CHAPTER 6

Existence and Stability of Spikes for the Gierer–Meinhardt System

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1. Introduction

It is a common belief that diffusion is a smoothing and trivializing process. Indeed, this is the case for a single diffusion equation. Consider the heat equation

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0, +\infty), \\ u(x,0) = u_0(x) \geqslant 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty). \end{cases} \tag{1.1}$$

Assume that $u_0(x)$ is continuous. It is known that $u(x,t)$ is smooth for $t > 0$ (*smoothing*), and $u(x,t) \to \frac{1}{|\Omega|} \int_\Omega u_0(x)\,dx$ as $t \to +\infty$ (*trivializing*). A similar result holds when a source/sink term (or a reaction term) is present. Namely, for the problem

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega \times (0, +\infty), \\ u(x,0) = u_0(x) \geqslant 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \tag{1.2}$$

it is known that when Ω is convex, the only stable solutions are constants [5,46]. Thus there are only trivial patterns (constant solutions) for single reaction–diffusion equations (on convex domains).

On the other hand, it is important to be able to use diffusion (and reaction) to model pattern formations in various branches of science (e.g., biology and chemistry). One important question is: can we get *non-trivial* patterns (stable non-trivial solutions) for systems of reaction–diffusion equations?

Let us consider the following system of reaction–diffusion equations:

$$\begin{cases} u_t = D_u \Delta u + f(u,v) & \text{in } \Omega \times (0, +\infty), \\ v_t = D_v \Delta v + g(u,v) & \text{in } \Omega \times (0, +\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty). \end{cases} \tag{1.3}$$

In 1957, Turing [68] proposed a mathematical model for morphogenesis, which describes the development of complex organisms from a single shell. He speculated that localized peaks in concentration of a chemical substance, known as an inducer or morphogen, could be responsible for a group of cells developing differently from the surrounding cells. He then demonstrated, with linear analysis, how a non-linear reaction diffusion system like (1.3) could possibly generate such isolated peaks. Later in 1972, Gierer and Meinhardt [21] demonstrated the existence of such solution numerically for the following (so-called Gierer–Meinhardt system)

$$\text{(GM)} \quad \begin{cases} \frac{\partial a}{\partial t} = \epsilon^2 \Delta a - a + \frac{a^p}{h^q}, & x \in \Omega, t > 0, \\ \tau \frac{\partial h}{\partial t} = D \Delta h - h + \frac{a^r}{h^s}, & x \in \Omega,\ t > 0, \\ \frac{\partial a}{\partial \nu} = \frac{\partial h}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Here, the unknowns $a = a(x,t)$ and $h = h(x,t)$ represent the respective concentrations at point $x \in \Omega \subset \mathbb{R}^N$ and at time t of the biochemical called an activator and an inhibitor; $\epsilon > 0$, $D > 0$, $\tau > 0$ are all positive constants; $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in $\mathbb{R}^N$; Ω is a smooth bounded domain in $\mathbb{R}^N$; $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$. The exponents (p,q,r,s) are assumed to satisfy the condition

$$p > 1, \quad q > 0, \quad r > 0, \quad s \geqslant 0, \quad \text{and} \quad \gamma := \frac{qr}{(p-1)(s+1)} > 1.$$

Gierer–Meinhardt system was used in [21] to model head formation in the hydra. *Hydra*, an animal of a few millimeters in length, is made up of approximately 100,000 cells of about fifteen different types. It consists of a “head” region located at one end along its length. Typical experiments on *hydra* involve removing part of the “head” region and transplanting it to other parts of the body column. Then, a new “head” will form if and only if the transplanted area is sufficiently far from the (old) head. These observations have led to the assumption of the existence of two chemical substances—a *slowly* diffusing (i.e., $\epsilon \ll 1$) activator a and a *fast* diffusing (i.e., $D \gg \epsilon$) inhibitor h.

To understand the dynamics of (GM), it is helpful to consider first its corresponding “kinetic system”

$$\begin{cases} a_t = -a + a^p/h^q, \\ \tau h_t = -h + a^r/h^s. \end{cases} \tag{1.4}$$

This system has a unique constant steady state $a \equiv 1$, $h \equiv 1$. For $0 < \tau < \frac{qr}{(p-1)(s+1)}$ it is easy to see that the constant solution $a \equiv 1, h \equiv 1$ is stable as a steady state of (ODE).

However, if $\frac{\epsilon}{\sqrt{D}}$ is small, it is not hard to see that the constant steady state $a \equiv 1, h \equiv 1$ of (GM) becomes unstable and bifurcation may occur. This phenomenon is generally referred to as *Turing's diffusion-driven instability*. (A general criteria for this can be found in Murray's book [47].)

There are many other reaction–diffusion systems which exhibit Turing's diffusion-driven instability: they include Gray–Scott model from chemical reactor theory, Schnakenberg model, Sel'kov model, Lengyl–Epstein model, Thomas model, Keener–Tyson model, Brusselator, Oregonator, etc. For introduction and discussion on these general Turing models, we refer to the book [47]. A survey of mathematical modeling of biological and chemical phenomena using reaction–diffusion systems is given in [38]. Mathematical modeling of patterns in biological morphogenesis using extensions of GM model are discussed in [36] and [48].

Several common characteristics of Turing type reaction–diffusion systems include: first, they are *non-variational*, i.e., they do not have Lyapunov or energy functional so standard variational (or energy) method cannot be applied; second, they are *non-cooperative*, i.e., they do not have Maximum Principles so sub-super-solution method cannot be applied; third, they support finite-amplitude spatial-temporal patterns of remarkable diversity and complexity, such as stable spikes, layers, stripes, spot-splitting, traveling waves, etc. (See [63].) The study of these RD systems not only increases our knowledge on Turing patterns, but also induces new tools and techniques to deal with other problems which may share similar characteristics.

The most interesting phenomena associated with (GM) is the existence of stable spikes and stripes. The numerical studies of [21] and more recent those of [31] have revealed that in the limit $\epsilon \to 0$, the (GM) system seems to have stable stationary solutions with the property that the activator concentration is localized around a finite number of points in $\bar{\Omega}$. Moreover, as $\epsilon \to 0$, the pattern exhibits a "*spike layer phenomenon*" by which we mean that the activator concentration is localized in narrower and narrower regions around some points and eventually shrinks to a certain number of points as $\epsilon \to 0$, whereas the maximum value of the activator concentration diverges to $+\infty$.

Such kind of point-condensation phenomena has generated a lot of interests both mathematically and biologically in recent years. The purpose of this paper is to report on the current trend and status of such studies (up to June, 2006). We shall not give most of proofs. For more details, please see the references and therein.

In the study of spiky patterns (or concentration phenomena), two fundamental methods emerge. The first one is the so-called "Localized Energy Method", or *LEM* in short. LEM is a combination of traditional Lyapunov–Schmidt reduction method with variational techniques. This is a very useful tool to construct solutions with various concentration behavior, such as spikes, layers, or vortices. The second method is the so-called "Nonlocal Eigenvalue Problem Method", or *NLEP* in short. This deals with eigenvalue problems which are non-selfadjoint. It plays fundamental role in the study of stability of spike patterns. In this survey, I shall illustrate these two methods in details in the hope that they may find applications in other problems.

Throughout this paper, unless otherwise stated, we always assume that

$$\epsilon \ll 1, \quad D \text{ is finite}, \quad \tau \geqslant 0. \tag{1.5}$$

2. Steady states in shadow system case

2.1. *Reduction to single equation*

In general, the full (GM) system is very difficult to study. A very useful idea, which goes back to Keener and Nishiura, is to consider the so-called *shadow system*. Namely, we let $D \to +\infty$ first. Suppose that the quantity $-h + a^p/h^q$ remains bounded, then we obtain

$$\Delta h \to 0, \quad \frac{\partial h}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Thus $h(x,t) \to \xi(t)$, a constant. To derive the equation for $\xi(t)$, we integrate both sides of the equation for h over Ω and then we obtain the following so-called shadow system

$$\begin{cases} a_t = \epsilon^2 \Delta a - a + a^p/\xi^q \quad \text{in } \Omega, \\ \tau \xi_t = -\xi + \frac{1}{|\Omega|} \int_\Omega a^r \, dx/\xi^s, \\ a > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial a}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

The advantage of shadow system is that by a simple scaling,

$$a = \xi^{\frac{q}{p-1}} u, \quad \xi = \left(\frac{1}{|\Omega|} \int_\Omega u^r \right)^{\frac{p-1}{(p-1)(s+1)-qr}}, \tag{2.3}$$

the stationary shadow system can be reduced to a single equation

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

whose energy functional is given by

$$J_\epsilon[u] := \int_\Omega \left(\frac{\epsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - \frac{1}{p+1} u_+^{p+1} \right) dx,$$
$$\text{where } u_+ = \max(u, 0), \tag{2.5}$$

for $u \in H^1(\Omega)$.

First we give some definitions on solutions to (2.4). A family of solutions $\{u_\epsilon\}$ to (2.4) are called *concentrated solutions* if there exists a subset $\Gamma \subset \bar{\Omega}$ such that $u_\epsilon \to 0$ in $C^0_{loc}(\bar{\Omega} \backslash \Gamma)$ and $\max_{x \in \Gamma} u_\epsilon(x) \geqslant c_0 > 0$. If Γ consists of only points in $\bar{\Omega}$, these kind solutions are called *point condensations*. Among point condensations, there are two kinds: *spikes* and *bubbles*. *Spikes* are those concentrated solutions such that $\max_{x \in \bar{\Omega}} u_\epsilon \leqslant C$, while *bubbles* are those with $\max_{x \in \bar{\Omega}} u_\epsilon \to +\infty$. If the dimension of Γ is positive, concentrated solutions are also called *layers*. (Similar definitions can also be given for solutions of the full Gierer–Meinhardt system by considering the activator a only.)

In the following, we discuss the existence of all kinds of *concentrated solutions* to (2.4).

2.2. *Subcritical case: spikes to (2.4)*

Let us assume first that $1 < p < (\frac{N+2}{N-2})_+$ ($= \frac{N+2}{N-2}$ if $N \geqslant 3$; $= +\infty$ when $N = 1, 2$). In this case, problem (2.4) can be studied by traditional variational methods, for example, Mountain-Pass method, or Nehari's solution manifold method. For Mountain-Pass method, by taking a function $e(x) \equiv k$ for some constant k in Ω, and choosing k large enough, we have $J_\epsilon(e) < 0$, for all $\epsilon \in (0, 1)$. Then for each $\epsilon \in (0, 1)$, we can define the so-called mountain-pass value

$$c_\epsilon = \inf_{h \in \Gamma} \max_{0 \leqslant t \leqslant 1} J_\epsilon\big[h(t)\big] \tag{2.6}$$

where $\Gamma = \{h\colon [0, 1] \to H^1(\Omega) \mid |h(t) \text{ is continuous}, \ h(0) = 0, \ h(1) = e\}$.

It is easy to see that (Lemma 2.1 of [57]), c_ϵ can be characterized by

$$c_\epsilon = \inf_{u \not\equiv 0, \ u \in H^1(\Omega)} \sup_{t > 0} J_\epsilon[tu], \tag{2.7}$$

which can be shown to be the least among all non-zero critical values of J_ϵ. (This formulation (2.7) is sometimes referred to as the Nehari manifold technique.) Moreover, c_ϵ is attained by some function u_ϵ which is then called a *least-energy solution*.

In a series of papers [57] and [58], Ni and Takagi studied the so-called *least energy* solutions and proved the following theorem

THEOREM 2.1. (See [57,58].) *For ϵ sufficiently small, there exists a mountain-pass solution u_ϵ which is also least-energy solution such that u_ϵ has only one local maximum point $P_\epsilon \in \partial\Omega$ and $u_\epsilon \to 0$ in $C^2_{\mathrm{loc}}(\bar{\Omega}\backslash\{P_\epsilon\})$. Moreover, as $\epsilon \to 0$,*

$$H(P_\epsilon) \to \max_{P\in\partial\Omega} H(P),$$

where $H(P)$ is the mean curvature function for $P \in \partial\Omega$, and $u_\epsilon(P_\epsilon + \epsilon y) \to w(y)$ uniformly in $\Omega_{\epsilon,P_\epsilon} = \{y \mid P_\epsilon + \epsilon y \in \Omega\}$, where $w(y)$ is the unique solution of the following

$$\begin{cases} \Delta w - w + w^p = 0, & w > 0 \text{ in } \mathbb{R}^N, \\ w(0) = \max_{y\in\mathbb{R}^N} w(y), & w \to 0 \text{ at } \infty. \end{cases} \tag{2.8}$$

REMARK 2.2.1. The existence of ground state to (2.8) is well known. The radial symmetry of w follows from the famous Gidas–Ni–Nirenberg theorem [22]. The uniqueness of w is proved in [39].

REMARK 2.2.2. The proof of Theorem 2.1 is by expansion of energy:

$$c_\epsilon = \epsilon^N\left[\frac{1}{2}I[w] - c_1\epsilon H(P_\epsilon) + o(\epsilon)\right] \tag{2.9}$$

where

$$I[w] = \int_{\mathbb{R}^N}\left(\frac{1}{2}\left(|\nabla w|^2 + w^2\right) - \frac{1}{p+1}w^{p+1}\right)$$

is the energy of the ground state. A further expansion of c_ϵ up to the ϵ^2 order is given by [90]

$$c_\epsilon = \epsilon^N\left[\frac{1}{2}I[w] - c_1\epsilon H(P_\epsilon) + \epsilon^2\left[c_2\left(H(P_\epsilon)\right)^2 + c_3R(P_\epsilon)\right] + o\left(\epsilon^2\right)\right] \tag{2.10}$$

where c_1, c_2, c_3 are generic constants and $R(P_\epsilon)$ is the scalar curvature at P_ϵ. In particular $c_1, c_3 > 0$. (When $N = 2$, a further expansion to the order of ϵ^3 is also given in [91].) Some applications of the formula (2.10) are given in [90].

Since then there has been a lot of studies on problem (2.4). A general principle is that boundary spike solutions are related to the boundary mean-curvature $H(P)$, $P \in \partial\Omega$, while interior spike solutions are related to the distance function $d(P, \partial\Omega)$. Note also that for boundary spike the order is usually $O(\epsilon)$ while for interior spikes the order is $O(e^{-\frac{d}{\epsilon}})$ for some $d > 0$.

Let me mention some results on multiple boundary and interior peaked solutions.

For single and multiple boundary spikes, Gui [26] first constructed multiple boundary spike solutions at multiple local maximum points of $H(P)$, using variational method. Wei [73], Wei and Winter [82,83] (independently by Bates, Dancer and Shi [4]) constructed single and multiple boundary spike solutions at multiple non-degenerate critical points of $H(P)$, using Lyapunov–Schmidt reduction method. Y.Y. Li [41], del Pino, Felmer and Wei [16] constructed single and multiple boundary spikes in the degeneracy case. Using Localized Energy method (LEM), a clustered solution is also constructed by Gui, Wei and Winter [29] (independently by Dancer and Yan [9]).

THEOREM 2.2. (See [9,29].) *Let Γ be a subset of $\partial\Omega$, where it holds*

$$\min_{\partial\Gamma} H(P) > \min_{\Gamma} H(P). \tag{2.11}$$

Then for any fixed positive integer k, there exists ϵ_k such that for $\epsilon < \epsilon_k$, problem (2.4) *has a solution u_ϵ with k boundary local maximum points $P_{j,\epsilon} \in \Gamma$. Furthermore, $H(P_{j,\epsilon}) \to \min_\Gamma H(P)$.*

The energy expansion for K-boundary spikes is

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{K}{2} I[w] - c_1 \epsilon \sum_{j=1}^{K} H(P_{j,\epsilon}) - \sum_{i \neq j} \big(\gamma_0 + o(1)\big) w\left(\frac{|P_{i,\epsilon} - P_{j,\epsilon}|}{\epsilon} \right) \right]. \tag{2.12}$$

For single and multiple interior peaked solutions, the situation is quite different, as the errors are *exponentially small*. Wei [79,74] first constructed single interior peak solution at a strictly local maximum point of $d(P, \partial\Omega)$. Gui and Wei [27] proved the following

THEOREM 2.3. (See [27].) *For any fixed positive integer k, there exists ϵ_k such that for $\epsilon < \epsilon_k$, problem* (2.4) *has a solution u_ϵ with k interior local maximum points $P_{j,\epsilon} \in \Omega$. Moreover, $(P_{1,\epsilon}, \dots, P_{k,\epsilon})$ approaches a limiting sphere-packing position, i.e.,*

$$\varphi_k(P_{1,\epsilon}, \dots, P_{k,\epsilon}) \to \max_{(P_1,\dots,P_k)\in\Omega^k} \varphi_k(P_1, \dots, P_k) \tag{2.13}$$

where

$$\varphi_k(P_1, \dots, P_k) = \min_{i,j,l,i\neq j} \big(|P_i - P_j|, 2d(P_l, \partial\Omega)\big). \tag{2.14}$$

The energy expansion for K-interior spikes is

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[K I[w] - \gamma_0 \sum_{j=1}^{K} e^{-\frac{2d(P_{j,\epsilon}, \partial\Omega)}{\epsilon}} \right. \\ \left. - \gamma_1 \sum_{i \neq j} w\left(\frac{|P_{i,\epsilon} - P_{j,\epsilon}|}{\epsilon} \right) \right]. \tag{2.15}$$

Grossi, Pistoia and Wei [30] further showed that there is an one-to-one correspondence between the (sub-differential) critical points of φ_k and k-interior peaked solutions.

Concerning the existence of mixed-boundary-interior-spikes, the following theorem gives a complete answer.

THEOREM 2.4. (See [28].) *For any two fixed positive integers k, l, there exists $\epsilon_{k,l}$ such that for $\epsilon < \epsilon_{k,l}$, problem* (2.4) *has a solution u_ϵ with k interior local maximum points and l boundary maximum points.*

Theorems 2.2, 2.3 and 2.4 imply that the number of solutions to (2.4) goes to infinity as $\epsilon \to 0$. Recently, the following lower bound on number of solutions is obtained:

THEOREM 2.5. (See [44].) *There exists an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ and for each integer K bounded by*

$$1 \leqslant K \leqslant \frac{\alpha_{N,\Omega,f}}{\epsilon^N (|\ln \epsilon|)^N}$$

where $\alpha_{N,\Omega,p}$ is a constant depending on N, Ω and p only, there exists a solution with K interior peaks. (*An explicit formula for $\alpha_{N,\Omega,p}$ is also given.*) *As a consequence, we obtain that for ϵ sufficiently small, there exists at least $[\frac{\alpha_{N,\Omega,p}}{\epsilon^N (|\ln \epsilon|)^N}]$ number of solutions. Moreover, for each $\beta \in (0, N)$ there exists solution with energy in the order of $\epsilon^{N-\beta}$.*

Theorems 2.2, 2.3, 2.4 and 2.5 can all be proved by the powerful method—*Localized Energy Method*—which was first introduced in [27]. We shall discuss it next.

2.3. *Localized energy method (LEM)*

We illustrate a general method in finding solutions with concentrating behavior—the so-called *Localized Energy Method*, or *LEM* in short. The advantage of such method is that it can be applied to subcritical, critical or supercritical problems, as long as the limiting solution is well analyzed. This method was introduced in Gui and Wei [27] in dealing with spikes.

In the following, we show how to prove Theorem 2.5 by LEM. We need to introduce some notation first.

Theorem 2.5 actually holds for a slightly more general equation than (2.4), namely,

$$\begin{cases} \epsilon^2\Delta u - u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

We will always assume that $f : \mathbb{R} \to \mathbb{R}$ is of class $C^{1+\sigma}$ for some $0 < \sigma \leqslant 1$ and satisfies the following conditions (f1)–(f2):

(f1) $f(u) \equiv 0$ for $u \leqslant 0$, $f(0) = f'(0) = 0$.

(f2) The following equation

$$\begin{cases} \Delta w - w + f(w) = 0, & w > 0 \text{ in } \mathbb{R}^N, \\ w(0) = \max_{y\in\mathbb{R}^N} w(y), & w \to 0 \text{ at } \infty, \end{cases} \tag{2.17}$$

has a unique solution $w(y)$ and w is non-degenerate, i.e.,

$$\operatorname{Kernel}\bigl(\Delta - 1 + f'(w)\bigr) = \operatorname{span}\left\{\frac{\partial w}{\partial y_1}, \ldots, \frac{\partial w}{\partial y_N}\right\}. \tag{2.18}$$

One typical example of f is: $f(u) = u^p - au^q$, where $a \geqslant 0$, $1 < q < p < (\frac{N+2}{N-2})_+$. For the uniqueness of w, see [39] and [40]. The proof of non-degeneracy is given in [58].

Without loss of generality, we may assume that $0 \in \Omega$. By the following rescaling:

$$x = \epsilon z, \quad z \in \Omega_\epsilon := \{z \mid |\epsilon z \in \Omega\}, \tag{2.19}$$

equation (2.16) becomes

$$\begin{cases} \Delta u - u + f(u) = 0 \quad \text{in } \Omega_\epsilon, \\ u > 0 \quad \text{in } \Omega_\epsilon, \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial\Omega_\epsilon. \end{cases} \tag{2.20}$$

For $u \in H^2(\Omega_\epsilon)$, we put

$$S_\epsilon[u] = \Delta u - u + f(u). \tag{2.21}$$

Then (2.20) is equivalent to

$$S_\epsilon[u] = 0, \quad u \in H^2(\Omega_\epsilon), \quad u > 0 \quad \text{in } \Omega_\epsilon, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\epsilon. \tag{2.22}$$

Associated with problem (2.20) is the following energy functional

$$\tilde{J}_\epsilon[u] = \frac{1}{2}\int_{\Omega_\epsilon} \bigl(|\nabla u|^2 + u^2\bigr) - \int_{\Omega_\epsilon} F(u), \quad u \in H^1(\Omega_\epsilon). \tag{2.23}$$

We define two inner products:

$$\langle u, v\rangle_\epsilon = \int_{\Omega_\epsilon} uv, \quad \text{for } u, v \in L^2(\Omega_\epsilon); \tag{2.24}$$

$$(u, v)_\epsilon = \int_{\Omega_\epsilon} (\nabla u \nabla v + uv), \quad \text{for } u, v \in H^1(\Omega_\epsilon). \tag{2.25}$$

Let σ be the Hölder exponent of f' and

$$M > \frac{6+2\sigma}{\sigma} K \tag{2.26}$$

be a fixed positive constant. Now we define a configuration space:

$$\Lambda := \left\{(Q_1, \dots, Q_K) \in \Omega^K \;\middle|\; \varphi_K(Q_1, \dots, Q_K) \geqslant M\epsilon |\ln \epsilon|\right\} \tag{2.27}$$

where φ_K is defined at (2.14).

Let w be the unique solution of (2.17). By the well-known result of Gidas, Ni and Nirenberg [22], w is radially symmetric: $w(y) = w(|y|)$ and strictly decreasing: $w'(r) < 0$ for $r > 0$, $r = |y|$. Moreover, we have the following asymptotic behavior of w:

$$\begin{aligned} w(r) &= A_N r^{-\frac{N-1}{2}} e^{-r}\left(1 + O\left(\frac{1}{r}\right)\right), \\ w'(r) &= -A_N r^{-\frac{N-1}{2}} e^{-r}\left(1 + O\left(\frac{1}{r}\right)\right), \end{aligned} \tag{2.28}$$

for r large, where $A_N > 0$ is a constant. Let $K(r)$ be the fundamental solution of $-\Delta + 1$ centered at 0. Then we have

$$\begin{aligned} w(r) &= \left(A_0 + O\left(\frac{1}{r}\right)\right) K(r), \\ w'(r) &= \left(-A_0 + O\left(\frac{1}{r}\right)\right) K(r), \quad \text{for } r \geqslant 1, \end{aligned} \tag{2.29}$$

where A_0 is a positive constant.

The idea of *LEM* is to look for solutions of (2.16) of the following type:

$$u = \sum_{j=1}^{K} w\left(z - \frac{Q_j}{\epsilon}\right) + \phi \tag{2.30}$$

where ϕ is solved first by Lyapunov–Schmidt reduction process, and $(Q_1, \dots, Q_K)$ are adjusted so as to achieve a solution. *LEM* is a method of reducing the infinite-dimensional problem of finding a critical point of $\tilde{J}_\epsilon$ to a finite-dimensional problem of $(Q_1, \dots, Q_K)$. In general, it consists of the following five steps:

STEP 1. Find out good approximate functions.

This step contains most of the important computations. The idea is to choose good approximate functions such that the error S_ϵ is small.

For $Q \in \Omega$, we define $w_{\epsilon,Q}$ to be the unique solution of

$$\Delta v - v + f\left(w\left(\cdot - \frac{Q}{\epsilon}\right)\right) = 0 \quad \text{in } \Omega_\epsilon, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\epsilon. \tag{2.31}$$

Let $\mathbf{Q} = (Q_1, \ldots, Q_K) \in \Lambda$. We then define the approximate solution as

$$w_{\epsilon,\mathbf{Q}} = \sum_{j=1}^{K} w_{\epsilon,Q_j}. \tag{2.32}$$

We first analyze $w_{\epsilon,Q}$. To this end, set

$$\varphi_{\epsilon,Q}(x) = w\left(\frac{|x-Q|}{\varepsilon}\right) - w_{\epsilon,Q}\left(\frac{x}{\epsilon}\right).$$

We state the following useful lemmas on the properties of $\varphi_{\epsilon,Q}$, whose proof can be found in [44].

LEMMA 2.6. *Assume that* $\frac{M}{2}\epsilon|\ln\epsilon| \leqslant d(Q,\partial\Omega) \leqslant \delta$ *where* δ *is sufficiently small. We have*

$$\varphi_{\epsilon,Q} = -\big(A_0 + o(1)\big)K\left(\frac{|x-Q^*|}{\epsilon}\right) + O\big(\epsilon^{\sqrt{2}M+N+1}\big) \tag{2.33}$$

where $K(r)$ *is the (radially symmetric) fundamental solution of* $-\Delta + 1$ *in* $\mathbb{R}^N$, $Q^* = Q + 2d(Q,\partial\Omega)\nu_{\bar{Q}}$, $\nu_{\bar{Q}}$ *denotes the unit outer normal at* $\bar{Q} \in \partial\Omega$ *and* $\bar{Q}$ *is the unique point on* $\partial\Omega$ *such that* $d(\bar{Q},Q) = d(Q,\partial\Omega)$.

The next lemma analyze $w_{\epsilon,\mathbf{Q}}$ in Ω_ϵ. To this end, we divide Ω_ϵ into $K+1$-parts:

$$\Omega_{\epsilon,j} = \left\{\left|z - \frac{Q_j}{\epsilon}\right| \leqslant \frac{1}{2\epsilon}\varphi_K(\mathbf{Q})\right\}, \quad j = 1, \ldots, K,$$

$$\Omega_{\epsilon,K+1} = \Omega_\epsilon \Big\backslash \bigcup_{j=1}^{K} \Omega_{\epsilon,j}. \tag{2.34}$$

LEMMA 2.7. *For* $z \in \Omega_{\epsilon,j}$, $j = 1, \ldots, K$, *we have*

$$w_{\epsilon,\mathbf{Q}} = w_{\epsilon,Q_j} + O\big(K\epsilon^{\frac{M}{2}}\big) = w\left(z - \frac{Q_j}{\epsilon}\right) + O\big(K\epsilon^{\frac{M}{2}}\big). \tag{2.35}$$

For $z \in \Omega_{\epsilon,K+1}$, we have

$$w_{\epsilon,\mathbf{Q}} = O\bigl(K\epsilon^{\frac{M}{2}}\bigr). \tag{2.36}$$

PROOF. For $k \neq j$ and $z \in \Omega_{\epsilon,j}$, we have

$$\begin{aligned} w_{\epsilon,Q_k} &= w\left(z - \frac{Q_k}{\epsilon}\right) - \varphi_{\epsilon,Q_k}(\epsilon z) \\ &= O\bigl(e^{-|z-\frac{Q_k}{\epsilon}|} + e^{-|z-\frac{Q_k^*}{\epsilon}|} + \epsilon^{M+N+1}\bigr) = O\bigl(\epsilon^{\frac{M}{2}}\bigr) \end{aligned}$$

and so

$$\sum_{k\neq j} w_{\epsilon,Q_k} = O\bigl(K\epsilon^{\frac{M}{2}}\bigr)$$

which proves (2.35). The proof of (2.36) is similar. □

Next we state a useful lemma about the interactions of two w's.

LEMMA 2.8. *For $\frac{|Q_1-Q_2|}{\epsilon}$ large, it holds*

$$\int_{\mathbb{R}^N} f\left(w\left(z - \frac{Q_1}{\epsilon}\right)\right) w\left(z - \frac{Q_2}{\epsilon}\right) = \bigl(\gamma_0 + o(1)\bigr) w\left(\frac{|Q_1 - Q_2|}{\epsilon}\right) \tag{2.37}$$

where

$$\gamma_0 = \int_{\mathbb{R}^N} f\bigl(w(y)\bigr)e^{-y_1}\,dy. \tag{2.38}$$

REMARK. Note that $\gamma_0 > 0$. See Lemma 4.7 of [61].

PROOF. By (2.28), we have for $|\epsilon y| \ll |Q_1 - Q_2|$,

$$\begin{aligned} w\left(y + \frac{Q_1 - Q_2}{\epsilon}\right) &= \bigl(A_N + o(1)\bigr)\left(\frac{\epsilon}{|\epsilon y + Q_1 - Q_2|}\right)^{\frac{N-1}{2}} e^{-|y+\frac{Q_1-Q_2}{\epsilon}|} \\ &= w\left(\frac{|Q_1 - Q_2|}{\epsilon}\right) e^{-\langle y, \frac{Q_1-Q_2}{|Q_1-Q_2|}\rangle + o(|y|)}. \end{aligned}$$

Thus by Lebesgue's Dominated Convergence Theorem

$$\int_{\mathbb{R}^N} f\left(w\left(z - \frac{Q_1}{\epsilon}\right)\right) w\left(z - \frac{Q_2}{\epsilon}\right)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} f\big(w(y)\big) w\left(y + \frac{Q_1 - Q_2}{\epsilon}\right) \\
&= \big(1 + o(1)\big) w\left(\frac{|Q_1 - Q_2|}{\epsilon}\right) \int_{\mathbb{R}^N} f\big(w(y)\big) e^{-\langle y, \frac{Q_1 - Q_2}{|Q_1 - Q_2|}\rangle} \, dy \\
&= \big(\gamma_0 + o(1)\big) w\left(\frac{|Q_1 - Q_2|}{\epsilon}\right).
\end{aligned}
$$

□

Let us define several quantities for later use:

$$
B_\epsilon(Q_j) = -\int_{\Omega_\epsilon} f(w_j)\varphi_{\epsilon, Q_j}, \quad B_\epsilon(Q_i, Q_j) = \int_{\Omega_\epsilon} f(w_i) w_j. \tag{2.39}
$$

Then we have

LEMMA 2.9. *For* $\mathbf{Q} = (Q_1, \ldots, Q_K) \in \Lambda$, *it holds*

$$
B_\epsilon(Q_j) = \big(\gamma_0 + o(1)\big) w\left(\frac{2d(Q_j, \partial\Omega)}{\epsilon}\right) + o\big(w\big(M|\ln\epsilon|\big)\big), \tag{2.40}
$$

$$
B_\epsilon(Q_i, Q_j) = \big(\gamma_0 + o(1)\big) w\left(\frac{|Q_i - Q_j|}{\epsilon}\right) + o\big(w\big(M|\ln\epsilon|\big)\big). \tag{2.41}
$$

PROOF. Note that

$$
A_0 K\left(\frac{|x - Q^*|}{\epsilon}\right) = \big(1 + o(1)\big) w\left(\frac{|x - Q^*|}{\epsilon}\right)
$$

and by Lemma 2.6

$$
\begin{aligned}
B_\epsilon(Q_j) &= \big(1 + o(1)\big) \int_{\Omega_\epsilon} f(w_j) w\left(z - \frac{Q_j^* - Q_j}{\epsilon}\right) + O\big(\epsilon^{\sqrt{2}M + N + 1}\big) \\
&= \big(\gamma + o(1)\big) w\left(\frac{|Q_j - Q_j^*|}{\epsilon}\right) + o\big(w(M|\ln\epsilon|)\big) \\
&= \big(\gamma + o(1)\big) w\left(\frac{2d(Q_j, \partial\Omega)}{\epsilon}\right) + o\big(w\big(M|\ln\epsilon|\big)\big).
\end{aligned}
$$

(2.40) follows from Lemma 2.6. To prove (2.41), we note that

$$
\begin{aligned}
B_\epsilon(Q_i, Q_j) = \int_{\mathbb{R}^N} f(w) w\left(y - \frac{Q_i - Q_j}{\epsilon}\right) \\
- \int_{\mathbb{R}^N \setminus \Omega_{\epsilon, Q_i}} f(w) w\left(y - \frac{Q_i - Q_j}{\epsilon}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (\gamma + o(1)) w\left(\frac{|Q_i - Q_j|}{\epsilon}\right) + O\left(e^{-(1+\frac{\sigma}{2})\frac{d(Q_i,\partial\Omega)}{\epsilon}} e^{-\frac{d(Q_j,\partial\Omega)}{\epsilon}}\right) \\
&= (\gamma + o(1)) w\left(\frac{|Q_i - Q_j|}{\epsilon}\right) + o\left(w\left(M|\ln\epsilon|\right)\right). \qquad \square
\end{aligned}
$$

We then have the following which provides the key estimates on the energy expansion and error estimates.

LEMMA 2.10. *For any* $\mathbf{Q} = (Q_1, \ldots, Q_K) \in \Lambda$ *and* ϵ *sufficiently small we have*

$$
\begin{aligned}
\tilde{J}_\epsilon\left[\sum_{i=1}^{K} w_{\epsilon,Q_j}\right] = KI[w] - \frac{1}{2}\sum_{i=1}^{K} B_\epsilon(Q_i) \\
- \frac{1}{2}\sum_{i,j=1,\ldots,K,\, i\neq j} B_\epsilon(Q_i, Q_j) + o\left(w\left(M|\ln\epsilon|\right)\right),
\end{aligned}
\tag{2.42}
$$

and

$$
\left\| S_\epsilon\left[\sum_{j=1}^{K} w_{\epsilon,Q_j}\right]\right\|_{L^q(\Omega_\epsilon)} \leqslant CK^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}} \tag{2.43}
$$

for any $q > \frac{N}{2}$.

The proof of Lemma 2.10 is technical and tedious. We refer to [44] for the computations.

STEP 2. Obtain a priori estimates for a linear problem.

This is the fundamental step in reducing an infinite-dimensional problem to finite-dimensional one. The key result we need here is the non-degeneracy assumption (f2).

Fix $\mathbf{Q} \in \Lambda$. We define the following functions

$$
\begin{aligned}
&Z_{i,j} = (\Delta - 1)\left[\frac{\partial w_i}{\partial z_j}\chi_i(z)\right], \quad \text{where } \chi_i(z) = \chi\left(\frac{2|\epsilon z - Q_i|}{(M-1)\epsilon|\ln\epsilon|}\right), \\
&i = 1, \ldots, K,\ j = 1, \ldots, N,
\end{aligned}
\tag{2.44}
$$

where $\chi(t)$ is a smooth cut-off function such that $\chi(t) = 1$ for $|t| < 1$ and $\chi(t) = 0$ for $|t| > \frac{M^2}{M^2-1}$. Note that the support of $Z_{i,j}$ belongs to $B_{\frac{M^2-1}{2M}|\ln\epsilon|}(\frac{Q_i}{\epsilon})$.

In this step, we consider the following linear problem: Given $h \in L^2(\Omega_\epsilon)$, find a function ϕ satisfying

$$\begin{cases} L_\epsilon[\phi] := \Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi = h + \sum_{k,l} c_{k,l} Z_{k,l}; \\ \langle \phi, Z_{i,j}\rangle_\epsilon = 0, \quad i = 1, \dots, K,\ j = 1, \dots, N, \quad \text{and} \\ \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \partial\Omega_\epsilon, \end{cases} \tag{2.45}$$

for some constants $c_{k,l}$, $k = 1, \dots, K$, $l = 1, \dots, N$.

To this purpose, we define two norms

$$\|\phi\|_* = \|\phi\|_{W^{2,q}(\Omega_\epsilon)}, \qquad \|f\|_{**} = \|f\|_{L^q(\Omega_\epsilon)}, \tag{2.46}$$

where $q > \frac{N}{2}$ is a fixed number.

We have the following result:

PROPOSITION 2.11. *Let ϕ satisfy* (2.45). *Then for ϵ sufficiently small and $\mathbf{Q} \in \Lambda$, we have*

$$\|\phi\|_* \leqslant C\|h\|_{**} \tag{2.47}$$

where C is a positive constant independent of ϵ, K and $\mathbf{Q} \in \Lambda$.

PROOF. Arguing by contradiction, assume that

$$\|\phi\|_* = 1; \qquad \|h\|_{**} = o(1). \tag{2.48}$$

We multiply (2.45) by $\frac{\partial w_i}{\partial z_j}\chi_i(z)$ and integrate over Ω_ϵ to obtain

$$\begin{aligned} &\sum_{k,l} c_{k,l}\left\langle Z_{k,l}, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon \\ &\quad = -\left\langle h, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon + \left\langle \Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon. \end{aligned} \tag{2.49}$$

From the exponential decay of w one finds

$$\left\langle h, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon = o(1).$$

Observe that $\frac{\partial w_i}{\partial z_j}\chi_i(z)$ satisfies

$$\begin{aligned} &\Delta\left(\frac{\partial w_i}{\partial z_j}\chi_i(z)\right) - \left(\frac{\partial w_i}{\partial z_j}\chi_i(z)\right) + f'(w_i)\left(\frac{\partial w_i}{\partial z_j}\chi_i(z)\right) \\ &\quad = 2\nabla_z \frac{\partial w_i}{\partial z_j}\nabla_z\chi_i + (\Delta\chi_i)\frac{\partial w_i}{\partial z_j}. \end{aligned} \tag{2.50}$$

Integrating by parts and using Lemma 2.7, we deduce

$$\begin{aligned}&\left\langle \Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon\\&\quad= \left\langle \left(f'(w_{\epsilon,\mathbf{Q}}) - f'(w_i)\right)\frac{\partial w_i}{\partial z_j}\chi_i(z), \phi\right\rangle_\epsilon + O\left(\epsilon^{\frac{M-1}{2}}\|\phi\|_*\right)\\&\quad= O\left(K^\sigma \epsilon^{\frac{M\sigma}{2}}\|\phi\|_*\right) = o\left(\|\phi\|_*\right) = o(1)\end{aligned}$$

where we have used the fact that $M > \frac{6+2\sigma}{\sigma}N$ and that

$$\left\|\left(f'(w_{\epsilon,\mathbf{Q}}) - f'(w_i)\right)\frac{\partial w_i}{\partial z_j}\chi_i\right\|_{**} \leqslant C\left\||w_{\epsilon,\mathbf{Q}} - w_i|^\sigma \left|\frac{\partial w_i}{\partial z_j}\chi_i\right|\right\|_* \leqslant K^\sigma \epsilon^{\frac{M\sigma}{2}}.$$

It is easy to see that

$$\left\langle Z_{i,j}, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon = -\int_{\mathbb{R}^N} f'(w)\left(\frac{\partial w}{\partial y_j}\right)^2 dy + o(1). \tag{2.51}$$

On the other hand, for $k \neq i$ we have

$$\left\langle Z_{k,l}, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon = 0 \tag{2.52}$$

and for $k = i$ and $l \neq j$, we have

$$\left\langle Z_{i,l}, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon = O\left(\epsilon^M\right). \tag{2.53}$$

The left hand side of (2.49) becomes

$$c_{i,j} + \sum_{l\neq j} O\left(\epsilon^M c_{i,l}\right) = o(1)$$

and hence

$$c_{i,j} = o(1), \quad i = 1, \ldots, K,\ j = 1, \ldots, N. \tag{2.54}$$

To obtain a contradiction, we define the following cut-off functions:

$$\phi_i = \phi\chi_i', \quad \text{where } \chi_i' = \chi\left(\frac{2|\epsilon z - Q_i|}{(M - M^{-1})\epsilon|\ln\epsilon|}\right),\ i = 1, \ldots, K. \tag{2.55}$$

Note that $\chi_i' = 1$ for $z \in B_{\frac{M^2-1}{2M}|\ln\epsilon|}(\frac{Q_i}{\epsilon})$ and the support of ϕ belongs to $B_{\frac{M}{2}|\ln\epsilon|}(\frac{Q_i}{\epsilon})$.

Then the conditions $\langle \phi, Z_{i,j}\rangle_\epsilon = 0$ is equivalent to

$$\langle \phi_i, Z_{i,j}\rangle_\epsilon = 0. \tag{2.56}$$

The equation for ϕ_i becomes

$$\Delta\phi_i - \phi_i + f'(w_{\epsilon,\mathbf{Q}})\phi_i = \sum_j c_{i,j} Z_{i,j} + h\chi_i' + 2\nabla\phi\nabla\chi_i' + \big(\Delta\chi_i'\big)\phi. \tag{2.57}$$

Lemma 2.7 yields

$$f'(w_{\epsilon,\mathbf{Q}})\phi_i = \big(f(w_i) + o\big(\epsilon^{M/2-N}\big)\big)\phi_i. \tag{2.58}$$

Using (2.56) and (2.58), a contradiction argument similar to that of Proposition 3.2 of [27] gives

$$\|\phi_i\|^q_{W^{2,q}(\Omega_\epsilon)} \leqslant C\big\|h\chi_i'\big\|^q_{L^q(\Omega_\epsilon)} + C\big\|2\nabla\phi\nabla\chi_i' + \big(\Delta\chi_i'\big)\phi\big\|^q_{L^q(\Omega_\epsilon)}. \tag{2.59}$$

Next, we decompose

$$\phi = \sum_{i=1}^K \phi_i + \Phi \tag{2.60}$$

where $\Phi = \phi(1 - \sum_{i=1}^K \chi_i')$. Then the equation for Φ becomes

$$\begin{aligned}&\Delta\Phi - \Phi + f'(w_{\epsilon,\mathbf{Q}})\Phi\\ &\quad= h\left(1 - \sum_{i=1}^K \chi_i'\right) - 2\sum_{i=1}^K \nabla\phi\nabla\chi_i' - \sum_{i=1}^K\big(\Delta\chi_i'\big)\phi.\end{aligned} \tag{2.61}$$

By Lemma 2.7, $f'(w_{\epsilon,\mathbf{Q}})\Phi = o(1)\Phi$. Standard regularity theorem gives

$$\begin{aligned}\|\Phi\|^q_{W^{2,q}(\Omega_\epsilon)} &\leqslant C\left\|h\left(1 - \sum_{i=1}^K \chi_i'\right)\right\|^q_{L^q(\Omega_\epsilon)}\\ &\quad + C\left\|2\sum_{i=1}^K \nabla\phi\nabla\chi_i' + \sum_{i=1}^K(\Delta\chi_i')\phi\right\|^q_{L^q(\Omega_\epsilon)}.\end{aligned} \tag{2.62}$$

(Observe that the constant C in the L^p-regularity is independent of $\epsilon < 1$. The case of Dirichlet boundary condition has been proved in Lemma 6.4 of [61]. The case of Neumann boundary condition can be proved similarly.)

Combining (2.60), (2.59) and (2.62), we obtain

$$\begin{aligned}
\|\phi\|^q_{W^{2,q}(\Omega_\epsilon)} &\leqslant C\left\|\sum_{i=1}^K \phi_i\right\|^q_{W^{2,q}(\Omega_\epsilon)} + C\|\Phi\|^q_{W^{2,q}(\Omega_\epsilon)} \\
&\leqslant C\sum_{i=1}^K \|\phi_i\|^q_{W^{2,q}(\Omega_\epsilon)} + C\|\Phi\|^q_{W^{2,q}(\Omega_\epsilon)} \\
&\leqslant C\left(\sum_{i=1}^K \|h\chi_i'\|^q_{L^q(\Omega_\epsilon)} + \left\|h\left(1-\sum_{i=1}^K \chi_i'\right)\right\|^q_{L^q(\Omega_\epsilon)}\right) \\
&\quad + C\sum_{i=1}^K \|2\nabla\phi\nabla\chi_i' + (\Delta\chi_i')\phi\|^q_{L^q(\Omega_\epsilon)} \\
&\leqslant C\|h\|^q_{L^q(\Omega_\epsilon)} + O\big(|\ln\epsilon|^{-1}\big)\|\phi\|^q_{W^{2,q}(\Omega_\epsilon)}
\end{aligned}$$

since

$$\sum_{i=1}^K (\chi_i')^q + \left(1-\sum_{i=1}^K \chi_i'\right)^q \leqslant 2, \qquad |\nabla\chi'| + |\Delta\chi'| \leqslant C\big(|\ln\epsilon|\big)^{-1}. \tag{2.63}$$

This gives

$$\|\phi\|_{W^{2,q}(\Omega_\epsilon)} = o(1). \tag{2.64}$$

A contradiction to (2.48). □

From Proposition 2.11, we derive the following existence result:

PROPOSITION 2.12. *There exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ the following property holds true. Given $h \in W^{2,q}(\Omega_\epsilon)$, there exist s a unique pair $(\phi, \mathbf{c}) = (\phi, \{c_{i,j}\}_{i=1,\dots,K,j=1,\dots,N})$ such that*

$$L_\epsilon[\phi] = h + \sum_{i,j} c_{i,j} Z_{i,j}, \tag{2.65}$$

$$\langle\phi, Z_{i,j}\rangle_\epsilon = 0, \quad i=1,\dots,K,\ j=1,\dots,N, \qquad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \partial\Omega_\epsilon. \tag{2.66}$$

Moreover, we have

$$\|\phi\|_* \leqslant C\|h\|_{**} \tag{2.67}$$

for some positive constant C.

PROOF. The bound in (2.67) follows from Proposition 2.11 and (2.54). Let us now prove the existence part. Set

$$\mathcal{H} = \{u \in H^1(\Omega_\epsilon) \mid (u, (\Delta - 1)^{-1} Z_{i,j})_\epsilon = 0\}$$

where we define the inner product on $H^1(\Omega_\epsilon)$ as

$$(u, v)_\epsilon = \int_{\Omega_\epsilon} (\nabla u \nabla v + uv).$$

Note that, integrating by parts, one has

$$\psi \in \mathcal{H} \quad \text{if and only if} \quad \langle \psi, Z_{i,j} \rangle_\epsilon = 0, \quad i = 1, \ldots, K, \ j = 1, \ldots, N.$$

Observe that ϕ solves (2.65) and (2.66) if and only if $\phi \in \mathcal{H}$ satisfies

$$\int_{\Omega_\epsilon} (\nabla \phi \nabla \psi + \phi \psi) - \langle f'(w_{\epsilon,\mathbf{Q}})\phi, \psi \rangle_\epsilon = \langle h, \psi \rangle_\epsilon, \quad \forall \psi \in \mathcal{H}.$$

This equation can be rewritten as

$$\phi + \mathcal{S}(\phi) = \bar{h} \quad \text{in } \mathcal{H}, \tag{2.68}$$

where $\bar{h}$ is defined by duality and $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is a linear compact operator.

Using Fredholm's alternative, showing that equation (2.68) has a unique solution for each $\bar{h}$, is equivalent to showing that the equation has a unique solution for $\bar{h} = 0$, which in turn follows from Proposition 2.11 and our proof is complete. □

In the following, if ϕ is the unique solution given in Proposition 2.12, we set

$$\phi = \mathcal{A}_\epsilon(h). \tag{2.69}$$

Note that (2.67) implies

$$\|\mathcal{A}_\epsilon(h)\|_* \leqslant C\|h\|_{**}. \tag{2.70}$$

STEP 3. A non-linear Lyapunov–Schmidt reduction.

For ϵ small and for $\mathbf{Q} \in \Lambda$, we are going to find a function $\phi_{\epsilon,\mathbf{Q}}$ such that for some constants $c_{i,j}$, $j = 1, \ldots, N$, the following equation holds true

$$\begin{cases} \Delta(w_{\epsilon,\mathbf{Q}} + \phi) - (w_{\epsilon,\mathbf{Q}} + \phi) + f(w_{\epsilon,\mathbf{Q}} + \phi) = \sum_{k,l} c_{k,l} Z_{k,l} & \text{in } \Omega_\epsilon, \\ \langle \phi, Z_{i,j} \rangle_\epsilon = 0, \quad j = 1, \ldots, N, \quad \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{2.71}$$

The first equation in (2.71) can be written as

$$\Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi = \big(-S_\epsilon[w_{\epsilon,\mathbf{Q}}]\big) + N_\epsilon[\phi] + \sum_{i,j} c_{i,j} Z_{i,j},$$

where

$$N_\epsilon[\phi] = -\big[f(w_{\epsilon,\mathbf{Q}} + \phi) - f(w_{\epsilon,\mathbf{Q}}) - f'(w_{\epsilon,\mathbf{Q}})\phi\big]. \tag{2.72}$$

LEMMA 2.13. *For* $\mathbf{Q} \in \Lambda$ *and* ϵ *sufficiently small, we have for* $\|\phi\|_* + \|\phi_1\|_* + \|\phi_2\|_* \leqslant 1$,

$$\big\|N_\epsilon[\phi]\big\|_{**} \leqslant C\|\phi\|_*^{1+\sigma}; \tag{2.73}$$

$$\big\|N_\epsilon[\phi_1] - N_\epsilon[\phi_2]\big\|_{**} \leqslant C\big(\|\phi_1\|_*^\sigma + \|\phi_2\|_*^\sigma\big)\|\phi_1 - \phi_2\|_*. \tag{2.74}$$

PROOF. Inequality (2.73) follows from the mean-value theorem. In fact, for all $z \in \Omega_\epsilon$ there holds

$$f(w_{\epsilon,\mathbf{Q}} + \phi) - f(w_{\epsilon,\mathbf{Q}}) = f'(w_{\epsilon,\mathbf{Q}} + \theta\phi)\phi.$$

Since f' is Hölder continuous with exponent σ, we deduce

$$\big|f(w_{\epsilon,\mathbf{Q}} + \phi) - f(w_{\epsilon,\mathbf{Q}}) - f'(w_{\epsilon,\mathbf{Q}})\phi\big| \leqslant C|\phi|^{1+\sigma},$$

which implies (2.73). The proof of (2.74) goes along the same way. □

PROPOSITION 2.14. *For* $\mathbf{Q} \in \Lambda$ *and* ϵ *sufficiently small, there exists a unique* $\phi = \phi_{\epsilon,\mathbf{Q}}$ *such that* (2.71) *holds. Moreover,* $\mathbf{Q} \mapsto \phi_{\epsilon,\mathbf{Q}}$ *is of class* C^1 *as a map into* $W^{2,q}(\Omega_\epsilon) \cap \mathcal{H}$, *and we have*

$$\|\phi_{\epsilon,\mathbf{Q}}\|_* \leqslant rK^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}} \tag{2.75}$$

for some constant $r > 0$.

PROOF. Let $\mathcal{A}_\epsilon$ be as defined in (2.69). Then (2.71) can be written as

$$\phi = \mathcal{A}_\epsilon\big[\big(-S_\epsilon[w_{\epsilon,\mathbf{Q}}]\big) + N_\epsilon[\phi]\big]. \tag{2.76}$$

Let r be a positive (large) number, and set

$$\mathcal{F}_r = \big\{\phi \in \mathcal{H} \cap W^{2,q}(\Omega_\epsilon)\colon\ \|\phi\|_* < rK^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}}\big\}.$$

Define now the map $\mathcal{G}_\epsilon : \mathcal{F}_r \to \mathcal{H} \cap W^{2,q}(\Omega_\epsilon)$ as

$$\mathcal{G}_\epsilon[\phi] = \mathcal{A}_\epsilon\big[\big(-S_\epsilon[w_{\epsilon,\mathbf{Q}}]\big) + N_\epsilon[\phi]\big].$$

Solving (2.71) is equivalent to finding a fixed point for $\mathcal{G}_\epsilon$. By Lemmas 2.10 and 2.13, for ϵ sufficiently small and r large we have

$$\begin{aligned}
&\left\|\mathcal{G}_\epsilon[\phi]\right\|_* \leqslant C\left\|S_\epsilon[w_{\epsilon,\mathbf{Q}}]\right\|_{**} + C\left\|N_\epsilon[\phi]\right\|_{**} < rK^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}},\\
&\left\|\mathcal{G}_\epsilon[\phi_1] - \mathcal{B}_\epsilon[\phi_2]\right\|_* \leqslant C\left\|N_\epsilon[\phi_1] - N_\epsilon[\phi_2]\right\|_* < \frac{1}{2}\|\phi_1 - \phi_2\|_*,
\end{aligned}$$

which shows that $\mathcal{G}_\epsilon$ is a contraction mapping on $\mathcal{F}_r$. Hence there exists a unique $\phi = \phi_{\epsilon,\mathbf{Q}} \in \mathcal{F}_r$ such that (2.71) holds.

Now we come to the differentiability of $\phi_{\epsilon,\mathbf{Q}}$. Consider the following map $H_\epsilon : \Lambda \times \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK} \to \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK}$ of class C^1

$$H_\epsilon(\mathbf{Q}, \phi, \mathbf{c}) = \begin{pmatrix} (\Delta - 1)^{-1}(S_\epsilon[w_{\epsilon,\mathbf{Q}} + \phi]) - \sum_{i,j} c_{i,j}(\Delta - 1)^{-1}Z_{i,j} \\ (\phi, (\Delta - 1)^{-1}Z_{1,1})_\epsilon \\ \vdots \\ (\phi, (\Delta - 1)^{-1}Z_{K,N})_\epsilon \end{pmatrix}. \tag{2.77}$$

Equation (2.71) is equivalent to $H_\epsilon(\mathbf{Q}, \phi, \mathbf{c}) = 0$. We know that, given $\mathbf{Q} \in \Lambda$, there is a unique local solution $\phi_{\epsilon,\mathbf{Q}}, c_{\epsilon,\mathbf{Q}}$ obtained with the above procedure. We prove that the linear operator

$$\left.\frac{\partial H_\epsilon(\mathbf{Q}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\right|_{(\mathbf{Q}, \phi_{\epsilon,\mathbf{Q}}, \mathbf{c}_{\epsilon,\mathbf{Q}})} : \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK} \to \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK}$$

is invertible for ϵ small. Then the C^1-regularity of $\mathbf{Q} \mapsto (\phi_{\epsilon,\mathbf{Q}}, c_{\epsilon,\mathbf{Q}})$ follows from the Implicit Function Theorem. Indeed we have

$$\begin{aligned}
&\left.\frac{\partial H_\epsilon(\mathbf{Q}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\right|_{(\mathbf{Q}, \phi_{\epsilon,\mathbf{Q}}, \mathbf{c}_{\epsilon,\mathbf{Q}})}[\psi, \mathbf{d}]\\
&\quad = \begin{pmatrix} (\Delta - 1)^{-1}(S'[w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}}](\psi)) - \sum_{i,j} d_{ij}(\Delta - 1)^{-1}Z_{i,j} \\ (\psi, (\Delta - 1)^{-1}Z_{1,1})_\epsilon \\ \vdots \\ (\psi, (\Delta - 1)^{-1}Z_{K,N})_\epsilon \end{pmatrix}.
\end{aligned}$$

Since $\|\phi_{\epsilon,\mathbf{Q}}\|_*$ is small, the same proof as in that of Proposition 2.11 shows that

$$\left.\frac{\partial H_\epsilon(\mathbf{Q}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\right|_{(\mathbf{Q}, \phi_{\epsilon,\mathbf{Q}}, \mathbf{c}_{\epsilon,\mathbf{Q}})}$$

is invertible for ϵ small.

This concludes the proof of Proposition 2.14. $\square$

In some cases (e.g., critical or nearly critical exponent problems), we need to obtain further differentiability of $\phi_{\epsilon,\mathbf{Q}}$ (e.g., C^2 in $\mathbf{Q}$). This will be achieved by further reduction. See [13,65] and [66] for such arguments.

STEP 4. A reduction lemma.

Fix $\mathbf{Q} \in \Lambda$. Let $\phi_{\epsilon,\mathbf{Q}}$ be the solution given by Proposition 2.14. We define a new functional

$$\mathcal{M}_\epsilon(\mathbf{Q}) = \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}}] : \Lambda \to R. \tag{2.78}$$

Then we have the following reduction lemma

LEMMA 2.15. *If $\mathbf{Q}_\epsilon$ is critical point of $\mathcal{M}_\epsilon(\mathbf{Q})$ in Λ, then $u_\epsilon = w_{\epsilon,\mathbf{Q}_\epsilon} + \phi_{\epsilon,\mathbf{Q}_\epsilon}$ is a critical point of $\tilde{J}_\epsilon[u]$.*

PROOF. By Proposition 2.14, there exists ϵ_0 such that for $0 < \epsilon < \epsilon_0$ we have a C^1 map which, to any $\mathbf{Q} \in \Lambda$, associates $\phi_{\epsilon,\mathbf{Q}}$ such that

$$\begin{aligned} &S_\epsilon[w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}}] = \sum_{k=1,\dots,K;\, l=1,\dots,N} c_{kl} Z_{k,l}, \\ &\langle \phi_{\epsilon,\mathbf{Q}}, Z_{i,j} \rangle_\epsilon = 0 \end{aligned} \tag{2.79}$$

for some constants $c_{kl} \in R^{KN}$.

Let $\mathbf{Q}^\epsilon \in \Lambda$ be a critical point of $\mathcal{M}_\epsilon$. Set $u_\epsilon = w_{\epsilon,\mathbf{Q}^\epsilon} + \phi_{\epsilon,\mathbf{Q}^\epsilon}$. Then we have

$$D_{Q_{i,j}}|_{Q_i = Q_i^\epsilon} \mathcal{M}_\epsilon(\mathbf{Q}^\epsilon) = 0, \quad i = 1, \dots, K, \ j = 1, \dots, N.$$

Hence we have

$$\begin{aligned} \int_{\Omega_\epsilon} \Bigg[&\nabla u_\epsilon \nabla \frac{\partial (w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}} \bigg|_{Q_i = Q_i^\epsilon} + u_\epsilon \frac{\partial (w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}} \bigg|_{Q_i = Q_i^\epsilon} \\ &- f(u_\epsilon) \frac{\partial (w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}} \bigg|_{Q_i = Q_i^\epsilon} \Bigg] = 0, \end{aligned}$$

which gives

$$\sum_{k=1,\dots,K;\ l=1,\dots,N} c_{kl} \int_{\Omega_\epsilon} Z_{k,l} \frac{\partial (w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}} \bigg|_{Q_i = Q_i^\epsilon} = 0. \tag{2.80}$$

We claim that (2.80) is a diagonally dominant system. In fact, since $\langle\phi_{\epsilon,\mathbf{Q}}, Z_{i,j}\rangle_\epsilon = 0$, we have that

$$\int_{\Omega_\epsilon} Z_{k,l}\frac{\partial\phi_{\epsilon,\mathbf{Q}^\epsilon}}{\partial Q^\epsilon_{i,j}} = -\int_{\Omega_\epsilon}\phi_{\epsilon,\mathbf{Q}^\epsilon}\frac{\partial Z_{k,l}}{\partial Q^\epsilon_{i,j}} = 0 \quad \text{if } k\neq i.$$

If $k=i$, we have

$$\begin{aligned}\int_{\Omega_\epsilon} Z_{k,l}\frac{\partial\phi_{\epsilon,\mathbf{Q}^\epsilon}}{\partial Q^\epsilon_{k,j}} &= -\int_{\Omega_\epsilon}\frac{\partial Z_{k,l}}{\partial Q^\epsilon_{k,j}}\phi_{\epsilon,\mathbf{Q}^\epsilon} = \left\|\frac{\partial Z_{k,l}}{\partial Q^\epsilon_{k,j}}\right\|_{**}\|\phi_{\epsilon,\mathbf{Q}^\epsilon}\|_{**}\\ &= O\big(K^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}-1}\big) = O\big(\epsilon^{\frac{M(1+\sigma)}{2}-(\frac{q+1}{q}+\sigma)N-1}\big)\\ &= O\big(\epsilon^{\frac{M}{2}}\big).\end{aligned}$$

For $k\neq i$, we have

$$\int_{\Omega_\epsilon} Z_{k,l}\frac{\partial w_{\epsilon,Q^\epsilon_i}}{\partial Q^\epsilon_{i,j}} = \int_{\Omega_\epsilon\cap B_{\frac{M}{2}|\ln\epsilon|}(\frac{Q^\epsilon_k}{\epsilon})} Z_{k,l}\frac{\partial w_{\epsilon,Q^\epsilon_i}}{\partial Q^\epsilon_{i,j}} = O\big(\epsilon^M\big).$$

For $k=i$, we have

$$\begin{aligned}\int_{\Omega_\epsilon} Z_{k,l}\frac{\partial w_{\epsilon,Q^\epsilon_k}}{\partial Q^\epsilon_{k,j}} &= \int_{\Omega_\epsilon\cap B_{\frac{M}{2}|\ln\epsilon|}(\frac{Q^\epsilon_k}{\epsilon})} Z_{k,l}\frac{\partial w_{\epsilon,Q^\epsilon_k}}{\partial Q^\epsilon_{k,j}}\\ &= -\epsilon^{-1}\delta_{lj}\int_{\mathbb{R}^N} f'(w)\left(\frac{\partial w}{\partial y_j}\right)^2 + O(1).\end{aligned}$$

For each (k,l), the off-diagonal term gives

$$O\big(\epsilon^{\frac{M}{2}}\big) + \sum_{k\neq i}\epsilon^M + \sum_{k=i,l\neq j} O(\epsilon) = O\big(\epsilon^{\frac{M}{2}} + K\epsilon^M + \epsilon\big) = o(1)$$

by our choice of $M > \frac{6+2\sigma}{\sigma}N$.

Thus equation (2.80) becomes a system of homogeneous equations for c_{kl} and the matrix of the system is non-singular. So $c_{kl}\equiv 0$, $k=1,\ldots,K$, $l=1,\ldots,N$.

Hence $u_\epsilon = \sum_{i=1}^K w_{\epsilon,Q^\epsilon_i} + \phi_{\epsilon,Q^\epsilon_1,\ldots,Q^\epsilon_K}$ is a solution of (2.20). □

STEP 5. Using variational arguments to find critical points for the finite-dimensional reduced problem.

By Lemma 2.15, we just need to find a critical point for the reduced energy functional $\mathcal{M}_\epsilon(\mathbf{Q})$. Depending on the asymptotic behavior of the reduced energy functional, one can use either local minimization, or local maximization [29], or saddle point techniques [66]. Here there is no compactness problem since the reduced problem is already finite-dimensional.

We first obtain an asymptotic formula for $\mathcal{M}_\epsilon(\mathbf{Q})$. In fact for any $\mathbf{Q} \in \Lambda$, we have

$$\begin{aligned}\mathcal{M}_\epsilon(\mathbf{Q}) &= \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}}] + \int_{\Omega_\epsilon} (\nabla w_{\epsilon,\mathbf{Q}} \nabla \phi_{\epsilon,\mathbf{Q}} + w_{\epsilon,\mathbf{Q}}\phi_{\epsilon,\mathbf{Q}}) \\ &\quad - \int_{\Omega_\epsilon} f(w_{\epsilon,\mathbf{Q}})\phi_{\epsilon,\mathbf{Q}} + O\big(\|\phi_{\epsilon,\mathbf{Q}}\|_*^2\big) \\ &= \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}}] + \int_{\Omega_\epsilon} \big(-S_\epsilon[w_{\epsilon,\mathbf{Q}}]\big)\phi_{\epsilon,\mathbf{Q}} + O\big(\|\phi_{\epsilon,\mathbf{Q}}\|_*^2\big) \\ &= \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}}] + O\big(\big\|S_\epsilon[w_{\epsilon,\mathbf{Q}}]\big\|_{**}\|\phi_{\epsilon,\mathbf{Q}}\|_*\big) + O\big(\|\phi_{\epsilon,\mathbf{Q}}\|_*^2\big) \\ &= \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}}] + O\big(K^{2+\frac{2}{q}+2\sigma}\epsilon^{M(1+\sigma)}\big) = \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}}] + o\big(w\big(M|\ln\epsilon|\big)\big)\end{aligned}$$

by Lemma 2.10, Proposition 2.14 and the choice of M at (2.26).

By Lemma 2.10, we obtain

$$\begin{aligned}\mathcal{M}_\epsilon(\mathbf{Q}) &= KI[w] - \frac{1}{2}\big(\gamma_0 + o(1)\big)\sum_{i=1}^{K} w\left(\frac{2d(Q_i, \partial\Omega)}{\epsilon}\right) \\ &\quad - \frac{1}{2}\big(\gamma_0 + o(1)\big)\sum_{i\neq j} w\left(\frac{|Q_i - Q_j|}{\epsilon}\right) + o\big(w\big(M|\ln\epsilon|\big)\big).\end{aligned} \tag{2.81}$$

We shall prove

PROPOSITION 2.16. *For ϵ small, the following maximization problem*

$$\max\{\mathcal{M}_\epsilon(\mathbf{Q}) \colon \mathbf{Q} \in \Lambda\} \tag{2.82}$$

has a solution $\mathbf{Q}^\epsilon \in \Lambda^\circ$—the interior of Λ.

PROOF. First, we obtain a lower bound for $\mathcal{M}_\epsilon$: Recall that $K_\Omega(r)$ is the maximum number of non-overlapping balls with equal radius r packed in Ω. Now we choose K such that

$$1 \leqslant K \leqslant K_\Omega\left(\frac{M+2N}{2}\epsilon|\ln\epsilon|\right). \tag{2.83}$$

Let $\mathbf{Q}^0 = (Q_1^0, \ldots, Q_K^0)$ be the centers of arbitrary K balls among those $K_\Omega(\frac{M+2N}{2} \times \epsilon|\ln\epsilon|)$ balls. Certainly $\mathbf{Q}^0 \in \Lambda$. Then we have

$$w\left(\frac{2d(Q_i^0, \partial\Omega)}{\epsilon}\right) \leqslant e^{-\frac{2d(Q_i^0,\partial\Omega)}{\epsilon}} \leqslant \epsilon^{M+2N}, \qquad w\left(\frac{|Q_i^0 - Q_j^0|}{\epsilon}\right) \leqslant \epsilon^{M+2N}$$

and hence

$$\begin{aligned}\mathcal{M}_\epsilon(\mathbf{Q}^\epsilon) &\geqslant \mathcal{M}_\epsilon(\mathbf{Q}^0) \geqslant KI[w] - \frac{K}{2}(\gamma_0 + o(1))\epsilon^{M+2N}\\ &\quad - \frac{K^2}{2}(\gamma_0 + o(1))\epsilon^{M+2N} + o(w(M|\ln\epsilon|))\\ &\geqslant KI[w] - K^2(\gamma_0 + o(1))\epsilon^{M+2N} + o(w(M|\ln\epsilon|)). \qquad (2.84)\end{aligned}$$

On the other hand, if $\mathbf{Q}^\epsilon \in \partial\Lambda$, then either there exists (i, j) such that $|Q_i^\epsilon - Q_j^\epsilon| = M\epsilon|\ln\epsilon|$, or there exists a k such that $d(Q_k^\epsilon, \partial\Omega) = \frac{M}{2}\epsilon|\ln\epsilon|$. In both cases we have

$$\mathcal{M}_\epsilon(\mathbf{Q}^\epsilon) \leqslant KI[w] - \frac{1}{2}(\gamma_0 + o(1))w(M|\ln\epsilon|) + o(w(M|\ln\epsilon|)). \qquad (2.85)$$

Combining (2.85) and (2.84), we obtain

$$w(M|\ln\epsilon|) \leqslant 2K^2\epsilon^{M+2N} \leqslant C\epsilon^M(|\ln\epsilon|)^{-2N} \qquad (2.86)$$

which is impossible.

We conclude that $\mathbf{Q}^\epsilon \in \Lambda$. This completes the proof of Proposition 2.16. ☐

COMPLETION OF PROOF OF THEOREM 2.5. Theorem 2.5 follows from Proposition 2.16 and the reduction Lemma 2.15. ☐

2.4. *Bubbles to (2.4): the critical case*

Let $p = \frac{N+2}{N-2}$. By suitable scaling, (2.4) becomes the following problem

$$\begin{cases}\Delta u - \mu u + u^{\frac{N+2}{N-2}} = 0 \quad \text{in } \Omega,\\ u > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega\end{cases} \qquad (2.87)$$

where $\mu = \frac{1}{\epsilon^2}$ is large.

It is well known that the solutions to

$$\Delta U + U^{\frac{N+2}{N-2}} = 0 \qquad (2.88)$$

are given by the following

$$U_{\Lambda,\xi} = c_N\left(\frac{1}{\Lambda^2 + |x-\xi|^2}\right)^{\frac{N-2}{2}}, \quad \text{where } \Lambda > 0,\ \xi \in \mathbb{R}^N. \qquad (2.89)$$

A notable difference here is that the linearized operator $\Delta + (\frac{N+2}{N-2})U_{\Lambda,\xi}^{\frac{4}{N-2}}$ has $(N+1)$-dimensional kernels. Namely,

$$\mathrm{Kernel}\left(\Delta + \frac{N+2}{N-2}U_{\Lambda,\xi}^{\frac{4}{N-2}}\right) = \mathrm{span}\left\{\frac{\partial U_{\lambda,\xi}}{\partial \Lambda}, \frac{\partial U_{\Lambda,\xi}}{\partial \xi_1}, \ldots, \frac{\partial U_{\lambda,\xi}}{\partial \xi_N}\right\}. \tag{2.90}$$

Thus when we apply LEM, we need also to take care of the scaling parameters. See [13,43,65,66] and the references therein.

Concerning boundary bubbles, the existence of mountain-pass solutions was first proved in Wang [69] and Adimurthi and Mancini [1]. Ni, Takagi and Pan [55] showed the least energy solutions develop a bubble at the maximum point of the mean curvature (thereby establishing results similar to Theorem 2.1). Local mountain-pass solutions concentrating on one or separated boundary points are established in [23]. At non-degenerate critical points of the positive mean curvature, single boundary bubbles exist [2]. Lin, Wang and Wei [43] established results similar to Theorem 2.2 for dimension $N \geqslant 7$, at a non-degenerate local minimum point of the mean curvature with positive value:

THEOREM 2.17. *Suppose the following two assumptions hold*:

(H1) $N \geqslant 7$,

(H2) $Q_0 = 0$ *is a non-degenerate minimum point of* $H(Q)$ *and* $H(Q_0) > 0$.

Let $K \geqslant 2$ *be a fixed integer. Then there exists a* $\mu_K > 0$ *such that for* $\mu > \mu_K$, *problem* (2.87) *has a non-trivial solution* u_μ *with the following properties*

(1)

$$u(x) = \sum_{j=1}^{K} U_{\frac{1}{\mu}\Lambda_j, Q_0 + \mu^{\frac{3-N}{N}}\hat{Q}_j^{\mu}} + O\left(\mu^{\frac{N-4}{2}}\right),$$

where $\Lambda_j \to \Lambda_0 := A_0 H(Q_0) > 0$, $j = 1, \ldots, K$, *and*

(2) $\hat{\mathbf{Q}}^\mu := (\hat{Q}_1^\mu, \ldots, \hat{Q}_K^\mu)$ *approach an optimal configuration in the following problem*:

(∗) *Find out the optimal configuration* $(\hat{Q}_1, \ldots, \hat{Q}_K)$ *that minimizes the functional* $R[\hat{Q}_1, \ldots, \hat{Q}_K]$.

Here for $\hat{\mathbf{Q}} = (\hat{Q}_1, \ldots, \hat{Q}_K) \in R^{(N-1)K}$, $\hat{Q}_i \neq \hat{Q}_j$, *we define*

$$R[\hat{Q}_1, \ldots, \hat{Q}_K] := c_1 \sum_{j=1}^{K} \varphi(\hat{Q}_j) + c_2 \sum_{i \neq j} \frac{1}{|\hat{Q}_i - \hat{Q}_j|^{N-2}} \tag{2.91}$$

where $\varphi(Q) = \sum_{k,l} \partial_k \partial_l H(Q_0) Q_k Q_l$, c_1 *and* c_2 *are two generic constants.*

Theorem 2.17 is proved by *LEM*. Here the computation is more complicated, since the interaction between bubbles is very involved.

Concerning interior bubbles, under some assumptions, it is proved in [24] and [64] that there are *no* interior bubble solutions. However interior bubble solutions can be recovered if one add the boundary layers. (The boundary layer solution has been constructed in [50] (see Section 2.6).) The following result establishes the existence of multiple interior bubbles in dimension $N = 3, 4, 5$.

THEOREM 2.18. (See [71,92].) *Let $N = 3, 4, 5$. For any fixed integer k, then problem* (2.87) *has a solution* (*at least along a subsequence $\epsilon_k \to 0$*) *with k interior bubbles and one boundary layer.*

2.5. *Bubbles to (2.4): slightly supercritical case*

In the slightly supercritical case, we let $p = \frac{N+2}{N-2} + \delta$ where $\delta > 0$. Consider

$$\begin{cases} \Delta u - \mu u + u^p = 0 & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.92}$$

The following result was proved by [66] and [14] through the use of LEM.

THEOREM 2.19. *Let $N \geqslant 3$. Then $\delta > 0$ sufficiently small, problem* (2.92) *admits a boundary bubble solution.*

In fact, in the slightly supercritical case, there is also the phenomena of *bubble-towers*. A bubble-tower is a sum of bubbles centered at the same point

$$\sum_{j=1}^{K} U_{\Lambda_j,\xi}, \quad \text{where } \Lambda_1, \ \frac{\Lambda_{j+1}}{\Lambda_j} \to +\infty, \ j = 1, \dots, K-1. \tag{2.93}$$

This has been discussed in [15] and [25].

It is completely open whether or not point condensation solutions exist for (2.92) when $p > \frac{N+2}{N-2} + \delta$. In fact, let Ω be the unit ball. Using Pohozaev's identity, it is not difficult to show that *there exists a positive constant c_0, independent of $\epsilon \leqslant 1$, such that*

$$\inf_{\Omega} u \geqslant c_0 \tag{2.94}$$

for all radial solution u of (2.4). This marks a *basic* difference between the behavior of solutions of these two cases $p \leqslant \frac{N+2}{N-2}$ and $p > \frac{N+2}{N-2}$. *It eliminates the possibility of the existence of a radial spiky solution which approaches zero in measure as ϵ approaches zero* in the supercritical case $p > \frac{N+2}{N-2}$.

2.6. *Concentration on higher-dimensional sets*

The following conjecture has been made by Ni [53,54].

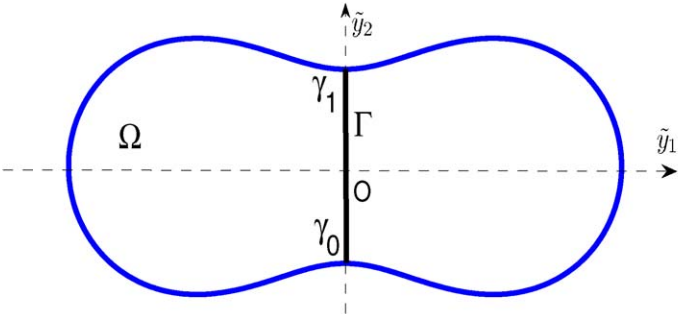

Fig. 1. Lines intersecting with $\partial\Omega$ orthogonally.

CONJECTURE. *Given any integer* $0 \leqslant k \leqslant n-1$, *there exists* $p_k \in (1,\infty)$ *such that for* $1 < p < p_k$, (2.4) *possesses a solution with* k*-dimensional concentration set*, *provided that* ϵ *is sufficiently small.*

Progress in this direction has only been made very recently. In [49] and [50], Malchiodi and Montenegro proved that for $N \geqslant 2$, there exists a sequence of numbers $\varepsilon_k \to 0$ such that problem (2.4) has a solution u_{ε_k} which concentrates at boundary of $\partial\Omega$ (or any component of $\partial\Omega$). Such a solution has the following energy bound

$$J_{\varepsilon_k}[u_{\varepsilon_k}] \sim \varepsilon_k^{N-1}. \tag{2.95}$$

In [48], Malchiodi showed the concentration phenomena for (2.4) along a closed non-degenerate geodesic of $\partial\Omega$ in three-dimensional smooth bounded domain Ω. F. Mahmoudi and A. Malchiodi in [51] prove a full general concentration of solutions along k-dimensional $(1 \leqslant k \leqslant n-1)$ non-degenerate minimal sub-manifolds of the boundary for $n \geqslant 3$ and $1 < p < \frac{n-k+2}{n-k-2}$. When $\Omega = B_1(0)$, there are also multiple (radially symmetric) clustered interfaces near the boundary [52].

For concentrations on lines intersecting with the boundary, Wei and Yang [93] made the first attempt in the two-dimensional case. Let $\Gamma \subset \Omega \subset \mathbb{R}^2$ be a curve satisfying the following assumptions: The curvature of Γ is zero and Γ intersects $\partial\Omega$ at exactly two points, saying, γ_1, γ_0 and at these points $\Gamma \perp \partial\Omega$. Let $-k_1$ and k_0 are the curvatures of the boundary $\partial\Omega$ at the points γ_1 and γ_0 respectively. A picture of Γ and Ω is as follows:

We define a geometric eigenvalue problem

$$\begin{aligned} &-f''(\theta) = \lambda f(\theta), \quad 0 < \theta < 1, \\ &f'(1) + k_1 f(1) = 0, \\ &f'(0) + k_0 f(0) = 0. \end{aligned} \tag{2.96}$$

We say that Γ is non-degenerate if (2.96) does not have a zero eigenvalue. This is equivalent to the following condition:

$$k_0 - k_1 + k_0 k_1 |\Gamma| \neq 0, \tag{2.97}$$

where $|\Gamma|$ denotes the length of Γ.

Moreover, we set up the gap condition that there exists a small constant $c > 0$

$$\left|\lambda_0 - \frac{k^2\pi^2}{|\Gamma|^2}\varepsilon^2\right| \geqslant c\varepsilon, \quad \forall k \in \mathbb{N}. \tag{2.98}$$

In [93], the following result was proved

THEOREM 2.20. *We assume that the line segment Γ satisfies the non-degenerate condition* (2.97). *Given a small constant c, there exists ε_0 such that for all $\varepsilon < \varepsilon_0$ satisfying the gap condition* (2.98), *problem* (2.4) *has a positive solution u_ε concentrating along a curve Γ_ϵ near Γ. Moreover, there exists some number c_0 such that u_ε satisfies globally,*

$$u_\varepsilon(x) \leqslant \exp\bigl[-c_0\varepsilon^{-1}\operatorname{dist}(x, \Gamma_\epsilon)\bigr]$$

and the curve Γ_ϵ will collapse to Γ as $\varepsilon \to 0$.

REMARK 2.6.1. The geometric eigenvalue problem (2.96) was first introduced by M. Kowalczyk in [37] where he constructed layered solution concentrating on a line for the Allen–Cahn equation.

REMARK 2.6.2. Theorem 2.20 is proved using the *infinite-dimensional Lyapunov–Schmidt reduction* technique introduced in [18].

REMARK 2.6.3. One can also constructed multiple clustered line concentrating solutions, using the Toda system. See [94]. This follows from earlier work in [19], where multiple clustered interfaces are constructed at non-minimizing lines for the Allen–Cahn equation. It is quite interesting to see the connection between Toda system

$$q_j'' + e^{q_j - q_{j+1}} - e^{q_{j-1} - q_j} = 0 \tag{2.99}$$

and clustered interfaces.

REMARK 2.6.4. It will be interesting to construct solutions concentrating on surfaces which intersect with $\partial\Omega$ orthogonally.

2.7. *Robin boundary condition*

Robin boundary conditions are particularly interesting in biological models where they often arise. We refer the reader to [10] for this aspect.

In [3], Berestycki and Wei discussed the existence and asymptotic behavior of least energy solution for following singularly perturbed problem with Robin boundary condition:

$$\begin{cases} \epsilon^2\Delta u - u + u^p = 0, u > 0 & \text{in } \Omega, \\ \epsilon\frac{\partial u}{\partial \nu} + \lambda u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.100}$$

where $\lambda > 0$. Similar to [57], we can define the following energy functional associated with (2.100):

$$J_\epsilon[u] := \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 - \int_\Omega F(u) + \frac{\epsilon\lambda}{2}\int_{\partial\Omega} u^2, \tag{2.101}$$

where $F(u) = \int_0^u f(s)\,ds$, $f(s) = s^p$, $u \in H^1(\Omega)$.

Similarly, for $\epsilon \in (0,1)$, we can define the so-called mountain-pass value

$$c_{\epsilon,\lambda} = \inf_{h\in\Gamma}\max_{0\leqslant t\leqslant 1} J_\epsilon\big[h(t)\big] \tag{2.102}$$

where $\Gamma = \{h\colon [0,1] \to H^1(\Omega) \mid h(t) \text{ is continuous}, h(0) = 0, h(1) = e\}$.

For fixed ϵ small, as λ moves from 0 (which is Neumann BC) to $+\infty$ (which is Dirichlet BC), by the results of [57,58] and [61], the asymptotic behavior of $u_{\epsilon,\lambda}$ changes dramatically: a boundary spike is displaced to become an interior spike. The question we shall answer is: where is the borderline of λ for spikes to move inwards?

Note that when $N = 1$, by ODE analysis, it is easy to see that the borderline is exactly at $\lambda = 1$. In fact, we may assume that $\Omega = (0,1)$, and a s $\epsilon \to 0$, the least energy solution converges to a homoclinic solution of the following ODE:

$$w'' - w + w^p = 0 \quad \text{in } \mathbb{R}^1, \qquad w(y) \to 0 \quad \text{as } |y| \to +\infty. \tag{2.103}$$

Then it follows that

$$(w')^2 = w^2 - \frac{2}{p+1} w^{p+1}, \qquad |w'| < w. \tag{2.104}$$

As $\epsilon \to 0$, the limiting boundary condition (2.100) becomes $w'(0) - \lambda w(0) = 0$. We see from (2.104) that this is possible if and only if $\lambda < 1$.

When $N = 2$, the situation changes dramatically. To understand the location of the spikes at the boundary, an essential role is played by the analogous problem in a half space with Robin boundary condition on the boundary. Thus we first consider

$$\begin{cases} \Delta u - u + f(u) = 0, u > 0 & \text{in } \mathbb{R}^N_+, \\ u \in H^1(\mathbb{R}^N_+), \quad \frac{\partial u}{\partial \nu} + \lambda u = 0 & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{2.105}$$

where $\mathbb{R}^N_+ = \{(y', y_N) \mid y_N > 0\}$ and ν is the outer normal on $\partial\mathbb{R}^N_+$.

Let

$$I_\lambda[u] = \int_{\mathbb{R}^+_N}\left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2\right) - \int_{\mathbb{R}^+_N} F(u) + \frac{\lambda}{2}\int_{\partial\mathbb{R}^+_N} u^2. \tag{2.106}$$

As before, we define a mountain-pass vale for I_λ:

$$c_\lambda = \inf_{v\not\equiv 0,\, v\in H^1(\mathbb{R}^+_N)} \sup_{t>0} I_\lambda[tv]. \tag{2.107}$$

Our first result deals with the half space problem:

THEOREM 2.21.

(1) *For* $\lambda \leqslant 1$, c_λ *is achieved by some function* w_λ, *which is a solution of* (2.105).

(2) *For* λ *large enough*, c_λ *is never achieved.*

(3) *Set*

$$\lambda_* = \inf\{\lambda \mid c_\lambda \text{ is achieved}\}. \tag{2.108}$$

Then $\lambda_* > 1$ *and for* $\lambda \leqslant \lambda_*$, c_λ *is achieved, and for* $\lambda > \lambda_*$, c_λ *is not achieved.*

The proof of Theorem 2.21 is by the method of *concentration-compactness*, and the method of *vanishing viscosity*.

Now consider the problem in a bounded domain.

THEOREM 2.22. *Let* $\lambda \leqslant \lambda_*$ *and* $u_{\epsilon,\lambda}$ *be a least energy solution of* (2.100). *Let* $x_\epsilon \in \Omega$ *be a point where* $u_{\epsilon,\lambda}$ *reaches its maximum value. Then after passing to a subsequence,* $x_\epsilon \to x_0 \in \partial\Omega$ *and*

(1) $d(x_\epsilon, \partial\Omega)/\epsilon \to d_0$, *for some* $d_0 > 0$,

(2) $v_{\epsilon,\lambda}(y) = u_{\epsilon,\lambda}(x_\epsilon + \epsilon y) \to w_\lambda(y)$ *in* C^1 *locally, where* w_λ *attains* c_λ *of* (2.107) (*and thus is a solution of* (2.105)),

(3) *the associated critical value can be estimated as follows*:

$$c_{\epsilon,\lambda} = \epsilon^N \bigl\{c_\lambda - \epsilon \bar{H}(x_0) + o(\epsilon)\bigr\} \tag{2.109}$$

where c_λ *is given by* (2.107), *and* $\bar{H}(x_0)$ *is given by the following*

$$\bar{H}(x_0) = \max_{w_\lambda \in \mathcal{S}_\lambda} \left[-\int_{\mathbb{R}^+_N} y' \cdot \nabla' w_\lambda \frac{\partial w_\lambda}{\partial y_N} H(x_0) \right] \tag{2.110}$$

where $\mathcal{S}_\lambda$ *is the set of all solutions of* (2.105) *attaining* c_λ, *and* $y' = (y_1, \ldots, y_{N-1})$, $\nabla' = (\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{N-1}})$,

(4) $\bar{H}(x_0) = \max_{x \in \partial\Omega} \bar{H}(x)$.

On the other hand, when $\lambda > \lambda_*$, a different asymptotic behavior appears.

THEOREM 2.23. *Let* $\lambda > \lambda_*$ *and* $u_{\epsilon,\lambda}$ *be a least energy solution of* (2.100). *Let* $x_\epsilon \in \Omega$ *be a point where* $u_{\epsilon,\lambda}$ *reaches its maximum value. Then after passing a subsequence, we have*

(1) $d(x_\epsilon, \partial\Omega) \to \max_{x \in \Omega} d(x, \partial\Omega)$,

(2) $v_{\epsilon,\lambda}(y) := u_{\epsilon,\lambda}(x_\epsilon + \epsilon y) \to w(y)$ *in* C^1 *locally, where* w *is the unique solution of* (2.8),

(3) *the associated critical value can be estimated as follows*:

$$c_{\epsilon,\lambda} = \epsilon^N \left[I[w] + \exp\left(-\frac{2d(x_\epsilon, \partial\Omega)}{\epsilon} \bigl(1 + o(1)\bigr) \right) \right]. \tag{2.111}$$

3. Stability and instability in the shadow system case

As we have already seen in Section 2 that there are *many* single and multiple spike solutions for the shadow system (2.2). The question is: are they all stable with respect to the shadow system (2.2)? Unfortunately, as we will show below, only one of them is stable.

Let u_ϵ be a (boundary or interior) spike solution. Then it is easy to see that $(a_\epsilon, \xi_\epsilon)$ defined by the following

$$a_\epsilon = \xi_\epsilon^{q/(p-1)} u_\epsilon, \qquad \xi_\epsilon = \left(\frac{1}{|\Omega|} \int_\Omega u_\epsilon^r \, dx \right)^{-(p-1)/(qr-(p-1)(s+1))} \tag{3.1}$$

is a solution pair of the stationary problem to the shadow system (2.2).

In this section, we analyze the following linearized eigenvalue problem

$$\begin{cases} \epsilon^2 \Delta \phi_\epsilon - \phi_\epsilon + p \frac{a_\epsilon^{p-1}}{\xi_\epsilon^q} \phi_\epsilon - q \frac{a_\epsilon^p}{\xi_\epsilon^{q+1}} \eta = \alpha_\epsilon \phi_\epsilon, \quad \frac{\partial \phi_\epsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \frac{r}{\tau|\Omega|} \int_\Omega \frac{a_\epsilon^{r-1}\phi_\epsilon}{\xi_\epsilon^s} \, dx - \frac{1+s}{\tau} \eta = \alpha_\epsilon \eta. \end{cases} \tag{3.2}$$

By using (3.1), it is easy to see that the eigenvalues of problem (3.2) in $H^2(\Omega) \times L^\infty(\Omega)$ are the same as the eigenvalues of the following eigenvalue problem

$$\begin{aligned} &\epsilon^2 \Delta \phi - \phi + p u_\epsilon^{p-1} \phi - \frac{qr}{s+1+\tau\alpha_\epsilon} \frac{\int_\Omega u_\epsilon^{r-1}\phi}{\int_\Omega u_\epsilon^r} u_\epsilon^p = \alpha_\epsilon \phi, \\ &\phi \in H^2(\Omega). \end{aligned} \tag{3.3}$$

A simple argument [8] shows that

THEOREM 3.1. *Any multiple-spike solution is linearly unstable for the shadow system* (2.2).

Let

$$\begin{aligned} &L_\epsilon(\phi) = \epsilon^2 \Delta \phi - \phi + p u_\epsilon^{p-1} \phi, \\ &\mathcal{L}_\epsilon(\phi) = L_\epsilon(\phi) - \frac{qr}{s+1+\tau\lambda} \frac{\int_\Omega u_\epsilon^{r-1}\phi}{\int_\Omega u_\epsilon^r} u_\epsilon^p. \end{aligned} \tag{3.4}$$

Thus we can only concentrate on the study of stability for single-spike solutions. The study of stability and instability of single spike solutions can be divided into two parts: *small eigenvalues* and *large eigenvalues*.

3.1. *Small eigenvalues for L_ϵ*

In [73], it was proved that single boundary spike must concentrate at a critical point of the mean curvature function $H(P)$. On the other hand, at a non-degenerate critical point of

$H(P)$, there is also a single boundary spike. Furthermore, in [76], it is proved that the single boundary spike at a non-degenerate critical point of $H(P)$ is actually non-degenerate.

Next we study the eigenvalue estimates associated with the linearized operator at u_ϵ: $L_\epsilon = \epsilon^2\Delta - 1 + pu_\epsilon^{p-1}$. (Here the domain of L_ϵ is $H^2(\Omega)$.) We first note the following result.

LEMMA 3.2. *The following eigenvalue problem*

$$\Delta\phi - \phi + pw^{p-1}\phi = \mu\phi \quad \text{in } \mathbb{R}^N,\ \phi \in H^1(\mathbb{R}^N) \tag{3.5}$$

admits the following set of eigenvalues:

$$\mu_1 > 0, \quad \mu_2 = \cdots = \mu_{N+1} = 0, \quad \mu_{N+2} < 0, \ldots. \tag{3.6}$$

Moreover, the eigenfunction corresponding to μ_1 is radial and of constant sign.

PROOF. This follows from Theorem 2.12 of [42] and Lemma 4.2 of [58]. ☐

The small eigenvalues for L_ϵ were characterized completely in [76].

THEOREM 3.3. (See [76].) *For ϵ sufficiently small, the following eigenvalue problem*

$$\begin{cases} \epsilon^2\Delta\phi_\epsilon - \phi_\epsilon + pu_\epsilon^{p-1}\phi_\epsilon = \tau_\epsilon\phi_\epsilon & \text{in } \Omega, \\ \frac{\partial\phi_\epsilon}{\partial\nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.7}$$

admits exactly $(N-1)$ eigenvalues $\tau_\epsilon^1 \leqslant \tau_\epsilon^2 \leqslant \cdots \leqslant \tau_\epsilon^{N-1}$ in the interval $[\frac{\mu_{N+1}}{2}, \frac{\mu_1}{2}]$, where μ_1 and μ_{N+1} are given by Lemma 3.2.

Moreover, we have the following asymptotic behavior of τ_ϵ^j:

$$\frac{\tau_\epsilon^j}{\epsilon^2} \to \eta_0\lambda_j, \quad j = 1, \ldots, N-1, \tag{3.8}$$

where $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{N-1}$ are the eigenvalues of the matrix $G_b(P_0) := (\partial_i\partial_j H(P_0))$, and

$$\eta_0 = \frac{N-1}{N+1}\frac{\int_{R_+^N}(w'(|z|))^2 z_N\,dz}{\int_{R_+^N}(\frac{\partial w}{\partial z_1})^2\,dz} > 0. \tag{3.9}$$

(*Here $w'(|z|)$ denotes the radial derivative of w with respect to $|z|$.*)

Furthermore the eigenfunction corresponding to τ_ϵ^j, $j = 1, \ldots, N-1$, is given by the following:

$$\phi_j^\epsilon = \sum_{i=1}^{N-1}\bigl(a_{ij} + o(1)\bigr)\frac{\partial w_{\epsilon,P_\epsilon}}{\partial\tau_i(P_\epsilon)} \tag{3.10}$$

where P_ϵ *is the local maximum point of* u_ϵ, $\vec{a}_j = (a_{1j}, \ldots, a_{(N-1)j})^T$ *is the eigenvector corresponding to* λ_j, *namely*

$$G_b(P_0)\vec{a}_j = \lambda_j \vec{a}_j, \quad j = 1, \ldots, N-1. \tag{3.11}$$

For single interior spikes, we obtain similar results. But it becomes more involved since now the error is exponentially small.

The existence of interior spike solutions depends highly on the geometry of the domain. In [74] and [75], the author first constructed a single interior spike solution. To state the result, we need to introduce some notations. Let

$$d\mu_{P_0}(z) = \lim_{\varepsilon \to 0} \frac{e^{-\frac{2|z-P_0|}{\varepsilon}}\,dz}{\int_{\partial\Omega} e^{-\frac{2|z-P_0|}{\varepsilon}}\,dz}. \tag{3.12}$$

It is easy to see that the support of $d\mu_{P_0}(z)$ is contained in $\bar{B}_{d(P_0,\partial\Omega)}(P_0) \cap \partial\Omega$.

A point P_0 is called "non-degenerate peak point" if the followings hold: there exists $a \in \mathbb{R}^N$ such that

$$\int_{\partial\Omega} e^{\langle z-P_0,a\rangle}(z-P_0)\,d\mu_{P_0}(z) = 0 \tag{H1}$$

and

$$\left(\int_{\partial\Omega} e^{\langle z-P_0,a\rangle}(z-P_0)_i(z-P_0)_j\,d\mu_{P_0}(z)\right) := G_i(P_0) \quad \text{is non-singular.} \tag{H2}$$

Such a vector a is unique. Moreover, $G_i(P_0)$ is a positive definite matrix. A geometric characterization of a non-degenerate peak point P_0 is the following:

$$P_0 \in \text{ interior (convex hull of support } \big(d\mu_{P_0}(z)\big).$$

For a proof of the above facts, see Theorem 5.1 of [74].

In [75] and [74], the author proved the following theorem.

THEOREM 3.4. *Suppose that* P_0 *is a non-degenerate peak point. Then for* $\epsilon \ll 1$, *there exists a single interior spike solution* u_ϵ *concentrating at* P_0. *Furthermore,* u_ϵ *is locally unique. Namely, if there are two families of single interior spike solutions* $u_{\epsilon,1}$ *and* $u_{\epsilon,2}$ *of* (2.4) *such that* $P_\epsilon^1 \to P_0$, $P_\epsilon^2 \to P_0$ *where*

$$u_{\epsilon,1}\big(P_\epsilon^1\big) = \max_{P\in\bar{\Omega}} u_\epsilon(P), \qquad u_{\epsilon,2}\big(P_\epsilon^2\big) = \max_{P\in\bar{\Omega}} u_{\epsilon,2}(P),$$

then $P_\epsilon^1 = P_\epsilon^2$, $u_{\epsilon,1} = u_{\epsilon,2}$. *Moreover,*

$$P_\epsilon^1 = P_\epsilon^2 = P_0 + \epsilon\left(\frac{1}{2}d(P_0,\partial\Omega)a + o(1)\right) \quad \text{as } \epsilon \to 0.$$

Let $w_{\epsilon,P}$ and $\varphi_{\epsilon,P}$ be defined as in Section 2.3. (It was proved in [75] and [74] that $-\epsilon\log[-\varphi_{\epsilon,P}(P)] \to 2d(P,\partial\Omega)$ as $\epsilon \to 0$.)

Similarly, we obtain the following eigenvalue estimates for u_ϵ

THEOREM 3.5. *The following eigenvalue problem*

$$\epsilon^2\Delta\phi - \phi + pu_\epsilon^{p-1}\phi = \tau^\epsilon\phi \quad \text{in } \Omega, \qquad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \partial\Omega \tag{3.13}$$

admits the following set of eigenvalues:

$$\begin{aligned}
&\tau_1^\epsilon = \mu_1 + o(1), \quad \tau_j^\epsilon = \big(c_0 + o(1)\big)\varphi_{\epsilon,P_0}(P_0)\lambda_{j-1}, \quad j = 2,\dots,N+1,\\
&\tau_l^\epsilon = \mu_l + o(1), \quad l \geqslant N+2,
\end{aligned}$$

where λ_j, $j = 1,\dots,N$, are the eigenvalues of $G_i(P_0)$ and

$$c_0 = 2d^{-2}(P_0,\partial\Omega)\frac{\int_{\mathbb{R}^N} pw^{p-1}w'u_*'(r)}{\int_{\mathbb{R}^N}(\frac{\partial w}{\partial y_1})^2\,dy} < 0, \tag{3.14}$$

where $u_*(r)$ is the unique radial solution of the following problem

$$\Delta u - u = 0, \qquad u(0) = 1, \qquad u = u(r) \quad \text{in } \mathbb{R}^N. \tag{3.15}$$

Furthermore, the eigenfunction (*suitably normalized*) *corresponding to τ_j^ϵ, $j = 2,\dots,N+1$, is given by the following*:

$$\phi_j^\epsilon = \sum_{l=1}^N \big(a_{j-1,l} + o(1)\big)\epsilon\frac{\partial w_{\epsilon,P}}{\partial P_l}\bigg|_{P=P_\epsilon}, \tag{3.16}$$

where $\vec{a}_j = (a_{j,1},\dots,a_{j,N})^t$ is the eigenvector corresponding to λ_j, namely

$$G_i(P_0)\vec{a}_j = \lambda_j\vec{a}_j, \quad j = 1,\dots,N.$$

3.2. *A reduction lemma*

Let α_ϵ be an eigenvalue of (3.3). Then the following holds. (The proof of it is routine. See Appendix of [77].)

LEMMA A.

(1) $\alpha_\epsilon = o(1)$ *if and only if* $\alpha_\epsilon = (1 + o(1))\tau_j^\epsilon$ *for some* $j = 2,\dots,N+1$, *where* τ_j^ϵ *is given by Theorem* 3.3 *or Theorem* 3.5.

(2) *If* $\alpha_\epsilon \to \alpha_0 \neq 0$. *Then* α_0 *is an eigenvalue of the following eigenvalue problem*

$$\begin{aligned} &\Delta\phi - \phi + pw^{p-1}\phi - \frac{qr}{s+1+\tau\alpha_0}\frac{\int_{\mathbb{R}^N} w^{r-1}\phi}{\int_{\mathbb{R}^N} w^r} w^p = \alpha_0\phi, \\ &\phi \in H^2(\mathbb{R}^N). \end{aligned} \tag{3.17}$$

A direct application of Theorem 3.5 is the following corollary.

COROLLARY 3.6. *For* $\epsilon \ll 1$, $(a_\epsilon, \xi_\epsilon)$ *is unstable with respect to the shadow system* (2.2).

3.3. *Large eigenvalues: NLEP method*

This section is devoted to the study of the non-local eigenvalue problem (3.17). By [77] and [78], if problem (3.17) admits an eigenvalue λ with positive real part, then all single point-condensation solutions are unstable, while if all eigenvalues of problem (3.17) have negative real part, then all single point-condensation solutions are either stable or metastable. (Here we say that a solution is metastable if the eigenvalues of the associated linearized operator either are exponentially small or have strictly negative real parts.) Therefore it is vital to study problem (3.17).

We first consider the simple case when $\tau = 0$. Namely, we study the following NLEP:

$$\Delta\phi - \phi + pw^{p-1}\phi - \gamma(p-1)\frac{\int_{\mathbb{R}^N} w^{r-1}\phi}{\int_{\mathbb{R}^N} w^r} w^p = \lambda\phi, \quad \phi \in H^2(\mathbb{R}^N), \tag{3.18}$$

where

$$\begin{aligned} &\gamma := \frac{qr}{(s+1)(p-1)}, \\ &\lambda \in \mathcal{C}, \quad \lambda \neq 0, \quad \phi(x) = \phi(|x|). \end{aligned} \tag{3.19}$$

For problem (3.18), it is known that when $\gamma = 0$, there exists an eigenvalue $\lambda = \mu_1 > 0$ (Lemma 3.2). An important property of (3.18) is that non-local term can push the eigenvalues of problem (3.18) to become negative so that the point-condensation solutions of the Gierer–Meinhardt system become stable or metastable.

A major difficulty in studying problem (3.18) is that the left-hand side operator is *not self-adjoint* if $r \neq p+1$. (In the classical Gierer–Meinhardt system, $r = 2$, $p = 2$.) Therefore it may have complex eigenvalues or Hopf bifurcations. Many traditional techniques do not work here.

In [78] and [77], the eigenvalues of problem (3.18) in the following two cases

$$r = 2, \quad \text{or} \quad r = p+1$$

are studied and the following results are proved.

THEOREM 3.7.

(1) *If* (p, q, r, s) *satisfies*

$$\text{(A)} \quad \gamma = \frac{qr}{(s+1)(p-1)} > 1,$$

and

$$\text{(B)} \quad r = 2, \quad 1 < p \leqslant 1 + \frac{4}{N} \quad \textit{or } r = p+1,\ 1 < p < \left(\frac{N+2}{N-2}\right)_+,$$

where $(\frac{N+2}{N-2})_+ = \frac{N+2}{N-2}$ *when* $N \geqslant 3$ *and* $(\frac{N+2}{N-2})_+ = +\infty$ *when* $N = 1, 2$.

Then $Re(\lambda) < -c_1 < 0$ *for some* $c_1 > 0$, *where* $\lambda \neq 0$ *is an eigenvalue of problem* (3.18).

(2) *If* $\gamma < 1$, *problem* (3.18) *has an eigenvalue* $\lambda_1 > 0$.

(3) *If*

$$\text{(C)} \quad r = 2, \quad p > 1 + \frac{4}{N} \quad \textit{and} \quad 1 < \gamma < 1 + c_0,$$

for some $c_0 > 0$. *Then problem* (3.18) *has an eigenvalue* $\lambda_1 > 0$.

We give a complete proof of Theorem 3.7 since this is the key element in all the stability result later on.

The proof of Theorem 3.7 is based on the following important inequalities which are new and interesting.

LEMMA 3.8. *Let* w *be the unique solution to* (2.8).

(1) *If* $1 < p < 1 + \frac{4}{N}$, *then there exists a positive constant* $a_1 > 0$ *such that*

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2\right) + \frac{2(p-1)\int_{\mathbb{R}^N} w\phi \int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2} \\ &\quad -(p-1)\frac{\int_{\mathbb{R}^N} w^{p+1}}{(\int_{\mathbb{R}^N} w^2)^2}\left(\int_{\mathbb{R}^N} w\phi\right)^2 \\ &\quad \geqslant a_1 d^2_{L^2(\mathbb{R}^N)}(\phi, X_1), \end{aligned} \tag{3.20}$$

for all $\phi \in H^1(\mathbb{R}^N)$, *where* $X_1 := \operatorname{span}\{w, \frac{\partial w}{\partial y_j},\ j = 1, \ldots, N\}$.

(2) *If* $p = 1 + \frac{4}{N}$, *then there exists a positive constant* $a_2 > 0$ *such that*

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2\right) + \frac{2(p-1)\int_{\mathbb{R}^N} w\phi \int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2} \\ &\quad - (p-1)\frac{\int_{\mathbb{R}^N} w^{p+1}}{(\int_{\mathbb{R}^N} w^2)^2}\left(\int_{\mathbb{R}^N} w\phi\right)^2 \end{aligned}$$

$$\geqslant a_2 d^2_{L^2(\mathbb{R}^N)}(\phi, X_2), \tag{3.21}$$

for all $\phi \in H^1(\mathbb{R}^N)$, *where* $X_2 := \operatorname{span}\{w,\ \frac{1}{p-1}w + \frac{1}{2}y\nabla w(y),\ \frac{\partial w}{\partial y_j},\ j = 1, \ldots, N\}$.

(3) *There exists a positive constant* $a_3 > 0$ *such that*

$$\int_{\mathbb{R}^N} \left(|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2\right) + \frac{(p-1)(\int_{\mathbb{R}^N} w^p\phi)^2}{\int_{\mathbb{R}^N} w^{p+1}}$$
$$\geqslant a_3 d^2_{L^2(\mathbb{R}^N)}(\phi, X_1), \quad \forall \phi \in H^1\left(\mathbb{R}^N\right). \tag{3.22}$$

PROOF OF LEMMA 3.8. To this end, we first introduce some notations and make some preparations. Set

$$L\phi := L_0\phi - \gamma(p-1)\frac{\int_{\mathbb{R}^N} w^{r-1}\phi}{\int_{\mathbb{R}^N} w^r} w^p, \quad \phi \in H^2\left(\mathbb{R}^N\right)$$

where $\gamma = \frac{qr}{(p-1)(s+1)}$ and $L_0 := \Delta - 1 + pw^{p-1}$. Note that L is not selfadjoint if $r \neq p+1$.

Let

$$X_0 := \operatorname{kernel}(L_0) = \operatorname{span}\left\{ \frac{\partial w}{\partial y_j} \;\middle|\; j = 1, \ldots, N \right\}.$$

Then

$$L_0 w = (p-1)w^p, \qquad L_0\left(\frac{1}{p-1}w + \frac{1}{2}x\nabla w\right) = w \tag{3.23}$$

and

$$\int_{\mathbb{R}^N} \left(L_0^{-1}w\right)w = \int_{\mathbb{R}^N} w\left(\frac{1}{p-1}w + \frac{1}{2}x\nabla w\right) = \left(\frac{1}{p-1} - \frac{N}{4}\right)\int_{\mathbb{R}^N} w^2, \tag{3.24}$$

$$\int_{\mathbb{R}^N} \left(L_0^{-1}w\right)w^p = \int_{\mathbb{R}^N} w^p\left(\frac{1}{p-1}w + \frac{1}{2}x\nabla w\right)$$
$$= \int_{\mathbb{R}^N} \left(L_0^{-1}w\right)\frac{1}{p-1}L_0 w = \frac{1}{p-1}\int_{\mathbb{R}^N} w^2. \tag{3.25}$$

Since L is not selfadjoint, we introduce a new operator as follows:

$$L_1\phi := L_0\phi - (p-1)\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^N} w^2}w^p - (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2}w$$
$$+ (p-1)\frac{\int_{\mathbb{R}^N} w^{p+1}\int_{\mathbb{R}^N} w\phi}{(\int_{\mathbb{R}^N} w^2)^2}w. \tag{3.26}$$

By (3.26), L_1 is selfadjoint. Next we compute the kernel of L_1. It is easy to see that $w, \frac{\partial w}{\partial y_j}, j = 1, \ldots, N, \in \mathrm{kernel}(L_1)$. On the other hand, if $\phi \in \mathrm{kernel}(L_1)$, then by (3.23)

$$\begin{aligned}L_0\phi &= c_1(\phi)w + c_2(\phi)w^p \\ &= c_1(\phi)L_0\left(\frac{1}{p-1}w + \frac{1}{2}x\nabla w\right) + c_2(\phi)L_0\left(\frac{w}{p-1}\right)\end{aligned}$$

where

$$\begin{aligned}c_1(\phi) &= (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2} - (p-1)\frac{\int_{\mathbb{R}^N} w^{p+1}\int_{\mathbb{R}^N} w\phi}{(\int_{R^N} w^2)^2}, \\ c_2(\phi) &= (p-1)\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^N} w^2}.\end{aligned}$$

Hence

$$\phi - c_1(\phi)\left(\frac{1}{p-1}w + \frac{1}{2}x\nabla w\right) - c_2(\phi)\frac{1}{p-1}w \in \mathrm{kernel}(L_0). \tag{3.27}$$

Note that

$$\begin{aligned}c_1(\phi) &= (p-1)c_1(\phi)\frac{\int_{\mathbb{R}^N} w^p(\frac{1}{p-1}w + \frac{1}{2}x\nabla w)}{\int_{\mathbb{R}^N} w^2} \\ &\quad - (p-1)c_1(\phi)\frac{\int_{\mathbb{R}^N} w^{p+1}\int_{\mathbb{R}^N} w(\frac{1}{p-1}w + \frac{1}{2}x\nabla w)}{(\int_{\mathbb{R}^N} w^2)^2} \\ &= c_1(\phi) - c_1(\phi)\left(\frac{1}{p-1} - \frac{N}{4}\right)\frac{\int_{\mathbb{R}^N} w^{p+1}}{\int_{\mathbb{R}^N} w^2}\end{aligned}$$

by (3.24) and (3.25). This implies that $c_1(\phi) = 0$. By (3.27) and Lemma 3.2, this shows that the kernel of L_1 is exactly X_1.

Now we prove (3.20). Suppose (3.20) is not true, then there exists (α, ϕ) such that (i) α is real and positive, (ii) $\phi \perp w$, $\phi \perp \frac{\partial w}{\partial y_j}$, $j = 1, \ldots, N$, and (iii) $L_1\phi = \alpha\phi$.

We show that this is impossible. From (ii) and (iii), we have

$$(L_0 - \alpha)\phi = (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2}w. \tag{3.28}$$

We first claim that $\int_{\mathbb{R}^N} w^p\phi \neq 0$. In fact if $\int_{\mathbb{R}^N} w^p\phi = 0$, then $\alpha > 0$ is an eigenvalue of L_0. By Lemma 3.2, $\alpha = \mu_1$ and ϕ has constant sign. This contradicts with the fact that $\phi \perp w$. Therefore $\alpha \neq \mu_1, 0$, and hence $L_0 - \alpha$ is invertible in $X_0^\perp$. So (3.28) implies

$$\phi = (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2}(L_0 - \alpha)^{-1}w.$$

Thus

$$\int_{\mathbb{R}^N} w^p\phi = (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2}\int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w\big)w^p,$$
$$\int_{\mathbb{R}^N} w^2 = (p-1)\int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w\big)w^p,$$
$$\int_{\mathbb{R}^N} w^2 = \int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w\big)\big((L_0-\alpha)w+\alpha w\big),$$
$$0 = \int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w\big)w. \tag{3.29}$$

Let $h_1(\alpha) = \int_{\mathbb{R}^N}((L_0-\alpha)^{-1}w)w$, then

$$h_1(0) = \int_{\mathbb{R}^N}\big(L_0^{-1}w\big)w = \int_{\mathbb{R}^N}\left(\frac{1}{p-1}w + \frac{1}{2}x\cdot\nabla w\right)w$$
$$= \left(\frac{1}{p-1} - \frac{N}{4}\right)\int_{\mathbb{R}^N} w^2 > 0$$

since $1 < p < 1 + \frac{4}{N}$. Moreover

$$h_1'(\alpha) = \int_{\mathbb{R}^N}\big((L_0-\alpha)^{-2}w\big)w = \int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w\big)^2 > 0.$$

This implies $h_1(\alpha) > 0$ for all $\alpha \in (0,\mu_1)$. Clearly, also $h_1(\alpha) < 0$ for $\alpha \in (\mu_1,\infty)$ (since $\lim_{\alpha\to+\infty} h_1(\alpha) = 0$). This is a contradiction to (3.29)!

This proves the inequality (3.20).

The proof of (3.21) is similar. In this case we have

$$\int_{\mathbb{R}^N}\big(L_0^{-1}w\big)w = \int_{\mathbb{R}^N} w\left(\frac{1}{p-1}w + \frac{1}{2}x\nabla w\right) = 0. \tag{3.30}$$

Thus the kernel of L_1 is X_2. The rest of the proof is exactly the same as before.

To prove (3.22), we introduce

$$L_3\phi := L_0\phi - (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^{p+1}}w^p. \tag{3.31}$$

Similar as before, the kernel of L_3 is exactly X_1.

Suppose (3.22) is not true, then there exists (α,ϕ) such that (a) α is real and positive, (b) $\phi \perp w, \phi \perp \frac{\partial w}{\partial y_j}$, $j = 1,\ldots,N$, and (c) $L_3\phi = \alpha\phi$.

We show that this is impossible. From (a) and (c), we have

$$(L_0-\alpha)\phi = \frac{(p-1)\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^{p+1}}w^p. \tag{3.32}$$

Similar to the proof of (3.20), we have that $\int_{\mathbb{R}^N} w^p\phi \neq 0, \alpha \neq \mu_1, 0$, and hence $L_0 - \alpha$ is invertible in $X_0^\perp$. So (3.32) implies

$$\phi = \frac{(p-1)\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^{p+1}}(L_0-\alpha)^{-1}w^p.$$

Thus

$$\int_{\mathbb{R}^N} w^p\phi = (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^{p+1}}\int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w^p\big)w^p,$$

$$\int_{\mathbb{R}^N} w^{p+1} = (p-1)\int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w^p\big)w^p. \tag{3.33}$$

Let

$$h_3(\alpha) = (p-1)\int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w^p\big)w^p - \int_{\mathbb{R}^N} w^{p+1},$$

then

$$h_3(0) = (p-1)\int_{\mathbb{R}^N}(L_0^{-1}w^p)w^p - \int_{\mathbb{R}^N} w^{p+1} = 0.$$

Moreover

$$h_3'(\alpha) = (p-1)\int_{\mathbb{R}^N}\big((L_0-\alpha)^{-2}w^p\big)w^p = (p-1)\int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w^p\big)^2 > 0.$$

This implies $h_3(\alpha) > 0$ for all $\alpha \in (0, \mu_1)$. Clearly, also $h_3(\alpha) < 0$ for $\alpha \in (\mu_1, \infty)$. A contradiction to (3.33)! □

Using Lemma 3.8, we can prove Theorem 3.7(i).

PROOF OF THEOREM 3.7(I). We divide the proof into three cases.

CASE 1. $r = 2, 1 < p < 1 + \frac{4}{N}$.

Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp \text{kernel}(L_0)$. Then we obtain two equations

$$L_0\phi_R - (p-1)\gamma\frac{\int_{\mathbb{R}^N} w\phi_R}{\int_{\mathbb{R}^N} w^2}w^p = \alpha_R\phi_R - \alpha_I\phi_I, \tag{3.34}$$

$$L_0\phi_I - (p-1)\gamma\frac{\int_{\mathbb{R}^N} w\phi_I}{\int_{\mathbb{R}^N} w^2}w^p = \alpha_R\phi_I + \alpha_I\phi_R. \tag{3.35}$$

Multiplying (3.34) by ϕ_R and (3.35) by ϕ_I and adding them together, we obtain

$$
\begin{aligned}
&-\alpha_R \int_{\mathbb{R}^N} (\phi_R^2 + \phi_I^2) \\
&\quad = L_1(\phi_R, \phi_R) + L_1(\phi_I, \phi_I) \\
&\qquad + (p-1)(\gamma-2) \frac{\int_{\mathbb{R}^N} w\phi_R \int_{\mathbb{R}^N} w^p \phi_R + \int_{\mathbb{R}^N} w\phi_I \int_{\mathbb{R}^N} w^p \phi_I}{\int_{\mathbb{R}^N} w^2} \\
&\qquad + (p-1) \frac{\int_{\mathbb{R}^N} w^{p+1}}{(\int_{\mathbb{R}^N} w^2)^2} \left[\left(\int_{\mathbb{R}^N} w\phi_R \right)^2 + \left(\int_{\mathbb{R}^N} w\phi_I \right)^2 \right].
\end{aligned}
$$

Multiplying (3.34) by w and (3.35) by w we obtain

$$
\begin{aligned}
&(p-1) \int_{\mathbb{R}^N} w^p \phi_R - \gamma(p-1) \frac{\int_{\mathbb{R}^N} w\phi_R}{\int_{\mathbb{R}^N} w^2} \int_{\mathbb{R}^N} w^{p+1} \\
&\quad = \alpha_R \int_{\mathbb{R}^N} w\phi_R - \alpha_I \int_{\mathbb{R}^N} w\phi_I,
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
&(p-1) \int_{\mathbb{R}^N} w^p \phi_I - \gamma(p-1) \frac{\int_{\mathbb{R}^N} w\phi_I}{\int_{\mathbb{R}^N} w^2} \int_{\mathbb{R}^N} w^{p+1} \\
&\quad = \alpha_R \int_{\mathbb{R}^N} w\phi_I + \alpha_I \int_{\mathbb{R}^N} w\phi_R.
\end{aligned}
\tag{3.37}
$$

Multiplying (3.36) by $\int_{\mathbb{R}^N} w\phi_R$ and (3.37) by $\int_{\mathbb{R}^N} w\phi_I$ and adding them together, we obtain

$$
\begin{aligned}
&(p-1) \int_{\mathbb{R}^N} w\phi_R \int_{\mathbb{R}^N} w^p \phi_R + (p-1) \int_{\mathbb{R}^N} w\phi_I \int_{\mathbb{R}^N} w^p \phi_I \\
&\quad = \left(\alpha_R + \gamma(p-1) \frac{\int_{\mathbb{R}^N} w^{p+1}}{\int_{\mathbb{R}^N} w^2} \right) \left(\left(\int_{\mathbb{R}^N} w\phi_R \right)^2 + \left(\int_{\mathbb{R}^N} w\phi_I \right)^2 \right).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&-\alpha_R \int_{\mathbb{R}^N} (\phi_R^2 + \phi_I^2) \\
&\quad = L_1(\phi_R, \phi_R) + L_1(\phi_I, \phi_I) \\
&\qquad + (p-1)(\gamma-2) \left(\frac{1}{p-1} \alpha_R + \gamma \frac{\int_{\mathbb{R}^N} w^{p+1}}{\int_{\mathbb{R}^N} w^2} \right) \frac{(\int_{\mathbb{R}^N} w\phi_R)^2 + (\int_{\mathbb{R}^N} w\phi_I)^2}{\int_{\mathbb{R}^N} w^2} \\
&\qquad + (p-1) \frac{\int_{\mathbb{R}^N} w^{p+1}}{(\int_{\mathbb{R}^N} w^2)^2} \left[\left(\int_{\mathbb{R}^N} w\phi_R \right)^2 + \left(\int_{\mathbb{R}^N} w\phi_I \right)^2 \right].
\end{aligned}
$$

Set

$$\phi_R = c_R w + \phi_R^{\perp}, \phi_R^{\perp} \perp X_1, \quad \phi_I = c_I w + \phi_I^{\perp}, \quad \phi_I^{\perp} \perp X_1.$$

Then

$$\int_{\mathbb{R}^N} w\phi_R = c_R \int_{\mathbb{R}^N} w^2, \qquad \int_{\mathbb{R}^N} w\phi_I = c_I \int_{\mathbb{R}^N} w^2,$$
$$d^2_{L^2(\mathbb{R}^N)}(\phi_R, X_1) = \|\phi_R^{\perp}\|^2_{L^2}, \qquad d^2_{L^2(\mathbb{R}^N)}(\phi_I, X_1) = \|\phi_I^{\perp}\|^2_{L^2}.$$

By some simple computations we have

$$\begin{aligned} &L_1(\phi_R, \phi_R) + L_1(\phi_I, \phi_I) \\ &\quad + (\gamma - 1)\alpha_R\big(c_R^2 + c_I^2\big) \int_{\mathbb{R}^N} w^2 + (p-1)(\gamma-1)^2\big(c_R^2 + c_I^2\big) \int_{\mathbb{R}^N} w^{p+1} \\ &\quad + \alpha_R\big(\|\phi_R^{\perp}\|^2_{L^2} + \|\phi_I^{\perp}\|^2_{L^2}\big) = 0. \end{aligned}$$

By Lemma 3.8 (1)

$$\begin{aligned} &(\gamma - 1)\alpha_R\big(c_R^2 + c_I^2\big) \int_{\mathbb{R}^N} w^2 \\ &\quad + (p-1)(\gamma-1)^2\big(c_R^2 + c_I^2\big) \int_{\mathbb{R}^N} w^{p+1} \\ &\quad + (\alpha_R + a_1)\big(\|\phi_R^{\perp}\|^2_{L^2} + \|\phi_I^{\perp}\|^2_{L^2}\big) \leqslant 0. \end{aligned}$$

Since $\gamma > 1$, we must have $\alpha_R < 0$, which proves Theorem 3.7 in Case 1.

CASE 2. $r = 2$, $p = 1 + \frac{4}{N}$.

Set

$$w_0 = \frac{1}{p-1} w + \frac{1}{2} x \nabla w. \tag{3.38}$$

We just need to take care of w_0.

Suppose that $\alpha_0 \neq 0$ is an eigenvalue of L. Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp \text{kernel}(L_0)$. Then similar to Case 1, we obtain two equations (3.34) and (3.35). We now decompose

$$\phi_R = c_R w + b_R w_0 + \phi_R^{\perp}, \quad \phi_R^{\perp} \perp X_1,$$
$$\phi_I = c_I w + b_I w_0 + \phi_I^{\perp}, \quad \phi_I^{\perp} \perp X_1.$$

Similar to Case 1, we obtain

$$L_1(\phi_R,\phi_R)+L_1(\phi_I,\phi_I)\\ +(\gamma-1)\alpha_R\bigl(c_R^2+c_I^2\bigr)\int_{\mathbb{R}^N}w^2+(p-1)(\gamma-1)^2\bigl(c_R^2+c_I^2\bigr)\int_{\mathbb{R}^N}w^{p+1}\\ +\alpha_R\left(b_R^2\left(\int_{\mathbb{R}^N}w_0^2\right)^2+b_I^2\left(\int_{\mathbb{R}^N}w_0^2\right)^2+\|\phi_R^\perp\|_{L^2}^2+\|\phi_I^\perp\|_{L^2}^2\right)\leqslant 0$$

By Lemma 3.8(2)

$$(\gamma-1)\alpha_R\bigl(c_R^2+c_I^2\bigr)\int_{\mathbb{R}^N}w^2+(p-1)(\gamma-1)^2\bigl(c_R^2+c_I^2\bigr)\int_{\mathbb{R}^N}w^{p+1}\\ +\alpha_R\left(b_R^2\left(\int_{\mathbb{R}^N}w_0^2\right)^2+b_I^2\left(\int_{\mathbb{R}^N}w_0^2\right)^2\right)+(\alpha_R+a_2)\bigl(\|\phi_R^\perp\|_{L^2}^2+\|\phi_I^\perp\|_{L^2}^2\bigr)\\ \leqslant 0.$$

If $\alpha_R\geqslant 0$, then necessarily we have

$$c_R=c_I=0,\quad \phi_R^\perp=0,\quad \phi_I^\perp=0.$$

Hence $\phi_R=b_Rw_0,\phi_I=b_Iw_0$. This implies that

$$b_RL_0w_0=(b_R-b_I)w_0,\quad b_IL_0w_0=(b_R+b_I)w_0,$$

which is impossible unless $b_R=b_I=0$. A contradiction!

CASE 3. $r=p+1$, $1<p<(\frac{N+2}{N-2})_+$.

Let $r=p+1$. L becomes

$$L=L_0-\frac{qr}{s+1}\frac{\int_{\mathbb{R}^N}w^p\cdot}{\int_{\mathbb{R}^N}w^{p+1}}w^p.$$

We will follow the proof of Case 1.

Let $\alpha_0=\alpha_R+i\alpha_I$ and $\phi=\phi_R+i\phi_I$. Since $\alpha_0\neq 0$, we can choose $\phi\perp\text{kernel}(L_0)$. Then similarly we obtain two equations

$$L_0\phi_R-(p-1)\gamma\frac{\int_{\mathbb{R}^N}w^p\phi_R}{\int_{\mathbb{R}^N}w^{p+1}}w^p=\alpha_R\phi_R-\alpha_I\phi_I, \tag{3.39}$$

$$L_0\phi_I-(p-1)\gamma\frac{\int_{\mathbb{R}^N}w^p\phi_I}{\int_{\mathbb{R}^N}w^{p+1}}w^p=\alpha_R\phi_I+\alpha_I\phi_R. \tag{3.40}$$

Multiplying (3.39) by ϕ_R and (3.40) by ϕ_I and adding them together, we obtain

$$-\alpha_R \int_{\mathbb{R}^N} (\phi_R^2 + \phi_I^2) = L_3(\phi_R, \phi_R) + L_3(\phi_I, \phi_I) + (p-1)(\gamma-1) \frac{(\int_{\mathbb{R}^N} w^p \phi_R)^2 + (\int_{\mathbb{R}^N} w^p \phi_I)^2}{\int_{\mathbb{R}^N} w^{p+1}}.$$

By Lemma 3.8(3)

$$\alpha_R \int_{\mathbb{R}^N} (\phi_R^2 + \phi_I^2) + a_3 d_{L^2}^2(\phi, X_1) + (p-1)(\gamma-1) \frac{(\int_{\mathbb{R}^N} w^p \phi_R)^2 + (\int_{\mathbb{R}^N} w^p \phi_I)^2}{\int_{\mathbb{R}^N} w^{p+1}} \leqslant 0$$

which implies $\alpha_R < 0$ since $\gamma > 1$.

Theorem 3.7(i) in Case 3 is thus proved. ☐

PROOF OF THEOREM 3.7(II). Assume that $\gamma < 1$. To prove Theorem 3.7(ii), we introduce the following function:

$$h_4(\lambda) := \int_{\mathbb{R}^N} w^r - \gamma(p-1) \int_{\mathbb{R}^N} \big((L_0 - \lambda)^{-1} w^p\big) w^{r-1}. \tag{3.41}$$

Note that $h_4(\lambda)$ is well defined in $(0, \mu_1)$, where μ_1 is the unique positive eigenvalue of L_0. Let us denote the corresponding eigenfunction by Φ_0. Since μ_1 is a principal eigenvalue, we may assume that $\Phi_0 > 0$.

It is easy to see that to prove Theorem 3.7(ii), it is enough to find a positive zero of $h_4(\lambda)$.

First we have

$$h_4(0) = \int_{\mathbb{R}^N} w^r - \gamma(p-1) \int_{\mathbb{R}^N} L_0^{-1} w^p w^{r-1} = (1-\gamma) \int_{\mathbb{R}^N} w^r > 0. \tag{3.42}$$

Set $\Phi_\lambda = (L_0 - \lambda)^{-1} w^p$. Then Φ_λ satisfies

$$(L_0 - \lambda)\Phi_\lambda = w^p. \tag{3.43}$$

Multiplying (3.43) by Φ_0 and integrating by parts, we have

$$(\mu_1 - \lambda) \int_{\mathbb{R}^N} \Phi_\lambda \Phi_0 = \int_{\mathbb{R}^N} \Phi_0 w^p,$$

which implies that

$$\int_{\mathbb{R}^N} \Phi_\lambda \Phi_0 = \frac{1}{\mu_1 - \lambda} \int_{\mathbb{R}^N} \Phi_0 w^p.$$

Let

$$\Phi_\lambda = \left(\frac{1}{(\mu_1-\lambda)\int_{\mathbb{R}^N}\Phi_0^2}\int_{\mathbb{R}^N}\Phi_0 w^p\right)\Phi_0 + \Phi_\lambda^\perp, \qquad \Phi_\lambda^\perp \perp \Phi_0. \tag{3.44}$$

Then as $\lambda \to \mu_1, \lambda < \mu_1$, we have that $\|\Phi_\lambda^\perp\|_{L^2(\mathbb{R}^N)}$ is uniformly bounded and by (3.44)

$$\int_{\mathbb{R}^N}\Phi_\lambda w^{r-1} \to +\infty,$$

which implies that

$$h_4(\lambda) \to -\infty \quad \text{as } \lambda \to \mu_1,\ \lambda < \mu_1. \tag{3.45}$$

By (3.42) and (3.45), there is a $\lambda_0 \in (0,\mu_1)$ such that $h_4(\lambda_0) = 0$.

This proves (ii) of Theorem 3.7. □

PROOF OF THEOREM 3.7(III). Similarly, we just need to find a zero of

$$h_5(\lambda) := \int_{\mathbb{R}^N} w^2 - \gamma(p-1)\int_{\mathbb{R}^N} w(L_0-\lambda)^{-1}w^p. \tag{3.46}$$

We write it as

$$\begin{aligned} h_5(\lambda) &= (1-\gamma)\int_{\mathbb{R}^N} w^2 - \gamma(p-1)\lambda\int_{\mathbb{R}^N} w\big[(L_0-\lambda)^{-1}(w)\big] \\ &= (1-\gamma)\int_{\mathbb{R}^N} w^2 - \gamma(p-1)\lambda\int_{\mathbb{R}^N} wL_0^{-1}(w) + O\big(\lambda^2\big). \end{aligned}$$

Since $\int_{\mathbb{R}^N} wL_0^{-1}(w) < 0$, we see that for $0 < \gamma - 1$ small, there is a small $\lambda_0 > 0$ such that $h_5(\lambda_0) > 0$. □

For general r, the author in [80] proved the following:

THEOREM 3.9.

(1) *If*

$$D(r) := \frac{(p-1)\int_{\mathbb{R}^N} L_0^{-1}w^{r-1}w^{r-1}\int_{\mathbb{R}^N} w^2}{(\int_{\mathbb{R}^N} w^r)^2} > 0 \tag{3.47}$$

where $L_0 = \Delta - 1 + pw^{p-1}$ (L_0^{-1} *exists in* $H_r^2(\mathbb{R}^N) = \{u \in H^2(\mathbb{R}^N) \mid u(x) = u(|x|)\}$) *and*

$$1 + \frac{1}{\sqrt{1+\rho_0}} < \gamma < 1 + \frac{1}{\sqrt{1-\rho_0}}, \tag{3.48}$$

where $\rho_0 > 0$ is given by

$$\rho_0 := \frac{\int_{\mathbb{R}^N} w^{p+1}}{\sqrt{\int_{\mathbb{R}^N} w^{2p} \int_{\mathbb{R}^N} w^2}} < 1. \tag{3.49}$$

Then for any non-zero eigenvalue λ of problem (3.18), *we have* $\mathrm{Re}(\lambda) < -c_1 < 0$ *for some* $c_1 > 0$.

(2) *If (p, q, r, s) satisfies*

$$1 + \frac{2r}{N} < p < \left(\frac{N+2}{N-2}\right)_+ \quad \text{and} \quad \gamma < 1 + c_0, \tag{3.50}$$

for some $c_0 > 0$. Then problem (3.18) *has a real eigenvalue* $\lambda_1 > 0$.

Generally speaking, $D(r)$ is very difficult to compute. A recent result of the author and L. Zhang partially solved this problem and moreover we obtained more general and explicit result. For example the following result are proved [81].

THEOREM 3.10. *Let*

$$F(r) = 1 - \frac{p-1}{2r} N.$$

Suppose $2 < r < p + 1$, $1 < p < 1 + \frac{2r}{N}$ *and*

$$F(r) \geqslant \frac{\gamma - 2}{\gamma} F(p+1) + \frac{|\gamma - 2|}{\gamma} \sqrt{F(p+1)(F(p+1) - F(2))}, \tag{3.51}$$

then for any non-zero eigenvalue λ of problem (3.18), *we have* $\mathrm{Re}(\lambda) < -c_1 < 0$ *for some* $c_1 > 0$.

REMARK. Condition (3.51) holds if $2 < r < p + 1$, $F(r) \geqslant 0$ (i.e., $1 < p \leqslant 1 + \frac{2r}{N}$) and $1 < \gamma \leqslant 2$. Thus in this case we obtain the stability of the non-zero eigenvalues of (3.18). This is the first explicit result for the case when $r \notin \{2, p+1\}$. For $\gamma > 2$, we need

$$F(r) \geqslant \frac{\gamma - 2}{\gamma} \big[F(p+1) - \sqrt{F(p+1)(F(p+1) - F(2))}\big].$$

Going back to the shadow system case, the following result was proved in [77].

THEOREM 3.11. *Assume that $\epsilon \ll 1$ and τ is small. If (p, q, r, s) satisfy* (A) *and* (B) *in Theorem* 3.7, *then*

(1) *single boundary spike solution at a non-degenerate local maximum point of mean curvature is stable, and*

(2) *single interior spike solution is metastable.*

Related work can also be found in [59] and [60].

3.4. *Uniqueness of Hopf bifurcations*

In Section 3.3, we have discussed the NLEP (3.17) when $\tau = 0$. It is easy to see that when τ small, results in Theorem 3.7 still hold. On the other hand, for τ large, it is easy to see that there is an unstable eigenvalue [8] to (3.17). (In fact, as $\tau \to +\infty$, there is a positive eigenvalue near $\mu_1 > 0$.) Therefore, as τ varies from 0 to ∞, Hopf bifurcation may occur. In this section, we show that in some special cases, Hopf bifurcation is actually unique.

We consider the following non-local eigenvalue problem (putting $r = p = 2, s = 0$ in (3.17))

$$L\phi := \Delta\phi - \phi + 2w\phi - \frac{\gamma}{1+\tau\lambda_0}\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^2} w^2} w^2 = \lambda_0\phi, \quad \phi \in H^2\big(\mathbb{R}^N\big). \tag{3.52}$$

THEOREM 3.12. *Let L be defined by* (3.52). *Assume that $N \leqslant 3$ and $\gamma > 1$. Then there exists a unique $\tau = \tau_1 > 0$ such that for $\tau < \tau_1$,* (3.52) *admits a positive eigenvalue, and for $\tau > \tau_1$, all non-zero eigenvalues of problem* (3.52) *satisfy* $\mathrm{Re}(\lambda) < 0$. *At $\tau = \tau_1$, L has a Hopf bifurcation.*

PROOF OF THEOREM 3.12. Let $\gamma > 1$. As in [8], we may consider radially symmetric functions only. By Theorem 1.4 of [77], for $\tau = 0$ (and by perturbation, for τ small), all eigenvalues lie on the left half plane. By [8], for τ large, there exist unstable eigenvalues.

Note that the eigenvalues will not cross through zero: in fact, if $\lambda_0 = 0$, then we have

$$L_0\phi - \gamma\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^N} w^2} w^2 = 0$$

which implies that

$$L_0\left(\phi - \gamma\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^N} w^2} w\right) = 0$$

and hence by Lemma 3.2

$$\phi - \gamma\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^N} w^2} w \in X_0.$$

This is impossible since ϕ is radially symmetric and $\phi \neq cw$ for all $c \in R$.

Thus there must be a point τ_1 at which L has a Hopf bifurcation, i.e., L has a purely imaginary eigenvalue $\alpha = \sqrt{-1}\alpha_I$. To prove Theorem 3.12, all we need to show is that τ_1 is unique. That is

LEMMA 3.13. *Let $\gamma > 1$. Then there exists a unique $\tau_1 > 0$ such that L has a Hopf bifurcation.*

PROOF. Let $\lambda_0 = \sqrt{-1}\alpha_I$ be an eigenvalue of L. Without loss of generality, we may assume that $\alpha_I > 0$. (Note that $-\sqrt{-1}\alpha_I$ is also an eigenvalue of L.) Let $\phi_0 = (L_0 - \sqrt{-1}\alpha_I)^{-1}w^2$. Then (3.52) becomes

$$\frac{\int_{\mathbb{R}^N} w\phi_0}{\int_{\mathbb{R}^N} w^2} = \frac{1+\tau\sqrt{-1}\alpha_I}{\gamma}. \tag{3.53}$$

Let $\phi_0 = \phi_0^R + \sqrt{-1}\phi_0^I$. Then from (3.53), we obtain the two equations

$$\frac{\int_{\mathbb{R}^N} w\phi_0^R}{\int_{\mathbb{R}^2} w^N} = \frac{1}{\gamma}, \tag{3.54}$$

$$\frac{\int_{\mathbb{R}^N} w\phi_0^I}{\int_{\mathbb{R}^2} w^N} = \frac{\tau\alpha_I}{\gamma}. \tag{3.55}$$

Note that (3.54) is independent of τ.

Let us now compute $\int_{\mathbb{R}^N} w\phi_0^R$. Observe that (ϕ_0^R, ϕ_0^I) satisfies

$$L_0\phi_0^R = w^2 - \alpha_I\phi_0^I, \quad L_0\phi_0^I = \alpha_I\phi_0^R.$$

So $\phi_0^R = \alpha_I^{-1}L_0\phi_0^I$ and

$$\phi_0^I = \alpha_I\bigl(L_0^2+\alpha_I^2\bigr)^{-1}w^2, \qquad \phi_0^R = L_0\bigl(L_0^2+\alpha_I^2\bigr)^{-1}w^2. \tag{3.56}$$

Substituting (3.56) into (3.54) and (3.55), we obtain

$$\frac{\int_{\mathbb{R}^N}[wL_0(L_0^2+\alpha_I^2)^{-1}w^2]}{\int_{\mathbb{R}^N} w^2} = \frac{1}{\gamma}, \tag{3.57}$$

$$\frac{\int_{\mathbb{R}^N}[w(L_0^2+\alpha_I^2)^{-1}w^2]}{\int_{\mathbb{R}^2} w^2} = \frac{\tau}{\gamma}. \tag{3.58}$$

Let

$$h_6(\alpha_I) = \frac{\int_{\mathbb{R}^N} wL_0(L_0^2+\alpha_I^2)^{-1}w^2}{\int_{\mathbb{R}^2} w^2}.$$

Then integration by parts gives

$$h_6(\alpha_I) = \frac{\int_{\mathbb{R}^N} w^2(L_0^2+\alpha_I^2)^{-1}w^2}{\int_{\mathbb{R}^N} w^2}.$$

Note that

$$h_6'(\alpha_I) = -2\alpha_I\frac{\int_{\mathbb{R}^N} w^2(L_0^2+\alpha_I^2)^{-2}w^2}{\int_{\mathbb{R}^N} w^2} < 0.$$

So since

$$h_6(0) = \frac{\int_{\mathbb{R}^N} w(L_0^{-1} w^2)}{\int_{\mathbb{R}^N} w^2} > 0,$$

$h_6(\alpha_I) \to 0$ as $\alpha_I \to \infty$, and $\gamma > 1$, there exists a unique $\alpha_I > 0$ such that (3.57) holds. Substituting this unique α_I into (3.58), we obtain a unique $\tau = \tau_1 > 0$.

Lemma 3.13 is thus proved. □

Theorem 3.12 now follows from Lemma 3.13. □

3.5. *Finite ϵ case*

In all the previous sections, it is always assumed that ϵ is small. However, in practical applications, it is vital to know how small ϵ should be. The finite ϵ case has been completely characterized in one-dimensional case by Wei and Winter [89]. We summarize the results here.

Without loss of generality, we may assume that $\Omega = (0, 1)$. That is, we consider

$$\begin{cases} a_t = \epsilon^2 a_{xx} - a + \frac{a^p}{\xi^q}, & 0 < x < 1,\ t > 0, \\ \tau \xi_t = -\xi + \xi^{-s} \int_0^1 a^r \, dx, \\ a > 0, & a_x(0,t) = a_x(1,t) = 0. \end{cases} \tag{3.59}$$

The steady-state problem of (3.59) is equivalent to the following problem for the transformed function u_ϵ given by $u_\epsilon(x) = \xi^{-\frac{q}{p-1}} a(x)$:

$$\xi^{1+s-\frac{qr}{p-1}} = \int_0^1 u^r(x)\, dx$$

and

$$\begin{aligned} &\epsilon^2 u_{xx} - u + u^p = 0, \\ &u_x(x) < 0,\ 0 < x < 1, \quad u_x(0) = u_x(1) = 0. \end{aligned} \tag{3.60}$$

Letting

$$L := \frac{1}{\epsilon} \tag{3.61}$$

and rescaling $u(x) = w_L(y)$, where $y = Lx$, we see that w_L satisfies the following ODE:

$$\begin{aligned} &w_L'' - w_L + w_L^p = 0, \\ &w_L'(y) < 0,\ 0 < y < L, \quad w_L'(0) = w_L'(L) = 0. \end{aligned} \tag{3.62}$$

Since (3.62) is an autonomous ODE, it is easy to see that a non-trivial solution exists if and only if

$$\epsilon < \frac{\sqrt{p-1}}{\pi} \quad \left(\text{or } L > \frac{\pi}{\sqrt{p-1}}\right). \tag{3.63}$$

The stability of steady-state solutions to (3.59) has been a subject of study in the last few years. A recent result of [56] (see Theorem 1.1 of [56]) says that a stable solution to (3.59) must be asymptotically monotone. More precisely, if $(A(x,t), \xi(t)), t \geqslant 0$ is a solution to (3.59) that is linearly neutrally stable, then there is a $t_0 > 0$ such that

$$a_x(x, t_0) \neq 0 \quad \text{for all } (x,t) \in (0,1) \times [t_0, +\infty). \tag{3.64}$$

Thus all *non-monotone* steady-state solutions are linearly unstable. Therefore we focus our attention on *monotone solutions*. There are two monotone solutions—the monotone increasing one and the monotone decreasing one. Since these two solutions differ by reflection, we consider the monotone decreasing function only. This solution is then called u_ϵ and it has the least energy among all positive solutions of (3.60), see [60]. If $L \leqslant \frac{\pi}{\sqrt{p-1}}$, then $w_L = 1$. We also denote the corresponding solutions to (3.59) by

$$a_L(x) = \xi_L^{\frac{q}{p-1}} w_L(Lx), \qquad \xi_L^{1+s-\frac{qr}{p-1}} = \int_0^1 w_L^r(Lx)\,dx. \tag{3.65}$$

Before stating our results, we first introduce some notation. Let $I = (0, L)$ and $\phi \in H^2(I)$. We define the following operator:

$$\mathcal{L}[\phi] = \phi'' - \phi + p w_L^{p-1} \phi. \tag{3.66}$$

It is proved [89] that $\mathcal{L}$ has the spectrum

$$\lambda_1 > 0, \quad \lambda_j < 0, \quad j = 2, 3, \ldots. \tag{3.67}$$

Hence for the map $\mathcal{L}$ from $H^2(I)$ to $L^2(I)$ we know that $\mathcal{L}^{-1}$ exists, where $\mathcal{L}^{-1}$ is the inverse of $\mathcal{L}$. This implies that $\mathcal{L}^{-1} w_L$ is well defined.

Then we have the following theorem

THEOREM 3.14. *Assume that* $L > \frac{\pi}{\sqrt{p-1}}$ *and either*

$$r = 2, \quad \int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0 \tag{3.68}$$

or

$$r = p + 1. \tag{3.69}$$

Then (a_L, ξ_L) *(given by* (3.65)*) is a linearly stable steady state to* (3.59) *for* τ *small.*

This theorem reduces the issue of stability to the computation of the integral

$$\int_0^L w_L \mathcal{L}^{-1} w_L \, dy.$$

This integral is quite difficult to compute for general L.

For τ finite, we have the following theorem.

THEOREM 3.15. *Let* (3.68) *be true and* $L > \frac{\pi}{\sqrt{p-1}}$. *Then there exists a unique* $\tau_c > 0$ *such that for* $\tau < \tau_c$, (a_L, ξ_L) *is stable and for* $\tau > \tau_c$, (a_L, ξ_L) *is unstable. At* $\tau = \tau_c$, *there exists a unique Hopf bifurcation. Furthermore, the Hopf bifurcation is transversal, namely, we have*

$$\left.\frac{d\lambda_R}{d\tau}\right|_{\tau=\tau_c} > 0, \tag{3.70}$$

where λ_R *is the real part of the eigenvalue.*

Using Weierstrass $p(z)$ functions and Jacobi elliptic integrals, one can show that $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0$ for all $L > \pi$ in the cases $r = 2$, $p = 2, 3$. The original Gierer–Meinhardt system ($(p, q, r, s) = (2, 1, 2, 0)$) falls into this class. Thus for the shadow system of the original Gierer–Meinhardt system, we have a complete picture of the stability of (a_L, ξ_L) for any $\tau > 0$ and any $L > 0$, by the following theorem

THEOREM 3.16. *Assume that* $L > \frac{\pi}{\sqrt{p-1}}$ *and* $r = 2$, $p = 2$ *or* 3. *Then there exists a unique* $\tau_c > 0$ *such that for* $\tau < \tau_c$, (a_L, ξ_L) *is stable and for* $\tau > \tau_c$, (A_L, ξ_L) *is unstable. At* $\tau = \tau_c$, *there exists a Hopf bifurcation. Furthermore, the Hopf bifurcation is transversal.*

Theorem 3.16 gives a complete picture of the stability of non-trivial monotone solutions in terms of L since for $L \leqslant \frac{\pi}{\sqrt{p-1}}$ we necessarily have $w_L \equiv 1$. Combining this with the results of [56], we have *completely* classified stability and instability of all steady-state solutions for *all* $\epsilon > 0$ for the shadow system of the original Gierer–Meinhardt system.

We do not know if the Hopf bifurcation in Theorem 3.15 is subcritical or supercritical. This is related to another interesting question: is there time-periodic solution $(a(x,t), \xi(x,t))$ to (3.59) at the Hopf bifurcation point $\tau = \tau_c$? If so, is it stable or unstable?

We can also extend this idea to general domains in $\mathbb{R}^N$, $N \geqslant 2$. Namely we consider

$$\begin{cases} a_t = \Delta a - a + \frac{a^p}{\xi^q}, & x \in \Omega_L, \quad t > 0, \\ \tau \xi_t = -\xi + \xi^{-s} \frac{1}{|\Omega_L|} \int_{\Omega_L} a^r, \\ a > 0, \quad \frac{\partial a}{\partial \nu} = 0 & \text{on } \partial \Omega_L, \end{cases} \tag{3.71}$$

where we have scaled the ϵ into the domain through $\Omega_L = \frac{1}{\epsilon}\Omega$. In this case, let us assume that $\Omega_L \subset \mathbb{R}^N$ is a smooth and bounded domain, and the exponents (p, q, r, s) satisfy the following condition

$$p > 1, \quad q > 0, \quad r > 0, \quad s \geqslant 0, \quad \gamma := \frac{qr}{(p-1)(s+1)} > 1,$$

and p is subcritical:

$$1 < p < \frac{N+2}{N-2} \quad \text{if } N \geqslant 3; \quad 1 < p < +\infty \quad \text{if } N = 2.$$

The steady state solution of (3.71) is given by

$$a = \xi^{\frac{q}{p-1}} u, \quad \xi^{1+s-\frac{qr}{p-1}} = \frac{1}{|\Omega_L|} \int_{\Omega_L} u^r \tag{3.72}$$

where u is a solution of the following problem:

$$\begin{cases} \Delta u - u + u^p = 0, \quad u > 0 & \text{in } \Omega_L, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_L. \end{cases} \tag{3.73}$$

We again consider the minimizer solution $w_L(x)$ which satisfies (3.73) and

$$E[w_L] = \inf_{u \in H^1(\Omega_L),\ u \not\equiv 0} E[u] \tag{3.74}$$

where

$$E[u] = \frac{\int_{\Omega_L} (|\nabla u|^2 + u^2)}{(\int_{\Omega_L} u^{p+1})^{\frac{2}{p+1}}}.$$

The corresponding steady-state solution to the shadow system (3.71) is denoted by

$$a_L = \xi_L^{\frac{q}{p-1}} w_L, \quad \xi_L^{1+s-\frac{qr}{p-1}} = \frac{1}{|\Omega_L|} \int_{\Omega_L} w_L^r. \tag{3.75}$$

Let

$$\mathcal{L}[\phi] = \Delta\phi - \phi + p w_L^{p-1} \phi.$$

Then we have the following lemma whose proof is similar to Lemma 3.2.

LEMMA 3.17. *Consider the following eigenvalue problem*

$$\begin{cases} \mathcal{L}\phi = \lambda\phi, & \text{in } \Omega_L, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega_L. \end{cases} \tag{3.76}$$

Then $\lambda_1 > 0$ *and* $\lambda_2 \leqslant 0$.

We now put two important assumptions:
We first assume that

$$\text{(A1)} \quad \mathcal{L}^{-1} \text{ exists.}$$

Under (A1), we assume that

$$\text{(A2)} \quad \int_{\Omega_L} w_L\bigl(\mathcal{L}^{-1} w_L\bigr) > 0.$$

We can now state the following theorem

THEOREM 3.18. *Assume that either*

$$r = p + 1, \quad \textit{and } \text{(A1)} \textit{ holds},$$

or

$$r = 2, \quad \textit{and } \text{(A1)} \textit{ and } \text{(A2)} \textit{ hold}.$$

Then (a_L, ξ_L) *is linearly stable for* τ *small.*

In the case of $r = 2$, *there exists a unique* $\tau = \tau_c$ *such that* (a_L, ξ_L) *is stable for* $\tau < \tau_c$, *unstable for* $\tau > \tau_c$, *and there is a Hopf bifurcation at* $\tau = \tau_c$. *Furthermore, the Hopf bifurcation is transversal.*

The proof of Theorem (3.18) is similar to the one-dimensional case.

It remains an interesting and difficult question as to verify (A1) and (A2) analytically. If L is large, the assumption (A1) is verified in [76] and assumption (A2) holds true if

$$1 < p < 1 + \frac{4}{N}. \tag{3.77}$$

This recovers the results of [77].

It is difficult to verify (A1) and (A2) in general domains. One may ask: does (A1) hold true for *generic* domains?

3.6. *The stability of boundary spikes for the Robin boundary condition*

The stability of least energy solution in the Robin boundary condition case is quite complicated. We state the following result which deals with one-dimensional case only:

THEOREM 3.19. (See [45].) *Consider the following*

$$\begin{cases} a_t = \epsilon^2 a_{xx} - a + \frac{a^p}{\xi^q}, \quad 0 < x < 1,\ t > 0, \\ \tau \xi_t = -\xi + \xi^{-s} \int_0^1 a^r \, dx, \\ a > 0, \quad \epsilon a_x(0,t) + \lambda a(0,t) = \epsilon a_x(1,t) + \lambda a(1,t) = 0, \\ h_x(0,t) = h_x(1,t) = 0. \end{cases} \tag{3.78}$$

Assume that $r = 2$, $1 < p \leqslant 3$ *or* $r = p + 1$, $1 < p < +\infty$. *Then for each* $\lambda \in (0, 1)$ *the least energy solution is stable for* $\tau < \tau_1$ *and unstable for* $\tau > \tau_1$. *At* τ_1, *there is a Hopf bifurcation.*

The main idea of the proof is similar to that of Theorem 3.14. Here we have to study an NLEP on a half line with Robin boundary condition:

$$\begin{cases} \phi'' - \phi + p w_{x_0}^{p-1} \phi - \gamma(p-1) \frac{\int_0^\infty w_{x_0} \phi}{\int_0^\infty w_{x_0}^2} w_{x_0}^p = \alpha \phi, \quad 0 < y < +\infty, \\ \phi'(0) - \lambda \phi(0) = 0 \end{cases} \tag{3.79}$$

where $w_{x_0} = w(y - x_0)$ with $w'(-x_0) = \lambda w(-x_0)$. Let $L_{x_0}(\phi) = \phi'' - \phi + p w_{x_0}^{p-1} \phi$. Then we need to show that

$$\int_0^\infty w_{x_0} \big[L_{x_0}^{-1}(w_{x_0}) \big] > 0. \tag{3.80}$$

By some lengthy computations, we can show that the function $\int_0^\infty w_{x_0}[L_{x_0}^{-1}(w_{x_0})]$ is an increasing function in x_0 when $p < 3$, and a constant when $p = 3$, and an decreasing function when $p > 3$.

REMARK 3.6.1. An interesting phenomena is the case of $3 < p < 5$. In this case, one can show that there exists a $a_0 \in (0, 1)$ such that the boundary spike is stable when $a \in (0, a_0)$ and unstable when $a \in (a_0, 1)$. It is quite interesting to see that the Robin boundary condition can also introduce some instability.

4. Full Gierer–Meinhardt system: One-dimensional case

In this section, we study the full Gierer–Meinhardt system in the one-dimensional case. Unlike the shadow system case, where one can reduce the existence of solutions to a variational elliptic problem, there is no variational structure for the full Gierer–Meinhardt system. This is the major problem, which is also the source of all interesting new phenomena.

We begin with the steady-state problem in the full space case.

4.1. *Bound states: the case of $\Omega = \mathbb{R}^1$*

Let $\Omega = \mathbb{R}^1$. By a change of variables the steady-state problem for (GM) can be conveniently written as follows

$$\begin{cases} \Delta a - a + \frac{a^p}{h^q} = 0, \ a > 0 & \text{in } \mathbb{R}^1, \\ \Delta h - \sigma^2 h + \frac{a^r}{h^s} = 0, \ h > 0 & \text{in } \mathbb{R}^1, \\ a(x), \ h(x) \to 0 & \text{as } |x| \to +\infty \end{cases} \tag{4.1}$$

where

$$\sigma^2 = \frac{\epsilon^2}{D} \ll 1.$$

The existence of multiple spikes solutions to (4.1) is referred to as "symmetry-breaking" phenomena. This was proved in [12] (by dynamical system techniques) and [7] (by PDE methods). We will sketch the PDE methods in Section 5.1.

THEOREM 4.1. (See [7,12].) *For each fixed positive integer k, there exists $\sigma_k > 0$ such that problem* (4.1) *has a solution (a_ϵ, h_ϵ) with the following properties*

$$a_\epsilon(x) \sim \frac{c_k}{\sigma}\left(\sum_{j=1}^{k} w\left(x - \xi_j^\sigma\right)\right) \tag{4.2}$$

where $c_k > 0$ is a generic constant and

$$\xi_j^\sigma = \left(j - \frac{k+1}{2}\right)\log\frac{1}{\sigma} + O\left(\log\log\frac{1}{\sigma}\right), \quad j = 1, \ldots, k. \tag{4.3}$$

4.2. *The bounded domain case: Existence of symmetric K-spikes*

Without loss of generality, we may assume that $\Omega = (-1, 1)$. We consider the following elliptic system

$$\begin{cases} \epsilon^2 a'' - a + \frac{a^p}{h^q} = 0, & -1 < x < 1, \\ D h'' - h + \frac{a^r}{h^s} = 0, & -1 < x < 1, \\ a'(\pm 1) = h'(\pm 1) = 0. \end{cases} \tag{4.4}$$

In this case, the existence of multiple-peaked solutions was first obtained by I. Takagi in [67].

THEOREM 4.2. (See [67].) *Fix any positive integer K. If $\frac{\epsilon}{\sqrt{D}}$ sufficiently small, there exists a K-peaked solution $(a_{\epsilon,K}, h_{\epsilon K})$ to* (4.4) *such that $(a_{\epsilon,K}, h_{\epsilon,K})$ has exactly K local* *maximum points* $-1 < x_1 < x_2 < \cdots < x_K < 1$ *which are equally distributed. In fact, we have*

$$x_j = -1 + \frac{2j-1}{K}, \quad j = 1, \ldots, K.$$

Takagi's proof uses the symmetry of the problems: by reflection, one can reduce the existence of multiple symmetric spikes solutions to studying the existence of one boundary spike solution. Namely, we just need to study the following system

$$\begin{cases} \epsilon^2 a'' - a + \frac{a^p}{h^q} = 0, & 0 < x < \frac{1}{2K}, \\ Dh'' - h + \frac{a^r}{h^s} = 0, & 0 < x < \frac{1}{2K}, \\ a(x) \sim \xi^{\frac{q}{p-1}} w(\frac{x}{\epsilon}), & h(0) = \xi, \\ a'(0) = a'(\frac{1}{2K}) = h'(0) = h'(\frac{1}{2K}) = 0. \end{cases} \tag{4.5}$$

For the one boundary spike solution, one can use the Implicit Function Theorem, since the linearized operator is invertible in the space of functions with Neumann boundary conditions. (The last statement follows from the fact that the kernel of the operator $\Delta - 1 + pw^{p-1}$ consists exactly those of partial derivatives of w. See Lemma 3.2.)

4.3. *The bounded domain case: existence of asymmetric K-spikes*

In the bounded domain case, as D is getting smaller, more and more new solutions appear. By using the same matched asymptotic analysis in [34], M. Ward and Wei in [70] discovered that for $D < D_K$, where D_K is given by (4.67) below, problem (4.4) has *asymmetric* K-peaked steady-state solutions. Such asymmetric solutions are generated by two types of peaks-called type **A** and type **B**, respectively. Type **A** and type **B** peaks have *different heights*. They can be arranged in any given order

ABAABBB...ABBBA...B

to form an K-peaked solution. The existence of such solutions is surprising. It shows that the solution structure of (4.4) is much more complicated than one would expect. The stability of such asymmetric K-peaked solutions is also studied in [70], through a formal approach. We remark that asymmetric patterns can also be obtained for the Gierer–Meinhardt system on the real line, see [12].

In this and next section, we present a *rigorous and unified theoretic foundation* for the existence and stability of general K-peaked (*symmetric* or *asymmetric*) solutions. In particular, the results of [34] and [70] are rigorously established. Moreover, we show that if the K peaks are separated, then they are generated by peaks of type **A** and type **B**, respectively. This implies that there are only two kinds of K-peaked patterns: symmetric K-peaked solutions constructed in [67] and asymmetric K-peaked patterns constructed in [70].

The existence proof is based on Lyapunov–Schmidt reduction. Stability is proved by first separating the problem into the case of large eigenvalues which tend to a non-zero limit and the case of small eigenvalues which tend to zero in the limit $\epsilon \to 0$. Large eigenvalues are then explored by studying non-local eigenvalue problems using results in Section 3.3 and employing an idea of Dancer [8]. Small eigenvalues are calculated explicitly by an asymptotic analysis with rigorous error estimates.

In this section, we present the existence part.

Before we state our main results, we introduce some notation. Let $G_D(x,z)$ be the Green function of

$$\begin{cases} DG''_D(x,z) - G_D(x,z) + \delta_z i(x) = 0 \quad \text{in } (-1,1), \\ G'_D(-1,z) = G'_D(1,z) = 0. \end{cases} \tag{4.6}$$

We can calculate explicitly

$$G_D(x,z) = \begin{cases} \frac{\theta}{\sinh(2\theta)} \cosh[\theta(1+x)]\cosh[\theta(1-z)], & -1 < x < z, \\ \frac{\theta}{\sinh(2\theta)} \cosh[\theta(1-x)]\cosh[\theta(1+z)], & z < x < 1 \end{cases} \tag{4.7}$$

where

$$\theta = D^{-1/2}.$$

We set

$$K_D\bigl(|x-z|\bigr) = \frac{1}{2\sqrt{D}} e^{-\frac{1}{\sqrt{D}}|x-z|}, \tag{4.8}$$

to be the singular part of $G_D(x,z)$ and by $G_D = K_D - H_D$ we define the regular part H_D of G_D. Note that H_D is C^∞ in both x and z.

Let $-1 < t_1^0 < \cdots < t_j^0 < \cdots < t_K^0 < 1$ be K points in $(-1,1)$ and w be the unique solution of (2.8).

Put

$$\xi_\epsilon := \left(\epsilon \int_R w^r(z)\,dz\right)^{\frac{p-1}{(p-1)(s+1)-qr}}. \tag{4.9}$$

We introduce several matrices for later use: For $\mathbf{t} = (t_1, \ldots, t_K) \in (-1,1)^K$, let

$$\mathcal{G}_D(\mathbf{t}) = \bigl(G_D(t_i, t_j)\bigr). \tag{4.10}$$

Let us denote $\frac{\partial}{\partial t_i}$ as ∇_{t_i}. When $i \neq j$, we can define $\nabla_{t_i} G(t_i, t_j)$ in the classical way. When $i = j$, $K_D(|t_i - t_j|) = K_D(0) = \frac{1}{2\sqrt{D}}$ is a constant and we define

$$\nabla_{t_i} G_D(t_i, t_i) := -\frac{\partial}{\partial x}\bigg|_{x=t_i} H(x, t_i).$$

Similarly, we define

$$\nabla_{t_i}\nabla_{t_j}G_D(t_i,t_j)=\begin{cases}\frac{\partial}{\partial x}\Big|_{x=t_i}\frac{\partial}{\partial y}\Big|_{y=t_i}H_D(x,y) & \text{if } i=j,\\ \nabla_{t_i}\nabla_{t_j}G_D(t_i,t_j) & \text{if } i\neq j.\end{cases} \tag{4.11}$$

Now the derivatives of $\mathcal{G}$ are defined as follows:

$$\nabla\mathcal{G}_D(\mathbf{t})=\big(\nabla_{t_i}G_D(t_i,t_j)\big), \tag{4.12}$$

$$\nabla^2\mathcal{G}_D(\mathbf{t})=\big(\nabla_{t_i}\nabla_{t_j}G_D(t_i,t_j)\big). \tag{4.13}$$

We now have our first assumption:
(H1) There exists a solution $(\hat{\xi}_1^0,\ldots,\hat{\xi}_N^0)$ of the following equation

$$\sum_{j=1}^{N}G_D\big(t_i^0,t_j^0\big)\big(\hat{\xi}_j^0\big)^{\frac{qr}{p-1}-s}=\hat{\xi}_i^0,\quad i=1,\ldots,N. \tag{4.14}$$

Next we introduce the following matrix

$$b_{ij}=G_D\big(t_i^0,t_j^0\big)\big(\hat{\xi}_j^0\big)^{\frac{qr}{p-1}-s-1},\quad \mathcal{B}=(b_{ij}). \tag{4.15}$$

Our second assumption is the following:
(H2) It holds that

$$\frac{p-1}{qr-s(p-1)}\notin\sigma(\mathcal{B}), \tag{4.16}$$

where $\sigma(\mathcal{B})$ is the set of eigenvalues of $\mathcal{B}$.

REMARK 4.3.1. Since the matrix $\mathcal{B}$ is of the form $\mathcal{G}_D\mathcal{D}$, where $\mathcal{G}_D$ is symmetric and $\mathcal{D}$ is a diagonal matrix, it is easy to see that the eigenvalues of $\mathcal{B}$ are real.

By the assumption (H2) and the implicit function theorem, for $\mathbf{t}=(t_1,\ldots,t_K)$ near $\mathbf{t}_0=(t_1^0,\ldots,t_K^0)$, there exists a unique solution $\hat{\xi}(\mathbf{t})=(\hat{\xi}_1(\mathbf{t}),\ldots,\hat{\xi}_K(\mathbf{t}))$ for the following equation

$$\sum_{j=1}^{K}G_D(t_i,t_j)\hat{\xi}_j^{\frac{qr}{p-1}-s}=\hat{\xi}_i,\quad i=1,\ldots,K. \tag{4.17}$$

Set

$$\mathcal{H}(\mathbf{t})=\big(\hat{\xi}_i(\mathbf{t})\delta_{ij}\big). \tag{4.18}$$

We define the following vector field:

$$F(\mathbf{t}) = \big(F_1(\mathbf{t}), \ldots, F_K(\mathbf{t})\big),$$

where

$$\begin{aligned} F_i(\mathbf{t}) &= \sum_{l=1}^{K} \nabla_{t_i} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s} \\ &= -\nabla_{t_i} H_D(t_i, t_i) \hat{\xi}_i^{\frac{qr}{p-1}-s} + \sum_{l \neq i} \nabla_{t_i} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s}, \\ & \quad i = 1, \ldots, K. \end{aligned} \tag{4.19}$$

Set

$$\mathcal{M}(\mathbf{t}) = \big(\hat{\xi}_i^{-1} \nabla_{t_j} F_i(\mathbf{t})\big). \tag{4.20}$$

Our final assumption concerns the vector field $F(\mathbf{t})$.
(H3) We assume that at $\mathbf{t}_0 = (t_1^0, \ldots, t_K^0)$:

$$F(\mathbf{t}_0) = 0, \tag{4.21}$$

$$\det\big(\mathcal{M}(\mathbf{t}_0)\big) \neq 0. \tag{4.22}$$

Let us now calculate $\mathcal{M}(\mathbf{t}^0)$: Therefore we first compute the derivatives of $\hat{\xi}$. It is easy to see that $\hat{\xi}(\mathbf{t})$ is C^1 in $\mathbf{t}$. We can calculate:

$$\begin{aligned} \nabla_{t_j} \hat{\xi}_i &= \left(\frac{qr}{p-1} - s\right) \sum_{l=1}^{K} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s-1} \nabla_{t_j} \hat{\xi}_l \\ &\quad + \sum_{l=1}^{K} \frac{\partial}{\partial t_j} \big(G_D(t_i, t_l)\big) \hat{\xi}_l^{\frac{qr}{p-1}-s}. \end{aligned}$$

For $i \neq j$, we have

$$\nabla_{t_j} \hat{\xi}_i = \left(\frac{qr}{p-1} - s\right) \sum_{l=1}^{N} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s-1} \nabla_{t_j} \hat{\xi}_l + \nabla_{t_j} G_D(t_i, t_j) \hat{\xi}_j^{\frac{qr}{p-1}-s}.$$

For $i = j$, we have

$$\nabla_{t_j} \hat{\xi}_i = \left(\frac{qr}{p-1} - s\right) \sum_{l=1}^{K} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s-1} \nabla_{t_i} \hat{\xi}_l + \sum_{l=1}^{K} \frac{\partial}{\partial t_i} \big(G_D(t_i, t_l)\big) \hat{\xi}_l^{\frac{qr}{p-1}-s}$$

$$
\begin{aligned}
&= \left(\frac{qr}{p-1} - s\right)\sum_{l=1}^{K} G_D(t_i,t_l)\hat{\xi}_l^{\frac{qr}{p-1}-s-1}\nabla_{t_i}\hat{\xi}_l + \nabla_{t_i}G_D(t_i,t_i)\hat{\xi}_i^{\frac{qr}{p-1}-s} \\
&\quad + \sum_{l=1}^{K}\nabla_{t_i}G_D(t_i,t_l)\hat{\xi}_l^{\frac{qr}{p-1}-s},
\end{aligned}
$$

since $\frac{\partial}{\partial t_i}G_D(t_i,t_i) = 2\nabla_{t_i}G_D(t_i,t_i)$.

Note that

$$
\left(\nabla_{t_j}G_D(t_i,t_j)\right) = (\nabla\mathcal{G}_D)^T.
$$

Therefore if we denote the matrix

$$
\nabla\xi = (\nabla_{t_j}\hat{\xi}_i) \tag{4.23}
$$

then we have

$$
\begin{aligned}
\nabla\xi(\mathbf{t}) &= \left(I - \left(\frac{qr}{p-1} - s\right)\mathcal{G}_D\mathcal{H}^{\frac{qr}{p-1}-s-1}\right)^{-1}(\nabla\mathcal{G}_D)^T\mathcal{H}^{\frac{qr}{p-1}-s} \\
&\quad + O\left(\sum_{j=1}^{K}\left|F_j(\mathbf{t})\right|\right).
\end{aligned} \tag{4.24}
$$

We can compute $\mathcal{M}(\mathbf{t}^0)$ by using (4.24):

$$
\begin{aligned}
\mathcal{M}(\mathbf{t}^0) &= \mathcal{H}^{-1}\nabla^2\mathcal{G}_D\mathcal{H}^{\frac{qr}{p-1}-s} \\
&\quad + \mathcal{H}^{-1}\left(\frac{qr}{p-1} - s\right)\nabla\mathcal{G}_D\mathcal{H}^{\frac{qr}{p-1}-s-1} \\
&\quad \times \left(I - \left(\frac{qr}{p-1} - s\right)\mathcal{G}_D\mathcal{H}^{\frac{qr}{p-1}-s-1}\right)^{-1}(\nabla\mathcal{G}_D)^T\mathcal{H}^{\frac{qr}{p-1}-s}.
\end{aligned} \tag{4.25}
$$

The existence result is as follows

THEOREM 4.3. (See [84].) *Assume that assumptions* (H1), (H2) *and* (H3) *are satisfied. Then for* $\epsilon \ll 1$, *problem* (4.4) *has an* K*-peaked solution which concentrates at* $t_1^\epsilon, \ldots, t_K^\epsilon$, *or more precisely*:

$$
a_\epsilon(x) \sim \sum_{j=1}^{K}\xi_\epsilon^{\frac{q}{p-1}}\left(\hat{\xi}_j^0\right)^{\frac{q}{p-1}}w\left(\frac{x - t_j^\epsilon}{\epsilon}\right), \tag{4.26}
$$

$$
h_\epsilon\left(t_i^\epsilon\right) \sim \xi_\epsilon\hat{\xi}_i^0, \quad i = 1, \ldots, K, \tag{4.27}
$$

$$
t_i^\epsilon \to t_i^0, \quad i = 1, \ldots, K. \tag{4.28}
$$

REMARK 4.3.2. In the case of symmetric K-peaked solutions, conditions (H2) and (H3) are not needed, as in the construction of solutions one can restrict the function space to the class of symmetric functions (see for example [67]). Note that for small ϵ (and not only in the limit $\epsilon \to 0$) the peaks are placed equidistantly.

REMARK 4.3.3. Our results here can be applied to give a *rigorous proof* for the existence and stability of K-peaked solutions consisting of peaks with *different heights*.

In [70], by using matched asymptotic analysis, Ward and the first author constructed such solutions and studied their stability. We now summarize their main ideas. First (4.4) is solved in a small interval $(-l,l)$:

$$\begin{cases} \epsilon^2 a'' - a + \frac{a^p}{h^q} = 0 & \text{in } (-l,l),\\ Dh'' - h + \frac{a^r}{h^s} = 0 & \text{in } (-l,l),\\ a(x) > 0, h(x) > 0 & \text{in } (-l,l),\\ a'(-l) = a'(l) = h'(-l) = h'(l) = 0. \end{cases} \tag{4.29}$$

Then the single interior symmetric spike solution is considered which was constructed by I. Takagi [67]. By some simple computations based on (4.6), we have that

$$h(l) \sim c(D) b\left(\frac{l}{\sqrt{D}}\right), \tag{4.30}$$

where $c(D)$ is some positive constant depending on D only and the function $b(z)$ is given by

$$b(z) := \frac{\tanh^{\alpha}(z)}{\cosh(z)}, \quad \alpha := \frac{(p-1)}{qr-(s+1)(p-1)}. \tag{4.31}$$

The idea now is that we fix l and try to find another $\bar{l} \neq l$ such that the following holds

$$b\left(\frac{l}{\sqrt{D}}\right) = b\left(\frac{\bar{l}}{\sqrt{D}}\right), \quad 0 < l < \bar{l} < 1, \tag{4.32}$$

which will imply that $h(l) = h(\bar{l})$. This shows that if there exists a solution to (4.32), we may match up $h(l)$ and $h(\bar{l})$. In other words, we may match up solutions of (4.29) in different intervals.

It turns out that for D small, (4.32) is always solvable. Now (4.32) has to be solved along with the following interval constraint:

$$K_1 l + K_2 \bar{l} = 1, \qquad K_1 + K_2 = K. \tag{4.33}$$

For a solution l of (4.60) and (4.33) and $j = 1, \dots, K$ we define

$$l_j = l \quad \text{or} \quad l_j = \bar{l} \tag{4.34}$$

where the number of j's such that $l_j = l$ is K_1 (and consequently the number of j's such that $l_j = \bar{l}$ is K_2). We call the small spike with $l_j = l$ type **A** and the large spike with $l_j = \bar{l}$ type **B**.

Then we choose t_j^0 such that

$$\left|t_j^0 - t_{j+1}^0\right| = l_j + l_{j-1}, \quad j = 0, \ldots, K,$$

where $t_0^0 = -1$, $t_{K+1}^0 = 1$.

By using matched asymptotics, we now have K_1 type **A** and K_2 type **B** peaks. This ends the short review of the ideas in [70]. Let us now use Theorem 4.3 to give a rigorous proof of results of [70]. In order to apply Theorem 4.3, we have to check the three assumptions (H1), (H2) and (H3).

To this end, let us set

$$\hat{\xi}_j^0 = (2\sqrt{D}) \tanh(\theta_j), \quad j = 1, \ldots, K, \tag{4.35}$$

where

$$\theta_j = \frac{l_j}{\sqrt{D}}. \tag{4.36}$$

It is difficult to check (H1) directly. Instead we note that $\mathcal{G}_D^{-1}$ is a tridiagonal matrix. (See [34] and [70].) More precisely, we calculate

$$\mathcal{G}_D^{-1} = (a_{ij}) = 2\sqrt{D} \begin{pmatrix} \gamma_1 & \beta_1 & 0 & \ddots & \ddots & 0 \\ \beta_1 & \gamma_2 & \beta_2 & \ddots & \ddots & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \beta_{j-1} & \gamma_j & \beta_j & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & \beta_{N-1} & \gamma_N \end{pmatrix}$$

where

$$\begin{aligned} \gamma_1 &= \coth(\theta_1 + \theta_2) + \tanh(\theta_1), \\ \gamma_j &= \coth(\theta_{j-1} + \theta_j) + \coth(\theta_j + \theta_{j+1}), \quad j = 2, \ldots, K-1, \\ \gamma_K &= \coth(\theta_{K-1} + \theta_K) + \tanh(\theta_K), \\ \beta_j &= -\operatorname{csch}(\theta_j + \theta_{j+1}), \quad j = 1, \ldots, N-1 \end{aligned}$$

and θ_j was defined in (4.36). Note that

$$a_{ij} = 2\sqrt{D}(\beta_j \delta_{i(j-1)} + \gamma_j \delta_{ij} + \beta_{j+1} \delta_{i(j+1)}). \tag{4.37}$$

Verifying (4.14) amounts to checking the following identity

$$\sum_{j=1}^{N} a_{ij}\hat{\xi}_j^0 = \left(\hat{\xi}_i^0\right)^{\frac{qr}{p-1}-s}, \tag{4.38}$$

which is an easy exercise.

It remains to verify (H2) and (H3).

To this end, we need to know the eigenvalues of $\mathcal{B}$ and $\mathcal{M}$. In the same way as for the matrix $\mathcal{G}_D$, one can show that $\mathcal{B}^{-1}$ is a tridiagonal matrix. However, it is almost impossible to obtain an explicit formula for the eigenvalues. Numerical software for solving eigenvalue problems of large matrices is indispensable. Then (H2) has to be checked explicitly. Numerical computations in [70] do suggest that assumption (H3) is always satisfied.

4.4. *Classification of asymmetric patterns*

A natural question is the following: Are all K-peaked solution generated by two types of peaks as the solutions which were constructed in [70]?

Our next theorem gives an affirmative answer. It completely classifies all K-peaked solutions, provided that the K peaks are separated.

THEOREM 4.4. (See [84].) *Suppose that for ϵ sufficiently small, there are solutions (a_ϵ, h_ϵ) of* (4.4) *such that*

$$a_\epsilon(x) \sim \sum_{j=1}^{K} \xi_\epsilon^{\frac{q}{p-1}} \left(\hat{\xi}_j^\epsilon\right)^{\frac{q}{p-1}} w\left(\frac{x-t_j^\epsilon}{\epsilon}\right), \tag{4.39}$$

and

$$h_\epsilon\left(t_i^\epsilon\right) \sim \xi_\epsilon \hat{\xi}_i^\epsilon, \quad i = 1, \dots, K, \tag{4.40}$$

where ξ_ϵ is given by (4.9),

$$\hat{\xi}_i^\epsilon \to \hat{\xi}_i^0 > 0, \quad t_i^\epsilon \to t_i^0, \quad t_i^0 \neq t_j^0, \quad i \neq j,\ i, j = 1, \dots, K. \tag{4.41}$$

Then necessarily, we have

$$l_i := t_i^0 - t_{i-1}^0 \in \{l, \bar{l}\}, \quad i = 1, \dots, K, \tag{4.42}$$

where $t_0^0 = -1$, l and $\bar{l}$ satisfy (4.32) *and* (4.33) *with K_1 being the number of i's for which $l_i = l$ and K_2 being the number of i's for which $l_i = \bar{l}$ (hence $K_1 + K_2 = K$).*

Theorem 4.4 shows that an K-peaked solution must be generated by exactly two types of peaks—type **A** with shorter length l and type **B** with larger length $\bar{l}$. This shows that the solutions constructed in [70] (through a formal approach) exhaust all possible separated K-peaked solutions. In particular, it shows that there are at most 2^K K-peaked solutions. If the assumptions (H1)–(H3) are met, then there are exactly 2^K K-peaked solutions.

PROOF OF THEOREM 4.4. First we make the following scaling

$$a_\epsilon = \xi_\epsilon^{\frac{q}{p-1}} \hat{a}_\epsilon, \quad h_\epsilon = \xi_\epsilon \hat{h}_\epsilon$$

where ξ_ϵ is defined at (4.9). Hence $(\hat{a}_\epsilon, \hat{h}_\epsilon)$ satisfies

$$\begin{cases} \epsilon^2 \Delta \hat{a}_\epsilon - \hat{a}_\epsilon + \frac{\hat{a}_\epsilon^p}{\hat{h}_\epsilon^q} = 0, & -1 < x < 1, \\ D \Delta \hat{h}_\epsilon - \hat{h}_\epsilon + c_\epsilon \frac{\hat{a}_\epsilon^r}{\hat{h}_\epsilon^s} = 0, & -1 < x < 1, \end{cases} \tag{4.43}$$

where c_ϵ is defined as $c_\epsilon = (\epsilon \int_R w^r)^{-1}$.

Now (4.39) and (4.40) imply that

$$\hat{a}_\epsilon \sim \sum_{j=1}^K (\hat{\xi}_j^\epsilon)^{\frac{q}{p-1}} w\left(\frac{x - t_j^\epsilon}{\epsilon}\right), \quad \hat{h}_\epsilon(t_j^\epsilon) = \hat{\xi}_j^\epsilon. \tag{4.44}$$

Letting $\epsilon \to 0$, we assume that

$$\hat{\xi}_j^\epsilon \to \hat{\xi}_j^0, \quad t_j^\epsilon \to t_j^0, \quad j = 1, \ldots, K.$$

We see that $\hat{h}_\epsilon \to h_0(x)$ where $h_0(x)$ satisfies

$$\begin{cases} D \Delta h_0 - h_0 + \sum_{j=1}^K (\hat{\xi}_j^0)^{\frac{qr}{p-1} - s} \delta(x - t_j^0) = 0, & -1 < x < 1, \\ h_0'(-1) = h_0'(1) = 0. \end{cases} \tag{4.45}$$

In other words, we have

$$h_0(x) = \sum_{j=1}^K (\hat{\xi}_j^0)^{\frac{qr}{p-1} - s} G_D(x, t_j^0). \tag{4.46}$$

Since $h_0(t_j^0) = \hat{\xi}_j^0$, $j = 1, \ldots, K$, we have from (4.46) that $(\hat{\xi}_1^0, \ldots, \hat{\xi}_K^0)$ must satisfy the following identity:

$$\sum_{j=1}^K G_D(t_i^0, t_j^0)(\hat{\xi}_j^0)^{\frac{qr}{p-1} - s} = \hat{\xi}_i^0, \quad i = 1, \ldots, K. \tag{4.47}$$

This is the same as (4.14).

Define

$$\tilde{a}_{\epsilon,j} = \hat{a}_\epsilon \chi\left(\frac{x - t_j^0}{\tilde{r}_0}\right)$$

where $\tilde{r}_0$ is a very small number. Then $\tilde{a}_{\epsilon,j}$ is supported in the interval $I_j^\epsilon = (-\tilde{r}_0 + t_j^\epsilon, \tilde{r}_0 + t_j^\epsilon)$. We may choose $\tilde{r}_0$ so small that $I_i^\epsilon \cap I_j^\epsilon = \emptyset$ for $i \neq j$. Then

$$\hat{a}_\epsilon = \sum_{j=1}^{K} \tilde{a}_{\epsilon,j} + \text{e.s.t.}$$

Now we multiply the first equation in (4.43) by $\tilde{a}'_{\epsilon,j}$ and integrate over $(-1, 1)$. We obtain

$$\begin{aligned}
0 &= \int_{-1}^{1} \left[\left(\frac{\hat{a}_\epsilon^{\,p}}{\hat{h}_\epsilon^{\,q}} \right) \tilde{a}'_{\epsilon,j} - \left(\frac{\hat{a}_\epsilon^p}{\hat{h}_\epsilon^q} \right)' \tilde{a}_{\epsilon,j} \right] \\
&= -2 \int_{I_j^\epsilon} \left(\frac{\hat{a}_\epsilon^p}{\hat{h}_\epsilon^q} \right)' \hat{a}_\epsilon + \text{e.s.t.} \\
&= -2 \int_{I_j^\epsilon} \left[\frac{p \hat{a}_\epsilon^p \hat{a}'_\epsilon}{\hat{h}_\epsilon^q} - \frac{q \hat{a}_\epsilon^{p+1} \hat{h}'_\epsilon}{\hat{h}_\epsilon^{q+1}} \right] + \text{e.s.t.} \\
&= \frac{q(p+2)}{p+1} \int_{I_j^\epsilon} \frac{\hat{a}_\epsilon^{p+1}}{\hat{h}_\epsilon^{q+1}} \hat{h}'_\epsilon + \text{e.s.t.}
\end{aligned} \tag{4.48}$$

By the equation for $\hat{h}_\epsilon$, we have that

$$\hat{h}_\epsilon(x) = c_\epsilon \int_{-1}^{1} G_D(x, z) \frac{\hat{a}_\epsilon^r}{\hat{h}_\epsilon^s}$$

and thus for $x \in I_j^\epsilon$,

$$\hat{h}_\epsilon(x) = \sum_{k=1}^{K} G_D\big(x, t_k^\epsilon\big) \big(\hat{\xi}_k^\epsilon\big)^{\frac{qr}{p-1} - s} + O(\epsilon)$$

and

$$\hat{H}'_\epsilon\big(t_j^\epsilon\big) = \sum_{k=1}^{K} \nabla_{t_j^\epsilon} G_D\big(t_j^\epsilon, t_k^\epsilon\big) \big(\hat{\xi}_k^\epsilon\big)^{\frac{qr}{p-1} - s} + O(\epsilon). \tag{4.49}$$

Substituting (4.49) into (4.48) and using (4.44), we obtain the following identity

$$\sum_{k=1}^{K} \nabla_{t_j^\epsilon} G_D\bigl(t_j^\epsilon, t_k^\epsilon\bigr)\bigl(\hat{\xi}_k^\epsilon\bigr)^{\frac{qr}{p-1}-s} = o(1) \tag{4.50}$$

and hence

$$\sum_{k=1}^{K} \nabla_{t_j^0} G_D\bigl(t_j^0, t_k^0\bigr)\bigl(\hat{\xi}_k^0\bigr)^{\frac{qr}{p-1}-s} = 0, \quad j = 1, \ldots, K, \tag{4.51}$$

which is the same as (4.21).

Note that by the expression for h_0 in (4.46), (4.51) is equivalent to the following

$$h_0'\bigl(t_j^0+\bigr) + h_0'\bigl(t_j^0-\bigr) = 0, \quad j = 1, \ldots, K, \tag{4.52}$$

where $h_0'(t_j^0+)$ is the right-hand derivative of h_0 at t_j^0 and $h_0'(t_j^0-)$ is the left-hand derivative of h_0 at t_j^0. On the other hand, from the equation for h_0, we have that

$$D\bigl(h_0'\bigl(t_j^0+\bigr) - h_0'\bigl(t_j^0-\bigr)\bigr) = -\bigl(\hat{\xi}_j^0\bigr)^{\frac{qr}{p-1}-s}, \quad j = 1, \ldots, K. \tag{4.53}$$

Solving (4.52) and (4.53), we have that

$$h_0'\bigl(t_j^0+\bigr) = -h_0'\bigl(t_j^0-\bigr) = -\frac{1}{2D}\bigl(\hat{\xi}_j^0\bigr)^{\frac{qr}{p-1}-s} < 0, \quad j = 1, \ldots, K. \tag{4.54}$$

Since h_0 satisfies $Dh_0'' = h_0 > 0$ in each interval (t_{j-1}^0, t_j^0), $j = 2, \ldots, K$, we see that there exists a unique point $s_{j-1} \in (t_{j-1}^0, t_j^0)$ such that $h_0'(s_{j-1}) = 0$. Since $h_0'(-1) = 0$, by using symmetry, we see that

$$\frac{s_{j-1} + s_j}{2} = t_j^0, \quad j = 1, \ldots, K, \tag{4.55}$$

where we take $s_0 = -1$, $s_K = 1$. Let $2l_j = s_j - s_{j-1}$, $j = 1, \ldots, K$.

Note that on each interval $(-l_j + t_j^0, l_j + t_j^0)$, h_0 satisfies

$$D\Delta h_0 - h_0 + \bigl(\hat{\xi}_j^0\bigr)^{\frac{qr}{p-1}-s}\delta\bigl(t - t_j^0\bigr) = 0$$

with Neumann boundary conditions at both ends. Thus from (4.6) it is easy to see that

$$\bigl(\hat{\xi}_j^0\bigr)^{\frac{qr}{p-1}-s-1} = 2\sqrt{D}\tanh\biggl(\frac{l_j}{\sqrt{D}}\biggr), \quad j = 1, \ldots, K, \tag{4.56}$$

$$h_0(l_j) = \frac{\hat{\xi}_j^0}{\cosh(\frac{l_j}{\sqrt{D}})}. \tag{4.57}$$

Since h_0 is continuous on $(-1, 1)$, we have

$$h_0(l_1) = h_0(l_2) = \cdots = h_0(l_K). \tag{4.58}$$

Using (4.56) and (4.57), we see that (4.58) is equivalent to

$$b\left(\frac{l_1}{\sqrt{D}}\right) = b\left(\frac{l_2}{\sqrt{D}}\right) = \cdots = b\left(\frac{l_K}{\sqrt{D}}\right), \tag{4.59}$$

where the function b was defined in (4.31). Suppose without loss of generality that $l_1 \leqslant l_2$, then we take $l_1 = l$ and (4.59) implies that $l_2 \in \{l, \bar{l}\}$ and that $l_j \in \{l, \bar{l}\}$ for $j = 2, \ldots, K$. Thus l must satisfy (4.60) and (4.33).

This finishes the proof of Theorem 4.4. ☐

4.5. *The stability of symmetric and asymmetric K-spikes*

In this section, we present the stability of the K-peaked solutions constructed in Theorem 4.3.

To this end, we need to study the following linearized eigenvalue problem

$$\mathcal{L}_\epsilon \begin{pmatrix} \phi_\epsilon \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \epsilon^2 \Delta\phi_\epsilon - \phi_\epsilon + p\frac{a_\epsilon^{p-1}}{H_\epsilon^q}\phi_\epsilon - q\frac{a_\epsilon^p}{h_\epsilon^{q+1}}\psi_\epsilon, \\ \frac{1}{\tau}\left(D\Delta\psi_\epsilon - \psi_\epsilon + r\frac{a_\epsilon^{r-1}}{h_\epsilon^s}\phi_\epsilon - s\frac{a_\epsilon^r}{h_\epsilon^{s+1}}\psi_\epsilon\right) \end{pmatrix} = \lambda_\epsilon \begin{pmatrix} \phi_\epsilon \\ \psi_\epsilon \end{pmatrix}, \tag{4.60}$$

where (a_ϵ, h_ϵ) is the solution constructed in Theorem 4.3 and $\lambda_\epsilon \in \mathcal{C}$—the set of complex numbers.

We say that (a_ϵ, h_ϵ) is *linearly stable* if the spectrum $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ lies in the left half plane $\{\lambda \in \mathcal{C}\colon \operatorname{Re}(\lambda) < 0\}$. (a_ϵ, h_ϵ) is called *linearly unstable* if there exists an eigenvalue λ_ϵ of $\mathcal{L}_\epsilon$ with $\operatorname{Re}(\lambda_\epsilon) > 0$. (From now on, we use the notations linearly stable and linearly unstable as defined above.)

THEOREM 4.5. *Let (a_ϵ, h_ϵ) be the solutions constructed in Theorem* 4.3. *Assume that* $\epsilon \ll 1$.

(1) (*Stability*) *If*

$$r = 2, \ < p < 5 \quad \text{or} \quad r = p + 1, \ < p < +\infty \tag{4.61}$$

and furthermore

$$\left(\frac{qr}{p-1} - s\right) \min_{\sigma \in \sigma(\mathcal{B})} \sigma > 1 \tag{4.62}$$

and

$$\sigma(\mathcal{M}) \subseteq \{\sigma \mid \operatorname{Re}(\sigma) > 0\}, \tag{4.63}$$

there exists $\tau_0 > 0$ *such that* (a_ϵ, h_ϵ) *is linearly stable for* $0 \leqslant \tau < \tau_0$.

(2) (*Instability*) *If*

$$\left(\frac{qr}{p-1} - s\right) \min_{\sigma \in \sigma(\mathcal{B})} \sigma < 1, \tag{4.64}$$

there exists $\tau_0 > 0$ *such that* (a_ϵ, h_ϵ) *is linearly unstable for* $0 \geqslant \tau < \tau_0$.

(3) (*Instability*) *If there exists*

$$\sigma \in \sigma(\mathcal{M}), \quad \mathrm{Re}(\sigma) < 0, \tag{4.65}$$

then (a_ϵ, h_ϵ) *is linearly unstable for all* $\tau > 0$.

REMARK 4.5.1. In the original Gierer–Meinhardt model, $(p, q, r, s) = (2, 1, 2, 0)$ or $(p, q, r, s) = (2, 4, 2, 0)$. This means that condition (4.61) is satisfied.

REMARK 4.5.2. By Theorems 4.3 and 4.5, the existence and stability of K-peaked solutions are completely determined by the two matrices $\mathcal{B}$ and $\mathcal{M}$. They are related to the asymptotic behavior of large eigenvalues which tend to a non-zero limit and small eigenvalues which tend to zero as $\epsilon \to 0$, respectively. The computations of these two matrices are by no means easy. We refer to [34] and [70] for exact computations and numerics. Combining the results here and the computations in [34], the stability of symmetric K-peaked solutions is completely characterized and the following result is established rigorously.

THEOREM 4.6. (See [34,84].) *Let* $(a_{\epsilon,K}, h_{\epsilon,K})$ *be the symmetric* K*-peaked solutions constructed in* [67]. *Assume that* $\epsilon \ll 1$.

(a) (*Stability*) *Assume that* $0 < \tau < \tau_0$ *for some* τ_0 *small and that*

$$r = 2,\ 1 < p < 5 \quad \textit{or} \quad r = p + 1,\ 1 < p < +\infty \tag{4.66}$$

and

$$D < D_K := \frac{1}{K^2 (\log(\sqrt{\alpha} + \sqrt{\alpha + 1}))^2}, \tag{4.67}$$

where α *is given by* (4.31), *then the symmetric* K*-peaked solution is linearly stable.*

(b) (*Instability*) *If*

$$D > D_K, \tag{4.68}$$

where D_K *is given by* (4.67), *then the symmetric* K*-peaked solution is linearly unstable for all* $\tau > 0$.

The proof of Theorem 4.5 consists of two parts: we have to compute both *small* and *large* eigenvalues. For large eigenvalues, we will arrive at the following system of nonlocal eigenvalue problems (NLEPs)

$$\begin{aligned}&\Phi'' - \Phi + pw^{p-1}\Phi \\ &\qquad - qr(I+s\mathcal{B})^{-1}\mathcal{B}\left(\int_{\mathbb{R}} w^{r-1}\Phi\right)\left(\int_{\mathbb{R}} w^{r}\right)^{-1} w^{p} = \lambda\Phi \end{aligned} \tag{4.69}$$

where $\mathcal{B}$ is given by (4.15) and

$$\Phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \vdots \\ \phi_K\end{pmatrix} \in \left(H^2(\mathbb{R})\right)^K.$$

By diagonalization, we may reduce it to K NLEPs of the form (3.17). Using the results of Theorem 3.7, we obtain the stability (or instability) of large eigenvalues.

For the study of small eigenvalues, we need to expand the eigenfunction up to the order $O(\epsilon^2)$ term. This computation is quite involved. In the end, the matrix $\mathcal{B}$ and $\mathcal{M}$ will appear.

A similar stability analysis for the Schnakenberg model has been carried out in [35].

5. The full Gierer–Meinhardt system: Two-dimensional case

Let us now consider the Gierer–Meinhardt system in a two-dimensional domain. The results are more complicated. To reduce the complexity and grasp the essential difficulties, we assume that $(p,q,r,s)=(2,1,2,0)$ in this section.

We start with the bound states.

5.1. *Bound states: spikes on polygons*

We first consider the case when $\Omega = \mathbb{R}^2$:

$$\begin{cases}\Delta a - a + \frac{a^2}{h} = 0, \quad a>0 & \text{in } \mathbb{R}^2,\\ \Delta h - \sigma^2 h + a^2 = 0, \quad h>0 & \text{in } \mathbb{R}^2,\\ a(x), h(x) \to 0 & \text{as } |x| \to +\infty.\end{cases} \tag{5.1}$$

As we will see, a notable feature of this ground-state problem in the plane is the presence of solutions with any prescribed number of bumps in the activator as the parameter σ gets smaller. These bumps are separated from each other at a distance $O(|\log\log\sigma|)$ and approach a single universal profile given by the unique radial solution of (2.8). The multi-bump solutions correspond respectively to bumps arranged at the vertices of a k-regular polygon and at those of two concentric regular polygons. These arrangements with one extra bump at the origin are also considered. This unveils a new side of the rich and complex structure of the solution set of the Gierer–Meinhardt system in the plane and gives rise to a number of questions.

Let us set

$$\tau_\sigma = \left(\frac{k}{2\pi}\log\frac{1}{\sigma}\int_{\mathbb{R}^2} w^2(y)\,dy\right)^{-1}. \tag{5.2}$$

THEOREM 5.1. (See [17].) *Let $k \geqslant 1$ be a fixed positive integer. There exists $\sigma_k > 0$ such that, for each $0 < \sigma < \sigma_k$, problem* (5.1) *admits a solution (a, h) with the following property*:

$$\lim_{\sigma\to 0}\left|\tau_\sigma a_\sigma(x) - \sum_{i=1}^{k} w(x-\xi_i)\right| = 0, \tag{5.3}$$

uniformly in $x \in \mathbb{R}^2$. Here the points ξ_i correspond to the vertices of a regular polygon centered at the origin, with sides of equal length l_σ satisfying

$$l_\sigma = \log\log\frac{1}{\sigma} + O\left(\log\log\log\frac{1}{\sigma}\right). \tag{5.4}$$

Finally, for each $1 \leqslant j \leqslant k$ we have

$$\lim_{\sigma\to 0}\left|\tau_\sigma h_\sigma(\xi_j + y) - 1\right| = 0,$$

uniformly on compact sets in y.

Our second result gives existence of a solution with bumps at vertices of two concentric polygons.

THEOREM 5.2. (See [17].) *Let $k \geqslant 1$ be a fixed positive integer. There exists $\sigma_k > 0$ such that, for each $0 < \sigma < \sigma_k$, problem* (5.1) *admits a solution (a, h) with the following property*:

$$\lim_{\sigma\to 0}\left|\tau_\sigma a_\sigma(x) - \sum_{i=1}^{k}\left[w(x-\xi_i) + w(x-\xi_i^*)\right]\right| = 0, \tag{5.5}$$

uniformly in $x \in \mathbb{R}^2$. Here the points ξ_i and ξ_i^* are the vertices of two concentric regular polygons. They satisfy

$$\xi_j = \rho_\sigma e^{\frac{2j\pi}{k}i}, \quad \xi_j^* = \rho_\sigma^* e^{\frac{2\pi j}{k}i}, \quad j = 1, \ldots, k,$$

where

$$\rho_\sigma = \frac{1}{|1-e^{\frac{2\pi i}{k}}|}\log\log\frac{1}{\sigma} + O\left(\log\log\log\frac{1}{\sigma}\right),$$

and

$$\rho_\sigma^* = \left(1+\frac{1}{|1-e^{\frac{2\pi i}{k}}|}\right)\log\log\frac{1}{\sigma} + O\left(\log\log\log\frac{1}{\sigma}\right).$$

A similar assertion to (5.4) *holds for* h_σ, *around each of the* ξ_i *and the* ξ_i^**'s.*

THEOREM 5.3. (See [17].) *Let* $k \geqslant 1$ *be given. Then there exists solutions which are exactly as those in Theorems* 5.1 *and* 5.2 *but with an additional bump at the origin. More precisely, with* $w(x)$ *added to* $\sum_{i=1}^k w(x-\xi_i)$ *in* (5.3) *and added to* $\sum_{i=1}^k [w(x-\xi_i) + w(x-\xi_i^*)]$ *in* (5.5).

The method employed in the proof of the above results consists of a Lyapunov–Schmidt type reduction. The basic idea of solving the second equation in (5.1) for h first and then working with a non-local elliptic PDE rather than directly with the system. Let $T(a^2)$ be the unique solution of the equation

$$\begin{aligned} &\Delta h - \sigma^2 h + a^2 = 0 && \text{in } \mathbb{R}^2, \\ &h(x) \to 0 && \text{as } |x| \to +\infty, \end{aligned} \tag{5.6}$$

for $a^2 \in L^2(\mathbb{R}^2)$. Equation (5.3) can be solved via sub-super-solution method. Solving the second equation for h in (5.1) we get $h = T(a^2)$, which leads to the non-local PDE for a

$$\Delta a - a + \frac{a^2}{T(a^2)} = 0. \tag{5.7}$$

Fixing m points which satisfy the constraints

$$\frac{2}{3}\log\log\frac{1}{\sigma} \leqslant |\xi_j - \xi_i| \leqslant 2\log\log\frac{1}{\sigma}, \quad \text{for all } i \neq j.$$

We look for solutions to (5.7) of the form

$$a(x) = \frac{1}{\tau_\sigma}(W+\phi), \quad \text{where } W = \sum_{j=1}^K w(x-\xi_j). \tag{5.8}$$

By using finite-dimensional reduction method, we first solve an auxiliary problem

$$\begin{cases} \Delta(W+\phi) - (W+\phi) + \frac{(W+\phi)^2}{T(\frac{1}{\tau_\sigma}(W+\phi)^2)} = \sum_{i,\alpha} c_{i\alpha}\frac{\partial W}{\partial \xi_{i,\alpha}} \\ \int_{\mathbb{R}^2} \phi \frac{\partial W}{\partial \xi_{i,\alpha}} = 0, \quad i = 1,\ldots,m, \ \alpha = 1,2. \end{cases} \tag{5.9}$$

Solutions satisfying the required conditions in Theorems 5.1–5.3 will be precisely those satisfying a non-linear system of equations of the form

$$c_{i\alpha}(\xi_1, \xi_2, \dots, \xi_m) = 0, \quad i = 1, \dots, m, \ \alpha = 1, 2,$$

where for such a class of points the functions $c_{i\alpha}$ satisfy

$$c_{i\alpha}(\xi_1, \dots, \xi_k) = \frac{\partial}{\partial \xi_{i\alpha}}\left[\sum_{i \neq j} F\big(|\xi_j - \xi_i|\big)\right] + \epsilon_{i\alpha}, \tag{5.10}$$

function $F : \mathbb{R}_+ \to \mathbb{R}$ is of the form

$$F(r) = \frac{c_7 \log r}{\log \frac{1}{\sigma}} + c_8 w(r),$$

c_7 and c_8 are universal constants and

$$\epsilon_{i\alpha} = O\left(\frac{1}{(\log \frac{1}{\sigma})^{1+\gamma}}\right),$$

for some $\gamma > 0$. Although (5.10) does not have a variational structure, solutions of the problem $c_{i\alpha} = 0$ are close to critical points of the functional $\sum_{i \neq j} F(|\xi_j - \xi_i|)$. In spite of the simple form of this functional, its critical points are highly degenerate because of the invariance under rotations and translations of the problem. Thus, to get solutions using degree theoretical arguments, we need to restrict ourselves to classes of points enjoying symmetry constraints. This is how Theorems 5.1–5.3 are established. On the other hand, we believe strongly that finer analysis may yield existence of more complex patterns, such as honey-comb patterns, or lattice patterns.

REMARK 5.1.1. Similar method can also be used to prove Theorem 4.1. In that case, we have

$$c_i(\xi_1, \dots, \xi_k) = \frac{\partial}{\partial \xi_i}\left[\sum_{i \neq j} F_1\big(|\xi_j - \xi_i|\big)\right] + O\big(\sigma^{1+\gamma}\big), \tag{5.11}$$

function $F_1 : \mathbb{R}_+ \to \mathbb{R}$ is of the form

$$F_1(r) = c_9 \sigma r + c_{10} w(r),$$

c_9 and c_{10} are universal constants. It is easy to see that the critical points of $\sum_{i \neq j} F_1(|\xi_i - \xi_j|)$ is *non-degenerate* (in the class of points with $\sum_{j=1}^{K} \xi_j = 0$).

5.2. *Existence of symmetric K-spots*

We look for solutions to the stationary GM on a two-dimensional domain with the following form

$$a_\epsilon(x) \sim \sum_{j=1}^{K} \xi_{\epsilon,j} w\left(\frac{x-P_j}{\epsilon}\right) \tag{5.12}$$

where P_j are the locations of the K-spikes and $\xi_{\epsilon,j}$ is the height of the spike at P_j.

If all the heights are *asymptotically equal*, i.e.

$$\lim_{\epsilon\to 0} \frac{\xi_{\epsilon,i}}{\xi_{\epsilon,j}} = 1, \quad \text{for } i \neq j, \tag{5.13}$$

such solutions are called symmetric K-spots. Otherwise, they are called asymmetric K-spots.

In this section, we discuss the existence of symmetric K-spots. It turns out in two-dimensional case, we have to discuss two cases: the strong coupling case, $D \sim O(1)$, and the weak coupling case, $D \gg 1$.

We first have the following existence result in the strong coupling case

THEOREM 5.4. (See [86].) *Let $\Omega \subset R^2$ be a bounded smooth domain and D be a fixed positive constant. Let $G_D(x,y)$ be the Green function of $-D\Delta + 1$ in Ω (with Neumann boundary condition). Let $H_D(x,y)$ be the regular part of $G_D(x,y)$ and set $h_D(P) = H_D(P,P)$.*

Set

$$F_D(P_1,\dots,P_K) = \sum_{i=1}^{K} H_D(P_i,P_i) - \sum_{j\neq l} G_D(P_j,P_l).$$

Assume that $(P_1,\dots,P_K) \in \Omega^K$ is a non-degenerate critical point of $F_D(P_1,\dots,P_K)$. Then for ϵ sufficiently small, problem (GM) *has a steady state solution (a_ϵ,h_ϵ) with the following properties*:

1. $a_\epsilon(x) = \xi_\epsilon(\sum_{j=1}^{K} w(\frac{x-P_j^\epsilon}{\epsilon}) + o(1))$ *uniformly for $x \in \bar\Omega$, $P_j^\epsilon \to P_j^0$, $j = 1,\dots,K$, as $\epsilon \to 0$, and w is the unique solution of the problem* (2.8).
2. $h_\epsilon(x) = \xi_\epsilon(1 + O(\frac{1}{|\log\epsilon|}))$ *uniformly for $x \in \bar\Omega$, where*
3. $\xi_\epsilon^{-1} = (\frac{1}{2\pi} + o(1))\epsilon^2 \log\frac{1}{\epsilon} \int_{\mathbb{R}^2} w^2$.

REMARK 5.2.1. Theorem 5.4 shows that interior peaks solutions are related to the Green function (contrast to shadow system case). Thus in the strong coupling case, the peaks are produced by a different mechanism. It seems that the equation for h controls everything.

REMARK 5.2.2. In a general domain, the function $F_D(\mathbf{P})$ always has a global maximum point $\mathbf{P}_0$ in $\Omega \times \cdots \times \Omega$. (A proof of this fact can be found in the Appendix of [86].)

The proof of Theorem 5.4 depends on fine estimates in the finite-dimensional reduction: the major problem is to sum up the errors of powers in terms of $\frac{1}{\log\frac{1}{\epsilon}}$.

Next, we discuss the *weak coupling* case. We assume that $\lim_{\epsilon\to 0} D = +\infty$. We first introduce a Green function G_0 which we need to formulate our main results.

Let $G_0(x,\xi)$ be the Green function given by

$$\begin{cases} \Delta G_0(x,\xi) - \frac{1}{|\Omega|} + \delta_\xi(x) = 0 & \text{in } \Omega, \\ \int_\Omega G_0(x,\xi)\,dx = 0, \\ \frac{\partial G_0(x,\xi)}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{5.14}$$

and let

$$H_0(x,\xi) = \frac{1}{2\pi}\log\frac{1}{|x-\xi|} - G_0(x,\xi) \tag{5.15}$$

be the regular part of $G_0(x,\xi)$.

Denote $\mathbf{P}\in\Omega^K$, where $\mathbf{P}$ is arranged such that

$$\mathbf{P} = (P_1, P_2, \dots, P_K)$$

with

$$P_i = (P_{i,1}, P_{i,2}) \quad \text{for } i = 1, 2, \dots, K.$$

For $\mathbf{P}\in\Omega^K$ we define

$$F_0(\mathbf{P}) = \sum_{k=1}^K H_0(P_k, P_k) - \sum_{i,j=1,\dots,K,\ i\neq j} G_0(P_i, P_j) \tag{5.16}$$

and

$$M_0(\mathbf{P}) = \bigl(\nabla^2_{\mathbf{P}} F_0(\mathbf{P})\bigr). \tag{5.17}$$

Here $M_0(\mathbf{P})$ is a $(2K)\times(2K)$ matrix with components $\frac{\partial^2 F(\mathbf{P})}{\partial P_{i,j}\partial P_{k,l}}$, $i,k = 1,\dots,K$, $j,l = 1,2$ (recall that $P_{i,j}$ is the jth component of P_i).

Set

$$D = \frac{1}{\beta^2}, \qquad \eta_\epsilon := \frac{\beta^2|\Omega|}{2\pi}\log\frac{1}{\epsilon}. \tag{5.18}$$

Then $D\to+\infty$ is equivalent to $\beta\to 0$.

The stationary system for (GM) is the following system of elliptic equations:

$$\begin{cases} \epsilon^2 \Delta a - a + \frac{a^2}{h} = 0, \quad a > 0 & \text{in } \Omega, \\ \Delta h - \beta^2 h + \beta^2 a^2 = 0, \quad h > 0 & \text{in } \Omega, \\ \frac{\partial a}{\partial \nu} = \frac{\partial h}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.19}$$

The following concerns the existence of symmetric K-peaked solutions in a two-dimensional domain which generalizes the one-dimensional result Theorem 4.2.

THEOREM 5.5. (See [87].) *Let* $\mathbf{P}^0 = (P_1^0, P_2^0, \ldots, P_K^0)$ *be a non-degenerate critical point of* $F_0(\mathbf{P})$ (*defined by* (5.16)). *Moreover, we assume that the following technical condition holds*

$$\text{if } K > 1, \text{ then } \lim_{\epsilon \to 0} \eta_\epsilon \neq K, \tag{5.20}$$

where η_ϵ *is defined by* (5.18).

Then for ϵ *sufficiently small and* $D = \frac{1}{\beta^2}$ *sufficiently large, problem* (5.19) *has a solution* (a_ϵ, h_ϵ) *with the following properties*:

(1) $a_\epsilon(x) = \xi_\epsilon(\sum_{j=1}^K w(\frac{x - P_j^\epsilon}{\epsilon}) + O(k(\epsilon, \beta)))$ *uniformly for* $x \in \bar{\Omega}$. *Here* w *is the unique solution of* (2.8) *and*

$$\xi_\epsilon = \begin{cases} \frac{1}{K} \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2} w^2(y)\,dy} & \text{if } \eta_\epsilon \to 0, \\ \frac{1}{\eta_\epsilon} \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2} w^2(y)\,dy} & \text{if } \eta_\epsilon \to \infty, \\ \frac{1}{K + \eta_0} \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2} w^2(y)\,dy} & \text{if } \eta_\epsilon \to \eta_0, \end{cases} \tag{5.21}$$

and

$$k(\epsilon, \beta) := \epsilon^2 \xi_\epsilon \beta^2. \tag{5.22}$$

(*By* (5.21), $k(\epsilon, \beta) = O(\min\{\frac{1}{\log \frac{1}{\epsilon}}, \beta^2\})$.)

Furthermore, $P_j^\epsilon \to P_j^0$ *as* $\epsilon \to 0$ *for* $j = 1, \ldots, K$.

(2) $h_\epsilon(x) = \xi_\epsilon(1 + O(k(\epsilon, \beta)))$ *uniformly for* $x \in \bar{\Omega}$.

5.3. *Existence of multiple asymmetric spots*

Similar to the on dimensional case, there are also multiple asymmetric spots in a two-dimensional domain. But the existence of such patterns is only restricted when

$$\lim_{\epsilon \to 0} \frac{D}{\log \frac{1}{\epsilon}} < +\infty. \tag{5.23}$$

We first derive the algebraic equations for the heights $(\xi_{\epsilon,1},\dots,\xi_{\epsilon,K})$.

For $\beta>0$ let $G_\beta(x,\xi)$ be the Green function given by

$$\begin{cases} \Delta G_\beta-\beta^2 G_\beta+\delta_\xi=0 & \text{in } \Omega, \\ \frac{\partial G_\beta}{\partial\nu}=0 & \text{on } \partial\Omega. \end{cases} \tag{5.24}$$

Recall that $\beta^2=\frac{1}{D}$ and therefore $\beta\sim\frac{1}{\sqrt{\log\frac{1}{\epsilon}}}$. Let $G_0(x,\xi)$ be the Green function defined in (5.14).

In Section 2 of [87] a relation between G_0 and G_β is derived as follows

$$G_\beta(x,\xi)=\frac{\beta^{-2}}{|\Omega|}+G_0(x,\xi)+O\bigl(\beta^2\bigr) \tag{5.25}$$

in the operator norm of $L^2\Omega)\to H^2(\Omega)$. (Note that the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$ is compact.)

We define cut-off functions as follows: Let $\mathbf{P}\in\Omega^K$. Introduce

$$\chi_{\epsilon,P_j}(x)=\chi\left(\frac{x-P_j}{\delta}\right),\quad x\in\Omega,\ j=1,\dots,Km, \tag{5.26}$$

where χ is a smooth cut-off function which is equal to 1 in $B_1(0)$ and equal to 0 in $R^2\setminus\overline{B_2(0)}$.

Let us assume the following ansatz for a multiple-spike solution (a_ϵ,h_ϵ) of (GM):

$$\begin{cases} a_\epsilon\sim\sum_{i=1}^K\xi_{\epsilon,i}w(\frac{x-P_i^\epsilon}{\epsilon})\chi_{\epsilon,P_i}(x), \\ h_\epsilon(P_i^\epsilon)\sim\xi_{\epsilon,i}, \end{cases} \tag{5.27}$$

where w is the unique solution of (2.8), $\xi_{\epsilon,i}$, $i=1,\dots,K$, are the heights of the peaks, to be determined later, and $\mathbf{P}^\epsilon=(P_1^\epsilon,\dots,P_K^\epsilon)$ are the locations of K peaks.

Then we can make the following calculations, which can be made rigorous with error terms of the order $O(\frac{1}{\log\frac{1}{\epsilon}})$ in $H^2(\Omega)$.

From the equation for h_ϵ,

$$\Delta h_\epsilon-\beta^2h_\epsilon+\beta^2a_\epsilon^2=0,$$

we get, using (5.25),

$$\begin{aligned} h_\epsilon\bigl(P_i^\epsilon\bigr)&=\int_\Omega G_\beta\bigl(P_i^\epsilon,\xi\bigr)\beta^2a_\epsilon^2(\xi)\,d\xi \\ &=\int_\Omega\left(\frac{\beta^{-2}}{|\Omega|}+G_0\bigl(P_i^\epsilon,\xi\bigr)+O\bigl(\beta^2\bigr)\right)\beta^2 \end{aligned}$$

$$\times\left(\sum_{j=1}^{K}\xi_{\epsilon,j}^2 w^2\left(\frac{\xi-P_j^\epsilon}{\epsilon}\right)\chi_{\epsilon,P_j}(\xi)\right)d\xi$$

$$=\int_\Omega\left(\frac{1}{|\Omega|}+\beta^2G_0\left(P_i^\epsilon,\xi\right)+O\left(\beta^4\right)\right)$$

$$\times\left(\sum_{j=1}^{K}\xi_{\epsilon,j}^2 w^2\left(\frac{\xi-P_j^\epsilon}{\epsilon}\right)\chi_{\epsilon,P_j}(\xi)\right)d\xi.$$

Thus

$$\begin{aligned}\xi_{\epsilon,i}&=\xi_{\epsilon,i}^2\frac{\epsilon^2}{|\Omega|}\int_{\mathbb{R}^2}w^2(y)\,dy+\xi_{\epsilon,i}^2\beta^2\int_\Omega G_0\left(P_i^\epsilon,\xi\right)w^2\left(\frac{\xi-P_i^\epsilon}{\epsilon}\right)\chi_{\epsilon,P_i}(\xi)\,d\xi\\&\quad+\sum_{j\neq i}\left(\frac{1}{|\Omega|}+\beta^2G_0\left(P_i^\epsilon,P_j^\epsilon\right)\right)\xi_{\epsilon,j}^2\epsilon^2\int_{\mathbb{R}^2}w^2(y)\,dy\\&\quad+\sum_{j=1}^{K}\xi_{\epsilon,j}^2\left(O\left(\beta^2\epsilon^4\right)+O\left(\beta^4\epsilon^2\right)\right).\end{aligned}\tag{5.28}$$

Here we have used that for $j\neq i$

$$\begin{aligned}&\int_\Omega G_0\left(P_i^\epsilon,\xi\right)w^2\left(\frac{\xi-P_j^\epsilon}{\epsilon}\right)\chi_{\epsilon,P_j}(\xi)\,d\xi\\&\quad=\epsilon^2\int_{\mathbb{R}^2}G_0\left(P_i^\epsilon,\epsilon y+P_j^\epsilon\right)w^2(y)\,dy+\text{e.s.t.}\\&\quad=\epsilon^2G_0\left(P_i^\epsilon,P_j^\epsilon\right)\int_{\mathbb{R}^2}w^2(y)\,dy\\&\qquad+\epsilon^3\sum_{l=1}^{K}\frac{\partial G_0(P_i^\epsilon,P_j^\epsilon)}{\partial P_{j,l}^\epsilon}\int_{\mathbb{R}^2}w^2(y)y_l\,dy+O\left(\epsilon^4\right)\\&\quad=\epsilon^2G_0\left(P_i^\epsilon,P_j^\epsilon\right)\int_{\mathbb{R}^2}w^2(y)\,dy+O\left(\epsilon^4\right).\end{aligned}$$

(Note that we have set $y=\frac{\xi-P_j^\epsilon}{\epsilon}$ and we have used the relation

$$\int_{\mathbb{R}^2}w^2(y)y_l\,dy=0$$

which holds since w is radially symmetric.)

Using (5.15) in (5.28) gives

$$\begin{aligned}\xi_{\epsilon,i} &= \xi_{\epsilon,i}^2 \frac{\epsilon^2}{|\Omega|}\int_{\mathbb{R}^2} w^2(y)\,dy \\ &\quad + \xi_{\epsilon,i}^2\beta^2 \int_\Omega \left(\frac{1}{2\pi}\log\frac{1}{|P_i^\epsilon-\xi|} - H_0\big(P_i^\epsilon,\xi\big)\right) w^2\left(\frac{\xi-P_i^\epsilon}{\epsilon}\right)\chi_{\epsilon,P_i^\epsilon}(\xi)\,d\xi \\ &\quad + \sum_{j\neq i}\left(\frac{1}{|\Omega|}+\beta^2 G_0\big(P_i^\epsilon,P_j^\epsilon\big)\right)\xi_{\epsilon,j}^2\epsilon^2\int_{\mathbb{R}^2} w^2(y)\,dy \\ &\quad + \sum_{j=1}^K \xi_{\epsilon,j}^2\big(O\big(\beta^2\epsilon^4\big)+O\big(\beta^4\epsilon^2\big)\big) \\ &= \xi_{\epsilon,i}^2\frac{\epsilon^2}{|\Omega|}\int_{\mathbb{R}^2} w^2(y)\,dy + \xi_{\epsilon,i}^2\frac{\beta^2}{2\pi}\epsilon^2\log\frac{1}{\epsilon}\int_{\mathbb{R}^2} w^2(y)\,dy \\ &\quad + \xi_{\epsilon,i}^2\frac{\beta^2}{2\pi}\left(\epsilon^2\int_{\mathbb{R}^2} w^2(y)\log\frac{1}{|y|}\,dy - \epsilon^2 H_0\big(P_i^\epsilon,P_i^\epsilon\big)\int_{\mathbb{R}^2} w^2(y)\,dy\right) \\ &\quad + \sum_{j\neq i}\left(\frac{1}{|\Omega|}+\beta^2 G_0\big(P_i^\epsilon,P_j^\epsilon\big)\right)\xi_{\epsilon,j}^2\epsilon^2\int_{\mathbb{R}^2} w^2(y)\,dy \\ &\quad + \sum_{j=1}^K \xi_{\epsilon,j}^2\big(O\big(\beta^2\epsilon^4\big)+O\big(\beta^4\epsilon^2\big)\big). \end{aligned} \tag{5.29}$$

Recall that $H_0 \in C^2(\bar{\Omega}\times\Omega)$.

Considering only the leading terms in (5.29) we get following

$$\begin{aligned}\xi_{\epsilon,i} &= \sum_{j=1}^K \xi_{\epsilon,j}^2\frac{\epsilon^2}{|\Omega|}\int_{\mathbb{R}^2} w^2(y)\,dy + \xi_{\epsilon,i}^2\frac{\beta^2}{2\pi}\epsilon^2\log\frac{1}{\epsilon}\int_{\mathbb{R}^2} w^2(y)\,dy \\ &\quad + \sum_{j=1}^K \xi_{\epsilon,j}^2 O\big(\beta^2\epsilon^2\big). \end{aligned} \tag{5.30}$$

Let us rescale

$$\xi_{\epsilon,i} = \xi_\epsilon\hat{\xi}_{\epsilon,i}, \quad \text{where } \xi_\epsilon = \frac{|\Omega|}{\epsilon^2\int_{\mathbb{R}^2} w^2}. \tag{5.31}$$

Then from (5.30) we get

$$\xi_{\epsilon,i} = \left(\frac{1}{|\Omega|}+\frac{\eta_\epsilon}{|\Omega|}\right)\xi_{\epsilon,i}^2\epsilon^2\int_{\mathbb{R}^2} w^2(y)\,dy$$

$$+\sum_{j\neq i}\xi_{\epsilon,j}^2\frac{\epsilon^2}{|\Omega|}\int_{\mathbb{R}^2}w^2(y)\,dy+\sum_{j=1}^{K}\xi_{\epsilon,j}^2 O\bigl(\beta^2\epsilon^2\bigr),$$

where η_ϵ was introduced in (5.18). Assuming that

$$\hat{\xi}_{\epsilon,i}\to\hat{\xi}_i,\quad \eta_\epsilon\to\eta_0, \tag{5.32}$$

we obtain the following system of algebraic equations

$$\hat{\xi}_{\epsilon,i}=\sum_{j=1}^{K}\hat{\xi}_{\epsilon,j}^2+\hat{\xi}_{\epsilon,i}^2\eta_0,\quad i=1,\ldots,K, \tag{5.33}$$

which can be determined completely.

In fact, let

$$\rho(t)=t-\eta_0t^2. \tag{5.34}$$

Then (5.33) is equivalent to

$$\rho(\hat{\xi}_i)=\sum_{j=1}^{K}\hat{\xi}_j^2,\quad i=1,\ldots,K, \tag{5.35}$$

which implies that

$$\rho(\hat{\xi}_i)=\rho(\hat{\xi}_j)\quad \text{for } i\neq j. \tag{5.36}$$

That is

$$(\hat{\xi}_i-\hat{\xi}_j)\bigl(1-\eta_0(\hat{\xi}_i+\hat{\xi}_j)\bigr)=0. \tag{5.37}$$

Hence for $i\neq j$ we have

$$\hat{\xi}_i-\hat{\xi}_j=0\quad \text{or}\quad \hat{\xi}_i+\hat{\xi}_j=\frac{1}{\eta_0}. \tag{5.38}$$

The case of symmetric solutions ($\hat{\xi}_i=\hat{\xi}_1$, $i=2,\ldots,N$) has been studied in [86] and [87]. Let us now consider asymmetric solutions, i.e., we assume that there exists an $i\in\{2,\ldots,N\}$ such that $\hat{\xi}_i\neq\hat{\xi}_1$. Without loss of generality, let us assume that

$$\hat{\xi}_2\neq\hat{\xi}_1,$$

which implies that

$$\hat{\xi}_1+\hat{\xi}_2=\frac{1}{\eta_0}. \tag{5.39}$$

Let us calculate $\hat{\xi}_j$, $j = 3, \ldots, K$. If $\hat{\xi}_j \neq \hat{\xi}_1$, then by (5.38), $\hat{\xi}_j + \hat{\xi}_1 = \frac{1}{\eta_0}$, which implies that $\hat{\xi}_j = \hat{\xi}_2$.

Thus for $j \geqslant 3$, we have either $\hat{\xi}_j = \hat{\xi}_1$ or $\hat{\xi}_j = \hat{\xi}_2$.

Let k_1 be the number of $\hat{\xi}_1$'s in $\{\hat{\xi}_1, \ldots, \hat{\xi}_K\}$ and k_2 be the number of $\hat{\xi}_2$'s in $\{\hat{\xi}_1, \ldots, \hat{\xi}_K\}$. Then we have $k_1 \geqslant 1$, $k_2 \geqslant 1$, $k_1 + k_2 = K$.

This gives

$$\hat{\xi}_1 - \eta_0 \hat{\xi}_1^2 = \sum_{j=1}^{K} \hat{\xi}_j^2 = k_1 \hat{\xi}_1^2 + k_2 \hat{\xi}_2^2, \tag{5.40}$$

$$\hat{\xi}_2 = \frac{1}{\eta_0} - \hat{\xi}_1. \tag{5.41}$$

Substituting (5.41) into (5.40), we obtain

$$\hat{\xi}_1 - \eta_0 \hat{\xi}_1^2 = k_1 \hat{\xi}_1^2 + k_2 \left(\frac{1}{\eta_0} - \hat{\xi}_1 \right)^2$$

and therefore

$$(k_1 + k_2 + \eta_0) \hat{\xi}_1^2 - \frac{2k_2 + \eta_0}{\eta_0} \hat{\xi}_1 + \frac{k_2}{\eta_0^2} = 0. \tag{5.42}$$

Equation (5.42) has a solution if and only if

$$\frac{(2k_2 + \eta_0)^2}{\eta_0^2} \geqslant 4 \frac{k_2}{\eta_0^2} (k_1 + k_2 + \eta_0). \tag{5.43}$$

The strict inequality of (5.43) is equivalent to

$$\eta_0 > 2\sqrt{k_1 k_2}. \tag{5.44}$$

It is easy to see that if (5.44) holds, then there are two different solutions to (5.42) which are given by $(\rho_\pm, \eta_\pm)$.

Therefore we arrive at the following conclusion.

LEMMA 5.6. *Let* $\eta_0 \geqslant 2\sqrt{k_1 k_2}$. *Then the solutions of* (5.33) *are given by* $(\hat{\xi}_1, \ldots, \hat{\xi}_N) \in (\{\rho_\pm, \eta_\pm\})^K$ *where the number of* $\rho_\pm$*'s is* k_1 *and the number of* $\eta_\pm$*'s is* k_2.

If $\eta_0 > 2\sqrt{k_1 k_2}$, *there exist two solutions* $(\rho_\pm, \eta_\pm)$.

If $\eta_0 = 2\sqrt{k_1 k_2}$, *there exists one solution* $(\rho_\pm, \rho_\pm)$.

If $\eta_0 < 2\sqrt{k_1 k_2}$, *there are no solutions* $(\rho_\pm, \rho_\pm)$.

Let $\eta_0 > 2\sqrt{k_1 k_2}$ where $k_1 + k_2 = K$, $k_1, k_2 \geqslant 1$. By Lemma 5.6, there are two solutions to (5.33). In fact, we can solve

$$\rho_+ = \frac{2k_2 + \eta_0 + \sqrt{\eta_0^2 - 4k_1 k_2}}{2\eta_0(\eta_0 + K)}, \qquad \rho_- = \frac{2k_2 + \eta_0 - \sqrt{\eta_0^2 - 4k_1 k_2}}{2\eta_0(\eta_0 + K)}, \tag{5.45}$$

$$\eta_+ = \frac{2k_1 + \eta_0 - \sqrt{\eta_0^2 - 4k_1 k_2}}{2\eta_0(\eta_0 + K)}, \qquad \eta_- = \frac{2k_1 + \eta_0 + \sqrt{\eta_0^2 - 4k_1 k_2}}{2\eta_0(\eta_0 + K)}. \tag{5.46}$$

Note that

$$\rho_+ + \eta_+ = \frac{1}{\eta_0}, \qquad \rho_- + \eta_- = \frac{1}{\eta_0}. \tag{5.47}$$

Let $(\rho, \eta) = (\rho_+, \eta_+)$ or $(\rho, \eta) = (\rho_-, \eta_-)$. We drop "±" if there is no confusion.

Let $(\hat\xi_1, \ldots, \hat\xi_K) \in R_+^K$ be such that

$$\hat\xi_j \in \{\rho, \eta\}, \text{ and the number of } \rho\text{'s in } (\hat\xi_1, \ldots, \hat\xi_K) \text{ is } k_1. \tag{5.48}$$

Then there are k_2 η's in $(\hat\xi_1, \ldots, \hat\xi_K)$.

Let $\mathbf{P} = (P_1, \ldots, P_K) \in \Omega^K$, where $\mathbf{P}$ is arranged such that

$$\mathbf{P} = (P_1, P_2, \ldots, P_K)$$

with

$$P_i = (P_{i,1}, P_{i,2}) \quad \text{for } i = 1, 2, \ldots, K.$$

For $\mathbf{P} \in \Omega^K$ we define

$$\hat F_0(\mathbf{P}) = \sum_{k=1}^{K} H_0(P_k, P_k)\hat\xi_k^4 - \sum_{i,j,=1,\ldots,K,\, i\neq j} G_0(P_i, P_j)\hat\xi_i^2\hat\xi_j^2 \tag{5.49}$$

and

$$\hat M_0(\mathbf{P}) = \nabla^2_{\mathbf{P}} \tilde F_0(\mathbf{P}). \tag{5.50}$$

Then we have the following theorem, which is on the existence of asymmetric K-peaked solutions.

THEOREM 5.7. (See [88].) *Let $K \geqslant 2$ be a positive integer. Let k_1, $k_2 \geqslant 1$ be two integers such that $k_1 + k_2 = K$. Let*

$$\beta^2 = \frac{1}{D}, \qquad \eta_\epsilon = \frac{\beta^2|\Omega|}{2\pi}\log\frac{\sqrt{|\Omega|}}{\epsilon},$$

where $|\Omega|$ denotes the area of Ω, Assume that $\eta_0 = \lim_{\epsilon\to 0}\eta_\epsilon > 2\sqrt{k_1 k_2}$,

$$\text{(T1)}\quad \eta_0 \neq K$$

and that

$$\text{(T2)}\quad \mathbf{P}^0 = \big(P_1^0, P_2^0, \dots, P_K^0\big) \text{ is a non-degenerate critical point of } \hat{F}_0(\mathbf{P})$$

(*defined by* (5.49)).

Then for ϵ *sufficiently small the stationary* (GM) *has a solution* (a_ϵ, h_ϵ) *with the following properties*:

(1) $a_\epsilon(x) = \sum_{j=1}^K \xi_{\epsilon,j}(w(\frac{x-P_j^\epsilon}{\epsilon}) + O(\frac{1}{D}))$ *uniformly for* $x \in \bar{\Omega}$, *where* w *is the unique solution of* (2.8) *and*

$$\xi_{\epsilon,j} = \xi_\epsilon \hat{\xi}_{\epsilon,j}, \quad \xi_\epsilon = \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2} w^2}. \tag{5.51}$$

Further, $(\hat{\xi}_{\epsilon,1}, \dots, \hat{\xi}_{\epsilon,K}) \to (\hat{\xi}_1, \dots, \hat{\xi}_K)$ *which is given by* (5.48).

(2) $h_\epsilon(P_j^\epsilon) = \xi_{\epsilon,j}(1 + \frac{1}{D})$ *in* $H^2(\Omega)$, $j = 1, \dots, K$.

(3) $P_j^\epsilon \to P_j^0$ *as* $\epsilon \to 0$ *for* $j = 1, \dots, K$.

5.4. *Stability of symmetric* K*-spots*

Next we study the stability and instability of the symmetric K-peaked solutions constructed in Theorems 5.4 and 5.5.

In the strong coupling case, it turns out all solutions are stable:

THEOREM 5.8. (See [86].) *Suppose* $D = O(1)$. *Let* $\mathbf{P}_0$ *and* (a_ϵ, h_ϵ) *be defined as in Theorem* 5.4. *Then for* ϵ *and* τ *sufficiently small* (a_ϵ, h_ϵ) *is stable if all eigenvalues of the matrix* $M_D(\mathbf{P}_0) = (\nabla^2_{\mathbf{P}_0} F_D(\mathbf{P}_0))$ *are negative.* (a_ϵ, h_ϵ) *is unstable if one of the eigenvalues of the matrix* $M_D(\mathbf{P}_0)$ *is positive.*

In the weak coupling case, the stability of symmetric K-peaked solutions in a bounded two-dimensional domain can be summarized as follows.

THEOREM 5.9. (See [87].) *Let* $\mathbf{P}^0$ *be a non-degenerate critical point of* $F_0(\mathbf{P})$ *and for* ϵ *sufficiently small and* $D = \frac{1}{\beta^2}$ *sufficiently large let* (a_ϵ, h_ϵ) *be the* K*-peaked solutions constructed in Theorem* 5.5 *whose peaks approach* $\mathbf{P}^0$.

Assume (5.20) *holds and further that*

$$(*)\quad \mathbf{P}^0 \text{ is a non-degenerate local maximum point of } F_0(\mathbf{P}).$$

Then we have

Table 1

	Case 1	Case 2	Case 3 ($\eta_0 < K$)	Case 3 ($\eta_0 > K$)
$K = 1$, τ small	stable	stable	stable	stable
$K = 1$, τ finite	?	stable	?	?
$K = 1$, τ large	unstable	stable	unstable	stable
$K > 1$, τ small	unstable	stable	unstable	stable
$K > 1$, τ finite	unstable	stable	unstable	?
$K > 1$, τ large	unstable	stable	unstable	stable

CASE 1. $\eta_\epsilon \to 0$ (*i.e.*, $\frac{2\pi D}{|\Omega|} \gg \log \frac{1}{\epsilon}$).

If $K = 1$ *then there exists a unique* $\tau_1 > 0$ *such that for* $\tau < \tau_1$, (a_ϵ, h_ϵ) *is linearly stable, while for* $\tau > \tau_1$, (a_ϵ, h_ϵ) *is linearly unstable.*

If $K > 1$, (a_ϵ, h_ϵ) *is linearly unstable for any* $\tau \geqslant 0$.

CASE 2. $\eta_\epsilon \to +\infty$ (*i.e.*, $\frac{2\pi D}{|\Omega|} \ll \log \frac{1}{\epsilon}$).

(a_ϵ, h_ϵ) *is linearly stable for any* $\tau > 0$.

CASE 3. $\eta_\epsilon \to \eta_0 \in (0, +\infty)$ (*i.e.*, $\frac{2\pi D}{|\Omega|} \sim \frac{1}{\eta_0} \log \frac{1}{\epsilon}$).

If $K > 1$ *and* $\eta_0 < K$, *then* (a_ϵ, h_ϵ) *is linearly unstable for any* $\tau > 0$.

If $\eta_0 > K$, *then there exist* $0 < \tau_2 \leqslant \tau_3$ *such that* (a_ϵ, h_ϵ) *is linearly stable for* $\tau < \tau_2$ *and* $\tau > \tau_3$.

If $K = 1$, $\eta_0 < 1$, *then there exist* $0 < \tau_4 \leqslant \tau_5$ *such that* (a_ϵ, h_ϵ) *is linearly stable for* $\tau < \tau_4$ *and linearly unstable for* $\tau > \tau_5$.

The statement of Theorem 5.9 is rather long. Let us therefore explain the results by the following remarks.

REMARK 5.4.1. Assuming that condition $(*)$ holds, then for ϵ small the stability behavior of (a_ϵ, h_ϵ) can be summarized in the following table:

REMARK 5.4.2. The condition $(*)$ on the locations $\mathbf{P}^0$ arises in the study of small ($o(1)$) eigenvalues. For any bounded smooth domain Ω, the functional $F_0(\mathbf{P})$, defined by (5.16), always admits a global maximum at some $\mathbf{P}^0 \in \Omega^K$. The proof of this fact is similar to the Appendix in [87]. We believe that in *generic* domains, this global maximum point $\mathbf{P}^0$ is non-degenerate.

It is an interesting open question to numerically compute the critical points of $F_0(\mathbf{P})$ and link them explicitly to the geometry of the domain Ω.

We believe that for other types of critical points of $F_0(\mathbf{P})$, such as saddle points, the solution constructed in Theorem 5.5 should be linearly unstable. We are not able to prove this at the moment, since the operator $\mathcal{L}_\epsilon$ is *not self-adjoint*.

REMARK 5.4.3. Case 1 and Case 3 with $\eta_0 < K$ resemble the *shadow system* and Case 2 and Case 3 with $\eta_0 > K$ are similar to the *strong coupling* case.

From Case 2 and Case 3 of Theorem 5.9, we see that for multiple spikes ($K > 1$) large τ may increase stability, provided that $\eta_0 > K$. This is a *new* phenomenon in R^2. It is known that in R^1, large τ implies linear instability for multiple spikes [8,34,59,60].

REMARK 5.4.4. We conjecture that in Case 3, $\tau_2 = \tau_3$. This will imply that for any $\tau \geqslant 0$ and $\eta_0 > K$, multiple spikes are stable, provided condition $(*)$ is satisfied. (It is possible to obtain explicit values for τ_2 and τ_3.)

REMARK 5.4.5. Roughly speaking, assuming that condition $(*)$ holds and that τ is small, then for $\epsilon \ll 1$, $D_K(\epsilon) = \frac{|\Omega|}{2\pi K} \log \frac{1}{\epsilon}$ is the critical threshold for the asymptotic behavior of the diffusion coefficient of the inhibitor which determines the stability of K-peaked solutions.

The proof of Theorem 5.9 is again divided by two parts: large eigenvalues and small eigenvalues. For small eigenvalues, we relate them to the functional $F(\mathbf{P})$. For large eigenvalues, we obtain a system of NLEPs:

$$\begin{aligned}
&\Delta\phi_i - \phi_i + 2w\phi_i \\
&\quad - \frac{2[(1+\eta_0(1+\tau\lambda_0)) \int_{\mathbb{R}^2} w\phi_i + \sum_{j\neq i} \int_{\mathbb{R}^2} w\phi_j]}{(K+\eta_0)(1+\tau\lambda_0)\int_{\mathbb{R}^2} w^2} w^2 = \lambda_0 \phi_i, \\
&\quad i = 1, \ldots, K.
\end{aligned} \tag{5.52}$$

By diagonalization, we obtain two NELPs:

$$\Delta\phi - \phi + 2w\phi - \frac{2\eta_0}{(K+\eta_0)\int_{\mathbb{R}^2} w^2} \left[\int_{\mathbb{R}^2} w(y)\phi(y)\,dy\right] w^2 = \lambda\phi, \tag{5.53}$$

and

$$\begin{aligned}
&\Delta\phi - \phi + 2w\phi - \frac{2(K+\eta_0(1+\tau\lambda_0))}{(K+\eta_0)(1+\tau\lambda_0)} \frac{\int_{\mathbb{R}^2} w\phi}{\int_{\mathbb{R}^2} w^2} w^2 = \lambda_0\phi, \\
&\quad \phi \in H^2(\mathbb{R}^2),
\end{aligned} \tag{5.54}$$

where $0 < \eta_0 < +\infty$ and $0 \leqslant \tau < +\infty$.

Problem (5.53) is the same as (3.7). For problem (5.54), we have the following result

THEOREM 5.10.

(1) *If $\eta_0 < K$, then for τ small problem* (5.54) *is stable while for τ large it is unstable.*

(2) *If $\eta_0 > K$, then there exists $0 < \tau_2 \leqslant \tau_3$ such that problem* (5.54) *is stable for $\tau < \tau_2$ or $\tau > \tau_3$.*

PROOF. Let us set

$$f(\tau\lambda) = \frac{2(K + \eta_0(1 + \tau\lambda))}{(K + \eta_0)(1 + \tau\lambda)}. \tag{5.55}$$

We note that

$$\lim_{\tau\lambda\to+\infty} f(\tau\lambda) = \frac{2\eta_0}{K + \eta_0} =: f_\infty.$$

If $\eta_0 < K$, then by Theorem 3.12(2), problem (3.52) with $\mu = f_\infty$ has a positive eigenvalue α_1. Now by perturbation arguments (similar to those in [8]), for τ large, problem (5.54) has an eigenvalue near $\alpha_1 > 0$. This implies that for τ large, problem (5.54) is unstable.

Now we show that problem (5.54) has no non-zero eigenvalues with non-negative real part, provided that either τ is small or $\eta_0 > K$ and τ is large. (It is immediately seen that $f(\tau\lambda) \to 2$ as $\tau\lambda \to 0$ and $f(\tau\lambda) \to \frac{2\eta_0}{\eta_0+K} > 1$ as $\tau\lambda \to +\infty$ if $\eta_0 > K$. Then Theorem 3.12 should apply. The problem is that we do not have control on $\tau\lambda$. Here we provide a rigorous proof.)

We apply the following inequality (Lemma 3.8(1)): for any (real-valued function) $\phi \in H_r^2(R^2)$, we have

$$\int_{\mathbb{R}^2} \left(|\nabla\phi|^2 + \phi^2 - 2w\phi^2\right) + 2\frac{\int_{\mathbb{R}^2} w\phi \int_{\mathbb{R}^2} w^2\phi}{\int_{\mathbb{R}^2} w^2} - \frac{\int_{\mathbb{R}^2} w^3}{(\int_{\mathbb{R}^2} w^2)^2}\left(\int_{\mathbb{R}^2} w\phi\right)^2 \geqslant 0, \tag{5.56}$$

where equality holds if and only if ϕ is a multiple of w.

Now let $\lambda_0 = \lambda_R + \sqrt{-1}\lambda_I$, $\phi = \phi_R + \sqrt{-1}\phi_I$ satisfy (5.54). Then we have

$$L_0\phi - f(\tau\lambda_0)\frac{\int_{\mathbb{R}^2} w\phi}{\int_{\mathbb{R}^2} w^2} w^2 = \lambda_0\phi. \tag{5.57}$$

Multiplying (5.57) by $\bar{\phi}$—the conjugate function of ϕ—and integrating over R^2, we obtain that

$$\int_{\mathbb{R}^2} \left(|\nabla\phi|^2 + |\phi|^2 - 2w|\phi|^2\right) = -\lambda_0 \int_{\mathbb{R}^2} |\phi|^2 - f(\tau\lambda_0)\frac{\int_{\mathbb{R}^2} w\phi}{\int_{\mathbb{R}^2} w^2} \int_{\mathbb{R}^2} w^2\bar{\phi}. \tag{5.58}$$

Multiplying (5.57) by w and integrating over $\mathbb{R}^2$, we obtain that

$$\int_{\mathbb{R}^2} w^2\phi = \left(\lambda_0 + f(\tau\lambda_0)\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\right) \int_{\mathbb{R}^2} w\phi. \tag{5.59}$$

Taking the conjugate of (5.59) we have

$$\int_{\mathbb{R}^2} w^2\bar{\phi} = \left(\bar{\lambda_0} + f(\tau\bar{\lambda}_0)\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\right)\int_{\mathbb{R}^2} w\bar{\phi}. \tag{5.60}$$

Substituting (5.60) into (5.58), we have that

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(|\nabla\phi|^2 + |\phi|^2 - 2w|\phi|^2\right)\\ &\quad = -\lambda_0\int_{\mathbb{R}^2}|\phi|^2 - f(\tau\lambda_0)\left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\right)\frac{|\int_{\mathbb{R}^2} w\phi|^2}{\int_{\mathbb{R}^2} w^2}.\end{aligned} \tag{5.61}$$

We just need to consider the real part of (5.61). Now applying the inequality (5.56) and using (5.60) we arrive at

$$\begin{aligned}-\lambda_R \geqslant{} & \mathrm{Re}\left(f(\tau\lambda_0)\left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\right)\right)\\ & - 2\,\mathrm{Re}\left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\right) + \frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2},\end{aligned}$$

where we recall $\lambda_0 = \lambda_R + \sqrt{-1}\lambda_I$ with $\lambda_R,\ \lambda_I \in R$.

Assuming that $\lambda_R \geqslant 0$, then we have

$$\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\left|f(\tau\lambda_0) - 1\right|^2 + \mathrm{Re}\left(\bar{\lambda}_0\left(f(\tau\lambda_0) - 1\right)\right) \leqslant 0. \tag{5.62}$$

By the usual Pohozaev's identity for (2.8) (multiplying (2.8) by $y\cdot\nabla w(y)$ and integrating by parts), we obtain that

$$\int_{\mathbb{R}^2} w^3 = \frac{3}{2}\int_{\mathbb{R}^2} w^2. \tag{5.63}$$

Substituting (5.63) and the expression (5.55) for $f(\tau\lambda)$ into (5.62), we have

$$\begin{aligned}&\frac{3}{2}\left|\eta_0 + K + (\eta_0 - K)\tau\lambda\right|^2 + \mathrm{Re}\left((\eta_0 + K)(1 + \tau\bar{\lambda}_0)\left((\eta_0 + K)\bar{\lambda}_0\right.\right.\\ &\qquad \left.\left. + (\eta_0 - K)\tau|\lambda_0|^2\right)\right) \leqslant 0\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{3}{2}(1 + \mu_0\tau\lambda_R)^2 + \lambda_R + \left(\mu_0\tau + \tau + \mu_0\tau^2|\lambda_0|^2\right)\lambda_R\\ &\qquad + \left(\frac{3}{2}\mu_0^2\tau^2 + \mu_0\tau - \tau\right)\lambda_I^2 \leqslant 0\end{aligned} \tag{5.64}$$

where we have introduced $\mu_0 := \frac{\eta_0 - K}{\eta_0 + K}$.

If $\eta_0 > K$ (i.e., $\mu_0 > 0$) and τ is large, then

$$\frac{3}{2}\mu_0^2\tau^2 + \mu_0\tau - \tau \geqslant 0. \tag{5.65}$$

So (5.64) does not hold for $\lambda_R \geqslant 0$.

To consider the case when τ is small, we have to obtain an upper bound for λ_I.

From (5.58), we have

$$\lambda_I \int_{\mathbb{R}^2} |\phi|^2 = \operatorname{Im}\left(-f(\tau\lambda_0)\frac{\int_{\mathbb{R}^2} w\phi}{\int_{\mathbb{R}^2} w^2}\int_{\mathbb{R}^2} w^2\bar{\phi}\right).$$

Hence

$$|\lambda_I| \leqslant \left|f(\tau\lambda_0)\right|\sqrt{\frac{\int_{\mathbb{R}^2} w^4}{\int_{\mathbb{R}^2} w^2}} \leqslant C \tag{5.66}$$

where C is independent of λ_0.

Substituting (5.66) into (5.64), we see that (5.64) cannot hold for $\lambda_R \geqslant 0$, if τ is small. □

5.5. *Stability of asymmetric K-spots*

Finally we study the stability or instability of the asymmetric K-peaked solutions constructed in Theorem 5.7.

THEOREM 5.11. *Let (a_ϵ, h_ϵ) be the K-peaked solutions constructed in Theorem* 5.7 *for ϵ sufficiently small, whose peaks are located near $\mathbf{P}^0$. Further assume that*

(∗) $\mathbf{P}^0$ *is a non-degenerate local maximum point of $\hat{F}(\mathbf{P})$.*

Then we have:

(a) (*Stability*)

Assume that

$$2\sqrt{k_1k_2} < \eta_0 < K \tag{5.67}$$

and

$$k_1 > k_2, \quad (\rho,\eta) = (\rho_+,\eta_+).$$

Then, *for τ small enough*, (a_ϵ, h_ϵ) *is stable.*

(b) (*Instability*)

Assume that either

$$\eta_0 > K$$

or

$$\tau \text{ is large enough.}$$

Then (a_ϵ, h_ϵ) *is linearly unstable.*

A consequence of Theorem 5.11 is *stable* asymmetric patterns can exist in a two-dimensional domain for a very narrow range of D, namely for

$$\frac{1}{2\pi K}\log\frac{\sqrt{|\Omega|}}{\epsilon} < \frac{D}{|\Omega|} < \frac{1}{4\pi\sqrt{k_1k_2}}\log\frac{\sqrt{|\Omega|}}{\epsilon} \tag{5.68}$$

and ϵ small enough, where k_1 and k_2 are two integers satisfying $k_1 + k_2 = K, k_1 \geqslant 1, k_2 \geqslant 1$. In most cases, asymmetric patterns are unstable.

6. High-dimensional case: $N \geqslant 3$

When $N \geqslant 3$, there are very few results on the full Gierer–Meinhardt system. The difference between $N \geqslant 3$ and $N \leqslant 2$ lies on the behavior of the Green function: when $N \leqslant 2$, the Green function is locally *constant* (when $N = 2$, it is locally ∞). The limiting problem is still a single equation (2.8). But when $N \geqslant 3$, the Green function is like $\frac{1}{|x-y|^{N-2}}$. The limiting problem when $N \geqslant 3$ becomes

$$\begin{cases} \Delta a - a + \frac{a^p}{h^q} = 0 & \text{in } \mathbb{R}^N, \\ \Delta h + \frac{a^r}{h^s} = 0 & \text{in } \mathbb{R}^N, \\ a, h > 0,\ a, h \to 0 & \text{as } |y| \to +\infty. \end{cases} \tag{6.1}$$

Problem (6.1) seems out of reach at this moment. We believe that there should a radially symmetric solution to (6.1) which is also stable.

As far as the author knows, the only result in higher-dimensional case is the existence of radially symmetric layer solutions [62].

Let $\Omega = B_R$ be a ball of radius R in $\mathbb{R}^N$. By scaling, we may take $D = 1$ and obtain formally the following elliptic system

$$\begin{cases} \epsilon^2\Delta a - a + \frac{a^p}{h^q} = 0 & \text{in } B_R, \\ \Delta h - h + \frac{a^m}{h^s} = 0 & \text{in } B_R \\ vsa > 0,\ h > 0 & \text{in } B_R, \\ \frac{\partial a}{\partial \nu} = \frac{\partial a}{\partial \nu} = 0 & \text{on } B_R, \end{cases} \tag{6.2}$$

where (p, q, m, s) satisfies

$$p > 1, \quad q > 0, \quad m > 0, \quad s \geqslant 0, \quad \frac{qm}{(p-1)(s+1)} > 1. \tag{6.3}$$

(The case of the whole $\mathbb{R}^N$ is also included here, by taking $R = +\infty$.)

Note that in (6.2), we have replaced a^r by a^m since we will use $r = |x|$ to denote the radial variable.

We first define two functions, to be used later: let $J_1(r)$ be the radially symmetric solutions of the following problem

$$J_1'' + \frac{N-1}{r} J_1' - J_1 = 0, \quad J'(0) = 0, \quad J_1(0) = 1, \quad J_1 > 0. \tag{6.4}$$

The second function, called $J_2(r)$, satisfies

$$J_2'' + \frac{N-1}{r} J_2' - J_2 + \delta_0 = 0, \quad J_2 > 0, \quad J_2(+\infty) = 0, \tag{6.5}$$

where δ_0 is the Dirac measure at 0.

The functions $J_1(r)$ and $J_2(r)$ can be written in terms of modified Bessel's functions. In fact

$$J_1(r) = c_1 r^{\frac{2-N}{2}} I_\nu(r), \qquad J_2(r) = c_2 r^{\frac{2-N}{2}} K_\nu(r), \quad \nu = \frac{N-2}{2} \tag{6.6}$$

where c_1, c_2 are two positive constants and I_ν, K_ν are modified Bessel functions of order ν. In the case of $N = 3$, J_1, J_2 can be computed explicitly:

$$J_1 = \frac{\sinh r}{r}, \qquad J_2(r) = \frac{e^{-r}}{4\pi r}. \tag{6.7}$$

Let $w(y)$ be the unique solution for ODE 2.103. Let $R > 0$ be a fixed constant. We define

$$J_{2,R}(r) = J_2(r) - \frac{J_2'(R)}{J_1'(R)} J_1(r) \tag{6.8}$$

and a Green function $G_R(r; r')$

$$G_R'' + \frac{N-1}{r} G_R' - G_R + \delta_{r'} = 0, \quad G_R'(0; r') = 0, \quad G_R'(R; r') = 0. \tag{6.9}$$

Note that

$$J_{2,R}'(R) = 0, \qquad \lim_{R \to +\infty} J_{2,R}(r) = J_2(r). \tag{6.10}$$

For $t \in (0, R)$, set

$$M_R(t) := \frac{(N-1)(p-1)}{qt} + \frac{J_1'(t)}{J_1(t)} + \frac{J_{2,R}'(t)}{J_{2,R}(t)}. \tag{6.11}$$

When $R = +\infty$, $J_{2,+\infty}(r) = J_2(r)$. We denote $G_{+\infty}(r; r')$ as $G(r; r')$ and $M_{+\infty}(t)$ as $M(t)$. That is,

$$G(r; r') = c_0(r')^{N-1} \begin{cases} J_2(r')J_1(r), & \text{for } r < r', \\ J_1(r')J_2(r), & \text{for } r > r', \end{cases} \tag{6.12}$$

$$M(t) := \frac{(N-1)(p-1)}{qt} + \frac{J_1'(t)}{J_1(t)} + \frac{J_2'(t)}{J_2(t)}. \tag{6.13}$$

Then we have the following existence result on layered solutions.

THEOREM 6.1. (See [62].) *Let $N \geqslant 2$. Assume that there exist two radii $0 < r_1 < r_2 < R$ such that*

$$M_R(r_1)M_R(r_2) < 0. \tag{6.14}$$

Then for ϵ sufficiently small, problem (6.2) *has a solution $(a_{\epsilon,R}, h_{\epsilon,R})$ with the following properties*:

(1) *$a_{\epsilon,R}, h_{\epsilon,R}$ are radially symmetric*,

(2) $a_{\epsilon,R}(r) = \xi_{\epsilon,R}^{\frac{q}{p-1}} w(\frac{r-t_\epsilon}{\epsilon})(1 + o(1))$,

(3) $a_{\epsilon,R}(r) = \xi_{\epsilon,R}(G_R(t_\epsilon; t_\epsilon))^{-1} G_R(r; t_\epsilon)(1 + o(1))$, *where $G_R(r; t_\epsilon)$ satisfies* (6.9), *$\xi_{\epsilon,R}$ is defined by the following*

$$\xi_{\epsilon,R} = \left(\epsilon\left(\int_{\mathbb{R}} w^m\right) G_R(t_\epsilon; t_\epsilon)\right)^{\frac{(1+s)(p-1)-qm}{qm}} \tag{6.15}$$

and $t_\epsilon \in (r_1, r_2)$ satisfies $\lim_{\epsilon\to 0} M_R(t_\epsilon) = 0$.

It remains to check condition (6.14), which can be verified numerically. Under some conditions on p, q, we can obtain the following corollary.

COROLLARY 6.2. *Assume that the following condition holds*:

$$\frac{(N-2)q}{N-1} + 1 < p < q + 1. \tag{6.16}$$

Then there exists an $R_0 > 0$ such that for $R > R_0$ and ϵ sufficiently small, problem (6.2) *has* two *radially symmetric solutions $(a_{\epsilon,R}^i, h_{\epsilon,R}^i)$ concentrating on sphere $\{r = t_i\}$ with $M_R(t_i) = 0$, $i = 1, 2$, and $0 < t_1 < t_2 < R$, $i = 1, 2$.*

We remark that Corollary 6.2 is the first rigorous result on the existence to (6.2) of positive solutions in dimension $N \geqslant 3$. Next we consider the existence of bound states. That is, we consider the following elliptic system in $\mathbb{R}^N$:

$$\begin{cases} \epsilon^2 \Delta a - a + \frac{a^p}{h^q} = 0 & \text{in } \mathbb{R}^N, \\ \Delta h - h + \frac{a^m}{h^s} = 0 & \text{in } \mathbb{R}^N, \\ a, h > 0, \quad a, h \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{6.17}$$

We have the following result.

THEOREM 6.3. (See [62].) *Let $N \geqslant 2$. Assume that there exist two radii $0 < r_1 < r_2 < +\infty$ such that*

$$M(r_1) M(r_2) < 0. \tag{6.18}$$

Then for ϵ sufficiently small, problem (6.17) *has a solution (a_ϵ, h_ϵ) with the following properties*:

(1) *a_ϵ, h_ϵ are radially symmetric,*

(2) $a_\epsilon(r) = \xi_\epsilon^{\frac{q}{p-1}} w(\frac{r - r_\epsilon}{\epsilon})(1 + o(1))$,

(3) $h_\epsilon(r) = \xi_\epsilon (G(r_\epsilon; r_\epsilon))^{-1} G(r; r_\epsilon)(1 + o(1))$, *where ξ_ϵ is defined at the following*

$$\xi_\epsilon = \left(\epsilon \left(\int_{\mathbb{R}} w^m \right) G(r_\epsilon; r_\epsilon) \right)^{\frac{(1+s)(p-1) - qm}{qm}} \tag{6.19}$$

and $r_\epsilon \in (r_1, r_2)$ satisfying $\lim_{\epsilon \to 0} M(r_\epsilon) = 0$.

Similarly we have the following corollary.

COROLLARY 6.4. *Assume that $N \geqslant 2$ and that the condition* (6.16) *holds. Then for ϵ sufficiently small, problem* (6.2) *has a radially symmetric bound state solution (a_ϵ, h_ϵ) which concentrates on a sphere $\{r = t_0\}$ where $M(t_0) = 0$.*

By using the same method, it is not difficult to generalize the results of Theorem 6.1 to other symmetric domains, such as annulus or the exterior of a ball. We omit the details.

Several interesting questions are left open. First, can multiple layered solutions to (6.2) exist? Second, it would be an interesting question to study the stability of these “ring-like” solutions. Numerical computations in two dimension indicate that the “ring-like” solutions constructed in Theorem 6.1 are unstable and will break into several spots due to angular fluctuations. Third, if we vary R from 0 to $+\infty$, what is the relation between the layered solution constructed in [52] for the single equation (2.4) and the solutions in Theorem 6.1?

7. Conclusions and remarks

In this chapter, I have surveyed the most recent results on the study of Gierer–Meinhardt system.

First, we consider the case $D = +\infty$. In this case, the state-state problem becomes a singularly perturbed elliptic Neumann problem (2.4). Using the LEM, we established various existence results on concentrating solutions. In particular, Theorem 2.5 gives a lower bound on the number of solutions to (2.4). Several interesting questions are associated with (2.4). First, is there a lower bound on the number of boundary spikes? What is the optimal bound on the number of solutions to (2.4)? The followings are just some related conjectures

CONJECTURE 1. *Suppose the mean curvature function $H(P)$ has l local minimum points. Then there is at least*

$$\frac{C}{\epsilon^{l(N-1)}}$$

number of boundary spikes to (2.4).

CONJECTURE 2. *Suppose the distance function $d(P, \partial\Omega)$ has l local maximum points. Then there is at least*

$$\frac{C}{\epsilon^{Nl}}$$

number of interior spikes to (2.4).

CONJECTURE 3. *Suppose we have the energy bound $J_\epsilon[u_\epsilon] \leqslant C\epsilon^m$ for some $m \leqslant N$. Assume that the concentration set $\Gamma_\epsilon = \{u_\epsilon > \frac{1}{2}\}$ is connected. Then the limiting set $\Gamma = \lim_{\epsilon\to 0} \Gamma_\epsilon$ has Hausdorff dimension $N - m$.*

Second, we consider the stability of spike solutions to the shadow system (2.2). By studying both small and large eigenvalues, we have completely characterized the stability (or instability) in the case of $r = 2$, $1 < p < 1 + \frac{4}{N}$ or $r = p + 1$. The study of the NLEP (3.52) is not complete yet. Many interesting questions are still open: the case of general r, the case of large τ, the uniqueness of Hopf bifurcation, etc. The non-linear metastability of interior spike solutions is studied in [6]. The stability of boundary spikes is studied in [32], through a formal approach. It can be proved that when $D > D_0(\epsilon) \gg 1$, the full Gierer–Meinhardt system converges to the shadow system [59,60,77,78]. However, the critical threshold $D_0(\epsilon)$ seems unknown.

Third, we consider the one- and two-dimensional Gierer–Meinhardt systems. For steady states, we established the existence of *symmetric* and *asymmetric* K-peaked spikes. In 1D, the bifurcation of asymmetric K-spikes occur when $D < D_K$. In 2D, the bifurcation of asymmetric K-spikes occur when $D \sim \log\frac{1}{\epsilon}$. We also obtain critical thresholds for the stability of K-peaked solutions: If $\epsilon \ll 1$ there are stability thresholds

$$D_1(\epsilon) > D_2(\epsilon) > D_3(\epsilon) > \cdots > D_K(\epsilon) > \cdots$$

such that if

$$\lim_{\epsilon \to 0} \frac{D_K(\epsilon)}{D} > 1$$

then the K-peaked solution is stable, and if

$$\lim_{\epsilon \to 0} \frac{D_K(\epsilon)}{D} < 1$$

then the K-peaked solution is unstable. In 1D, the critical threshold is $D_K \sim \frac{1}{K^2}$. In 2D, the critical threshold is $\frac{\log \frac{\sqrt{|\Omega|}}{\epsilon}}{2\pi K}$. In 1D, the *small* eigenvalues determine the critical thresholds, while in 2D, the *large* eigenvalues give the critical thresholds. An interesting question is to obtain the next order term in the critical threshold for 2D (which should be $O(1)$ and location-dependent). The dynamics of multiple spikes in 1D and 2D is completely open. In 1D, the dynamical equation for the positions of the spikes is a system of algebraic-differential-equations (ADE). A matched asymptotic analysis is given in [33]. In 2D, the dynamics of two well-separated spots is studied in [20] and it is shown that the two spots will repel each other, provided that the initial distance between the two spots is large enough. In a general two-dimensional domain, the dynamics of multiple spots should be governed by $\nabla F_D(\mathbf{P})$ or $\nabla F_0(\mathbf{P})$.

Finally, it is almost completely open as regards to three-dimensional Gierer–Meinhardt system. The main difficulty is the study of the coupled system (6.1) which requires some new insights. A layered bound state is constructed, but most likely it is unstable. An interesting question is to generalize Theorem 6.1 to general domains.

Although the analysis in this paper was carried out for the Gierer–Meinhardt system, the results can certainly be generalized to a much wide class of non-local reaction diffusion systems that have localized spike solutions.

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