UNLIKELY INTERSECTIONS IN ARITHMETIC DYNAMICS

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Combining ideas of Ihara-Serre-Tate, Lang [5] proved the following natural result. If a (complex, irreducible) plane curve $C \subset \mathbb{A}^2$ contains infinitely many points with both coordinates roots of unity, then C is the zero locus of an equation of the form $x^a y^b = \zeta$, where $a, b \in \mathbb{Z}$ and ζ is a root of unity. In other words, if $F \in \mathbb{C}[x, y]$ is an irreducible polynomial for which there exist infinitely many pairs (μ, ν) of roots of unity such that $F(\mu, \nu) = 0$, then (modulo multiplying by a constant) $F(x, y)$ is either a polynomial of the form $x^a - \zeta y^b$, or of the form $x^a y^b - \zeta$, where a and b are non-negative integers and ζ is a root of unity. In particular, Lang's result [5] provided the first instance when the Manin-Mumford Conjecture was proven: if a curve $C \subset \mathbb{G}_m^2$ contains a Zariski dense set of torsion points, then C is a torsion translate of a 1-dimensional torus. The proof of Ihara-Serre-Tate-Lang is a clever combination of various tools from mathematics (not only from number theory, but even basic complex analysis is used); for example, any graduate student in mathematics would benefit from reading their proof for the special case when the curve C is the graph of a polynomial.

The result of [5] is the first instance of the principle of *unlikely intersections* in arithmetic geometry since Lang's result may be interpreted as follows: if an *unlikely* event (such as the existence on the plane curve C of a point with both coordinates roots of unity) occurs infinitely often, then this must be explained by a *global*, geometric condition satisfied by C (in this case, C is a translate of a 1-dimensional algebraic subgroup of $\mathbb{G}_m^2$ by a torsion point). Other famous conjectures in number theory, such as the Mordell-Lang, the Bogomolov, or the André-Oort conjectures may be coined in the same terminology of unlikely intersections, which is also paraphrased as "*special points and special subvarieties*" since the *special points* in this case are the ones with both coordinates roots of unity and the only subvarieties containing a Zariski dense set of special points are the *special subvarieties*, which are torsion translates of algebraic subgroups.

It is also natural to formulate a dynamical analogue of Lang's result [5] by interpreting the roots of unity as preperiodic points under the action of the squaring map $z \mapsto z^2$. We recall that given a self-map f on any set X, a point a is preperiodic if its orbit under the action of $z \mapsto f(z)$ is finite, i.e., $f^m(a) = f^n(a)$ for some positive integers $m < n$ (where in dynamics, unless otherwise noted, f^ℓ always represents the ℓ-th compositional iterate of f for each positive integer ℓ). So, the dynamical reformulation of Lang's result yields the following: if a plane curve C contains infinitely many preperiodic points for the map $F : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ given by $F(x, y) = (x^2, y^2)$, then C must be preperiodic under the action of F. Furthermore, one can replace the squaring map by an arbitrary polynomial (or more generally a rational function) and ask whether the same result holds; this was conjectured in early 90's by Zhang, with the conjecture being formally published in [11]. After a series of partial results obtained by several authors, Zhang's conjecture was proven in [3] for all plane curves.

The Dynamical Manin-Mumford theorem for split endomorphisms of $(\mathbb{P}^1)^2$. *Let* $f \in \mathbb{C}[x]$ *be a rational function of degree* $d > 1$ *and let* $F : (\mathbb{P}^1)^2 \longrightarrow (\mathbb{P}^1)^2$ *be the endomorphism given by* $F(x, y) = (f(x), f(y))$. *If the curve* $C \subset (\mathbb{P}^1)^2$ *contains infinitely many points preperiodic under the action of* F*, then* C *itself must be preperiodic.*

Actually, in [3] we proved a more general result valid for the action of two *different* rational functions f_1 and f_2 (of same degree $d > 1$) on the two coordinate axes of $(\mathbb{P}^1)^2$; however, one needs to impose additional conditions when f_1 and f_2 are different Lattés maps of same degree corresponding to a CM elliptic curve (for more details, see [4]). Also, we note that the hypothesis that the two rational functions have the same degree (larger than 1) is absolutely necessary since otherwise there are obvious counterexamples, as seen by considering the diagonal line $\Delta \subset \mathbb{A}^2$ under the coordinatewise action of the two polynomials $z \mapsto z^2$ and respectively, $z \mapsto z^3$; indeed, Δ contains infinitely many points with both coordinates roots of unity, but Δ is not preperiodic under the action of $(x, y) \mapsto (x^2, y^3)$.

The Manin-Mumford conjecture (along with its further generalization, the Pink-Zilber conjecture) served as inspiration also for the following result of Masser-Zannier [7]. Given the Legendre family of elliptic curves: $y^2 = x(x-1)(x-t)$ parametrized by $t \in \mathbb{C} \setminus \{0, 1\}$, there exist at most finitely many $t_0 \in \mathbb{C}$ such that both the points $P_{t_0} := \left(2, \sqrt{2(2-t_0)}\right)$ and $Q_{t_0} := \left(3, \sqrt{6(3-t_0)}\right)$ are torsion for E_{t_0}. More generally, as shown in [8], given a curve C and an elliptic surface $\mathcal{E} \longrightarrow C$ endowed with two sections $P, Q : C \longrightarrow \mathcal{E}$, if there exist infinitely many $t \in C(\mathbb{C})$ such that both $P_t := P(t)$ and $Q_t := Q(t)$ are torsion points on the elliptic fiber E_t, then P and Q are linearly dependent (globally) on the elliptic surface. In their proof, Masser and Zannier use the results of Pila-Wilkie [6] regarding the number of rational points on an analytic curve. Intrinsic to the method employed in [7, 8] is the fact that there exists a global analytic uniformization map for the family of elliptic curves.

The result of Masser-Zannier inspired Baker and DeMarco [1] to prove the following (first) result about unlikely intersections in arithmetic dynamics. Given an integer $d > 1$ and given complex numbers a and b, if there exist infinitely many $t \in \mathbb{C}$ such that *both* a and b are preperiodic under the action of $z \mapsto z^d + t$, then $a^d = b^d$. In other words, if the unlikely event that a and b behave similarly from a dynamics perspective (i.e., having finite orbit) with respect to the same map $f_{t_0}(z) := z^d + t_0$ occurs infinitely often, then a and b are mapped to the same point globally, by the entire family of maps f_t. Several papers followed, finally settling the generalization of the result from [1] in which $f_t(z)$ is allowed now to be any algebraic family of polynomials parametrized by $\mathbb{P}^1$, while the starting points $a(t)$ and $b(t)$ are allowed to be polynomial in t (not necessarily constant); the conclusion is the same: the existence of infinitely many parameters $t_0 \in \mathbb{C}$ such that both $a(t_0)$ and $b(t_0)$ are preperiodic under the action of f_{t_0} yields that $a(t)$ and $b(t)$ satisfy an algebraic relation given by polynomials commuting (globally) with $f_t(z)$. If one attempts to generalize further the result from [1] to families $f_t(z)$ of rational functions parametrized by an arbitrary curve C (not necessarily $\mathbb{P}^1$), one encounters significant technical difficulties; see [2] for one of the very few known instances when such an extension was established. The strategy from [1, 2, 3] (along with all of the other papers on the new topic of unlikely intersections in arithmetic dynamics) employs powerful equidistribution theorems for points of small height with respect to metrized line bundles (see [10], for example) in order to prove that if there exist infinitely many parameters t_0 such that both $a(t_0)$ and $b(t_0)$ are preperiodic under the action of f_{t_0} (and also assuming neither $a(t)$ nor $b(t)$ is persistently preperiodic for the family $f_t(z)$), then for *each* parameter t, we have that $a(t)$ is preperiodic if and only if $b(t)$ is preperiodic. Finally, in order to go from the aforementioned *if and only if condition* to the *exact relation* between the starting points $a(t)$ and $b(t)$, one uses complex analysis (sometimes along with nontrivial real analysis) combined with the complete characterization of all invariant plane curves under a split polynomial action, as provided by Medvedev-Scanlon [9].

References

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