POSITIVE SOLUTIONS OF THE GROSS–PITAEVSKII EQUATION FOR ENERGY CRITICAL AND SUPERCRITICAL NONLINEARITIES

DMITRY E.PELINOVSKY, JUNCHENG WEI, AND YUANZE WU

Abstract. We consider positive and spatially decaying solutions to the following Gross–Pitaevskii equation with a harmonic potential:

$$-\Delta u + |x|^2 u = \omega u + |u|^{p-2}u \quad \text{in } \mathbb{R}^d,$$

where $d \geq 3$, $p > 2$ and $\omega > 0$. For $p = \frac{2d}{d-2}$ (energy-critical case) there exists a ground state u_ω if and only if $\omega \in (\omega_*, d)$, where $\omega_* = 1$ for $d = 3$ and $\omega_* = 0$ for $d \geq 4$. We give a precise description on asymptotic behaviors of u_ω as $\omega \to \omega_*$ up to the leading order term for different values of $d \geq 3$. When $p > \frac{2d}{d-2}$ (energy-supercritical case) there exists a singular solution u_∞ for some $\omega \in (0, d)$. We compute the Morse index of u_∞ in the class of radial functions and show that the Morse index of u_∞ is infinite in the oscillatory case, is equal to 1 or 2 in the monotone case for p not large enough and is equal to 1 in the monotone case for p sufficiently large.

Keywords: Gross–Pitaevskii equation; Critical and supercritical nonlinearity; Positive solutions; Asymptotic behavior; Morse index.

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1. Introduction

In this paper, we consider positive and spatially decaying solutions to the following stationary Gross–Pitaevskii equation with a harmonic potential:

$$-\Delta u + |x|^2 u = \omega u + |u|^{p-2}u \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

where $d \geq 3$, $p > 2$ and $\omega > 0$.

The stationary equation (1.1) is a classical model to describe the Bose–Einstein condensate with attractive inter-particle interactions under magnetic trap (cf. [38]) if $d = 1, 2, 3$ and $p = 4$ (the cubic case) or $p = 6$ (the quintic case). In this context, $\psi(t, x) = e^{-i\omega t}u(x)$ is a standing wave solution of the time-dependent Gross–Pitaevskii equation

$$i\partial_t \psi = -\Delta \psi + |x|^2 \psi - |\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^d,$$

where ψ stands for the macroscopic wave function, $|x|^2$ is an isotropic trapping potential that confines the Bose–Einstein condensate, and the nonlinear term corresponds to attractive inter-atomic interactions. Positive and spatially decaying solutions are called the bright solitons in the physics literature.

Since the operator $-\Delta+|x|^2$ is compact in $L^2(\mathbb{R}^d)$, the energy-subcritical case $2<p<\frac{2d}{d-2}$ can be studied by classical variational methods or bifurcation methods (cf. [17,25,33]). On the other hand, energy-critical $p=\frac{2d}{d-2}$ and energy-supercritical $p>\frac{2d}{d-2}$ cases were less investigated in the literature (cf. [33–35], [4,14,29], and [10]).

In the energy-critical case $p=\frac{2d}{d-2}$ with $d\geq 3$, based on the well-known Gidas-Ni-Nirenberg theorem (cf. [18]), the existence of positive and spatially decaying solutions of the stationary equation (1.1) has been shown in [33,34,36] for $\omega\in(\omega_*,d)$, where

$$\omega_*=\begin{cases}1, & d=3,\\ 0, & d\geq 4.\end{cases} \tag{1.2}$$

However, besides the existence and nonexistence of solutions, it is also interesting for critical elliptic equations to study the concentration phenomenon and the asymptotic behavior of solutions for $\omega\to\omega_*$. These problems were initialed by Brezís *et al.* in [5–7] in the context of the following Dirichlet problem

$$\begin{cases} -\Delta u+a(x)u=\omega u+|u|^{\frac{4}{d-2}}u & \text{in } \Omega,\\ u(x)=0, & \text{on } \partial\Omega,\end{cases} \tag{1.3}$$

where $\Omega\subset\mathbb{R}^d$ ($d\geq 3$) is a bounded domain with smooth boundary and $a(x)$ is a smooth weight (cf. [1, 8, 11–13, 15, 16, 20–24, 26–28, 30–32]). The concentration phenomenon of solutions of the Dirichlet problem (1.3) depends on the geometry of the domain Ω. More precisely, solutions concentrate around the critical points of the Robin function of the domain Ω. To our best knowledge, the concentration phenomenon and the asymptotic behavior of positive and spatially decaying solutions of the stationary equation (1.1) in the energy-critical case $p=\frac{2d}{d-2}$ have not been studied yet. *The first purpose of this paper* is to give a precise description of the latter problems.

We shall now introduce some notations and definitions to state our main results. Let $X\subset L^2(\mathbb{R}^d)$ be the form domain space for the operator $-\Delta+|x|^2$ equipped with the norm

$$\|u\|_X:=\left(\int_{\mathbb{R}^d}(|\nabla u|^2+|x|^2|u|^2)dx\right)^{\frac{1}{2}},$$

We also introduce the energy space $\Sigma:=X\cap L^{\frac{2d}{d-2}}(\mathbb{R}^d)$. For fixed $\omega\in(\omega_*,d)$, we define

$$\mathcal{I}_\omega=\inf_{v\in\Sigma}\left\{I_\omega(v):\quad \|v\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}=1\right\},\quad I_\omega(v):=\|v\|_X^2-\omega\|v\|^2_{L^2(\mathbb{R}^d)}. \tag{1.4}$$

By the method of Lagrange's multipliers and the scaling transformation, $u=(\mathcal{I}_\omega)^{\frac{d-2}{4}}v$ is a nontrivial solution of the stationary equation (1.1) if v is a minimizer of the variational problem (1.4). Based on the above observations, we can introduce the following definition.

Definition 1.1. *We say u_ω is a ground state of the stationary equation* (1.1) *if $v_\omega\in\Sigma$ is a minimizer of the variational problem (1.4) such that $I_\omega(v_\omega)=\mathcal{I}_\omega$ and $u_\omega:=(\mathcal{I}_\omega)^{\frac{d-2}{4}}v_\omega$.*

Let

$$U_\varepsilon(x) = \varepsilon^{\frac{d-2}{2}}[d(d-2)]^{\frac{d-2}{4}}\left(\frac{1}{\varepsilon^2+|x|^2}\right)^{\frac{d-2}{2}}, \quad \varepsilon > 0 \tag{1.5}$$

be a family of the algebraic solitons (also called the Aubin-Talanti bubbles [2,37]) which satisfy the elliptic problem

$$-\Delta u = u^{\frac{d+2}{d-2}}, \quad u \in D^{1,2}(\mathbb{R}^d), \tag{1.6}$$

where $D^{1,2}(\mathbb{R}^d)$ denotes the space of closure of $C_0^\infty(\mathbb{R}^d)$ under the norm $\|\nabla\cdot\|_{L^2(\mathbb{R}^d)}$.

For the sake of simplicity, we also denote $U_{\varepsilon=1}$ by U. It is well known (cf. [2,37]) that U_ε for every $\varepsilon > 0$ attains the best constant of the Sobolev embedding

$$\|u\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \le \mathcal{S}^{-\frac{1}{2}}\|\nabla u\|_{L^2(\mathbb{R}^d)}, \tag{1.7}$$

where $\mathcal{S}$ is given by

$$\mathcal{S} = \inf_{v\in D^{2,1}(\mathbb{R}^d)}\{\|\nabla v\|^2_{L^2(\mathbb{R}^d)} : \quad \|v\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = 1\}. \tag{1.8}$$

By the scaling transformation, if v is a minimizer of the variational problem (1.8), then $u := (\mathcal{S})^{\frac{d-2}{4}}v$ is a solution of the elliptic problem (1.6) given by the family of algebraic solutions (1.5) up to spatial translations in $\mathbb{R}^d$.

Since the operator $-\Delta + |x|^2 - \omega_*$ is positive in X, we can define the unique solution of the following inhomogeneous equation

$$-\Delta u + (|x|^2 - \omega_*)u = U_\varepsilon^{\frac{d+2}{d-2}}, \quad u \in X, \tag{1.9}$$

denoted by PU_ε. Moreover, since $U_\varepsilon > 0$, by the positivity of the operator $-\Delta + |x|^2 - \omega_*$ and the maximum principle, we know that $PU_\varepsilon > 0$ in $\mathbb{R}^d$.

Let G be the Green function of the positive operator $-\Delta + |x|^2 - \omega_*$,

$$\begin{cases} -\Delta G + (|x|^2 - \omega_*)G = (d-2)|\mathbb{S}^{d-1}|\delta_0 & \text{in } \mathbb{R}^d, \\ G(x) \to 0 & \text{as } |x| \to +\infty, \end{cases} \tag{1.10}$$

where δ_0 is the Dirac measure supported at $x = 0$ and $|\mathbb{S}^{d-1}|$ is the Lebesgue measure of the unit sphere in $\mathbb{R}^d$. This gives the unique normalization of the Green function such that $G = |x|^{2-d} - H$, where H is a regular part of G satisfying the following equation:

$$\begin{cases} -\Delta H + (|x|^2 - \omega_*)H = (|x|^2 - \omega_*)|x|^{2-d} & \text{in } \mathbb{R}^d, \\ H(x) \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{1.11}$$

By uniqueness of solutions to the elliptic problems (1.10) and (1.11), G and H are radially symmetric. Our main results in the energy-critical case $p = \frac{2d}{d-2}$ can be stated as follows.

Theorem 1.1. *Let $d \ge 3$, $p = \frac{2d}{d-2}$, and u_ω be the ground state solution of the stationary equation (1.1) for $\omega \in (\omega_*, d)$, where ω_* is given by (1.2). There exsts $\varepsilon_\omega > 0$ such that*

- $u_\omega = PU_{\varepsilon_\omega} + \hat{u}_\omega$ *for* $3 \leq d \leq 6$
- $u_\omega = U_{\varepsilon_\omega} + \hat{u}_\omega$ *for* $d \geq 7$,

with $\varepsilon_\omega \to 0$ *and* $\|\hat{u}_\omega\|_X \to 0$ *as* $\omega \to \omega_*$. *Moreover,* $\mathcal{I}_\omega < \mathcal{S}$ *for* $\omega \in (\omega_*, d)$, $\mathcal{I}_\omega \mapsto \mathcal{S}$ *as* $\omega \to \omega_*$, *and there exist positive constants* a_d *and* b_d *which only depend on the dimension* d, *such that the concentration rate* ε_ω *and the ground state energy* $\mathcal{I}_\omega$ *satisfy*

- *for* $d = 3$
$$a_d = \lim_{\omega\to 1} \frac{\varepsilon_\omega}{(\omega-1)\|G\|^2_{L^2(\mathbb{R}^3)}}, \qquad b_d = \lim_{\omega\to 1} \frac{\mathcal{S}-\mathcal{I}_\omega}{((\omega-1)\|G\|^2_{L^2(\mathbb{R}^3)})^2},$$
- *for* $d = 4$,
$$a_d = \lim_{\omega\to 0} \frac{\omega|\log\varepsilon_\omega|}{H(0)\|U\|^3_{L^3(\mathbb{R}^4)}}, \qquad b_d = \lim_{\omega\to 0} \frac{\omega|\log(\mathcal{S}-\mathcal{I}_\omega) - \log(c_d H(0)\|U\|^3_{L^3(\mathbb{R}^4)})|}{H(0)\|U\|^3_{L^3(\mathbb{R}^4)}},$$
- *for* $d = 5$,
$$a_d = \lim_{\omega\to 0} \frac{H(0)\|U\|^{\frac{7}{3}}_{L^{\frac{7}{3}}(\mathbb{R}^d)}\varepsilon_\omega}{\|U\|^2_{L^2(\mathbb{R}^5)}\omega}, \qquad b_d = \lim_{\omega\to 0} \frac{(H(0)\|U\|^{\frac{7}{3}}_{L^{\frac{7}{3}}(\mathbb{R}^d)})^2(\mathcal{S}-\mathcal{I}_\omega)}{\|U\|^6_{L^2(\mathbb{R}^5)}\omega^3},$$
- *for* $d = 6$,
$$a_d = \lim_{\omega\to 0} \frac{|\log\omega|\varepsilon_\omega^2}{\|U\|^2_{L^2(\mathbb{R}^6)}\omega}, \qquad b_d = \lim_{\omega\to 0} \frac{|\log\omega|(\mathcal{S}-\mathcal{I}_\omega)}{\|U\|^4_{L^2(\mathbb{R}^6)}\omega^2},$$
- *for* $d \geq 7$,
$$\frac{1}{2} = \lim_{\omega\to 0} \frac{\||x|U\|^2_{L^2(\mathbb{R}^d)}\varepsilon_\omega^2}{\|U\|^2_{L^2(\mathbb{R}^d)}\omega}, \qquad b_d = \lim_{\omega\to 0} \frac{\||x|U\|^2_{L^2(\mathbb{R}^d)}(\mathcal{S}-\mathcal{I}_\omega)}{\|U\|^4_{L^2(\mathbb{R}^d)}\omega^2}.$$

Remark 1.1. *To our best knowledge, Theorem 1.1 is the first result on the concentration phenomenon and the asymptotic behavior of solutions of the stationary Gross–Pitaevskii equation* (1.1) *in the energy-critical case* $p = \frac{2d}{d-2}$. *It is worth pointing out that a formal and brief calculation on the upper bounds of* $\mathcal{I}_\omega$ *is obtained in [33, Section 5] to ensure the existence of minimizers of* $\mathcal{I}_\omega$. *These upper bounds of* $\mathcal{I}_\omega$ *are calculated in a standard way by choosing the Aubin-Talanti bubbles as test functions of* $\mathcal{I}_\omega$, *as that in [6]. However, the main difficulty in proving Theorem 1.1 is to obtain a good lower bound of* $\mathcal{I}_\omega$ *which will match the upper bound generated by the Aubin-Talanti bubbles up to the leading order terms. To achieve this, we need to further employ the ideas in literature [7, 12, 13, 15, 16, 20, 21, 31, 32], that is, splitting of* u_ω *into two parts in* X *and estimating of these two parts precisely up to the leading order term. We remark that, due to the growth of the harmonic potential at infinity and the unboundedness of* $\mathbb{R}^d$, *the regular part of the Green function of the operator* $-\Delta + |x|^2 - \omega_*$ *is no longer bounded for all* $d \geq 3$. *Thus, we need to modify the arguments of the proofs in a nontrivial way to capture the leading order terms of* ε_ω *and* $\mathcal{I}_\omega$ *for all* $d \geq 3$, *which also makes the concentration phenomenon of positive solutions of the stationary equation* (1.1) *to be more complicated than that of the Dirichlet problem* (1.3).

Remark 1.2. *One can use parameter ε in the family of algebraic solitons (1.5) to parameterize the family of the ground states (ω, u_ω) of the stationary equation (1.1). It follows from Theorem 1.1 that the asymptotic behavior of the mapping $\varepsilon \mapsto \omega$ as $\varepsilon \to 0$ depends on the dimension $d \geq 3$ and satisfies*

$$\omega - \omega_* \sim \begin{cases} \varepsilon & \text{for } d = 3, \\ |\log \varepsilon|^{-1} & \text{for } d = 4, \\ \varepsilon & \text{for } d = 5, \\ \varepsilon^2 |\log \varepsilon| & \text{for } d = 6, \\ \varepsilon^2 & \text{for } d \geq 7. \end{cases}$$

In the energy-supercritical case $p > \frac{2d}{d-2}$, we will fix $p = 4$ to simplify the computations similarly to what was adopted in [4, 29]. The energy-supercritical case for $p = 4$ corresponds to $d \geq 5$ and the stationary equation (1.1) is reduced to

$$-\Delta u + |x|^2 u = \omega u + u^3, \quad \text{in } \mathbb{R}^d. \tag{1.12}$$

It has been proved in [35] (see also [4] for a different proof), that there exists a singular radial solution u_∞ of the stationary equation (1.12) for some $\omega_\infty \in (d-4, d)$ satisfying

$$u_\infty(x) = \frac{\sqrt{d-3}}{|x|} \left[1 + \mathcal{O}(|x|^2)\right] \quad \text{as } |x| \to 0. \tag{1.13}$$

Moreover, by [4, Theorem 1.1], for every $b > 0$, there exists a positive radial solution u_b of the stationary equation (1.12) for some $\omega_b \in (d-4, d)$ satisfying $u_b(0) = b$. By [35, Theorem 1.2], it is known that $u_b \to u_\infty$ strongly in Σ and $\omega_b \to \omega_\infty$ as $b \to +\infty$. The precise asymptotic behavior of ω_b as $b \to +\infty$ is obtained in [4, Theorem 1.3] under some nondegeneracy assumptions. By [4, Theorem 1.3], ω_b is oscillatory around ω_∞ as $b \to +\infty$ for $5 \leq d \leq 12$ and ω_b converges to ω_∞ monotonically as $b \to +\infty$ for $d \geq 13$. These results suggest that the Morse index of u_∞ in the class of radial functions is infinite for $5 \leq d \leq 12$ and finite for $d \geq 13$, where the Morse index of u_∞ is defined as follows.

Definition 1.2. *Let u_∞ be the singular radial solution of the stationary equation (1.12) for some $\omega_\infty \in (d-4, d)$ satisfying (1.13) and consider the linearized operator*

$$L_\infty := -\Delta + |x|^2 - \omega_\infty - 3u_\infty^2$$

in $X_{\rm rad} := \{f \in X : f \text{ is radial}\}$. The Morse index of u_∞ denoted by $\mathfrak{m}(u_\infty)$ is the number of negative eigenvalues of L_∞ in $X_{\rm rad}$.

It was proven in [29] that the Morse index of u_b for large b coincides with the Morse index of u_∞ for $d \geq 13$. It was conjectured in [29] based on numerical evidences that $\mathfrak{m}(u_\infty) = 1$ for $d \geq 13$. *The second purpose of this paper* is to estimate $\mathfrak{m}(u_\infty)$ which is done in the following main result.

Theorem 1.2. *Let $p = 4$, $d \geq 5$ and u_∞ be the singular radial solution of the stationary equation* (1.12) *for some $\omega_\infty \in (d-4, d)$ satisfying (1.13). Then*

$$\mathfrak{m}(u_\infty) = \begin{cases} \infty, & 5 \leq d \leq 12, \\ 1 \text{ or } 2, & 13 \leq d \leq 15, \\ 1, & d \geq 16. \end{cases}$$

Remark 1.3.

(*i*) *To prove Theorem 1.2 for $5 \leq d \leq 12$, we shall mainly follow the ideas in [19]. The oscillation of ω_b around ω_∞ as $b \to +\infty$ is obtained in [4, Theorem 1.3] under some nondegeneracy assumptions, which is hard to verify. In order to avoid making these nondegeneracy assumptions, we need to modify the arguments in [19].*

(*ii*) *In proving Theorem 1.2 for $d \geq 13$, we consider the limiting spectral problem*

$$-\Delta u + |x|^2 u - \frac{3(d-3)}{|x|^2} u = \sigma u, \quad u \in X_{\text{rad}}, \tag{1.14}$$

whose eigenvalues $\{\sigma_n\}_{n\in\mathbb{N}}$ are completely known in the literature from the confluent hypergeometric equation [39]. We compare $\omega_\infty + 3u_\infty^2$ and $\sigma_3 + \frac{3(d-3)}{r^2}$ to control $\mathfrak{m}(u_\infty)$ by the Morse index of the radial eigenfunctions of the spectral problem (1.14)*. As a by-product, we also prove that u_∞ is nondegenerate for $d \geq 16$, this avoids the nondegeneracy assumptions of [29]. See Remark 3.3 for more details.*

Notations. Throughout this paper, C and C' are indiscriminately used to denote various positive constants. Notation $a \lesssim b$ means that there exists $C > 0$ such that $a \leq Cb$. Notation $a = \mathcal{O}(b)$ means that there exist $C, C' > 0$ such that $C'b \leq a \leq Cb$. Notation $a = o(b)$ means that $\lim\limits_{b\to 0} a/b = 0$. Notation $a \sim b$ as $b \to 0$ means that $\lim\limits_{b\to 0} a/b = 1$ (the same convention is used if $b \to \infty$).

2. The energy-critical case

2.1. **Preliminaries.** It has been proved in [33, Section 5], without the statement of theorems, that $\mathcal{I}_\omega$ is attained for $\omega \in (\omega_*, d)$. On the other hand, by the Pohozaev identity, see, e.g., [4, Proposition 2.2], we know that the stationary equation (1.1) has no solutions in Σ for $\omega \leq 0$ which implies that $\mathcal{I}_\omega$ can not be attained for $\omega \leq 0$. Moreover, since d is the first eigenvalue of $-\Delta + |x|^2$ in X, by multiplying (1.1) with the first eigenfunction of the operator $-\Delta + |x|^2$ on both sides and integrating by parts, see, e.g., [4, Proposition 2.1], we know that the stationary equation (1.1) has no positive solutions for $\omega \geq d$. This implies that $\mathcal{I}_\omega$ can not be attained for $\omega \geq d$ either since minimizers of the variational problem (1.4) are positive and radially symmetric. In addition, by [34, Theorem 3] or [36, Theorem 7], the stationary equation (1.1) also has no positive solutions for $\omega \leq 1$ in the case of $d = 3$. Thus, we know that $\mathcal{I}_\omega$ is attained if and only if $\omega \in (\omega_*, d)$.

Since $\mathcal{I}_\omega$ is attained for $\omega \in (\omega_*, d)$, it can be proven by a standard way that $\mathcal{I}_\omega$ is strictly decreasing for $\omega \in [\omega_*, d]$ with $\mathcal{I}_{\omega=\omega_*} = \mathcal{S}$ and $\mathcal{I}_{\omega=d} = 0$, where $\mathcal{S}$ is the best constant of the Sobolev embedding given by the variational problem (1.8). The monotone property was first pointed out by Brezís and Nirenberg in [6, Remark 1.5]. The detailed proofs were recently given in [9, Lemma 2.1] and [41, Lemma 3.3]. Hence, we have

$$0<\mathcal{I}_\omega<\mathcal{S}=\mathcal{I}_{\omega_*}\quad \text{for all } \omega\in(\omega_*,d). \tag{2.1}$$

Let v_ω be the minimizer of the variational problem (1.4) for $\omega\in(\omega_*,d)$. Then, $u_\omega:=(\mathcal{I}_\omega)^{\frac{d-2}{4}}v_\omega$ is the ground state solution of the stationary equation (1.1). Since we are interested in $\omega\to\omega_*$ with $\omega_*<d$, it is standard to show that $\{u_\omega\}$ is bounded in X. By the compactness of the embedding from X to $L^2(\mathbb{R}^d)$ due to the harmonic potential $|x|^2$, we may assume that there exists $u_*\in\Sigma$ such that $u_\omega\rightharpoonup u_*$ weakly in Σ and $u_\omega\to u_*$ strongly in $L^2(\mathbb{R}^d)$ as $\omega\to\omega_*$. We claim that $u_*=0$. Indeed, if $u_*\neq 0$, then $u_*\in\Sigma$ satisfies

$$-\Delta u_*+|x|^2u_*=\omega_*u_*+|u_*|^{\frac{4}{d-2}}u_* \tag{2.2}$$

in the weak sense, which, together with (2.1), implies

$$I_{\omega_*}(u_*)=\|u_*\|^{\frac{4}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}\leq\|u_\omega\|^{\frac{4}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}+o(1)=I_\omega(u_\omega)+o(1)\leq I_{\omega_*}(u_*)+o(1). \tag{2.3}$$

Thus, u_* corresponds to the minimizer v_* with $I_{\omega_*}(v_*)=\mathcal{I}_{\omega_*}$ by $u_*:=(\mathcal{I}_{\omega_*})^{\frac{d-2}{4}}v_*$ so that u_* is positive and radially symmetric. This contradicts the previously reviewed results, from which no positive and radially symmetric solution of the stationary equation (2.2) exists in Σ with ω_* given by (1.2). Therefore, we must have $u_*=0$ and $u_\omega\rightharpoonup 0$ weakly in X and $u_\omega\to 0$ strongly in $L^2(\mathbb{R}^d)$ as $\omega\to\omega_*$. Moreover, since v_ω be the minimizer of the variational problem (1.4) and $u_\omega=(\mathcal{I}_\omega)^{\frac{d-2}{4}}v_\omega$, by (2.1) and (2.3),

$$\|u_\omega\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}=(\mathcal{I}_\omega)^{\frac{d}{2}}\|v_\omega\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}=\mathcal{S}^{\frac{d}{2}}+o(1)\quad\text{as}\quad\omega\to\omega_*. \tag{2.4}$$

Since u_ω is also the ground state solution of the stationary equation (1.1), by multiplying (1.1) with u_ω on both sides and integrating by parts, we also have

$$\|u_\omega\|^2_X=\omega\|u_\omega\|^2_{L^2(\mathbb{R}^d)}+\|u_\omega\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}=\mathcal{S}^{\frac{d}{2}}+o(1)\quad\text{as}\quad\omega\to\omega_* \tag{2.5}$$

since $u_\omega\to 0$ strongly in $L^2(\mathbb{R}^d)$ as $\omega\to\omega_*$. On the other hand, it follows from (1.7), (2.4), and (2.5) that

$$\|\nabla u_\omega\|^2_{L^2(\mathbb{R}^d)}=\mathcal{S}\|u_\omega\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}=\mathcal{S}^{1+\frac{d-2}{2}}+o(1)=\mathcal{S}^{\frac{d}{2}}+o(1)\quad\text{as}\quad\omega\to\omega_*,$$

which implies that

$$\|xu_\omega\|^2_{L^2(\mathbb{R}^d)}=o(1)\quad\text{as}\quad\omega\to\omega_*. \tag{2.6}$$

2.2. **Expansions of u_ω.** Since u_ω is a ground state of the stationary equation (1.1) related to a minimizer of the variational problem (1.4), the moving-plane method (cf. [18]) or the Schwarz symmetrization (cf. [42]) imply that u_ω is radial, positive and strictly decreasing in $r = |x|$. The following lemma clarifies the construction of PU_ε from solutions of the inhomogeneous equation (1.9).

Lemma 2.1. *Let $3 \le d \le 6$, then*

$$PU_\varepsilon = U_\varepsilon - \varepsilon^{\frac{d-2}{2}}[d(d-2)]^{\frac{d-2}{4}} H - \eta_\varepsilon, \quad |x| \lesssim 1 \tag{2.7}$$

and

$$PU_\varepsilon(x) \lesssim \varepsilon^{\frac{d+2}{2}} |x|^{-(4+d)} \quad \text{for } |x| \gtrsim 1, \tag{2.8}$$

where H is defined by (1.11) and the correction term η_ε satisfies

$$\|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \lesssim \varepsilon^{\frac{d+2}{2}} \quad \text{for } 3 \le d \le 5, \tag{2.9}$$

and

$$\|\eta_\varepsilon\|_{W^{2,\frac{3}{2}}(\mathbb{R}^6)} \lesssim \varepsilon^4, \quad \text{for } d = 6. \tag{2.10}$$

Proof. Since it follows from (1.5) that

$$U_\varepsilon(x) \sim \varepsilon^{\frac{d-2}{2}} |x|^{2-d} \quad \text{for } |x| \gtrsim 1, \tag{2.11}$$

the classical L^p-theory of elliptic equations and the Sobolev embedding theorem imply that the unique solution of the inhomogeneous equation (1.9) exists and satisfies $PU_\varepsilon \in L^\infty_{loc}(\mathbb{R}^d \backslash \{0\})$. In particular, $PU_\varepsilon \lesssim 1$ for $|x| \gtrsim 1$ and $\varepsilon \lesssim 1$. Since

$$-\Delta |x|^{-(4+d)} + (|x|^2 - \omega_*)|x|^{-(4+d)} \sim |x|^{-(2+d)} \quad \text{for } |x| \gtrsim 1,$$

it follows from (2.11) that $\varepsilon^{\frac{d+2}{2}}|x|^{-(4+d)}$ is a supersolution of equation (1.9) for $|x| \gtrsim 1$. Now, by the fact that $PU_\varepsilon \lesssim 1$ for $|x| \gtrsim 1$ and $\varepsilon \lesssim 1$, the fact that $PU_\varepsilon \to 0$ and $\varepsilon^{\frac{d+2}{2}}|x|^{-(4+d)} \to 0$ as $|x| \to +\infty$ and the maximum principle, we obtain (2.8).

To obtain (2.7), we write

$$\varphi_\varepsilon := U_\varepsilon - PU_\varepsilon, \tag{2.12}$$

then by (1.6) and (1.9), φ_ε is the unique solution of the following equation:

$$-\Delta u + (|x|^2 - \omega_*)u = (|x|^2 - \omega_*)U_\varepsilon, \quad u \in X. \tag{2.13}$$

By (1.2) and the maximum principle, $\varphi_\varepsilon > 0$ in $\mathbb{R}^d$ for $d \ge 4$. For $d = 3$, since $PU_\varepsilon > 0$ in $\mathbb{R}^3$, there exists a unique $r_0 > 0$ such that φ_ε is strictly increasing with respect to $r = |x|$ in $[0, r_0)$ and is strictly decreasing in $[r_0, +\infty)$. Moreover, it follows from (1.10) by using the maximum principle that

$$|G(x)| \lesssim e^{-\sigma|x|^2} \quad \text{for some } \sigma > 0, \tag{2.14}$$

so that $H(x) = |x|^{2-d} + \mathcal{O}(e^{-\sigma|x|^2})$ as $|x| \to \infty$. Thus, by (1.2) and the classical L^p-theory of elliptic equations, we know that $H \in W^{2,s}_{loc}(\mathbb{R}^d)$ for $1 < s < 3$ in the case of $d = 3$, $1 < s < +\infty$ in the case of $d = 4$ and $1 < s < \frac{d}{d-4}$ in the case of $d \ge 5$. It follows from the

Sobolev embedding theorem that $H \in L^\infty(\mathbb{R}^d) \cap C^\alpha_{loc}(\mathbb{R}^d)$ for $3 \le d \le 5$ with $\forall \alpha \in (0,1)$ and $H \in L^{\frac{3s}{3-s}}_{loc}(\mathbb{R}^6)$ for $1 < s < 3$. Next we define

$$\eta_\varepsilon := \varphi_\varepsilon - \varepsilon^{\frac{d-2}{2}}[d(d-2)]^{\frac{d-2}{4}} H. \tag{2.15}$$

It follows from (1.11) and (2.13) that η_ε is the unique solution of the following equation:

$$\begin{cases} -\Delta u + (|x|^2 - \omega_*)u = \varepsilon^{\frac{d-2}{2}}[d(d-2)]^{\frac{d-2}{4}}(|x|^2 - \omega_*)g_\varepsilon & \text{in } \mathbb{R}^d, \\ u(x) \to 0 & \text{as } |x| \to +\infty, \end{cases}$$

where $g_\varepsilon = (\varepsilon^2 + |x|^2)^{\frac{2-d}{2}} - |x|^{2-d}$ satisfies

$$g_\varepsilon(x) \sim \begin{cases} -|x|^{2-d}, & |x| \le \frac{\varepsilon}{\sqrt{2}}, \\ -\varepsilon^2 |x|^{-d}, & |x| \ge \frac{\varepsilon}{\sqrt{2}}. \end{cases} \tag{2.16}$$

As in the previous estimates, by the classical L^p-theory of elliptic equations, the Sobolev embedding theorem and the maximum principle, we obtain

$$|\eta_\varepsilon(x)| \lesssim \varepsilon^{\frac{d+2}{2}} |x|^{-(2+d)} \quad \text{for } |x| \gtrsim 1. \tag{2.17}$$

Let $h_\varepsilon = \varepsilon^{\frac{d-2}{2}}[d(d-2)]^{\frac{d-2}{4}}(|x|^2 - \omega_*)g_\varepsilon$. It follows from (2.16) that

$$\|h_\varepsilon\|_{L^s_{loc}(\mathbb{R}^d)} \lesssim \begin{cases} \varepsilon^{\frac{3}{s} - \frac{1}{2}}, & d = 3, \\ \varepsilon^{2 + \frac{d}{s} - \frac{d-2}{2}}, & 4 \le d \le 6. \end{cases} \tag{2.18}$$

Thus, by (2.17) and the classical L^p-theory of elliptic equations, we know that $\eta_\varepsilon \in W^{2,s}(\mathbb{R}^d)$ for $1 < s < 3$ in the case of $d = 3$, $1 < s < +\infty$ in the case of $d = 4$ and $1 < s < \frac{d}{d-4}$ in the case of $d \ge 5$. The Sobolev embedding theorem implies that $\eta_\varepsilon \in L^\infty(\mathbb{R}^d) \cap C^\alpha_{loc}(\mathbb{R}^d)$ for $3 \le d \le 5$ with $\forall \alpha \in (0,1)$ and $\eta_\varepsilon \in L^{\frac{3s}{3-s}}(\mathbb{R}^6)$ for $1 < s < 3$. Representation (2.7) follows from (2.12) and (2.15). Estimates (2.9) and (2.10) follow from (2.17), (2.18), the classical L^p-theory and the Sobolev embedding theorem by choosing $s = 2$ for $d = 3$ and $s = \frac{d}{d-2}$ for $d = 4, 5$. □

By (2.6) and Lions' theorem (cf. [40, Theorem 1.41]), there exists $\{\varepsilon_\omega\} \subset \mathbb{R}_+$ such that $u_\omega \to U_{\varepsilon_\omega}$ strongly in $D^{1,2}(\mathbb{R}^d)$ as $\omega \to \omega_*$. Since $u_\omega \to 0$ strongly in $L^2_{loc}(\mathbb{R}^d)$ as $\omega \to \omega_*$, it is easy to see that $\varepsilon_\omega \to 0$ as $\omega \to \omega_*$. The following lemma specifies a precise decomposition of u_ω near U_{ε_ω}.

Lemma 2.2. *As $\omega \to \omega_*$, there exists $\varepsilon_\omega > 0$ such that*

$$u_\omega = \begin{cases} PU_{\varepsilon_\omega} + \hat{u}_\omega & \text{for } 3 \le d \le 6, \\ U_{\varepsilon_\omega} + \hat{u}_\omega & \text{for } d \ge 7, \end{cases} \tag{2.19}$$

where $\varepsilon_\omega \to 0$ and $\hat{u}_\omega \to 0$ in X as $\omega \to \omega_*$ and where $\hat{u}_\omega \in \mathcal{M}_\omega^\perp$ defined by

$$\mathcal{M}_\omega = \begin{cases} \{PU_{\varepsilon_\omega}, \partial_{\varepsilon_\omega} PU_{\varepsilon_\omega}, \partial_{x_1} PU_{\varepsilon_\omega}, \cdots, \partial_{x_d} PU_{\varepsilon_\omega}\} & \text{for } 3 \le d \le 6, \\ \{U_{\varepsilon_\omega}, \partial_{\varepsilon_\omega} U_{\varepsilon_\omega}, \partial_{x_1} U_{\varepsilon_\omega}, \cdots, \partial_{x_d} U_{\varepsilon_\omega}\} & \text{for } d \ge 7, \end{cases}$$

and the orthogonality holds simultaneously in X and $L^2(\mathbb{R}^d)$.

Proof. It follows from the explicit formula (1.5) for $d \geq 7$ that

$$\int_{\mathbb{R}^d} |x|^2 U_\varepsilon^2 dx = \varepsilon^4 \int_{\mathbb{R}^d} |x|^2 U^2 dx, \quad \int_{\mathbb{R}^d} U_\varepsilon^2 dx = \varepsilon^2 \int_{\mathbb{R}^d} U^2 dx. \tag{2.20}$$

Moreover, for all $d \geq 3$,

$$\int_{\mathbb{R}^d} U_\varepsilon^q dx = \varepsilon^{d - \frac{(d-2)q}{2}} \int_{\mathbb{R}^d} U^q dx \quad \text{for } q > \frac{d}{d-2} \tag{2.21}$$

and

$$\int_{B_1} U_\varepsilon^{\frac{d}{d-2}} dx \sim \varepsilon^{\frac{d}{2}} |\log \varepsilon|. \tag{2.22}$$

Thus, by the fact that $u_\omega \to U_{\varepsilon_\omega}$ strongly in $D^{1,2}(\mathbb{R}^d)$ as $\omega \to \omega_*$ and (2.6), we have

$$\|u_\omega - U_{\varepsilon_\omega}\|_X^2 \to 0 \quad \text{as } \omega \to \omega_*. \tag{2.23}$$

On the other hand, since $H, \eta_{\varepsilon_\omega} \in L^\infty(\mathbb{R}^d) \cap C^\alpha_{loc}(\mathbb{R}^d)$ for $3 \leq d \leq 5$ with $\forall \alpha \in (0,1)$ and $H, \eta_{\varepsilon_\omega} \in L^{\frac{3s}{3-s}}(\mathbb{R}^6)$ for $1 < s < 3$ by Lemma 2.1, it follows from (2.7), (2.8), (2.9), and (2.10) that $PU_{\varepsilon_\omega} \to U_{\varepsilon_\omega}$ strongly in $D^{1,2}(\mathbb{R}^d)$ as $\omega \to \omega_*$. Thus, it is also easy to see that

$$\|u_\omega - PU_{\varepsilon_\omega}\|_X^2 \to 0 \quad \text{as } \omega \to \omega_*. \tag{2.24}$$

Now, we define

$$e(\omega) := \begin{cases} \inf_{\varepsilon \in \mathbb{R}_+} \|u_\omega - PU_\varepsilon\|_X^2 & \text{for } 3 \leq d \leq 6, \\ \inf_{\varepsilon \in \mathbb{R}_+} \|u_\omega - U_\varepsilon\|_X^2 & \text{for } d \geq 7. \end{cases}$$

By (2.23) and (2.24), it is standard (cf. [3, 16, 32]) to show that $e(\omega) = o_\omega(1)$ is attained by some ε_ω satisfying $\varepsilon_\omega \to 0$ as $\omega \to \omega_*$, which implies that (2.19) hold with $\hat{u}_\omega \to 0$ in X as $\omega \to \omega_*$. The orthogonality conditions in X for $\hat{u}_\omega \in \mathcal{M}_\omega^\perp$ are obtained from

$$\frac{d}{d\varepsilon} \|u_\omega - PU_\varepsilon\|_X^2 |_{\varepsilon = \varepsilon_\omega} = 0 \quad \text{for } 3 \leq d \leq 6, \qquad \frac{d}{d\varepsilon} \|u_\omega - U_\varepsilon\|_X^2 |_{\varepsilon = \varepsilon_\omega} = 0 \quad \text{for } d \geq 7.$$

The orthogonality conditions in $L^2(\mathbb{R}^d)$ follows from the fact that the eigenfunctions of $-\Delta + |x|^2$ is a orthogonal basis of $L^2(\mathbb{R}^d)$. □

2.3. **Estimates on $\hat{u}_\omega$.** By [32, Appendix D],

$$\int_{\mathbb{R}^d} \left(|\nabla v|^2 - (2^* - 1) U_{\varepsilon_\omega}^{2^*-2} |v|^2 \right) dx \geq \frac{4}{d+4} \int_{\mathbb{R}^d} |v|^2 dx \tag{2.25}$$

for all $v \in D^{1,2}(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} \nabla v \nabla U_{\varepsilon_\omega} dx = \int_{\mathbb{R}^d} \nabla v \nabla \partial_{\varepsilon_\omega} U_{\varepsilon_\omega} dx = \int_{\mathbb{R}^d} \nabla v \nabla \partial_{x_l} U_{\varepsilon_\omega} dx = 0$$

where $l = 1, 2, \cdots, d$. By Lemma 2.2, we have

$$\int_{\mathbb{R}^d} \nabla \hat{u}_\omega \nabla U_{\varepsilon_\omega} dx = \int_{\mathbb{R}^d} \nabla \hat{u}_\omega \nabla \partial_{\varepsilon_\omega} U_{\varepsilon_\omega} dx = \int_{\mathbb{R}^d} \nabla \hat{u}_\omega \nabla \partial_{x_l} U_{\varepsilon_\omega} dx = o(1)$$

for all $l = 1, 2, \cdots, d$ as $\omega \to \omega_*$. Thus,

$$\int_{\mathbb{R}^d} \left(|\nabla \hat{u}_\omega|^2 - (2^* - 1)U_{\varepsilon_\omega}^{2^*-2}|\hat{u}_\omega|^2\right) dx \geq \left(\frac{4}{d+4} + o(1)\right) \int_{\mathbb{R}^d} |\hat{u}_\omega|^2 dx \tag{2.26}$$

for $d \geq 7$. On the other hand, by (1.2), (2.7)–(2.10), (2.25) and $\hat{u}_\omega \in \mathcal{M}_\omega^\perp$, it is also standard (cf. [32, Appendix D]) to show that

$$\int_{\mathbb{R}^d} \left(|\nabla \hat{u}_\omega|^2 + (|x|^2 - \omega_*)|\hat{u}_\omega|^2 - (2^* - 1)PU_{\varepsilon_\omega}^{2^*-2}|\hat{u}_\omega|^2\right) dx \gtrsim \int_{\mathbb{R}^d} |\hat{u}_\omega|^2 dx \tag{2.27}$$

for $3 \leq d \leq 6$. For $d = 3$, we need to use the fact that $\omega_* = 1$ and $\lambda = 3$ is the first eigenvalue of the operator $-\Delta + |x|^2$ in $L^2(\mathbb{R}^3)$.

The following lemma gives the asymptotic estimate on the X norm of $\hat{u}_\omega$. The proofs are simpler for $d \geq 7$ but get more technically involved for $3 \leq d \leq 6$.

Lemma 2.3. *Let $d \geq 3$, Then as $\omega \to \omega_*$,*

$$\|\hat{u}_\omega\|_X \lesssim \begin{cases} (\omega - 1)\varepsilon^{\frac{1}{2}} + \varepsilon & \text{for } d = 3, \\ \omega \varepsilon_\omega^{\frac{d}{2}-\sigma} + \varepsilon_\omega^{\frac{d-2}{2}+\sigma} & \text{for } 4 \leq d \leq 6, \\ \omega \varepsilon_\omega^2 & \text{for } d \geq 7, \end{cases} \tag{2.28}$$

where $\sigma > 0$ is a small constant.

Proof. For $3 \leq d \leq 6$, we obtain from (1.1), (1.9), and (2.19) that $\hat{u}_\omega$ satisfies

$$\begin{cases} -\Delta \hat{u}_\omega + (|x|^2 - \omega_*)\hat{u}_\omega - \frac{d+2}{d-2} PU_{\varepsilon_\omega}^{\frac{4}{d-2}} \hat{u}_\omega = E_\omega + N_\omega(\hat{u}_\omega) & \text{in } \mathbb{R}^d, \\ \hat{u}_\omega(x) \to 0 & \text{as } |x| \to 0 \end{cases} \tag{2.29}$$

where the nonhomogeneous term is

$$E_\omega := (\omega - \omega_*)PU_{\varepsilon_\omega} + PU_{\varepsilon_\omega}^{\frac{d+2}{d-2}} - U_{\varepsilon_\omega}^{\frac{d+2}{d-2}}$$

and the nonlinear term satisfies

$$|N_\omega(\hat{u}_\omega)| \lesssim PU_{\varepsilon_\omega}^{\frac{6-d}{d-2}} |\hat{u}_\omega|^2 + |\hat{u}_\omega|^{\frac{d+2}{d-2}}. \tag{2.30}$$

It follows from (2.8) and (2.11) that

$$|E_\omega| \lesssim \varepsilon_\omega^{\frac{d+2}{2}} \left((\omega - \omega_*)|x|^{-(4+d)} + |x|^{-(2+d)}\right) \quad \text{for } |x| \gtrsim 1. \tag{2.31}$$

For $|x| \lesssim 1$, it follows from (2.7) for $4 \leq d \leq 6$ (for which $\omega_* = 0$) that

$$\begin{aligned} |E_\omega| \lesssim\ & \omega U_{\varepsilon_\omega} + U_{\varepsilon_\omega}^{\frac{4}{d-2}} (\varepsilon_\omega^{\frac{d-2}{2}} |H| + |\eta_{\varepsilon_\omega}|) + U_{\varepsilon_\omega}^{\frac{6-d}{d-2}} (\varepsilon_\omega^{d-2} |H|^2 + |\eta_{\varepsilon_\omega}|^2) \\ & + \varepsilon_\omega^{\frac{d+2}{2}} |H|^{\frac{d+2}{d-2}} + |\eta_{\varepsilon_\omega}|^{\frac{d+2}{d-2}}, \end{aligned}$$

where $\varphi_{\varepsilon_\omega} > 0$ is given by (2.12) and (2.15). By Lemma 2.1, we obtain from (2.21) for $4 \le d \le 5$ and $R > 0$ sufficiently large,

$$\begin{aligned}
\left|\int_{B_R} E_\omega \hat{u}_\omega dx\right| \lesssim & \omega \|\hat{u}_\omega\|_{L^{\frac{2d}{d-2}}(B_R)} \|U_{\varepsilon_\omega}\|_{L^{\frac{d}{d-2}+\sigma}(B_R)} + \int_{B_R} U_{\varepsilon_\omega}^{\frac{4}{d-2}} (\varepsilon_\omega^{\frac{d-2}{2}} |H| + |\eta_{\varepsilon_\omega}|) \hat{u}_\omega dx \\
& + \int_{B_R} U_{\varepsilon_\omega}^{\frac{6-d}{d-2}} (\varepsilon_\omega^{d-2} |H|^2 + |\eta_{\varepsilon_\omega}|^2) \hat{u}_\omega dx + \int_{B_R} (\varepsilon_\omega^{\frac{d+2}{2}} |H|^{\frac{d+2}{d-2}} + |\eta_{\varepsilon_\omega}|^{\frac{d+2}{d-2}}) \hat{u}_\omega dx \\
\lesssim & (\omega \|U_{\varepsilon_\omega}\|_{L^{\frac{d}{d-2}+\sigma}(B_R)} + \varepsilon_\omega^{\frac{d+2}{2}}) \|\hat{u}_\omega\|_{L^{\frac{2d}{d-2}}(B_R)} \\
& + \int_{B_R} U_{\varepsilon_\omega}^{\frac{4}{d-2}} (\varepsilon_\omega^{\frac{d-2}{2}} |H| + |\eta_{\varepsilon_\omega}|) \hat{u}_\omega dx + \int_{B_R} U_{\varepsilon_\omega}^{\frac{6-d}{d-2}} (\varepsilon_\omega^{d-2} |H|^2 + |\eta_{\varepsilon_\omega}|^2) \hat{u}_\omega dx \\
\lesssim & (\omega \varepsilon_\omega^{\frac{d}{2}-\sigma} + \varepsilon_\omega^{\frac{d+2}{2}}) \|\hat{u}_\omega\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} + |I|, \qquad (2.32)
\end{aligned}$$

where

$$I = \int_{B_R} U_{\varepsilon_\omega}^{\frac{4}{d-2}} (\varepsilon_\omega^{\frac{d-2}{2}} |H| + |\eta_{\varepsilon_\omega}|) \hat{u}_\omega dx + \int_{B_R} U_{\varepsilon_\omega}^{\frac{6-d}{d-2}} (\varepsilon_\omega^{d-2} |H|^2 + |\eta_{\varepsilon_\omega}|^2) \hat{u}_\omega dx.$$

Note that $H, \eta_{\varepsilon_\omega} \in L^\infty(\mathbb{R}^d)$ for $4 \le d \le 5$ and $H, \eta_{\varepsilon_\omega} \in L^{\frac{3s}{3-s}}(\mathbb{R}^d)$ for $d = 6$ with $1 < s < 3$ by Lemma 2.1. Thus, for $4 \le d \le 5$, it follows from (2.9) and (2.21) that

$$\begin{aligned}
|I| &\lesssim \varepsilon_\omega^{\frac{d-2}{2}} \|U_{\varepsilon_\omega}\|_{L^{\frac{8d}{d^2-4}}(B_R)}^{\frac{4}{d-2}} \|\hat{u}_\omega\|_{L^{\frac{2d}{d-2}}(B_R)} + \varepsilon_\omega^{d-2} \|U_{\varepsilon_\omega}\|_{L^{\frac{2d(6-d)}{d^2-4}}(B_R)}^{\frac{6-d}{d-2}} \|\hat{u}_\omega\|_{L^{\frac{2d}{d-2}}(B_R)} \\
&\lesssim \varepsilon_\omega^{\frac{d-2}{2}+\frac{4}{d+2}} \|\hat{u}_\omega\|_{L^{\frac{2d}{d-2}}(B_R)} \\
&\lesssim \varepsilon_\omega^{\frac{d-2}{2}+\sigma} \|\hat{u}_\omega\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)},
\end{aligned}$$

and for $d = 6$, it follows by (2.10) that

$$\begin{aligned}
|I| &\lesssim \varepsilon_\omega^2 \|U_{\varepsilon_\omega}\|_{L^{\frac{p}{p-1}}(B_R)} \|\hat{u}_\omega\|_{L^3(B_R)} + \varepsilon_\omega^4 \|\hat{u}_\omega\|_{L^3(B_R)} \\
&\lesssim \varepsilon_\omega^{2+\sigma} \|\hat{u}_\omega\|_{L^3(\mathbb{R}^d)}.
\end{aligned}$$

It follows from (2.31) and (2.32) that for $4 \le d \le 6$,

$$\left|\int_{\mathbb{R}^d} E_\omega \hat{u}_\omega dx\right| \lesssim (\omega \varepsilon_\omega^{\frac{d}{2}-\sigma} + \varepsilon_\omega^{\frac{d-2}{2}+\sigma}) \|\hat{u}_\omega\|_{L^{2^*}(\mathbb{R}^d)}. \qquad (2.33)$$

For $d = 3$, the estimates are similar to that of $d = 4, 5$. The difference is that $\omega_* = 1$ and we do not know if $\varphi_{\varepsilon_\omega} > 0$ in $\mathbb{R}^3$. Thus, we write

$$\begin{aligned}
|E_\omega| \lesssim & (\omega - 1) U_{\varepsilon_\omega} + U_{\varepsilon_\omega}^{\frac{4}{d-2}} (\varepsilon_\omega^{\frac{d-2}{2}} |H| + |\eta_{\varepsilon_\omega}|) + U_{\varepsilon_\omega}^{\frac{6-d}{d-2}} (\varepsilon_\omega^{d-2} |H|^2 + |\eta_{\varepsilon_\omega}|^2) \\
& + \varepsilon_\omega^{\frac{d+2}{2}} |H|^{\frac{d+2}{d-2}} + |\eta_{\varepsilon_\omega}|^{\frac{d+2}{d-2}} + (\omega - 1) 3^{\frac{1}{4}} \varepsilon_\omega^{\frac{1}{2}} |H| + (\omega - 1) |\eta_{\varepsilon_\omega}|,
\end{aligned}$$

which implies that

$$\left|\int_{\mathbb{R}^d} E_\omega \hat{u}_\omega dx\right| \lesssim ((\omega-1)\varepsilon_\omega^{\frac{1}{2}} + \varepsilon_\omega)\|\hat{u}_\omega\|_{L^{2^*}(\mathbb{R}^d)}. \tag{2.34}$$

Estimates (2.33) and (2.34), together with (2.27), (2.29) and (2.30), imply (2.28) for $3 \le d \le 6$.

For $d \ge 7$, we obtain from (1.1), (1.6), and (2.19) that $\hat{u}_\omega$ satisfies

$$\begin{cases} -\Delta \hat{u}_\omega + |x|^2 \hat{u}_\omega - \frac{d+2}{d-2} U_{\varepsilon_\omega}^{\frac{4}{d-2}} \hat{u}_\omega = E_\omega + N_\omega(\hat{u}_\omega) & \text{in } \mathbb{R}^d, \\ \hat{u}_\omega(x) \to 0 & \text{as } |x| \to 0, \end{cases} \tag{2.35}$$

where $E_\omega := \omega U_{\varepsilon_\omega}$ and

$$|N_\omega(\hat{u}_\omega)| \lesssim |\hat{u}_\omega|^{\frac{d+2}{d-2}}. \tag{2.36}$$

It follows from (2.11) that

$$\begin{aligned}\left|\int_{B_R} \widehat{E}_\omega \hat{u}_\omega dx\right| &\lesssim \omega \|U_{\varepsilon_\omega}\|_{L^2(\mathbb{R}^d)} \|\hat{u}_\omega\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \omega \varepsilon_\omega^2 \|\hat{u}_\omega\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

which, together with (2.26), (2.35) and (2.36) implies (2.28) for $d \ge 7$. □

2.4. **Asymptotic behaviors of $\mathcal{I}_\omega$ and ε_ω as $\omega \to \omega_*$.** It follows from (1.1) and (1.4) that if $u_\omega = (\mathcal{I}_\omega)^{\frac{d-2}{4}} v_\omega$, then

$$\mathcal{I}_\omega = \frac{\|u_\omega\|_X^2 - \omega \|u_\omega\|_{L^2(\mathbb{R}^d)}^2}{\|u_\omega\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}} = \|u_\omega\|^{\frac{4}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}, \tag{2.37}$$

which yields

$$\mathcal{I}_\omega = (\|u_\omega\|_X^2 - \omega \|u_\omega\|_{L^2(\mathbb{R}^d)}^2)^{\frac{2}{d}}. \tag{2.38}$$

The following four lemmas give details in the derivation of Theorem 1.1 for different values of $d \ge 3$. The derivation is simpler for $d \ge 7$ and becomes computationally challenging for $3 \le d \le 6$ due to different leading order terms in the expansion of $\mathcal{I}_\omega$ and due to different regularity of the non-singular part H of Green's function. Some similar computations can be found in [5, 6, 15, 16, 32] for $d \ge 4$ and in [12, 13, 16, 20] for $d = 3$.

Lemma 2.4. *For $d \ge 7$, we have*

$$\mathcal{I}_\omega = \mathcal{S} - \mathcal{S}^{-\frac{d-2}{2}} \frac{\|U\|^4_{L^2(\mathbb{R}^d)}}{2d\|xU\|^2_{L^2(\mathbb{R}^d)}} \omega^2 + o(\omega^2) \tag{2.39}$$

and

$$\varepsilon_\omega = \left(\frac{\|U\|^2_{L^2(\mathbb{R}^d)}}{2\|xU\|^2_{L^2(\mathbb{R}^d)}} \omega\right)^{\frac{1}{2}} + o(\omega^{\frac{1}{2}}) \tag{2.40}$$

as $\omega \to 0$.

Proof. By (2.19), (2.20), and the estimates of Lemma 2.3, we obtain from (2.38):

$$\begin{aligned}\mathcal{I}_\omega &= \left(\|U_{\varepsilon_\omega}\|_X^2 - \omega\varepsilon_\omega^2\|U\|^2_{L^2(\mathbb{R}^d)} + o(\omega\varepsilon_\omega^2)\right)^{\frac{2}{d}}\\ &= \left(\mathcal{S}^{\frac{d}{2}} + \varepsilon_\omega^4\|xU_{\varepsilon_\omega}\|^2_{L^2(\mathbb{R}^d)} - \omega\varepsilon_\omega^2\|U\|^2_{L^2(\mathbb{R}^d)} + o(\omega\varepsilon_\omega^2)\right)^{\frac{2}{d}}\\ &= \mathcal{S} + \frac{2}{d}\mathcal{S}^{-\frac{d-2}{2}}(\varepsilon_\omega^4\|xU\|^2_{L^2(\mathbb{R}^d)} - \omega\varepsilon_\omega^2\|U\|^2_{L^2(\mathbb{R}^d)}) + o(\omega\varepsilon_\omega^2). \end{aligned} \tag{2.41}$$

On the other hand, by using $\{U_\varepsilon\}_{\varepsilon>0}$ as a test function of $\mathcal{I}_\omega$ for $d \geq 7$, we obtain

$$\begin{aligned}\mathcal{I}_\omega &\leq \frac{\|U_\varepsilon\|_X^2 - \omega\|U_\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{\|U_\varepsilon\|^2_{L^{2^*}(\mathbb{R}^d)}}\\ &= \mathcal{S} + \frac{2}{d}\mathcal{S}^{-\frac{d-2}{2}}(\varepsilon^4\|xU\|^2_{L^2(\mathbb{R}^d)} - \omega\varepsilon^2\|U\|^2_{L^2(\mathbb{R}^d)}).\end{aligned} \tag{2.42}$$

Minimizing the right hand side of (2.42) in terms of ε implies that

$$\mathcal{I}_\omega \leq \mathcal{S} - \mathcal{S}^{-\frac{d-2}{2}}\frac{\|U\|^4_{L^2(\mathbb{R}^d)}}{2d\|xU\|^2_{L^2(\mathbb{R}^d)}}\omega^2. \tag{2.43}$$

Thus, combining (2.41) and (2.43), we have (2.39) and (2.40). □

Lemma 2.5. *For $d = 6$, we have*

$$\mathcal{I}_\omega = \mathcal{S} - \mathcal{S}^{-2}\frac{\|U\|^4_{L^2(\mathbb{R}^d)}\omega^2}{24^3|\mathbb{S}^5||\log\omega|} + o\left(\frac{\omega^2}{\log\omega}\right) \tag{2.44}$$

and

$$\varepsilon_\omega = \left(\frac{\|U\|^2_{L^2(\mathbb{R}^d)}\omega}{12\times 24^2|\mathbb{S}^5||\log\omega|}\right)^{\frac{1}{2}} + o\left(\frac{\omega}{|\log\omega|}\right) \tag{2.45}$$

as $\omega \to 0$.

Proof. With $d = 6$, expression (2.38) becomes

$$\mathcal{I}_\omega = (\|u_\omega\|_X^2 - \omega\|u_\omega\|^2_{L^2(\mathbb{R}^6)})^{\frac{1}{3}}.$$

By Lemmas 2.2 and 2.3, we have

$$\|u_\omega\|_X^2 = \|PU_{\varepsilon_\omega}\|_X^2 + \mathcal{O}(\omega^2\varepsilon_\omega^{6-\sigma} + \varepsilon_\omega^{4+\sigma}), \tag{2.46}$$

where $\sigma > 0$ is a small constant given by Lemma 2.3 and if necessary, σ can be taken arbitrary small. By Lemma 2.1, we obtain from (1.9) that

$$\begin{aligned}\|PU_{\varepsilon_\omega}\|_X^2 &= \int_{\mathbb{R}^6} U^2_{\varepsilon_\omega} PU_{\varepsilon_\omega}dx\\ &= \mathcal{S}^3 - 24\varepsilon_\omega^2\int_{B_R} U^2_{\varepsilon_\omega} H dx + \mathcal{O}(\varepsilon_\omega^4),\end{aligned} \tag{2.47}$$

where $R > 0$ is sufficient large. Since $H \in L^{\infty}_{loc}(\mathbb{R}^6\backslash\{0\})$, we need to further expand H in B_R. Since $\Delta(\log|x|) = \frac{4}{|x|^2}$ in $\mathbb{R}^6$ in the sense of distributions, it follows from (1.11) that $\widehat{H} := H + \frac{1}{4}\log|x|$ satisfies the following equation:

$$-\Delta\widehat{H} + |x|^2\widehat{H} = |x|^2\log|x| \quad \text{in } \mathbb{R}^6,$$

in the sense of distributions. Since $|x|^2\log|x| \in W^{1,\infty}_{loc}(\mathbb{R}^6)$, by the classical elliptic regularity, $\widehat{H} \in C^2_{loc}(\mathbb{R}^6)$. It follows that $H = -\frac{1}{4}\log|x| + \mathcal{O}(1)$ in B_R as $R \to \infty$. Thus, we obtain from (2.22) that

$$\begin{aligned}\int_{B_R} U^2_{\varepsilon_\omega} H dx &= -\frac{1}{4}\int_{B_R} U^2_{\varepsilon_\omega}\log|x|dx + \mathcal{O}\left(\int_{B_R} U^2_{\varepsilon_\omega} dx\right)\\ &= 144|\mathbb{S}^5|\varepsilon^2_\omega|\log\varepsilon_\omega| + \mathcal{O}(\varepsilon^{2+\sigma}_\omega),\end{aligned}$$

which, together with (2.46) and (2.47), implies

$$\|u_\omega\|^2_X = \mathcal{S}^3 - 6\times 24^2|\mathbb{S}^5|\varepsilon^4_\omega|\log\varepsilon_\omega| + \mathcal{O}(\varepsilon^4_\omega). \tag{2.48}$$

Similarly, by Lemmas 2.1 and 2.3 and the expansion $H = -\frac{1}{4}\log|x| + O(1)$ in B_R for any sufficiently large $R > 0$, we have

$$\int_{\mathbb{R}^6} PU^2_{\varepsilon_\omega} dx = \varepsilon^2_\omega\|U\|^2_{L^2(\mathbb{R}^d)} + o(\varepsilon^2_\omega). \tag{2.49}$$

Thus, by (2.48) and (2.49), we have

$$\mathcal{I}_\omega = \mathcal{S} + \frac{1}{3}\mathcal{S}^{-2}(6\times 24^2|\mathbb{S}^5|\varepsilon^4_\omega|\log\varepsilon_\omega| - \omega\varepsilon^2_\omega\|U\|^2_{L^2(\mathbb{R}^d)} + o(|\varepsilon^4_\omega|\log\varepsilon_\omega| + \omega\varepsilon^2_\omega)). \tag{2.50}$$

On the other hand, by using $W_\varepsilon := (U_\varepsilon - 24\varepsilon^2 H)\phi_R$, where $\phi_R \in [0,1]$ is a smooth cut-off function such that $\phi_R = 1$ for $|x| \leq R$ and $\phi_R = 0$ for $|x| \geq R+1$, as a test function of $\mathcal{I}_\omega$ for $d = 6$, we have

$$\mathcal{I}_\omega \quad\leq\quad \frac{\|W_\varepsilon\|^2_X - \omega\|W_\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{\|W_\varepsilon\|^2_{L^{2^*}(\mathbb{R}^d)}},$$

which implies

$$\mathcal{I}_\omega \leq \mathcal{S} + \frac{1}{3}\mathcal{S}^{-2}(6\times 24^2|\mathbb{S}^5|\varepsilon^4|\log\varepsilon| - \omega\varepsilon^2\|U\|^2_{L^2(\mathbb{R}^6)} + o(|\varepsilon^4|\log\varepsilon| + \omega\varepsilon^2)). \tag{2.51}$$

Minimizing the right hand side of (2.51) in terms of ε implies that

$$\mathcal{I}_\omega \leq \mathcal{S} - \mathcal{S}^{-2}\frac{\|U\|^4_{L^2(\mathbb{R}^d)}\omega^2}{12\times 24^2|\mathbb{S}^5||\log\omega|} + o(\frac{\omega^2}{|\log\omega}|). \tag{2.52}$$

Thus, by combining (2.50) and (2.52), we have (2.44) and (2.45). □

Lemma 2.6. *For $d = 4, 5$, we have*

$$\mathcal{I}_\omega = \begin{cases} \mathcal{S} - \sqrt{2}\mathcal{S}^{-2} H(0)\|U\|^3_{L^3(\mathbb{R}^d)} e^{\frac{3\sqrt{2}H(0)\|U\|^3_{L^3(\mathbb{R}^d)}}{2\omega|\mathbb{S}^3|}} + o(e^{-\frac{1}{\omega}}), & d = 4, \\ \mathcal{S} - \mathcal{S}^{-\frac{5}{2}} \dfrac{54\|U\|^6_{L^2(\mathbb{R}^5)}}{1715 \times 15^{\frac{3}{2}} (H(0)\|U\|^{\frac{7}{3}}_{L^{\frac{7}{3}}(\mathbb{R}^d)})^2} \omega^3 + o(\omega^3), & d = 5, \end{cases} \tag{2.53}$$

and

$$\varepsilon_\omega = \begin{cases} e^{-\frac{3\sqrt{2}H(0)\|U\|^3_{L^3(\mathbb{R}^d)}}{4\omega|\mathbb{S}^3|}} + o(e^{-\frac{1}{\omega}}), & d = 4, \\ \dfrac{3\|U\|^2_{L^2(\mathbb{R}^5)}}{7 \times 15^{\frac{3}{4}} H(0)\|U\|^{\frac{7}{3}}_{L^{\frac{7}{3}}(\mathbb{R}^d)}} \omega + o(\omega), & d = 5 \end{cases} \tag{2.54}$$

as $\omega \to 0$.

Proof. Different from the proofs of Lemmas 2.4 and 2.5, we use the relation (2.37). Moreover, we note that $H(0)$ is the maximum of $H(x)$ by the maximum principle, which implies from (1.11) for $d = 4, 5$ that $H(0) > 0$. Recall that $H, \eta_{\varepsilon_\omega} \in L^\infty(\mathbb{R}^d) \cap C^\alpha_{loc}(\mathbb{R}^d)$ for $d = 4, 5$ with $\forall \alpha \in (0, 1)$ by Lemma 2.1. Then by Lemma 2.3, we have

$$\|u_\omega\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = \|PU_{\varepsilon_\omega}\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} + \frac{2d}{d-2}\int_{\mathbb{R}^d} PU^{\frac{d+2}{d-2}}_{\varepsilon_\omega} \hat{u}_\omega dx + \mathcal{O}(\omega^2 \varepsilon_\omega^{d-\sigma} + \varepsilon_\omega^{d-2+\sigma}).$$

By Lemma 2.2, it follows from (1.9) that

$$\int_{\mathbb{R}^d} PU^{\frac{d+2}{d-2}}_{\varepsilon_\omega} \hat{u}_\omega dx = \int_{\mathbb{R}^d} (PU^{\frac{d+2}{d-2}}_{\varepsilon_\omega} - U^{\frac{d+2}{d-2}}_{\varepsilon_\omega}) \hat{u}_\omega dx.$$

Then by similar calculations for (2.32) and using (2.7), (2.19) and (2.28) and Lemma 2.2 similar to that of $d = 6$,

$$\begin{aligned} \|u_\omega\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} &= \|PU_{\varepsilon_\omega}\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} + \mathcal{O}(\omega^2 \varepsilon_\omega^{d-\sigma} + \varepsilon_\omega^{d-2+\sigma}) \\ &= \mathcal{S}^{\frac{d}{2}} + \mathcal{O}(\varepsilon_\omega^4) - \frac{2d}{d-2}\int_{B_R} U^{\frac{d+2}{d-2}}_{\varepsilon_\omega} (\varepsilon_\omega^{\frac{d-2}{2}} [d(d-2)]^{\frac{d-2}{4}} H + \eta_{\varepsilon_\omega}) dx \\ &\quad + \mathcal{O}(\int_{B_R} U^{\frac{4}{d-2}}_{\varepsilon_\omega} (\varepsilon_\omega^{\frac{d-2}{2}} H + \eta_{\varepsilon_\omega})^2 + ((\varepsilon_\omega^{\frac{d-2}{2}} H + \eta_{\varepsilon_\omega})^{\frac{2d}{d-2}}) dx) \\ &= \mathcal{S}^{\frac{d}{2}} - \frac{2d^{\frac{d+2}{4}}}{(d-2)^{\frac{6-d}{4}}} \varepsilon_\omega^{\frac{d-2}{2}} \int_{B_R} U^{\frac{d+2}{d-2}}_{\varepsilon_\omega} H dx + \mathcal{O}(\varepsilon_\omega^{d-2+\sigma}), \end{aligned} \tag{2.55}$$

where we have used (2.21) and $R > 0$ is a large constant. By the regularity of H for $d = 4, 5$, $H(x) = H(0) + \mathcal{O}(|x|^\alpha)$ near $|x| = 0$. Therefore, we can choose $\rho > 0$ sufficiently small and obtain

$$\begin{aligned}\int_{B_R} U_{\varepsilon_\omega}^{\frac{d+2}{d-2}} H dx &= H(0)\int_{B_\rho} U_{\varepsilon_\omega}^{\frac{d+2}{d-2}} dx + \mathcal{O}(\int_{B_\rho} U_{\varepsilon_\omega}^{\frac{d+2}{d-2}} |x|^\alpha dx) + \mathcal{O}(\int_{B_R\setminus B_\rho} U_{\varepsilon_\omega}^{\frac{d+2}{d-2}} dx)\\ &= \varepsilon_\omega^{\frac{d-2}{2}} H(0)\|U\|_{L^{\frac{d+2}{d-2}}(\mathbb{R}^d)}^{\frac{d+2}{d-2}} + \mathcal{O}(\varepsilon_\omega^{\frac{d-2}{2}+\alpha}). \end{aligned} \tag{2.56}$$

It follows from (2.37), (2.55) and (2.56) that

$$\mathcal{I}_\omega = \mathcal{S} - \frac{2}{d}\mathcal{S}^{-\frac{d-2}{2}}\varepsilon_\omega^{d-2}[d(d-2)]^{\frac{d-2}{4}} H(0)\|U\|_{L^{\frac{d+2}{d-2}}(\mathbb{R}^d)}^{\frac{d+2}{d-2}} + \mathcal{O}(\varepsilon_\omega^{d-2+\sigma}), \tag{2.57}$$

where $\sigma > 0$ is a small constant given by Lemma 2.3 and if necessary, σ can be taken arbitrary small. On the other hand, by (1.1), we have

$$\|u_\omega\|_X^2 - \omega\|u_\omega\|_{L^2(\mathbb{R}^d)}^2 = \|u_\omega\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^{\frac{2d}{d-2}}. \tag{2.58}$$

Moreover, by using (2.7), (2.9), (2.19) and (2.28), as in Lemma 2.5 we obtain

$$\begin{aligned}\|u_\omega\|_X^2 &= \|PU_{\varepsilon_\omega}\|_X^2 + \mathcal{O}(\omega^2\varepsilon_\omega^{d-\sigma} + \varepsilon_\omega^{d-2+\sigma})\\ &= \int_{\mathbb{R}^d} U_{\varepsilon_\omega}^{\frac{d+2}{d-2}} PU_{\varepsilon_\omega} dx + \mathcal{O}(\omega^2\varepsilon_\omega^{d-\sigma} + \varepsilon_\omega^{d-2+\sigma})\\ &= \mathcal{S}^{\frac{d}{2}} + \varepsilon_\omega^{d-2}[d(d-2)]^{\frac{d-2}{4}} H(0)\|U\|_{L^{\frac{d+2}{d-2}}(\mathbb{R}^d)}^{\frac{d+2}{d-2}} + \mathcal{O}(\varepsilon_\omega^{\frac{d-2}{2}+\alpha})\end{aligned}$$

and

$$\begin{aligned}\|u_\omega\|_{L^2(\mathbb{R}^d)}^2 &= \|PU_{\varepsilon_\omega}\|_{L^2(\mathbb{R}^d)}^2 + 2\int_{\mathbb{R}^d} PU_{\varepsilon_\omega}\hat{u}_\omega dx + \mathcal{O}(\omega^2\varepsilon_\omega^{d-\sigma} + \varepsilon_\omega^{d-2+\sigma})\\ &= \begin{cases} 8|\mathbb{S}^3|\varepsilon_\omega^2|\log\varepsilon_\omega| + o(|\varepsilon_\omega^2|\log\varepsilon_\omega|), & d=4,\\ \varepsilon_\omega^2\|U\|_{L^2(\mathbb{R}^5)}^2 + o(\varepsilon_\omega^2), & d=5.\end{cases}\end{aligned}$$

By using (2.58), we obtain

$$\begin{aligned}&\frac{d+2}{d-2}\varepsilon_\omega^{d-2}[d(d-2)]^{\frac{d-2}{4}} H(0)\|U\|_{L^{\frac{d+2}{d-2}}(\mathbb{R}^d)}^{\frac{d+2}{d-2}} + o(\varepsilon_\omega^{d-2})\\ &= \begin{cases} 8\omega|\mathbb{S}^3|\varepsilon_\omega^2|\log\varepsilon_\omega| + o(|\varepsilon_\omega^2|\log\varepsilon_\omega|), & d=4,\\ \omega\varepsilon_\omega^2\|U\|_{L^2(\mathbb{R}^5)}^2 + o(\varepsilon_\omega^2), & d=5,\end{cases}\end{aligned}$$

which, together with (2.57), implies (2.53) and (2.54). □

Remark 2.1. *Two methods have been used to compute ε_ω. The first one is to use the variational formula (1.4) which works for minimizers. In this method, one need to use test functions in the variational formula to determine ε_ω. This method is used for $d \geq 6$ in Lemmas 2.4 and 2.5. The other one is to use the equation (1.1) which works for solutions (not necessary to be minimizers of variational problems) satisfying the decompositions in Lemmas 2.2 and 2.3. In this method, we expand the equation (2.58) and determine ε_ω. This method is used for $d = 3, 4, 5$ in Lemmas 2.6 and 2.7.*

Lemma 2.7. *For $d = 3$, we have*

$$\mathcal{I}_\omega = \mathcal{S} - \mathcal{S}^{-\frac{3}{2}} \frac{3^{\frac{3}{4}} \|G\|^4_{L^2(\mathbb{R}^3)}}{40\pi} (\omega - 1)^2 + o((\omega - 1)^2) \tag{2.59}$$

and

$$\varepsilon_\omega = \frac{3^{\frac{5}{4}} \|G\|^2_{L^2(\mathbb{R}^3)}}{20\pi} (\omega - 1) + o(\omega - 1) \tag{2.60}$$

as $\omega \to 1$.

Proof. Since the proof is similar to that of Lemma 2.6, we only sketch it. We still use the relations (2.37) and (2.38) rewritten for $d = 3$ as

$$\mathcal{I}_\omega = \frac{\|u_\omega\|^2_X - \omega \|u_\omega\|^2_{L^2(\mathbb{R}^3)}}{\|u_\omega\|^2_{L^6(\mathbb{R}^3)}} = \|u_\omega\|^4_{L^6(\mathbb{R}^3)}. \tag{2.61}$$

and

$$\|u_\omega\|^2_X - \omega \|u_\omega\|^2_{L^2(\mathbb{R}^3)} = \|u_\omega\|^6_{L^6(\mathbb{R}^3)}. \tag{2.62}$$

As that in the proofs of Lemma 2.5 and 2.6, by (2.28), we have

$$\|u_\omega\|^2_{L^2(\mathbb{R}^3)} = \|PU_{\varepsilon_\omega}\|^2_{L^2(\mathbb{R}^3)} + 2 \int_{\mathbb{R}^3} PU_{\varepsilon_\omega} \hat{u}_\omega dx + \mathcal{O}((\omega - 1)^2 \varepsilon_\omega^{3-\sigma} + \varepsilon_\omega^{4+\sigma}).$$

By using (2.7), (2.9), (2.14) and (2.19), we obtain

$$\begin{aligned}
\|PU_{\varepsilon_\omega}\|^2_{L^2(\mathbb{R}^3)} &= \int_{B_{\frac{1}{\varepsilon_\omega}}} (U_{\varepsilon_\omega} - \varepsilon_\omega^{\frac{1}{2}} 3^{\frac{1}{4}} H - \eta_{\varepsilon_\omega})^2 + \mathcal{O}(\varepsilon_\omega^{\frac{5}{2}}) \\
&= \int_{B_{\frac{1}{\varepsilon_\omega}}} (U_{\varepsilon_\omega} - \varepsilon_\omega^{\frac{1}{2}} 3^{\frac{1}{4}} H)^2 dx + \mathcal{O}(\varepsilon_\omega^{2-\sigma}) \\
&= \varepsilon_\omega 3^{\frac{1}{2}} \int_{B_{\frac{1}{\varepsilon_\omega}}} G^2 dx - 2\varepsilon_\omega^{\frac{1}{2}} 3^{\frac{1}{4}} \int_{B_{\frac{1}{\varepsilon_\omega}}} H(U_{\varepsilon_\omega} - \varepsilon_\omega^{\frac{1}{2}} 3^{\frac{1}{4}} |x|^{-1}) dx \\
&\quad + \int_{B_{\frac{1}{\varepsilon_\omega}}} U^2_{\varepsilon_\omega} - \varepsilon_\omega 3^{\frac{1}{2}} |x|^{-2} dx + \mathcal{O}(\varepsilon_\omega^{2-\sigma}) \\
&= \varepsilon_\omega 3^{\frac{1}{2}} \int_{\mathbb{R}^3} G^2 dx + \mathcal{O}(\varepsilon_\omega^{2-\sigma}).
\end{aligned}$$

Moreover, similar to that of (2.55), by (2.7), (2.9), (2.28) and Lemmas 2.2 and 2.3,

$$\begin{aligned}
\|u_\omega\|^6_{L^6(\mathbb{R}^3)} &= \|PU_{\varepsilon_\omega}\|^6_{L^6(\mathbb{R}^3)} + \mathcal{O}((\omega - 1)^2 \varepsilon_\omega + \varepsilon_\omega^2) \\
&= \mathcal{S}^{\frac{3}{2}} - \int_{B_R} 6\sqrt[4]{3} \varepsilon_\omega^{\frac{1}{2}} U^5_{\varepsilon_\omega} H - 15\sqrt{3} \varepsilon_\omega^2 U^4_{\varepsilon_\omega} H^2 dx + \mathcal{O}((\omega - 1)^2 \varepsilon_\omega + \varepsilon_\omega^2).
\end{aligned} \tag{2.63}$$

Since H is not C^1, we need to expand $H(x)$ as that in [16]. We define $\psi = H(x) - \frac{1}{2}|x|$, then by (1.11), ψ satisfies

$$-\Delta\psi + (|x|^2 - 1)\psi = \frac{1}{2}|x|(1 - |x|^2) \quad \text{in } \mathbb{R}^3.$$

Since the data $\frac{|x|-|x|^3}{2}$ belongs to $W^{1,\infty}_{loc}(\mathbb{R}^3)$. Thus, by the classical regularity theory, $\psi \in C^{2,\alpha}_{loc}(\mathbb{R}^3)$ for some $\alpha \in (0,1)$, which, together with ψ being radial, implies $\nabla\psi(0) = 0$. It follows that

$$H(x) = H(0) + \frac{1}{2}|x| + \mathcal{O}(|x|^2) \quad \text{near } |x| = 0.$$

Therefore, we obtain

$$\int_{B_R} U^5_{\varepsilon_\omega} H dx = \frac{4\pi}{3} H(0)\varepsilon_\omega^{\frac{1}{2}} - \frac{4\pi}{3}\varepsilon_\omega^{\frac{3}{2}} + \mathcal{O}(\varepsilon_\omega^{\frac{5}{2}}|\log\varepsilon_\omega|) \tag{2.64}$$

and

$$\int_{B_R} U^4_{\varepsilon_\omega} H^2 dx = H(0)^2\pi^2\varepsilon_\omega + \mathcal{O}(\varepsilon_\omega^2|\log\varepsilon_\omega|). \tag{2.65}$$

Now, as that in the proof of Lemma 2.6, we obtain by using (2.62) and (2.63),

$$\begin{aligned}&\int_{B_R} 5\sqrt[4]{3}\varepsilon_\omega^{\frac{1}{2}} U^5_{\varepsilon_\omega} H - 15\sqrt{3}\varepsilon_\omega U^4_{\varepsilon_\omega} H^2 dx + o(\varepsilon_\omega^2)\\ &= -(\omega - 1)\varepsilon_\omega 3^{\frac{1}{2}} \int_{\mathbb{R}^3} G^2 dx + o((\omega - 1)\varepsilon_\omega).\end{aligned} \tag{2.66}$$

Dividing ε_ω on both sides of (2.66) and letting $\omega \to 1$, we have $H(0) = 0$ by (2.64) and (2.65). Thus, by (2.64) and (2.65) once more, (2.66) is reduced to

$$\frac{20\sqrt[4]{3}\pi}{3}\varepsilon_\omega^2 + o(\varepsilon_\omega^2) = (\omega - 1)\varepsilon_\omega 3^{\frac{1}{2}} \int_{\mathbb{R}^3} G^2 dx + o((\omega - 1)\varepsilon_\omega),$$

which, together with (2.61), implies that (2.59) and (2.60). □

Remark 2.2. *We note that $H(0)$ is a global minimum of $H(x)$ in $\mathbb{R}^3$. Indeed, by the maximum principle, it is easy to see that there exists $r_0 \geq 1$ such that $H(r)$ is strictly increasing in $[0, r_0]$ and is strictly decreasing in $[r_0, +\infty)$. Thus, by $H(0) = 0$ and $H(x) \to 0$ as $|x| \to +\infty$, we have that $H(0)$ is actually a global minimum of $H(x)$.*

The proof of Theorem 1.1 follows immediately from Lemmas 2.4–2.7. □

3. The energy-supercritical case

3.1. **Preliminaries.** Let u_∞ be the singular solution of the stationary equation (1.12) for some $\omega_\infty \in (d-4, d)$ satisfying (1.13) for $d \geq 5$. Let L_∞ be the associated linear operator given by

$$L_\infty := -\Delta + |x|^2 - \omega_\infty - 3u_\infty^2.$$

Since $u_\infty(r) = \mathcal{O}(r^{-1})$ as $r \to 0$, $u_\infty \in C^\infty(0,\infty)$, and $u_\infty(r) \to 0$ exponentially fast as $r \to +\infty$, we consider L_∞ in the form domain $X_{\rm rad} := \{f \in X : f \text{ is radial}\}$. The singular potential is controlled in the form domain by using the following Hardy inequality for every $d \geq 3$:

$$\||\cdot|^{-1} f\|_{L^2(\mathbb{R}^d)} \leq \frac{2}{d-2}\|\nabla f\|_{L^2(\mathbb{R}^d)}, \quad \forall f \in D^{1,2}(\mathbb{R}^d). \tag{3.1}$$

where $D^{1,2}(\mathbb{R}^d)$ is the same as in (1.6).

In order to justify the definition of Morse index $\mathfrak{m}(u_\infty)$ according to Definition 1.2, we show that the linear operator L_∞ has a compact resolvent, which implies that its spectrum of L_∞ in $X_{\rm rad}$ is purely discrete and consists of isolated (simple) eigenvalues.

Lemma 3.1. *For every $d \geq 5$, the linear operator L_∞ has a compact resolvent in $X_{\rm rad}$.*

Proof. Consider the following variational problem:

$$\tau_1 = \inf_{\phi \in X_{\rm rad}} \frac{\int_{\mathbb{R}^d}(|\nabla\phi|^2 + (|x|^2 - 3u_\infty^2)|\phi|^2)dx}{\int_{\mathbb{R}^d}|\phi|^2 dx}.$$

Since $F(r) := r u_\infty(r)$ is monotonically decreasing (cf. [4, 35]), we have $F(r) < F(0) = \sqrt{d-3}$, which implies that $u_\infty(r) < \frac{\sqrt{d-3}}{r}$ for every $r > 0$. By Hardy's inequality (3.1), we obtain

$$\int_{\mathbb{R}^d} 3u_\infty^2|\phi|^2 dx \leq \int_{\mathbb{R}^d}\frac{3(d-3)}{|x|^2}|\phi|^2 dx \leq \frac{12(d-3)}{(d-2)^2}\|\nabla\phi\|_{L^2(\mathbb{R}^d)}.$$

By classical variational arguments and the fact that X is compactly embedded into $L^2(\mathbb{R}^d)$, we can see that $\tau_1 > -\infty$ is attained. Since the linear operator

$$L_\infty + \omega_\infty - \tau_1 + 1 = -\Delta + |x|^2 - 3u_\infty^2 - \tau_1 + 1$$

is strictly positive in $X_{\rm rad}$, the linear equation

$$-\Delta\psi + (|x|^2 - 3u_\infty^2 - \tau_1 + 1)\psi = \varphi \quad \text{in } X_{\rm rad}, \tag{3.2}$$

is unique solvable for every $\varphi \in X_{\rm rad}$. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be bounded in $X_{\rm rad}$, then it follows by the compactness of the embedding from X to $L^2(\mathbb{R}^d)$ that $\varphi_n \to \varphi_*$ as $n \to \infty$ strongly in $L^2(\mathbb{R}^d)$. Since the equation (3.2) is linear, we may assume that $\varphi_* = 0$. By the positivity of $L_\infty + \omega_\infty - \tau_1 + 1$ in $X_{\rm rad}$, $\{\psi_n\}_{n\in\mathbb{N}}$ is bounded in $X_{\rm rad}$. Since $\varphi_n \to 0$ strongly in $L^2(\mathbb{R}^d)$ as $n \to \infty$, then $\psi_n \to 0$ as $n \to \infty$ strongly in $X_{\rm rad}$. Therefore, $L_\infty + \omega_\infty - \tau_1 + 1$ has a compact resolvent in $X_{\rm rad}$, and so does L_∞. □

Remark 3.1. *The mapping $d \mapsto \frac{12(d-3)}{(d-2)^2}$ is monotonically decreasing for $d \geq 5$. Since $\frac{12(d-3)}{(d-2)^2} < 1$ for $d \geq 13$, we have $\tau_1 > 0$ for $d \geq 13$. However, $\tau_1 < 0$ for $5 \leq d \leq 12$.*

Let $\mathfrak{m}(u_\infty)$ be the Morse index of u_∞ in $X_{\rm rad}$ according to Definition 1.2. It is well-defined for $d \geq 5$ because L_∞ has a purely discrete spectrum of isolated (simple) eigenvalues by Lemma 3.1.

3.2. **Morse index in the oscillatory case.** The following lemma shows that the Morse index of u_∞ is infinite for $5 \le d \le 12$, for which ω_b oscillates near ω_∞ as $b \to \infty$.

Lemma 3.2. *For $5 \le d \le 12$, we have $\mathfrak{m}(u_\infty) = \infty$.*

Proof. We consider the following two cases:

(1) There exists $b_n \to +\infty$ as $n \to \infty$ such that $\omega_{b_n} - \omega_\infty > 0$.

(2) $\omega_b \le \omega_\infty$ for $b > 0$ sufficiently large.

Case (1)**.** By using equations (5.4), (6.30), and (6.47) from [4], we obtain

$$u_\infty(r) = \frac{\sqrt{d-3}}{r} - \frac{\omega_\infty \sqrt{d-3}}{4d-10} r + \mathcal{O}(r^3) \tag{3.3}$$

and

$$u_{b_n}(r) = \frac{\sqrt{d-3}}{r} + C(\omega_{b_n} - \omega) r^{-\beta-1} \sin(\alpha \log r + \delta) + \mathcal{O}(b_n^{-2(1-a)} + \varepsilon^2), \tag{3.4}$$

for $r = \mathcal{O}(b_n^{a-1})$, where $|\omega_{b_n} - \omega_\infty| = \mathcal{O}(\varepsilon b_n^{-\beta(1-a)})$, $C \in \mathbb{R}$, $\delta \in \mathbb{R}$, $\varepsilon > 0$ is sufficiently small, $a \in (0,1)$ and

$$\beta = \frac{d-4}{2}, \qquad \alpha = \frac{\sqrt{-d^2 + 16d - 40}}{2}.$$

Let $\varphi_{b_n} := u_\infty - u_{b_n}$. Since $u_{b_n}(0) = b_n$ and $\omega_{b_n} - \omega_\infty > 0$, it follows from (3.3) and (3.4) that there exists $r_{b_n} \to 0$ such that $\varphi_{b_n}(r) > 0$ for $r \in (0, r_{b_n})$ and $\varphi_{b_n}(r_{b_n}) = 0$. It follows from (1.12) that φ_{b_n} satisfies for $r \in (0, r_{b_n})$:

$$\begin{aligned} -\Delta \varphi_{b_n} + |x|^2 \varphi_{b_n} &= (\omega_\infty + u_\infty^2) \varphi_{b_n} - (u_{b_n}^2 - u_\infty^2 + \omega_{b_n} - \omega_\infty) u_{b_n} \\ &= (\omega_\infty + 3u_\infty^2) \varphi_{b_n} - (2u_\infty^2 - u_\infty u_{b_n} - u_{b_n}^2) \varphi_{b_n} - (\omega_{b_n} - \omega_\infty) u_{b_n} \\ &< (\omega_\infty + 3u_\infty^2) \varphi_{b_n}. \end{aligned} \tag{3.5}$$

Let

$$\widetilde{\varphi}_{b_n} = \begin{cases} \varphi_{b_n}, & 0 < r < r_{b_n}, \\ 0, & r \ge r_{b_n}. \end{cases}$$

Then by multiplying (3.5) with $\widetilde{\varphi}_{b_n}$ on both sides and integrating by parts, we have

$$\int_{\mathbb{R}^d} \left(|\nabla \widetilde{\varphi}_{b_n}|^2 + |x|^2 |\widetilde{\varphi}_{b_n}|^2 \right) dx < \int_{\mathbb{R}^d} (\omega_\infty + 3u_\infty^2) |\widetilde{\varphi}_{b_n}|^2 dx.$$

Since $r_{b_n} \to 0$ as $n \to \infty$, $\{\widetilde{\varphi}_{b_n}\}$ is linearly independent up to a subsequence. Hence, $\mathfrak{m}(u_\infty) = \infty$.

Case (2)**.** We follow the idea in [19]. Let $W_b = u_b(e^t)$ and $W_\infty = u_\infty(e^t)$, then $Z_b = \frac{W_b}{W_\infty}$ satisfies

$$Z_b'' + (d - 2 + \frac{2W_\infty'}{W_\infty}) Z_b' + e^{2t} Z_b (\omega_b - \omega_\infty + W_\infty^2 (Z_b^2 - 1)) = 0. \tag{3.6}$$

It follows from the convergence $u_b \to u_\infty$ in Σ by [35, Theorem 1.2] that $Z_{b_n}(t) \to 1$ as $n \to +\infty$ for every fixed t. Moreover, by classical elliptic regularity, we also have

$u_b \to u_\infty$ in $C^{1,\alpha}_{loc}(\mathbb{R}^d \backslash \{0\})$ as $b \to +\infty$. We claim that there exists $b_n \to +\infty$ as $n \to \infty$ such that $1 - Z_{b_n}(t)$ has at least n zeros, say $t_{n,n} < \cdots < t_{2,n} < t_{1,n}$, such that $t_{n,n} \to 0$ as $n \to +\infty$. In other words, we claim that Z_{b_n} is oscillatory around 1 as $n \to \infty$ on $(-\infty, 0)$, in agreement with (3.4).

Suppose the contrary. Then for every sequence $\{b_n\}$ satisfying $b_n \to +\infty$ as $n \to \infty$, there exists $N > 0$, independent of n, such that $1 - Z_{b_n}(t)$ has at most N zeros for all n. Since $Z_b(t) = \mathcal{O}(e^t)$ as $t \to -\infty$ by [4, (3.9)] for every $b > 0$ there exists $t_0 > 0$, independent of n, such that $0 < Z_{b_n}(t) < 1$ for all $t < t_0$ and n. If $V_{b_n} = 1 - Z_{b_n}$, then $0 < V_{b_n}(t) < 1$ for $t < t_0$. Moreover, by (3.6), V_{b_n} satisifes

$$V''_{b_n} + (d - 2 + \frac{2W'_\infty}{W_\infty})V'_{b_n} - e^{2t} Z_{b_n}(\omega_{b_n} - \omega_\infty - W^2_\infty (Z_{b_n} + 1)V_{b_n}) = 0.$$

Since $u_{b_n} \to u_\infty$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^d \backslash \{0\})$ as $n \to \infty$, we know that $Z_{b_n}(t) \to 1$ as $n \to \infty$ uniformly in every compact set of the interval $(-\infty, t_0]$. Note that we also have $e^{2t} W^2_\infty \to (d-3)$ as $t \to -\infty$, thus, there exists $t'_0 < t_0$ which is independent of n, such that $e^{2t} W^2_\infty = (d-3) + o(1)$ for $t < t'_0$ where $o(1) \to 0$ as $t'_0 \to -\infty$. Thus, without loss of generality, we may assume that $e^{2t} Z_{b_n} W^2_\infty (Z_{b_n} + 1) = 2(d-3) + o(1)$ uniformly in every compact set of the interval $(-\infty, t'_0]$, where $o(1)$ could be arbitrary small if necessary by taking t'_0 sufficiently close to $-\infty$ and n sufficiently large. Note that by (3.3),

$$\frac{2W'_\infty(t)}{W_\infty(t)} \to -2, \quad \text{as } \ t \to -\infty.$$

Since $\omega_{b_n} \leq \omega_\infty$ is obtained from the assumption, in this case, we can write the equation of V_{b_n} as follows:

$$V''_{b_n} + (d - 4 + o(1))V'_{b_n} + (2(d-3) + o(1))V_{b_n} \leq 0$$

in every compact set of the interval $(-\infty, t'_0]$ by taking t'_0 sufficiently close to $-\infty$ if necessary. Since $5 \leq d \leq 12$, the fundamental solution of the linear equation,

$$\phi'' + (d-4)\phi' + 2(d-3)\phi = 0,$$

is given by $\phi = Ce^{-\beta t} \sin(\alpha t + \delta)$ for some $C \in \mathbb{R}$ and $\delta \in \mathbb{R}$. By the Sturm–Liouville theorem, V_{b_n} must have zeros in a sufficiently large compact set of the interval $(-\infty, t'_0]$. It contradicts the assumption that $V_{b_n}(t) > 0$ for all $t < t'_0$. Thus, there exists $b_n \to +\infty$ as $n \to \infty$ such that $1 - Z_{b_n}(t)$ has at least n zeros for $t \ll -1$. We assume the zeros of Z_{b_n} by $0 < a_{1,n} < a_{2,n} < \cdots < a_{k_n,n}$ with $k_n \geq n$. For the sake of simplicity, we also denote $a_{0,n} = 0$. Then we can define

$$\widehat{\varphi}_{n,j} = \begin{cases} 0, & 0 < r \leq a_{j-1,n}, \\ u_\infty - u_{b_n}, & a_{j-1,n} < r < a_{j,n}, \\ 0, & r \geq a_{j,n} \end{cases}$$

and by the convexity of t^3, we have

$$\int_{\mathbb{R}^d} \left(|\nabla \widehat{\varphi}_{n,j}|^2 + |x|^2 |\widehat{\varphi}_{n,j}|^2 \right) dx < \int_{\mathbb{R}^d} (\omega_\infty + 3u^2_\infty) |\widehat{\varphi}_{n,j}|^2 dx.$$

It follows from $k_n \to \infty$ as $n \to \infty$ that $\mathfrak{m}(u_\infty) = \infty$. □

3.3. **Morse index in the monotone case.** By Theorems 1.1 and 1.2 in [29], the Morse index of u_∞ is finite for $d \geq 13$ for which ω_b converges to ω_∞ monotonically as $b \to \infty$. Here we will give a more precise estimates on $\mathfrak{m}(u_\infty)$ for $d \geq 13$.

Let us consider the confluent hypergeometric function, which is also called Kummer's function, given by

$$M(a;b;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!},$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ are Pochhammer symbols. It is well known (cf. [39]) that $M(a;b;x)$ is a solution of the confluent hypergeometric differential equation, which is also called the Kummer equation:

$$x\frac{d^2u}{dx^2} + (b-x)\frac{du}{dx} + au = 0.$$

Let

$$W_{a,l}(r) = r^l e^{-\frac{r^2}{2}} M(a; l + \frac{d}{2}; r^2),$$

then it can be directly verified that W satisfies

$$-W''_{a,l} - \frac{d-1}{r}W'_{a,l} + \frac{l(l+d-2)}{r^2}W_{a,l} + r^2 W_{a,l} = (d-4a+2l)W_{a,l}.$$

Let

$$l_\pm = \frac{2-d \pm \sqrt{d^2-16d+40}}{2} \tag{3.7}$$

then $W_{a,l_\pm}$ satisfies

$$-\Delta W_{a,l_\pm} + |x|^2 W_{a,l_\pm} - \frac{3(d-3)}{|x|^2} W_{a,l_\pm} = (d-4a+2l_\pm)W_{a,l_\pm}. \tag{3.8}$$

Remark 3.2. *It is easy to see that $b = l_\pm + \frac{d}{2} \neq 0, -1, -2, \cdots$. Otherwise, we have*

$$\frac{d^2-16d+40}{4} - p^2 = 0$$

for some $p \in \mathbb{Z}$, which implies

$$d = 2(4 \pm \sqrt{p^2+6}) \in \mathbb{N}.$$

It follows that $(\frac{q}{2})^2 - p^2 = 6$ for some $q \in \mathbb{Z}$. Thus, either $\frac{q}{2} - p = 2k$ or $\frac{q}{2} + p = 2k$ for some $k \in \mathbb{N}$, which implies $4k^2 \pm 4pk = 6$. It is impossible since $2k^2$ is even but $3 \pm 2kp$ is odd.

If $a \neq 0, -1, -2, \cdots$ then

$$M(a; l_\pm + \frac{d}{2}; r^2) \sim \sum_{n=0}^{\infty} n^{l_\pm + \frac{d}{2} - a} \frac{r^{2n}}{n!} \gtrsim \sum_{n=0}^{\infty} \frac{(\frac{2}{3} r^2)^n}{n!} = e^{\frac{2}{3} r^2}.$$

If $-a \in \mathbb{N}$, then $M(-n; l_\pm + \frac{d}{2}; r^2) = P_n(r^2)$ is a polynomial of order $2n$. Therefore, $W_{a,l_\pm} \in L^2(\mathbb{R}^d)$ if and only if $-a \in \mathbb{N}$. On the other hand, if $W_{a,l_\pm} \in L^2(\mathbb{R}^d)$ is a eigenfunction of the operator $-\Delta + |x|^2 - \frac{3(d-3)}{|x|^2}$ in $L^2(\mathbb{R}^d)$, then $W_{a,l_\pm} \in L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ by the Hardy inequality for $d \geq 13$. However, as $r \to 0$,

$$|r^{l_-} e^{-\frac{r^2}{2}} M(-n; l_- + \frac{d}{2}; r^2)|^{2^*} \sim r^{2^* l_-} \sim r^{-d - \frac{d\sqrt{d^2-16d+40}}{d-2}} > r^{-d}.$$

Thus, by (3.8),

$$W_{-n,l_+} = r^{l_+} e^{-\frac{r^2}{2}} M(-n; l_+ + \frac{d}{2}; r^2)$$

is the only eigenfunctions of the operator $-\Delta + |x|^2 - \frac{3(d-3)}{|x|^2}$ in X_{rad} with eigenvalues $(d + 4n + 2l_+)$, for all $n \in \mathbb{N}$. By (3.7), the third eigenvalue σ_3 is given by

$$\sigma_3 = 10 + \sqrt{d^2 - 16d + 40} \tag{3.9}$$

and the fourth eigenvalue σ_4 is given by

$$\sigma_4 = 14 + \sqrt{d^2 - 16d + 40}. \tag{3.10}$$

The following lemma gives the estimate on $\mathfrak{m}(u_\infty)$ for $d \geq 13$.

Lemma 3.3. *For $d \geq 13$, we have*

$$\mathfrak{m}(u_\infty) = \begin{cases} 1 \text{ or } 2, & 13 \leq d \leq 15, \\ 1, & d \geq 16. \end{cases}$$

Proof. **Case $d \geq 16$.** Since $F(r) := r u_\infty(r)$ is monotonically decreasing (cf. [4, 35]), we have $F(r) < F(0) = \sqrt{d-3}$, which implies that $u_\infty(r) < \frac{\sqrt{d-3}}{r}$ for every $r > 0$. Note that $\omega_\infty \in (d-4, d)$ by [4, Theorem 1.2]. Then by $\sigma_3 > d$ for $d \geq 16$, as is clear from (3.9), we have

$$\omega_\infty + 3u_\infty^2 < \sigma_3 + 3\frac{d-3}{r^2} \quad \text{in } \mathbb{R}^d \text{ for } d \geq 16. \tag{3.11}$$

Since L_∞ has a compact resolvent in X_{rad} by Lemma 3.1, the spectrum of $-\Delta + |x|^2 - 3u_\infty^2$ in X_{rad} consists of isolated (simple) eigenvalues $\{\tau_j\}_{j \in \mathbb{N}}$ such that $\tau_j \to \infty$ as $j \to \infty$. For each simple eigenvalue τ_j, there exists a unique eigenfunction $\phi_j \in X_{\text{rad}}$ (up to scalar multiplication) which satisfies

$$-\Delta \phi_j + |x|^2 \phi_j - 3u_\infty^2 \phi_j = \tau_j \phi_j \quad \text{in } \mathbb{R}^d.$$

Moreover, ϕ_j has exact $j-1$ zeros. Since

$$\int_{\mathbb{R}^d} \left(|\nabla u_\infty|^2 + |x|^2 u_\infty^2\right) dx = \int_{\mathbb{R}^d} \left(\omega_\infty u_\infty^2 + u_\infty^4\right) dx < \int_{\mathbb{R}^d} \left(\omega_\infty u_\infty^2 + 3u_\infty^4\right) dx,$$

we have $\mathfrak{m}(u_\infty) \geq 1$ so that $\tau_1 < \omega_\infty$. Suppose that $\tau_2 \leq \omega_\infty$, then it follows from (3.11) that

$$\tau_2 + 3u_\infty^2 < \sigma_3 + 3\frac{d-3}{r^2} \quad \text{in } \mathbb{R}^d \text{ for } d \geq 16. \tag{3.12}$$

Recall that ϕ_2 has exact one zero on $(0, \infty)$ so that we can define

$$\phi_{2,f} = \begin{cases} \phi_2, & 0 \leq r < r_u, \\ 0, & r \geq r_u, \end{cases} \quad \text{and} \quad \phi_{2,l} = \begin{cases} 0, & 0 \leq r < r_u, \\ \phi_2, & r \geq r_u, \end{cases}$$

where r_u is the unique zero of ϕ_2. Then by (3.12), we have

$$\int_{\mathbb{R}^d} \left(|\nabla\phi_1|^2 + |x|^2\phi_1^2\right) dx = \int_{\mathbb{R}^d} (\tau_1 + 3u_\infty^2)\phi_1^2 dx < \int_{\mathbb{R}^d} (\sigma_3 + 3\frac{d-3}{r^2})\phi_1^2 dx,$$

$$\int_{\mathbb{R}^d} \left(|\nabla\phi_{2,f}|^2 + |x|^2\phi_{2,f}^2\right) dx = \int_{\mathbb{R}^d} (\tau_2 + 3u_\infty^2)\phi_{2,f}^2 dx < \int_{\mathbb{R}^d} (\sigma_3 + 3\frac{d-3}{r^2})\phi_{2,f}^2 dx$$

and

$$\int_{\mathbb{R}^d} \left(|\nabla\phi_{2,l}|^2 + |x|^2\phi_{2,l}^2\right) dx = \int_{\mathbb{R}^d} (\tau_2 + 3u_\infty^2)\phi_{2,l}^2 dx < \int_{\mathbb{R}^d} (\sigma_3 + 3\frac{d-3}{r^2})\phi_{2,l}^2 dx.$$

Since ϕ_1 is sign-constant and $\phi_{2,f}$ and $\phi_{2,l}$ share the unique zero at r_u, the functions ϕ_1, $\phi_{2,f}$ and $\phi_{2,l}$ are linearly independent. Indeed, if there exists c_1, $c_{2,f}$ and $c_{2,l}$ such that

$$c_1\phi_1 + c_{2,f}\phi_{2,f} + c_{2,l}\phi_{2,l} \equiv 0 \quad \text{in } \mathbb{R}^d,$$

then by $\phi_{2,f}(r_u) = \phi_{2,l}(r_u) = 0$, we have $c_1 = 0$. On the other hand, since $\phi_{2,f}\phi_{2,l} \equiv 0$, then we also have $c_{2,f} = c_{2,l} = 0$, which implies ϕ_1, $\phi_{2,f}$ and $\phi_{2,l}$ are linearly independent. However, σ_3 is the third eigenvalue of the operator $-\Delta + |x|^2 - 3\frac{d-3}{r^2}$ in X_{rad}, thus, $\mathfrak{m}(W_{-2,l_+}) = 2$, which is a contradiction. Therefore, $\tau_2 > \omega_\infty$ for $d \geq 16$, which implies $\mathfrak{m}(u_\infty) = 1$.

Case $13 \leq d \leq 15$**.** We use the same idea to show that $1 \leq \mathfrak{m}(u_\infty) \leq 2$. Indeed, since $\sigma_4 > \omega_\infty$ for $13 \leq d \leq 15$, as follows from (3.10), we have

$$\omega_\infty + 3u_\infty^2 < \sigma_4 + 3\frac{d-3}{r^2} \quad \text{in } \mathbb{R}^d \text{ for } 13 \leq d \leq 15. \tag{3.13}$$

If $\tau_3 \leq \omega_\infty$, then by (3.13),

$$\tau_3 + 3u_\infty^2 < \sigma_4 + 3\frac{d-3}{r^2} \quad \text{in } \mathbb{R}^d \text{ for } 13 \leq d \leq 15. \tag{3.14}$$

The third eigenfunction ϕ_3, corresponding to τ_3, has exact two zeros $\widetilde{r}_f < \widetilde{r}_l$. Moreover, by the Stum-Liouville theorem, it is well known that $\widetilde{r}_f < r_u < \widetilde{r}_l$. Let

$$\phi_{3,f} = \begin{cases} \phi_3, & 0 \leq r < \widetilde{r}_f, \\ 0, & r \geq \widetilde{r}_f, \end{cases} \qquad \phi_{3,l} = \begin{cases} 0, & 0 \leq r < \widetilde{r}_l, \\ \phi_3, & r \geq \widetilde{r}_l, \end{cases}$$

and

$$\phi_{3,m} = \begin{cases} 0, & 0 \le r < \widetilde{r}_f, \\ \phi_3, & r_f \le r < \widetilde{r}_l, \\ 0, & r \ge \widetilde{r}_l. \end{cases}$$

Then by similar arguments as used above, we can show from (3.14) that $\mathfrak{m}(W_{-3,l_+}) \ge 6$, which contradicts the fact that $\mathfrak{m}(W_{-3,l_+}) = 3$. Thus, we must have $\tau_3 > \omega_\infty$ for $13 \le d \le 15$, which implies that $\mathfrak{m}(u_\infty) \le 2$. □

Remark 3.3. *As a by-product, the proof of Lemma 3.3 shows that $\tau_1 < \omega_\infty < \tau_2$ for $d \ge 16$. Therefore, the homogeneous equation $L_\infty Z = 0$ has only trivial solutions in $X_{\rm rad}$ for $d \ge 16$. This implies that u_∞ is nondegenerate in $X_{\rm rad}$ for $d \ge 16$ in the following sense. The radial Z satisfies $Z = \mathcal{O}(r^{\omega_\infty - d} e^{-\frac{r^2}{2}})$ as $r \to +\infty$ and there exists $L_- \ne 0$ such that*

$$Z = L_- r^{l_-} + O(r^{l_+}, r^{l_-+2}) \quad \text{as } r \to 0.$$

This argument verifies the non-degeneracy Assumption 2.2 in [29] for $d \ge 16$. It is not clear if this assumption can be verified for $13 \le d \le 15$.

The proof of Theorem 1.2 follows immediately from Lemmas 3.2 and 3.3. □

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Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1
E-mail address: dmpeli@math.mcmaster.ca

Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z2
E-mail address: jcwei@math.ubc.ca

School of Mathematics, China University of Mining and Technology, Xuzhou, 221116, P.R. China
E-mail address: wuyz850306@cumt.edu.cn