Lecture 7

Adding marked points In order to count curves with condition imposed, we need a way of specifying that our stable maps satisfy the conditions. This is done by including marked points in the domain.

$$\overline{M}_{g,n}(X,\beta) = \left\{ \begin{array}{l} f:(C,x_1,\dots,x_n) \longrightarrow X, \quad C \text{ is connected, possibly nodal} \\ \text{curve of arithmetic genus } g, \; x_i \in C \text{ are } \underline{\text{distinct}} \text{ non-nodal} \\ \text{points}, \; f_*[C]=\beta \quad |\mathrm{Aut}(f:(C,x_1,\dots,x_n)\to X)| < \infty \end{array} \right\}$$

$$\mathrm{vdim}\, \overline{M}_{g,n}(X,\beta) = -K_X\cdot\beta + (\dim X - 3)(1-g) + n$$

stability $\Rightarrow$ every genus 0 collapsing component must have 3 or more special points (marked or nodal).

$\overline{M}_{g,n} = \overline{M}_{g,n}(pt,0)$ Deligne-Mumford moduli space of stable curves. Non-empty for $2g+n \geq 3$ smooth orbifold of dimension $3g-3+n$. $\overline{M}_{0,n}$ is a manifold.

examples $\overline{M}_{0,3} = pt$

upto isomorphism $(\mathbb{P}^1, x_1, x_2, x_3) \cong (\mathbb{P}^1, 0, 1, \infty)$

$\overline{M}_{0,4} = \mathbb{P}^1 \qquad M_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$

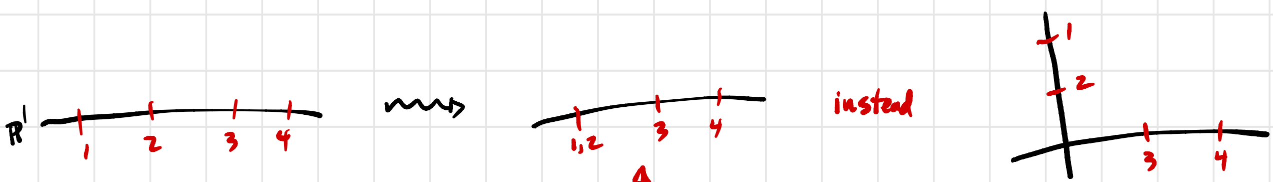
not $0, 1,$ or ∞ since $x_i \neq x_j$

given $(\mathbb{P}^1, x_1, x_2, x_3, x_4) \in M_{0,4}$ we get the cross-ratio: $\lambda = \dfrac{(x_3-x_1)(x_4-x_2)}{(x_3-x_2)(x_4-x_1)}$

cross-ratio is invariant under Mobius transformations and so gives

a well defined map $M_{0,4} \longrightarrow \mathbb{P}^1 - \{0,1,\infty\}$

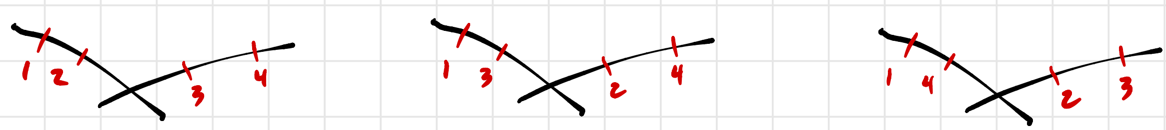
what happens when points come together?



not in moduli space

unique stable curve with this topological type.

$\overline{M}_{0,4} = \mathbb{P}^1$ the 3 points in the boundary $\overline{M}_{0,4} - M_{0,4}$ correspond to

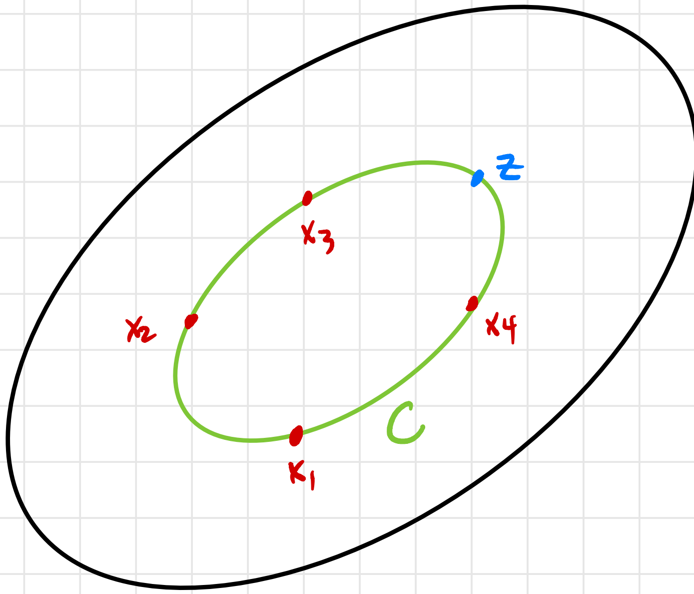


$\overline{M}_{0,5} = Bl_{4pts}\mathbb{P}^2$ to see this, choose 4 distinct points

$x_1, x_2, x_3, x_4 \in \mathbb{P}^2$, no 3 of which are colinear (up to automorphisms of $\mathbb{P}^2$ this is unique).

We define a map

$$Bl_{\{x_1, \dots x_4\}}\mathbb{P}^2 \longrightarrow \overline{M}_{0,5} \quad \text{as follows}$$

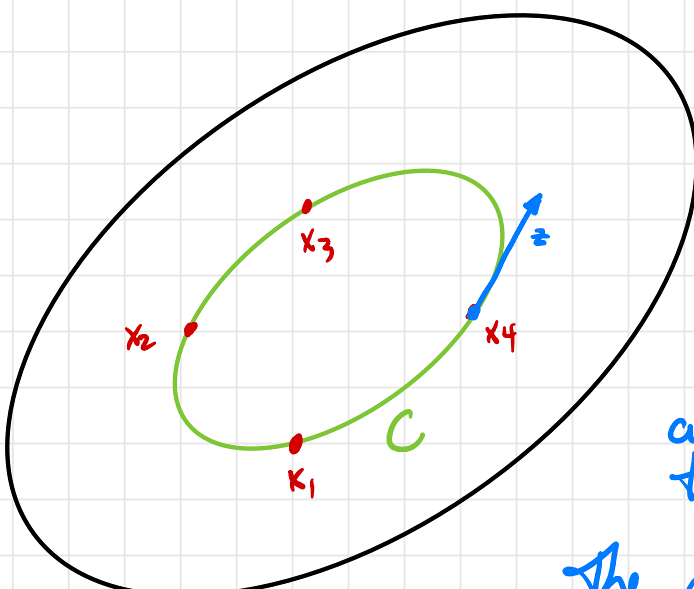


$Bl_{\{x_1,\dots,x_4\}}\mathbb{P}^2$

C is the unique conic passing through $(x_1,\dots,x_4,z)$

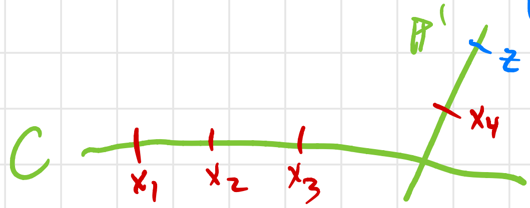
$$Bl_{\{x_1\dots x_4\}}\mathbb{P}^2 \longrightarrow \overline{M}_{0,5}$$

$$z \longmapsto (C, x_1,\dots,x_4,z)$$



When $z \to x_i$ the blow up includes $\mathbb{P}^1 = \mathbb{P}(T_{x_i}\mathbb{P}^2)$ i.e z is x_i plus a tangent direction.
C is then the unique conic passing through $x_1,\dots,x_4$ and having the specified tangent direction at x_i

The corresponding stable curve is



Try to convince yourself that this map is bijective. What do the boundary components look like and what do they correspond to in $Bl_{4pts}\mathbb{P}^2$?

Lecture 8

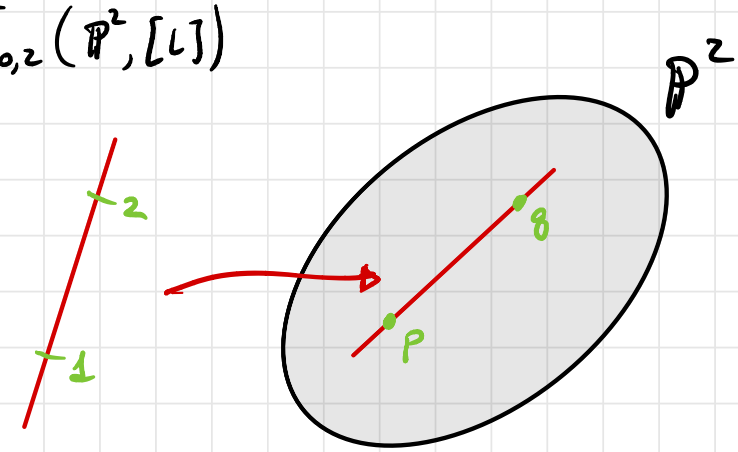
Gromov-Witten Invariants The moduli spaces with marked points have evaluation maps

$$\overline{M}_{g,n}(X,\beta) \xrightarrow{ev_1 \times \cdots \times ev_n} X \times \cdots \times X$$

$$[f:(C,x_1 \ldots x_n) \to X] \longmapsto (f(x_1), \ldots, f(x_n))$$

Suppose we wanted to count the number of lines passing through points p & $q \in \mathbb{P}^2$. We could look at

$$\underbrace{ev_1^{-1}(p) \cap ev_2^{-1}(q)}_{\substack{\text{maps such that} \\ \text{the first marked point} \\ \text{goes to } p \text{ and the second} \\ \text{to } q}} \subset \overline{M}_{0,2}(\mathbb{P}^2, [L])$$



More generally, if $A_1, \ldots, A_n \subset X$ are submanifolds and $ev_1^{-1}(A_1) \cap \cdots \cap ev^{-1}(A_n) \subset \overline{M}_{g,n}(X,\beta)$ is finite, then it is the number of genus g curves of degree β meeting the cycles $A_1, \ldots, A_n$.

Recall that intersection is dual to cup product under Poincare Duality.

If $\overline{M}_{g,n}(X,\beta)$ is a smooth manifold and $ev_i^{-1}(A_i)$ are submanifolds intersecting transversely, then the # of points in $\bigcap_i ev_i^{-1}(A_i)$ is given by

pairing between coh and homology

$$N^{GW}_{g,\beta}(A_1,\dots,A_n) = \int_{[\overline{M}_{g,n}(X,\beta)]} ev_1^*(PD(A_1)) \cup \cdots \cup ev_n^*(PD(A_n))$$

fundamental class $\in H_{2\dim \overline{M}_{g,n}(X,\beta)}(\overline{M}_{g,n}(X,\beta))$

The above makes sense even if $ev_i^{-1}(A_i)$ do not intersect transversely, but it does require the fundamental class $[\overline{M}_{g,n}(X,\beta)]$ which a priori requires $\overline{M}_{g,n}(X,\beta)$ to be smooth. We've seen examples where $\overline{M}_{g,n}(X,\beta)$ has multiple components of different dimensions and it isn't clear what we should do in that case.

<u>Theorem</u> There exists a class $[\overline{M}_{g,n}(X,\beta)]^{vir} \in H_*(\overline{M}_{g,n}(X,\beta);\mathbb{Q})$ (the <u>virtual fundamental class</u>) of degree $2\,\mathrm{virdim}(\overline{M}_{g,n}(X,\beta))$

This is not really a theorem without specifying the desired properties of this class, but lets vaguely say that it behaves "as if" it were the fundamental class. In particular, if $\overline{M}_{g,n}(X,\beta)$ is smooth and of the expected dim, then $[\ ]^{vir} =$ usual fund. class

also $$N^{GW}_{g,\beta}(A_1,\dots,A_n) := \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} ev_1^*(PD(A_1)) \cup \cdots \cup ev_n^*(PD(A_n))$$ $\boxed{\text{is a deformation invariant}}$

Here is the way to think of it:

Model case: suppose M is defined as the zero locus of a section of a vector bundle:

$$M = s^{-1}(0) \xhookrightarrow{i} Y \qquad \begin{array}{c} E \\ s \uparrow \downarrow \\ Y \end{array}$$

E ← rank r bundle on smooth ambient manifold Y of dim d.

Then M has expected dimension $d-r$:

If s is transverse to the zero section, then M is smooth and dim $d-r$. If s is not transverse, then M is singular and/or larger than expected dimension.

In the transverse case, the fundamental class satisfies

$$i_*[M] = \underbrace{PD(\underbrace{c_r(E)}_{\in H^{2r}(Y)})}_{\in H_{2d-2r}(Y)}$$

In general, the class $PD(c_r(E)) \in H_{2\text{vdim}}(Y)$ does still come from a class on M: there is a class $[M]^{vir} \in H_{2\text{vdim}}(M)$ s.t.

$$i_*[M]^{vir} = PD(c_r(E))$$

The coefs of $[\overline{m}_{g,n}(X,\beta)]^{vir}$ are in $\mathbb{Q}$ because even if it is smooth, it may be an orbifold (think manifold quotiented by a finite group).

$$N^{GW}_{g,\beta}(A_1 \cdots A_N) = 0 \quad \text{unless} \quad -K_X\cdot\beta + (\dim X - 3)(1-g) + n = \sum_i \operatorname{codim} A_i$$

$$\Rightarrow \quad -K_X\cdot\beta + (\dim X - 3)(1-g) = \sum_{i=1}^{n} (\operatorname{codim} A_i - 1)$$

each cycle A_i imposes $\operatorname{codim} A_i - 1$ conditions

e.g. • Divisors impose no conditions, intersection can be determined cohomologically
• each point on a surface imposes 1 condition.

On a CY3, virtual dim is always 0 so all we have are the invariants with no insertions:

$$N^{GW}_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{vir}} 1 \in \mathbb{Q} \longleftarrow \text{often not "enumerative", regard as a virtual count.}$$

GW invariants are the closest to the enumerative interpretation when $\overline{M}_{g,n}(X,\beta)$ is smooth and of the expected dimension. For example, if $g=0$ and $X=\mathbb{P}^N$ then $H^1(C,f^*TX)=0$ for all stable maps $[f:C\to\mathbb{P}^N]\in\overline{M}_{0,n}(\mathbb{P}^N,\beta)$ and so moduli space is smooth. Kontsevich used GW theory of $\mathbb{P}^2$ to solve for

$$N_d = \#\text{ rational curves of deg } d \text{ in } \mathbb{P}^2 \text{ passing through } 3d-1 \text{ points}$$

$$= N^{GW,\mathbb{P}^2}_{0,d[L]}(\underbrace{pt,\dots,pt}_{3d-1}) = \int_{[\overline{M}_{0,3d-1}(\mathbb{P}^2,d[L])]} ev_1^*(PD(pt))\cup\cdots\cup ev_{3d-1}^*(PD(pt))$$

$\longleftarrow$ no need for vir here

genus 0 GW invariants can be packaged together into Quantum Cohomology

Quantum Cohomology is a deformation of the usual cup product $H^*(X)\otimes H^*(X)\to H^*(X)$ which depends and various parameters and is built from genus 0 GW invariants. The fact that quantum cohomology is <u>associative</u> comes from a non-trivial relation among the GW invariants (the WDVV equation) which comes from $\overline{M}_{0,4}$.

Lecture 9

Genus 0 GW potential Assume $H^*(X;\mathbb{Q}) = H^{ev}(X;\mathbb{Q})$ for simplicity

Let $\underset{\nearrow 1}{T_0}, \underbrace{T_1, \cdots, T_p}_{H^2}, \cdots\cdot T_m \in H^*(X;\mathbb{Q})$ be a basis

H^2 choosen so $\beta_i := T_i \cdot \beta \in \mathbb{N}$ for any effective curve class β.

Let $g_{ij} = \int_X T_i \cup T_j$ Poincare pairing g^{ij} inverse matrix.

Let $\{T^j\}$ be dual basis: $T^j = g^{ij} T_i$ (summation convention).

We use "correlator" notation borrowed from physics:

$$N^{GW}_{0,\beta}(\alpha_1 \cdots \alpha_n) = \langle \alpha_1, \cdots, \alpha_n \rangle^X_{0,\beta}$$

These are symmetric in the entries and multilinear so we use monomial notation:

for example $\langle \underbrace{pt, \cdots, pt}_{3d-1} \rangle^{\mathbb{P}^2}_{0,d[L]} = \langle pt^{3d-1} \rangle^{\mathbb{P}^2}_{0,d[L]}$

or more generally

$$\langle \underbrace{T_0, \cdots, T_0}_{n_0}, \underbrace{T_1, \cdots, T_1}_{n_1}, \cdots, \underbrace{T_m \cdots T_m}_{n_m} \rangle^X_{0,\beta} = \langle T_0^{n_0} T_1^{n_1} \cdots T_m^{n_m} \rangle_{0,\beta}$$

Consider formal variables $t_0, \cdots, t_m$, $q_1, \cdots, q_p$ and let $\gamma = \sum_{i=0}^m t_i T_i$

Def'n The genus 0 GW potential is

$$F := \sum_\beta \langle \exp(\gamma) \rangle^X_{0,\beta}\, q^\beta \qquad \text{where } q^\beta := q_1^{\beta_1} \cdots q_p^{\beta_p} \qquad \beta_i = T_i \cdot \beta$$

$$F := \sum_{\beta} \langle \exp(\gamma) \rangle_{0,\beta}^{X} \, q^{\beta} \qquad \text{where } q^{\beta} := q_1^{\beta_1} \cdots q_p^{\beta_p} \qquad \beta_i = T_i \cdot \beta$$

$$= \sum_{\beta} \left\langle \prod_{i=0}^{n} e^{t_i T_i} \right\rangle_{0,\beta}^{X} q^{\beta}$$

$$= \sum_{\beta}' \left\langle \sum_{k_0, k_1, \dots, k_m}' \frac{t_0^{k_0} T_0^{k_0}}{k_0!} \cdot \ldots \cdot \frac{t_m^{k_m} T_m^{k_m}}{k_m!} \right\rangle_{0,\beta}^{X} q^{\beta}$$

$$= \sum_{\beta}' \sum_{k_0 \cdots k_m}' \langle T_0^{k_0} \cdots T_m^{k_m} \rangle_{0,\beta}^{X} \frac{t_0^{k_0} \cdots t_m^{k_m}}{k_0! \cdots k_m!} q^{\beta} \qquad \in \mathbb{Q}[\![q_1, \dots, q_p, t_0, \dots, t_m]\!]$$

generating function for all possible genus 0 GW invariants.

This is a formal power series encoding all possible genus 0 invariants. The variables t_i keep track of insertions, the variables q_i keep track of degree (homology class of the curve).

- Even though this is a formal function, in physics it has meaning as an actual function and so we expect some convergence properties.

The formalism is designed so that we can extract individual invariants by taking derivatives:

$$\frac{\partial^{\ell} F}{\partial t_{i_1} \cdots \partial t_{i_\ell}} = \sum_{\beta} \langle \exp(\gamma) T_{i_1} \cdots T_{i_\ell} \rangle_{0,\beta}^{X} q^{\beta} \qquad \text{so}$$

$$\left. \frac{\partial^{\ell} F}{\partial t_{i_1} \cdots \partial t_{i_\ell}} \right|_{t=0} = \sum_{\beta} \langle T_{i_1}, \dots, T_{i_\ell} \rangle_{0,\beta} \, q^{\beta}$$

What about $g=0$? The constant coef in the g's corresponds to the $\beta=0$ invariants

$$\langle \gamma_1, \dots, \gamma_n \rangle^X_{0,0} = \int_{[\overline{M}_{0,n}(X,0)]^{vir}} ev_1^*(\gamma_1) \cup \dots \cup ev_n^*(\gamma_n)$$

$\overline{M}_{0,n}(X,0) = X \times \overline{M}_{0,n}$ $\quad ev_i = \pi_X$ projection onto X

$\text{vdim}\, \overline{M}_{0,n}(X,0) = \dim X - 3 + n = \dim X + n - 3$

so $\overline{M}_{0,n}(X,0)$ is smooth and of the expected dim $\Rightarrow [\overline{M}_{0,n}(X,0)]^{vir} = [X \times \overline{M}_{0,n}]$

$$\langle \gamma_1, \dots, \gamma_n \rangle_{0,0} = \int_{[X \times \overline{M}_{0,n}]} \pi_X^*(\gamma_1) \cup \dots \cup \pi_X^*(\gamma_n)$$

$$= \begin{cases} 0 & \text{if } n \neq 3 \\ \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3 & \text{if } n = 3 \end{cases}$$

$$F(t,g)\big|_{g=0} = \frac{1}{6} \int_X (t_0 T_0 + \dots + t_m T_m)^3 \qquad \text{cubic polynomial}$$

$$F_{\alpha\beta\gamma} := \frac{\partial^3 F}{\partial t_\alpha \partial t_\beta \partial t_\gamma} \quad \text{satisfies} \quad F_{\alpha\beta\gamma}\big|_{g=0} = \int_X T_\alpha \cup T_\beta \cup T_\gamma$$

so $F_{0\beta\gamma}\big|_{g=0} = g_{\beta\gamma}$ (poincaré pairing) which allows us to recover cup product purely from F:

Lemma: $T_\alpha \cup T_\beta = F_{\alpha\beta\varepsilon}\big|_{q=0}\, g^{\varepsilon\varepsilon'} T_{\varepsilon'}$ (using summation convention)

proof: Since the poincare pairing is non-degenerate, to show the above holds it suffices to show equality holds after applying $\int_{[X]} T_\gamma \cup (-)$ to both sides for any T_γ:

$$\int_{[X]} T_\alpha \cup T_\beta \cup T_\gamma \overset{?}{=} F_{\alpha\beta\varepsilon}\big|_{q=0}\, g^{\varepsilon\varepsilon'} \int_{[X]} T_{\varepsilon'} \cup T_\gamma$$

$$= F_{\alpha\beta\varepsilon}\big|_{q=0}\, g^{\varepsilon\varepsilon'} g_{\varepsilon'\gamma} = F_{\alpha\beta\gamma}\big|_{q=0} \quad \text{true.}$$

Lecture 10

Def'n We define the quantum product $\star$ on $H^*(X) \otimes \mathbb{Q}[\![q_1, \dots q_r, t_0, \dots, t_m]\!]$ by the formula

$$T_\alpha \star T_\beta = F_{\alpha\beta\gamma}\, g^{\varepsilon\varepsilon'} T_{\varepsilon'}$$

Above shows that when $q=0$, $\star = \cup$. $\star$ is obviously commutative

Theorem $\star$ is an associative product.

$$(T_\alpha \star T_\beta) \star T_\gamma = (F_{\alpha\beta\varepsilon}\, g^{\varepsilon\varepsilon'} T_{\varepsilon'}) \star T_\gamma$$

$$= F_{\alpha\beta\varepsilon}\, g^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta}\, g^{\delta\delta'} T_{\delta'}$$

$$(T_\beta \star T_\gamma) \star T_\alpha = F_{\beta\gamma\varepsilon}\, g^{\varepsilon\varepsilon'} F_{\varepsilon'\alpha\delta}\, g^{\delta\delta'} T_{\delta'}$$

$\star$ Associative $\iff F_{\alpha\beta\varepsilon}\, g^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta} = F_{\beta\gamma\varepsilon}\, g^{\varepsilon\varepsilon'} F_{\varepsilon'\alpha\beta}$

$\iff \boxed{F_{\alpha\beta\varepsilon}\, g^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta} \text{ is symmetric in } (\alpha,\beta,\gamma,\delta)}$ $\star$

Theorem (*) Holds and is called the WDVV equation.

We will prove this by pulling back the obvious relation on $\overline{M}_{0,4}$. First

Let's prove some simple relations and study some examples.

String Equation: $\langle \alpha_1 \cdots \alpha_n T_0 \rangle_{0,\beta} = 0$ unless $\beta = 0$ $n = 2$

pf recall $T_0 = 1$ so

$$\langle \alpha_1 \cdots \alpha_n T_0 \rangle_{0,\beta} = \int_{[\overline{M}_{0,n+1}(X,\beta)]^{vir}} ev_1^*(\alpha_1) \cup \cdots \cup ev_n^*(\alpha_n) \cup \underbrace{ev_{n+1}^*(T_0)}_{1}$$

so integrand pulls back from $\overline{M}_{0,n}(X,\beta)$ whenever the "forgetful" map

$\overline{M}_{0,n+1}(X,\beta) \xrightarrow{\pi} \overline{M}_{0,n}(X,\beta)$ exists (which is always except when $\beta = 0, n = 2$)

If $\overline{M}_{0,n}(X,\beta)$ is of the expected dimension then $ev_1^*(\alpha_1) \cup \cdots \cup ev_n^*(\alpha_n)$ must be zero since it is a class of degree vdim $\overline{M}_{0,n+1}(X,\beta)$ but pulls back from $\overline{M}_{0,n}(X,\beta)$, a space of dimension 1 less. In general we need a property of the virtual class.

Only non-zero invariant $\langle \alpha_1 \cdots \alpha_n T_0 \rangle_{0,\beta}$ is for $n = 2$ $\beta = 0$ and $\langle \alpha_1 \alpha_2 T_0 \rangle_{0,0}^X = \int_{[X]} \alpha_1 \cup \alpha_2$

so $$\frac{\partial F}{\partial t_0} = \sum_\beta{}' \langle \exp(\gamma) T_0 \rangle_{0,\beta} q^\beta = \frac{1}{2} \langle \gamma^2 T_0 \rangle_{0,0} = \frac{1}{2} \int_{[X]} \gamma^2 = \frac{1}{2} \sum_{i,j} t_i t_j g_{ij}$$

so $F_{0\alpha\beta} = g_{\alpha\beta} \Rightarrow T_0$ is the identity for $*$.

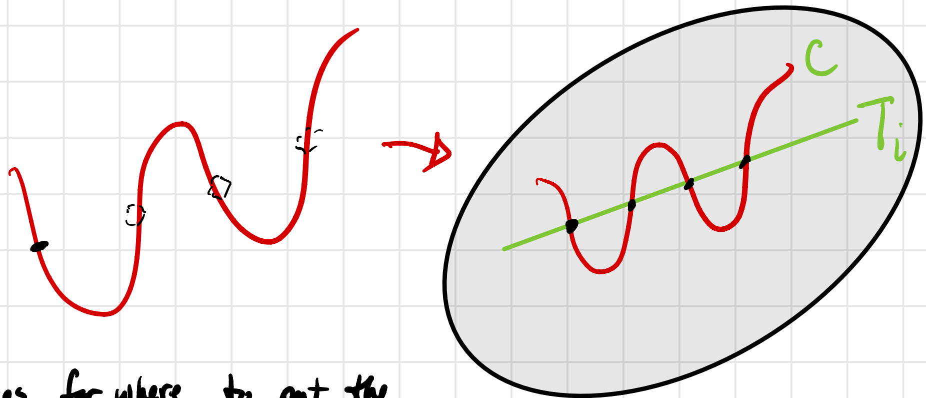
Divisor Equation for $i \in \{1, 2, \cdots, p\}$ $\langle \gamma^n, T_i \rangle_{0,\beta} = (T_i \cdot \beta) \langle \gamma^n \rangle_{0,0}$

divisor class

we called this β_i $\quad q^\beta = q_1^{\beta_1} \cdots q_p^{\beta_p}$

unless $n=2$ $\beta = 0$.

This formula is at least clear in the geometric interpretation of insersions.



$\beta \cdot T_i$ choices for where to put the $(n+1)$st marked point.

Divisor Equation $\Rightarrow$ $\dfrac{\partial F}{\partial t_i} = \sum_\beta' \langle \exp(\gamma) T_i \rangle_{0,\beta}\, q^\beta = \beta_i \sum_\beta \langle \exp(\gamma) \rangle_{0,\beta}\, q^\beta + \frac{1}{2} \langle \gamma^2 T_i \rangle_{0,0}$

$i \in \{1, \cdots, p\}$

$$= q_i \frac{\partial F}{\partial q_i} + \frac{1}{2} \int_{[X]} \gamma^2 T_i$$

$$\Rightarrow \left(\frac{\partial}{\partial t_i} - q_i \frac{\partial}{\partial q_i} \right) F_{\alpha\beta\gamma} = 0 \qquad i \in \{1, \cdots, p\}$$

Except for cubic terms in t when $g=0$, the dependence on q_i, t_i is as a function of $q_i e^{t_i}$

Lecture 11

__example__ $\mathbb{P}^1$ $\qquad T_0 = 1 \quad T_1 = [pt]^\vee$

$\qquad\qquad\qquad t_0 \qquad t_1 \qquad q$

$\beta = 0$ only non-zero invariant is $\langle T_0 T_0 T_1 \rangle_{0,0} = 1$

$\beta \neq 0 \quad \beta = d[\mathbb{P}^1] \qquad \langle T_1^\ell \rangle_{d[\mathbb{P}^1]} = d^\ell \langle \; \rangle_{d[\mathbb{P}^1]} = \begin{cases} 0 & d \neq 1 \\ 1 & d = 1 \end{cases}$ since vdim $\overline{M}_0(\mathbb{P}^1, d[\mathbb{P}^1]) = 2d-2$

so $$F = \frac{t_0^2 t_1}{2} + q \sum_\ell \frac{1}{\ell!} \langle T_1^\ell \rangle_{[\mathbb{P}^1]} t_1^\ell = \frac{t_0^2 t_1}{2} + q e^{t_1}$$

$F_{001} = 1$, $F_{111} = q e^{t_1}$, all other triple derivatives are zero.

$$T_1 * T_1 = F_{11\varepsilon} g^{\varepsilon\varepsilon'} T_{\varepsilon'} = \cancel{F_{110}}^{\,0} T_1 + F_{111} T_0 = q e^{t_1} T_0$$

Since T_0 = identity in ordinary and quantum coh. we write it as $\mathbb{1}$.

$$H^*(\mathbb{P}^1) \cong \mathbb{Q}[T_1]/T_1^2, \qquad QH^*(\mathbb{P}^1) = \mathbb{Q}[\![t_1, q]\!][T_1]/(T_1^2 - q e^{t_1})$$

"small quantum cohomology" is the $t \to 0$ limit

$$qH^*(\mathbb{P}^1) = \mathbb{Q}[T, q]/(T^2 - q)$$

we deformed a nilpotent ring to a semi-simple one.

Example $\mathbb{P}^2$

$H^*(\mathbb{P}^2) = \mathbb{Q}[H]/H^3$ generated by

$$\begin{array}{ccc} 1, & H, & H^2 = pt \\ T_0 & T_1 & T_2 \\ t_0 & t_1 & t_2 \\ & q & \end{array}$$

$\langle H, H, 1 \rangle_{0,0} = 1 \qquad \langle pt, 1, 1 \rangle = 1$

$$F = \frac{1}{2} t_0 t_1^2 + \frac{1}{2} t_0^2 t_2 + \sum_{d=1}^{\infty}{}' \sum_{n_1, n_2}{}' \langle H^{n_1} pt^{n_2} \rangle_{0, d[H]} \, q^d \frac{t_1^{n_1}}{n_1!} \frac{t_2^{n_2}}{n_2!}$$

$$= \frac{1}{2} t_0 t_1^2 + \frac{1}{2} t_0^2 t_2 + \sum_{d=1}^{\infty}{}' q^d \sum_{n_1}{}' \frac{d^{n_1} t_1^{n_1}}{n_1!} \langle pt^{3d-1} \rangle_{0, d[H]} \frac{t_2^{3d-1}}{(3d-1)!}$$

$$= \frac{1}{2} t_0 t_1^2 + \frac{1}{2} t_0^2 t_2 + \sum_{d=1}^{\infty} (q e^{t_1})^d \frac{t_2^{3d-1}}{(3d-1)!} N_d$$

$N_d = \langle pt^{3d-1} \rangle_{0, d[H]} = \#$ of degree d rational curves passing through $3d-1$ pts.

> **Homework:** Compute the quantum products $H * H$, $H * pt$, $pt * pt$.
>
> Show that $qH^*(\mathbb{P}^2) = \mathbb{Q}[H, q]/(H^3 - q)$

Associativity and Kontsevich's formula

Recall that associativity is equivalent to the expression

$(\alpha\beta|\gamma\delta) := F_{\alpha\beta\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta}$ being symmetric in the indices.

we study $(11|22) = (12|12)$ noting that $g_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = g^{ij}$

$$\underbrace{F_{110}}_{1} F_{222} + F_{111} F_{122} + F_{112} \underbrace{F_{022}}_{0} = \underbrace{F_{120}}_{0} F_{212} + F_{121} F_{112} + F_{122} \underbrace{F_{120}}_{0}$$

$$F_{222} = F_{112}^2 - F_{111} F_{122}$$

Let $Q = qe^{t_1}$ then

$$F_{222} = \sum_{d=2}^{\infty}{}' Q^d \frac{t_2^{3d-4}}{(3d-4)!} N_d \qquad F_{112} = \sum_{d=1}^{\infty}{}' N_d\, d^2 Q^d \frac{t_2^{3d-2}}{(3d-2)!}$$

$$F_{122} = \sum_{d=1}^{\infty}{}' N_d\, d Q^d \frac{t_2^{3d-3}}{(3d-3)!} \qquad F_{111} = \sum_{d=1}^{\infty}{}' N_d\, d^3 Q^d \frac{t_2^{3d-1}}{(3d-1)!}$$

Set $Q=1$ then $F_{222} = F_{112}^2 - F_{111}F_{122}$ becomes:

$$\sum_{d=2}^{\infty}{}' N_d \frac{t_2^{3d-4}}{(3d-4)!} = \sum_{d_1=1}^{\infty}{}' \sum_{d_2=1}^{\infty}{}' N_{d_1} N_{d_2} \left[\frac{d_1^2 d_2^2\, t_2^{3(d_1+d_2)-4}}{(3d_1-2)!\,(3d_2-2)!} - \frac{d_1^3 d_2\, t_2^{3(d_1+d_2)-4}}{(3d_1-1)!\,(3d_2-3)!} \right]$$

so $$N_d = \sum_{\substack{d_1+d_2=d \\ d_i>0}}{}' N_{d_1} N_{d_2} \left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right] \qquad \blacksquare$$

Lecture 12

Sketch of proof of WDVV eq'n in the case where $\overline{M}_{0,n}(X,\beta)$ is smooth $\forall n$.

Recall that we need to prove that the expression $(ij|kl) = F_{ija}\, g^{ab}\, F_{bkl}$ is symmetric in the indices (i,j,k,l)

Consider the integral

$$(*)_{ijkl,n,\beta} := \frac{1}{n!} \int_{[\overline{M}_{0,n+4}(X,\beta)]} \rho^*(pt^\vee)\, ev_1^*(T_i) \cup ev_2^*(T_j) \cup ev_3^*(T_k) \cup ev_4^*(T_l) \cup ev_5^*(\gamma) \cup \cdots \cup ev_{n+4}^*(\gamma)$$

assume smooth

$\gamma = \sum t_i T_i$

where $\rho : \overline{M}_{0,n+4}(X,\beta) \longrightarrow \overline{M}_{0,4}$ $\leftarrow \cong \mathbb{P}^1$

$$[f:(C,x_1,\dots,x_{n+4}) \to X] \longmapsto (C,x_1,\dots,x_4)_{st}$$

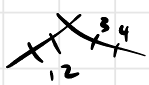
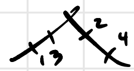
$(*)_{ijkl,n,\beta} \in \mathbb{Q}[[t_0 \cdots t_n]]$

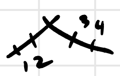
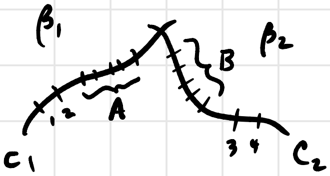
Since we are assuming that everything is smooth $\rho^*(pt^\vee) = (\rho^{-1}(pt))^\vee$ and so

$$(*)_{ijk\ell,n,\beta} = \frac{1}{n!}\int_{[\rho^{-1}(pt)]} ev_1^*(T_i)\cup\cdots\cup ev_4^*(T_\ell)\cup ev_5^*(\gamma)\cup\cdots\cup ev_{n+4}^*(\gamma)$$

Aside: we will use the following if $S\subset M$ then $\int_M S^{PD}\cup\alpha = \int_S \alpha$

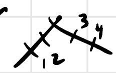
Since they are all homologous, we can choose any point in $\overline{M}_{0,4}\cong\mathbb{P}^1$, e.g.

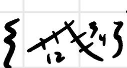
$pt = \{$  $\}$ or $pt = \{$  $\}$

$\rho^{-1}\{$  $\} = \{ f:$ 

$\longrightarrow X \}$ $\quad \beta_1+\beta_2=\beta$, $A\cup B = \{5,6,\dots,n+4\}$, $|A|+|B|=n$

$$\subset \bigcup_{\substack{A\cup B=\{5,\dots,n+4\}\\ \beta_1+\beta_2=\beta}} \overline{M}_{0,|A|+3}(X,\beta_1)\times\overline{M}_{0,n_2+3}(X,\beta_2)$$

Specifically $\rho^{-1}\{$  $\} = \bigcup_{\substack{A,B\\ \beta_1,\beta_2}} \left({}_Lev_{|A|+3}\times{}_Rev_{|B|+3}\right)^{-1}(\Delta)$ $\quad \Delta\subset X\times X$

Since everything is smooth $\rho^*(\{$  $\}^{PD}) = \sum_{\substack{A,B\\ \beta_1\beta_2}} \left({}_Lev_{|A|+3}\times{}_Rev_{|B|+3}\right)^*(\Delta^{PD})$

Fact in $H^*(X\times X) = H^*(X)\otimes H^*(X)$ $\quad \Delta^{PD} = T_a\otimes T^a$

$$(*)_{ijk\ell,\beta,n} = \sum_{\substack{A,B\\ \beta_1+\beta_2=\beta}}{}^{'} \frac{1}{n!}\int_{[\overline{M}_{0,|A|+3}(X,\beta_1)\times\overline{M}_{0,|B|+3}(X,\beta_2)]} {}_Lev^*_{|A|+3}(T_a)\cup{}_Rev^*_{|B|+3}(T^a)\cup{}_Lev_1^*(T_i)\cup{}_Lev_2^*(T_j)\cup{}_Lev_3^*(\gamma)\cup\cdots\cup{}_Lev^*_{|A|+3}(\gamma)$$
$$\cup\,{}_Rev_1^*(T_k)\cup{}_Rev_2^*(T_\ell)\cup{}_Rev_3^*(\gamma)\cup\cdots\cup{}_Rev^*_{|B|+3}(\gamma)$$

$$=\sum_{\substack{n_1+n_2=n\\ \beta_1+\beta_2=\beta}} \frac{1}{n!}\binom{n}{n_1}\int_{[\overline{M}_{0,n_1+3}(X,\beta_1)\times\overline{M}_{0,n_2+3}(X,\beta_2)]} {}_Lev^*_{n_1+3}(T_a)\cup{}_Rev^*_{n_2+3}(T^a)\cup{}_Lev_1^*(T_i)\cup{}_Lev_2^*(T_j)\cup{}_Lev_3^*(\gamma)\cup\cdots\cup{}_Lev^*_{n_1+3}(\gamma)$$
$$\cup\,{}_Rev_1^*(T_k)\cup{}_Rev_2^*(T_\ell)\cup{}_Rev_3^*(\gamma)\cup\cdots\cup{}_Rev^*_{n_2+3}(\gamma)$$

$$= \sum_{\substack{n_1+n_2=n \\ \beta_1+\beta_2=\beta}}' \frac{1}{n_1! n_2!} \langle \gamma^{n_1} T_i T_j T_a \rangle_{\beta_1} \langle \gamma^{n_2} T^a T_k T_\ell \rangle_{\beta_2}$$

$$\text{so} \quad \sum_{n,\beta}' (*)_{ijk\ell,n,\beta}\, q^\beta = \Big(\sum_{\beta_1} \langle \exp(\gamma) T_i T_j T_a \rangle_{\beta_1} q^{\beta_1} \Big)\Big(\sum_{\beta_2} \langle \exp(\gamma) \overset{g^{ab}T_b}{T^a} T_k T_\ell \rangle_{\beta_2} \Big)$$

$$= F_{ija}\, g^{ab}\, F_{bk\ell}$$

Since the original definition of $(*)_{ijk\ell,n,\beta}$ was symmetric in $(ijk\ell)$ we find the above is which proves the theorem.

Lecture 13

Return to the main targets of interest: CY3s. Since $\text{vdim}\, \overline{M}_g(X,\beta)=0$ there are no interesting insertions (we can stick in divisors but they are determined by zero insertion invariants). All the information is contained in the $t=0$ series.

Associativity in quantum cohomology (WDVV) tells us nothing:

$$\begin{array}{ccccccc} H^*(X) = & H^0(X) \oplus & H^2(X) & \oplus & H^4(X) & \oplus & H^6(X) \\ & T_0 & T_1 \cdots T_p & & T_1^\vee, \ldots, T_p^\vee & & 1^\vee \\ & 1 & \text{divisors} & & \text{curve classes} & & \text{pt.} \end{array}$$

Since $\text{vdim}\, \overline{M}_g(X,\beta)=0$ any curve insersion or pt insersion is zero if $\beta \neq 0$.

Thus F_{ijk} = const if any of i,j,k are indices corresponding to curves or points

$\int_{[X]} T_i \cup T_j \cup T_k \leftarrow$ only triple integrals involving pts or curves are $\int_{[X]} \text{pt}^\vee \cup 1 \cup 1$ or $\int_{[X]} \text{curve}^\vee \cup \text{divisor}^\vee \cup 1$ so has a 1

WDVV: $(ijk\ell) = F_{ija}\, g^{ab}\, F_{bk\ell} \longleftarrow$ either a or b will be a point or divisor index.

WDVV: $(ijk\ell) = F_{ija} g^{ab} F_{bk\ell}$ ← either a or b will be a point or divisor index.

$\Rightarrow$ we get 0 unless one of $ijk\ell$ is the 0 index, but then

$$(0\,ijk) = F_{0ia} g^{ab} F_{bjk} = g_{ia} g^{ab} F_{bjk} = F_{ijk}$$ so symmetry tells us nothing.

Since insertions tell us nothing new, we only use the $t=0$ series.

<u>**Def'n**</u> The genus g GW potential of a CY3 X is

$$F^g_X = \sum_\beta N_{g,\beta}(X)\, v^\beta \qquad N_{g,\beta}(X) = \langle\ \rangle^X_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{vir}} 1$$

$v_1^{\beta_1} \cdots v_p^{\beta_p}$ for $\beta_i = \beta \cdot T_i$ $\quad T_1, \dots, T_p$ pos. basis for $H^2(X)$

(previously g_i)

<u>**Def'n**</u> The all genus potential is $F_X = \sum_{g \geq 0}' F^g \lambda^{2g-2} \quad \in \mathbb{Q}[\![v]\!]((\lambda))$

Laurent series in λ, the string coupling constant.

<u>**Def'n**</u> The GW <u>partition function</u> is given by

$$Z = \exp(F)$$

<u>HW</u> Show that if we write

$$Z = \sum_\chi \sum_\beta N^\bullet_{\chi,\beta}\, \lambda^{-2\chi}\, v^\beta$$

Then $N^\bullet_{\chi,\beta}$ can be interpreted as <u>possibly disconnected</u> GW invariants:

$$N^\bullet_{\chi,\beta} = \int_{[\overline{m}^\bullet_\chi(X,\beta)]^{vir}} 1 \qquad \overline{M}^\bullet_\chi(X,\beta) : \left\{ f: C \to X,\ C \text{ is possibly disconnected, at worst nodal curve, } f_*[C] = \beta,\ |\mathrm{Aut}(f)| < \infty,\ \chi(\mathcal{O}_C) = \chi = \#\text{ of connected components} - \text{sum of genus} \right\}$$

The relationship between enumerative geometry (literal curve counting) and GW theory is very complicated on a CY3. To illustrate, we take some classical enumerative facts and study them in the context of GW theory.

Let $X = X_{(5)} \subset \mathbb{P}^4$ be a generic smooth quintic 3-fold

- There are 2875 lines on X (By Lefschetz hyperplane thm $H^2(X) \cong H^2(\mathbb{P}^4) \cong \mathbb{Z}$ so $H_2(X) \cong \mathbb{Z}$ generated by class of a line)
- There are 609,250 quadric curves (all genus 0)
- There are no curves of genus > 0 in degree 1 or 2.

What about the corresponding GW invariants? For $\beta = d[\text{line}]$ we just write d

$N_{g,d}(X)$ for $(g,d) = (0,1), (0,2), (1,1)$

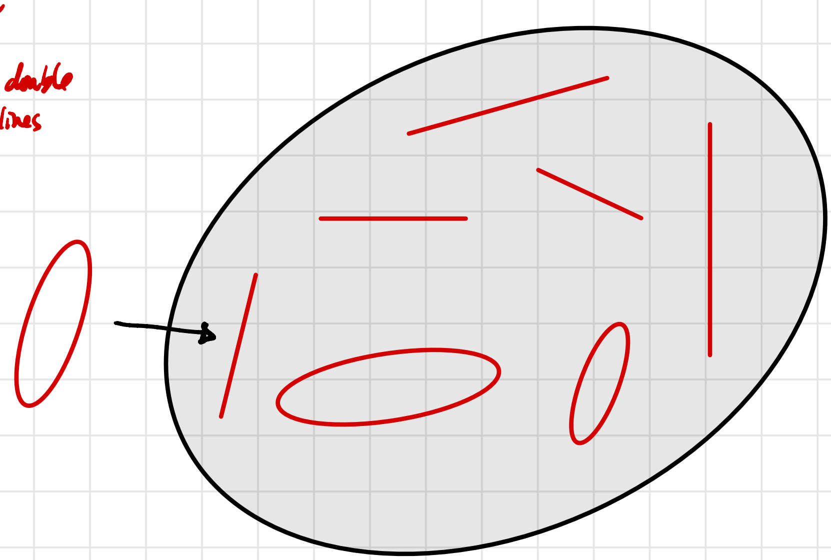
$N_{0,1} = 2875$ the maps must be isomorphisms onto their image

$$N_{0,2} = 609{,}250 + \underbrace{\tfrac{1}{8}(2875)}$$

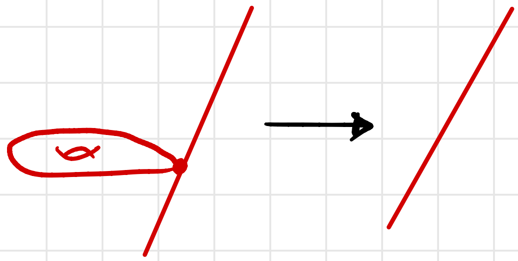
isomorphisms onto image where image is smooth conic curve

maps are double covers of lines

Contribution of the set of double covers of a fixed line is $\frac{1}{8}$ (not a trivial computation).



$N_{1,1} = \frac{1}{12}\cdot 2875$ (not zero) each of the lines contribute $\frac{1}{12}$ coming from maps



"degenerate map contribution"

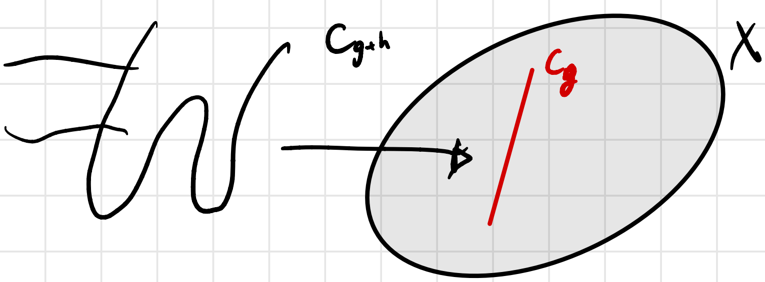
$\overline{M}_1(X,[\text{Line}]) = \underbrace{\overline{M}_{1,1}\times\mathbb{P}^1 \cup\cdots\cup \overline{M}_{1,1}\times\mathbb{P}^1}_{2875 \text{ copies}}$ on each component $[\overline{M}_{1,1}\times\mathbb{P}^1]^{vir} = \frac{1}{12}[pt]$.

GW invariants of a CY3 have contributions coming from multiple covers and degenerate maps (having collapsing components).

Lecture 14

Fundamental Problem what is the contribution of an isolated curve of genus g and degree β in X to $N_{g+h,\, d\beta}$ (and to what extent does the question make sense?).

Def'n A smooth genus g curve $C_g \subset X$ is called superrigid if $\forall\, h,d$ $\overline{M}_{g+h}(C_g, d[C])$ is a union of connected components of $\overline{M}_{g+h}(X, d[C])$. In otherwords $C_h \subset X$ doesn't deform and more over no multiple of C_g (i.e. multiple cover) deforms (even infinitesimally).



$[f: C_{g+h} \to C_g \subset X] \in \overline{M}_{g+h}(C_g, d[C_g]) \subset \overline{M}_{g+h}(X, d[C_g])$

Since $\overline{M}_g(C_0, d[C_0]) \subset \overline{M}_g(X, d[C_0])$ is a union of connected components it makes sense to restrict the virtual class

$$N^{GW}_{g+h,d}(C_g \subset X) = \int_{[\overline{M}_{g+h}(X, d[C_g])]^{vir}\big|_{[\overline{M}_{g+h}(C_g, d[C_g])}} 1$$

$$= \int_{[\overline{M}_{g+h}(C_g, d[C_g])]^{vir}} c_D(Ob)$$

$\longleftarrow$ virdim $= D = (2-2g)d + 2(g+h) - 2$

$= 2h + (2-2g)(d-1)$

$c_D(Ob)$ is D-th chern class of Obstruction sheaf. Fibers of obstruction sheaf are $H^1(C_{g+h}, f^* N_{C_g/X})$

Lecture 15

Example: if $C_0 \subset X$ has $C_0 \cong \mathbb{P}^1$ and $N_{C_0/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ then C_0 is super rigid.

local $\mathbb{P}^1$, a.k.a. resolved conifold

$$N_{h,d}(C_0 \subset X) = \int_{[\overline{M}_h(\mathbb{P}^1, d[\mathbb{P}^1])]^{vir}} c_D(Ob) \longleftarrow = N_{h,d}\left(\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\right)$$

This can be computed using the $\mathbb{C}^\times$ action: the $\mathbb{C}^\times$ action on target $\mathbb{P}^1$ induces an action of $\mathbb{C}^\times$ on the moduli space by composition. $\lambda \in \mathbb{C}^\times$ then

$$\lambda \cdot [f: C \to \mathbb{P}^1] = [C \xrightarrow{f} \mathbb{P}^1 \xrightarrow{\lambda} \mathbb{P}^1]$$

Integration on a smooth proj. manifold with a $\mathbb{C}^\times$ can be done by Atiyah-Bott localization. Integral can be computed purely by contributions from the $\mathbb{C}^\times$ fixed locus.

(pairing coh classes against the fundamental class)