Higher Order Conformally Invariant Equations in $\mathbb{R}^3$ with Prescribed Volume

Ali Hyder*

Department of Mathematics, University of British Columbia, Vancouver BC V6T1Z2, Canada

ali.hyder@math.ubc.ca

Juncheng Wei†

Department of Mathematics, University of British Columbia, Vancouver BC V6T1Z2, Canada

jcwei@math.ubc.ca

September 25, 2018

Abstract

In this paper we study the following conformally invariant poly-harmonic equation

$$\Delta^m u = -u^{\frac{3+2m}{3-2m}} \quad \text{in } \mathbb{R}^3, \quad u > 0,$$

with $m = 2, 3$. We prove the existence of radial solutions with prescribed volume $\int_{\mathbb{R}^3} u^{\frac{6}{3-2m}} dx$. We show that the set of all possible values of the volume is a bounded interval $(0, \Lambda^*]$ for $m = 2$, and it is $(0, \infty)$ for $m = 3$. This is in sharp contrast to $m = 1$ case in which the volume $\int_{\mathbb{R}^3} u^{\frac{6}{3-2m}} dx$ is a fixed value.

1 Introduction to the problem

We consider the negative exponent problem

$$\Delta^m u = -u^{\frac{3+2m}{3-2m}} \quad \text{in } \mathbb{R}^3, \quad u > 0, \tag{1}$$

where m is either 2 or 3. Geometrically, if u is a smooth solution to (1) then the conformal metric $g_u := u^{\frac{4}{3-2m}} |dx|^2$ ($|dx|^2$ is the Euclidean metric on $\mathbb{R}^3$) has constant Q-curvature on $\mathbb{R}^3$, see [1, 3, 5, 19]. Moreover, the volume of the metric g_u is

$$\int_{\mathbb{R}^3} dV_{g_u} = \int_{\mathbb{R}^3} \sqrt{|g_u|} dx = \int_{\mathbb{R}^3} u^{\frac{6}{3-2m}} dx,$$

*The author is supported by the Swiss National Science Foundation, Grant No. P2BSP2-172064

†The research is partially supported by NSERC

which is invariant under the scaling $u_\lambda(x) := \lambda^{\frac{3-2m}{2}} u(\lambda x)$ with $\lambda > 0$.

Equation (1) belongs to the class of conformally invariant equations. When $m = 1$ this is called Yamabe equation; while for $m = 2$ it is $Q-$curvature equation. In recent years Problem (1) has been extensively studied in [3, 4, 7, 10, 11, 15, 18] for $m = 2$, in [6, 5] for $m = 3$ and in [12, 16] for higher order case (but to an integral equation). We recall that radial solutions to (1) with $m = 2$ has either exact liner growth or exact quadratic growth at infinity, that is,

$$\lim_{r\to\infty} \frac{u(r)}{r} \in (0,\infty) \quad \text{or} \quad \lim_{r\to\infty} \frac{u(r)}{r^2} \in (0,\infty).$$

The solution with exact linear growth is unique (up to a scaling) and is given by

$$U_0(r) = \sqrt{\sqrt{1/15} + r^2}. \tag{2}$$

However, there are infinitely many (radial or nonradial) solutions with quadratic growth, see [4, 7, 10]. For $m = 3$, radial solutions grows either cubically or quatrically at infinity, that is,

$$\lim_{r\to\infty} \frac{u(r)}{r^3} \in (0,\infty) \quad \text{or} \quad \lim_{r\to\infty} \frac{u(r)}{r^4} \in (0,\infty).$$

In this case also we have an explicit solution which grows cubically at infinity, namely

$$U_1(r) = \left(315^{-\frac{1}{3}} + r^2\right)^{\frac{3}{2}}.$$

It is worth pointing out that both solutions U_0 and U_1 can be obtained by pulling back the round metric of S^3 via stereographic projection, and they satisfy an integral equation of the form

$$U(x) = c_m \int_{\mathbb{R}^3} |x-y|^p U^{\frac{3+2m}{3-2m}}(y)dy,$$

where $p = 1$ for $m = 2$ and $p = 3$ for $m = 3$. Nevertheless, U_1 is not unique (up to scaling) among the radial solutions having exact cubic growth at infinity.

We now state our main results concerning the existence of radial solutions to (1) with prescribed volume. For $m = 2$ we prove:

Theorem 1.1 *There exists a radial solution to*

$$\Delta^2 u = -\frac{1}{u^7} \quad \text{in } \mathbb{R}^3, \quad u > 0, \quad \Lambda_u := \int_{\mathbb{R}^3} \frac{dx}{u^6(x)} \tag{3}$$

if and an only if $\Lambda_u \in (0, \Lambda^*]$, where Λ^* is the volume of the metric g_{U_0}, that is,

$$\Lambda^* := \int_{\mathbb{R}^3} \frac{dx}{U_0^6(x)} = \int_{\mathbb{R}^3} \frac{dx}{(\sqrt{1/15} + |x|^2)^3}. \tag{4}$$

Moreover, if $\Lambda_u = \Lambda^*$ then up to a re-scaling we have $u = U_0$.

For $m = 3$ we prove existence of radial solution for every prescribed volume.

Theorem 1.2 *For every $\Lambda > 0$ there exists a positive radial solution to*

$$\Delta^3 u = -\frac{1}{u^3} \quad \text{in } \mathbb{R}^3 \tag{5}$$

such that

$$\int_{\mathbb{R}^3} \frac{dx}{u^2(x)} = \Lambda. \tag{6}$$

A similar phenomena has already been exhibited in a higher order Liouville equation, namely

$$(-\Delta)^{\frac{n}{2}} u = (n-1)! e^{nu} \quad \text{in } \mathbb{R}^n, \quad V := \int_{\mathbb{R}^n} e^{nu} dx < \infty. \tag{7}$$

(Here V is the volume of the conformal metric $g_u = e^{2u}|dx|^2$). More precisely, if u is a solution to (7) with $n = 4$ then necessarily $V \in (0, V^*]$, and $V = V^*$ if and only if u is a spherical solution, that is, for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ we have

$$u(x) = u_{\lambda, x_0}(x) := \log\left(\frac{2\lambda}{1 + \lambda^2 |x - x_0|^2}\right).$$

However, if $n \geq 5$ then for every $V \in (0, \infty)$ there exists a radial solution to (7). See [2, 8, 9, 13, 14, 17] and the references therein.

Finally, we remark that the upper bound of V in (7) with $n = 4$ comes from a Pohozaev type identity, and it holds for every solutions to (7) (radial and non-radial). However, from a similar Pohozaev type identity one does not get the same conclusion on the volume of the metric $g_u := u^{\frac{4}{3-2m}}$, compare [10, Lemma 2.3].

Notations For a radially symmetric function u we will write $u(|x|)$ to denote the same function $u(x)$.

2 Proof of the theorems

We shall use the following comparison lemma of two radial solutions to $\Delta^n u = f(u)$, whose proof is quite elementary.

Lemma 2.1 *Let f be a continuous and monotone increasing function on $(0, \infty)$. Let $u_1, u_2 \in C^{2k}([0, R))$ be two positive solutions of*

$$\begin{cases} \Delta^k u = f(u) & \text{on } (0, R) \\ \Delta^j u_1(0) \geq \Delta^j u_2(0) & \text{for every } j \in J \\ (\Delta^j u_1)'(0) = (\Delta^j u_2)'(0) = 0 & \text{for every } j \in J, \end{cases}$$

where $J := \{0, 1, \ldots, k-1\}$. Then $\Delta^j u_1 \geq \Delta^j u_2$ and $(\Delta^j u_1)' \geq (\Delta^j u_2)'$ on $(0, R)$ for every $j \in J$. Moreover, if $\Delta^j u_1(0) > \Delta^j u_2(0)$ for some $j \in J$ then $\Delta^j u_1 > \Delta^j u_2$ and $(\Delta^j u_1)' > (\Delta^j u_2)'$ on $(0, R)$ for every $j \in J$.

With the help of above comparison lemma and the fact that $\Delta U_0(\infty) = 0$ we prove Theorem 1.1.

Proof of Theorem 1.1 For $\rho \in \mathbb{R}$ we consider the solution u_ρ to the initial value problem

$$\begin{cases} \Delta^2 u_\rho = -\frac{1}{u_\rho^7} \\ u_\rho(0) = U_0(0) + \rho \\ \Delta u_\rho(0) = \Delta U_0(0) \\ u'_\rho(0) = (\Delta u_\rho)'(0) = 0. \end{cases} \tag{8}$$

It follows from Lemma 2.1 that u_ρ exists and $u_\rho > U_0$ on $(0,\infty)$ for every $\rho > 0$. In fact, $u_\rho(r) \geq \rho + U_0(r)$ on $(0,\infty)$, which implies that

$$\lim_{\rho\to\infty} \int_{\mathbb{R}^3} \frac{dx}{u_\rho^6(x)} = 0.$$

Hence, from the continuity of the map $\rho \mapsto \int_{\mathbb{R}^3} u_\rho^{-6} dx$ on $(0,\infty)$ we conclude that for every $\Lambda \in (0, \Lambda^*]$ there exists a solution u to (3) with $\Lambda = \Lambda_u$.

To prove the converse we let u to be a solution of (3) for some $\Lambda_u > 0$. We set $\bar{u}(x) := \lambda^{\frac{-1}{2}} u(\lambda x)$ where $\lambda > 0$ is such that $\Delta \bar{u}(0) = \Delta U_0(0)$. Then we have $\Lambda_u = \Lambda_{\bar{u}}$, and $\bar{u} = u_\rho$ for some $\rho \in \mathbb{R}$ where u_ρ is the solution to (8). We claim that $\rho \geq 0$. In order to prove the claim we assume by contradiction that $\rho < 0$. Then we have $u_\rho < U_0$ on $(0,\infty)$. Hence, using the identity

$$w(r) = w(0) + \frac{1}{4\pi} \int_0^r \frac{1}{t^2} \int_{B_t} \Delta w(x) dx dt \quad \text{for } w \in C^2_{rad}, \tag{9}$$

we obtain for $r \geq 1$

$$\Delta u_\rho(r) \leq \Delta U_0(r) - \varepsilon, \quad \varepsilon := \frac{1}{4\pi} \int_0^1 \frac{1}{t^2} \int_{B_t} \left(\frac{1}{u_\rho^7(x)} - \frac{1}{U_0^7(x)} \right) dx dt > 0.$$

Therefore, as $\Delta U_0(\infty) = 0$, we have $\Delta u_\rho(r) \leq -\frac{\varepsilon}{2}$ on (R, ∞) for some $R >> 1$. Using this in (9) we get $u_\rho(r) \leq C - C_\varepsilon r^2$ on $(0,\infty)$ for some $C_\varepsilon > 0$, a contradiction as $u_\rho > 0$ on $\mathbb{R}^3$.

Thus $\rho \geq 0$, and hence by Lemma 2.1 we have $\bar{u} \geq U_0$ on $(0,\infty)$. This in turn implies that $\Lambda_{\bar{u}} \leq \Lambda^*$, and $\Lambda_{\bar{u}} = \Lambda^*$ if and only if $\bar{u} = U_0$. $\square$

We now move to the proof of Theorem 1.2. We start with the following lemma.

Lemma 2.2 *For k large and $\varepsilon \in (0,1)$ there exists a positive entire radial solution to*

$$\begin{cases} \Delta^3 u = -\frac{1}{u^3} \\ u(0) = k \\ \Delta u(0) = -\varepsilon \\ \Delta^2 u(0) = 1 \\ u'(0) = (\Delta u)'(0) = (\Delta^2 u)'(0) = 0. \end{cases} \tag{10}$$

Moreover, if u is a positive entire radial solution to (10) *for some $\varepsilon \in \mathbb{R}$ then necessarily $\varepsilon \leq \sqrt{\frac{6k}{5}}$, and the solution u satisfies*

$$k - \frac{\varepsilon}{6}r^2 \leq u(r) \leq k - \frac{\varepsilon}{6}r^2 + \frac{r^4}{120} \quad \textit{on } (0, \infty). \tag{11}$$

Proof. It follows from the ODE local existence theorem that for every $\varepsilon > 0$ there exists a unique positive solution to (10) in a neighborhood of the origin. We let $(0, \delta)$ to be the maximum interval of existence. From the identity (9) we see that $\Delta^2 u$ is strictly monotone decreasing on $(0, \delta)$. Let $\bar{\delta} \in (0, \delta]$ be the largest number such that

$$\Delta^2 u \geq \frac{1}{2} \quad \text{on } (0, \bar{\delta}). \tag{12}$$

Using this lower bound in (9) with $w = \Delta u$ one obtains

$$\Delta u(r) \geq -\varepsilon + \frac{1}{12}r^2 \quad \text{for } r \in (0, \bar{\delta}).$$

Again by (9) with $w = u$ we obtain for $r \in (0, \bar{\delta})$

$$u(r) \geq k - \frac{\varepsilon}{6}r^2 + \frac{r^4}{240} \geq \frac{k}{2} + \frac{r^4}{250}, \tag{13}$$

for k sufficiently large and for every $\varepsilon \in (0, 1)$. Using this lower bound of u we obtain a lower bound of $\Delta^2 u$. Indeed, for $r \in (0, \bar{\delta})$ and for k sufficiently large, we have

$$\begin{aligned} \Delta^2 u(r) &\geq 1 - \frac{1}{4\pi} \int_0^r \frac{1}{t^2} \int_{B_t} \frac{dx}{(\frac{k}{2} + \frac{|x|^4}{250})^3} dt \\ &\geq 1 - \frac{1}{4\pi} \int_0^\infty \frac{1}{t^2} \int_{B_t} \frac{dx}{(\frac{k}{2} + \frac{|x|^4}{250})^3} dt \\ &\geq \frac{2}{3}. \end{aligned} \tag{14}$$

Thus, from the definition of $\bar{\delta}$ we get $\bar{\delta} = \delta$, and going back to (13) we conclude that $\delta = \infty$. This proves the first part of the lemma.

Now we let u be the positive entire radial solution to (10) for some $\varepsilon \in \mathbb{R}$. As $\Delta^2 u$ is strictly monotone decreasing on $(0, \infty)$ we have

$$0 \leq \Delta^2 u(\infty) \leq \Delta^2 u \leq 1 \quad \text{on } (0, \infty).$$

This implies that Δu is monotone increasing on $(0, \infty)$, and a repeated use of (9) gives (11). Finally, the upper bound of u in (11) and the positivity of u implies that $\varepsilon \leq \sqrt{\frac{6k}{5}}$.

We conclude the lemma. $\square$

As a consequence of the above lemma the number ε_k^* given by (for k large)

$$\varepsilon_k^* := \sup\{\varepsilon > 0 : (10) \text{ has a positive entire solution}\}$$

exists, and it satisfies the estimate $\varepsilon_k^* \leq \sqrt{\frac{6k}{5}}$. Moreover, for every $\varepsilon \in (-\infty, \varepsilon_k^*)$ there exists a positive entire solution to (10), thanks to Lemma 2.1.

Lemma 2.3 *For k large* (10) *has a positive entire solution with $\varepsilon = \varepsilon_k^*$.*

Proof. For simplicity we ignore the subscript k and write ε^* instead of ε_k^*. Let u be the solution to (10) with $\varepsilon = \varepsilon^*$, and let $(0, R)$ be the maximum interval of existence. We assume by contradiction that $R < \infty$. Then necessarily we have

$$\lim_{r \to R-} u(r) = 0.$$

It follows from the definition of ε^* that there exists a sequence of positive entire solutions (u_n) to (10) with $\Delta u_n(0) \downarrow -\varepsilon^*$. Then, from the continuous dependence on the initial data, we have that $u_n \to u$ locally uniformly in $[0, R)$. In particular, there exists $x_n \to R$ such that $u_n(x_n) \to 0$. We claim that there exists $C > 0$ such that

$$u_n(r) \leq u_n(x_n) + C(r - x_n) \quad \text{for } x_n \leq r \leq x_n + 1. \tag{15}$$

Indeed, as $0 < \Delta^2 u_n \leq 1$ on $(0, \infty)$, by (9) we obtain

$$-\varepsilon^* \leq \Delta u_n(r) \leq r^2 \quad \text{on } (0, \infty).$$

This gives $|u_n'| \leq C$ on $(0, R+3)$ for some $C > 0$, and hence we have (15). Therefore, by (9) and together with (15) we get

$$\begin{aligned}
\Delta^2 u_n(R+3) &\leq 1 - \frac{1}{4\pi} \int_{R+2}^{R+3} \frac{1}{t^2} \int_{x_n < |x| < x_n + 1} \frac{dx}{(u_n(x_n) + C(|x| - x_n))^3} dt \\
&\leq 1 - \frac{1}{4\pi} \frac{1}{(R+3)^2} \int_{x_n < |x| < x_n + 1} \frac{dx}{(u_n(x_n) + C(|x| - x_n))^3} \\
&\xrightarrow{n \to \infty} -\infty,
\end{aligned}$$

a contradiction as $\Delta^2 u_n > 0$ on $(0, \infty)$.

We conclude the lemma. $\square$

Lemma 2.4 *Let u be a positive entire radial solution to* (5)*. Assume that $\Delta^2 u(\infty) > 0$. Then there exists a positive entire radial solution v to* (5) *such that*

$$v(0) = u(0), \quad \Delta v(0) < \Delta u(0) \quad \text{and } \Delta^2 v(0) = \Delta^2 u(0).$$

Proof. For $\rho > 0$ small we consider the initial value problem

$$\begin{cases} \Delta^3 v = -\frac{1}{v^3} \\ v(0) = u(0) \\ \Delta v(0) = \Delta u(0) - \rho \\ \Delta^2 v(0) = \Delta^2 u(0) \\ v'(0) = (\Delta v)'(0) = (\Delta^2 v)'(0) = 0. \end{cases} \tag{16}$$

Since $\Delta^2 u(\infty) > 0$, it follows that $u(r) \geq \delta r^4$ at infinity for some $\delta > 0$. We fix $R_1 >> 1$ such that

$$\int_{R_1}^{\infty} \frac{1}{t^2} \int_{B_t} \frac{dx}{u^3(x)} dt < \varepsilon,$$

where $\varepsilon > 0$ will be chosen later. By continuous dependence on the initial data we can choose $\rho > 0$ sufficiently small such that the solution $v = v(\rho, u)$ to (16) exists on $(0, R_1)$ and

$$u - v \leq \varepsilon \quad \text{on } (0, R_1) \quad \text{and } \rho r^2 \leq \frac{1}{6} u(r) \quad \text{on } (0, \infty).$$

We claim that for such $\rho > 0$ the solution v exists entirely.

In order to prove the claim we let $R_2 > 0$ (possibly the largest one) be such that $v \geq \frac{u}{2}$ on $(0, R_2)$. (Note that $v \leq u$ on the iterval of existence, and for $\varepsilon > 0$ small enough we have $R_2 > R_1$). Then for $0 < r < R_2$ we have

$$\begin{aligned} \Delta^2 v(r) - \Delta^2 u(r) &= O(1) \int_0^r \frac{1}{t^2} \int_{B_t} \frac{u(x) - v(x)}{u(x) v^3(x)} dx dt \\ &\geq -C_1 \varepsilon - O(1) \int_{R_1}^{R_2} \frac{1}{t^2} \int_{B_t} \frac{dx}{u^3(x)} dt \\ &\geq -C_2 \varepsilon. \end{aligned}$$

The above estimate and a repeated use of (9) leads to

$$v(r) \geq u(r) - \frac{\rho}{6} r^2 - C_3 \varepsilon r^4.$$

Now we fix $\varepsilon > 0$ sufficiently small so that $C_3 \varepsilon r^4 < \frac{1}{6} u(r)$ on $(0, \infty)$. Then we have

$$v(r) \geq \frac{2}{3} u(r) \quad \text{on } (0, R_2).$$

This proves the claim. $\square$

Proof of Theorem 1.2 Let (u_k) be a sequence of positive entire radial solutions to (10) with $\varepsilon = \varepsilon_k^*$ as given by Lemma 2.3. We claim that (u_k) satisfies

$$\int_{\mathbb{R}^3} \frac{dx}{u_k^2(x)} \xrightarrow{k \to \infty} \infty. \tag{17}$$

In order to prove the claim first we note that by Lemma 2.4 we have $\Delta^2 u_k(\infty) = 0$, that is

$$1 = \frac{1}{4\pi} \int_0^\infty \frac{1}{t^2} \int_{B_t} \frac{dx}{u_k^3(x)} dt. \tag{18}$$

Moreover, $u_k \to \infty$ locally uniformly in $[0, \infty)$, thanks to (11) and the estimate $\varepsilon_k^* \leq \sqrt{\frac{6k}{5}}$. We consider the following two cases, and we show that (17) holds in each case.

Case 1 $\min_{(0,\infty)} u_k \to \infty$.

Since $u_k \to \infty$ locally uniformly in $\mathbb{R}^3$, from (18) we obtain

$$1 = o(1) + \frac{1}{4\pi} \int_1^\infty \frac{1}{t^2} \int_{B_t} \frac{dx}{u_k^3(x)} dt \leq o(1) + \frac{1}{4\pi \min_{\mathbb{R}^3} u_k} \int_{\mathbb{R}^3} \frac{dx}{u_k^2(x)},$$

which gives (17).

Case 2 $\min_{(0,\infty)} u_k =: u_k(x_k) \leq C$.

We claim that

$$u_k(x_k + r) \leq u_k(x_k) + 1 \quad \text{for } 0 \leq r \leq \frac{1}{x_k}.$$

In order to prove the claim we first note that $u_k' \geq 0$ on $[x_k, \infty)$ and $u_k'(x_k) = 0$. Moreover, as $\Delta^2 u_k \leq \Delta^2 u_k(0) = 1$, by (9) we have

$$u_k''(x_k + r) + \frac{2}{x_k + r} u_k'(x_k + r) = \Delta u_k(x_k + r) \leq \frac{1}{6}(x_k + r)^2.$$

Hence, $u_k''(x_k + r) \leq \frac{1}{6}(x_k + r)^2$, and by a Taylor expansion, we have our claim. Therefore, as $x_k \to \infty$, we get

$$\begin{aligned} \int_{x_k < |x| < x_k + \frac{1}{x_k}} \frac{dx}{u_k^2(x)} &\geq \frac{1}{(1 + u_k(x_k))^2} \left((x_k + \frac{1}{x_k})^3 - x_k^3 \right) \\ &\geq \frac{3x_k}{(1 + u_k(x_k))^2} \\ &\xrightarrow{k\to\infty} \infty. \end{aligned}$$

This proves (17).

Theorem 1.2 follows immediately as the integral in (17) depends continuously on the initial data, and

$$\int_{\mathbb{R}^3} \frac{dx}{u_{\rho,k}^2(x)} \xrightarrow{\rho\to\infty} 0,$$

where $u_{\rho,k}$ is the solution to (10) with $\Delta u_{\rho,k}(0) = \rho > -\varepsilon_k^*$. $\square$

References

[1] T. P. Branson: *Group representations arising from Lorentz conformal geometry*, J. Funct. Anal. **74** (1987) 199-291.

[2] S-Y. A. Chang, W. Chen: *A note on a class of higher order conformally covariant equations*, Discrete Contin. Dynam. Systems **63** (2001), 275-281.

[3] Y. S. Choi, X. Xu: *Nonlinear biharmonic equations with negative exponents*, J. Diff. Equations, **246** (2009) 216-234.

[4] T. V. Duoc, Q. A. Ngô: *A note on positive radial solutions of $\Delta^2 u + u^{-q} = 0$ in $\mathbb{R}^3$ with exactly quadratic growth at infinity*, Diff. Int. Equations **30** (2017), no. 11-12, 917-928.

[5] T. V. Duoc, Q. A. Ngô: *Exact growth at infinity for radial solutions to $\Delta^3 u + u^{-q} = 0$ in $\mathbb{R}^3$*, Preprint (2017), `ftp://file.viasm.org/Web/TienAnPham-17/Preprint_1702.pdf`

[6] X. Feng, X. Xu: *Entire solutions of an integral equation in $\mathbb{R}^5$*, ISRN Math. Anal. (2013), Art. ID 384394, 17 pp.

[7] I. Guerra: *A note on nonlinear biharmonic equations with negative exponents*, J. Differential Equations **253** (2012) 3147-3157.

[8] X. Hunag, D. Ye: *Conformal metrics in $\mathbb{R}^{2m}$ with constant Q-curvature and arbitrary volume*, Calc. Var. Partial Differential Equations **54** (2015), 3373-3384.

[9] A. Hyder: *Conformally Euclidean metrics on $\mathbb{R}^n$ with arbitrary total Q-curvature*, Anal. PDE **10** (2017), no. 3, 635-652.

[10] A. Hyder, J. Wei: *Non-radial solutions to a biharmonic equation with negative exponent*, Preprint (2018), `http://www.math.ubc.ca/~ali.hyder/W/HW.pdf`

[11] B. Lai: *A new proof of I. Guerra's results concerning nonlinear biharmonic equations with negative exponents*, J. Math. Anal. Appl. **418** (2014) 469-475.

[12] Y. Li: *Remarks on some conformally invariant integral equations: The method of moving spheres*, J. Eur. Math. Soc. **6** (2004) 1-28.

[13] C. S. Lin: *A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$*, Comment. Math. Helv. **73** (1998) 206-231.

[14] L. Martinazzi: *Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature and large volume*, Ann. Inst. H. Poincar Anal. Non Linaire **30** (2013), no. 6, 969-982.

[15] P. J. McKenna, W. Reichel: *Radial solutions of singular nonlinear biharmonic equations and applications to conformal geometry*, Electron. J. Differential Equations **37** (2003) 1-13.

[16] Q. A. Ngô: *Classification of entire solutions of $(-\Delta)^n u + u^{4n-1} = 0$ with exact linear growth at infinity in $\mathbb{R}^{2n-1}$*, Proc. Amer. Math. Soc. **146** (2018), no. 6, 2585-2600.

[17] J. Wei, D. Ye: *Nonradial solutions for a conformally invariant fourth order equation in* $\mathbb{R}^4$, Calc. Var. Partial Differential Equations **32** (2008), no. 3, 373-386.

[18] X. Xu: *Exact solutions of nonlinear conformally invariant integral equations in* $\mathbb{R}^3$, Adv. Math. **194** (2005) 485-503.

[19] P. Yang, M. Zhu: *On the Paneitz energy on standard three sphere*, ESAIM Control Optim. Calc. Var. **10** (2004), no. 2, 211-223.