DENSITY OF ORBITS OF DOMINANT REGULAR SELF-MAPS OF SEMIABELIAN VARIETIES

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ABSTRACT. We prove a conjecture of Medvedev and Scanlon [MS14] in the case of regular morphisms of semiabelian varieties. That is, if G is a semiabelian variety defined over an algebraically closed field K of characteristic 0, and $\varphi\colon G \to G$ is a dominant regular self-map of G which is not necessarily a group homomorphism, we prove that one of the following holds: either there exists a non-constant rational fibration preserved by φ, or there exists a point $x \in G(K)$ whose φ-orbit is Zariski dense in G.

1. INTRODUCTION

For any self-map Φ on a set X, and any non-negative integer n, we denote by Φ^n the n-th compositional power, where Φ^0 is the identity map. For any $x \in X$, we denote by $\mathcal{O}_\Phi(x)$ its orbit under the action of Φ, i.e., the set of all iterates $\Phi^n(x)$ for $n \geq 0$.

Our main result is the following.

Theorem 1.1. *Let G be a semiabelian variety defined over an algebraically closed field K of characteristic 0 and let $\varphi : G \longrightarrow G$ be a dominant regular self-map which is not necessarily a group homomorphism. Then either there exists $x \in G(K)$ such that $\mathcal{O}_\varphi(x)$ is Zariski dense in G, or there exists a nonconstant rational function $f \in K(G)$ such that $f \circ \varphi = f$.*

Theorem 1.1 answers affirmatively the following conjecture raised by Medvedev and Scanlon in [MS14] for the case of regular morphisms of semiabelian varieties.

Conjecture 1.2 ([MS14, Conjecture 7.14])**.** *Let X be a quasiprojective variety defined over an algebraically closed field K of characteristic 0 and let $\varphi : X \dashrightarrow X$ be a rational self-map. Then either there exists $x \in X(K)$ whose orbit under φ is Zariski dense in X, or φ preserves a nonconstant fibration, i.e., there exists a nonconstant rational function $f \in K(X)$ such that $f \circ \varphi = f$.*

The origin of [MS14, Conjecture 7.14] lies in a much older conjecture formulated by Zhang in the early 1990's (and published in [Zha10, Conjecture 4.1.6]). Zhang asked that for each polarizable endomorphism φ of a projective variety X defined over $\overline{\mathbb{Q}}$ there must exist a $\overline{\mathbb{Q}}$-point with Zariski dense orbit under φ. Medvedev and Scanlon [MS14] conjectured that as long as φ does not preserve a nonconstant fibration, then a Zariski dense orbit must exist; the hypothesis concerning polarizability of φ already implies that no nonconstant fibration is preserved by φ. In [MS14], they also prove their conjecture in the special case $X = \mathbb{A}^n$ and φ is given by the coordinatewise action of n one-variable polynomials $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$;

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their result was established over an arbitrary field K of characteristic 0 which is not necessarily algebraically closed.

In [AC08], Amerik and Campana proved Conjecture 1.2 for all uncountable algebraically closed fields K (see also [BRS10] for a proof of the special case of this result when φ is an automorphism). In fact, Conjecture 1.2 is true even in positive characteristic, as long as the field K is uncountable (see [BGR17, Corollary 6.1]); on the other hand, when the transcendence degree of K over $\mathbb{F}_p$ is smaller than the dimension of X, there are counterexamples to the corresponding variant of Conjecture 1.2 in characteristic p (as shown in [BGR17, Example 6.2]).

With the notation as in Conjecture 1.2, it is immediate to see that if φ preserves a nonconstant fibration, then there is no Zariski dense orbit. So, the real difficulty in Conjecture 1.2 lies in finding a Zariski dense orbit for a self-map φ of X, when the algebraically closed field K is *countable*; in this case, there are only a handful of results known, as we will briefly describe below.

- In [ABR11], Conjecture 1.2 was proven assuming there is a point $x \in X(K)$ which is fixed by φ and moreover, the induced action of φ on the tangent space of X at x has multiplicatively independent eigenvalues.
- Conjecture 1.2 is known for varieties X of positive Kodaira dimension, see for example [BGRS, Proposition 2.3].
- In [Xie15], Conjecture 1.2 was proven for all birational automorphisms of surfaces (see also [BGT15] for an independent proof of the case of automorphisms). Later, Xie [Xie] established the validity of Conjecture 1.2 for all polynomial endomorphisms of $\mathbb{A}^2$.
- In [BGRS], the conjecture was proven all smooth minimal 3-folds of Kodaira dimension 0 with sufficiently large Picard number, contingent on certain conjectures in the Minimal Model Program.
- In [GS17], Conjecture 1.2 was proven for all abelian varieties.
- In [GX], it was proven that if Conjecture 1.2 holds for the dynamical system (X, φ), then it also holds for the dynamical system $(X \times \mathbb{A}^n, \psi)$, where $\psi : X \times \mathbb{A}^n \dashrightarrow X \times \mathbb{A}^n$ is given by $(x, y) \mapsto (\varphi(x), A(x)y)$ and $A \in \mathrm{GL}_n(K(X))$.

Our Theorem 1.1 extends the main result of [GS17] where Conjecture 1.2 was shown for abelian varieties. There are numerous examples in arithmetic geometry when one needed to overcome significant difficulties to extend a known result for abelian varieties to the case of semiabelian varieties: the case of non-split semiabelian varieties presented intrinsic complications in each of the classical conjectures of Mordell-Lang, Bogomolov, or Pink-Zilber. In the case of the Medvedev-Scanlon conjecture, the major technical obstacle we face is the absence of Poincaré's Reducibility Theorem: if A is an abelian variety and $B \subset A$ is an abelian subvariety, then there exists an abelian subvariety $C \subset A$ such that $A = B + C$ and $B \cap C$ is finite, i.e. A/B is isogenous to an abelian subvariety of A. The corresponding version of this result is false for semiabelian varieties. Since Poincaré's Reducibility Theorem is used throughout [GS17], our proof of Theorem 1.1 requires significant conceptual changes, specifically in the proofs of the main results of Subsections 3.1, 3.2, and 4. Also, the absence of Poincaré's Reducibility Theorem in the case of non-split semiabelian varieties G makes it impossible for one to use a similar strategy as in [GS17] in order to prove a generalization of Theorem 1.1 when the action of φ is replaced by the action of a finitely generated commutative monoid of regular self-maps on G; for more details, see Remark 4.4.

The plan of our paper is as follows. In Section 2 we introduce our notation and state the various useful facts about semiabelian varieties which we will employ in our proof. We continue in Section 3 by proving several reductions and auxilliary statements to be used in the proof of our main result. Finally, we conclude by proving Theorem 1.1 in Section 4.

2. Properties of semiabelian varieties

2.1. **Notation.** We start by introducing the necessary notation for our paper.

Let G_1 and G_2 be abelian groups and let $G = G_1 \times G_2$. By an abuse of notation, we identify G_1 as a subgroup of G through the inclusion map $x \mapsto (x, 0)$; similarly, we identiy G_2 with a subgroup of G through the inclusion map $x \mapsto (0, x)$. Also, viewing G as $G_1 \oplus G_2$, then for any $x_1 \in G_1$ and $x_2 \in G_2$, we use either the notation (x_1, x_2) or $x_1 \oplus x_2$ for the element $(x_1, x_2) \in G$. For any group G, we denote by G_{tor} its torsion subgroup; also, if G is abelian, then (unless otherwise noted) we denote its group operation by "+".

2.2. **Semiabelian varieties.** We continue by stating some useful facts regarding semiabelian varieties. Unless, otherwise noted, G denotes a semiabelian variety defined over an algebraically closed field K of characteristic 0.

The following structure result for regular self-maps on semiabelian varieties is proven in [NW16, Theorem 5.1.37].

Fact 2.1. *Let G_1 and G_2 be semiabelian varieties and let $\varphi : G_1 \longrightarrow G_2$. Then there exists a group homomorphism $\tau : G_1 \longrightarrow G_2$ and there exists $y \in G_2$ such that $\varphi(x) = \tau(x) + y$ for each $x \in G_1$.*

By definition (see [NW16, Definition 5.1.20] and [BBP16, Fact 2.4]), a semiabelian variety over K is a commutative algebraic group G over K for which there is an algebraic torus T, an abelian variety A, and a short exact sequence of algebraic groups over K:

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0 \tag{2.1.1}$$

We often say that T is the *toric part* of G, while A is the *associated abelian variety* of G. When the short exact sequence (2.1.1) splits, we say that G is a *split* semiabelian variety.

The next fact will be used several times in our proof.

Fact 2.2. *There is no nontrivial group homomorphism between an algebraic torus and an abelian variety.*

As a consequence, we have the following: suppose $\sigma\colon G_1 \longrightarrow G_2$ is a group homomorphism of semiabelian varieties and

$$0 \longrightarrow T_i \longrightarrow G_i \xrightarrow{p_i} A_i \longrightarrow 0$$

is a short exact sequence with T_i the toric part of G_i and A_i the associated abelian variety of G_i. Then $p_2(\sigma(T_1)) = 0$, so we have:

Fact 2.3. *Let G_1 and G_2 be semiabelian varieties with toric parts T_1 and T_2, respectively associated abelian varieties A_1 and A_2. Then for any group homomorphism*

$\sigma : G_1 \longrightarrow G_2$, the restriction $\sigma|_{T_1}$ induces a group homomorphism between T_1 and T_2; furthermore there is an induced group homomorphism $\overline{\sigma} : A_1 \longrightarrow A_2$.

Thus, we see that morphisms of semiabelian varieties induce morphisms of their corresponding tori and associated abelian varieties. There is a converse to this statement as well. If G is a semiabelian variety and $p\colon G \to A$ is the quotient map to its associated abelian variety, then p is a T-torsor, and hence G is the relative spectrum of $\bigoplus_{m\in M} \mathcal{L}_m$ where $\mathcal{L}_m$ are line bundles and M is the character lattice of T. One shows, see e.g. [Lan08, Corollary 3.1.4.4], that for all $m, m' \in M$ we have $\mathcal{L}_m \otimes \mathcal{L}_{m'} \simeq \mathcal{L}_{m+m'}$ and each $\mathcal{L}_m \in \mathrm{Pic}^0(A) = A^\vee$. In other words, we have a group homomorphism $c\colon M \to A^\vee$. If $\sigma\colon G' \to G$ is a group homomorphism, then from Fact 2.3 we have homomorphisms $\sigma|_{T'}\colon T' \to T$ and $\psi = \overline{\sigma}\colon A' \to A$ between the toric parts and associated abelian varieties. This in turn, induces a homomorphism $\phi\colon M \to M'$ between the character lattices of T and T', and a homomorphism $\psi^\vee\colon A^\vee \to (A')^\vee$ between dual abelian varieties. Via these constructions, we obtain an equivalence of categories:

Fact 2.4 ([Lan08, Proposition 3.1.5.1])**.** *The category of semiabelian varieties is anti-equivalent to the following category: objects are group homomorphisms $c\colon M \to A^\vee$ where M is a finitely generated free abelian group and A is an abelian variety; morphisms of objects $(c\colon M \to A^\vee) \to (c'\colon M' \to (A')^\vee)$ consist of commutative diagrams*

$$\begin{array}{ccc} M & \xrightarrow{c} & A^\vee \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi^\vee} \\ M' & \xrightarrow{c'} & (A')^\vee \end{array}$$

where ϕ is a group homomorphism and $\psi\colon A' \to A$ is a homomorphism of abelian varieties.

From Fact 2.4, we see that if G is a semiabelian variety corresponding to the homomorphism $c\colon M \to A^\vee$, then $\mathrm{End}(G)$ is the subring of $\mathrm{End}(T) \times \mathrm{End}(A)$ consisting of pairs (α, ψ) such that $c \circ \alpha^\vee = \psi^\vee \circ c$, where $\alpha^\vee \in \mathrm{End}(M)$ is the endomorphism of the character lattice induced by α. So we have the following.

Fact 2.5. *With the notation as in* (2.1.1)*, we let* $\mathrm{End}(T)$*,* $\mathrm{End}(G)$*, and* $\mathrm{End}(A)$ *be the endomorphism rings of the corresponding algebraic groups. Then the endomorphism ring* $\mathrm{End}(G)$ *embeds into* $\mathrm{End}(T) \times \mathrm{End}(A)$*. In particular,* $\mathrm{End}(G)$ *is a finitely generated $\mathbb{Z}$-module.*

Fact 2.6. *Let G be a semiabelian variety and $\varphi : G \longrightarrow G$ be a group homomorphism. Then there exists a monic polynomial $f \in \mathbb{Z}[z]$ of degree at most equal to $2\dim(G)$ such that $f(\varphi(x)) = 0$ for all $x \in G$.*

Moreover, for any $x \in G(K)$ and any regular self-map $\varphi : G \longrightarrow G$, the orbit $\mathcal{O}_\varphi(x)$ is contained in a finitely generated subgroup of G.

Proof. For the first part, by Fact 2.5 it is enough to show that each $(\phi, \psi) \in \mathrm{End}(T) \times \mathrm{End}(A)$ satisfies a monic polynomial of degree at most $2\dim(G)$. Letting $d = \dim(T)$, we have $\mathrm{End}(T) \xrightarrow{\sim} M_d(\mathbb{Z})$ the ring of d-by-d matrices with integer entries. Then the matrix corresponding to ϕ satisfies its characteristic polynomial $g(z)$ which has degree d. By [GS17, Fact 3.3], we know that ψ satisfies a monic polynomial $h(z)$ of degree at most $2\dim(A)$, so we can take $f = gh$.

We now prove the "moreover" statement. By Fact 2.3, there exists $y \in G(K)$ and $\tau \in \mathrm{End}(G)$ such that $\varphi(x) = \tau(x) + y$ for any $x \in G$. Then for all $n \in \mathbb{N}$, we have

$$\varphi^n(x) = \tau^n(x) + y + \tau(y) + \cdots + \tau^{n-1}(y).$$

Since there exists a monic polynomial $f \in \mathbb{Z}[z]$ of degree at most $2\dim(G)$ such that $f(\tau) = 0$, we conclude that $\mathcal{O}_\varphi(x)$ is contained in the finitely generated subgroup of $G(K)$ spanned by $\tau^i(x)$ and $\tau^i(y)$ for $0 \le i \le 2\dim(G) - 1$. $\square$

For each positive integer n, we let $G[n]$ be the group of torsion points of G killed by the multiplication-by-n map on G. Then, as shown in [BBP16, Fact 2.9], $G[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{\dim(T)+2\dim(A)}$ where T and A are the toric part, respectively the associated abelian variety of G; see (2.1.1). Therefore, similar to the case of abelian varieties (see [GS17, Fact 3.10]), we obtain the following result.

Fact 2.7. *Let G be a semiabelian variety defined over a field K_0 of characteristic 0. Then the group $\mathrm{Gal}(K_0(G_{\mathrm{tor}})/K_0)$ embeds as a closed subgroup of $\mathrm{GL}_{\dim(T)+2\dim(A)}(\widehat{\mathbb{Z}})$, where T and A are the toric part, respectively the associated abelian variety of G, and $\widehat{\mathbb{Z}}$ is the ring of finite adéles.*

The following result, proven by Faltings [Fal94] for abelian varieties and by Vojta [Voj96] for semiabelian varieties, was known as the Mordell-Lang conjecture.

Fact 2.8 (Vojta [Voj96]). *Let $V \subset G$ be an irreducible subvariety of the semiabelian variety G defined over an algebraically closed field K of characteristic 0. Assume there exists a finitely generated subgroup $\Gamma \subset G(K)$ such that $V(K) \cap \Gamma$ is Zariski dense in V. Then V is a coset of a semiabelian subvariety of G.*

Combining Fact 2.6 with Fact 2.8, we obtain:

Fact 2.9. *Let $\varphi : G \longrightarrow G$ be a self-map and let $x \in G(K)$. The Zariski closure of $\mathcal{O}_\varphi(x)$ is a finite union of cosets of semiabelian subvarieties of G.*

Proof. Using the "moreover" part provided by Fact 2.6, we see $\mathcal{O}_\varphi(x)$ is contained in a finitely generated subgroup Γ of $G(K)$. Letting V be the closure of $\mathcal{O}_\varphi(x)$ we see $V(K) \cap \Gamma$ is Zariski dense in V. Fact 2.8 then tells us that each irreducible component of V is a coset of a semiabelian subvariety of G, which finishes the proof. $\square$

Finally, we end with the following easy observation which will be used in Section 3.

Fact 2.10. *Let*

$$0 \longrightarrow T \longrightarrow G \xrightarrow{p} A \longrightarrow 0$$

be a short exact sequence of algebraic groups with T being a torus and A an abelian variety. If $H \subset G$ is an algebraic subgroup such that $A = p(H)$, then G/H is an algebraic torus.

Proof. We obtain the following diagram where the rows are short exact sequences and the vertical arrows are inclusions:

$$\begin{tikzcd}
1 \arrow[r] & H \cap T \arrow[r] \arrow[d, hook] & H \arrow[r] \arrow[d, hook] & H/(H \cap T) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & T \arrow[r] & G \arrow[r, "p"] & A \arrow[r] & 1
\end{tikzcd}$$

Since $p(H) = A$, we see $H/(H \cap T) = A$. So, we have an isomorphism $T/(H \cap T) \simeq G/H$ which finishes the proof since quotients of tori are tori. $\square$

3. Useful results

In following subsections, we prove several propositions which will then be used in order to derive Theorem 1.1.

3.1. **Minimal dominating semiabelian subvarieties.**

Lemma 3.1. *It suffices to prove Theorem 1.1 for a conjugate $\sigma^{-1} \circ \varphi \circ \sigma$ of the self-map $\varphi : G \longrightarrow G$ under some automorphism $\sigma \colon G \longrightarrow G$.*

Proof. This is [GS17, Lemma 5.4]; the proof goes verbatim not only when G is a semiabelian variety, but also for any quasiprojective variety. $\square$

Definition 3.2. *Let G be a semiabelian variety and*

$$0 \longrightarrow T \longrightarrow G \xrightarrow{p} A \longrightarrow 0 \tag{3.2.1}$$

the corresponding short exact sequence. We say $H \subset G$ is a minimal dominating semiabelian subvariety of G *if: (i) H is a semiabelian subvariety with $p(H) = A$ and (ii) for any semiabelian subvariety $H' \subset G$ with $p(H') = A$, we have $H \subset H'$.*

We show the existence of minimal dominating semiabelian subvarieties, after allowing for an isogeny.

Lemma 3.3. *For every semiabelian variety G, there exists an isogeny $f \colon G' \to G$ such that G' has a minimal dominating semiabelian subvariety.*

Moreover, if $G = G_1 \times G_2$ with the G_i semiabelian varieties, then there exist isogenies $f_i \colon G_i' \to G_i$ such that $G_1' \times G_2'$ has a minimal dominating semiabelian subvariety.

Proof. By Fact 2.4, the semiabelian variety G corresponds to a morphism $c \colon M \to A^\vee$ where M is the character lattice of T. To begin, notice that a semiabelian subvariety $H_0 \subset G$ has $p(H_0) = A$ (see (3.2.1)) if and only if it induces a diagram

$$\begin{tikzcd} 1 \arrow[r] & T_0 \arrow[r] \arrow[d, hook] & H_0 \arrow[r] \arrow[d, hook] & A \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & T \arrow[r] & G \arrow[r] & A \arrow[r] & 1 \end{tikzcd}$$

where the rows are short exact. By Fact 2.4, this is equivalent to factoring c as $M \to M_0 \to A^\vee$ with $M \to M_0$ a surjection of free abelian groups. Therefore, a minimal dominating semiabelian subvariety exists if and only if c factors as $M \xrightarrow{q} \overline{M} \xrightarrow{\overline{c}} A^\vee$ such that (i) q is a surjection of free abelian group, and (ii) for all factorizations $M \xrightarrow{q_0} M_0 \xrightarrow{c_0} A^\vee$ of c, there exists a surjection $q_1 \colon M_0 \to \overline{M}$ such that $q = q_1 \circ q_0$ and $c_0 = \overline{c} \circ q_1$. In particular, if the image $\mathrm{Im}(c)$ is torsion-free, then a minimal dominating semiabelian subvariety exists.

Since $\mathrm{Im}(c) \subset A^\vee$ is a subgroup, we see that the torsion part $\mathrm{Im}(c)_{\mathrm{tors}}$ is a finite subgroup of $A^\vee$. Let Γ be any finite subgroup of $A^\vee$ that contains $\mathrm{Im}(c)_{\mathrm{tors}}$. Then $\mathrm{Im}(c)_{\mathrm{tors}} = \mathrm{Im}(c) \cap \Gamma$. Since Γ is a finite subgroup, $\pi \colon A^\vee \to A^\vee/\Gamma$ is an isogeny of abelian varieties, and by construction the image of the map $\pi \circ c \colon M \to A^\vee/\Gamma$ is equal to $\mathrm{Im}(c)/\mathrm{Im}(c)_{\mathrm{tors}}$ which is torsion-free. Letting $A' = (A^\vee/\Gamma)^\vee$ and $\psi = \pi^\vee$, we have $\pi = \psi^\vee$ and $\psi\colon A' \to A$ is an isogeny, see e.g. [Mil, Theorem 9.1]. By Fact 2.4, we have a morphism of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & T \arrow[r] \arrow[d, equal] & G' \arrow[r] \arrow[d, "f"] & A' \arrow[r] \arrow[d, "\psi"] & 1 \\
1 \arrow[r] & T \arrow[r] & G \arrow[r] & A \arrow[r] & 1
\end{tikzcd}$$

where G' is defined by $\pi \circ c\colon M \to A^\vee/\Gamma = (A')^\vee$. We see then that f is an isogeny. Since the image of $\pi \circ c$ is torsion-free, G' has a minimal dominating semiabelian subvariety.

Finally, it remains to handle the case when $G = G_1 \times G_2$. Here, G_i is defined by a map $c_i\colon M_i \to A_i^\vee$ with M_i finitely generated free abelian groups and A_i abelian varieties. Then G is defined by the map $c = (c_1, c_2)\colon M_1 \oplus M_2 \to A_1^\vee \times A_2^\vee = (A_1 \times A_2)^\vee$. Then $\mathrm{Im}(c) = \mathrm{Im}(c_1) \oplus \mathrm{Im}(c_2)$ so $\mathrm{Im}(c)_{\mathrm{tors}} = \mathrm{Im}(c_1)_{\mathrm{tors}} \oplus \mathrm{Im}(c_2)_{\mathrm{tors}}$. We can then choose $\Gamma = \Gamma_1 \times \Gamma_2 \subset A_1^\vee \times A_2^\vee$ where $\Gamma_i \subset A_i^\vee$ is a finite subgroup containing $\mathrm{Im}(c_i)$. The resulting isogeny $\psi\colon A' \to A_1 \times A_2$ defined by Γ in the previous paragraph is then of the form $\psi = \psi_1 \times \psi_2$ where $\psi_i\colon A_i' \to A_i$ is the isogeny defined by Γ_i. $\square$

Lemma 3.4. *For $i = 1, 2$, let G_i be a semiabelian variety fitting into a short exact sequence*

$$0 \longrightarrow T_i \longrightarrow G_i \xrightarrow{p_i} A_i \longrightarrow 0$$

with T_i a torus and A_i an abelian variety. Let $G = G_1 \times G_2$ and $p = (p_1, p_2)\colon G \longrightarrow A_1 \times A_2$. If $H \subset G$ is an algebraic subgroup with $T_1 \subset H$ and $p(H) = A_1 \times A_2$, then $G_1 \subset H$.

Proof. To prove the lemma, it suffices to replace H by the connected component of the identity of H, and so we can assume H is a semiabelian subvariety of G. By Fact 2.4, we know that G_i corresponds to a group homomorphism $c_i\colon M_i \longrightarrow A_i^\vee$ where M_i is the character lattice of T_i. Then G corresponds to the homomorphism $c = (c_1, c_2)\colon M_1 \oplus M_2 \longrightarrow A_1^\vee \times A_2^\vee$. Since H is a semiabelian subvariety of G and $p(H) = A_1 \times A_2$, then as in the proof of Lemma 3.3, we know that H corresponds to a factorization c' of c through a quotient of $M_1 \oplus M_2$. Moreover since $T_1 \subset H$, the quotient is of the following form: there is a surjection $\pi\colon M_2 \longrightarrow M'$ and H corresponds to a group homomorphism $c'\colon M_1 \oplus M' \longrightarrow A_1^\vee \times A_2^\vee$ such that $c = c' \circ (\mathrm{id}, \pi)$ where id is the identity map on M_1. Consider the following diagram

$$\begin{tikzcd}
M_1 \oplus M_2 \arrow[r, "c"] \arrow[d, "(\mathrm{id},\pi)"] & A_1^\vee \times A_2^\vee \arrow[d, equal] \\
M_1 \oplus M' \arrow[r, "c'"] \arrow[d, "\pi'"] & A_1^\vee \times A_2^\vee \arrow[d, "q"] \\
M_1 \arrow[r, "c_1"] & A_1^\vee
\end{tikzcd}$$

where π' and q are the natural projections. Since (id, π) is surjective and $c_1 \circ \pi' \circ (\mathrm{id}, \pi) = q \circ c$, it follows that $c_1 \circ \pi' = q \circ c'$. Since c_1 corresponds to the semiabelian subvariety $G_1 \subset G$, we see $G_1 \subset H$. $\square$

Proposition 3.5. *Let G_1 and G_2 be semiabelian varieties defined over an algebraically closed field K of characteristic 0, let $G = G_1 \oplus G_2$ and $\pi_i \colon G \to G_i$ be the natural projection maps. If $\Gamma \subset G_2(K)$ is a finitely generated subgroup, then there exists $x_1 \in G_1(K)$ with the following property: for any proper algebraic subgroup $H \subset G$ and for any $\gamma \in \Gamma$, if $(x_1, \gamma) \in H$ then $\pi_2(H)$ is a proper algebraic subgroup of G_2.*

In our proof for Proposition 3.5 we will use the following related result.

Lemma 3.6. *Let T be an algebraic torus, let $T_0 \subset T$ be a subtorus, and let $\Gamma_0 \subset T(K)$ be a finitely generated subgroup. Then there exists $y_0 \in T_0(K)$ such that given any algebraic subgroup $H_0 \subset T$, if there exists $\gamma_0 \in \Gamma_0$ such that $y_0 \cdot \gamma_0 \in H_0(K)$ then $T_0 \subset H_0$.*

Proof. Since K is algebraically closed, then T splits and so, without loss of generality, we may assume $T = \mathbb{G}_m^n$ and $T_0 = \mathbb{G}_m^{n_0}$ for some integers $n_0 \le n$. We let $\Gamma_{0,0} \subset \mathbb{G}_m(K)$ be the finitely generated subgroup spanned by all the coordinates of a finite set of generators of Γ_0. Then we simply pick $y_0 := (y_{0,1}, \dots, y_{0,n_0}) \in \mathbb{G}_m^{n_0}(K)$ with the property that for any nontorsion $\gamma_{0,0} \in \Gamma_{0,0}$ (i.e., $\gamma_{0,0}$ is not a root of unity) we have that $y_{0,1}, \dots, y_{0,n_0}, \gamma_{0,0}$ are multiplicatively independent. Since $\Gamma_{0,0}$ has finite rank, while $\mathbb{G}_m(K)$ has infinite rank, we can always do this.

Now, any algebraic subgroup $H_0 \subset \mathbb{G}_m^n$ is the zero locus of finitely many equations of the form

$$x_1^{m_1} \cdot \dots \cdot x_n^{m_n} = 1. \tag{3.6.1}$$

for some integers $m_1, \dots, m_n$. Now, if there exists some $\gamma_0 \in \Gamma_0$ such that $y_0 \cdot \gamma_0 \in H_0(K)$, then (3.6.1) yields that

$$y_{0,1}^{m_1} \cdots y_{0,n_0}^{m_{n_0}} \in \Gamma_{0,0}. \tag{3.6.2}$$

Our choice of $y_{0,1}, \dots, y_{0,n_0}$ yields that $m_1 = \cdots = m_{n_0} = 0$; therefore $\mathbb{G}_m^{n_0} \subset H_0$, as desired. $\square$

Proof of Proposition 3.5. We first observe that it is enough to prove the desired conclusion when each G_i is replaced by a finite cover.

Lemma 3.7. *It suffices to prove Proposition 3.5 after replacing each G_i (for $i = 1, 2$) by a finite cover.*

Proof of Lemma 3.7. For each $i = 1, 2$, we let $\widetilde{G}_i$ be a semiabelian variety, let $\sigma_i : \widetilde{G}_i \longrightarrow G_i$ be an isogeny, and $\widetilde{G} := \widetilde{G}_1 \oplus \widetilde{G}_2$. We also let $\sigma := (\sigma_1, \sigma_2) : \widetilde{G} \longrightarrow G$ and $\widetilde{\pi}_i : \widetilde{G} \longrightarrow \widetilde{G}_i$ be the natural projections maps onto each coordinate.

Since σ_2 is an isogeny, $\widetilde{\Gamma} := \sigma_2^{-1}(\Gamma)$ is a finitely generated subgroup of $\widetilde{G}_2(K)$. We assume that the conclusion of Proposition 3.5 holds for $\widetilde{G} = \widetilde{G}_1 \oplus \widetilde{G}_2$ and $\widetilde{\Gamma}$. Thus there exists $\widetilde{x}_1 \in \widetilde{G}_1(K)$ such that for any proper algebraic subgroup $\widetilde{H}$ of $\widetilde{G}$, if there exists some $\widetilde{\gamma} \in \widetilde{\Gamma}$ with $(\widetilde{x}_1, \widetilde{\gamma}) \in H(K)$, then $\widetilde{\pi}_2(\widetilde{H})$ is a proper algebraic subgroup of $\widetilde{G}_2$. We claim that $x_1 := \sigma_1(\widetilde{x}_1) \in G_1(K)$ satisfies the conclusion of Proposition 3.5.

Indeed, assume there exists some proper algebraic subgroup H of G containing (x_1, γ) for some $\gamma \in \Gamma$. Then letting $\widetilde{H} := \sigma^{-1}(H)$, we see that $(\widetilde{x}_1, \widetilde{\gamma}) \in \widetilde{H}(K)$ and moreover, since σ is an isogeny, $\widetilde{H}$ is also a proper algebraic subgroup of $\widetilde{G}$. Using the property satisfied by $\widetilde{x}_1$, it follows that $\widetilde{\pi}_2(\widetilde{H})$ is a proper algebraic subgroup of $\widetilde{G}_2$. Since $\pi_2 \circ \sigma = \sigma_2 \circ \widetilde{\pi}_2$ and σ_2 is an isogeny, we see $\pi_2(H)$ must be a proper algebraic subgroup of G_2, as desired. $\square$

For $i = 1, 2$, we let

$$0 \longrightarrow T_i \longrightarrow G_i \xrightarrow{p_i} A_i \longrightarrow 0$$

be a short exact sequence where T_i is an algebraic tori and A_i is an abelian variety. We also let $T := T_1 \times T_2$ and let $p := (p_1, p_2) : G \longrightarrow A$, where $A := A_1 \times A_2$.

Using Lemmas 3.3 and 3.7, after replacing G_1 and G_2 by finite covers, if necessary, we can assume that G admits a minimal dominant semiabelian subvariety H_0.

We let $\overline{\Gamma} := p_2(\Gamma) \subset A_2(K)$. Then applying [GS17, Lemma 5.5], there exists $\overline{x}_1 \in A_1(K)$ with the following property: given any algebraic subgroup $\overline{H} \subset A = A_1 \oplus A_2$ for which there exists some $\overline{\gamma} \in \overline{\Gamma}$ with $(\overline{x}_1, \overline{\gamma}) \in \overline{H}$, we must have that $A_1 \subset \overline{H}$.

We let $x_{1,0} \in G_1(K)$ such that $p_1(x_{1,0}) = \overline{x}_1$. We let $f : G \longrightarrow G/H_0$; since $p(H_0) = A$, then G/H_0 is an algebraic torus by Fact 2.10. We let Γ' be the finitely generated subgroup of $G(K)$ spanned by $x_{1,0}$ and Γ and let $\Gamma_0 := f(\Gamma')$. We also let $U := G/H_0$ and $U_0 \subset U$ be the algebraic subtorus $f(T_1)$. According to Lemma 3.6, there exists $y_0 \in U_0(K)$ such that for any algebraic subgroup $V \subset U$, if there exists $\gamma_0 \in \Gamma_0$ such that $y_0 + \gamma_0 \in V(K)$ then we must have that $U_0 \subset V$. We let $t_0 \in T_1(K)$ such that $f(t_0) = y_0$; we show next that $x_1 := t_0 + x_{1,0}$ satisfies the conclusion of Proposition 3.5.

So, let $H \subset G = G_1 \oplus G_2$ be a proper algebraic subgroup containing (x_1, γ) for some $\gamma \in \Gamma$. We argue by contradiction and therefore assume $\pi_2(H) = G_2$. Since $p_1(x_1) = p_1(x_{1,0}) = \overline{x}_1$, we obtain that $p(H)$ is an algebraic subgroup of A containing $(\overline{x}_1, p_2(\gamma))$. Notice that $p_2(\gamma) \in \overline{\Gamma}$. If $p(H)$ were a proper subgroup of $A = A_1 \oplus A_2$, then the hypothesis satisfied by $\overline{x}_1$ shows that $\overline{\pi}_2(p(H))$ is a proper algebraic subgroup of A_2, where $\overline{\pi}_2 : A \longrightarrow A_2$ is the projection of $A = A_1 \oplus A_2$ onto its second factor. However, $p_2(\pi_2(H)) = \overline{\pi}_2(p(H))$ which contradicts our assumption that $\pi_2(H) = G_2$; it follows that $p(H) = A$. Using the minimality of H_0, we get that $H_0 \subset H$.

Next we consider the projection map $f : G \longrightarrow G/H_0 = U$. We have

$$f(x_1 \oplus \gamma) = f(t_0) + f(x_{1,0}) + f(\gamma) = y_0 + f(x_{1,0}) + f(\gamma) \in y_0 + \Gamma_0;$$

on the other hand, $x_1 \oplus \gamma \in H$ so $y_0 + f(x_{1,0}) + f(\gamma) = f(x_1 \oplus \gamma)$ is contained in the subgroup $V := f(H)$ of U. Our choice of y_0 yields that $U_0 = f(T_1) \subset V$; taking inverse images under f, we have $T_1 \subset H + H_0 = H$. Since we also know that $p(H) = A = A_1 \times A_2$, we see from Lemma 3.4 that $G_1 \subset H$.

Finally, since H is a proper algebraic subgroup of $G = G_1 \times G_2$ containing G_1 (as shown above) and also projecting dominantly onto G_2 under the natural projection map π_2 (according to our assumption), we obtain a contradiction. Therefore $\pi_2(H)$ must be a proper algebraic subgroup of G_2. This concludes our proof of Proposition 3.5. $\square$

3.2. **Constructing topological generators.** The following is the main result of this subsection.

Proposition 3.8. *Let K be an algebraically closed field of characteristic 0. Let $\psi : B \longrightarrow C$ be a group homomorphism of semiabelian varieties defined over K, and let $y \in C(K)$. If the algebraic subgroup generated by $\psi(B)$ and y is C itself,* *then there exists $x \in B(K)$ such that the Zariski closure of the cyclic subgroup generated by $\psi(x) + y$ is C.*

We first prove a variant of Proposition 3.8: when B is an algebraic torus, but the algebraic group generated by $\psi(B)$ and y is not necessarily equal to C. This result, proven in Proposition 3.9, will then be used to derive Proposition 3.8.

Proposition 3.9. *Let K be an algebraically closed field of characteristic 0, let T be an algebraic torus and C be a semiabelian variety. Let $\psi : T \longrightarrow C$ be a homomorphism of algebraic groups defined over K, and let $y \in C(K)$. Then there exists $x \in T(K)$ such that the Zariski closure of the cyclic subgroup generated by $\psi(x) + y$ is the algebraic group generated by $\psi(T)$ and y.*

Proof. Our argument follows the proof of [GS17, Lemma 5.1].

Let K_0 be a finitely generated subfield of K such that T, C, and ψ are defined over K_0, and moreover, $y \in C(K_0)$. So, without loss of generality, we may assume K is the algebraic closure of K_0.

We let $T = T_1 \oplus \cdots \oplus T_m$ be written as a direct sum of 1-dimensional algebraic tori; at the expense of replacing K_0 by a finite extension, we may assume each T_i is defined over K_0. Then

$$\psi(T) = \sum_{i=1}^{m} \psi(T_i)$$

and moreover, each $\psi(T_i)$ is either trivial or a 1-dimensional algebraic torus. Our strategy is to find an algebraic point $z_i \in \psi(T_i)$ such that if $z := \sum_{i=1}^{m} z_i$, then the Zariski closure of the cyclic group generated by $z + y$ is the algebraic group generated by $\psi(T)$ and y. If for some i we have that $\psi(T_i) = \{0\}$ is trivial, then we simply pick $z_i = 0$. Now consider those $i \in \{1, \dots, m\}$ such that $\psi(T_i)$ is nontrivial. For each such i, we will show there exist $z_i \in \psi(T_i)$ such that for any positive integer n we have

$$nz_i \notin (\psi(T_i))\left(K_0\left(C_{\mathrm{tor}}, z_1, \dots, z_{i-1}\right)\right). \tag{3.9.1}$$

Claim 3.10. If the above condition (3.9.1) holds for each $i = 1, \dots, m$ such that $\psi(T_i) \neq \{0\}$, then the Zariski closure of the cyclic group generated by $z + y$ is the algebraic subgroup generated by $\psi(T)$ and y.

Proof of Claim 3.10. First, we note that if (3.9.1) holds, then $z_i \neq 0$; therefore, $z_i = 0$ automatically implies that $\psi(T_i) = \{0\}$.

Now, assume there exists some algebraic subgroup $D \subset C$ (not necessarily connected) such that $z + y \in D(K)$. Let $i \leq m$ be the largest integer such that $z_i \neq 0$; then we have

$$z_i \in ((-y - z_1 - \cdots - z_{i-1}) + D) \cap \psi(T_i).$$

Assume first that $\psi(T_i) \cap D$ is a proper algebraic subgroup of $\psi(T_i)$. Since $\psi(T_i)$ is a 1-dimensional torus, we see $D \cap \psi(T_i)$ is a 0-dimensional algebraic subgroup of C; hence there exists a nonzero integer n such that $n \cdot (D \cap \psi(T_i)) = \{0\}$. Then nz_i is the only (geometric) point of the subvariety $n \cdot (((-y - z_1 - \cdots - z_{i-1}) + D) \cap \psi(T_i))$ which is thus rational over $K_0(C_{\mathrm{tor}}, z_1, \dots, z_{i-1})$. But by our construction,

$$nz_i \notin \psi(T_i)(K_0(C_{\mathrm{tor}}, z_1, \dots, z_{i-1}))$$

which is a contradiction. Therefore $\psi(T_i) \subset D$ if i is the largest index in $\{1, 2, \dots, m\}$ such that $z_i \neq 0$, or equivalently, if i is the largest index for which $\psi(T_i) \neq 0$.

Now note that $z+y=\sum_{j\le i} z_j+y$ and $z_i\in\psi(T_i)\subset D$ so $z'+y=z+y-z_i\in D$, where $z':=z_1+\cdots+z_{i-1}$. Repeating the exact same argument as above for the next positive integer $i_1<i$ for which $\psi(T_{i_1})\neq\{0\}$, and then arguing inductively we obtain that each $\psi(T_j)$ is contained in D, and therefore $\psi(T)\subset D$. But then $z\in\psi(T)\subset D$ and so, $y\in D$ as well, which yields that the Zariski closure of the cyclic group generated by $z+y$ is the algebraic subgroup of C generated by $\psi(T)$ and y, as desired. ☐

We just have to show that we can choose z_i satisfying (3.9.1). So, the problem reduces to the following: L is a finitely generated field of characteristic 0, φ is an algebraic group homomorphism between an algebraic torus U and some semiabelian variety C all defined over L, φ has finite kernel, and we want to find $x\in U(\overline{L})$ such that for each positive integer n, we have

$$n\varphi(x)\notin\varphi(U)\left(L\left(C_{\mathrm{tor}}\right)\right). \tag{3.10.1}$$

Indeed, with the above notation, $U:=T_i$ (for each $i=1,\dots,m$), L is the extension of K_0 generated by z_j (for $j=1,\dots,i-1$), and φ is the homomorphism ψ restricted to $U=T_i$ for which $\psi(T_i)$ is nontrivial.

Let d be the degree of the isogeny $\varphi':U\longrightarrow\varphi(U)\subset C$. In particular, this means that for each $z\in C(\overline{L})$ and each $x\in U(\overline{L})$ for which $\varphi(x)=z$ we have

$$[L(x):L]\le d\cdot[L(z):L]. \tag{3.10.2}$$

For any subfield $M\subset\overline{L}$, we let $M^{(d)}$ be the compositum of all extensions of M of degree at most equal to d.

Claim 3.11. Let L be a finitely generated field of characteristic 0, let C be a semiabelian variety defined over L, let $L_{\mathrm{tor}}:=L(C_{\mathrm{tor}})$, and let d be a positive integer. Then there exists a normal extension of $L_{\mathrm{tor}}^{(d)}$ whose Galois group is not abelian.

Proof of Claim 3.11. The proof is identical to the one from [GS17, Claim 5.3]. Note that $L(C_{\mathrm{tor}})$ is Hilbertian since we can still apply [Tho13, Theorem, p. 238] due to Fact 2.7. ☐

Claim 3.11 yields that there exists a point $x\in U(\overline{L})$ which is not defined over an abelian extension of $L\left(C_{\mathrm{tor}}\right)^{(d)}$; i.e., $nx\notin U\left(L\left(C_{\mathrm{tor}}\right)^{(d)}\right)$ for all positive integers n. Hence, $n\varphi(x)\notin\varphi(U)\left(L\left(C_{\mathrm{tor}}\right)\right)$ (see (3.10.2)), which concludes the proof of Proposition 3.9. ☐

Proof of Proposition 3.8. Let

$$0\longrightarrow T_1\longrightarrow B\xrightarrow{p_1}A_1\longrightarrow 0$$
$$0\longrightarrow T_2\longrightarrow C\xrightarrow{p_2}A_2\longrightarrow 0$$

be two short exact sequences of algebraic groups with T_i tori and A_i abelian varieties. We let $\overline{y}:=p_2(y)$. By Fact 2.3, the endomorphism $\psi:B\longrightarrow C$ induces an endomorphism of abelian varieties $\overline{\psi}:A_1\longrightarrow A_2$. Using [GS17, Lemma 5.1], we conclude that there exists $x_0\in A_1(K)$ such that the Zariski closure of the cyclic group generated by $\overline{\psi}(x_0)+\overline{y}$ equals the algebraic subgroup generated by $\overline{\psi}(A_1)$ and $\overline{y}$. Since the algebraic subgroup generated by $\psi(B)$ and y equals C, we conclude that the algebraic subgroup generated by $\overline{\psi}(A_1)$ and $\overline{y}$ equals A_2. So, the cyclic subgroup generated by $\overline{\psi}(x_0)+\overline{y}$ is Zariski dense in A_2.

Choose a point $x_1 \in B(K)$ such that $p_1(x_1) = x_0$ and let $y_1 := \psi(x_1)+y \in C(K)$. Using Proposition 3.9, we can find $t \in T_1(K)$ such that the Zariski closure H of the cyclic group generated by $\psi(t) + y_1$ is equal to the algebraic group generated by $\psi(T_1)$ and y_1. We claim that the point $x := x_1 + t$ satisfies the conclusion of Proposition 3.8. Since $\psi(x)+y = \psi(t)+\psi(x_1)+y = \psi(t)+y_1$, it therefore suffices to prove the following:

Lemma 3.12. *With the above notation, $H = C$.*

Proof of Lemma 3.12. We let U be the algebraic subgroup which is the Zariski closure of the cyclic group generated by y_1. By our choice of x_0, x_1 and t, we know that

(i) $\psi(T_1) \subset H$;
(ii) $U \subset H$; and
(iii) $p_2(U) = A_2$.

Statements (i) and (ii) follow directly from the definitions. Statement (iii) holds because $p_2(y_1) = p_2(\psi(x_1)) + p_2(y) = \overline{\psi}(p_1(x_1)) + p_2(y) = \overline{\psi}(x_0) + \overline{y}$ and by the fact that the Zariski closure of the cyclic group generated by $\overline{\psi}(x_0) + \overline{y}$ equals A_2. Our hypothesis that the algebraic subgroup generated by $\psi(B)$ and y is C itself yields that $\psi(B) + U = C$. Our goal is to show that $\psi(T_1) + U = C$.

Using property (iii) above and Fact 2.10, we see C/U is an algebraic torus. Since $\psi(B)/(\psi(B) \cap U) \simeq (\psi(B) + U)/U = C/U$, we see

$$\psi(B)/(\psi(B) \cap U) \text{ is an algebraic torus.} \tag{3.12.1}$$

Since $\psi(T_1)$ is the toric part of $\psi(B)$, we obtain that

$$\psi(T_1)/(\psi(T_1) \cap U) \text{ is the toric part of } \psi(B)/(\psi(B) \cap U). \tag{3.12.2}$$

Equations (3.12.1) and (3.12.2) yield that

$$\psi(B)/(\psi(B) \cap U) \xrightarrow{\sim} \psi(T_1)/(\psi(T_1) \cap U) \text{ and therefore,} \tag{3.12.3}$$

$$(\psi(B) + U)/U \xrightarrow{\sim} (\psi(T_1) + U)/U. \tag{3.12.4}$$

Equation (3.12.4) yields that $\dim(\psi(B) + U) = \dim(\psi(T_1) + U)$ and because $C = \psi(B) + U$ is connected, we conclude that $H = \psi(T_1) + U = C$, as desired. $\square$

This concludes our proof of Proposition 3.8. $\square$

3.3. Conditions to guarantee the existence of a Zariski dense orbit.

Lemma 3.13. *Let K be an algebraically closed field of characteristic 0, let G be a semiabelian variety defined over K, let $y_1, \dots, y_r \in G(K)$, and let $P_1, \dots, P_r \in \mathbb{Q}[z]$ such that $P_i(n) \in \mathbb{Z}$ for each $n \geq 1$ and for each $i = 1, \dots, r$, while $\deg(P_r) > \cdots > \deg(P_1) > 0$. For an infinite subset $S \subset \mathbb{N}$, let $V := V(S; P_1, \dots, P_r; y_1, \dots, y_r)$ be the Zariski closure of the set*

$$\{P_1(n)y_1 + \cdots + P_r(n)y_r : n \in S\}.$$

Then there exist nonzero integers $\ell_1, \dots, \ell_r$ such that V contains a coset of the subgroup Γ generated by $\ell_1 y_1, \dots, \ell_r y_r$.

Proof. The proof is almost identical with the proof of [GS17, Lemma 5.6]; however, since that proof employed (though, in a non-essential way) Poincaré's Reducibility Theorem for abelian varieties, we include a proof for our present lemma in the context of semiabelian varieties which, of ocurse, does not use the Poincaré's Reducibility Theorem.

Let Γ_0 be the subgroup of G generated by $y_1, \dots, y_r$. Since $V(K) \cap \Gamma_0$ is Zariski dense in V, then by Fact 2.8 we see that V is a finite union of cosets of algebraic subgroups of G. So, at the expense of replacing S by an infinite subset, we may assume $V = z + C$, for some $z \in G(K)$ and some irreducible algebraic subgroup C of G. Hence $\{-z + P_1(n)y_1 + \cdots + P_r(n)y_r\}_{n\in S} \subset C(K)$. We will show there exist nonzero integers ℓ_i such that $\ell_i y_i \in C(K)$ for each $i = 1, \dots, r$.

We proceed by induction on r. We first handle the base case when $r = 1$. Then $\{P_1(n)\}_{n\in S}$ takes infinitely many distinct integer values as $\deg(P_1) \geq 1$, and in particular there exist $n_0, n \in S$ with $\ell := P_1(n) - P_1(n_0)$ non-zero. Since $C(K)$ is a subgroup of $G(K)$, we see $\ell y_1 = (-z + P_1(n)y_1) - (-z + P_1(n_0)y_1) \in C(K)$.

Next let $s \geq 2$. Assume the statement holds for all $r < s$, we prove it for $r = s$. Let $n_0 \in S$. Letting $P_i' := P_i - P_i(n_0)$, we see $\{P_1'(n)y_1 + \cdots + P_s'(n)y_s\}_{n\in S} \subset C(K)$. Since $\deg(P_1') \geq 1$ there exists $n_1 \in S$ such that $P_1'(n_1) \neq 0$. For each $i = 2, \dots, s$ we let

$$Q_i(z) := P_1'(n_1)P_i'(n) - P_1'(n)P_i'(n_1).$$

Since $C(K)$ is a subgroup of $G(K)$ and $\sum_{i=2}^s P_i'(n)y_i \in C(K)$ it follows that $\sum_{i=2}^s P_i'(n)P_1'(n_1)y_i \in C(K)$. Similarly, $\sum_{i=2}^s P_1'(n)P_i'(n_1)y_i \in C(K)$. Subtracting, we have

$$\left\{\sum_{i=2}^s Q_i(n)y_i\right\}_{n\in S} \subset C(K).$$

Since $\deg(Q_i) = \deg(P_i)$ for each $i = 2, \dots, s$, we can use the induction hypothesis and conclude that there exist nonzero integers $\ell_2, \dots, \ell_s$ such that $\ell_i y_i \in C(K)$ for each $i \geq 2$. Let $\ell_1 := P_1'(n_1) \cdot \prod_{i=2}^s \ell_i$ which is non-zero since $P_1'(n_1)$ is. Since $P_1'(n_1)y_1 + \cdots + P_s'(n_1)y_s \in C(K)$, we see $\ell_1 y_1 = \left(P_1'(n_1) \cdot \prod_{i=2}^s \ell_i\right) y_1 \in C(K)$. This concludes our proof. $\square$

Lemma 3.13 has the following important consequence for us.

Lemma 3.14. *Let K be an algebraically closed field of characteristic 0, let G be a semiabelian variety defined over K, let $\tau \in \mathrm{End}(G)$ with the property that there exists a positive integer r such that $(\tau - \mathrm{id})^r = 0$, let $y \in G(K)$, let $\varphi : G \longrightarrow G$ be a self-map such that $\varphi(x) = \tau(x) + y$ for each $x \in G$.*

Let $x \in G(K)$ and let $c + C$ be a coset of an algebraic subgroup $C \subset G$ with the property that there exists an infinite set S of positive integers such that $\{\varphi^n(x) \colon n \in S\} \subset c + C$. Then there exists a positive integer ℓ such that $\ell \cdot (\beta(x) + y) \in C(K)$, where $\beta := \tau - \mathrm{id}$.

Moreover, if the cyclic group generated by $\beta(x) + y$ is Zariski dense in G, then $C = G$ and therefore, the set $\{\varphi^n(x) \colon n \in S\}$ is Zariski dense in G.

Proof. The proof is identical with the derivation of [GS17, Lemma 5.7] from [GS17, Lemma 5.6]; this time, one employs Lemma 3.13 in order to derive the desired conclusion.

For the "moreover" part in Lemma 3.14, one argues as in the proof of [GS17, Corollary 5.8]; note that if a cyclic subgroup of $G(K)$ is Zariski dense, then any infinite subgroup of it is also Zariski dense (see also [GS17, Lemma 3.9]). $\square$

4. Proof of our main result

Proof of Theorem 1.1. By Fact 2.1, there exists a dominant group endomorphism $\tau : G \longrightarrow G$, and there exists $y \in G(K)$ such that $\varphi(x) = \tau(x) + y$ for all $x \in G$. By [BGRS, Lemma 2.1], it suffices to prove Theorem 1.1 for an iterate φ^n with $n > 0$. Replacing φ by φ^n replaces y by $\sum_{i=0}^{n-1} \tau^i(y)$ and τ by τ^n. As a result, we may assume

$$\dim \ker(\tau^m - \mathrm{id}) = \dim(\ker(\tau - \mathrm{id})) \text{ for all } m \in \mathbb{N}. \tag{4.0.1}$$

Letting $f \in \mathbb{Z}[t]$ be the minimal polynomial of $\tau \in \mathrm{End}(G)$, we may therefore assume that 1 is the only root of unity which is a root of f.

Let r be the order of vanishing at 1 of f, and let $f_1 \in \mathbb{Z}[t]$ such that $f(t) = f_1(t) \cdot (t-1)^r$. Then f_1 is also a monic polynomial. Let $G_1 := (\tau - \mathrm{id})^r(G)$ and let $G_2 := f_1(\tau)(G)$, where $f_1(\tau) \in \mathrm{End}(G)$ and id is the identity map on G. By definition, both G_1 and G_2 are connected algebraic subgroups of G, hence they are both semiabelian subvarieties of G. By definition, the restriction $\tau|_{G_1} \in \mathrm{End}(G_1)$ has minimal polynomial equal to f_1 whose roots are not roots of unity. On the other hand, $(\tau - \mathrm{id})^r|_{G_2} = 0$. Furthermore, as shown in [GS17, Lemma 6.1],

$$G = G_1 + G_2 \text{ and } G_1 \cap G_2 \text{ is finite.} \tag{4.0.2}$$

Even though [GS17, Lemma 6.1] was written in the context of abelian varieties, it uses no specific properties of abelian varieties; instead it is valid for any commutative algebraic group. So, G is isogenuous with the direct product $G_1 \times G_2$.

We let $y_1 \in G_1$ and $y_2 \in G_2$ such that $y = y_1 + y_2$. We denote by τ_i the induced action of τ on each G_i. Since the minimal polynomial f_1 of $\tau_1 \in \mathrm{End}(G_1)$ does not have the root 1, it follows that $(\mathrm{id} - \tau_1) : G_1 \longrightarrow G_1$ is an isogeny. As a result, there exists $y_0 \in G_1(K)$ such that $(\mathrm{id} - \tau_1)(y_0) = y_1$. Using Lemma 3.1, it suffices to prove Theorem 1.1 for $T_{-y_0} \circ \varphi \circ T_{y_0}$, where T_z represents the translation-by-z automorphism of G (for any given point $z \in G$). We may therefore assume that $y_1 = 0$.

Let $\varphi_i : G_i \longrightarrow G_i$ be given by $\varphi_1(x) = \tau_1(x)$ and $\varphi_2(x) = \tau_2(x) + y_2$; then for each $x_1 \in G_1(K)$ and $x_2 \in G_2(K)$ we have that

$$\varphi(x_1 + x_2) = \varphi_1(x_1) + \varphi_2(x_2). \tag{4.0.3}$$

We let $\beta := (\tau_2 - \mathrm{id})|_{G_2} \in \mathrm{End}(G_2)$; then $\beta^r = 0$. Let B be the Zariski closure of the subgroup of G_2 generated by $\beta(G_2)$ and y_2; then B is an algebraic subgroup of G_2.

Lemma 4.1. *Assume $B \neq G_2$. Then $C := G_1 + B$ is a proper algebraic subgroup of G and moreover, letting $f : G \longrightarrow G/C$ be the natural quotient homomorphism, we have that $f \circ \varphi = f$.*

Proof of Lemma 4.1. Since G_2 is connected and B is assumed to be a proper algebraic subgroup, we have $\dim(B) < \dim(G_2)$. As a result, (4.0.2) tells us that $C = G_1 + B$ is also a proper algebraic subgroup of G. Then the quotient map $f : G \longrightarrow G/C$ is a dominant morphism to a nontrivial semiabelian variety and moreover, we claim that $f \circ \varphi = f$. Indeed, for each $x \in G$, we let $x_i \in G_i$ (for $i = 1, 2$) such that $x = x_1 + x_2$ (see (4.0.2)) and then we get

$$\begin{aligned} f(\varphi(x)) &= f(\varphi_1(x_1) + \varphi_2(x_2)) \text{ by (4.0.3)} \\ &= f(\varphi_2(x_2)) \text{ because } \varphi_1(x_1) \in G_1 \subset C \\ &= f(x_2 + \beta(x_2) + y_2) \text{ by definition of } \varphi_2 \text{ and } \beta \\ &= f(x_2) \text{ because } \beta(x_2), y_2 \in B \subset C \\ &= f(x_1 + x_2) \text{ because } x_1 \in G_1 \\ &= f(x), \end{aligned}$$

as desired. □

By Lemma 4.1, if $B \neq G_2$ then φ preserves a non-constant fibration and so Theorem 1.1 holds. As a result, we may assume that $B = G_2$. We will prove in this case that there exists $x \in G(K)$ with a Zariski dense orbit under the action of φ. In order to do this, we first show that we may also assume G is the direct product $G_1 \oplus G_2$. Indeed, we construct

$$\widetilde{\varphi} := (\varphi_1, \varphi_2) : G_1 \oplus G_2 \longrightarrow G_1 \oplus G_2,$$

where (as before) $\varphi_1(x_1) = \tau_1(x_1)$ for each $x_1 \in G_1$ and $\varphi_2(x_2) = \tau_2(x_2) + y_2$ for each $x_2 \in G_2$. We also let $\sigma : G_1 \oplus G_2 \longrightarrow G$ given by $\sigma(x_1 \oplus x_2) = \iota_1(x_1) + \iota_2(x_2)$, where $\iota_j : G_j \longrightarrow G$ are the inclusion maps.

Lemma 4.2. *If there exists $(x_1, x_2) \in (G_1 \oplus G_2)(K)$ with a Zariski dense orbit under the action of $\widetilde{\varphi}$, then $x := \sigma(x_1, x_2) \in G(K)$ has a Zariski dense orbit under φ.*

Proof of Lemma 4.2. Indeed, identifying each G_j with its image $\iota_j(G_j)$ inside G, (4.0.3) yields that

$$\sigma \circ \widetilde{\varphi} = \varphi \circ \sigma. \tag{4.2.1}$$

Then equation (4.2.1) yields $\sigma \circ \widetilde{\varphi}^n = \varphi^n \circ \sigma$ for each $n \in \mathbb{N}$, which means that if there exists a Zariski dense orbit $\mathcal{O}_{\widetilde{\varphi}}(x_1 \oplus x_2) \subset (G_1 \oplus G_2)(K)$, then $\mathcal{O}_{\varphi}(x_1 + x_2) \subset G(K)$ is also a Zariski dense orbit; note that σ is a dominant homomorphism. □

So, from now on, we may assume $G = G_1 \oplus G_2$ and that $\varphi : G \longrightarrow G$ is given by the action $(x_1, x_2) \mapsto (\varphi_1(x_1), \varphi_2(x_2))$.

In order to prove the existence of a K-point in G with a Zariski dense orbit, we first prove there exists $x_2 \in G_2(K)$ such that $\mathcal{O}_{\varphi_2}(x_2)$ is Zariski dense in G_2. Since we assumed that the group generated by $\beta(G_2)$ and y_2 is Zariski dense in G_2, Proposition 3.8 yields the existence of $x_2 \in G_2(K)$ such that the cyclic group generated by $\beta(x_2) + y_2$ is Zariski dense in G_2. Then Lemma 3.14 yields that any infinite subset of $\mathcal{O}_{\varphi_2}(x_2)$ is Zariski dense in G_2. If G_1 is trivial, then $G_2 = G$ and $\varphi = \varphi_2$ and so, Theorem 1.1 is proven. Hence, from now on, assume that $\dim(G_1) > 0$.

Let π_i (for $i = 1, 2$) be the projection of G onto each of its two factors G_i. Let Γ be the $\mathrm{End}(G_2)$-submodule of $G_2(K)$ generated by x_2 and y_2. By Fact 2.5, Γ is a finitely generated subgroup of $G_2(K)$. Using Proposition 3.5, we may find $x_1 \in G_1(K)$ with the property that if there exists a proper algebraic subgroup $H \subset G = G_1 \oplus G_2$ such that $x_1 \in \Gamma + H$ (or equivalently, there exists $\gamma \in \Gamma$ such that $(x_1, \gamma) \in H \subset G_1 \oplus G_2$), then $\pi_2(H)$ is a proper algebraic subgroup of G_2.

Let $x := x_1 \oplus x_2$ (or equivalently, $x = (x_1, x_2)$); we will prove that $\mathcal{O}_\varphi(x)$ is Zariski dense in G.

Let V be the Zariski closure of $\mathcal{O}_\varphi(x)$. Then Fact 2.9 yields that V is a finite union of cosets of algebraic subgroups of G. So, if $V \neq G$, then there exists a coset $c + H$ of a proper algebraic subgroup $H \subset G$ which contains $\{\varphi^n(x)\}_{n \in S}$ for some infinite subset $S \subset \mathbb{N}$. In particular, for any integers $n > m$ from S, we have that

$$H \text{ contains } \varphi^n(x) - \varphi^m(x). \tag{4.2.2}$$

Using the fact that $G = G_1 \oplus G_2$, we construct $\mu : G \longrightarrow G$ as

$$\mu(z_1, z_2) := (\tau_1^n(z_1) - \tau_1^m(z_1), z_2)\,.$$

Recall that the minimal polynomial f_1 of $\tau_1 = \tau|_{G_1}$ does not have eigenvalues which are roots of unity, and so $\left(\tau_1^{n-m} - \mathrm{id}\right)$ is an isogeny on G_1. Because τ_1 is also an isogeny on G_1, we see μ is an isogeny on G. Since

$$\begin{aligned} \varphi^n(x) - \varphi^m(x) &= (\tau_1^n(x_1) - \tau_1^m(x_1)) \oplus (\varphi_2^n(x_2) - \varphi_2^m(x_2)) \\ &\text{and } \varphi_2^n(x_2) - \varphi_2^m(x_2) \in \Gamma, \end{aligned} \tag{4.2.3}$$

we obtain that there exists $\gamma \in \Gamma$ such that $\mu(x_1, \gamma) \in H$. In particular, this yields that $(x_1, \gamma) \subset \mu^{-1}(H)$; furthermore, $\mu^{-1}(H)$ is a proper algebraic subgroup of G since μ is an isogeny. By our choice of x_1, we conclude that $\pi_2(\mu^{-1}(H))$ is a proper algebraic subgroup of G_2. However, since $\mu|_{G_2}$ is the identity map, we get that $\pi_2(H)$ is a proper algebraic subgroup of G_2. On the other hand, using (4.2.2) and (4.2.3), we get that for any integers $n > m$ from S,

$$\pi_2(H) \text{ contains } \varphi_2^n(x_2) - \varphi_2^m(x_2). \tag{4.2.4}$$

As a result, if we fix $m_0 \in S$ we see that there are infinitely many n for which $\varphi_2^n(x_2) - \varphi_2^{m_0}(x_2) \in H$. That is, the coset $\varphi_2^{m_0}(x_2) + \pi_2(H)$ contains infinitely many points of the form $\varphi_2^n(x_2)$. Notice that $\varphi_2(x) = \tau_2(x) + y_2$ and $\beta = \tau_2 - \mathrm{id}$ is nilpotent. Furthermore, the cyclic subgroup generated by $\beta(x_2) + y_2$ is Zariski dense in G_2. As a result, Lemma 3.14 tells us $G_2 = \pi_2(H)$, which is a contradiction. Hence $\mathcal{O}_\varphi(x)$ is Zariski dense in G, which concludes our proof. ☐

Remark 4.3. As shown in the proof of Theorem 1.1 (see Lemma 4.1 specifically), we obtain that there exists a positive integer n such that if φ preserves a nonconstant fibration, then actually there exists a proper algebraic subgroup C such that

$$f \circ \varphi^n = f, \text{ where } f : G \longrightarrow G/C \tag{4.3.1}$$

is the usual quotient homomorphism. Also, one cannot expect that n can be taken to be equal to 1 in (4.3.1), as shown by the following example. If $\varphi : \mathbb{G}_m \longrightarrow \mathbb{G}_m$ is given by $\varphi(x) = 1/x$, then φ^2 is the identity on $\mathbb{G}_m$, and so, with the above notation, $n = 2$ and $C = \{1\}$ is the trivial subgroup of $\mathbb{G}_m$. On the other hand, φ does not preserve a nonconstant power map on $\mathbb{G}_m$; instead φ preserves the nonconstant rational function $f(x) := x + 1/x$.

So, the most one can get for the self-map φ itself is that there exists a finite collection of proper algebraic subgroups $C_1, \dots, C_\ell$ of G such that if φ preserves a nonconstant fibration, then each orbit of a point in G is contained in a finite union of cosets $c_i + C_i$ (for some $c_i \in G$). The subgroups C_i are precisely the subgroups appearing in the orbit under φ of the subgroup C from (4.3.1); note that equation (4.3.1) yields that C is fixed by φ^n and so, there exist finitely many subgroups C_i in the orbit of C under the action of φ.

Remark 4.4. One could ask whether our arguments can be adapted to yield a generalization of Theorem 1.1 in which the action of the cyclic monoid generated by φ is replaced by the action of a finitely generated commutative monoid S of regular self-maps on the semiabelian variety G. The corresponding statement for abelian varieties was proven in [GS17, Theorem 1.3], essentially using the same strategy as in the case of a cyclic monoid (i.e., [GS17, Theorem 1.2]), combined with some results regarding commutative monoids and linear algebra. However, in the proof from [GS17, Theorem 1.3] (see the bottom of [GS17, page 462]), one uses Poincaré's Reducibility Theorem in a crucial way by finding a complement of a given algebraic subgroup of an abelian variety. In our proof of Theorem 1.1 we can construct such a complement (see (4.0.2)) even in the absence of Poincaré's Reducibility Theorem, but that strategy fails when one deals with an arbitrary finitely generated commutative monoid S; choosing a decomposition of G as a sum of two semiabelian subvarieties as in (4.0.2) which works simultaneously for all maps from S is not possible unless either S is cyclic (as in Theorem 1.1), or G is a split semiabelian variety (and therefore Poincaré's Reducibility Theorem applies). So, for a non-split semiabelian variety G, in the absence of Poincaré's Reducibility Theorem, one would need a completely new strategy for proving the generalization of Theorem 1.1 regarding a finitely generated commutative monoid of regular self-maps acting on G.

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