psi_1, \psi_2 \rangle_X, \qquad \Phi_i = (\phi_i, \psi_i)^T \in Z, i = 1, 2. \tag{1.11}$$

We denote

$$H_N^2([-1,1]) := \{\phi \in H^2([-1,1]) : \phi_x(\pm 1) = 0\}. \tag{1.12}$$

We will use frequently the following facts, which are the basics for the treatment of spike type solutions.

Proposition 1.1. *The problem*

$$\begin{cases} w_{yy} - w + w^2 = 0, & w > 0 \quad \text{in } \mathbb{R}, \\ w(0) = \max_{y \in \mathbb{R}} w(y), & \\ w(y) \to 0, & \text{as } |y| \to \infty, \end{cases} \tag{1.13}$$

has a unique solution. Moreover, the eigenvalue problem

$$L_0\phi := \phi_{yy} - \phi + 2w\phi = \mu\phi, \qquad \phi \in H^2(\mathbb{R}), \tag{1.14}$$

admits the set of eigenvalues

$$\mu_1 > 0, \quad \mu_2 = 0, \quad \mu_3 < 0, \cdots. \tag{1.15}$$

The eigenfunction ϕ_1 corresponding to μ_1 can be made positive and even; the space of eigenfunctions corresponding to the eigenvalue 0 is

$$K_0 := span\{w_y\}. \tag{1.16}$$

For the proof of this proposition we refer to Theorem 2.1 of [18] and Lemma C of [21]. In fact

$$w(y) = \frac{3}{2}\operatorname{sech}^2\left(\frac{y}{2}\right). \tag{1.17}$$

Note that the nontrivial eigenfunctions corresponding to the eigenvalue 0 are odd functions.

Direct integration yields

$$\int_{\mathbb{R}} w\, dy = 6. \tag{1.18}$$

Integrating the equation for w over $\mathbb{R}$ then yields

$$\int_{\mathbb{R}} w^2 dy = \int_{\mathbb{R}} w\, dy = 6. \tag{1.19}$$

By the Pohozaev identity we have

$$\int_{\mathbb{R}} w^3\, dy = \frac{36}{5}, \qquad \int_{\mathbb{R}} w_y^2\, dy = \frac{6}{5}. \tag{1.20}$$

The Green's function and its properties make some of our calculations simple. In our particular one spatial dimensional case, the Green's function can be written explicitly. But we don't need the explicit form of the Green's function, only it properties as presented below.

Let $G_0(x,\xi)$ be the Green's function satisfying

$$\begin{cases} (G_0)_{xx}(x,\xi) - \dfrac{1}{2} + \delta(x-\xi) = 0 & \text{in } (-1,1), \\ (G_0)_x(x,\xi) = 0, & \text{for } x = -1, 1, \\ \displaystyle\int_{-1}^{1} G_0(x,\xi)dx = 0. & \end{cases} \tag{1.21}$$

For a complex number $\beta \in \mathbb{C}$ such that $\frac{d^2}{dx^2} - \beta^2 I : H_N^2([-1,1]) \to L^2([-1,1])$ is invertible, we let $G_\beta(x,\xi)$ be the Green's function given by

$$\begin{cases} (G_\beta)_{xx} - \beta^2 G_\beta + \delta(x-\xi) = 0 & \text{in } [-1,1], \\ (G_\beta)_x(x,\xi) = 0, & \text{for } x = -1, 1, \end{cases} \tag{1.22}$$

We can relate G_β and G_0 as follows. From (1.22) we get

$$\int_{-1}^{1} G_\beta(x,\xi)dx = \beta^{-2}.$$

Set

$$G_\beta(x,\xi) = \frac{1}{2}\beta^{-2} + \bar{G}_\beta(x,\xi).$$

Then

$$\begin{cases} (\bar{G}_\beta)_{xx} - \beta^2 \bar{G}_\beta - \dfrac{1}{2} + \delta(x-\xi) = 0 & \text{in } [-1,1], \\ \displaystyle\int_{-1}^{1} \bar{G}_\beta(x,\xi)dx = 0, & \\ (\bar{G}_\beta(x,\xi))_x = 0 & \text{for } x = -1, 1. \end{cases} \tag{1.23}$$

(1.21) and (1.23) imply that

$$\begin{aligned}\bar{G}_\beta(x,\xi) &= \left(\frac{d^2}{dx^2}-\beta^2 I\right)^{-1}\left(\frac{1}{2}-\delta(x-\xi)\right)\\ &= \left(\frac{d^2}{dx^2}-\beta^2 I\right)^{-1}\left[\left(\frac{d^2}{dx^2}-\beta^2 I\right)G_0(x,\xi)+\beta^2 G_0(x,\xi)\right]\\ &= G_0(x,\xi)+\beta^2\left(\frac{d^2}{dx^2}-\beta^2 I\right)^{-1}G_0(x,\xi).\end{aligned}$$

Since $G_0(\cdot,\xi)\in L^2([-1,1])$, we have

$$\beta^2\left(\frac{d^2}{dx^2}-\beta^2 I\right)^{-1}G_0(x,\xi)=O(1)$$

in the operator norm of $L^2([-1,1])\to H^2([-1,1])$. Hence

$$\begin{aligned}G_\beta(x,\xi) &= \frac{1}{2}\beta^{-2}+G_0(x,\xi)+\beta^2\left(\frac{d^2}{dx^2}-\beta^2 I\right)^{-1}G_0\\ &= \frac{1}{2}\beta^{-2}+G_0(x,\xi)+O(1)\end{aligned} \tag{1.24}$$

in the operator norm of $L^2([-1,1])\to H^2([-1,1])$.

The remainder of this paper is organized as follows. In Section 2 we summarize important properties of the stationary single spike solution $(u^S_\varepsilon,v^S_\varepsilon)^T$ of (1.3) for $0<\epsilon\ll 1$, and derive the associated leading order NLEP as $\varepsilon\to 0$. Sections 4 and 5 are dedicated to the analysis the spectral properties of the perturbed problem (2.4) (and (2.20)) for ϵ sufficiently small. This is followed by Sections 6 where we apply, set-up, and state the Hopf bifurcation theorem. Section 7 is devoted to the theoretical investigation of the stability of the Hopf bifurcations of the previous section. Finally, in Section 8 we numerically compute an unknown quantity whose sign dictates the criticality of the Hopf bifurcation, while in Section 9 we perform some numerical simulations which illustrate the theoretical predictions.

2. The steady states and the associated nonlocal eigenvalue problems

In this section, we study the steady states of the two systems, as well as the linear stabilities of these steady states.

2.1. The Gray-Scott system.

For this system, $u\equiv 0, v\equiv 1$ is the only constant stationary solution in the parameter regime $\varepsilon\in(0,4A^{-2})$. A simple analysis finds this constant solution is linearly stable.

In this paper, we consider the Gray-Scott system in the weak coupling (shadow limiting) case. This means we assume $D_\varepsilon\to\infty$ as $\varepsilon\to 0$. While the speed of this limiting is not required for the results in this paper to hold, we assume for convenience that there exists $\sigma\in(0,\infty)$ such that

$$D_\varepsilon = O\left((-\log\varepsilon)^\sigma\right)\quad \text{as } \varepsilon\to 0. \tag{2.1}$$

Using the reduction techniques of [34], one can easily prove that there exist two spiky stationary solutions $(u^\pm_\varepsilon,v^\pm_\varepsilon)$ of (1.8), for each $A>\sqrt{12}$ and $0<\varepsilon\ll 4A^{-2}$, with the properties

$$u^\pm_\varepsilon(x)\sim\begin{cases}\dfrac{1}{Av^\pm_\varepsilon}w\left(\dfrac{x}{\varepsilon}\right), & |x|\ll O(\varepsilon),\\ 0, & |x|\gg O(\varepsilon),\end{cases} \tag{2.2}$$

$$v_\varepsilon^\pm \sim v_0^\pm := \frac{1 \pm \sqrt{1-12A^{-2}}}{2}. \tag{2.3}$$

To study the linear stability of these solutions, we consider the following eigenvalue problem

$$\mathcal{L}_\varepsilon(\tau_\varepsilon)\Phi_\varepsilon = \lambda_\varepsilon \Phi_\varepsilon, \tag{2.4}$$

where

$$\mathcal{L}_\varepsilon(\tau_\varepsilon) = \begin{pmatrix} \varepsilon^2 \frac{d^2}{dx^2} + 2Au_\varepsilon^S v_\varepsilon^S - 1 & A(u_\varepsilon^S)^2 \\ -2\tau_\varepsilon^{-1}\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S & \tau_\varepsilon^{-1}[D_\varepsilon \frac{d^2}{dx^2} - \varepsilon^{-1}(u_\varepsilon^S)^2 - 1] \end{pmatrix}, \tag{2.5}$$

and

$$\Phi_\varepsilon = \begin{pmatrix} \phi_\varepsilon \\ \psi_\varepsilon \end{pmatrix} \in H_N^2([-1,1]) \times H_N^2([-1,1]).$$

Here

$$(u_\varepsilon^S, v_\varepsilon^S) = (u_\varepsilon^+, v_\varepsilon^+), \qquad \text{or} \qquad (u_\varepsilon^S, v_\varepsilon^S) = (u_\varepsilon^-, v_\varepsilon^-).$$

Or equivalently, the eigenvalue problem

$$\begin{cases} (\phi_\varepsilon)_{yy} - \phi_\varepsilon + 2Au_\varepsilon^S v_\varepsilon^S \phi_\varepsilon + A(u_\varepsilon^S)^2 \psi_\varepsilon = \lambda_\varepsilon \phi_\varepsilon, \\ D_\varepsilon (\psi_\varepsilon)_{yy} - \varepsilon^2(1+\tau_\varepsilon\lambda_\varepsilon)\psi_\varepsilon - \varepsilon(u_\varepsilon^S)^2\psi_\varepsilon = 2\varepsilon u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon, \\ (\phi_\varepsilon)_y(\pm 1/\varepsilon) = (\psi_\varepsilon)_y(\pm 1/\varepsilon) = 0, \end{cases} \tag{2.6}$$

where $(u_\varepsilon^S, v_\varepsilon^S) = (u_\varepsilon^+, v_\varepsilon^+)$, or $(u_\varepsilon^S, v_\varepsilon^S) = (u_\varepsilon^-, v_\varepsilon^-)$ is a (spike) stationary solution of (1.8).

Using the assumptions on D_ε and the techniques from [4] we have, subject to a subsequence if necessary, that $\tau_\varepsilon \to \tau_0$, $\lambda_\varepsilon \to \lambda_0$, $\phi_\varepsilon \to \phi_0$, $\psi_\varepsilon \to \psi_0$, as $\varepsilon \to 0$; moreover we have,

$$(\phi_0)_{yy} - \phi_0 + 2w\phi_0 + \frac{1}{Av_0^2} w^2 \psi_0 = \lambda_0 \phi_0, \tag{2.7}$$

and

$$\psi_0 = -\frac{Av_0^2}{3 + A^2 v_0^2(1+\tau_0\lambda_0)} \int_{-\infty}^{\infty} w\phi_0 \, dy, \tag{2.8}$$

where $v_0 = v_0^+$, or $v_0 = v_0^-$. Therefore we are led to consider the following nonlocal eigenvalue problem (NLEP):

$$(\phi_0)_{yy} - \phi_0 + 2w\phi_0 - \chi(\tau_0\lambda_0) w^2 \int_{-\infty}^{\infty} w\phi_0 \, dy = \lambda_0 \phi_0, \tag{2.9}$$

where

$$\chi(\tau_0\lambda_0) = \frac{1}{3 + A^2 v_0^2(1+\tau_0\lambda_0)} = \frac{\frac{6}{3+A^2v_0^2}}{1 + \frac{A^2 v_0^2}{3+A^2v_0^2}\tau_0\lambda_0} \frac{1}{\int_{-\infty}^{\infty} w^2 \, dy}. \tag{2.10}$$

We also need the adjoint eigenvalue problem

$$\mathcal{L}_\varepsilon^*(\tau_\varepsilon)\Phi_\varepsilon^* = \lambda_\varepsilon^* \Phi_\varepsilon^*, \tag{2.11}$$

where

$$\mathcal{L}_\varepsilon^*(\tau_\varepsilon) = \begin{pmatrix} \varepsilon^2 \frac{d^2}{dx^2} + 2Au_\varepsilon^S v_\varepsilon^S - 1 & -2\tau_\varepsilon^{-1}\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S \\ A(u_\varepsilon^S)^2 & \tau_\varepsilon^{-1}[D_\varepsilon \frac{d^2}{dx^2} - \varepsilon^{-1}(u_\varepsilon^S)^2 - 1] \end{pmatrix}, \tag{2.12}$$

and

$$\Phi_\varepsilon^* = \begin{pmatrix} \phi_\varepsilon^* \\ \psi_\varepsilon^* \end{pmatrix} \in H_N^2([-1,1]) \times H_N^2([-1,1]),$$

or equivalently,

$$\text{(2.13)} \qquad \begin{cases} (\phi_\varepsilon^*)_{yy} + 2Au_\varepsilon^S v_\varepsilon^S \phi_\varepsilon^* - \phi_\varepsilon^* - 2\tau_\varepsilon^{-1}\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S \psi_\varepsilon^* = \lambda_\varepsilon^* \phi_\varepsilon^*, \\ \dfrac{D_\varepsilon}{\varepsilon^2}(\psi_\varepsilon^*)_{yy} - (1+\tau_\varepsilon\lambda_\varepsilon^*)\psi_\varepsilon^* - \varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon^* = -A\tau_\varepsilon(u_\varepsilon^S)^2\phi_\varepsilon^*, \\ (\phi_\varepsilon^*)_y(\pm 1/\varepsilon) = (\psi_\varepsilon^*)_y(\pm 1/\varepsilon) = 0. \end{cases}$$

As $\varepsilon \to 0$, we have $\tau_\varepsilon \to \tau_0$, $\lambda_\varepsilon^* \to \lambda_0^*$, $\phi_\varepsilon^* \to \phi_0^*$, and in particular,

$$\text{(2.14)} \qquad \varepsilon^{-1}\psi_\varepsilon^* \to \frac{A\tau_0}{6 + 2A^2 v_0^2(1+\tau_0\lambda_0^*)} \int_{-\infty}^{\infty} w^2 \phi_0^* \, dy.$$

This leads us to consider the adjoint problem of (2.9):

$$\text{(2.15)} \qquad (\phi_0^*)_{yy} - \phi_0^* + 2w\phi_0^* - \chi(\tau_0\lambda_0^*) w \int_{-\infty}^{\infty} w^2\phi_0^* \, dy = \lambda_0^*\phi_0^*,$$

where

$$\text{(2.16)} \qquad \chi(\tau_0\lambda_0^*) = \frac{1}{3 + A^2 v_0^2(1+\tau_0\lambda_0^*)}.$$

We note here that

$$\lambda_0^* = \overline{\lambda_0}, \qquad \lambda_\varepsilon^* = \overline{\lambda_\varepsilon}.$$

2.2. **The Schnakenberg system.** For the Schnakenberg system (1.9), clearly

$$\text{(2.17)} \qquad u \equiv \varepsilon, \qquad v \equiv \varepsilon^{-1}$$

is the unique constant stationary solution. It is not difficulty to find out, after a simple analysis, that this constant stationary solution is linearly stable.

Other than this constant stationary solution, (1.9) has a spike stationary solution for each $0 < \varepsilon \ll 1$.

Proposition 2.1. *Assume there exists* $\sigma \in (0, \infty)$ *such that*

$$\text{(2.18)} \qquad D_\varepsilon = O\left((-\log\varepsilon)^\sigma\right) \quad \textit{as } \varepsilon \to 0.$$

Then for each $0 < \varepsilon \ll 1$*, there exists a positive stationary solution* $(u_\varepsilon^S, v_\varepsilon^S)$ *of* (1.9)*, satisfying*

$$\text{(2.19)} \qquad u_\varepsilon^S(x) \sim \frac{1}{3} w\left(\frac{x}{\varepsilon}\right), \qquad v_\varepsilon^S(x) \sim 3.$$

Similar to the case for the Gray-Scott system, this proposition can be easily proved by the reduction technique. We refer readers to [40, 14, 33] for the details.

Next we consider the linear stability of the spike solution described in Proposition 2.1. Therefore we study the eigenvalue problem

$$\text{(2.20)} \qquad \mathcal{L}_\varepsilon(\tau_\varepsilon)\Phi_\varepsilon = \lambda_\varepsilon\Phi_\varepsilon,$$

where

$$\text{(2.21)} \qquad \mathcal{L}_\varepsilon(\tau_\varepsilon) = \begin{pmatrix} \varepsilon^2 \frac{d^2}{dx^2} + 2u_\varepsilon^S v_\varepsilon^S - 1 & (u_\varepsilon^S)^2 \\ -2\tau_\varepsilon^{-1}\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S & \tau_\varepsilon^{-1}[D_\varepsilon \frac{d^2}{dx^2} - \varepsilon^{-1}(u_\varepsilon^S)^2] \end{pmatrix}.$$

Or equivalently,

$$\text{(2.22)} \qquad \begin{cases} (\phi_\varepsilon)_{yy} - \phi_\varepsilon + 2u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon + (u_\varepsilon^S)^2\psi_\varepsilon = \lambda_\varepsilon\phi_\varepsilon, \\ D_\varepsilon(\psi_\varepsilon)_{xx} - \tau_\varepsilon\lambda_\varepsilon\psi_\varepsilon - \varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon = 2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon, \\ (\phi_\varepsilon)_y(\pm 1/\varepsilon) = (\psi_\varepsilon)_x(\pm 1) = 0. \end{cases}$$

Using the assumptions on D_ε and the techniques from [4] we have, subject to a subsequence if necessary, that $\tau_\varepsilon \to \tau_0$, $\lambda_\varepsilon \to \lambda_0$, $\phi_\varepsilon \to \phi_0$, $\psi_\varepsilon \to \psi_0$, as $\varepsilon \to 0$; moreover we have,

$$(\phi_0)_{yy} - \phi_0 + 2w\phi_0 + \frac{1}{9}w^2\psi_0 = \lambda_0\phi_0, \tag{2.23}$$

and

$$\psi_0 = -\frac{3}{(1+3\tau_0\lambda_0)}\int_{-\infty}^{\infty} w\phi_0\,dy. \tag{2.24}$$

Therefore we are led to consider the following NLEP:

$$(\phi_0)_{yy} - \phi_0 + 2w\phi_0 - \chi(\tau_0\lambda_0)w^2\int_{-\infty}^{\infty} w\phi_0\,dy = \lambda_0\phi_0, \tag{2.25}$$

where

$$\chi(\tau_0\lambda_0) = \frac{1}{3+9\tau_0\lambda_0} = \frac{2}{1+3\tau_0\lambda_0}\,\frac{1}{\int_{-\infty}^{\infty} w^2\,dy}. \tag{2.26}$$

We also consider the adjoint eigenvalue problem

$$\mathcal{L}_\varepsilon^*(\tau_\varepsilon)\Phi_\varepsilon^* = \lambda_\varepsilon^*\Phi_\varepsilon^*, \tag{2.27}$$

where

$$\mathcal{L}_\varepsilon^*(\tau_\varepsilon) = \begin{pmatrix} \varepsilon^2\frac{d^2}{dx^2} + 2u_\varepsilon^S v_\varepsilon^S - 1 & -2\tau_\varepsilon^{-1}\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S \\ (u_\varepsilon^S)^2 & \tau_\varepsilon^{-1}[D_\varepsilon\frac{d^2}{dx^2} - \varepsilon^{-1}(u_\varepsilon^S)^2] \end{pmatrix}, \tag{2.28}$$

or equivalently,

$$\begin{cases} (\phi_\varepsilon^*)_{yy} + 2u_\varepsilon^S v_\varepsilon^S\phi_\varepsilon^* - \phi_\varepsilon^* - 2\tau_\varepsilon^{-1}\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S\psi_\varepsilon^* = \lambda_\varepsilon^*\phi_\varepsilon^*, \\ D_\varepsilon(\psi_\varepsilon^*)_{xx} - \tau_\varepsilon\lambda_\varepsilon^*\psi_\varepsilon^* - \varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon^* = -\tau_\varepsilon(u_\varepsilon^S)^2\phi_\varepsilon^*, \\ (\phi_\varepsilon^*)_y(\pm 1/\varepsilon) = (\psi_\varepsilon^*)_x(\pm 1) = 0. \end{cases} \tag{2.29}$$

Then as $\varepsilon \to 0$, subject to a subsequence, we have $\tau_\varepsilon \to \tau_0$, $\lambda_\varepsilon^* \to \lambda_0^*$, $\phi_\varepsilon^* \to \phi_0^*$, and in particular,

$$\varepsilon^{-1}\psi_\varepsilon^* \to \frac{\tau_0}{6+18\tau_0\lambda_0^*}\int_{-\infty}^{\infty} w^2\phi_0^*\,dy. \tag{2.30}$$

This leads us to consider the adjoint problem of (2.25):

$$(\phi_0^*)_{yy} - \phi_0^* + 2w\phi_0^* - \chi(\tau_0\lambda_0^*)w\int_{-\infty}^{\infty} w^2\phi_0^*\,dy = \lambda_0^*\phi_0^*, \tag{2.31}$$

where

$$\chi(\tau_0\lambda_0^*) = \frac{1}{3+9\tau_0\lambda_0^*} = \frac{2}{1+3\tau_0\lambda_0^*}\,\frac{1}{\int_{-\infty}^{\infty} w\,dy}. \tag{2.32}$$

We note that

$$\lambda_0^* = \overline{\lambda_0}, \qquad \lambda_\varepsilon^* = \overline{\lambda_\varepsilon}.$$

Remark 2.2. *The more detailed proofs of the limiting processes in this section will be represented in Section 4.*

3. The transversality condition for the limiting equation

The NLEPs (2.9) and (2.25) play essential role in this paper. Since they have a similar structure, in this section we treat them together. Let $\rho > 0$ be a fixed constant and $\hat{\tau}$ be a nonnegative parameter. We consider the following nonlocal eigenvalue problem

$$L\phi_0 := \phi_0'' + \phi_0 + 2w\phi_0 - \frac{\rho}{1+\hat{\tau}\lambda_0}\frac{\int_{-\infty}^{\infty} w\phi_0}{\int_{-\infty}^{\infty} w^2}w^2 = \lambda_0\phi_0, \qquad \phi_0 \in H^2(\mathbb{R}), \tag{3.1}$$

and its adjoint problem

$$L^*\phi_0^* := (\phi_0^*)'' + \phi_0^* + 2w\phi_0^* - \frac{\rho}{1+\hat{\tau}\lambda_0^*}\frac{\int_{-\infty}^{\infty} w^2\phi_0}{\int_{-\infty}^{\infty} w}w = \lambda_0^*\phi_0^*, \qquad \phi_0^* \in H^2(\mathbb{R}). \tag{3.2}$$

Lemma 3.1. *Let $\rho > 0, \hat{\tau} \geq 0$ and let L be defined as in* (3.1)*.*

(i) Suppose that $\rho < 1$. Then L admits a positive eigenvalue $\lambda_0 > 0$.

(ii) Suppose that $\rho > 1$. Then there exists a unique $\hat{\tau} = \hat{\tau}_h > 0$, such that for $\hat{\tau} < \hat{\tau}_h$, (3.1) *admits a positive eigenvalue, and for $\hat{\tau} > \hat{\tau}_h$, all nonzero eigenvalues of problem* (3.1) *satisfies $Re(\lambda_0) < 0$. At $\hat{\tau} = \hat{\tau}_h$,* (3.1) *has a pair of pure imaginary eigenvalues $\lambda_0(\hat{\tau}_h) = \pm i\alpha_I$ with $\alpha_I \in (0,\infty)$ uniquely determined by $\hat{\tau}_h$. Moreover, the following transversality condition holds.*

$$Re(\lambda_0'(\hat{\tau}_h)) \neq 0. \tag{3.3}$$

Proof. For the proof of part (i), we refer to Lemma 2.3 of [34]. The existence and uniqueness result of part (ii) is essentially part of Theorem 2.2 and Lemma 2.4 of [34], which treats interior spike solutions in a two-dimensional space. The proof found there can be applied here almost without modification but for the sake of completeness we reproduce it here. The transversality condition (3.3) and its proof, which plays a vital role in this paper, is new.

Note we here only consider even functions. By Theorem 1.4 of [35], for $\hat{\tau} = 0$ and by perturbation for $\hat{\tau}$ small, all eigenvalues lie on the left half-plane. By [4], for $\hat{\tau}$ large, there exist unstable eigenvalues. Therefore, for an intermediate value of $\hat{\tau} = \hat{\tau}_h$ an eigenvalue λ_0 must cross the imaginary axis into the positive real-part half-plane. We first show that this eigenvalue may not cross through the origin, and then we show the value of $\hat{\tau}_h$ must be unique.

Suppose that there is a zero-eigenvalue crossing, $\lambda_0 = 0$, when $\hat{\tau} = \hat{\tau}_h$. Let

$$L_0\phi_0 \equiv (\phi_0)_{yy} - \phi_0 + 2w\phi_0,$$

so that at the zero-eigenvalue crossing the NLEP (3.1) becomes

$$L_0\phi_0 - \rho\frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2}w^2 = 0,$$

and hence

$$L_0\left(\phi_0 - \rho\frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2}w\right) = 0.$$

Thus

$$\phi_0 - \rho\frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2}w \in K_0 (= \operatorname{span}\{w_y\}),$$

and since ϕ_0 is even one must have

$$\phi_0 - \rho\frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2}w = 0. \tag{3.4}$$

It follows from $\phi_0 \not\equiv 0$ that

$$\int_{\mathbb{R}} w\phi_0 \neq 0.$$

But on the other hand, multiplying (3.4) by w and integrating over $\mathbb{R}$, we arrive at

$$\int_{\mathbb{R}} w\phi_0 = \rho \int_{\mathbb{R}} w\phi_0.$$

It follows that $\rho = 1$, which contradicts to the assumption $\rho > 1$.

From the preceding argument we conclude that there must exist some $\hat{\tau}_h \in (0, \infty)$ at which L has a pair of pure imaginary eigenvalues

$$\lambda_0(\hat{\tau}_h) = \pm\alpha_I i,$$

where $i = \sqrt{-1}$ and $\alpha_I > 0$. Next we show that $\hat{\tau}_h$ is unique. From

$$(L_0 - \lambda_0)\phi_0 = \frac{\rho}{1+\hat{\tau}\lambda_0} \frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2} w^2,$$

we obtain for $\lambda_0 = \alpha_I i$ that

$$\phi_0 = \frac{\rho}{1+\hat{\tau}\lambda_0} \frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2} (L_0 - \lambda_0)^{-1} w^2,$$

and hence $\alpha_I i$ is a simple eigenvalue in the sense that

$$\mathrm{Ker}(L - \alpha_I i) = \mathrm{span}\{(L_0 - \alpha_I i)^{-1} w^2\}.$$

Thus we may assume that $\phi_0 = (L_0 - \alpha_I i)^{-1} w^2$ whence (3.1) becomes

$$\int_{\mathbb{R}} w\phi_0 = \frac{1+\hat{\tau}\alpha_I i}{\rho} \int_{\mathbb{R}} w^2. \tag{3.5}$$

Let $\phi_0 = \phi_0^R + \phi_0^I i$. Then from (3.5) we obtain

$$\int_{\mathbb{R}} w\phi_0^R = \frac{1}{\rho} \int_{\mathbb{R}} w^2,$$

and

$$\int_{\mathbb{R}} w\phi_0^I = \frac{\hat{\tau}\alpha_I}{\rho} \int_{\mathbb{R}} w^2.$$

But from

$$\phi_0 = (L_0 - \alpha_I i)^{-1} w^2 = (L_0 + \alpha_I i)(L_0^2 + \alpha_I^2)^{-1} w^2,$$

we have

$$\phi_0^R = L_0 (L_0^2 + \alpha_I^2)^{-1} w^2, \qquad \phi_0^I = \alpha_I (L_0^2 + \alpha_I^2)^{-1} w^2.$$

It follows that

$$\int_{\mathbb{R}} [w L_0 (L_0^2 + \alpha_I^2)^{-1} w^2] = \frac{1}{\rho} \int_{\mathbb{R}} w^2, \tag{3.6}$$

$$\int_{\mathbb{R}} [w (L_0^2 + \alpha_I^2)^{-1} w^2] = \frac{\hat{\tau}}{\rho} \int_{\mathbb{R}} w^2. \tag{3.7}$$

Let $h(\alpha_I) \equiv \int_{\mathbb{R}} [w L_0 (L_0^2 + \alpha_I^2)^{-1} w^2]$. Then

$$h'(\alpha_I) = -2\alpha_I \int_{\mathbb{R}} [w L_0 (L_0^2 + \alpha_I^2)^{-2} w^2].$$

By integration by parts, the last equation yields

$$h'(\alpha_I) = -2\alpha_I \int_{\mathbb{R}} [w^2 (L_0^2 + \alpha_I^2)^{-2} w^2] < 0.$$

Since

$$h(0) = \int_{\mathbb{R}} w(L_0^{-1} w^2) = \int_{\mathbb{R}} w^2, \quad \text{and} \quad h(\alpha_I) \to 0 \qquad \text{as} \quad \alpha_I \to \infty,$$

there exists a unique $\alpha_I \in (0,\infty)$ such that (3.6) holds. The unique value of $\hat{\tau} = \hat{\tau}_h \in (0,\infty)$ then comes from (3.7).

It is left to show that (3.3) holds. Setting $\lambda_0(\hat{\tau}) = \lambda_R(\hat{\tau}) + i\lambda_I(\hat{\tau})$ we have the system of equations

$$\begin{cases} \dfrac{1+\hat{\tau}\lambda_R}{\rho}\displaystyle\int_{\mathbb{R}} w^2 = \int_{\mathbb{R}} w \frac{L_0-\lambda_R}{(L_0-\lambda_R)^2+\lambda_I^2} w^2, \\ \dfrac{\hat{\tau}}{\rho}\displaystyle\int_{\mathbb{R}} w^2 = \int_{\mathbb{R}} w \frac{1}{(L_0-\lambda_R)^2+\lambda_I^2} w^2, \end{cases} \tag{3.8}$$

Suppose that $\frac{\partial(\lambda_R)}{\partial\hat{\tau}}(\hat{\tau}_h) = 0$ and differentiate the second equation of (3.8) with respect to $\hat{\tau}$ and evaluate it at $\hat{\tau} = \hat{\tau}_h$. We obtain

$$\frac{1}{\rho}\int_{\mathbb{R}} w^2 = -2\lambda_I(\hat{\tau}_h)\frac{\partial(\lambda_I)}{\partial\hat{\tau}}(\hat{\tau}_h)\int_{\mathbb{R}} w[L_0^2+\lambda_I^2(\hat{\tau}_h)]^{-2}w^2, \tag{3.9}$$

where we have used $\lambda_R(\hat{\tau}_h) = 0$. This implies that

$$\frac{\partial(\lambda_I)}{\partial\hat{\tau}}(\hat{\tau}_h) \neq 0.$$

If we now differentiate the first equation of (3.8) with respect to $\hat{\tau}$ we will obtain

$$0 = -\frac{\partial(\lambda_I^2)}{\partial\hat{\tau}}(\hat{\tau}_h)\int_{\mathbb{R}} wL_0[L_0^2+\lambda_I^2(\hat{\tau}_h)]^{-2}w^2. \tag{3.10}$$

However $\frac{\partial(\lambda_I^2)}{\partial\hat{\tau}}(\hat{\tau}_h) \neq 0$ and integrating by parts we see also that

$$\int_{\mathbb{R}} wL_0[L_0^2+\lambda_I^2(\hat{\tau}_h)]^{-2}w^2 = \int_{\mathbb{R}}[w^2(L_0^2+\alpha_I^2)^{-2}w^2] > 0,$$

which yields a contradiction. Therefore $\frac{\partial(\lambda_R)}{\partial\hat{\tau}}(\hat{\tau}_h) \neq 0$.

□

In the Gray-Scott system,

$$\rho = \frac{6}{3+A^2v_0^2}, \qquad \text{with } v_0 = v_0^{\pm} = \frac{1\pm\sqrt{1-12A^{-2}}}{2}.$$

We can easily verify that $v_0 = v_0^+ = \frac{1+\sqrt{1-12A^{-2}}}{2}$ implies $\rho < 1$, and $v_0 = v_0^- = \frac{1-\sqrt{1-12A^{-2}}}{2}$ implies $\rho > 1$. Hence we have the following corollary.

Corollary 3.2. *There exists an $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0,\varepsilon_1]$, the stationary solution $(u_\varepsilon^+, v_\varepsilon^+)$ of the Gray-Scott system is linearly stable for all $\tau_\varepsilon \in (0,\infty)$.*

Therefore for the Hopf bifurcation we focus on the solution $(u_\varepsilon^-, v_\varepsilon^-)$ and write it as $(u_\varepsilon^S, v_\varepsilon^S)$ throughout the rest of the paper. Note in this case $v_0 = v_0^-$.

We have more information on the transversality condition at $\hat{\tau}_h$.

Lemma 3.3. *Let $\lambda_0(\hat{\tau}_h) = \pm\alpha_I i$ be the unique imaginary eigenvalue pair described in Lemma 3.1. Then*

$$Re(\lambda_0'(\hat{\tau}_h)) > 0. \tag{3.11}$$

Proof. Consider the eigenvalue problem

$$L_0\phi_0 - \frac{\rho}{1+\hat{\tau}\lambda_0}\frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2}w^2 = \lambda_0\phi_0. \tag{3.12}$$

As in the proof of the transversality condition Lemma 3.1 we have

$$\phi_0 = \frac{\rho}{1+\hat{\tau}\lambda_0}\frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2}(L_0-\lambda_0)^{-1}w^2,$$

so that multiplying by w and integrating gives

$$\frac{1+\hat{\tau}\lambda_0}{\rho}\int_{\mathbb{R}} w^2 = \int_{\mathbb{R}} w(L_0-\lambda_0)^{-1}w^2. \tag{3.13}$$

Differentiating (3.13) with respect to $\hat{\tau}$ we obtain

$$\frac{\lambda_0+\hat{\tau}\lambda_0'}{\rho}\int_{\mathbb{R}} w^2 = \lambda_0'\int_{\mathbb{R}} w(L_0-\lambda_0)^{-2}w^2, \tag{3.14}$$

or equivalently

$$\lambda_0' = \lambda_0\frac{\int_{\mathbb{R}} w^2}{\rho}\left(\int_{\mathbb{R}} w(L_0-\lambda_0)^{-2}w^2 - \frac{\hat{\tau}}{\rho}\int_{\mathbb{R}} w^2\right)^{-1}. \tag{3.15}$$

Letting $\hat{\tau}=\hat{\tau}_h$ and using $Re(\lambda_0(\hat{\tau}_h))=0$ we obtain

$$Re(\lambda_0'(\hat{\tau}_h)) = -Im(\lambda_0(\hat{\tau}_h))\frac{\int_{\mathbb{R}} w^2}{\rho}Im\left[\left(\int_{\mathbb{R}} w(L_0-\lambda_0(\hat{\tau}_h))^{-2}w^2 - \frac{\hat{\tau}_h}{\rho}\int_{\mathbb{R}} w^2\right)^{-1}\right]. \tag{3.16}$$

Denote

$$\int_{\mathbb{R}} w(L_0-\lambda_0(\hat{\tau}_h))^{-2}w^2 = a+ib, \qquad c=\frac{\hat{\tau}_h}{\rho}\int_{\mathbb{R}} w^2, \qquad \text{with } a,b,c\in\mathbb{R}.$$

Then we have

$$\begin{aligned} &Im\left[\left(\int_{\mathbb{R}} w(L_0-\lambda_0(\hat{\tau}_h))^{-2}w^2 - \frac{\hat{\tau}_h}{\rho}\int_{\mathbb{R}} w^2\right)^{-1}\right] \\ &= Im\left[(a+bi-c)^{-1}\right] \\ &= \frac{-b}{(a-c)^2+b^2}. \end{aligned} \tag{3.17}$$

On the other hand

$$\int_{\mathbb{R}} w(L_0-\lambda_0(\hat{\tau}_h))^{-2}w^2 = \int_{\mathbb{R}} w\frac{L_0^2-\lambda_I(\hat{\tau}_h)^2+2i\lambda_I(\hat{\tau}_h)L_0}{(L_0^2+\lambda_I(\hat{\tau}_h)^2)^2}w^2, \tag{3.18}$$

and consequently by integration by parts we obtain

$$\begin{aligned} b &= 2\lambda_I(\hat{\tau}_h)\int_{\mathbb{R}} w\frac{L_0}{(L_0^2+\lambda_I(\hat{\tau}_h)^2)^2}w^2 \\ &= 2\lambda_I(\hat{\tau}_h)\int_{\mathbb{R}}(L_0w)(L_0^2+\lambda_I(\hat{\tau}_h)^2)^{-2}w^2 \\ &= 2\lambda_I(\hat{\tau}_h)\int_{\mathbb{R}} w^2(L_0^2+\lambda_I(\hat{\tau}_h)^2)^{-2}w^2. \end{aligned}$$

Hence

$$Re(\lambda_0'(\hat{\tau}_h)) = \frac{2}{\rho}\frac{\lambda_I(\hat{\tau}_h)^2\int_{\mathbb{R}} w^2}{(a-c)^2+b^2}\int_{\mathbb{R}} w^2(L_0^2+\lambda_I(\hat{\tau}_h)^2)^{-2}w^2 > 0. \tag{3.19}$$

□

We give an alternative representation of $\lambda_0'(\hat\tau_h)$. In (3.1) we write $\hat\mu_0 = \hat\tau\lambda_0$ and differentiate the equation with respect to $\hat\tau$

$$L_0\phi_0' - \frac{\rho}{1+\hat\mu_0}\frac{\int_{\mathbb{R}} w\phi_0'}{\int_{\mathbb{R}} w^2}w^2 + \frac{\rho\hat\mu_0'}{(1+\hat\mu_0)^2}\frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2}w^2 = \left(-\frac{\hat\mu_0}{\hat\tau^2} + \frac{\hat\mu_0'}{\hat\tau}\right)\phi_0 + \frac{\hat\mu_0}{\hat\tau}\phi_0'.$$

Multiplying by the conjugate of the adjoint eigenfunction $\overline{\phi_0^*}$ and integrating over $\mathbb{R}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}[\overline{\phi_0^*}L_0\phi_0'] - \frac{\rho}{1+\hat\mu_0}\frac{\int_{\mathbb{R}} w\phi_0'}{\int_{\mathbb{R}} w^2}\int_{\mathbb{R}} w^2\overline{\phi_0^*} + \frac{\rho\hat\mu_0'}{(1+\hat\mu_0)^2}\frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2}\int_{\mathbb{R}} w^2\overline{\phi_0^*} \\
&= \left(-\frac{\hat\mu_0}{\hat\tau^2} + \frac{\hat\mu_0'}{\hat\tau}\right)\int_{\mathbb{R}}\phi_0\overline{\phi_0^*} + \frac{\hat\mu_0}{\hat\tau}\int_{\mathbb{R}}\overline{\phi_0^*}\phi_0'.
\end{aligned} \tag{3.20}$$

Taking conjugate of (3.2) and recalling that $\lambda_0^* = \overline{\lambda_0}$ we obtain

$$L_0\overline{\phi_0^*} - \frac{\rho}{1+\hat\mu_0}\frac{\int_{\mathbb{R}} w^2\overline{\phi_0^*}}{\int_{\mathbb{R}} w^2}w = \frac{\hat\mu_0}{\hat\tau}\overline{\phi_0^*}.$$

Multiplying by ϕ_0' and integrating over $\mathbb{R}$, we obtain

$$\int_{\mathbb{R}}[\phi_0'L_0\overline{\phi_0^*}] - \frac{\rho}{1+\hat\mu_0}\frac{\int_{\mathbb{R}} w^2\overline{\phi_0^*}}{\int_{\mathbb{R}} w^2}\int_{\mathbb{R}} w\phi_0' = \frac{\hat\mu_0}{\hat\tau}\int_{\mathbb{R}}\overline{\phi_0^*}\phi_0'. \tag{3.21}$$

Note that by integration by parts,

$$\int_{\mathbb{R}}[\overline{\phi_0^*}L_0\phi_0'] = \int_{\mathbb{R}}[\phi_0'L_0\overline{\phi_0^*}].$$

We obtain from (3.20) and (3.21) that

$$\frac{\rho\hat\mu_0'}{(1+\hat\mu_0)^2}\frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2}\int_{\mathbb{R}} w^2\overline{\phi_0^*} = \left(-\frac{\hat\mu_0}{\hat\tau^2} + \frac{\hat\mu_0'}{\hat\tau}\right)\int_{\mathbb{R}}\phi_0\overline{\phi_0^*}. \tag{3.22}$$

Therefore we have the formula

$$\hat\mu_0'(\hat\tau_h) = \frac{\lambda_0(\hat\tau_h)\int_{\mathbb{R}}\phi_0\overline{\phi_0^*}}{\int_{\mathbb{R}}\phi_0\overline{\phi_0^*} - \frac{\rho\hat\tau_h}{[1+\hat\tau_h\lambda_0(\hat\tau_h)]^2\int_{\mathbb{R}} w^2}\int_{\mathbb{R}} w\phi_0\int_{\mathbb{R}} w^2\overline{\phi_0^*}}. \tag{3.23}$$

Remark 3.4. *In the case of Gray-Scott system,* $\hat\tau = \frac{A^2v_0^2}{3+A^2v_0^2}\tau_0$, $\rho = \frac{6}{3+A^2v_0^2}$. *In the case of Schnakenberg system,* $\hat\tau = 3\tau_0$, $\rho = 2$.

Finally we have the following bound estimates for the spectrum of (3.1) which will play a key role in showing the unperturbed linear operator is sectorial.

Lemma 3.5. *Let* λ_0 *be an eigenvalue of* (3.1). *Then one of the following alternative cases happens:*

(i) $Im(\lambda_0) = 0$ *and* $\lambda_0 \le \mu_1$, *where* $\mu_1 > 0$ *is the first eigenvalue of* L_0, *or*
(ii) $Im(\lambda_0) \neq 0$ *and* $|\hat\tau Re(\lambda_0) + 1| \le 2\rho$, $|\hat\tau Im(\lambda_0)| \le \rho$.

Proof. Multiplying (3.1) by w and integrating over $\mathbb{R}$, we obtain

$$\int_{\mathbb{R}} w^2\phi_0 = \left(\lambda_0 + \frac{\rho}{1+\hat\tau\lambda_0}\frac{\int_{\mathbb{R}} w^3}{\int_{\mathbb{R}} w^2}\right)\int_{\mathbb{R}} w\phi_0. \tag{3.24}$$

It follows that

$$\int_{\mathbb{R}} w^2\phi_0 = \left(\lambda_0 + \frac{6\rho}{5(1+\hat\tau\lambda_0)}\right)\int_{\mathbb{R}} w\phi_0. \tag{3.25}$$

Taking the conjugate gives

$$\int_{\mathbb{R}} w^2\overline{\phi_0} = \left(\overline{\lambda_0} + \frac{6\rho}{5(1+\hat{\tau}\overline{\lambda_0})}\right)\int_{\mathbb{R}} w\overline{\phi_0}. \tag{3.26}$$

Multiplying (3.1) by $\overline{\phi_0}$ and integrating over $\mathbb{R}$, we obtain that

$$\int_{\mathbb{R}}(|(\phi_0)_y|^2 + |\phi_0|^2 - 2w|\phi_0|^2) = -\lambda_0\int_{\mathbb{R}}|\phi_0|^2 - \frac{\rho}{1+\hat{\tau}\lambda_0}\frac{\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2}\int_{\mathbb{R}} w^2\overline{\phi_0}. \tag{3.27}$$

Combining (3.26) and (3.27) we obtain

$$\int_{\mathbb{R}}(|(\phi_0)_y|^2 + |\phi_0|^2 - 2w|\phi_0|^2) = -\lambda_0\int_{\mathbb{R}}|\phi_0|^2 - \left(\frac{\rho\overline{\lambda_0}}{1+\hat{\tau}\lambda_0} + \frac{6\rho^2}{5|1+\hat{\tau}\lambda_0|^2}\right)\frac{|\int_{\mathbb{R}} w\phi_0|^2}{\int_{\mathbb{R}} w^2}. \tag{3.28}$$

Writing

$$\lambda_0 = \lambda_R + i\lambda_I, \qquad \phi_0 = \phi_R + i\phi_I,$$

and considering the imaginary part of (3.28) we obtain

$$\lambda_I\int_{\mathbb{R}}|\phi_0|^2 = \frac{\rho\lambda_I(1+2\hat{\tau}\lambda_R)}{(1+\hat{\tau}\lambda_R)^2+\hat{\tau}^2\lambda_I^2}\frac{|\int_{\mathbb{R}} w\phi_0|^2}{\int_{\mathbb{R}} w^2}. \tag{3.29}$$

We first consider the case that $\lambda_I \neq 0$. In this case we have

$$\int_{\mathbb{R}}|\phi_0|^2 = \frac{\rho(1+2\hat{\tau}\lambda_R)}{(1+\hat{\tau}\lambda_R)^2+\hat{\tau}^2\lambda_I^2}\frac{|\int_{\mathbb{R}} w\phi_0|^2}{\int_{\mathbb{R}} w^2}.$$

Using the Schwartz inequality

$$|\int_{\mathbb{R}} w\phi_0|^2 \leq \int_{\mathbb{R}} w^2\int_{\mathbb{R}}|\phi_0|^2,$$

we get

$$\frac{\rho(1+2\hat{\tau}\lambda_R)}{(1+\hat{\tau}\lambda_R)^2+\hat{\tau}^2\lambda_I^2} \geq 1.$$

It follows that

$$|1+\hat{\tau}\lambda_R| \leq 2\rho, \tag{3.30}$$

and

$$\hat{\tau}^2\lambda_I^2 \leq 2\rho(1+2\lambda_R) - (1+2\lambda_R)^2 \leq \rho^2. \tag{3.31}$$

Hence case (ii) happens.

Now assume that $\lambda_I = 0$. If $\hat{\tau}\lambda_R + 1 = 0$, then

$$\lambda_0 = \lambda_R = -\frac{1}{\hat{\tau}} < 0 < \mu_1.$$

If $\hat{\tau}\lambda_R + 1 \neq 0$, we then use the Rayleigh's formula

$$\int_{\mathbb{R}}|(\phi_0)_y|^2 + \int_{\mathbb{R}}|\phi_0|^2 - 2\int_{\mathbb{R}} w|\phi_0|^2 \geq -\mu_1\int_{\mathbb{R}}|\phi_0|^2,$$

and (3.28) to get that

$$\lambda_R\int_{\mathbb{R}}|\phi_0|^2 + \left(\frac{\rho\lambda_R}{1+\hat{\tau}\lambda_R} + \frac{6\rho^2}{5|1+\hat{\tau}\lambda_R|^2}\right)\frac{|\int_{\mathbb{R}} w\phi_0|^2}{\int_{\mathbb{R}} w^2} \leq \mu_1\int_{\mathbb{R}}|\phi_0|^2.$$

The case $\lambda_R \leq 0$ is trivial. If $\lambda_R > 0$, we then have

$$\lambda_R\int_{\mathbb{R}}|\phi_0|^2 \leq \mu_1\int_{\mathbb{R}}|\phi_0|^2.$$

Hence case (i) happens. □

4. Spectral analysis of the perturbed problems

4.1. **The Gray-Scott system.** We want to show that the operator $\mathcal{L}_\varepsilon$ is an infinitesimal generator of a strongly continuous and analytical semigroup. Since it suffices to show that $\mathcal{L}_\varepsilon$ is a sectorial operator this naturally leads us to study the following eigenvalue problem

$$\begin{cases} (\phi_\varepsilon)_{yy} - \phi_\varepsilon + 2Au_\varepsilon^S v_\varepsilon^S \phi_\varepsilon + A(u_\varepsilon^S)^2 \psi_\varepsilon = \lambda_\varepsilon \phi_\varepsilon, \\ \dfrac{1}{\beta^2}(\psi_\varepsilon)_{xx} - \psi_\varepsilon - \varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon - 2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon = \tau_\varepsilon \lambda_\varepsilon \psi_\varepsilon, \end{cases} \tag{4.1}$$

where $y = \varepsilon^{-1}x$, $D_\varepsilon = \beta^{-2}$, λ_ε is some complex number, and

$$\phi_\varepsilon \in H_N^2([-\varepsilon^{-1}, \varepsilon^{-1}]), \qquad\qquad \psi_\varepsilon \in H_N^2([-1,1]). \tag{4.2}$$

The second equation in (4.1) is equivalent to

$$(\psi_\varepsilon)_{xx} - \beta_{\lambda_\varepsilon}^2 \psi_\varepsilon - \beta^2\varepsilon^{-1}[(u_\varepsilon^S)^2\psi_\varepsilon + 2u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon] = 0. \tag{4.3}$$

where

$$\beta_{\lambda_\varepsilon}^2 \equiv \beta^2(1 + \tau_\varepsilon \lambda_\varepsilon). \tag{4.4}$$

We may assume that $\|\phi_\varepsilon\|_{H^2([-\varepsilon^{-1},\varepsilon^{-1}])} = 1$.

Let χ be a smooth cut-off function which is equal to 1 in $[-\frac{1}{2}, \frac{1}{2}]$ and equal to 0 in $\mathbb{R} \setminus [-1,1]$. Let

$$\chi_\varepsilon(y) = \chi(\varepsilon y), \qquad y \in [-\varepsilon^{-1}, \varepsilon^{-1}]. \tag{4.5}$$

Define the cut-off of ϕ_ε:

$$\phi_\varepsilon^c(y) = \phi_\varepsilon(y)\chi_\varepsilon(y), \tag{4.6}$$

where $x = \varepsilon y$. Then if $Re(1 + \lambda_\varepsilon) > c$, or $|Im(\lambda_\varepsilon)| > c$, for a small constant $c > 0$, we have

$$\phi_\varepsilon^c = \phi_\varepsilon + e.s.t. \qquad \text{in } \; H^2([-\varepsilon^{-1}, \varepsilon^{-1}]). \tag{4.7}$$

Then by the standard procedure, we extend ϕ_ε^c to a function defined on $\mathbb{R}$ such that

$$\begin{aligned} &\|\phi_\varepsilon^c\|_{L^2(\mathbb{R})} \le C_0 \|\phi_\varepsilon\|_{L^2([-\varepsilon^{-1},\varepsilon^{-1}])}, \\ &\|(\phi_\varepsilon^c)_y\|_{L^2(\mathbb{R})} \le C_0 \|(\phi_\varepsilon)_y\|_{L^2([-\varepsilon^{-1},\varepsilon^{-1}])}, \\ &\|(\phi_\varepsilon^c)_{yy}\|_{L^2(\mathbb{R})} \le C_0 \|(\phi_\varepsilon)_{yy}\|_{L^2([-\varepsilon^{-1},\varepsilon^{-1}])}, \end{aligned} \tag{4.8}$$

for a constant $C_0 > 1$. Since $\|\phi_\varepsilon\|_{H^2([-\varepsilon^{-1},\varepsilon^{-1}])} = 1$, we have $\|\phi_\varepsilon^c\|_{H^2(\mathbb{R})} \le C_0$.

It is very easy to prove that $(\psi_\varepsilon)_x \to 0$ as $\varepsilon \to 0$. Using the Green's function introduced in Section 1 we write

$$\psi_\varepsilon(x) = -\int_{-1}^{1} G_{\beta_{\lambda_\varepsilon}}(x,\xi)\beta^2\varepsilon^{-1}[(u_\varepsilon^S(\xi))^2\psi_\varepsilon(\xi) + 2u_\varepsilon^S(\xi)v_\varepsilon^S(\xi)\phi_\varepsilon(\xi/\varepsilon)]d\xi. \tag{4.9}$$

As $\varepsilon \to 0$, we calculate at $x = 0$:

$$
\begin{aligned}
\psi_\varepsilon(0) &= -\frac{\beta^2\psi_\varepsilon(0)}{A^2 v_0^2 \varepsilon}\int_{-1}^{1} G_{\beta_{\lambda_\varepsilon}}(0,\xi) w^2(\xi/\varepsilon)\,d\xi \\
&\quad - \frac{2\beta^2}{A\varepsilon}\int_{-1}^{1} G_{\beta_{\lambda_\varepsilon}}(0,\xi) w(\xi/\varepsilon))\,\phi_\varepsilon^c(\xi/\varepsilon)d\xi + o(1) \\
&= -\frac{\beta^2\psi_\varepsilon(0)}{A^2 v_0^2 \varepsilon}\int_{-1}^{1}\left(\frac{(\beta_{\lambda_\varepsilon})^{-2}}{2} + G_0(0,\xi) + O(1)\right) w^2(\xi/\varepsilon)\,d\xi \\
&\quad - \frac{2\beta^2}{A\varepsilon}\int_{-1}^{1}\left(\frac{(\beta_{\lambda_\varepsilon})^{-2}}{2} + G_0(0,\xi) + O(1)\right) w(\xi/\varepsilon))\,\phi_\varepsilon^c(\xi/\varepsilon)d\xi + o(1) \\
&= -\frac{\psi_\varepsilon(0)}{2A^2 v_0^2(1+\tau_\varepsilon\lambda_\varepsilon)}\int_{-\infty}^{\infty} w^2(y)dy + O(\beta^2)\psi_\varepsilon(0) \\
&\quad - \frac{1+O(\beta^2)}{A(1+\tau_\varepsilon\lambda_\varepsilon)}\int_{-\infty}^{\infty} w(y)\phi_\varepsilon^c(y)dy + o(1).
\end{aligned}
\tag{4.10}
$$

Using $\int_{-\infty}^{\infty} w^2(y)dy = 6$, we obtain

$$
\psi_\varepsilon(0) = -\frac{A v_0^2}{3 + A^2 v_0^2(1+\tau_\varepsilon\lambda_\varepsilon)}\int_{-\infty}^{\infty} w(y)\phi_\varepsilon^c dy + h.o.t. \tag{4.11}
$$

Substituting (4.11) into the first equation of (4.1) we arrive at

$$
(\phi_\varepsilon)_{yy} - \phi_\varepsilon + 2w\phi_\varepsilon - \frac{w^2}{3 + A^2 v_0^2(1+\tau_\varepsilon\lambda_\varepsilon)}\int_{\mathbb{R}} w\phi_\varepsilon^c dy = \lambda_\varepsilon[1+o(1)]\phi_\varepsilon \tag{4.12}
$$

As in the proof of Theorem 1 in [4] one obtains

$$
\tau_\varepsilon \to \tau_0, \qquad \lambda_\varepsilon \to \lambda_0, \qquad \phi_\varepsilon(y) \to \phi_0(y) \qquad \text{in } H^2_{loc}(\mathbb{R}), \qquad \text{as } \varepsilon \to 0, \tag{4.13}
$$

where (λ_0, ϕ_0) is an eigenpair of the NLEP (2.9).

We can now prove the following spectral result for the eigenvalue problem (4.1).

Lemma 4.1. *If $\varepsilon > 0$ is sufficiently small then there exists a unique value $\tau_\varepsilon = \tau_\varepsilon^h$ for which (4.1) has a pair of purely imaginary eigenvalues $\lambda_\pm^\varepsilon = \pm i\alpha_I^\varepsilon$ with $\alpha_I^\varepsilon > 0$. Moreover this pair is unique in the sense that if $i\beta_I^\varepsilon$ is an eigenvalue of (4.1), then $\beta_I^\varepsilon = \alpha_I^\varepsilon$ or $\beta_I^\varepsilon = -\alpha_I^\varepsilon$. Furthermore at this value of $\tau_\varepsilon = \tau_\varepsilon^h$ all other eigenvalues have negative real parts.*

Proof. For $\varepsilon > 0$ sufficiently small, as in the proof of Lemma 3.1 all eigenvalues of (4.1) have negative real parts when $\tau > 0$ is small, whereas there exist eigenvalues with positive real part when $\tau > 0$ is sufficiently large. Furthermore, we can show that there are no zero eigenvalues for any $\tau > 0$. Thus, there exist a $\tau_\varepsilon^h \in (0,\infty)$ such that (4.1) has a pair of pure imaginary eigenvalues.

The uniqueness comes from the fact that for $Re(\lambda_\varepsilon) > -c$ we define $h_\varepsilon(\lambda_I^\varepsilon) := \int_{\mathbb{R}} wRe(\phi_\varepsilon^c)$ for the unperturbed problem (4.12) so that subject to a subsequence, $\alpha_I^\varepsilon \to \alpha_I$ and $\phi_\varepsilon \to \phi_0$ as $\varepsilon \to 0$ we have

$$
h_\varepsilon'(\lambda_I^\varepsilon) \to h'(\lambda_I) < 0 \qquad \text{as } \varepsilon \to 0, \tag{4.14}
$$

according to the calculation in the proof of Lemma 3.1 and the uniform continuity of $h'(\lambda_I)$ in λ_I.

□

The following two lemmas establish the semigroup framework.

Lemma 4.2. *Let $\lambda_\varepsilon \in \mathbb{C}$ be an eigenvalue of problem (4.1). Then for sufficiently small $\varepsilon > 0$, one of the following cases happens:*

(i) $Im(\lambda_\varepsilon)=0$ *and* $\lambda_\varepsilon \le 2\mu_1$*, or*
(ii) $Im(\lambda_\varepsilon)\neq 0$ *and* $|\tau_\varepsilon Re(\lambda_\varepsilon)+1|\le 4\rho$, $|\tau_\varepsilon Im(\lambda_\varepsilon)|\le 2\rho$.

Proof. We may assume that the constant $C_0>1$ in (4.8) is arbitrarily close to 1. Multiplying (4.12) by $\overline{\phi_\varepsilon^c}$ and integrating over $\mathbb{R}$ we get

$$-\int_{\mathbb{R}}|(\phi_\varepsilon^c)_y|^2-\int_{\mathbb{R}}|\phi_\varepsilon^c|^2+2\int_{\mathbb{R}}w|\phi_\varepsilon^c|^2-\frac{\rho[1+o(1)]}{1+\tau_\varepsilon\lambda_\varepsilon}\frac{\int_{\mathbb{R}}w\phi_\varepsilon^c}{\int_{\mathbb{R}}w^2}\int_{\mathbb{R}}w^2\overline{\phi_\varepsilon^c}=\lambda_\varepsilon[1+o(1)]\int_{\mathbb{R}}|\phi_\varepsilon^c|^2. \tag{4.15}$$

Multiplying (4.12) by w and integrating over $\mathbb{R}$ we get

$$[1+o(1)]\lambda_\varepsilon\int_{\mathbb{R}}w\phi_\varepsilon^c=\int_{\mathbb{R}}[w_{yy}-w+2w^2]\phi_\varepsilon^c-\frac{\rho[1+o(1)]}{1+\tau_\varepsilon\lambda_\varepsilon}\frac{\int_{\mathbb{R}}w^3}{\int_{\mathbb{R}}w^2}\int_{\mathbb{R}}w\phi_\varepsilon^c. \tag{4.16}$$

Using (1.19) and (1.20) we obtain

$$\int_{\mathbb{R}}w^2\phi_\varepsilon^c=[1+o(1)]\left(\lambda_\varepsilon+\frac{6\rho}{5(1+\tau_\varepsilon\lambda_\varepsilon)}\right)\int_{\mathbb{R}}w\phi_\varepsilon^c. \tag{4.17}$$

From (4.15) and (4.17) we obtain

$$[1+o(1)]\int_{\mathbb{R}}(|(\phi_\varepsilon^c)_y|^2+|\phi_\varepsilon^c|^2-2w|\phi_\varepsilon^c|^2)=-\lambda_\varepsilon\int_{\mathbb{R}}|\phi_\varepsilon^c|^2-\left(\frac{\rho\overline{\lambda_\varepsilon}}{1+\tau_\varepsilon\lambda_\varepsilon}+\frac{6\rho^2}{5|1+\tau_\varepsilon\lambda_\varepsilon|^2}\right)\frac{|\int_{\mathbb{R}}w\phi_\varepsilon^c|^2}{\int_{\mathbb{R}}w^2}. \tag{4.18}$$

Consider the imaginary part of (4.18) we get

$$[1+o(1)]\lambda_I^\varepsilon\int_{\mathbb{R}}|\phi_\varepsilon^c|^2=\frac{\rho\lambda_I^\varepsilon(1+2\tau_\varepsilon\lambda_R^\varepsilon)}{(1+\tau_\varepsilon\lambda_R^\varepsilon)^2+\tau_\varepsilon^2(\lambda_I^\varepsilon)^2}\frac{|\int_{\mathbb{R}}w\phi_\varepsilon^c|^2}{\int_{\mathbb{R}}w^2}. \tag{4.19}$$

If $\lambda_I^\varepsilon\neq 0$, we have

$$\frac{\rho(1+2\tau_\varepsilon\lambda_R^\varepsilon)}{(1+\tau_\varepsilon\lambda_R^\varepsilon)^2+\tau_\varepsilon^2(\lambda_I^\varepsilon)^2}\ge\frac{1}{2}\qquad\text{for sufficiently small }\ \varepsilon>0,$$

Therefore for $0<\varepsilon\ll 1$,

$$\frac{1+\tau_\varepsilon\lambda_R^\varepsilon}{(1+\tau_\varepsilon\lambda_R^\varepsilon)^2+\tau_\varepsilon^2(\lambda_I^\varepsilon)^2}\ge\frac{1}{4\rho}.$$

From here we obtain the coarse bounds

$$|\tau_\varepsilon\lambda_R^\varepsilon+1|\le 4\rho. \tag{4.20}$$

Moreover, we have

$$(\tau_\varepsilon\lambda_I^\varepsilon)^2\le 4\rho(1+\tau_\varepsilon\lambda_R^\varepsilon)-(1+\tau_\varepsilon\lambda_R^\varepsilon)^2=4\rho^2-(1-2\rho+\tau_\varepsilon\lambda_R^\varepsilon)^2, \tag{4.21}$$

and hence

$$|\tau_\varepsilon\lambda_I^\varepsilon|\le 2\rho. \tag{4.22}$$

If $\lambda_I^\varepsilon=0$, then $\lambda_\varepsilon=\lambda_R^\varepsilon$, and (4.18) becomes

$$[1+o(1)]\int_{\mathbb{R}}(|(\phi_\varepsilon^c)_y|^2+|\phi_\varepsilon^c|^2-2w|\phi_\varepsilon^c|^2)=-\lambda_R^\varepsilon\int_{\mathbb{R}}|\phi_\varepsilon^c|^2-\left(\frac{\rho\lambda_R^\varepsilon}{1+\tau_\varepsilon\lambda_R^\varepsilon}+\frac{6\rho^2}{5|1+\tau_\varepsilon\lambda_R^\varepsilon|^2}\right)\frac{|\int_{\mathbb{R}}w\phi_\varepsilon^c|^2}{\int_{\mathbb{R}}w^2}.$$

Using the inequality

$$\int_{\mathbb{R}}(|(\phi_\varepsilon^c)_y|^2+|\phi_\varepsilon^c|^2-2w|\phi_\varepsilon^c|^2)\ge-\mu_1\int_{\mathbb{R}}|\phi_\varepsilon^c|^2,$$

we obtain that for $\varepsilon>0$ sufficiently small

$$-2\mu_1\int_{\mathbb{R}}|\phi_\varepsilon^c|^2\le-\lambda_R^\varepsilon\int_{\mathbb{R}}|\phi_\varepsilon^c|^2-\left(\frac{\rho\lambda_R^\varepsilon}{1+\tau_\varepsilon\lambda_R^\varepsilon}+\frac{6\rho^2}{5|1+\tau_\varepsilon\lambda_R^\varepsilon|^2}\right)\frac{|\int_{\mathbb{R}}w\phi_\varepsilon^c|^2}{\int_{\mathbb{R}}w^2}. \tag{4.23}$$

Then $\lambda_R^\varepsilon \le 0$, or $\lambda_R^\varepsilon > 0$. In the case $\lambda_R^\varepsilon > 0$, we obtain from (4.23) that

$$\lambda_R^\varepsilon \int_{\mathbb{R}} |\phi_\varepsilon^c|^2 \le 2\mu_1 \int_{\mathbb{R}} |\phi_\varepsilon^c|^2,$$

and hence

$$\lambda_R^\varepsilon \le 2\mu_1. \tag{4.24}$$

This finishes the proof of the lemma. □

In view of Lemma 4.2, there exist constants $\varepsilon_0 > 0$, $a > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that the sector

$$S_{a,\theta} := \{\lambda \in \mathbb{C} : |\arg(\lambda - a)| < \theta\} \cup \{a\} \tag{4.25}$$

is contained in the resolvent set of $\mathcal{L}_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$.

Lemma 4.3. *The operator $\mathcal{L}_\varepsilon$ is a sectorial operator and hence generate a strongly continuous and analytic semigroup on the space Z. Moreover, for $\lambda \in S_{a,\theta}$ with $a \gg 1$, the operator $\mathcal{R}(\lambda, a) = (\lambda - \mathcal{L}_\varepsilon)^{-1}$ is compact as an operator mapping Z into itself and there exists a constant $M > 0$ such that*

$$\|\mathcal{R}(\lambda, a)\| \le \frac{M}{|\lambda - a|}, \qquad \text{for } \lambda \in S_{a,\theta}. \tag{4.26}$$

Proof. For any $\lambda \in S_{a,\theta}$ we consider the resolvent equation

$$(\mathcal{L}_\varepsilon - \lambda)\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}f_1\\ f_2\end{pmatrix}, \tag{4.27}$$

namely,

$$\begin{cases} \varepsilon^2(\phi_\varepsilon)_{xx} - \phi_\varepsilon + 2Au_\varepsilon^S v_\varepsilon^S \phi_\varepsilon + A(u_\varepsilon^S)^2\psi_\varepsilon = \lambda\phi_\varepsilon + f_1,\\ \dfrac{1}{\beta^2}(\psi_\varepsilon)_{xx} - \psi_\varepsilon - \varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon - 2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S\phi_\varepsilon = \tau_\varepsilon\lambda\psi_\varepsilon + \tau_\varepsilon f_2. \end{cases} \tag{4.28}$$

Put $\beta_\lambda^2 = \beta^2(1 + \tau_\varepsilon\lambda)$. Then from the second equation of (4.28) we obtain

$$\begin{aligned} \psi_\varepsilon(x) &= -\int_{-1}^1 G_{\beta_\lambda}(x,\xi)\beta^2\left[\varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon + 2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S\phi_\varepsilon + \tau_\varepsilon f_2\right]d\xi\\ &= -\int_{-1}^1\left(\frac{1}{2(1+\tau_\varepsilon\lambda)} + \beta^2 G_0(x,\xi) + O(\beta^2)\right)\left[\varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon + 2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S\phi_\varepsilon + \tau_\varepsilon f_2\right]d\xi\\ &= -\frac{\psi_\varepsilon(0)}{2A^2v_0^2(1+\tau_\varepsilon\lambda)}\int_{-\infty}^\infty w^2(y)dy - \frac{1}{A(1+\tau_\varepsilon\lambda)}\int_{-\infty}^\infty w(y)\phi_\varepsilon^c(y)dy\\ &\quad - \frac{\tau_\varepsilon}{2(1+\tau_\varepsilon\lambda)}\int_{-1}^1 f_2(x)dx + h.o.t. \end{aligned} \tag{4.29}$$

Using $\int_{-\infty}^\infty w^2 = 6$, we obtain

$$\psi_\varepsilon(x) \sim \psi_\varepsilon(0) = -\frac{Av_0^2}{3 + A^2v_0^2(1+\tau_\varepsilon\lambda)}\int_{-\infty}^\infty w(y)\phi_\varepsilon^c(y)dy - \frac{Av_0^2\tau_\varepsilon}{6 + 2A^2v_0^2(1+\tau_\varepsilon\lambda)}\int_{-1}^1 f_2(x)dx. \tag{4.30}$$

We assume $a \gg 1$ and θ be fixed. Then from the first equation in (4.28) we get

$$\phi_\varepsilon = \left[\varepsilon^2\frac{d^2}{dx^2} - (1+\lambda) + 2Au_\varepsilon^S v_\varepsilon^S\right]^{-1}\left(f_1 - A(u_\varepsilon^S)^2\psi_\varepsilon\right) \tag{4.31}$$

Since for ε small,

$$\max_{[-1,1]} Au_\varepsilon^S v_\varepsilon^S \le 2w(0) = 2\max_{\mathbb{R}} w,$$

there exists, by the resolvent estimate, a constant $M > 0$, such that

$$\|\phi_\varepsilon\|_{L^2([-1,1])} \leq \frac{M}{|\lambda + 1 - 4w(0)|}\left(\frac{w^2(0)}{Av_0^2}\|\psi_\varepsilon\|_{L^2([-1,1])} + \|f_1\|_{L^2([-1,1])}\right).$$

While

$$\begin{aligned}\|\psi_\varepsilon\|_{L^2([-1,1])} &\leq \frac{4Av_0^2(\int_{-\infty}^{\infty} w^2 dy)^{1/2}}{|3 + A^2v_0^2(1+\tau_\varepsilon\lambda)|}\|\phi_\varepsilon^c\|_{L^2(\mathbb{R})} + \frac{4Av_0^2\tau_\varepsilon}{|3 + A^2v_0^2(1+\tau_\varepsilon\lambda)|}\|f_2\|_{L^2([-1,1])} \\ &\leq \frac{4Av_0^2\max\{\sqrt{6},\tau_\varepsilon\}}{|3 + A^2v_0^2(1+\tau_\varepsilon\lambda)|}(\|\phi_\varepsilon\|_{L^2([-1,1])} + \|f_2\|_{L^2([-1,1])}).\end{aligned} \tag{4.32}$$

Let $a > 0$ be sufficiently large, then if $\lambda \in S_{a,\theta}$, we have

$$\frac{4Mw^2(0)\max\{\sqrt{6},\tau_\varepsilon\}}{|3 + A^2v_0^2(1+\tau_\varepsilon\lambda)||\lambda + 1 - 4w(0)|} < \frac{1}{2},$$

and hence

$$\|\phi_\varepsilon\|_{L^2([-1,1])} \leq \frac{CM}{|\lambda - a|}\left(\|f_1\|_{L^2([-1,1])}) + \|f_2\|_{L^2([-1,1])}\right). \tag{4.33}$$

From (4.33) we then have

$$\|\psi_\varepsilon\|_{L^2([-1,1])} \leq \frac{CM}{|\lambda - a|}\left(\|f_1\|_{L^2([-1,1])}) + \|f_2\|_{L^2([-1,1])}\right), \tag{4.34}$$

and therefore

$$\|\mathcal{R}(\lambda, a)\| \leq \frac{CM}{|\lambda - a|}, \qquad \text{for } \lambda \in S_{a,\varepsilon}. \tag{4.35}$$

The compactness of $(\lambda - \mathcal{L}_\varepsilon)^{-1}$ is obvious. This finishes the proof of the lemma. □

4.2. **The Schnakenberg system.** We only present here the derivation of the nonlocal eigenvalue problem, the existence and uniqueness of the critical τ_ε and the pair of pure imaginary eigenvalues, as well as the proof of the sectorial operator part is similar to the Gray-Scott system.

Let us consider the eigenvalue problem

$$\begin{cases}(\phi_\varepsilon)_{yy} - \phi_\varepsilon + 2u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon + (u_\varepsilon^S)^2\psi_\varepsilon = \lambda_\varepsilon\phi_\varepsilon, \\ D_\varepsilon(\psi_\varepsilon)_{xx} - \tau_\varepsilon\lambda_\varepsilon\psi_\varepsilon - \varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon = 2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S\phi_\varepsilon, \\ (\phi_\varepsilon)_y(\pm 1/\varepsilon) = (\psi_\varepsilon)_x(\pm 1) = 0.\end{cases} \tag{4.36}$$

Set $D_\varepsilon = \beta^{-2}$. The assumption on D_ε implies $\beta \to 0$. Set $\beta_{\lambda_\varepsilon}^2 = \beta^2\tau_\varepsilon\lambda_\varepsilon$. It is very easy to prove that $(\phi_\varepsilon)_x \to 0$ as $\varepsilon \to 0$. Using the Green's function introduced in Section 1 we write

$$\psi_\varepsilon(x) = -\int_{-1}^{1} G_{\beta_{\lambda_\varepsilon}}(x,\xi)\beta^2\varepsilon^{-1}[(u_\varepsilon^S(\xi))^2\psi_\varepsilon(\xi) + 2u_\varepsilon^S(\xi)v_\varepsilon^S(\xi)\phi_\varepsilon(\xi/\varepsilon)]d\xi. \tag{4.37}$$

As $\varepsilon \to 0$, we calculate at $x = 0$

$$\begin{aligned} \psi_\varepsilon(0) &= -\frac{\beta^2 \psi_\varepsilon(0)}{9\varepsilon} \int_{-1}^{1} G_{\beta_{\lambda_\varepsilon}}(0,\xi) w^2(\xi/\varepsilon)\, d\xi \\ &\quad - \frac{2\beta^2}{\varepsilon} \int_{-1}^{1} G_{\beta_{\lambda_\varepsilon}}(0,\xi) w(\xi/\varepsilon))\, \phi_\varepsilon^c(\xi/\varepsilon) d\xi + o(1) \\ &= -\frac{\beta^2 \psi_\varepsilon(0)}{9\varepsilon} \int_{-1}^{1} \left(\frac{(\beta_{\lambda_\varepsilon})^{-2}}{2} + G_0(0,\xi) + O(1) \right) w^2(\xi/\varepsilon)\, d\xi \\ &\quad - \frac{2\beta^2}{\varepsilon} \int_{-1}^{1} \left(\frac{(\beta_{\lambda_\varepsilon})^{-2}}{2} + G_0(0,\xi) + O(1) \right) w(\xi/\varepsilon))\, \phi_\varepsilon^c(\xi/\varepsilon) d\xi + o(1) \\ &= -\frac{\psi_\varepsilon(0)}{18\tau_\varepsilon\lambda_\varepsilon} \int_{-\infty}^{\infty} w^2(y) dy + O(\beta^2)\psi_\varepsilon(0) \\ &\quad - \frac{1 + O(\beta^2)}{\tau_\varepsilon\lambda_\varepsilon} \int_{-\infty}^{\infty} w(y)\phi_\varepsilon^c(y) dy + o(1). \end{aligned} \tag{4.38}$$

Using $\int_{-\infty}^{\infty} w^2(y)dy = 6$, we obtain

$$\psi_\varepsilon(0) = -\frac{3}{(1 + 3\tau_\varepsilon\lambda_\varepsilon)} \int_{-\infty}^{\infty} w(y)\phi_\varepsilon^c dy + h.o.t. \tag{4.39}$$

Substituting (4.39) into the first equation of (4.36) we arrive at

$$(\phi_\varepsilon)_{yy} - \phi_\varepsilon + 2w\phi_\varepsilon - \frac{w^2}{3 + 9\tau_\varepsilon\lambda_\varepsilon} \int_{\mathbb{R}} w\phi_\varepsilon^c dy = \lambda_\varepsilon[1 + o(1)]\phi_\varepsilon. \tag{4.40}$$

As in the proof of Theorem 1 in [4] one obtains

$$\tau_\varepsilon \to \tau_0, \qquad \lambda_\varepsilon \to \lambda_0, \qquad \phi_\varepsilon(y) \to \phi_0(y) \qquad \text{in } H^2_{loc}(\mathbb{R}), \qquad \text{as } \varepsilon \to 0, \tag{4.41}$$

where (λ_0, ϕ_0) is an eigenpair of the NLEP (2.25).

Lemmas 4.1 and 4.2 also works for the Schankenberg system. Therefore there exist constants $\varepsilon_0 > 0$, $a > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that the sector

$$S_{a,\theta} := \{\lambda \in \mathbb{C} : |\arg(\lambda - a)| < \theta\} \cup \{a\} \tag{4.42}$$

is contained in the resolvent set of $\mathcal{L}_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$.

Lemma 4.4. *The operator $\mathcal{L}_\varepsilon$ is a sectorial operator and hence generate a strongly continuous and analytic semigroup on the space Z. Moreover, for $\lambda \in S_{a,\theta}$ with $a \gg 1$, the operator $\mathcal{R}(\lambda, a) = (\lambda - \mathcal{L}_\varepsilon)^{-1}$ is compact as an operator mapping Z into itself and there exists a constant $M > 0$ such that*

$$\|\mathcal{R}(\lambda, a)\| \le \frac{M}{|\lambda - a|}, \qquad \textit{for } \lambda \in S_{a,\theta}. \tag{4.43}$$

Proof. For any $\lambda \in S_{a,\theta}$ we consider the resolvent equation

$$(\mathcal{L}_\varepsilon - \lambda) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{4.44}$$

namely,

$$\begin{cases} \varepsilon^2(\phi_\varepsilon)_{xx} - \phi_\varepsilon + 2u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon + (u_\varepsilon^S)^2 \psi_\varepsilon = \lambda\phi_\varepsilon + f_1, \\ \dfrac{1}{\beta^2}(\psi_\varepsilon)_{xx} - \varepsilon^{-1}(u_\varepsilon^S)^2 \psi_\varepsilon - 2\varepsilon^{-1} u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon = \tau_\varepsilon\lambda\psi_\varepsilon + \tau_\varepsilon f_2. \end{cases} \tag{4.45}$$

Put $\beta_\lambda^2 = \beta^2\tau_\varepsilon\lambda$. Then from the second equation of (4.45) we obtain

$$
\begin{aligned}
\psi_\varepsilon(x) &= -\int_{-1}^{1} G_{\beta_\lambda}(x,\xi)\beta^2\left[\varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon + 2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S\phi_\varepsilon + \tau_\varepsilon f_2\right]d\xi \\
&= -\int_{-1}^{1}\left(\frac{1}{2\tau_\varepsilon\lambda} + \beta^2 G_0(x,\xi) + O(\beta^2)\right)\left[\varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon + 2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S\phi_\varepsilon + \tau_\varepsilon f_2\right]d\xi \\
&= -\frac{\psi_\varepsilon(0)}{18\tau_\varepsilon\lambda}\int_{-\infty}^{\infty} w^2(y)dy - \frac{1}{\tau_\varepsilon\lambda}\int_{-\infty}^{\infty} w(y)\phi_\varepsilon^c(y)dy \\
&\quad - \frac{1}{2\lambda}\int_{-1}^{1} f_2(x)dx + h.o.t.
\end{aligned}
\tag{4.46}
$$

Using $\int_{-\infty}^{\infty} w^2 = 6$, we obtain

$$
\psi_\varepsilon(x) \sim \psi_\varepsilon(0) = -\frac{3}{1+3\tau_\varepsilon\lambda}\int_{-\infty}^{\infty} w(y)\phi_\varepsilon^c(y)dy - \frac{3\tau_\varepsilon}{6+2\tau_\varepsilon\lambda}\int_{-1}^{1} f_2(x)dx. \tag{4.47}
$$

We assume $a \gg 1$ and θ be fixed. Then from the first equation in (4.28) we get

$$
\phi_\varepsilon = \left[\varepsilon^2\frac{d^2}{dx^2} - (1+\lambda) + 2u_\varepsilon^S v_\varepsilon^S\right]^{-1}\left(f_1 - (u_\varepsilon^S)^2\psi_\varepsilon\right) \tag{4.48}
$$

Since for ε small,

$$
\max_{[-1,1]} u_\varepsilon^S v_\varepsilon^S \le 2w(0) = 2\max_{\mathbb{R}} w,
$$

there exists, by the resolvent estimate, a constant $M > 0$, such that

$$
\|\phi_\varepsilon\|_{L^2([-1,1])} \le \frac{M}{|\lambda + 1 - 4w(0)|}\left(\frac{w^2(0)}{3}\|\psi_\varepsilon\|_{L^2([-1,1])} + \|f_1\|_{L^2([-1,1])}\right).
$$

While

$$
\begin{aligned}
\|\psi_\varepsilon\|_{L^2([-1,1])} &\le \frac{3(\int_{-\infty}^{\infty} w^2 dy)^{1/2}}{|1+3\tau_\varepsilon\lambda|}\|\phi_\varepsilon^c\|_{L^2(\mathbb{R})} + \frac{3\tau_\varepsilon}{|1+3\tau_\varepsilon\lambda|}\|f_2\|_{L^2([-1,1])} \\
&\le \frac{3\max\{\sqrt{6},\tau_\varepsilon\}}{|1+3\tau_\varepsilon\lambda|}(\|\phi_\varepsilon\|_{L^2([-1,1])} + \|f_2\|_{L^2([-1,1])}).
\end{aligned}
\tag{4.49}
$$

Let $a > 0$ be sufficiently large, then if $\lambda \in S_{a,\theta}$, we have

$$
\frac{4Mw^2(0)\max\{\sqrt{6},\tau_\varepsilon\}}{|1+3\tau_\varepsilon\lambda||\lambda+1-4w(0)|} < \frac{1}{2},
$$

and hence

$$
\|\phi_\varepsilon\|_{L^2([-1,1])} \le \frac{CM}{|\lambda - a|}\left(\|f_1\|_{L^2([-1,1])}) + \|f_2\|_{L^2([-1,1])}\right). \tag{4.50}
$$

From (4.50) we then have

$$
\|\psi_\varepsilon\|_{L^2([-1,1])} \le \frac{CM}{|\lambda - a|}\left(\|f_1\|_{L^2([-1,1])}) + \|f_2\|_{L^2([-1,1])}\right), \tag{4.51}
$$

and therefore

$$
\|\mathcal{R}(\lambda,a)\| \le \frac{CM}{|\lambda - a|}, \qquad \text{for } \lambda \in S_{a,\varepsilon}. \tag{4.52}
$$

The compactness of $(\lambda - \mathcal{L}_\varepsilon)^{-1}$ is obvious. This finishes the proof of the lemma. □

In both the Gray-Scott system and the Schnakenberg system, the semigroup generated by $\mathcal{L}_\varepsilon$ is defined by the formula

$$T(t) = e^{\mathcal{L}_\varepsilon t} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\mathcal{R}(\lambda, a)d\lambda, \tag{4.53}$$

where Γ is a smooth curve in $S_{a,\theta}$ that connects $\infty e^{-\theta i}$ and $\infty e^{\theta i}$.

5. The transversality condition for the perturbed systems

5.1. **The Gray-Scott system.** We begin from the eigenvalue problem

$$\begin{cases} (\phi_\varepsilon)_{yy} - \phi_\varepsilon + 2Au_\varepsilon^S v_\varepsilon^S \phi_\varepsilon + A(u_\varepsilon^S)^2\psi_\varepsilon = \lambda_\varepsilon\phi_\varepsilon, \\ \dfrac{1}{\beta^2}(\psi_\varepsilon)_{xx} - \psi_\varepsilon - \varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon - 2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S\phi_\varepsilon = \tau_\varepsilon\lambda_\varepsilon\psi_\varepsilon, \end{cases} \tag{5.1}$$

where $y = \varepsilon^{-1}x$, $D_\varepsilon = \varepsilon\beta^{-2}$, λ_ε is some complex number, and

$$\phi_\varepsilon \in H_N^2([-\varepsilon^{-1}, \varepsilon^{-1}]), \qquad \psi \in H_N^2([-1,1]). \tag{5.2}$$

We let $\mu_\varepsilon = \tau_\varepsilon\lambda_\varepsilon$. Then (5.1) is equivalent to the following eigenvalue problem

$$\begin{cases} \tau_\varepsilon\{(\phi_\varepsilon)_{yy} - \phi_\varepsilon + 2Au_\varepsilon^S v_\varepsilon^S \phi_\varepsilon + A(u_\varepsilon^S)^2\psi_\varepsilon\} = \mu_\varepsilon\phi_\varepsilon, \\ \dfrac{1}{\beta^2}(\psi_\varepsilon)_{xx} - \psi_\varepsilon - \varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon - 2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S\phi_\varepsilon = \mu_\varepsilon\psi_\varepsilon. \end{cases} \tag{5.3}$$

i.e.,

$$\mathscr{L}_\varepsilon \begin{pmatrix}\phi_\varepsilon \\ \psi_\varepsilon\end{pmatrix} = \mu_\varepsilon \begin{pmatrix}\phi_\varepsilon \\ \psi_\varepsilon\end{pmatrix}, \tag{5.4}$$

with $\mathscr{L}_\varepsilon = \tau_\varepsilon\mathcal{L}_\varepsilon$. We note that $\mathscr{L}_\varepsilon^* = \tau_\varepsilon\mathcal{L}_\varepsilon^*$.

Let τ_ε be the parameter value from Lemma 4.1, so that $Re(\lambda_\varepsilon(\tau_\varepsilon^h)) = 0$. Then, via the relationship

$$\mu_\varepsilon'(\tau_\varepsilon) = \tau_\varepsilon\lambda_\varepsilon'(\tau_\varepsilon) + \lambda_\varepsilon(\tau_\varepsilon), \tag{5.5}$$

we obtain that $Re(\mu_\varepsilon'(\tau_\varepsilon^h)) = \tau_\varepsilon^h Re(\lambda_\varepsilon'(\tau_\varepsilon^h))$. We now show that $\mu_\varepsilon'(\tau_\varepsilon^h) > 0$ for $\varepsilon > 0$ sufficiently small.

Let $\Phi_\varepsilon = (\phi_\varepsilon, \psi_\varepsilon)^T$ be a nontrivial eigenfunction of $\mathscr{L}_\varepsilon$ corresponding to μ_ε and $\Phi_\varepsilon^* = (\phi_\varepsilon^*, \psi_\varepsilon^*)^T$ be a nontrivial eigenfunction of $\mathscr{L}_\varepsilon^*$ corresponding to μ_ε^*. We have by definition

$$\langle \Phi_\varepsilon, \overline{\Phi_\varepsilon^*}\rangle = \langle \overline{\Phi_\varepsilon}, \Phi_\varepsilon^*\rangle = 0. \tag{5.6}$$

Since λ_0 is a simple eigenvalue, μ_ε is simple. Moreover we also have

$$\langle \Phi_\varepsilon, \Phi_\varepsilon^*\rangle = \langle \overline{\Phi_\varepsilon}, \overline{\Phi_\varepsilon^*}\rangle \neq 0. \tag{5.7}$$

Using the Green's function introduced in Section 1 we obtain

$$\psi_\varepsilon(x) = -\frac{Av_0^2}{3 + A^2v_0^2(1+\mu_\varepsilon)}\int_{-\infty}^{\infty} w(y)\phi_\varepsilon^c dy + h.o.t. \tag{5.8}$$

Similarly, from

$$\mathscr{L}_\varepsilon^* \begin{pmatrix}\phi_\varepsilon^* \\ \psi_\varepsilon^*\end{pmatrix} = \mu_\varepsilon^* \begin{pmatrix}\phi_\varepsilon^* \\ \psi_\varepsilon^*\end{pmatrix}, \tag{5.9}$$

i.e.,

$$
(5.10)\qquad \begin{cases} (\phi_\varepsilon^*)_{yy} + 2Au_\varepsilon^S v_\varepsilon^S \phi_\varepsilon^* - \phi_\varepsilon^* - 2\tau_\varepsilon^{-1}\varepsilon^{-1} u_\varepsilon^S v_\varepsilon^S \psi_\varepsilon^* = \dfrac{\mu_\varepsilon^*}{\tau_\varepsilon}\phi_\varepsilon^*, \\ \dfrac{1}{\beta^2}(\psi_\varepsilon^*)_{xx} - (1+\mu_\varepsilon^*)\psi_\varepsilon^* - \varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon^* = -A\tau_\varepsilon (u_\varepsilon^S)^2\phi_\varepsilon^*, \\ (\phi_\varepsilon^*)_y(\pm 1/\varepsilon) = (\psi_\varepsilon^*)_x(\pm 1) = 0, \end{cases}
$$

We have

$$
(5.11)\qquad \psi_\varepsilon^*(x) = -\int_{-1}^{1} G_{\beta_{\lambda_\varepsilon^*}}(x,\xi)\beta^2[\varepsilon^{-1}(u_\varepsilon^S(\xi))^2\psi_\varepsilon^*(\xi) - A\tau_\varepsilon (u_\varepsilon^S(\xi))^2\phi_\varepsilon^*(\xi/\varepsilon)]d\xi.
$$

As $\varepsilon \to 0$, we calculate at $x = 0$,

$$
(5.12)\qquad \begin{aligned} \varepsilon^{-1}\psi_\varepsilon^*(0) &= -\frac{\beta^2\varepsilon^{-1}\psi_\varepsilon^*(0)}{A^2v_0^2\varepsilon}\int_{-1}^{1} G_{\beta_{\lambda_\varepsilon^*}}(0,\xi) w^2\left(\xi/\varepsilon\right) d\xi \\ &\quad + \frac{\tau_\varepsilon\beta^2}{Av_0^2\varepsilon}\int_{-1}^{1} G_{\beta_{\lambda_\varepsilon^*}}(0,\xi) w^2\left(\xi/\varepsilon\right))\,(\phi_\varepsilon^*)^c(\xi/\varepsilon) d\xi + o(1) \\ &= -\frac{\beta^2\varepsilon^{-1}\psi_\varepsilon^*(0)}{A^2v_0^2\varepsilon}\int_{-1}^{1}\left(\frac{(\beta_{\lambda_\varepsilon^*})^{-2}}{2} + G_0(0,\xi) + O(1)\right) w^2\left(\xi/\varepsilon\right) d\xi \\ &\quad + \frac{\tau_\varepsilon\beta^2}{Av_0^2\varepsilon}\int_{-1}^{1}\left(\frac{(\beta_{\lambda_\varepsilon^*})^{-2}}{2} + G_0(0,\xi) + O(1)\right) w^2(\xi/\varepsilon)(\phi_\varepsilon^*)^c(\xi/\varepsilon) d\xi + o(1) \\ &= -\frac{\varepsilon^{-1}\psi_\varepsilon(0)}{2A^2v_0^2(1+\tau_\varepsilon\lambda_\varepsilon^*)}\int_{-\infty}^{\infty} w^2(y)dy + O(\beta^2)\varepsilon^{-1}\psi_\varepsilon(0) \\ &\quad + \frac{\tau_\varepsilon[1+O(\beta^2)]}{2Av_0^2(1+\tau_\varepsilon\lambda_\varepsilon^*)}\int_{-\infty}^{\infty} w^2(y)(\phi^*)_\varepsilon^c(y)dy + o(1). \end{aligned}
$$

Using $\int_{-\infty}^{\infty} w^2(y)dy = 6$, we obtain

$$
(5.13)\qquad \varepsilon^{-1}\psi_\varepsilon^*(0) = \frac{A\tau_\varepsilon}{2[3 + A^2v_0^2(1+\tau_\varepsilon\lambda_\varepsilon^*)]}\int_{-\infty}^{\infty} w^2(y)(\phi^*)_\varepsilon^c dy + h.o.t.
$$

Substituting (5.13) into the first equation of (5.10) we arrive at

$$
(5.14)\qquad (\phi_\varepsilon^*)_{yy} - \phi_\varepsilon^* + 2w\phi_\varepsilon^* - \frac{w}{3 + A^2v_0^2(1+\mu_\varepsilon^*)}\int_{\mathbb{R}} w^2(\phi_\varepsilon^*)^c dy = [1+o(1)]\frac{\mu_\varepsilon^*}{\tau_\varepsilon}\phi_\varepsilon^*
$$

Differentiating (5.4) with respect to $\tau = \tau_\varepsilon$ we find that

$$
(5.15)\qquad \frac{\partial \mathscr{L}_\varepsilon}{\partial\tau}\Phi_\varepsilon + \mathscr{L}_\varepsilon\frac{\partial\Phi_\varepsilon}{\partial\tau} = \frac{\partial\mu_\varepsilon}{\partial\tau}\Phi_\varepsilon + \mu_\varepsilon\frac{\partial\Phi_\varepsilon}{\partial\tau}.
$$

Taking the inner product with Φ_ε^* gives

$$
(5.16)\qquad \langle\frac{\partial \mathscr{L}_\varepsilon}{\partial\tau}\Phi_\varepsilon, \Phi_\varepsilon^*\rangle + \langle\mathscr{L}_\varepsilon\frac{\partial\Phi_\varepsilon}{\partial\tau}, \Phi_\varepsilon^*\rangle = \langle\frac{\partial\mu_\varepsilon}{\partial\tau}\Phi_\varepsilon, \Phi_\varepsilon^*\rangle + \langle\mu_\varepsilon\frac{\partial\Phi_\varepsilon}{\partial\tau}, \Phi_\varepsilon^*\rangle,
$$

Using

$$
\langle\mathscr{L}_\varepsilon\frac{\partial\Phi_\varepsilon}{\partial\tau}, \Phi_\varepsilon^*\rangle = \mu_\varepsilon\langle\frac{\partial\Phi_\varepsilon}{\partial\tau}, \Phi_\varepsilon^*\rangle,
$$

we then obtain at τ_ε^h that

$$
(5.17)\qquad \mu_\varepsilon'(\tau_\varepsilon^h) = \frac{\partial\mu_\varepsilon}{\partial\tau}(\tau_\varepsilon^h) = \frac{\langle\frac{\partial\mathscr{L}_\varepsilon}{\partial\tau}\Phi_\varepsilon, \Phi_\varepsilon^*\rangle}{\langle\Phi_\varepsilon, \Phi_\varepsilon^*\rangle} = \frac{\mu_\varepsilon}{\tau_\varepsilon^h}\frac{\int_{-1}^{1}\phi_\varepsilon\overline{\phi_\varepsilon^*}}{\langle\Phi_\varepsilon, \Phi_\varepsilon^*\rangle}.
$$

We compute

$$\int_{-1}^{1} \phi_\varepsilon \overline{\phi_\varepsilon^*}\, dx = \varepsilon \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} \phi_\varepsilon(y) \overline{\phi_\varepsilon^*}(y)\, dy = \varepsilon[1+o(1)] \int_{\mathbb{R}} \phi_0 \overline{\phi_0^*}\, dy, \tag{5.18}$$

and

$$\int_{-1}^{1} \psi_\varepsilon \overline{\psi_\varepsilon^*}\, dx = -\varepsilon[1+o(1)] \frac{A^2 v_0^2 \tau_\varepsilon^h}{[3 + A^2 v_0^2 (1 + \tau_\varepsilon^h \lambda_\varepsilon(\tau_\varepsilon^h))]^2} \int_{\mathbb{R}} w^2 \overline{\phi_0^*} \int_{\mathbb{R}} w\phi_0, \tag{5.19}$$

so that in view of (3.23) and Remark 3.4, we obtain

$$\mu_\varepsilon'(\tau_\varepsilon^h) = \frac{[1+o(1)]\lambda_0(\tau_h) \int_{\mathbb{R}} \phi_0 \overline{\phi_0^*}}{\int_{\mathbb{R}} \phi_0 \overline{\phi_0^*} - \frac{A^2 v_0^2 \tau_h}{[3 + A^2 v_0^2 (1 + \tau_h \lambda_0(\tau_h))]^2} \int_{\mathbb{R}} w\phi_0 \int_{\mathbb{R}} w^2 \overline{\phi_0^*}} = [1+o(1)] \hat{\mu}_0'(\hat{\tau}_h). \tag{5.20}$$

As a consequence of Lemma 3.3 we therefore have

$$Re(\lambda_\varepsilon'(\tau_\varepsilon^h)) = \frac{1}{\tau_\varepsilon^h} Re(\mu_\varepsilon'(\tau_\varepsilon^h)) > 0, \tag{5.21}$$

for sufficiently small $\varepsilon > 0$.

5.2. **The Schnakenberg system.** We begin from the eigenvalue problem

$$\begin{cases} (\phi_\varepsilon)_{yy} - \phi_\varepsilon + 2u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon + (u_\varepsilon^S)^2 \psi_\varepsilon = \lambda_\varepsilon \phi_\varepsilon, \\ \dfrac{1}{\beta^2} (\psi_\varepsilon)_{xx} - \tau_\varepsilon \lambda_\varepsilon \psi_\varepsilon - \varepsilon^{-1} (u_\varepsilon^S)^2 \psi_\varepsilon = 2\varepsilon^{-1} u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon, \\ (\phi_\varepsilon)_y(\pm 1/\varepsilon) = (\psi_\varepsilon)_x(\pm 1) = 0. \end{cases} \tag{5.22}$$

We let $\mu_\varepsilon = \tau_\varepsilon \lambda_\varepsilon$. Then (5.22) is equivalent to the following eigenvalue problem

$$\begin{cases} \tau_\varepsilon \{(\phi_\varepsilon)_{yy} - \phi_\varepsilon + 2u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon + (u_\varepsilon^S)^2 \psi_\varepsilon\} = \mu_\varepsilon \phi_\varepsilon, \\ \dfrac{1}{\beta^2} (\psi_\varepsilon)_{xx} - \varepsilon^{-1} (u_\varepsilon^S)^2 \psi_\varepsilon - 2\varepsilon^{-1} u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon = \mu_\varepsilon \psi_\varepsilon. \end{cases} \tag{5.23}$$

i.e.,

$$\mathscr{L}_\varepsilon \begin{pmatrix} \phi_\varepsilon \\ \psi_\varepsilon \end{pmatrix} = \mu_\varepsilon \begin{pmatrix} \phi_\varepsilon \\ \psi_\varepsilon \end{pmatrix}, \tag{5.24}$$

with $\mathscr{L}_\varepsilon = \tau_\varepsilon \mathcal{L}_\varepsilon$. We note that $\mathscr{L}_\varepsilon^* = \tau_\varepsilon \mathcal{L}_\varepsilon^*$.

Let τ_ε be the parameter value from Lemma 4.1, so that $Re(\lambda_\varepsilon(\tau_\varepsilon^h)) = 0$. Then, via the relationship

$$\mu_\varepsilon'(\tau_\varepsilon) = \tau_\varepsilon \lambda_\varepsilon'(\tau_\varepsilon) + \lambda_\varepsilon(\tau_\varepsilon), \tag{5.25}$$

we obtain that $Re(\mu_\varepsilon'(\tau_\varepsilon^h)) = \tau_\varepsilon^h Re(\lambda_\varepsilon'(\tau_\varepsilon^h))$. We now show that $\mu_\varepsilon'(\tau_\varepsilon^h) > 0$ for $\varepsilon > 0$ sufficiently small.

Let $\Phi_\varepsilon = (\phi_\varepsilon, \psi_\varepsilon)^T$ be a nontrivial eigenfunction of $\mathscr{L}_\varepsilon$ corresponding to μ_ε and $\Phi_\varepsilon^* = (\phi_\varepsilon^*, \psi_\varepsilon^*)^T$ be a nontrivial eigenfunction of $\mathscr{L}_\varepsilon^*$ corresponding to μ_ε^*. We have by definition

$$\langle \Phi_\varepsilon, \overline{\Phi_\varepsilon^*} \rangle = \langle \overline{\Phi_\varepsilon}, \Phi_\varepsilon^* \rangle = 0. \tag{5.26}$$

Since λ_ε is a simple eigenvalue, μ_ε is simple. Moreover we also have

$$\langle \Phi_\varepsilon, \Phi_\varepsilon^* \rangle = \langle \overline{\Phi_\varepsilon}, \overline{\Phi_\varepsilon^*} \rangle \neq 0. \tag{5.27}$$

Using the Green's function introduced in Section 1 we obtain

$$\psi_\varepsilon(x) = -\frac{3}{(1+3\mu_\varepsilon)} \int_{-\infty}^{\infty} w(y) \phi_\varepsilon^c dy + h.o.t. \tag{5.28}$$

Similarly, from

$$\mathscr{L}_\varepsilon^* \begin{pmatrix} \phi_\varepsilon^* \\ \psi_\varepsilon^* \end{pmatrix} = \mu_\varepsilon^* \begin{pmatrix} \phi_\varepsilon^* \\ \psi_\varepsilon^* \end{pmatrix}, \tag{5.29}$$

i.e.,

$$\begin{cases} (\phi_\varepsilon^*)_{yy} - \phi_\varepsilon^* + 2u_\varepsilon^S v_\varepsilon^S \phi_\varepsilon^* - 2\tau_\varepsilon^{-1}\varepsilon^{-1} u_\varepsilon^S v_\varepsilon^S \psi_\varepsilon^* = \dfrac{\mu_\varepsilon^*}{\tau_\varepsilon}\phi_\varepsilon^*, \\ \dfrac{1}{\beta^2}(\psi_\varepsilon^*)_{xx} - \varepsilon^{-1}(u_\varepsilon^S)^2\psi_\varepsilon^* + \tau_\varepsilon (u_\varepsilon^S)^2 \phi_\varepsilon^* = \mu_\varepsilon^* \psi_\varepsilon^*. \end{cases} \tag{5.30}$$

We have

$$\psi_\varepsilon^*(x) = -\int_{-1}^1 G_{\beta_{\lambda_\varepsilon^*}}(x,\xi)\beta^2[\varepsilon^{-1}(u_\varepsilon^S(\xi))^2\psi_\varepsilon^*(\xi) - \tau_\varepsilon(u_\varepsilon^S(\xi))^2\phi_\varepsilon^*(\xi/\varepsilon)]d\xi. \tag{5.31}$$

As $\varepsilon \to 0$, we calculate at $x = 0$,

$$\begin{aligned} \varepsilon^{-1}\psi_\varepsilon^*(0) = &-\frac{\beta^2\varepsilon^{-1}\psi_\varepsilon^*(0)}{9\varepsilon}\int_{-1}^1 G_{\beta_{\lambda_\varepsilon^*}}(0,\xi) w^2\left(\xi/\varepsilon\right)d\xi \\ &+ \frac{\tau_\varepsilon\beta^2}{9\varepsilon}\int_{-1}^1 G_{\beta_{\lambda_\varepsilon^*}}(0,\xi)w^2\left(\xi/\varepsilon)\right)(\phi_\varepsilon^*)^c(\xi/\varepsilon)d\xi + o(1) \\ = &-\frac{\beta^2\varepsilon^{-1}\psi_\varepsilon^*(0)}{9\varepsilon}\int_{-1}^1 \left(\frac{(\beta_{\lambda_\varepsilon^*})^{-2}}{2} + G_0(0,\xi) + O(1)\right) w^2\left(\xi/\varepsilon\right)d\xi \\ &+ \frac{\tau_\varepsilon\beta^2}{9\varepsilon}\int_{-1}^1 \left(\frac{(\beta_{\lambda_\varepsilon^*})^{-2}}{2} + G_0(0,\xi) + O(1)\right) w^2\left(\xi/\varepsilon)\right)(\phi_\varepsilon^*)^c(\xi/\varepsilon)d\xi + o(1) \\ = &-\frac{\varepsilon^{-1}\psi_\varepsilon(0)}{18\tau_\varepsilon\lambda_\varepsilon^*}\int_{-\infty}^\infty w^2(y)dy + O(\beta^2)\varepsilon^{-1}\psi_\varepsilon(0) \\ &+ \frac{[1+O(\beta^2)]}{18\lambda_\varepsilon^*}\int_{-\infty}^\infty w^2(y)(\phi_\varepsilon^*)^c(y)dy + o(1). \end{aligned} \tag{5.32}$$

Using $\int_{-\infty}^\infty w^2(y)dy = 6$, we obtain

$$\varepsilon^{-1}\psi_\varepsilon^*(0) = \frac{\tau_\varepsilon}{6 + 18\tau_\varepsilon\lambda_\varepsilon^*}\int_{-\infty}^\infty w^2(y)(\phi_\varepsilon^*)^c dy + h.o.t. \tag{5.33}$$

Substituting (5.33) into the first equation of (5.30) we arrive at

$$(\phi_\varepsilon^*)_{yy} - \phi_\varepsilon^* + 2w\phi_\varepsilon^* - \frac{w}{3 + 9\mu_\varepsilon^*}\int_{\mathbb{R}} w^2(\phi_\varepsilon^*)^c dy = [1 + o(1)]\frac{\mu_\varepsilon^*}{\tau_\varepsilon}\phi_\varepsilon^* \tag{5.34}$$

Differentiating (5.24) with respect to $\tau = \tau_\varepsilon$ we find that

$$\frac{\partial \mathscr{L}_\varepsilon}{\partial \tau}\Phi_\varepsilon + \mathscr{L}_\varepsilon\frac{\partial \Phi_\varepsilon}{\partial \tau} = \frac{\partial \mu_\varepsilon}{\partial \tau}\Phi_\varepsilon + \mu_\varepsilon\frac{\partial \Phi_\varepsilon}{\partial \tau}. \tag{5.35}$$

Taking the inner product with Φ_ε^* gives

$$\langle\frac{\partial \mathscr{L}_\varepsilon}{\partial \tau}\Phi_\varepsilon, \Phi_\varepsilon^*\rangle + \langle\mathscr{L}_\varepsilon\frac{\partial \Phi_\varepsilon}{\partial \tau}, \Phi_\varepsilon^*\rangle = \langle\frac{\partial \mu_\varepsilon}{\partial \tau}\Phi_\varepsilon, \Phi_\varepsilon^*\rangle + \langle\mu_\varepsilon\frac{\partial \Phi_\varepsilon}{\partial \tau}, \Phi_\varepsilon^*\rangle, \tag{5.36}$$

Using

$$\langle\mathscr{L}_\varepsilon\frac{\partial \Phi_\varepsilon}{\partial \tau}, \Phi_\varepsilon^*\rangle = \mu_\varepsilon\langle\frac{\partial \Phi_\varepsilon}{\partial \tau}, \Phi_\varepsilon^*\rangle,$$

we then obtain at τ_ε^h that

$$\mu_\varepsilon'(\tau_\varepsilon^h) = \frac{\partial \mu_\varepsilon}{\partial \tau}(\tau_\varepsilon^h) = \frac{\langle\frac{\partial \mathscr{L}_\varepsilon}{\partial \tau}\Phi_\varepsilon, \Phi_\varepsilon^*\rangle}{\langle\Phi_\varepsilon, \Phi_\varepsilon^*\rangle} = \frac{\mu_\varepsilon}{\tau_\varepsilon^h}\frac{\int_{-1}^1 \phi_\varepsilon\overline{\phi_\varepsilon^*}}{\langle\Phi_\varepsilon, \Phi_\varepsilon^*\rangle}. \tag{5.37}$$

We compute

$$\int_{-1}^{1}\phi_\varepsilon\overline{\phi_\varepsilon^*}\,dx = \varepsilon\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}}\phi_\varepsilon(y)\overline{\phi_\varepsilon^*}(y)\,dy = \varepsilon[1+o(1)]\int_{\mathbb{R}}\phi_0\overline{\phi_0^*}\,dy, \tag{5.38}$$

and

$$\int_{-1}^{1}\psi_\varepsilon\overline{\psi_\varepsilon^*}\,dx = -\frac{[1+o(1)]\tau_\varepsilon^h\varepsilon}{[1+3\tau_\varepsilon^h\lambda_\varepsilon(\tau_\varepsilon^h)]^2}\int_{\mathbb{R}}w^2\overline{\phi_0^*}\int_{\mathbb{R}}w\phi_0, \tag{5.39}$$

so that in view of (3.23) and Remark 3.4, we obtain

$$\mu_\varepsilon'(\tau_\varepsilon^h) = \frac{[1+o(1)]\lambda_0(\tau_h)\int_{\mathbb{R}}\phi_0\overline{\phi_0^*}}{\int_{\mathbb{R}}\phi_0\overline{\phi_0^*} - \frac{\tau_\varepsilon^h}{[1+3\tau_\varepsilon^h\lambda_\varepsilon(\tau_\varepsilon^h)]^2}\int_{\mathbb{R}}w\phi_0\int_{\mathbb{R}}w^2\overline{\phi_0^*}} = [1+o(1)]\hat{\mu}_0'(\hat{\tau}_h). \tag{5.40}$$

As a consequence of Lemma 3.3 we therefore have

$$Re(\lambda_\varepsilon'(\tau_\varepsilon^h)) = \frac{1}{\tau_\varepsilon^h}Re(\mu_\varepsilon'(\tau_\varepsilon^h)) > 0, \tag{5.41}$$

for sufficiently small $\varepsilon > 0$.

6. Hopf bifurcation: existence, uniqueness and symmetry

We have now established all the assumptions of the Hopf bifurcation theorem of [15]. Indeed, the relevant spectral and semigroup assumptions on the linearization $D_\Phi\mathcal{F}_\varepsilon = \mathcal{L}_\varepsilon$ at $\tau = \tau_\varepsilon^h$ were established in Sections 4 and 5. Furthermore, with $X = H_N^2([0,1])\times H_N^2([0,1])$ and $Z = L^2([0,1])\times L^2([0,1]$, the map $\mathcal{F}_\varepsilon : X\to Z$ satisfies the required regularity assumptions. We introduce the spaces

$$C_{2\pi\rho}^{\gamma}(\mathbb{R},X) := \Big\{\Phi:\mathbb{R}\to X\,\Big|\,\Phi(t+2\pi\rho)=\Phi(t)\quad t\in\mathbb{R}, \quad ||\Phi||_{X,\gamma} := \max_{t\in\mathbb{R}}||\Phi(t)||_X + \sup_{s\neq t}\frac{||\Phi(t)-\Phi(s)||_X}{|t-s|^\gamma} < \infty\Big\}, \tag{6.1}$$

and

$$C_{2\pi\rho}^{1+\gamma}(\mathbb{R},Z) := \Big\{\Phi:\mathbb{R}\to Z\,\Big|\,\Phi\in C_{2\pi\rho}^{\gamma}(\mathbb{R},Z),\ \tfrac{d\Phi}{dt}\in C_{2\pi\rho}^{\gamma}(\mathbb{R},Z), \quad ||\Phi||_{Z,1+\gamma} := ||\Phi||_{Z,\gamma} + ||\tfrac{d\Phi}{dt}||_{Z,\gamma} < \infty\Big\}, \tag{6.2}$$

where $\gamma\in(0,1]$ is the Hölder exponent. The relevant space for solutions to (1.3) is $Y \equiv C_{2\pi\rho}^{\gamma}(\mathbb{R},X)\cap C_{2\pi\rho}^{1+\gamma}(\mathbb{R},Z)$ with the norm

$$||\Phi||_Y \equiv ||\Phi||_{X,\gamma} + ||\tfrac{d\Phi}{dt}||_{Z,\gamma}. \tag{6.3}$$

The Hopf bifurcation theorem thus applies and yields the following result.

Theorem 6.1. *There exists an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ there are numbers $\delta_\varepsilon, \eta_\varepsilon > 0$ and continuously differentiable functions $\rho_\varepsilon(s)$, $\tau_\varepsilon(s)$, and $(u_\varepsilon(s), v_\varepsilon(s))\in Y$ defined in $-\eta_\varepsilon < s < \eta_\varepsilon$ such that $(u_\varepsilon(s), v_\varepsilon(s))$ is a $2\pi\rho_\varepsilon(s)$-periodic solution to (1.3) and*

$$\tau_\varepsilon(0) = \tau_\varepsilon^h,\quad \rho_\varepsilon(0) = 1/\alpha_I^\varepsilon,\quad u_\varepsilon(0) = u_\varepsilon^S,\quad v_\varepsilon(0) = v_\varepsilon^S.$$

In addition the solutions are nontrivial in that $(u_\varepsilon(s), v_\varepsilon(s)) \neq (u_\varepsilon^S, v_\varepsilon^S)$ for $0 < |s| < \eta_\varepsilon$. Furthermore we have uniqueness in the sense that if $(\tau_{\varepsilon_1}, u_{\varepsilon,1}, v_{\varepsilon,1})$ is a $2\pi\rho_{\varepsilon,1}$-periodic solution of (1.3) with $|\rho_{\varepsilon,1} - 1/\alpha_I^\varepsilon| < \delta_\varepsilon$, $|\tau_{\varepsilon,1} - \tau_\varepsilon^h| < \delta_\varepsilon$, and $||(u_{\varepsilon,1}, v_{\varepsilon,1}) - (u_\varepsilon, v_\varepsilon)||_Y < \delta_\varepsilon$, then there exist *numbers* $s \in [0, \eta_\varepsilon)$ *and* $\theta \in [0, 2\pi\rho_{\varepsilon,1})$ *so that* $\tau_{\varepsilon,1} = \tau_\varepsilon(s)$ *and the solution* $(u_{\varepsilon,1}, v_{\varepsilon,1})$ *is obtained from a* θ*-phase shift of* $(u_\varepsilon(s), v_\varepsilon(s))$*, i.e.*

$$(u_{\varepsilon,1}, v_{\varepsilon,1})(t) = [S_\theta(u_\varepsilon(s), v_\varepsilon(s))](t) \equiv (u_\varepsilon(s), v_\varepsilon(s))(t+\theta) \qquad \text{for all} \quad t \in \mathbb{R}.$$

Finally, the bifurcating solutions have the following symmetry property

$$(u_\varepsilon(-s), v_\varepsilon(-s)) = S_{\pi\rho_\varepsilon(s)}(u_\varepsilon(s), v_\varepsilon(s)), \quad \tau_\varepsilon(-s) = \tau_\varepsilon(s), \quad \rho_\varepsilon(-s) = \rho_\varepsilon(s),$$

for all $-\eta_\varepsilon < s < \eta_\varepsilon$*.*

7. Linear stability of the Hopf bifurcations

In this section we investigate the linear stability of the periodic solutions obtained in Theorem 6.1 from the previous section. This stability analysis is carried out in the context of a generalization of Floquet Theory from ODEs to semilinear parabolic PDEs and we refer here to Section I.12 of [15]. We briefly summarize the main aspects of this theory so that our stability result may be accurately stated.

Suppose $A(t)$ is a time-dependent linear operator which is p-periodic in t and consider the problem

$$\frac{dw}{dt} - A(t)w = 0. \tag{7.1}$$

The Floquet multipliers of (7.1) are the eigenvalues of $\Phi(p)$, where $w(t) = \Phi(t)x$ is the solution of (7.1) satisfying $w(0) = x$. We say that κ is a Floquet exponent of (7.1) if and only if $e^{-p\kappa}$ is a Floquet multiplier, or equivalently if κ is an eigenvalue of $\frac{d}{dt} - A(t)$ in the space of p-periodic functions.

The concepts of Floquet Theory arise in the study of periodic solutions as follows. If u is a p-periodic solution of the nonlinear problem

$$\frac{du}{dt} = g(u), \tag{7.2}$$

then the linearization about this periodic solution results in the variational equation

$$\tfrac{dv}{dt} - g_u(u(t))v = 0, \tag{7.3}$$

from which the Floquet multipliers and exponents are defined as for (7.1) with $A(t) = g_u(u(t))$. If $\dot{u} = \frac{du}{dt} \not\equiv 0$, formally differentiating (7.2) shows that

$$\frac{d\dot{u}}{dt} = g_u(u(t))\dot{u},$$

so that 0 is always a Floquet exponent and 1 is a Floquet multiplier for u. It has been shown that the stability properties of a periodic solution to (7.2) are determined by the moduli of its Floquet multipliers (see Section 8.2 of [12]). Specifically, if the Floquet exponent $\kappa = 0$ is simple and all remaining Floquet exponents have positive real parts, then the p-periodic solution u is linearly stable.

The Floquet exponent for the $2\pi\rho_\varepsilon(s)$-periodic solutions $\Phi_\varepsilon(s) = (u_\varepsilon(s) - u_\varepsilon^S, v_\varepsilon(s) - v_\varepsilon^S)$ from Theorem 6.1 are therefore numbers κ such that the problem

$$\frac{1}{\rho_\varepsilon(s)}\frac{dw}{dt} - \big(\mathcal{L}_\varepsilon + R_\Phi(\tau_\varepsilon(s), \Phi_\varepsilon(s)(\rho_\varepsilon(s)t))\big)w = \kappa w, \qquad w(0) = w(2\pi) \tag{7.4}$$

has a nontrivial solution. At $s = 0$, (7.4) becomes

$$\alpha_I^\varepsilon \frac{dw}{dt} - \mathcal{L}_\varepsilon w = \kappa w, \qquad w(0) = w(2\pi). \tag{7.5}$$

The set of values of κ for which (7.5) has a nontrivial solution is $\{\alpha_I^\varepsilon ni - \sigma(\mathcal{L}_\varepsilon) : n = \pm1, \pm2, , \ldots\}$, so the corresponding multipliers are $e^{2\pi\sigma(\mathcal{L}_\varepsilon)/\alpha_I^\varepsilon}$. Thus, 1 is clearly a Floquet multiplier with multiplicity two corresponding to the double eigenvalue $\kappa = 0$ inherited from $\pm i\alpha_I \in \sigma(\mathcal{L}_\varepsilon)$. On the other hand, Lemma 4.1 implies that the remaining eigenvalues of $\mathcal{L}_\varepsilon$ at $s = 0$ have negative real part and therefore the remaining Floquet exponents have positive real parts.

Since a zero Floquet exponent persists for all values of $-\eta_\varepsilon < s < \eta_\varepsilon$, it is a second, nontrivial, Floquet exponent, $\kappa_\varepsilon(s)$, with $\kappa_\varepsilon(0) = 0$ which determines the linear stability of the periodic solution. Specifically, if $Re(\kappa_\varepsilon(s)) > 0$ then the periodic solution is linearly stable in the sense of [12], and is otherwise unstable. With $\cdot$ denoting a derivative with respect to s, Theorem I.12.2 of [15] implies that $\dot\kappa_\varepsilon(0) = 0$ and $\dot\tau_\varepsilon(0) = 0$. Moreover, formula (I.12.34) of [15] relates the second derivatives according to

$$\ddot\kappa_\varepsilon(0) = 2\ddot\tau_\varepsilon(0)Re(\lambda_\varepsilon'(\tau_\varepsilon^h)).$$

From Section 5 we know $Re(\lambda_\varepsilon'(\tau_\varepsilon^h)) > 0$ and therefore the first part of Corollary I.12.3, or the *Principle of Exchange of Stability*, of [15] applies.

Theorem 7.1. *Let the hypotheses of Theorem 6.1 be satisfied. Then*

$$sgn(\tau_\varepsilon(s) - \tau_\varepsilon^h) = sgn(\kappa_\varepsilon(s)) \quad for \quad -\eta_\varepsilon < s < \eta_\varepsilon.$$

Hence, the bifurcating periodic solutions of Theorem 6.1 are linearly stable (resp. unstable) if the bifurcation is supercritical (resp. subcritical).

To conclude the stability question it remains therefore to determine the sign of $\ddot\tau_\varepsilon(0)$. For this we use the formula (see equation (I.9.11) of [15])

$$\ddot\kappa_\varepsilon(0) = \frac{1}{Re(\lambda_\varepsilon'(\tau_\varepsilon))}Re(K(\varepsilon)). \tag{7.6}$$

7.1. **The Gray-Scott system.** Recall that for the stability of the Hopf bifurcation, we need to compute the sign of

$$\ddot\kappa_\varepsilon(0) = \frac{1}{Re(\lambda_\varepsilon'(\tau_\varepsilon))}Re(K(\varepsilon)). \tag{7.7}$$

Here $Re(\lambda_\varepsilon'(\tau_\varepsilon)) \sim Re(\lambda'(\tau_h)) > 0$.

$$\begin{aligned} K(\varepsilon) &= -\langle \partial_{\Phi\Phi\Phi}R_\varepsilon(\tau_\varepsilon, 0)[\Phi_\varepsilon, \Phi_\varepsilon, \overline{\Phi_\varepsilon}], \Phi_\varepsilon^*\rangle \\ &\quad + \langle \partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon, 0)[\overline{\Phi_\varepsilon}, (\mathcal{L}_\varepsilon(\tau_\varepsilon) - 2\alpha_I^\varepsilon i)^{-1}\partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon, 0)[\Phi_\varepsilon, \Phi_\varepsilon]], \Phi_\varepsilon^*\rangle \\ &\quad + 2\langle \partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon, 0)[\Phi_\varepsilon, (\mathcal{L}_\varepsilon(\tau_\varepsilon))^{-1}\partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon, 0)[\Phi_\varepsilon, \overline{\Phi_\varepsilon}]], \Phi_\varepsilon^*\rangle \\ &= K_1(\varepsilon) + K_2(\varepsilon) + K_3(\varepsilon), \end{aligned} \tag{7.8}$$

where $\Phi_\varepsilon = (\phi_\varepsilon, \psi_\varepsilon)$ is a nontrivial eigefunction of $\mathcal{L}_\varepsilon(\tau_\varepsilon)$ corresponding to the eigenvalue $\alpha_I^\varepsilon i$, and $\Phi_\varepsilon^* = (\phi_\varepsilon^*, \psi_\varepsilon^*)$ is a nontrivial eigenfunction of $\mathcal{L}_\varepsilon^*(\tau_\varepsilon)$ corresponding to the eigenvalue $-\alpha_I^\varepsilon i$, moreover,

$$\langle \Phi_\varepsilon, \Phi_\varepsilon^*\rangle = 1. \tag{7.9}$$

We have

$$\begin{aligned} \langle \Phi_\varepsilon, \Phi_\varepsilon^*\rangle &= \int_{-1}^1 \phi_\varepsilon\overline{\phi_\varepsilon^*}\,dx + \int_{-1}^1 \psi_\varepsilon\overline{\psi_\varepsilon^*}\,dx \\ &= \varepsilon[1 + o_\varepsilon(1)]\left[\int_{\mathbb{R}} \phi_0\overline{\phi_0^*} - \frac{A^2v_0^2\tau_h}{[3 + A^2v_0^2(1 + \tau_h\alpha_I i)]^2}\int_{\mathbb{R}} w\phi_0 \int_{\mathbb{R}} w^2\overline{\phi_0^*}\right]. \end{aligned} \tag{7.10}$$

Therefore, we choose constants $c_1(\varepsilon), c_2(\varepsilon)$, so that when

$$\phi_0 = c_1(\varepsilon)(L_0 - \alpha_I i)^{-1}w^2, \qquad \phi_0^* = c_2(\varepsilon)(L_0 + \alpha_I i)^{-1}w, \tag{7.11}$$

we have

$$\int_{\mathbb{R}} \phi_0\overline{\phi_0^*} - \frac{A^2v_0^2\tau_h}{[3 + A^2v_0^2(1 + \tau_h\alpha_I i)]^2}\int_{\mathbb{R}} w\phi_0 \int_{\mathbb{R}} w^2\overline{\phi_0^*} = 1/\varepsilon. \tag{7.12}$$

With ϕ_0 and ϕ_0^* so chosen, we put

$$R_\varepsilon(\tau_\varepsilon,\Phi)=\begin{pmatrix}R_{1\varepsilon}(\tau_\varepsilon,\Phi)\\ R_{2\varepsilon}(\tau_\varepsilon,\Phi)\end{pmatrix}=\begin{pmatrix}Av_\varepsilon^S\phi^2+2Au_\varepsilon^S\phi\psi+A\phi^2\psi\\ -\tau_\varepsilon^{-1}\varepsilon^{-1}[v_\varepsilon^S\phi^2+2u_\varepsilon^S\phi\psi+\phi^2\psi]\end{pmatrix}. \tag{7.13}$$

For functions

$$k=\begin{pmatrix}k_1\\ k_2\end{pmatrix},\quad h=\begin{pmatrix}h_1\\ h_2\end{pmatrix},\quad l=\begin{pmatrix}l_1\\ l_2\end{pmatrix}\qquad\in Z,$$

we have

$$\partial_{\Phi\Phi}R_{1\varepsilon}(\tau_\varepsilon,0)[k,h]=2Av_\varepsilon^S k_1h_1+2Au_\varepsilon^S(k_1h_2+k_2h_1),$$

$$\partial_{\Phi\Phi}R_{2\varepsilon}(\tau_\varepsilon,0)[k,h]=-2\tau_\varepsilon^{-1}[\varepsilon^{-1}v_\varepsilon^S k_1h_1+\varepsilon^{-1}u_\varepsilon^S(k_1h_2+k_2h_1)],$$

$$\partial_{\Phi\Phi\Phi}R_{1\varepsilon}(\tau_\varepsilon,0)[k,h,l]=2A(k_1h_1l_2+k_1h_2l_1+k_2h_1l_1),$$

$$\partial_{\Phi\Phi\Phi}R_{2\varepsilon}(\tau_\varepsilon,0)[k,h,l]=-2\tau_\varepsilon^{-1}\varepsilon^{-1}(k_1h_1l_2+k_1h_2l_1+k_2h_1l_1).$$

Therefore,

$$\begin{aligned}\varepsilon^{-1}K_1(\varepsilon)&=-\varepsilon^{-1}\langle\partial_{\Phi\Phi\Phi}R_\varepsilon(\tau_\varepsilon,0)[\Phi_\varepsilon,\Phi_\varepsilon,\overline{\Phi_\varepsilon}],\Phi_\varepsilon^*\rangle\\ &=2\tau_\varepsilon^{-1}\varepsilon^{-2}\int_{-1}^{1}(\phi_\varepsilon^2\overline{\psi_\varepsilon}+2|\phi_\varepsilon|^2\psi_\varepsilon)\overline{\psi_\varepsilon^*}\,dx-2A\varepsilon^{-1}\int_{-1}^{1}(\phi_\varepsilon^2\overline{\psi_\varepsilon}+2|\phi_\varepsilon|^2\psi_\varepsilon)\overline{\phi_\varepsilon^*}\,dx\\ &=\frac{1}{3+A^2v_0^2(1+\tau_h\alpha_Ii)}\int_{-\infty}^{\infty}w^2\overline{\phi_0^*}\,dy\int_{-\infty}^{\infty}(\phi_0^2\overline{\psi_0}+2|\phi_0|^2\psi_0)\,dy\\ &\quad-2A\int_{-\infty}^{\infty}(\phi_0^2\overline{\psi_0}+2|\phi_0|^2\psi_0)\overline{\phi_0^*}\,dy+o_\varepsilon(1),\end{aligned}\tag{7.14}$$

where

$$\psi_0=-\frac{Av_0^2}{3+A^2v_0^2(1+\tau_h\alpha_Ii)}\int_{-\infty}^{\infty}w\phi_0\,dy.$$

Next we evaluate $K_2(\varepsilon)$. Define

$$\begin{pmatrix}z_1^\varepsilon\\ z_2^\varepsilon\end{pmatrix}=(\mathcal{L}_\varepsilon(\tau_\varepsilon)-2\alpha_I^\varepsilon i)^{-1}\partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon^0,0)[\Phi_\varepsilon,\Phi_\varepsilon]=(\mathcal{L}_\varepsilon(\tau_\varepsilon)-2\alpha_I^\varepsilon i)^{-1}\begin{pmatrix}2A[v_\varepsilon^S\phi_\varepsilon^2+2u_\varepsilon^S\phi_\varepsilon\psi_\varepsilon]\\ -2\tau_\varepsilon^{-1}\varepsilon^{-1}[v_\varepsilon^S\phi_\varepsilon^2+2u_\varepsilon^S\phi_\varepsilon\psi_\varepsilon]\end{pmatrix},$$

namely,

$$\begin{cases}\varepsilon^2(z_1^\varepsilon)_{xx}+(2Au_\varepsilon^Sv_\varepsilon^S-1-2\alpha_I^\varepsilon i)z_1^\varepsilon+A(u_\varepsilon^S)^2z_2^\varepsilon=2Av_\varepsilon^S\phi_\varepsilon^2+4Au_\varepsilon^S\phi_\varepsilon\psi_\varepsilon\\ D_\varepsilon(z_2^\varepsilon)_{xx}-(\varepsilon^{-1}(u_\varepsilon^S)^2+1+2\tau_\varepsilon\alpha_I^\varepsilon i)z_2^\varepsilon-2\varepsilon^{-1}u_\varepsilon^Sv_\varepsilon^Sz_1^\varepsilon=-2\varepsilon^{-1}(v_\varepsilon^S\phi_\varepsilon^2+2u_\varepsilon^S\phi_\varepsilon\psi_\varepsilon),\\ (z_1^\varepsilon)_x(\pm1)=(z_2^\varepsilon)_x(\pm1)=0.\end{cases}\tag{7.15}$$

By the discussions in precious sections, we can derive a limit equation

$$\begin{cases}(z_1)_{yy}-(1+2\alpha_Ii)z_1+2wz_1+\dfrac{1}{Av_0^2}w^2z_2=2Av_0\phi_0^2+\dfrac{4w}{v_0}\phi_0\psi_0,\\ z_2=\dfrac{1}{3+A^2v_0^2(1+2\tau_h\alpha_Ii)}\left[A^2v_0^3\displaystyle\int_{-\infty}^{\infty}\phi_0^2\,dy+2Av_0\psi_0\int_{-\infty}^{\infty}w\phi_0\,dy-Av_0^2\int_{-\infty}^{\infty}wz_1\,dy\right].\end{cases}\tag{7.16}$$

We have

$$
\begin{aligned}
\varepsilon^{-1}K_2(\varepsilon) &= \varepsilon^{-1}\langle \partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon,0)[\overline{\Phi_\varepsilon},(\mathcal{L}_\varepsilon(\tau_\varepsilon)-2\alpha_I^\varepsilon i)^{-1}\partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon,0)[\Phi_\varepsilon,\Phi_\varepsilon]],\Phi_\varepsilon^*\rangle \\
&= 2A\varepsilon^{-1}\int_{-1}^{1}\left[v_\varepsilon^S\overline{\phi_\varepsilon}z_1^\varepsilon + u_\varepsilon^S(\overline{\phi_\varepsilon}z_2^\varepsilon+\overline{\psi_\varepsilon}z_1^\varepsilon)\right]\overline{\phi_\varepsilon^*}\,dx \\
&\quad - 2\tau_\varepsilon^{-1}\varepsilon^{-2}\int_{-1}^{1}\left[v_\varepsilon^S\overline{\phi_\varepsilon}z_1^\varepsilon + u_\varepsilon^S(\overline{\phi_\varepsilon}z_2^\varepsilon+\overline{\psi_\varepsilon}z_1^\varepsilon)\right]\overline{\psi_\varepsilon^*}\,dx \\
&= 2A\int_{-\infty}^{\infty}[v_0\overline{\phi_0}z_1+(Av_0)^{-1}w(\overline{\phi_0}z_2+\overline{\psi_0}z_1)]\overline{\phi_0^*}\,dy \\
&\quad - \frac{A\int_{-\infty}^{\infty}w^2\overline{\phi_0^*}\,dy}{3+A^2v_0^2(1+\tau_h\alpha_I i)}\int_{-\infty}^{\infty}[v_0\overline{\phi_0}z_1+(Av_0)^{-1}w(\overline{\phi_0}z_2+\overline{\psi_0}z_1)]\,dy \\
&\quad + o_\varepsilon(1).
\end{aligned}
\tag{7.17}
$$

To evaluate $K_3(\varepsilon)$ we define

$$
\begin{pmatrix} h_1^\varepsilon \\ h_2^\varepsilon \end{pmatrix} = (\mathcal{L}_\varepsilon(\tau_\varepsilon))^{-1}\partial_{\Phi\Phi}R(\tau_\varepsilon^0,0)[\Phi_\varepsilon,\overline{\Phi_\varepsilon}] = (\mathcal{L}_\varepsilon(\tau_\varepsilon))^{-1}\begin{pmatrix} 2A[v_\varepsilon^S|\phi_\varepsilon|^2+u_\varepsilon^S(\phi_\varepsilon\overline{\psi_\varepsilon}+\overline{\phi_\varepsilon}\psi_\varepsilon)] \\ -2\tau_\varepsilon^{-1}\varepsilon^{-1}[v_\varepsilon^S|\phi_\varepsilon|^2+u_\varepsilon^S(\phi_\varepsilon\overline{\psi_\varepsilon}+\overline{\phi_\varepsilon}\psi_\varepsilon)]\end{pmatrix},
$$

namely,

$$
\begin{cases}
\varepsilon^2(h_1^\varepsilon)_{xx}+(2Au_\varepsilon^Sv_\varepsilon^S-1)h_1^\varepsilon+A(u_\varepsilon^S)^2h_2^\varepsilon = 2A[v_\varepsilon^S|\phi_\varepsilon|^2+u_\varepsilon^S(\phi_\varepsilon\overline{\psi_\varepsilon}+\overline{\phi_\varepsilon}\psi_\varepsilon)], \\
D_\varepsilon(h_2^\varepsilon)_{xx}-(\varepsilon^{-1}(u_\varepsilon^S)^2+1)h_2^\varepsilon-2\varepsilon^{-1}u_\varepsilon^Sv_\varepsilon^Sh_1^\varepsilon = -2\varepsilon^{-1}[v_\varepsilon^S|\phi_\varepsilon|^2+u_\varepsilon^S(\phi_\varepsilon\overline{\psi_\varepsilon}+\overline{\phi_\varepsilon}\psi_\varepsilon)], \\
(h_1^\varepsilon)_x(\pm1)=(h_2^\varepsilon)_x(\pm1)=0.
\end{cases}
\tag{7.18}
$$

As before we derive the limit equation

$$
\begin{cases}
(h_1)_{yy}-h_1+2wh_1+\dfrac{1}{Av_0^2}w^2h_2 = 2Av_0|\phi_0|^2+\dfrac{2w}{v_0}(\phi_0\overline{\psi_0}+\psi_0\overline{\phi_0}), \\
h_2 = \dfrac{1}{3+A^2v_0^2}\left[A^2v_0^3\displaystyle\int_{-\infty}^{\infty}|\phi_0|^2\,dy+Av_0\left(\overline{\psi_0}\int_{-\infty}^{\infty}w\phi_0\,dy+\psi_0\int_{-\infty}^{\infty}w\overline{\phi_0}\,dy\right)-Av_0^2\int_{-\infty}^{\infty}wh_1\,dy\right].
\end{cases}
\tag{7.19}
$$

Then we have

$$
\begin{aligned}
\varepsilon^{-1}K_3(\varepsilon) &= 2\varepsilon^{-1}\langle \partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon,0)[\Phi_\varepsilon,(\mathcal{L}_\varepsilon(\tau_\varepsilon))^{-1}\partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon,0)[\Phi_\varepsilon,\overline{\Phi}_\varepsilon]],\Phi_\varepsilon^*\rangle \\
&= 4A\varepsilon^{-1}\int_{-1}^{1}\left[v_\varepsilon^S\phi_\varepsilon h_1^\varepsilon+u_\varepsilon^S(\phi_\varepsilon h_2^\varepsilon+\psi_\varepsilon h_1^\varepsilon)\right]\overline{\phi_\varepsilon^*}\,dx \\
&\quad - 4\tau_\varepsilon^{-1}\varepsilon^{-2}\int_{-1}^{1}\left[v_\varepsilon^S\phi_\varepsilon h_1^\varepsilon+u_\varepsilon^S(\phi_\varepsilon h_2^\varepsilon+\psi_\varepsilon h_1^\varepsilon)\right]\overline{\psi_\varepsilon^*}\,dx \\
&= 4A\int_{-\infty}^{\infty}[v_0\phi_0h_1+(Av_0)^{-1}w(\phi_0h_2+\psi_0h_1)]\phi_0^*\,dy \\
&\quad - \frac{2A\int_{-\infty}^{\infty}w^2\overline{\phi_0^*}\,dy}{3+A^2v_0^2(1+\tau_h\alpha_I i)}\int_{-\infty}^{\infty}[v_0\phi_0h_1+(Av_0)^{-1}w(\phi_0h_2+\psi_0h_1)]\,dy \\
&\quad + o_\varepsilon(1).
\end{aligned}
\tag{7.20}
$$

7.2. **The Schnakenberg system.** For the stability of the Hopf bifurcation, we need to compute the sign of

$$
\ddot{\kappa}_\varepsilon(0) = \frac{1}{Re(\lambda_\varepsilon'(\tau_\varepsilon^h))}Re(K(\varepsilon)).
\tag{7.21}
$$

Here $Re(\lambda'_\varepsilon(\tau_\varepsilon^h)) \sim Re(\lambda'(\tau_h)) > 0$.

$$
\begin{aligned}
K(\varepsilon) &= -\langle \partial_{\Phi\Phi\Phi} R_\varepsilon(\tau_\varepsilon, 0)[\Phi_\varepsilon, \Phi_\varepsilon, \overline{\Phi_\varepsilon}], \Phi_\varepsilon^* \rangle \\
&\quad + \langle \partial_{\Phi\Phi} R_\varepsilon(\tau_\varepsilon, 0)[\overline{\Phi_\varepsilon}, (\mathcal{L}_\varepsilon(\tau_\varepsilon) - 2\alpha_I^\varepsilon i)^{-1} \partial_{\Phi\Phi} R_\varepsilon(\tau_\varepsilon, 0)[\Phi_\varepsilon, \Phi_\varepsilon]], \Phi_\varepsilon^* \rangle \\
&\quad + 2\langle \partial_{\Phi\Phi} R_\varepsilon(\tau_\varepsilon, 0)[\Phi_\varepsilon, (\mathcal{L}_\varepsilon(\tau_\varepsilon))^{-1} \partial_{\Phi\Phi} R_\varepsilon(\tau_\varepsilon, 0)[\Phi_\varepsilon, \overline{\Phi}_\varepsilon]], \Phi_\varepsilon^* \rangle \\
&= K_1(\varepsilon) + K_2(\varepsilon) + K_3(\varepsilon),
\end{aligned} \tag{7.22}
$$

where $\Phi_\varepsilon = (\phi_\varepsilon, \psi_\varepsilon)$ is a nontrivial eigefunction of $\mathcal{L}_\varepsilon(\tau_\varepsilon)$ corresponding to the eigenvalue $\alpha_I^\varepsilon i$, and $\Phi_\varepsilon^* = (\phi_\varepsilon^*, \psi_\varepsilon^*)$ is a nontrivial eigenfunction of $\mathcal{L}_\varepsilon^*(\tau_\varepsilon)$ corresponding to the eigenvalue $-\alpha_I^\varepsilon i$,

$$
R_\varepsilon(\tau_\varepsilon, \Phi) = \begin{pmatrix} v_\varepsilon^S \phi^2 + 2u_\varepsilon^S \phi\psi + \phi^2\psi \\ -\tau_\varepsilon^{-1}\varepsilon^{-1}[v_\varepsilon^S \phi^2 + 2u_\varepsilon^S \phi\psi + \phi^2\psi] \end{pmatrix}, \tag{7.23}
$$

moreover,

$$
\langle \Phi_\varepsilon, \Phi_\varepsilon^* \rangle = 1. \tag{7.24}
$$

We have

$$
\begin{aligned}
\langle \Phi_\varepsilon, \Phi_\varepsilon^* \rangle &= \int_{-1}^1 \phi_\varepsilon \overline{\phi_\varepsilon^*}\, dx + \int_{-1}^1 \psi_\varepsilon \overline{\psi_\varepsilon^*}\, dx \\
&= \varepsilon[1 + o_\varepsilon(1)] \left[\int_{\mathbb{R}} \phi_0 \overline{\phi_0^*} - \frac{\tau_h}{[1 + 3\tau_h \lambda_0(\tau_h)]^2} \int_{\mathbb{R}} w\phi_0 \int_{\mathbb{R}} w^2 \overline{\phi_0^*} \right].
\end{aligned} \tag{7.25}
$$

Therefore, we choose constants $c_1(\varepsilon), c_2(\varepsilon)$, so that when

$$
\phi_0 = c_1(\varepsilon)(L_0 - \alpha_I i)^{-1} w^2, \qquad \phi_0^* = c_2(\varepsilon)(L_0 + \alpha_I i)^{-1} w, \tag{7.26}
$$

we have

$$
\int_{\mathbb{R}} \phi_0 \overline{\phi_0^*} - \frac{\tau_h}{[1 + 3\tau_h \lambda_0(\tau_h)]^2} \int_{\mathbb{R}} w\phi_0 \int_{\mathbb{R}} w^2 \overline{\phi_0^*} = 1/\varepsilon. \tag{7.27}
$$

With ϕ_0 and ϕ_0^* so chosen, we put

$$
R_\varepsilon(\tau_\varepsilon, 0) = \begin{pmatrix} R_{1\varepsilon}(\tau_\varepsilon, 0) \\ R_{2\varepsilon}(\tau_\varepsilon, 0) \end{pmatrix}. \tag{7.28}
$$

For functions

$$
k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad l = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \qquad \in Z,
$$

we have

$$
\begin{aligned}
\partial_{\Phi\Phi} R_{1\varepsilon}(\tau_\varepsilon, 0)[k, h] &= 2v_\varepsilon^S k_1 h_1 + 2u_\varepsilon^S (k_1 h_2 + k_2 h_1), \\
\partial_{\Phi\Phi} R_{2\varepsilon}(\tau_\varepsilon, 0)[k, h] &= -2\tau_\varepsilon^{-1}\varepsilon^{-1}[v_\varepsilon^S k_1 h_1 + u_\varepsilon^S (k_1 h_2 + k_2 h_1)], \\
\partial_{\Phi\Phi\Phi} R_{1\varepsilon}(\tau_\varepsilon, 0)[k, h, l] &= 2(k_1 h_1 l_2 + k_1 h_2 l_1 + k_2 h_1 l_1), \\
\partial_{\Phi\Phi\Phi} R_{2\varepsilon}(\tau_\varepsilon, 0)[k, h, l] &= -2\tau_\varepsilon^{-1}\varepsilon^{-1}(k_1 h_1 l_2 + k_1 h_2 l_1 + k_2 h_1 l_1).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\varepsilon^{-1} K_1(\varepsilon) &= -\varepsilon^{-1} \langle \partial_{\Phi\Phi\Phi} R_\varepsilon(\tau_\varepsilon, 0)[\Phi_\varepsilon, \Phi_\varepsilon, \overline{\Phi_\varepsilon}], \Phi_\varepsilon^* \rangle \\
&= 2\tau_\varepsilon^{-1}\varepsilon^{-2} \int_{-1}^1 (\phi_\varepsilon^2 \overline{\psi_\varepsilon} + 2|\phi_\varepsilon|^2 \psi_\varepsilon) \overline{\psi_\varepsilon^*}\, dx - 2\varepsilon^{-1} \int_{-1}^1 (\phi_\varepsilon^2 \overline{\psi_\varepsilon} + 2|\phi_\varepsilon|^2 \psi_\varepsilon) \overline{\phi_\varepsilon^*}\, dx \\
&= \frac{1}{3 + 9\tau_h \alpha_I i} \int_{-\infty}^{\infty} w^2 \overline{\phi_0^*}\, dy \int_{-\infty}^{\infty} (\phi_0^2 \overline{\psi_0} + 2|\phi_0|^2 \psi_0)\, dy \\
&\quad - 2 \int_{-\infty}^{\infty} (\phi_0^2 \overline{\psi_0} + 2|\phi_0|^2 \psi_0) \overline{\phi_0^*}\, dy + o_\varepsilon(1),
\end{aligned} \tag{7.29}
$$

where

$$\psi_0 = -\frac{3}{1+3\tau_h\alpha_I i}\int_{-\infty}^{\infty} w\phi_0\,dy.$$

Next we evaluate $K_2(\varepsilon)$. Define

$$\begin{pmatrix} z_1^\varepsilon \\ z_2^\varepsilon \end{pmatrix} = (\mathcal{L}_\varepsilon(\tau_\varepsilon)-2\alpha_I^\varepsilon i)^{-1}\partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon^0,0)[\Phi_\varepsilon,\Phi_\varepsilon] = (\mathcal{L}_\varepsilon(\tau_\varepsilon)-2\alpha_I^\varepsilon i)^{-1}\begin{pmatrix} 2[v_\varepsilon^S\phi_\varepsilon^2+2u_\varepsilon^S\phi_\varepsilon\psi_\varepsilon] \\ -2\tau_\varepsilon^{-1}\varepsilon^{-1}[v_\varepsilon^S\phi_\varepsilon^2+2u_\varepsilon^S\phi_\varepsilon\psi_\varepsilon]\end{pmatrix},$$

namely,

$$\begin{cases} \varepsilon^2(z_1^\varepsilon)_{xx}+(2u_\varepsilon^S v_\varepsilon^S-1-2\alpha_I^\varepsilon i)z_1^\varepsilon+(u_\varepsilon^S)^2 z_2^\varepsilon = 2v_\varepsilon^S\phi_\varepsilon^2+4u_\varepsilon^S\phi_\varepsilon\psi_\varepsilon \\ D_\varepsilon(z_2^\varepsilon)_{xx}-(\varepsilon^{-1}(u_\varepsilon^S)^2+2\tau_\varepsilon\alpha_I^\varepsilon i)z_2^\varepsilon-2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S z_1^\varepsilon = -2\varepsilon^{-1}(v_\varepsilon^S\phi_\varepsilon^2+2u_\varepsilon^S\phi_\varepsilon\psi_\varepsilon), \\ (z_1^\varepsilon)_x(\pm1)=(z_2^\varepsilon)_x(\pm1)=0. \end{cases} \tag{7.30}$$

By the discussions in precious sections, we can derive a limit equation

$$\begin{cases} (z_1)_{yy}-(1+2\alpha_I i)z_1+2wz_1+\frac{1}{9}w^2 z_2 = 6\phi_0^2+\frac{4w}{3}\phi_0\psi_0, \\ z_2 = \frac{3}{1+6\tau_h\alpha_I i}\left[3\int_{-\infty}^{\infty}\phi_0^2\,dy+\frac{2}{3}\psi_0\int_{-\infty}^{\infty} w\phi_0\,dy-\int_{-\infty}^{\infty} wz_1\,dy\right]. \end{cases} \tag{7.31}$$

We have

$$\begin{aligned} \varepsilon^{-1}K_2(\varepsilon) &= \varepsilon^{-1}\langle\partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon,0)[\overline{\Phi_\varepsilon},(\mathcal{L}_\varepsilon(\tau_\varepsilon)-2\alpha_I^\varepsilon i)^{-1}\partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon,0)[\Phi_\varepsilon,\Phi_\varepsilon]],\Phi_\varepsilon^*\rangle \\ &= 2\varepsilon^{-1}\int_{-1}^{1}\left[v_\varepsilon^S\overline{\phi_\varepsilon}z_1^\varepsilon+u_\varepsilon^S(\overline{\phi_\varepsilon}z_2^\varepsilon+\overline{\psi_\varepsilon}z_1^\varepsilon)\right]\overline{\phi_\varepsilon^*}\,dx \\ &\quad -2\tau_\varepsilon^{-1}\varepsilon^{-2}\int_{-1}^{1}\left[v_\varepsilon^S\overline{\phi_\varepsilon}z_1^\varepsilon+u_\varepsilon^S(\overline{\phi_\varepsilon}z_2^\varepsilon+\overline{\psi_\varepsilon}z_1^\varepsilon)\right]\overline{\psi_\varepsilon^*}\,dx \\ &= 2\int_{-\infty}^{\infty}\left[3\overline{\phi_0}z_1+\frac{w}{3}(\overline{\phi_0}z_2+\overline{\psi_0}z_1)\right]\overline{\phi_0^*}\,dy \\ &\quad -\frac{\int_{-\infty}^{\infty}w^2\overline{\phi_0^*}\,dy}{3+9\tau_h\alpha_I i}\int_{-\infty}^{\infty}\left[3\overline{\phi_0}z_1+\frac{w}{3}(\overline{\phi_0}z_2+\overline{\psi_0}z_1)\right]dy \\ &\quad +o_\varepsilon(1). \end{aligned} \tag{7.32}$$

To evaluate $K_3(\varepsilon)$ we define

$$\begin{pmatrix} h_1^\varepsilon \\ h_2^\varepsilon \end{pmatrix} = (\mathcal{L}_\varepsilon(\tau_\varepsilon))^{-1}\partial_{\Phi\Phi}(\tau_\varepsilon^0,0)[\Phi_\varepsilon,\overline{\Phi_\varepsilon}] = (\mathcal{L}_\varepsilon(\tau_\varepsilon))^{-1}\begin{pmatrix} 2[v_\varepsilon^S|\phi_\varepsilon|^2+u_\varepsilon^S(\phi_\varepsilon\overline{\psi_\varepsilon}+\psi_\varepsilon\overline{\phi_\varepsilon})] \\ -2\tau_\varepsilon^{-1}\varepsilon^{-1}[v_\varepsilon^S|\phi_\varepsilon|^2+u_\varepsilon^S u_\varepsilon^S(\phi_\varepsilon\overline{\psi_\varepsilon}+\psi_\varepsilon\overline{\phi_\varepsilon})]\end{pmatrix},$$

namely,

$$\begin{cases} \varepsilon^2(h_1^\varepsilon)_{xx}+(2u_\varepsilon^S v_\varepsilon^S-1)h_1^\varepsilon+(u_\varepsilon^S)^2 h_2^\varepsilon = 2[v_\varepsilon^S|\phi_\varepsilon|^2+u_\varepsilon^S(\phi_\varepsilon\overline{\psi_\varepsilon}+\psi_\varepsilon\overline{\phi_\varepsilon}) \\ D_\varepsilon(h_2^\varepsilon)_{xx}-\varepsilon^{-1}(u_\varepsilon^S)^2 h_2^\varepsilon-2\varepsilon^{-1}u_\varepsilon^S v_\varepsilon^S h_1^\varepsilon = -2\varepsilon^{-1}[v_\varepsilon^S|\phi_\varepsilon|^2+u_\varepsilon^S(\phi_\varepsilon\overline{\psi_\varepsilon}+\psi_\varepsilon\overline{\phi_\varepsilon})], \\ (h_1^\varepsilon)_x(\pm1)=(h_2^\varepsilon)_x(\pm1)=0. \end{cases} \tag{7.33}$$

As before we derive the limit equation

$$\begin{cases} (h_1)_{yy}-h_1+2wh_1+\frac{1}{9}w^2 h_2 = 6|\phi_0|^2+\frac{2w}{3}(\phi_0\overline{\psi_0}+\phi_0\overline{\psi_0}), \\ h_2 = 9\int_{-\infty}^{\infty}|\phi_0|^2\,dy+\left(\overline{\psi_0}\int_{-\infty}^{\infty}w\phi_0\,dy+\psi_0\int_{-\infty}^{\infty}w\overline{\phi_0}\,dy\right)-3\int_{-\infty}^{\infty}wh_1\,dy. \end{cases} \tag{7.34}$$

Then we have

$$
\begin{aligned}
\varepsilon^{-1}K_3(\varepsilon) &= 2\varepsilon^{-1}\langle \partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon,0)[\Phi_\varepsilon,(\mathcal{L}_\varepsilon(\tau_\varepsilon))^{-1}\partial_{\Phi\Phi}R_\varepsilon(\tau_\varepsilon,0)[\Phi_\varepsilon,\overline{\Phi}_\varepsilon]],\Phi_\varepsilon^*\rangle \\
&= 4\varepsilon^{-1}\int_{-1}^{1}\left[v_\varepsilon^S\phi_\varepsilon h_1^\varepsilon + u_\varepsilon^S(\phi_\varepsilon h_2^\varepsilon+\psi_\varepsilon h_1^\varepsilon)\right]\overline{\phi_\varepsilon^*}\,dx \\
&\quad - 4\tau_\varepsilon^{-1}\varepsilon^{-2}\int_{-1}^{1}\left[v_\varepsilon^S\phi_\varepsilon h_1^\varepsilon + u_\varepsilon^S(\phi_\varepsilon h_2^\varepsilon+\psi_\varepsilon h_1^\varepsilon)\right]\overline{\psi_\varepsilon^*}\,dx \\
&= 4\int_{-\infty}^{\infty}\left[3\phi_0h_1+\frac{w}{3}(\phi_0h_2+\psi_0h_1)\right]\overline{\phi_0^*}\,dy \\
&\quad - \frac{2\int_{-\infty}^{\infty}w^2\overline{\phi_0^*}\,dy}{3+9\tau_h\alpha_I i}\int_{-\infty}^{\infty}\left[3\phi_0h_1+\frac{w}{3}(\phi_0h_2+\psi_0h_1)\right]dy \\
&\quad + o_\varepsilon(1).
\end{aligned}
\tag{7.35}
$$

8. Numerical Computation of $Re(K)$

It remains only to calculate the sign of Re K for the Gray-Scott and Schnakenberg systems. Since the calculations are nearly identical we will state results for both systems, but only give details of the calculation for the Gray-Scott system.

8.1. **Gray-Scott Model.**

First we have to compute the Hopf bifurcation threshold τ_h and corresponding purely imaginary eigenvalue $\lambda_0 = i\alpha_I$ along with the corresponding eigenfunction ϕ_0. We can reduce the NLEP (2.9) to an algebraic equation by writing $\phi_0 = (L_0 - i\alpha_I)^{-1}w_0$ thus obtaining

$$
\frac{1}{\chi(i\tau_h\alpha_I)} - \int_{-\infty}^{\infty} w(L_0 - i\alpha_I)^{-1}w^2dy = 0.
$$

Using the specific form of $\chi(\tau\lambda)$ we can equate real and imaginary parts to obtain

$$
3 + A^2v_0^2 - \mathrm{Re}\left[\int_{-\infty}^{\infty} w(y)(L_0-i\alpha_I)^{-1}[w(y)]^2dy\right] = 0, \tag{8.1a}
$$

$$
A^2v_0^2\tau_h\alpha_I - \mathrm{Im}\left[\int_{-\infty}^{\infty} w(y)(L_0-i\alpha_I)^{-1}[w(y)]^2dy\right] = 0. \tag{8.1b}
$$

The integral term is computed by numerically solving the boundary-value-problem $(L_0-i\alpha_I)\xi = w^2$, with $\xi'(0) = 0$ and $\xi(y) \to 0$ as $y \to \pm\infty$ on a truncated domain with appropriate boundary conditions. Specifically we used the solve_bvp routine from the scipy library on the interval $[0, 500]$. For each given value of A we can then solve (8.1a) for α_I using the brentq routine from the scipy library and use this value in (8.1b) to calculate the corresponding threshold τ_h. The values of $\alpha_I = \alpha_I(A)$ and $\tau_h = \tau_h(A)$ are plotted in Figures 1a and 1b. The corresponding eigenfunction ϕ_0 and its adjoint ϕ_0^* are calculated by setting

$$
\phi_0 = (L_0 - i\alpha_I)^{-1}w^2, \qquad \phi_0^* = \overline{\beta}(L_0 + i\alpha_I)^{-1}w,
$$

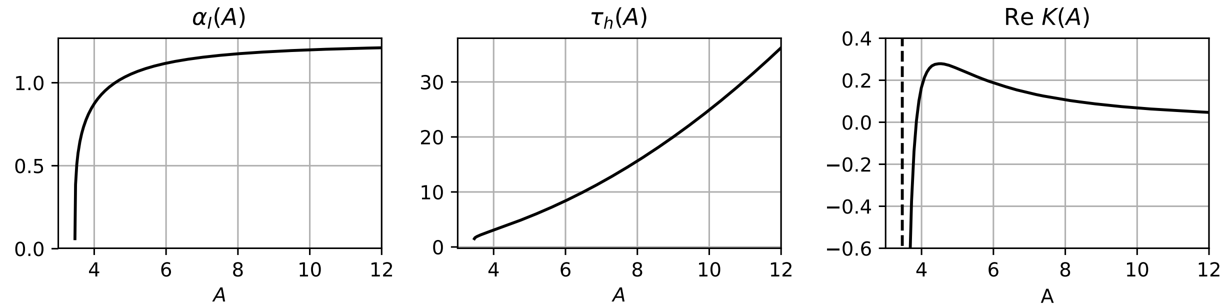
and choosing the multiplier β to satisfy the normalization constraint (7.9). It is then straightforward to calculate ψ_0 and ψ_0^* by numerical integration.

To calculate the remaining axillary functions z_1 and h_1 we observe that the solution to the non-local problem

$$
(L_0 - i\kappa)\xi(y) = f(y) + g(y)\int_{-\infty}^{\infty} h(s)\xi(s)ds,
$$

can be found by first solving the boundary value problems

$$
(L_0 - i\kappa)\xi_1 = f, \quad \text{and} \quad (L_0 - i\kappa)\xi_2 = g,
$$



(A) (B) (C)

FIGURE 1. Plots of the numerically computed values of (A) the Hopf bifurcation eigenvalue $\alpha_I(A)$, (B) the Hopf bifurcation threshold $\tau_h(A)$, and (C) of Re $K(A)$ for the Gray-Scott system.

and then setting

$$\xi(y) = \xi_1(y) + \frac{\int_{-\infty}^{\infty} h(s)\xi_1(s)ds}{1 - \int_{-\infty}^{\infty} h(s)\xi_2(s)ds}\xi_2(y)$$

With the appropriate choices of functions f, g, and h as dictated by (7.16) and (7.19) we can numerically compute the functions z_1 and h_1 for every value of A. It is then straightforward to use numerical integration to obtain the corresponding values of z_2 and h_2.

Finally we use the asymptotic expressions for K_1, K_2, and K_3 given by (7.14), (7.17), and (7.20) to obtain the leading order behaviour of $K = K(A)$. The real part of $K(A)$ is shown in Figure 1c. From this Figure we observe that a distinctive feature of the Gray-Scott system is that the theory predicts a change from a subcritical to a supercritical Hopf bifurcation as the parameter A extends beyond a value of $A \approx 3.85$. We will illustrate this theoretical prediction with full numerical simulations in a subsequent section. We further remark that the numerics indicate that Re $K(A)$ tends to $-\infty$ as A approaches $\sqrt{12}$ and otherwise tends towards zero as A tends to $+\infty$.

8.2. **The Schnakenberg Model.** We summarize here some of the key quantities calculated for the Schnakenberg system. First we find that the Hopf bifurcation threshold τ_h and corresponding eigenvalue $\lambda_0 = i\alpha_I$ are given by

$$\tau_h = 0.25702, \qquad \alpha_I = 1.2376. \tag{8.2}$$

We then proceed in a similar fashion to the Gray-Scott system to calculate

$$K_1(\varepsilon) = 17.357 - 6.8642i + o(1)$$
$$K_2(\varepsilon) = 10.928 + 16.520i + o(1)$$
$$K_3(\varepsilon) = -21.566 - 2.0317i + o(1),$$

and hence

$$K(\varepsilon) = 6.7201 + 7.6246i + o(1). \tag{8.3}$$

Therefore the Hopf bifurcation is supercritical for the Schnakenberg system.

9. Numerical Simulations

To illustrate the theoretical predictions of the preceding sections we here perform full numerical simulations of both the Gray-Scott and the Schnakenberg systems. In both systems we will fix the parameter values $\varepsilon = 0.002$ and $D = 100,000$. Additionally we discretize the interval

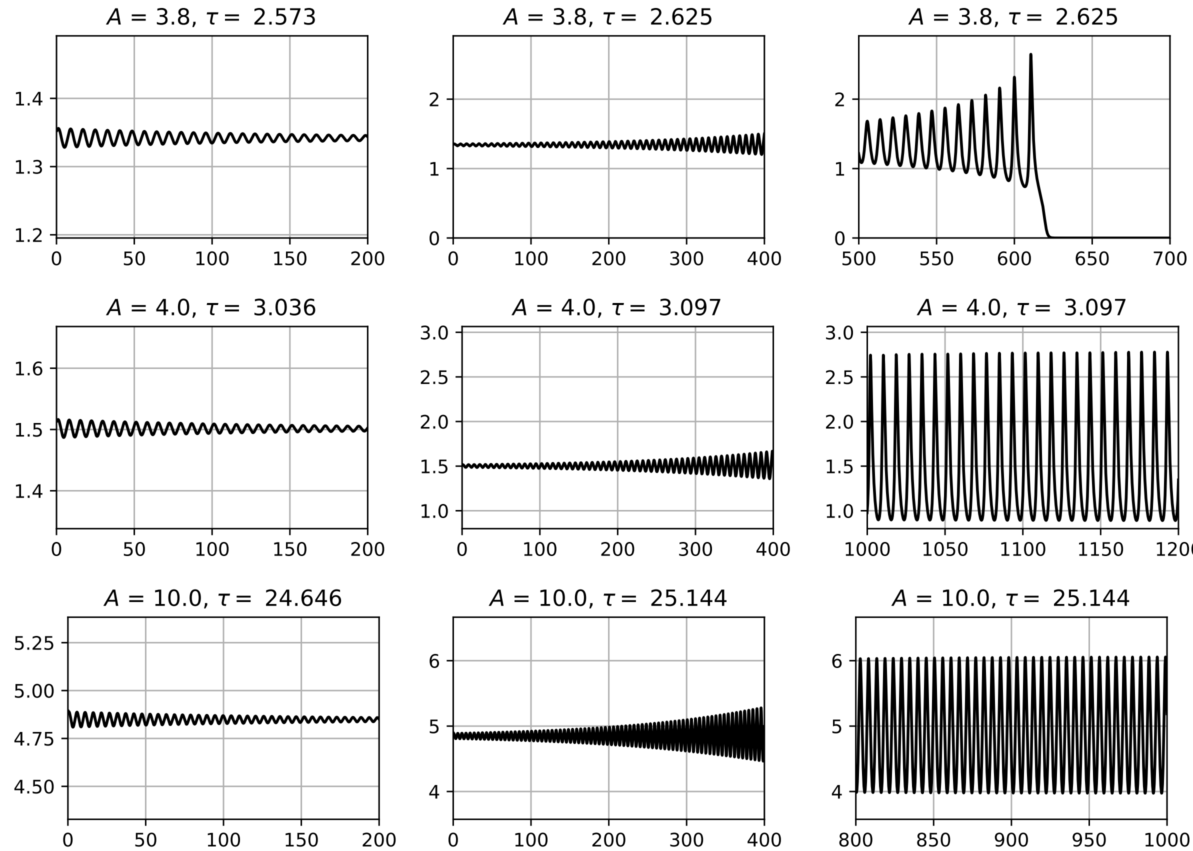


FIGURE 2. Numerical simulations for Gray-Scott system with $D = 10^5$ and $\varepsilon = 0.002$. Each row of plots corresponds to a distinct value of A. The leftmost column uses a value of $\tau = 0.99\tau_h(A)$ while the remaining two columns use $\tau = 1.01\tau_h$. In the last two columns we have shown the behaviour of the solution for the first 400 seconds and then jumped ahead to a time range where the resulting subcritical or supercritical characteristic of the Hopf bifurcation is best represented.

$[0, 1]$ into $5,000$ equally spaced mesh points and use a finite-difference approximation for the second order spatial derivatives. Furthermore we use a second-oder semi-implicit backwards difference (2-SBDF) implicit-explicit (IMEX) time-stepping scheme with a time-step size of 0.001 to solve the resulting large system of nonlinear ODEs (see [24] for details).

We focus first on the Gray-Scott system. To illustrate the change in criticality of the Hopf bifurcation we choose parameter values of A shown, along with the corresponding values of τ_h and Re $K(A)$, in Table 1. Starting with an initial condition given by a 1% perturbation away from the asymptotically calculated equilibrium (2.2) and (2.3) we then numerically compute the solution for values of $\tau = 0.99\tau_h(A)$ and $\tau = 1.01\tau_h(A)$. The resulting spike heights, $u(0, t)$, are plotted in Figure 2. In each plot we observe that the equilibrium solution is stable when $\tau = 0.99\tau_h(A)$ but is unstable otherwise. Furthermore we observe that for $A = 3.8$ the Hopf bifurcation appears to be subcritical, whereas for $A = 4.0$ and $A = 10.0$ the Hopf bifurcation is seen to be supercritical, leading to the emergence of a stable limit cycle. These numerical simulations thus support, and illustrate, the theoretical predictions from the preceding sections.

For the Schnaknberg system we only need to vary the time-constant τ. Thus we choose values of $\tau = 0.9\tau_h$ and $\tau = 1.1\tau_h$ and numerically calculate the solution of the full Schnakenberg

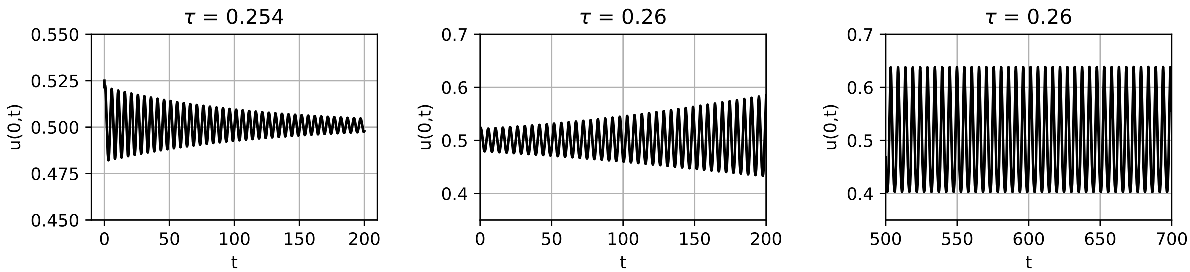


FIGURE 3. Plots of the spike height, $u(0,t)$, for numerically computed solutions of the Schnakenberg system with values of τ below and above the Hopf bifurcation threshold of $\tau_h = 0.257$. The remaining system parameters are given by $D = 10^5$ and $\varepsilon = 0.002$. For $\tau = 0.26$ the middle plot shows the onset of the Hopf instability while the rightmost plot shows the resulting stable limit cycle.

system starting with an initial condition that is a 5% perturbation away from the single-spike equilibrium calculated in (2.19). The resulting spike heights, $u(0,t)$ are plotted in Figure 3 where we observe that the solution is stable for $\tau = 0.9\tau_h$ but is unstable and leads to a stable limit cycle for $\tau = 1.1\tau_h$.

A	τ_h	Re $K(A)$
3.8	2.5987	-0.11664
4.0	3.0665	0.1675
10	24.895	0.068284

TABLE 1. Parameter values for numerical calculations Gray Scott

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Department of Mathematics, The University of British Columbia, Vancouver, BC Canada V6T 1Z2

E-mail address: dagubc@math.ubc.ca

Department of Mathematics, Henan Normal University, Xinxiang 453007, P.R. China

E-mail address: mlf@htu.edu.cn

Department of Mathematics, The University of British Columbia, Vancouver, BC Canada V6T 1Z2

E-mail address: jcwei@math.ubc.ca