DECOMPOSITION OF POLYHARMONIC OPERATOR AND CLASSIFICATION OF HOMOGENEOUS STABLE SOLUTIONS

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Abstract. In this paper, we provide a unified framework to classify homogeneous stable solutions of arbitrary order polyharmonic Lane-Emden equations. The key idea is the spherical decomposition of polyharmonic operator into differential operators.

1. Introduction and main results

The polyharmonic Lane-Emden equations

$$(-\Delta)^m u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^n \tag{1.1}$$

have attracted a lot of attention in the past few decades as important semi-linear PDEs. Here $n \geq 2, m \geq 1, m \in \mathbb{N}^+$ and $p > 1$.

When $m = 1$, a celebrated result of Gidas and Spruck [14] states that the equation has no positive classical solutions (radial or non-radial) if

$$1 < p < p_S := \begin{cases} +\infty & n \leq 2, \\ \frac{n+2}{n-2} & n \geq 3. \end{cases}$$

When $p = p_S$, radially symmetric solution exists. Caffarelli, Gidas and Spruck [6] proved that all positive solutions are radially symmetric around some point. For non-radial sign-changing solutions we refer to del Pino, Musso, Pacard and Pistoia [7, 8], Musso and Wei [22] and the references therein.

When $p > p_S$ there are very few classification results. A seminal work by Farina [13] proves that there is no nontrivial stable solutions if p is below the Joseph-Lundgren exponent [17]

$$1 < p < p_{JL} := \begin{cases} \infty & 3 \leq n \leq 10, \\ 1 + \frac{4}{n-4-2\sqrt{n-1}} & n \leq 11. \end{cases}$$

For the general nonlinearity and more results on stable solutions to elliptic equations, we refer to Dupaigne-Farina [11] and the monograph by Dupaigne [12]. For stable solutions on bounded domains, we refer to Cabré [1, 3], Cabré and Ros-Oton [2], and the references therein. For recent results on stable solutions of second order elliptic equations see Cabré and Poggesi [4] and Cabré, Figalli, Ros-Oton and Serra [5] in which they solved a longstanding conjecture on the optimal dimension for regularity of stable solutions in bounded domains with general nonlinearity.

When $m \geq 2$ and $p \leq \frac{n+2m}{n-2m}$, the classification of positive solutions has been given by Wei and Xu [24]. When $p > \frac{n+2m}{n-2m}$, the classification of stable solutions requires new set of ideas as the Moser iteration techniques as in Farina [13] do not work when $m \geq 2$. In Davila, Dupaigne, Wang and Wei [9], a new scheme of proof is derived. It consists of two steps. First they obtained a monotonicity formula. Then by dimension reduction method, they reduced the problem to classification of homogeneous stable solutions. We recall that a solution u is called homogeneous if it has the form $u(x) = |x|^k w(\frac{x}{|x|})$ for some $k \in \mathbb{R}$ and some function w. In particular, by some algebraic computation, a solution to (1.1) is called homogeneous if $u = |x|^{-\frac{2m}{p-1}} w(\frac{x}{|x|})$ for some function w. Combining these two steps they gave a complete classification result for stable or finite Morse index solutions to (1.1) in the biharmonic case. These two-step approaches have been extended to $m = 3$

in [19] and [16], and to $m=4$ in [20]. This paper concerns about the second step, i.e., classification of homogeneous stable solutions to (1.1). For the case of $0<m<1, m=1, 1<m<2$ and $m=2$ such classification are given by Davila, Dupaigne and Wei in [10], Farina in [13], Fazly and Wei in [15] and Davila, Dupaigne, Wang and Wei in [9], respectively. Recently, the monotonicity formula and classification of stable solutions for general $m\geq 3$ in large dimensions are obtained by the authors in [21].

We first introduce the higher dimensional Joseph-Lundgren exponent. For $m\geq 3$, it is well-known that the radial solution of (1.1) is stable under the following condition

$$p\frac{\Gamma(\frac{n}{2}-\frac{m}{p-1})\Gamma(m+\frac{m}{p-1})}{\Gamma(\frac{m}{p-1})\Gamma(\frac{n-2m}{2}-\frac{m}{p-1})}\leq\frac{\Gamma(\frac{n+2m}{4})^2}{\Gamma(\frac{n-2m}{4})^2},\quad \text{where } 0<m<\frac{n}{2},\ n\in\mathbb{N}^+. \tag{1.2}$$

In [18] we have shown that there exists a unique exponent $p_{JL}(n,m)>\frac{n+2m}{n-2m}$ such that (1.2) is equivalent to

$$p\geq p_{JL}(n,m). \tag{1.3}$$

In this paper we give a complete classification of homogeneous stable (radial or non-radial) solutions for the polyharmonic Lane-Emden equations (1.1) for any $m\geq 3$ and $p<p_{JL}(n,m)$, i.e.,

$$p\frac{\Gamma(\frac{n}{2}-\frac{m}{p-1})\Gamma(m+\frac{m}{p-1})}{\Gamma(\frac{m}{p-1})\Gamma(\frac{n-2m}{2}-\frac{m}{p-1})}>\frac{\Gamma(\frac{n+2m}{4})^2}{\Gamma(\frac{n-2m}{4})^2},\quad \text{where } 0<m<\frac{n}{2},\ n\in\mathbb{N}^+. \tag{1.4}$$

To simplify the notations in polyharmonic equations, we use the following notations

$$\nabla^j\circ w=\begin{cases}\Delta^{\frac{j}{2}}w, & j \text{ is even},\\ \nabla\Delta^{\frac{j-1}{2}},w & j \text{ is odd},\end{cases} \tag{1.5}$$

$$\nabla_\theta^j\circ w=\begin{cases}\Delta_\theta^{\frac{j}{2}}w, & j \text{ is even},\\ \nabla_\theta\Delta_\theta^{\frac{j-1}{2}}w, & j \text{ is odd},\end{cases}$$

where $\theta=\frac{x}{|x|}$, $\Delta_\theta=\Delta_{\mathbb{S}^{n-1}}$ and ∇_θ denotes the co-variant derivative on $\mathbb{S}^{n-1}$.

We recall that a solution to (1.1) is called stable if

$$\int_{\mathbb{R}^n}|\nabla^m\circ\varphi|^2dx-p\int_{\mathbb{R}^n}|u|^{p-1}\varphi^2dx\geq 0,\ \text{for } \forall\varphi\in C_0^\infty(\mathbb{R}^n). \tag{1.6}$$

In the literature, the basic idea to classify the homogeneous stable solutions of polyharmonic Lane-Emden equations is by choosing "right" test functions and then making the refined energy estimates. For any fixed m, we may perform tedious calculations to obtain some classification results. However the amount of the calculations would be exponentially growth with the order of polyharmonic order. See the case $m=3$ in [19] and [16], and the case $m=4$ in [20]. On the other hand, the sharp estimates of $p_{JL}(n,m)$ are also needed for every known result when $m=1,2,3$. In this paper, we develop a new, simple and unified method of classification stable solutions to homogeneous solutions by using the inner symmetry and monotonicity properties of the corresponding symmetric differential operator.

Next we state the main results of the paper. The first one establishes a priori estimate of the homogeneous stable solutions by introducing a symmetric function. The second one gives the classification of the supercritical homogeneous stable solutions to (1.1).

Theorem 1.1. *Let $u\in W^{m,2}_{loc}(\mathbb{R}^n\setminus\{0\}), |u|^{p+1}\in L^1(\mathbb{R}^n\setminus\{0\})$ be a homogeneous, stable solution of equations (1.1). Then the following inequality holds:*

$$\sum_{j=0}^m\int_{\mathbb{S}^{n-1}}\Big(pJ_{j,m}(\frac{2m}{p-1})-J_{j,m}(\frac{n-2m}{2})\Big)|\nabla_\theta^j\circ w|^2\leq 0.$$

Here the symmetric function $J_{j,m}(x)$ is defined by (see (2.1))

$$J_{t,m}(x) = \sum_{0\le i_1<i_2<\cdots<i_{m-t}\le m-1} \Pi_{j=1}^{m-t} a(x+2i_j),\ a(x)=x(n-2-x).$$

Theorem 1.2. *Assume that $n > 2m$. Let $u \in W^{m,2}_{loc}(\mathbb{R}^n \setminus \{0\}), |u|^{p+1} \in L^1(\mathbb{R}^n \setminus \{0\})$ be a homogeneous stable solution of the polyharmonic Lane-Emden equations (1.1). If $\frac{n+2m}{n-2m} < p < p_{JL}(n,m)$, then $u \equiv 0$.*

Remark 1.1. *The condition $p < p_{JL}(n,m)$ is sharp since there is a radial stable solution for $p \ge p_{JL}(n,m)$, see* [18].

The proofs of Theorems 1.1-1.2 depend on two key observations: first we can write $(-\Delta)^m$ in symmetric operators. Second, we prove monotonicity properties for these corresponding symmetric functions (due to [23]). In the rest of the paper we prove these two observations.

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2. Decomposition of polyharmonic operator and new combinatorial operators

Let us introduce some combinatorial operators which will be frequently used in our analysis.

First, let $a(x) := x(n-2-x)$, where n is the dimension. $a(x)$ is naturally related to the radial Laplacian operator $\partial_{rr} + \frac{n-1}{r}\partial_r$. In fact, $(\partial_{rr} + \frac{n-1}{r}\partial_r)r^{-x} = a(x)r^{-x-2}$.

Next, for $m \ge 1$, we define the symmetric function

$$J_{t,m}(x) := \sum_{0\le i_1<i_2<\cdots<i_{m-t}\le m-1} \Pi_{j=1}^{m-t} a(x+2i_j), \tag{2.1}$$

which is associated with the following symmetric differential operator

$$P_{t,m} := \sum_{all\ \ the\ \ different\ \ arrangements\ \ of} (\underbrace{r^{-2},\cdots,r^{-2}}_{t},\underbrace{\partial_{rr}+\frac{n-1}{r}\partial_r,\cdots,\partial_{rr}+\frac{n-1}{r}\partial_r}_{m-t}). \tag{2.2}$$

To unify the notations, it is natural to assume that $P_{t,m} = 0$ if $t > m$ or $t < 0$. For example, when $t = 1, m = 4$, we have

$$\begin{aligned}P_{1,4} &= (\partial_{rr} + \frac{n-1}{r}\partial_r)^2(r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)) + (\partial_{rr} + \frac{n-1}{r}\partial_r)^3 r^{-2}\\ &+(\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)^2) + r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)^3.\end{aligned}$$

The differential operator $P_{t,m}$ and the symmetric function $J_{t,m}(x)$ are related by

$$P_{t,m} \circ r^{-x} = (-1)^{m-t} J_{t,m}(x) r^{-x-2(m-t)}. \tag{2.3}$$

By definition (2.2) it is easy to see that we have the following recursing relation.

Proposition 2.1.

$$P_{j,m+1} = (\partial_{rr} + \frac{n-1}{r}\partial_r)P_{j,m} + r^{-2}P_{j-1,m}.$$

Next we turn to the spherical decomposition of the polyharmonic operator.

Proposition 2.2. *(Decomposition of polyharmonic operator)*

$$\Delta^m = \sum_{j=0}^{m} \Delta_\theta^j P_{j,m},\ \ m = 1,2,3\cdots. \tag{2.4}$$

Proof. We prove it by induction. For the case $m=1$, since $P_{0,1}=\partial_{rr}+\frac{n-1}{r}\partial_r, P_{1,1}=r^{-2}$ and $\Delta=\partial_{rr}+\frac{n-1}{r}\partial_r+r^{-2}\Delta_\theta$, (2.4) holds immediately.

Suppose that $\Delta^m=\sum_{j=0}^m\Delta_\theta^jP_{j,m}$. Let us consider Δ^{m+1}:

$$\begin{aligned}\Delta^{m+1}=\Delta\Delta^m&=(\partial_{rr}+\frac{n-1}{r}\partial_r+r^{-2}\Delta_\theta)\sum_{j=0}^m\Delta_\theta^jP_{j,m}\\&=\sum_{j=0}^m\Delta_\theta^j(\partial_{rr}+\frac{n-1}{r}\partial_r)P_{j,m}+\sum_{j=0}^m\Delta_\theta^{j+1}r^{-2}P_{j,m}\\&=\sum_{j=0}^m\Delta_\theta^j(\partial_{rr}+\frac{n-1}{r}\partial_r)P_{j,m}+\sum_{j=1}^{m+1}\Delta_\theta^jr^{-2}P_{j-1,m}\\&=\sum_{j=0}^{m+1}\Delta_\theta^jP_{j,m+1}.\end{aligned}$$

Here we have used Proposition 2.1. Therefore by induction, one gets (2.4). ☐

3. Proof of Theorem 1.1

In this Section, we consider a homogeneous stable solution to (1.1), i.e., $u=r^{-\frac{2m}{p-1}}w(\theta)$ and prove Theorem 1.1.

By Proposition (2.2), we have the following

$$\begin{aligned}\Delta^m(r^{-k}w(\theta))&=\sum_{j=0}^m\Delta_\theta^jP_{j,m}\circ(r^{-k}w(\theta))\\=&\sum_{j=0}^mP_{j,m}\circ r^{-k}\Delta_\theta^jw(\theta)=\sum_{j=0}^m(-1)^{m-j}J_{j,m}(k)\Delta_\theta^jw(\theta)r^{-k-2m}.\end{aligned}\tag{3.1}$$

As in [9], we use the cut off function $\varphi(r,\theta)=r^{-q}w(\theta)\eta_\varepsilon(r)$, $q=\frac{n-2m}{2}$. Here $\eta_\varepsilon\in C_0^\infty(\frac{\varepsilon}{2},\frac{2}{\varepsilon})$ and $\eta_\varepsilon=1$ in $(\varepsilon,\frac{1}{\varepsilon})$.

Depending on m is odd or even, the analysis will be slightly different. We divide the proofs into two cases.

Case 1: m is even. In this case we compute

$$\Delta^{\frac{m}{2}}\varphi=r^{-\frac{n}{2}}\eta_\varepsilon(r)\sum_{j=0}^{\frac{m}{2}}(-1)^{\frac{m}{2}-j}J_{j,\frac{m}{2}}(q)\Delta_\theta^jw+\sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{m-2i}C_{j,i}r^{-\frac{n}{2}+j}\eta_\varepsilon^{(j)}\Delta_\theta^iw(\theta)\tag{3.2}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}|\Delta^{\frac{m}{2}}\varphi|^2=&\int_0^{+\infty}r^{-1}\eta_\varepsilon^2(r)dr\Big(\int_{\mathbb{S}^{n-1}}\sum_{i=1}^{\frac{m}{2}}q_{2i}(m)|\Delta_\theta^iw|^2+q_{2i-1}(m)|\nabla_\theta\Delta_\theta^{i-1}w|^2\Big)\\&+\Big(\int_0^{+\infty}\sum_{1\le i+j\le 2m;i,j\ge0}C_{i,j}r^{i+j-1}\eta_\varepsilon^{(i)}\eta_\varepsilon^{(j)}dr\Big)\Big(\int_{\mathbb{S}^{n-1}}\sum_{i=1}^{\frac{m}{2}}(|\Delta_\theta^iw|^2+|\nabla_\theta\Delta_\theta^{i-1}w|^2)\Big),\end{aligned}$$

where

$$\begin{aligned}q_{2i}(m)&:=J_{i,\frac{m}{2}}^2(q)+2\sum_{0\le s\le i-1}J_{s,\frac{m}{2}}(q)J_{2i-s,\frac{m}{2}}(q),i=1,2,\cdots,\frac{m}{2};\\q_{2i-1}(m)&:=2\sum_{0\le s\le i-\frac{1}{2}}J_{s,\frac{m}{2}}(q)J_{2i-1-s,\frac{m}{2}}(q),i=1,2,\cdots,\frac{m}{2}.\end{aligned}$$

One can rewrite $q_{2i}(m)$ and $q_{2i-1}(m)$ in a unified way as

$$q_i(m) = \sum_{s\geq 0} J_{s,\frac{m}{2}}(q)J_{i-s,\frac{m}{2}}(q), i = 1, 2, \cdots, m. \tag{3.3}$$

Here $J_{t,m}(x) = 0$ if $t < 0$ or $t > m$.

Case 2: m is odd. In this case, making the use of the fact that $|\nabla u|^2 = \frac{1}{r^2}|\nabla_\theta u|^2 + |\partial_r u|^2$ and (3.2), we have

$$\begin{aligned}
&|\nabla\Delta^{\frac{m-1}{2}}(r^{-q}w(\theta)\eta_\varepsilon(r))|^2 = \frac{1}{r^2}|\nabla_\theta\Delta^{\frac{m-1}{2}}(r^{-q}w(\theta)\eta_\varepsilon(r))|^2\\
&\quad + |\frac{\partial}{\partial r}\Delta^{\frac{m-1}{2}}(r^{-q}w(\theta)\eta_\varepsilon(r))|^2\\
&= |r^{-\frac{n}{2}}\eta_\varepsilon(r)\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}-j}J_{j,\frac{m-1}{2}}(q)\nabla_\theta\Delta_\theta^j w|^2\\
&+ (\frac{n-2}{2})^2 r^{-n}\eta_\varepsilon^2(r)|\sum_{j=0}^{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}-j}J_{j,\frac{m-1}{2}}(q)\Delta_\theta^j w|^2 \ +\text{Remainder} \quad \text{terms}.
\end{aligned}$$

The remainder terms satisfy

$$\begin{aligned}
|\int_{\mathbb{R}^n}\text{Remainder} \quad \text{terms}| \leq C\int_0^{+\infty} r^{i+j-1}\eta_\varepsilon^{(i)}\eta_\varepsilon^{(j)}dr\\
\cdot\Big(\int_{\mathbb{S}^{n-1}}\sum_{i=1}^{\frac{m}{2}}(|\Delta_\theta^i w|^2 + |\nabla_\theta\Delta_\theta^{i-1}w|^2)\Big).
\end{aligned} \tag{3.4}$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}^n}|\nabla\Delta^{\frac{m-1}{2}}\varphi|^2 = \int_0^{+\infty} r^{-1}\eta_\varepsilon^2(r)dr\Big(\int_{\mathbb{S}^{n-1}}\sum_{j=0}^{m} q_j(m)|\nabla_\theta^j\circ w|^2\Big)\\
+\int_{\mathbb{R}^n}\text{Remainder} \quad \text{terms}.
\end{aligned} \tag{3.5}$$

Here

$$q_j(m) = (\frac{n-2}{2})^2\sum_{s\geq 0}J_{s,\frac{m-1}{2}}(q)J_{j-s,\frac{m-1}{2}}(q) + \sum_{t\geq 0}J_{t,\frac{m-1}{2}}(q)J_{j-1-t,\frac{m-1}{2}}(q), j = 0, 1, \cdots, m. \tag{3.6}$$

Substituting (3.4) and (3.5) into the stability condition for u in (1.6), one obtains that

$$p\int_{\mathbb{S}^{n-1}}|w|^{p+1}\cdot\int_0^{+\infty} r^{-1}\eta_\varepsilon^2(r)dr \leq \int_{\mathbb{R}^n}|\nabla^m\circ\varphi|^2.$$

Notice that

$$\int_0^{+\infty} r^{-1}\eta_\varepsilon^2(r)dr \geq |\log\varepsilon| \to +\infty, \ \text{ as } \ \varepsilon\to 0^+,$$

$$\int_0^{+\infty}\sum_{1\leq i+j\leq 2m, i,j\geq 0} r^{i+j-1}|\eta_\varepsilon^{(i)}(r)\eta_\varepsilon^{(j)}(r)|dr \leq C, C \text{ is a constant independent of } \varepsilon.$$

Hence we obtain that

$$p\int_{\mathbb{S}^{n-1}}|w|^{p+1}d\theta \leq \int_{\mathbb{S}^{n-1}}\sum_{j=0}^{m} q_j(m)|\nabla_\theta^j\circ w|^2. \tag{3.7}$$

From the polyharmonic equations (1.1), setting $u = r^{-k}w(\theta)$, $k = \frac{2m}{p-1}$ and combining with (2.2), one has

$$\sum_{j=0}^{m}(-1)^j J_{j,m}(k)\Delta_\theta^j w(\theta) = |w|^{p-1}(\theta)w(\theta). \tag{3.8}$$

Testing (3.8) with $w(\theta)$ and integrating by parts on the sphere $\mathbb{S}^{n-1}$, one gets that

$$\sum_{j=0}^{m}\int_{\mathbb{S}^{n-1}} J_{j,m}(k)|\nabla_\theta^j \circ w(\theta)|^2 = \int_{\mathbb{S}^{n-1}}|w|^{p+1}.$$

Therefore, in view of (3.7), we obtain that

$$\sum_{j=0}^{m}\int_{\mathbb{S}^{n-1}}\Big(pJ_{j,m}(k) - q_j(m)\Big)|\nabla_\theta^j \circ w|^2 \le 0. \tag{3.9}$$

Here $q_j(m)$ is given in (3.3) and (3.6) for m being even and odd respectively.

This proves Theorems 1.1.

4. Properties of the symmetric function $J_{j,m}(x)$

In this section, we discuss the properties of the symmetric function $J_{j,m}(x)$, defined in (2.1), in details. Observe that the energy inequality (3.9) has the precise and unified expression for coefficients of terms $\int_{\mathbb{S}^{n-1}}|\nabla_\theta^j\circ w|^2, j = 1, 2, \cdots, m$ by the combinatorial operators and the corresponding symmetric functions. Therefore, if we can prove that $pJ_{j,m}(k) - q_j(m) > 0$ for $j = 1, 2, \cdots, m$, then $w \equiv 0$, which then implies that the homogeneous stable solution must be trivial.

Proposition 4.1 (Symmetry). *If $x + \overline{x} = n - 2m$, then*

$$J_{j,m}(x) = J_{j,m}(\overline{x}). \tag{4.1}$$

Proof. By the definition of $J_{j,m}(x)$ in (2.1), we know that $J_{j,m}(x)$ is a symmetric function of the set

$$\{a(x+2j)\}_{j=0}^{m-1}, \text{ where, } a(x+2j) = (x+2j)(n-2-x-2j).$$

Notice that when $\overline{x} = n - 2m - x$, then the set below is the same as above, that is,

$$\{a(\overline{x}+2j)\}_{j=0}^{m-1} = \{a(x+2j)\}_{j=0}^{m-1}.$$

Therefore by definition, we have $J_{j,m}(x) = J_{j,m}(\overline{x})$. □

Proposition 4.2 (Additivity). *For any nonnegative integers t, p, q, $t \le p + q$ and any $x \in \mathbb{R}$,*

$$\begin{aligned} J_{t,p+q}(x) &= \sum_{i+j=t} J_{j,p}(x)J_{i,q}(x+2p) \\ &= \sum_{i+j=t} J_{i,q}(x)J_{j,p}(x+2q). \end{aligned}$$

In particular,

$$\begin{aligned} J_{t,m}(x) &= J_{t-1,m-1}(x)J_{1,1}(x+2m-2) + J_{t,m-1}(x)J_{0,1}(x+2m-2) \\ &= J_{t-1,m-1}(x) + (x+2m-2)(n-x-2m)J_{t,m-1}(x), \end{aligned}$$

and for m is even

$$J_{t,m}(x) = \sum_{s\ge 0} J_{s,\frac{m}{2}}(x)J_{t-s,\frac{m}{2}}(x+m).$$

Proof. Applying formula (3.1), we have

$$(-\Delta)^{p+q}(r^{-k}w(\theta)) = \sum_{t=0}^{m} J_{t,p+q}(k) r^{-k-2p-2q}(-\Delta_{\mathbb{S}^{n-1}})^t w(\theta), \tag{4.2}$$

and

$$\begin{aligned}(-\Delta)^{p+q}(r^{-k}w(\theta)) &= (-\Delta)^p \circ (-\Delta)^q (r^{-k}w(\theta)) \\ &= (-\Delta)^q \circ \Big(\sum_{j=0}^{m} J_{j,p}(k) r^{-k-2p}(-\Delta_{\mathbb{S}^{n-1}})^j w(\theta)\Big) \\ &= \sum_{i,j} J_{j,p}(k) J_{i,q}(k+2p) r^{-k-2p-2q}(-\Delta_{\mathbb{S}^{n-1}})^{i+j} w(\theta).\end{aligned} \tag{4.3}$$

Comparing the coefficients of the term $r^{-k-2p-2q}(-\Delta_{\mathbb{S}^{n-1}})^t w(\theta)$ in equations (4.2) and (4.3), we get the desired identity. □

Proposition 4.3. *For m is even, it holds that*

$$\sum_{s\geq 0} J_{s,\frac{m}{2}}(\frac{n-2m}{2}) J_{j-s,\frac{m}{2}}(\frac{n-2m}{2}) = J_{j,m}(\frac{n-2m}{2}). \tag{4.4}$$

Proof. By Propositions 4.1 and 4.2, we have

$$\begin{aligned}&\sum_{s\geq 0} J_{s,\frac{m}{2}}(\frac{n-2m}{2}) J_{j-s,\frac{m}{2}}(\frac{n-2m}{2}) \\ =&\sum_{s\geq 0} J_{s,\frac{m}{2}}(\frac{n-2m}{2}) J_{j-s,\frac{m}{2}}(\frac{n-2m}{2}+m) \\ =&J_{j,m}(\frac{n-2m}{2}).\end{aligned}$$

□

Corollary 4.1. *Let m be even and $q_j(m)$ be defined in (3.3). We have*

$$q_j(m) = J_{j,m}(\frac{n-2m}{2}) = J_{j,m}(q). \tag{4.5}$$

Hence for m is even,

$$pJ_{j,m}(k) - q_j(m) = pJ_{j,m}(k) - J_{j,m}(q).$$

Proposition 4.4 (Recursion formula)**.** *For m is odd and any $x \in \mathbb{R}$, we have*

$$\begin{aligned}J_{s,m}(x) =&a(x+m-1)\cdot \sum_{s\geq 0} J_{s,\frac{m-1}{2}}(x) J_{j-s,\frac{m-1}{2}}(x+m+1) \\ &+ \sum_{s\geq 0} J_{t,\frac{m-1}{2}}(x) J_{j-1-s,\frac{m-1}{2}}(x+m+1).\end{aligned}$$

Proof. Since $J_{j,m}(x)$ is a symmetric function of the set

$$\{a(x+2j)\}_{j=0}^{m-1}, \text{ where, } a(x+2j) = (x+2j)(n-2-x-2j).$$

We divide above set into three parts as follows,

$$\{a(x+2j)\}_{j=0}^{\frac{m-1}{2}-1}; a(x+m-1); \{a(x+m-1+2j)\}_{j=0}^{\frac{m-1}{2}-1};$$

Regrouping the terms by the criteria of involving the term $a(x+m-1)$ or not, one gets the terms $a(x+m-1)\cdot\sum_{s\geq 0} J_{s,\frac{m-1}{2}}(x) J_{j-s,\frac{m-1}{2}}(x+m+1)$ and $\sum_{s\geq 0} J_{t,\frac{m-1}{2}}(x) J_{j-1-s,\frac{m-1}{2}}(x+m+1)$ respectively, therefore the conclusion follows. □

Corollary 4.2. *For m is odd and $q=\frac{n-2m}{2}$, we have*

$$(\frac{n-2}{2})^2\sum_{s\geq 0}J_{s,\frac{m-1}{2}}(q)J_{j-s,\frac{m-1}{2}}(q)+\sum_{t\geq 0}J_{t,\frac{m-1}{2}}(q)J_{j-1-t,\frac{m-1}{2}}(q)=J_{j,m}(q).$$

Therefore, when m is odd, the $q_j(m)$ defined in (3.3) can be rewritten as

$$q_j(m)=J_{j,m}(q).$$

Proof. Taking $x=\frac{n-2m}{2}$ in the recursion formula as in Proposition 4.4, and noting that $a(x+m-1)=(\frac{n-2}{2})^2$, one has

$$\begin{aligned}J_{j,m}(q)=&(\frac{n-2}{2})^2\cdot\sum_{s\geq 0}J_{s,\frac{m-1}{2}}(q)J_{j-s,\frac{m-1}{2}}(q+m+1)\\&+\sum_{s\geq 0}J_{t,\frac{m-1}{2}}(q)J_{j-1-s,\frac{m-1}{2}}(q+m+1).\end{aligned}\tag{4.6}$$

On the other hand by Proposition 4.1, one also has

$$J_{j-1-s,\frac{m-1}{2}}(q+m+1)=J_{j-1-s,\frac{m-1}{2}}(q),\tag{4.7}$$

Therefore by (4.6) and (4.7), one completes the proof.

□

Proposition 4.5 (Monotonicity and concavity)**.** *The function $J_{j,m}(x)$ is concave on the interval $[0,n-2m]$; monotonically increasing on the interval $[\frac{n-2m}{2},n-2m]$, and monotonically decreasing on the interval $[\frac{n-2m}{2},n-2m]$.*

Proof. Let $a_i:=a(x+2i)$, then $J_{j,m}(x)=\sigma_{m-j}(a_0,\cdots,a_{m-1})$, where σ_j is the elementary symmetric function. Hence

$$\begin{aligned}\frac{d^2}{dx^2}J_{j,m}(x)&=\sum_{i,t}\frac{\partial^2\sigma_{m-j}(a)}{\partial a_i\partial a_t}\frac{da_i}{dx}\frac{da_t}{dx}+\sum_i\frac{\partial\sigma_{m-j}(a)}{\partial a_i}\frac{d^2a_i}{dx^2},\\&\leq -2\sum_i\frac{\partial\sigma_{m-j}(a)}{\partial a_i},\\&<0.\end{aligned}$$

Here we have used the fact that for $x\in(0,n-2m)$,$a_i>0$, $a=(a_i)$ belongs to the cone $\Gamma(m)$, and

$$a_i>0,a=(a_i)\in\text{ the cone }\Gamma(m),\text{then }\{\frac{\partial^2\sigma_{m-j}(a)}{\partial a_i\partial a_t}\}\leq 0\text{ on the cone }\Gamma(m).$$

See Wang [23].

Therefore $J_{j,m}(x)$ is concave on the interval $[0,n-2m]$. On the other hand, by Proposition 4.1, we have $J_{j,m}(x)$ is symmetric about the axis $x=\frac{n-2m}{2}$. Therefore one concludes that $J_{j,m}(x)$ is monotonically increasing on the interval $[0,\frac{n-2m}{2}]$, and monotonically decreasing on the interval $[\frac{n-2m}{2},n-2m]$. □

Remark 4.1. *A direct calculation $\frac{d}{dx}J_{j,m}(x)=\sum_{i=0}^{m-1}\frac{\partial\sigma_{m-j}(a)}{\partial a_i}\frac{da_i}{dx}$ doesn't work for the proof of the monotonicity on interval $[0,\frac{n-2m}{2}]$ or $[\frac{n-2m}{2},n-2m]$, since $\frac{da_i}{dx}=n-2-4i-2x$ may change sign when $i\geq\frac{m}{2}$. So here we use the second derivative of $J_{j,m}(x)$.*

Corollary 4.3. *Let $k=\frac{2m}{p-1}$, $q=\frac{n-2m}{2}$. For $k<q$, it holds that*

$$J_{j,m}(k)<J_{j,m}(q).$$

Proposition 4.6 (Monotonicity and convexity)**.** *The function $\frac{J_{j,m}(x)}{J_{0,m}(x)}$ is convex on the interval $[0,n-2m]$; monotonically decreasing on the interval $[0,\frac{n-2m}{2}]$, and monotonically increasing on the interval $[\frac{n-2m}{2},n-2m]$.*

Remark 4.2. *This can be viewed as a dual version of Proposition 4.5, see the formula below.*

Proof. We adopt the same notations as in Proposition 4.5. Noting that

$$-\frac{J_{j,m}(x)}{J_{0,m}(x)} = -\frac{\sigma_{m-j}(a_0,\cdots,a_{m-1})}{\sigma_m(a_0,\cdots,a_{m-1})} = -\sigma_j(\frac{1}{a_0},\cdots,\frac{1}{a_{m-1}}).$$

Then a similar analysis as in Proposition 4.5 leads to the conclusion. □

5. Proof of Theorem 1.2

Using the properties of $J_{j,m}(k)$ proved in the previous section, we give the proof of Theorem 1.2.

By Corollaries 4.1 and 4.2, we have the simplification of $pJ_{j,m}(k) - q_j(m)$, that is

$$pJ_{j,m}(k) - q_j(m) = pJ_{j,m}(k) - J_{j,m}(q).$$

Then the stability inequality (3.9) reads as

$$\sum_{j=0}^{m}\int_{\mathbb{S}^{n-1}}\Big(pJ_{j,m}(k) - J_{j,m}(q)\Big)|\nabla_\theta^j \circ w|^2 \le 0. \tag{5.1}$$

A straightforward calculation shows that

$$pJ_{0,m}(k) - J_{0,m}(q) = p\frac{\Gamma(\frac{n}{2}-\frac{m}{p-1})\Gamma(m+\frac{m}{p-1})}{\Gamma(\frac{m}{p-1})\Gamma(\frac{n-2m}{2}-\frac{m}{p-1})} - \frac{\Gamma(\frac{n+2m}{4})^2}{\Gamma(\frac{n-2m}{4})^2}. \tag{5.2}$$

Therefore by the classification results in [18], we have that

$$pJ_{0,m}(k) - J_{0,m}(q) > 0 \Leftrightarrow p < p_{JL}(n,m). \tag{5.3}$$

Another interesting observation here is that when $j = m$, $pJ_{m,m}(k) - J_{m,m}(q) = p - 1$. Hence it is positive automatically. Next we show that

Lemma 5.1. *Assume that $p > \frac{n+2m}{n-2m}$. If $pJ_{0,m}(k) - J_{0,m}(q) > 0$, then $pJ_{j,m}(k) - J_{j,m}(q) > 0$ for $j = 1, 2, \cdots, m-1$.*

Proof. Since $pJ_{0,m}(k) - J_{0,m}(q) > 0$, one gets that $p > \frac{J_{0,m}(q)}{J_{0,m}(k)}$ directly. Then $pJ_{j,m}(k) - J_{j,m}(q) > 0$ if

$$\frac{J_{0,m}(q)}{J_{0,m}(k)}J_{j,m}(k) - J_{j,m}(q) \ge 0. \tag{5.4}$$

Inequality (5.4) equivalents to

$$\frac{J_{j,m}(k)}{J_{0,m}(k)} \ge \frac{J_{j,m}(q)}{J_{0,m}(q)}. \tag{5.5}$$

On the other hand, (5.5) is followed by the monotonically property of $\frac{J_{j,m}(x)}{J_{0,m}(x)}$ on the interval $[0, \frac{n-2m}{2}]$ in Proposition 4.6 since $0 < k < q = \frac{n-2m}{2}$ $(p > \frac{n+2m}{n-2m})$. □

Therefore combining the stability equality (5.1) and the critical exponent in (5.3) with Lemma 5.1, we obtain that when $p < p_{JL}(n,m)$, then $w(\theta) \equiv 0$, which completes the proof of Theorem 1.2.

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