PREPERIODIC POINTS FOR FAMILIES OF POLYNOMIALS

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Abstract. Let $a(\lambda), b(\lambda) \in \mathbb{C}[\lambda]$ and let $f_\lambda(x) \in \mathbb{C}[x]$ be a one-parameter family of polynomials indexed by all $\lambda \in \mathbb{C}$. We study whether there exist infinitely many $\lambda \in \mathbb{C}$ such that both $a(\lambda)$ and $b(\lambda)$ are preperiodic for f_λ.

1. Introduction

The classical Manin-Mumford conjecture for abelian varieties (now a theorem due to Raynaud [26, 27]) predicts that the set of torsion points of an abelian variety A defined over $\mathbb{C}$ is not Zariski dense in a subvariety V of A, unless V is a translate of an algebraic subgroup of A by a torsion point. Pink and others have suggested extending the Manin-Mumford conjecture to a more general question regarding unlikely intersections between a subvariety V of a semiabelian scheme A and algebraic subgroups of the fibers of A having codimension greater than the dimension of V (see [9, 18, 21, 22, 24]). Here we state a special case of the question when V is a curve.

Question 1.1. *Let $\mathcal{S}$ be a semiabelian scheme over a variety $\mathcal{Y}$ defined over $\mathbb{C}$, and let $V \subset \mathcal{S}$ be a curve which is not contained in any proper algebraic subgroup of $\mathcal{S}$. We define*

$$\mathcal{S}^{[2]} := \bigcup_{y \in \mathcal{Y}} B_y,$$

where B_y is the union of all algebraic subgroups of the fibre $\mathcal{S}_y$ of codimension at least equal to 2. Must the intersection of V with $\mathcal{S}^{[2]}$ be finite?

Bertrand [8] recently showed that the answer to Question 1.1 is sometimes "no". The question may, however, have a positive answer in many instances. For example, in [21, 22], Masser and Zannier study Question 1.1 when $\mathcal{S}$ is the square of the Legendre family of elliptic curves E_λ (over the base $\mathbb{A}^1 \setminus \{0, 1\}$) given by the equation $y^2 = x(x-1)(x-\lambda)$. They show that for any two independent points P and Q on the generic fiber, there are at most finitely many $\lambda \in \mathbb{C}$ such that the specializations P_λ and Q_λ are both torsion points for E_λ. Their work thus gives a positive answer to Question 1.1 in this special case.

The result of Masser and Zannier has a distinctive dynamical flavor. Indeed, one may consider the following more general problem. Let $\{X_\lambda\}$ be an algebraic family

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of quasiprojective varieties defined over $\mathbb{C}$, let $\Phi_\lambda : X_\lambda \longrightarrow X_\lambda$ be an algebraic family of endomorphisms, and let $P_\lambda \in X_\lambda$ and $Q_\lambda \in X_\lambda$ be two algebraic families of points. Under what conditions do there exist infinitely many λ such that both P_λ and Q_λ are preperiodic for Φ_λ? Indeed, the problem from [21, 22] fits into this general dynamical framework by letting $X_\lambda = E_\lambda$ be the Legendre family of elliptic curves, and letting Φ_λ be the multiplication-by-2-map on each elliptic curve in this family.

In [2], Baker and DeMarco study an interesting special case of the above general dynamical question, first suggested by Zannier at an American Institute of Mathematics workshop in 2008. Given complex numbers a and b and an integer $d \geq 2$, when do there exist infinitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic for the action of $f_\lambda(x) := x^d + \lambda$ on $\mathbb{C}$? They show that this happens if and only if $a^d = b^d$. We prove the following generalization of the main result of [2].

Theorem 1.2. *Let $f \in \mathbb{C}[x]$ be any polynomial of degree $d \geq 2$, and let $a, b \in \mathbb{C}$. Then there exist infinitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic for $f(x) + \lambda$ if and only if $f(a) = f(b)$.*

We will derive Theorem 1.2 from a more technical result, Theorem 2.3, which also treats the case of "non-constant starting points" **a** and **b**, a topic that was raised in [2].

One might hope to formulate a general dynamical version of Question 1.1 for polarizable endomorphisms of projective varieties more general than multiplication-by-m maps on abelian varieties (an endomorphism Φ of a projective variety X is polarizable, if there exists $d \geq 2$ and a line bundle $\mathcal{L}$ on X such that $\Phi^*(\mathcal{L})$ is linearly equivalent to $\mathcal{L}^{\otimes d}$ in $\mathrm{Pic}(X)$) by using the analogy between abelian subschemes and preperiodic subvarieties. Because of the results of Baker and DeMarco, along with the results of this paper, we believe it is reasonable to ask the following dynamical analog of Question 1.1.

Question 1.3. *Let Y be any quasiprojective curve defined over $\mathbb{C}$, and let F be the function field of Y. Let $\mathbf{a}, \mathbf{b} \in \mathbb{P}^1(F)$, and let $V \subset \mathcal{X} := \mathbb{P}^1_F \times_F \mathbb{P}^1_F$ be the $\mathbb{C}$-curve $(\mathbf{a}, \mathbf{b})$. Let $\mathbf{f} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a rational map of degree $d \geq 2$ defined over F. Then for all but finitely many $\lambda \in Y$, $\mathbf{f}$ induces a well-defined rational map $f_\lambda : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ defined over $\mathbb{C}$. If there exist infinitely many $\lambda \in Y$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic points of $\mathbb{P}^1(\mathbb{C})$ under the action of f_λ, then must V be contained in a proper preperiodic subvariety of $\mathcal{X}$ under the action of $\Phi := (\mathbf{f}, \mathbf{f})$?*

Theorem 1.2 is a special case of Question 1.3 for $f_\lambda(x) = f(x) + \lambda$ and constant starting points $\mathbf{a}(\lambda) = a$ and $\mathbf{b}(\lambda) = b$. Theorem 2.3 also allows us to prove some other special cases of Question 1.3, such as the following.

Theorem 1.4. *Let $f \in \mathbb{C}[x]$ be any polynomial of degree $d \geq 2$, let $g \in \mathbb{C}[x]$ be any nonconstant polynomial, and let $c \in \mathbb{C}^*$. Then there exist at most finitely many $\lambda \in \mathbb{C}$ such that either*

(1) *both $g(\lambda)$ and $g(\lambda + c)$ are preperiodic for $f(x) + \lambda$, or*
(2) *both $g(\lambda)$ and $g(\lambda) + c$ are preperiodic for $f(x) + \lambda$.*

The next result is for the case when the family of maps $\mathbf{f}$ is constant.

Theorem 1.5. *Let $f \in \mathbb{C}[x]$ be a polynomial of degree $d \geq 2$, and let $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ be two polynomials of same degree and with the same leading coefficient. If there* *exist infinitely many* $\lambda \in \mathbb{C}$ *such that both* $\mathbf{a}(\lambda)$ *and* $\mathbf{b}(\lambda)$ *are preperiodic for* f*, then* $\mathbf{a} = \mathbf{b}$.

A special case of Theorem 1.5 is that for any fixed $c \in \mathbb{C}^*$ there can be only finitely many $\lambda \in \mathbb{C}$ such that both λ and $\lambda + c$ are preperiodic for f. In fact, more generally it provides a positive answer to a special case of Zhang's Dynamical Manin-Mumford Conjecture, which states that for a polarizable endomorphism $\Phi : X \longrightarrow X$ on a projective variety, the only subvarieties of X containing a dense set of preperiodic points are those subvarieties which are themselves preperiodic under f (see [33, Conjecture 2.5] or [34, Conjecture 1.2.1, Conjecture 4.1.7] for details). This conjecture turns out to be false in general (see [17]), but it may be true in many cases. For example, let $X := \mathbb{P}^1 \times \mathbb{P}^1$, and $\Phi(x, y) := (f(x), f(y))$ for a polynomial f of degree $d \geq 2$, and Y be the Zariski closure in X of the set $\{(\mathbf{a}(z), \mathbf{b}(z)) : z \in \mathbb{C}\}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{C}[x]$ are polynomials of same degree and with the same leading coefficient; Theorem 1.5 implies that if Y contains infinitely many points preperiodic under Φ, then Y is the diagonal subvariety of X, and thus is itself preperiodic under Φ. Theorem 1.5 also has consequences for a case of a revised Dynamical Manin-Mumford Conjecture [17, Conjecture 1.4] (see Section 11 for details).

The plan of our paper is as follows. In Section 2 we state our main result (Theorem 2.3) and some of its consequences, and then describe the method of our proof. In Section 3 we set up our notation, while in Section 4 we give a brief overview of Berkovich spaces. Then, in Section 5 we introduce some basic preliminaries regarding the iterates of a generic starting point $\mathbf{c}$ under a family of maps $\mathbf{f}$. Section 6 contains computations of the capacities of the generalized v-adic Mandelbrot sets associated to a generic point $\mathbf{c}$ under the action of $\mathbf{f}$. In Section 7 we prove an explicit formula for the Green's function for the generalized v-adic Mandelbrot sets when v is an archimedean valuation. We proceed with our proof of the direct implication in Theorem 2.3 in Section 8 (for the case $f_\lambda \in \overline{\mathbb{Q}}[x]$ and $\mathbf{a}, \mathbf{b} \in \overline{\mathbb{Q}}[x]$) and in Section 10 (for the general case). In Section 9 we prove the converse implication from Theorem 2.3. Then, in Section 11 we conclude our paper by proving Corollary 2.7 and discussing the connections between our Question 1.3 and the Dynamical Manin-Mumford Conjecture formulated by Ghioca, Tucker, and Zhang in [17].

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2. Statement of the main results

A special case of Question 1.3 is when $Y = \mathbb{A}^1$, $\mathbf{f} \in R[x]$ where $R = \mathbb{C}[\lambda]$, and $\mathbf{a}, \mathbf{b} \in R$. In Theorem 2.3 we provide a positive answer to Question 1.3 for any family of polynomials of the form

$$f_\lambda(x) = x^d + \sum_{i=0}^{d-2} c_i(\lambda) x^i \quad \text{where } c_i(\lambda) \in \mathbb{C}[\lambda] \text{ for } i = 0, \dots, d-2, \tag{2.1}$$

together with some mild restriction on the polynomials $\mathbf{a}$ and $\mathbf{b}$.

We say that a polynomial $f(x)$ of degree d is in *normal form* if it is monic and its coefficient of x^{d-1} equals 0 (note that any polynomial of degree $d>1$ can be put in normal form after a change of coordinates). As a matter of notation, we rewrite (2.1) as

$$f_\lambda(x) = P(x) + \sum_{i=1}^{r} Q_i(x)\cdot \lambda^{m_i}, \tag{2.2}$$

for some polynomial $P\in\mathbb{C}[x]$ in normal form of degree d, and some nonnegative integer r and integers $m_0 := 0 < m_1 \cdots < m_r$, and some polynomials $Q_i\in\mathbb{C}[x]$ of degrees $0\le e_i\le d-2$. We do not exclude the case $r=0$, in which case the sum in the sigma notation is empty and $\{f_\lambda\}_\lambda$ is a constant family of polynomials.

Let $\mathbf{a}(\lambda),\mathbf{b}(\lambda)\in\mathbb{C}[\lambda]$. If $\mathbf{a}$ is preperiodic for $\mathbf{f}$, i.e. $\mathbf{f}^k(\mathbf{a})=\mathbf{f}^\ell(\mathbf{a})$ for some $k\ne\ell$, then for each $\mathbf{b}$ one can show that there are infinitely many $\lambda\in\mathbb{C}$ such that $\mathbf{b}(\lambda)$ (and thus also $\mathbf{a}(\lambda)$) is preperiodic for f_λ (see also Proposition 9.1). Therefore, we may assume that $\mathbf{a}$ and $\mathbf{b}$ are not preperiodic for $\mathbf{f}$. Assuming there exist infinitely many $\lambda\in\mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ, then Question 1.3 predicts that there exist φ_1,φ_2 commuting with $\mathbf{f}$ such that $\varphi_1(\mathbf{a})=\varphi_2(\mathbf{b})$. A natural possibility is for φ_1 and φ_2 be iterates of $\mathbf{f}$; under a mild condition on $\mathbf{a}$ and $\mathbf{b}$ we prove that this is the *only* possibility.

Theorem 2.3. *Let $\mathbf{f}:=f_\lambda$ be the family of one-parameter polynomials (indexed by all $\lambda\in\mathbb{C}$) given by*

$$f_\lambda(x) := x^d + \sum_{i=0}^{d-2} c_i(\lambda)x^i = P(x) + \sum_{j=1}^{r} Q_j(x)\cdot\lambda^{m_j},$$

as above (see (2.1) and (2.2)). Let $\mathbf{a},\mathbf{b}\in\mathbb{C}[\lambda]$, and assume there exist nonnegative integers k and ℓ such that the following conditions hold

(i) *$f_\lambda^k(\mathbf{a}(\lambda))$ and $f_\lambda^\ell(\mathbf{b}(\lambda))$ have the same degree and the same leading coefficient as polynomials in λ; and*
(ii) *if $m=\deg_\lambda(f_\lambda^k(\mathbf{a}(\lambda)))=\deg_\lambda(f_\lambda^\ell(\mathbf{b}(\lambda)))$, then $m\ge m_r$.*

Then there exist infinitely many $\lambda\in\mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic points for f_λ if and only if $f_\lambda^k(\mathbf{a}(\lambda))=f_\lambda^\ell(\mathbf{b}(\lambda))$.

Remarks 2.4. (a) In Theorem 2.3, the 1-dimensional $\mathbb{C}$-scheme $(\mathbf{a},\mathbf{b})\subset\mathcal{X}:=\mathbb{P}^1_{\mathbb{C}(\lambda)}\times_{\mathbb{C}(\lambda)}\mathbb{P}^1_{\mathbb{C}(\lambda)}$ is contained in the 2-dimensional $\mathbb{C}$-subscheme $\mathcal{Y}$ of $\mathcal{X}$ given by the equation:

$$\mathbf{f}^k(x)=\mathbf{f}^\ell(y),$$

where (x,y) are the coordinates of $\mathcal{X}$. Such a $\mathcal{Y}$ is fixed by the action of $(\mathbf{f},\mathbf{f})$ on $\mathcal{X}$, as predicted by Question 1.3.
(b) It follows from the Lefschetz Principle that the same statements in Theorem 2.3 hold if we replace $\mathbb{C}$ by any other algebraically closed complete valued field of characteristic 0.
(c) We note that if $\mathbf{c}\in\mathbb{C}[\lambda]$ has the property that there exists $k\in\mathbb{N}$ such that $\deg_\lambda(f_\lambda^k(\mathbf{c}(\lambda)))=m$ has the property (ii) from Theorem 2.3, then $\mathbf{c}$ is not preperiodic for $\mathbf{f}$ (see Lemma 5.2).
(d) If $\mathbf{f}$ is not a constant family, then it follows from Benedetto's theorem [6] that $\mathbf{c}\in\mathbb{C}[\lambda]$ is not preperiodic for $\mathbf{f}$ if and only if there exists $k\in\mathbb{N}$ such that

$\deg_\lambda(f_\lambda^k(\mathbf{c}(\lambda))) \ge m_r$. On the other hand, if $\mathbf{f}$ is a constant family of polynomials defined over $\mathbb{C}$, i.e. $r = 0$ and $m_0 = 0$ in Theorem 2.3, then implicitly $m > 0$ (otherwise the conclusion holds trivially).

Theorem 2.3 generalizes known results regarding "unlikely intersections" in the dynamical setting, including the dynamical Manin-Mumford questions (see Section 11). Firstly, Theorem 2.3 generalizes the main result of [2] in two ways. On the one hand, in the case $\mathbf{a}$ and $\mathbf{b}$ are both constant we can prove a generalization of the main result from [2] as follows.

Theorem 2.5. *Let $a, b \in \mathbb{C}$, let $d \ge 2$, and let $c_0, \cdots, c_{d-2} \in \mathbb{C}[\lambda]$ such that $\deg(c_0) > \deg(c_i)$ for each $i = 1, \dots, d-2$. If there are infinitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic for*

$$f_\lambda(x) := x^d + \sum_{i=0}^{d-2} c_i(\lambda) x^i,$$

then $f_\lambda(a) = f_\lambda(b)$.

Proof. We apply Theorem 2.3 for $\mathbf{a}(\lambda) := f_\lambda(a)$ and $\mathbf{b}(\lambda) := f_\lambda(b)$. ☐

Consequently, Theorem 2.5 yields the proof of Theorem 1.2.

Proof of Theorem 1.2. Note that in this case we may drop the hypothesis that $f(x)$ is in normal form since, we may conjugate $f(x)$ by some linear polynomial $\delta \in \mathbb{C}[x]$ such that $g := \delta^{-1} \circ f \circ \delta + \delta^{-1}(\lambda)$ is a family of polynomials in normal form. Then apply Theorem 2.5 to the pair of points $\delta^{-1}(a)$ and $\delta^{-1}(b)$. ☐

On the other hand, using our Theorem 2.3 we are able to treat the case where the pair of points $\mathbf{a}$ and $\mathbf{b}$ depend algebraically on the parameter. This answers a question raised by Silverman mentioned in [2, § 1.1]. For instance, as an application of Theorem 2.3, by taking $\mathbf{f} = f(x) + \lambda$ for *any* nonconstant polynomial $f(x) \in \mathbb{C}[x]$ of degree at least 2 we have the following.

Corollary 2.6. *Let $f \in \mathbb{C}[x]$ be any polynomial of degree $d \ge 2$, and let $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ be polynomials such that $\mathbf{a}$ and $\mathbf{b}$ have the same degree and the same leading coefficient. Then there are infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic under the action of $f(x) + \lambda$ if and only if $\mathbf{a}(\lambda) = \mathbf{b}(\lambda)$.*

Proof. Firstly, the theorem is vacuously true if $\mathbf{a}$ and $\mathbf{b}$ are constant polynomials, since then they are automatically equal because they have the same leading coefficient. So, we may assume that $\deg(\mathbf{a}) = \deg(\mathbf{b}) \ge 1$.

Secondly, we conjugate $f(x)$ by some linear polynomial $\delta \in \mathbb{C}[x]$ such that $g := \delta^{-1} \circ f \circ \delta$ is a polynomial in normal form. Then we apply Theorem 2.3 to the family of polynomials $g(x) + \delta^{-1}(\lambda)$ and to the starting points $\delta^{-1}(\mathbf{a}(\lambda))$ and $\delta^{-1}(\mathbf{b}(\lambda))$. Since $\mathbf{a}$ and $\mathbf{b}$ are polynomials of same positive degree and same leading coefficient, it is immediate to check that conditions (i)-(ii) of Theorem 2.3 hold for $k = \ell = 0$. Therefore, $\mathbf{a}(\lambda) = \mathbf{b}(\lambda)$ as desired. ☐

An important special case of Corollary 2.6 is Theorem 1.4. Also, using Theorem 2.3 in the case $\mathbf{f}$ is a constant family of polynomials, we obtain a proof of Theorem 1.5.

Proof of Theorem 1.5. The result is an immediate consequence of Theorem 2.3 once we observe, as before, that we may replace f with a conjugate $\delta^{-1} \circ f \circ \delta$ of itself which is a polynomial in normal form. (Note that in this case we also replace $\mathbf{a}$ and $\mathbf{b}$ by $\delta^{-1}(\mathbf{a})$ and respectively $\delta^{-1}(\mathbf{b})$ which are also polynomials in λ of same degree and same leading coefficient.) □

On the other hand, assuming each c_i and also $\mathbf{a}$ and $\mathbf{b}$ have algebraic coefficients, the exact same proof we have yields stronger statements of Theorems 2.3, 1.5 and Corollary 2.6 allowing us to replace the hypothesis that there are infinitely many $\lambda \in \overline{\mathbb{Q}}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ with the weaker condition that there exists an infinite sequence of $\lambda_n \in \overline{\mathbb{Q}}$ such that

$$\lim_{n\to\infty} \widehat{\mathrm{h}}_{f_{\lambda_n}}(\mathbf{a}(\lambda_n)) + \widehat{\mathrm{h}}_{f_{\lambda_n}}(\mathbf{b}(\lambda_n)) = 0,$$

where for each $\lambda \in \overline{\mathbb{Q}}$, $\widehat{\mathrm{h}}_{f_\lambda}$ is the canonical height constructed with respect to the polynomial f_λ (for the precise definition of the canonical height with respect to a polynomial map, see Section 3). Therefore we can prove a special case of Zhang's Dynamical Bogomolov Conjecture (see [34]).

Corollary 2.7. *Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve which admits a parameterization given by $(\mathbf{a}(z), \mathbf{b}(z))$ for $z \in \mathbb{C}$, where $\mathbf{a}, \mathbf{b} \in \overline{\mathbb{Q}}[x]$ are polynomials of same degree and with the same leading coefficient. Let $f \in \overline{\mathbb{Q}}[x]$ be a polynomial of degree at least equal to 2, and let $\Phi(x, y) := (f(x), f(y))$ be the diagonal action of f on $\mathbb{P}^1 \times \mathbb{P}^1$. If there exist an infinite sequence of points $(x_n, y_n) \in Y(\overline{\mathbb{Q}})$ such that*

$$\lim_{n\to\infty} \widehat{\mathrm{h}}_f(x_n) + \widehat{\mathrm{h}}_f(y_n) = 0,$$

then $\mathbf{a} = \mathbf{b}$. In particular, Y is the diagonal subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ and thus is preperiodic under the action of Φ.

Remark 2.8. In fact, this result holds not only over $\overline{\mathbb{Q}}$ but over the algebraic closure of any global function field L (whose subfield of constants is K), as long as f is not conjugate to a polynomial with coefficients in $\overline{K}$.

Note that the second author, together with Baker, proved a similar result [3, Theorem 8.10] in the case Y is a line; i.e., if a line in $\mathbb{P}^1 \times \mathbb{P}^1$ contains an infinite set of points of small canonical height with respect to the coordinatewise action of the polynomial f on $\mathbb{P}^1 \times \mathbb{P}^1$, then the line Y is preperiodic under the action of (f, f) on $\mathbb{P}^1 \times \mathbb{P}^1$.

Laura DeMarco communicated to us that our Theorem 2.3 yields the proof of the first case of a conjecture she made as a dynamical analogue of the André-Oort Conjecture. Essentially, the Dynamical André-Oort Conjecture envisioned by DeMarco aims of characterizing subvarieties in the moduli space M_d of complex rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ (of degree $d > 1$) that contain a Zariski dense subset of post-critically finite rational maps. A rational map is *post-critically finite* (PCF) if all its critical points are preperiodic. The PCF rational maps play an important role in complex dynamics; for example, the Lattès maps are PCF.

Our Theorem 2.3 has the following consequence. Let $\mathbf{f} = f_\lambda$ be a family of polynomials in normal form of degree d, with polynomial coefficients in λ. Furthermore, assume the critical points $\mathbf{c}_1(\lambda), \dots, \mathbf{c}_{d-1}(\lambda)$ of f_λ are also polynomials in λ. Let I be the collection of indices i such that $\mathbf{c}_i$ is not preperiodic for $\mathbf{f}$. Suppose for each $i \in I$ there exist iterates $f_\lambda^{m_i}(\mathbf{c}_i(\lambda))$ with the same degree and leading coefficients in λ, and that this degree is large enough (i.e. satisfying the hypothesis from Theorem 2.3). Then there are infinitely many PCF maps in this family if and only if all $\mathbf{f}^{m_i}(\mathbf{c}_i)$ (for $i \in I$) are equal.

We prove Theorem 2.3 first for the case when both $\mathbf{a}$ and $\mathbf{b}$, and also each of the c_i's have algebraic coefficients, and then we extend our proof to the general case. For the extension to $\mathbb{C}$, we use a result of Benedetto [6] (see also Baker's extension [1] to arbitrary rational maps) which states that for a polynomial f of degree at least equal to 2 defined over a function field K of finite transcendence degree over a subfield K_0, if f is not isotrivial (i.e., f is not conjugate to a polynomial defined over $\overline{K_0}$), then each $x \in \overline{K}$ is preperiodic if and only if its canonical height $\widehat{\mathrm{h}}_f(x)$ equals 0. Strictly speaking, Benedetto's result is stated for function fields of transcendence degree 1, but a simple inductive argument on the transcendence degree yields the result for function fields of arbitrary finite transcendence degree (see also [1, Corollary 1.8] where Baker extends Benedetto's result to rational maps defined over function fields of arbitrary finite transcendence degree).

Our results and proofs are inspired by the results of [2] so that the strategy for the proof of Theorem 2.3 essentially follows the ideas in the paper [2]. However there are significantly more technical difficulties in our proofs. The plan of our proof is to use the v-adic generalized Mandelbrot sets introduced in [2] for the family of polynomials f_λ, and then use the equidistribution result of Baker-Rumely from [4]. A key ingredient is Proposition 6.8 which says that the canonical local height of the point in question at the place v is a constant multiple of the Green's function associated to the v-adic generalized Mandelbrot set. Then the condition that $\mathbf{a}(\lambda)$ ($\mathbf{b}(\lambda)$) are preperiodic is translated to the condition that the heights $h_{\mathbb{M}_{\mathbf{a}}}(\lambda)$ ($h_{\mathbb{M}_{\mathbf{b}}}(\lambda)$ respectively) are zero, for the corresponding parameter λ. Therefore the equidistribution result of Baker-Rumely can thus be applied to conclude that the v-adic generalized Mandelbrot sets for $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are the same for each place v. Finally, we need to use an explicit formula for the Green's function associated to the v-adic generalized Mandelbrot set corresponding to an archimedean valuation v to conclude that the desired equality of $f_\lambda^k(\mathbf{a}(\lambda))$ and $f_\lambda^\ell(\mathbf{b}(\lambda))$ holds. Extra work is needed for the explicit description of the Green's function for a v-adic generalized Mandelbrot set (when v is an archimedean place) due to the fact that in our case the polynomial f_λ has arbitrary (finitely) many critical points which vary with λ in contrast to the family of polynomials $x^d + \lambda$ from [2], which has only one critical point for the entire family.

3. Notation and preliminary

For any quasiprojective variety X endowed with an endomorphism Φ, we call a point $x \in X$ preperiodic if there exist two distinct nonnegative integers m and n such that $\Phi^m(x) = \Phi^n(x)$, where by Φ^i we always denote the i-iterate of the endomorphism Φ. If $n = 0$ then, by convention, Φ^0 is the identity map.

Let K be a field of characteristic 0 equipped with a set of inequivalent absolute values (places) Ω_K, normalized so that the product formula holds; more precisely, for each $v \in \Omega_K$ there exists a positive integer N_v such that for all $\alpha \in K^*$ we have $\prod_{v\in\Omega} |\alpha|_v^{N_v} = 1$ where for $v \in \Omega_K$, the corresponding absolute value is denoted by $|\cdot|_v$. Let $\mathbb{C}_v$ be a fixed completion of the algebraic closure of a completion of $(K, |\cdot|_v)$. When v is an archimedean valuation, then $\mathbb{C}_v = \mathbb{C}$. We fix an extension of $|\cdot|_v$ to an absolute value of $(\mathbb{C}_v, |\cdot|_v)$. Examples of product formula fields (or *global fields*) are number fields and function fields of projective varieties which are regular in codimension 1 (see [20, § 2.3] or [10, § 1.4.6]).

Let $f \in \mathbb{C}_v[x]$ be any polynomial of degree $d \geq 2$. Following Call and Silverman [12], for each $x \in \mathbb{C}_v$, we define the *local canonical height* of x as follows

$$\widehat{\mathrm{h}}_{f,v}(x) := \lim_{n\to\infty} \frac{\log^+ |f^n(x)|_v}{d^n},$$

where by $\log^+ z$ we always denote $\log\max\{z,1\}$ (for any real number z).

It is immediate that $\widehat{\mathrm{h}}_{f,v}(f^i(x)) = d^i\,\widehat{\mathrm{h}}_{f,v}(x)$, and thus $\widehat{\mathrm{h}}_{f,v}(x) = 0$ whenever x is a preperiodic point for f. If v is nonarchimedean and $f(x) = \sum_{i=0}^d a_i x^i$, then $|f(x)|_v = |a_d x^d|_v > |x|_v$ when $|x|_v > r_v$, where

$$r_v := \max\left\{ |a_d|_v^{-\frac{1}{d-1}}, \max\left\{ \left|\frac{a_i}{a_d}\right|^{\frac{1}{d-i}} \right\}_{0\leq i<d} \right\}. \tag{3.1}$$

Moreover, if $|x|_v > r_v$, then $\widehat{\mathrm{h}}_v(x) = \log|x|_v + \frac{\log|a_d|_v}{d-1} > 0$. For more details see [16] and [19] (although the results from [16, 19] are for canonical heights associated to Drinfeld modules, all the proofs go through for any local canonical height associated to any polynomial with respect to any nonarchimedean place).

Now, if v is archimedean, again it is easy to see that if $|x|_v$ is sufficiently large then $|f(x)|_v \gg |x|_v^d$ and moreover, $|f^n(x)|_v \to \infty$ as $n \to \infty$.

We fix an algebraic closure $\overline{K}$ of K, and for each $v \in \Omega_K$ we fix an embedding $\overline{K} \hookrightarrow \mathbb{C}_v$. Assume $f \in \overline{K}[x]$. In [12], Call and Silverman also defined the *global canonical height* $\widehat{\mathrm{h}}(x)$ for each $x \in \overline{K}$ as

$$\widehat{\mathrm{h}}_f(x) = \lim_{n\to\infty} \frac{h(f^n(x))}{d^n},$$

where h is the usual (logarithmic) Weil height on $\overline{K}$. Call and Silverman show that the global canonical height decomposes into a sum of the corresponding local canonical heights.

For each $\sigma \in \mathrm{Gal}(\overline{K}/K)$, we denote by $\widehat{\mathrm{h}}_{f^\sigma}$ the global canonical height computed with respect to f^σ, which is the polynomial obtained by applying σ to each coefficient of f. Similarly, for each $v \in \Omega_K$ we denote by $\widehat{\mathrm{h}}_{f^\sigma,v}$ the corresponding local canonical height constructed with respect to the polynomial f^σ. For $x \in \overline{K}$, we have $\widehat{\mathrm{h}}_f(x) = 0$ if and only if $\widehat{\mathrm{h}}_{f^\sigma}(x^\sigma) = 0$ for all $\sigma \in \mathrm{Gal}(\overline{K}/K)$. More precisely, for $x \in \overline{K}$ we have

$$\widehat{\mathrm{h}}_f(x) = 0 \text{ if and only if } \widehat{\mathrm{h}}_{f^\sigma,v}(x^\sigma) = 0 \text{ for all } v \in \Omega_K \text{ and all } \sigma \in \mathrm{Gal}(\overline{K}/K). \tag{3.2}$$

Essentially, (3.2) says that $\widehat{\mathrm{h}}_f(x) = 0$ if and only if the orbits of x^σ under each polynomial f^σ (for $\sigma \in \mathrm{Gal}(\overline{K}/K)$) are bounded with respect to each absolute value $|\cdot|_v$ for $v \in \Omega_K$.

In [6], Benedetto proved that if a polynomial f defined over a function field K (endowed with a set Ω_K of absolute values) is not *isotrivial* (that is, it cannot be conjugated to a polynomial defined over the constant subfield of K) then each point $c \in \overline{K}$ is preperiodic for f if and only if its global canonical height (computed with respect to f) equals 0. In particular, if $c \in \overline{K}$, then c is preperiodic if and only if

$$\widehat{\mathrm{h}}_{f^\sigma,v}(c^\sigma) = 0 \text{ for all } \sigma \in \mathrm{Gal}(\overline{K}/K) \text{ and for all places } v \in \Omega_K. \tag{3.3}$$

Let $\mathbf{f} = f_\lambda := x^d + \sum_{i=0}^{d-2} c_i(\lambda)x^i$, where $c_i(\lambda) \in \mathbb{C}[\lambda]$ for $i = 0, \ldots, d-2$, and let $\mathbf{c}(\lambda) \in \mathbb{C}[\lambda]$. We let K be the field extension of $\mathbb{Q}$ generated by all coefficients of each $c_i(\lambda)$ and of $\mathbf{c}(\lambda)$. Assume K is a global field, i.e. it has a set Ω_K of inequivalent absolute values with respect to which the nonzero elements of K satisfy a product formula. For each place $v \in \Omega_K$ we define the v-adic Mandelbrot set $M_{\mathbf{c},v}$ for $\mathbf{c}$ with respect to the family of polynomials $\mathbf{f}$ as the set of all $\lambda \in \mathbb{C}_v$ such that $\widehat{\mathrm{h}}_{f_\lambda,v}(\mathbf{c}(\lambda)) = 0$, i.e. the set of all $\lambda \in \mathbb{C}_v$ such that the iterates $f_\lambda^n(\mathbf{c}(\lambda))$ are bounded with respect to the v-adic absolute value.

4. Berkovich spaces

In this section we introduce the Berkovich spaces, and state the equidistribution theorem of Baker and Rumely [4] which will be key for the proofs of Theorems 2.3 and 2.5.

Let K be a global field of characteristic 0, and let Ω_K be the set of its inequivalent absolute values. For each $v \in \Omega_K$, we let $\mathbb{C}_v$ be the completion of an algebraic closure of the completion of K at v. Let $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ denote the Berkovich affine line over $\mathbb{C}_v$ (see [7] or [4, § 2.1] for details). Then $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ is a locally compact, Hausdorff, path-connected space containing $\mathbb{C}_v$ as a dense subspace (with the topology induced from the v-adic absolute value). As a topological space, $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ is the set consisting of all multiplicative seminorms, denoted by $[\cdot]_x$, on $\mathbb{C}_v[T]$ extending the absolute value $|\cdot|_v$ on $\mathbb{C}_v$ endowed with the weakest topology such that the map $z \mapsto [f]_z$ is continuous for all $f \in \mathbb{C}_v[T]$. It follows from the Gelfand-Mazur theorem that if $\mathbb{C}_v$ is the field of complex numbers $\mathbb{C}$ then $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}}$ is homeomorphic to $\mathbb{C}$. In the following, we will also use $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ to denote the complex line $\mathbb{C}$ whenever $\mathbb{C}_v = \mathbb{C}$. If $(\mathbb{C}_v, |\cdot|_v)$ is nonarchimedean then the set of seminorms can be described as follows. If $\{D(a_i, r_i)\}_i$ is any decreasing nested sequence of closed disks $D(c_i, r_i)$ centered at points $c_i \in \mathbb{C}_v$ of radius $r_i \geq 0$, then the map $f \mapsto \lim_{i\to\infty} [f]_{D(c_i,r_i)}$ defines a multiplicative seminorm on $\mathbb{C}_v[T]$ where $[f]_{D(c_i,r_i)}$ is the sup-norm of f over the closed disk $D(a_i, r_i)$. Berkovich's classification theorem says that there are exactly four types of points, Type I, II, III and IV. The first three types of points can be described in terms of closed disks $\zeta = D(c, r) = \cap D(c_i, r_i)$ where $c \in \mathbb{C}_v$ and $r \geq 0$. The corresponding multiplicative seminorm is just $f \mapsto [f]_{D(c,r)}$ for $f \in \mathbb{C}_v[T]$. Then, ζ is of Type I, II or III if and only if $r = 0, r \in |\mathbb{C}_v^*|_v$ or $r \notin |\mathbb{C}_v^*|_v$ respectively. As for Type IV points, they correspond to sequences of decreasing nested disks $D(c_i, r_i)$ such that $\cap D(c_i, r_i) = \emptyset$ and the multiplicative seminorm is $f \mapsto \lim_{i\to\infty}[f]_{D(c_i,r_i)}$ as described above. For details, see [7] or [4]. For $\zeta \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, we sometimes write $|\zeta|_v$ instead of $[T]_\zeta$.

In order to apply the main equidistribution result from [4, Theorem 7.52], we recall the potential theory on the affine line over $\mathbb{C}_v$. We will focus on the case $\mathbb{C}_v$ is a nonarchimedean field; the case $\mathbb{C}_v = \mathbb{C}$ is classical (we refer the reader to [25]). The right setting for nonarchimedean potential theory is the potential theory on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ developed in [4]. We quote part of a nice summary of the theory from [2, § 2.2 and 2.3] without going into details. We refer the reader to [2, 4] for all the details and proofs. Let E be a compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. Then analogous to the complex case, the logarithmic capacity $\gamma(E) = e^{-V(E)}$ and the Green's function G_E of E relative to ∞ can be defined where $V(E)$ is the infimum of the *energy integral* with respect to all possible probability measures supported on E. More precisely,

$$V(E) = \inf_{\mu} \int \int_{E\times E} -\log \delta(x,y) d\mu(x) d\mu(y),$$

where the infimum is computed with respect to all probability measures μ supported on E, while for $x, y \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, the function $\delta(x,y)$ is the *Hsia kernel* (see [4, Proposition 4.1]):

$$\delta(x,y) := \limsup_{\substack{z,w\in\mathbb{C}_v \\ z\to x, w\to y}} \mid z-w \mid_v .$$

The following are basic properties of the logarithmic capacity of E.

- If E_1, E_2 are two compact subsets of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ such that $E_1 \subset E_2$ then $\gamma(E_1) \le \gamma(E_2)$.
- If $E = \{\zeta\}$ where ζ is a Type II or III point corresponding to a closed disk $D(c,r)$ then $\gamma(E) = r > 0$. [4, Example 6.3]. (This can be viewed as analogue of the fact that a closed disk $D(c,r)$ of positive radius r in $\mathbb{C}_v$ has logarithmic capacity $\gamma(D(c,r)) = r$.)

If $\gamma(E) > 0$, then the exists a unique probability measure μ_E attaining the infimum of the energy integral. Furthermore, the support of μ_E is contained in the boundary of the unbounded component of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \setminus E$.

The Green's function $G_E(z)$ of E relative to infinity is a well-defined nonnegative real-valued subharmonic function on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ which is harmonic on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \setminus E$ (in the sense of [4, Chapter 8] for the nonarchimedean setting; see [25] for the archimedean case). If $\gamma(E) = 0$, then there exists no Green's function associated to the set E (see [25, Exercise 1, page 115] in the case $|\cdot|_v$ is archimedean; a similar argument works in the case $|\cdot|_v$ is nonarchimedean). Indeed, as shown in [4, Proposition 7.17, page 151], if $\gamma(\partial E) = 0$ then there exists no nonconstant harmonic function on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \setminus E$ which is bounded below (this is the Strong Maximum Principle for harmonic functions defined on Berkovich spaces). The following result is [2, Lemma 2.2 and 2.5], and it gives a characterization of the Green's function of the set E.

Lemma 4.1. *Let $(\mathbb{C}_v, |\cdot|_v)$ be either an archimedean or a nonarchimedean field. Let E be a compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ and let U be the unbounded component of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \setminus E$.*

(1) *If $\gamma(E) > 0$ (i.e. $V(E) < \infty$), then $G_E(z) = V(E) + \log|z|_v + o(1)$ for all $z \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ such that $|z|_v$ is sufficiently large. Furthermore, the $o(1)$-term may be omitted if v is nonarchimedean.*
(2) *If $G_E(z) = 0$ for all $z \in E$, then G_E is continuous on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, $\mathrm{Supp}(\mu_E) = \partial U$ and $G_E(z) > 0$ if and only if $z \in U$.*
(3) *If $G : \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \to \mathbb{R}$ is a continuous subharmonic function which is harmonic on U, identically zero on E, and such that $G(z) - \log^+|z|_v$ is bounded, then $G = G_E$. Furthermore, if $G(z) = \log|z|_v + V + o(1)$ (as $|z|_v \to \infty$) for some $V < \infty$, then $V(E) = V$ and so, $\gamma(E) = e^{-V}$.*

To state the equidistribution result from [4], we consider the compact *Berkovich adelic sets* which are of the following form

$$\mathbb{E} := \prod_{v\in\Omega} E_v$$

where E_v is a non-empty compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ for each $v \in \Omega$ and where E_v is the closed unit disk $\mathcal{D}(0,1)$ in $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ for all but finitely many $v \in \Omega$. The *logarithmic capacity* $\gamma(\mathbb{E})$ of $\mathbb{E}$ is defined as follows

$$\gamma(\mathbb{E}) = \prod_{v\in\Omega} \gamma(E_v)^{N_v},$$

where the positive integers N_v are the ones associated to the product formula on the global field K. Note that this is a finite product as for all but finitely many $v \in \Omega$, $\gamma(E_v) = \gamma(\mathcal{D}(0,1)) = 1$. Let $G_v = G_{E_v}$ be the Green's function of E_v relative to ∞ for each $v \in \Omega$. For every $v \in \Omega$, we fix an embedding $\overline{K} \hookrightarrow \mathbb{C}_v$. Let $S \subset \overline{K}$ be any finite subset that is invariant under the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$. We define the height $h_{\mathbb{E}}(S)$ of S relative to $\mathbb{E}$ by

$$h_{\mathbb{E}}(S) = \sum_{v\in\Omega} N_v \left(\frac{1}{|S|} \sum_{z\in S} G_v(z) \right). \tag{4.2}$$

Note that this definition is independent of any particular embedding $\overline{K} \hookrightarrow \mathbb{C}_v$ that we choose at $v \in \Omega$. The following is a special case of the equidistribution result [4, Theorem 7.52] that we need for our application.

Theorem 4.3. *Let $\mathbb{E} = \prod_{v\in\Omega} E_v$ be a compact Berkovich adelic set with $\gamma(\mathbb{E}) = 1$. Suppose that S_n is a sequence of $\mathrm{Gal}(\overline{K}/K)$-invariant finite subsets of $\overline{K}$ with $|S_n| \to \infty$ and $h_{\mathbb{E}}(S_n) \to 0$ as $n \to \infty$. For each $v \in \Omega$ and for each n let δ_n be the discrete probability measure supported equally on the elements of S_n. Then the sequence of measures $\{\delta_n\}$ converges weakly to μ_v the equilibrium measure on E_v.*

5. General results about the dynamics of polynomials f_λ

In this Section we work with a family of polynomials f_λ as given in Section 2, i.e.

$$f_\lambda(x) = x^d + \sum_{i=0}^{d-2} c_i(\lambda) x^i,$$

with $c_i(\lambda) \in \mathbb{C}[\lambda]$ for $i = 0, \dots, d-2$. As before, we may rewrite our family of polynomials as

$$f_\lambda(x) = P(x) + \sum_{j=1}^{r} Q_j(x) \cdot \lambda^{m_j},$$

where $P(x)$ is a polynomial of degree d in normal form, each Q_i has degree at most equal to $d-2$, while r is a nonnegative integer and $m_0 := 0 < m_1 \cdots < m_r$. Let $\mathbf{c}(\lambda) \in \mathbb{C}[\lambda]$ be given, and let K be the field extension of $\mathbb{Q}$ generated by all the coefficients of $c_i(\lambda), i = 0, \dots, d-2$ and of $\mathbf{c}(\lambda)$. We define $g_{\mathbf{c},n}(\lambda) := f_\lambda^n(\mathbf{c}(\lambda))$ for each $n \in \mathbb{N}$. Assume $m := \deg(\mathbf{c})$ satisfies the property (ii) from Theorem 2.3, i.e.,

$$m = \deg(\mathbf{c}) \geq m_r. \tag{5.1}$$

Furthermore, if $r = 0$ we assume $m \geq 1$ (see also Remark 2.4 (c)). We let q_m be the leading coefficient of $\mathbf{c}(\lambda)$. In the next Lemma we compute the degrees of all polynomials $g_{\mathbf{c},n}$ for all positive integers n.

Lemma 5.2. *With the above hypothesis, the polynomial $g_{\mathbf{c},n}(\lambda)$ has degree $m \cdot d^n$ and leading coefficient $q_m^{d^n}$ for each $n \in \mathbb{N}$.*

Proof. The assertion follows easily by induction on n, using (5.1), since the term of highest degree in λ from $g_{\mathbf{c},n}(\lambda)$ is $\mathbf{c}(\lambda)^{d^n}$. □

We immediately obtain as a corollary of Lemma 5.2 the fact that $\mathbf{c}$ is not preperiodic for $\mathbf{f}$. We denote by $\mathrm{Prep}(\mathbf{c})$ the set of all $\lambda \in \mathbb{C}$ such that $\mathbf{c}(\lambda)$ is preperiodic for f_λ. The following result is an immediate consequence of Lemma 5.2.

Corollary 5.3. $\mathrm{Prep}(\mathbf{c}) \subset \overline{K}$.

6. Capacities of Generalized Mandelbrot sets

We continue with the notation from Sections 4 and 5. Let $\mathbf{c} = \mathbf{c}(\lambda) \in \mathbb{C}[\lambda]$ be a nonconstant polynomial, and let K be a product formula field containing the coefficients of each $c_i(\lambda), i = 0, \dots, d-2$ and of $\mathbf{c}$. We let Ω_K be the set of inequivalent absolute values of the global field K, and let $v \in \Omega_K$. Assume that $\mathbf{c}(\lambda) = q_m\lambda^m + \dots$ (lower terms), where $m = \deg(\mathbf{c})$ satisfies the condition (5.1).

Our goal is to compute the logarithmic capacities of the v-adic generalized Mandelbrot sets $M_{\mathbf{c},v}$ defined in Section 3. Following [2], we extend the definition of our v-adic Mandelbrot set $M_{\mathbf{c},v}$ to be a subset of the affine Berkovich line $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ as follows:

$$M_{\mathbf{c},v} := \{\lambda \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} : \sup_n [g_{\mathbf{c},n}(T)]_\lambda < \infty\}.$$

Note that if $\mathbb{C}_v$ is a nonarchimedean field, then our present definition for $M_{\mathbf{c},v}$ yields more points than our definition from Section 3. Let $\lambda \in \mathbb{C}_v$ and recall the local canonical height $\widehat{\mathrm{h}}_{\lambda,v}(x)$ of $x \in \mathbb{C}_v$ is given by the formula

$$\widehat{\mathrm{h}}_{\lambda,v}(x) := \widehat{\mathrm{h}}_{f_\lambda,v}(x) = \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(x)|_v}{d^n}.$$

Notice that $\widehat{\mathrm{h}}_{\lambda,v}(x)$ is a continuous function of both λ and x (see [11, Prop.1.2] for polynomials over complex numbers; the proof for the nonarchimedean case is similar). As $\mathbb{C}_v$ is a dense subspace of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, continuity in λ implies that the canonical local height function $\widehat{\mathrm{h}}_{\lambda,v}(\mathbf{c}(\lambda))$ has a natural extension on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ (note that the topology on $\mathbb{C}_v$ is the restriction of the weak topology on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, then any continuous function on $\mathbb{C}_v$ will automatically have a unique extension to $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$). In the following, we will view $\widehat{\mathrm{h}}_{\lambda,v}(\mathbf{c}(\lambda))$ as a continuous function on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. It follows from the definition of $M_{\mathbf{c},v}$ that $\lambda \in M_{\mathbf{c},v}$ if and only if $\widehat{\mathrm{h}}_{\lambda,v}(\mathbf{c}(\lambda)) = 0$. Thus, $M_{\mathbf{c},v}$ is a closed subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. In fact, the following is true.

Proposition 6.1. *$M_{\mathbf{c},v}$ is a compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$.*

We already showed that $M_{\mathbf{c},v}$ is a closed subset of the locally compact space $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, and thus in order to prove Proposition 6.1 we only need to show that $M_{\mathbf{c},v}$ is a bounded subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. If $\mathbf{f}$ is a constant family of polynomials, then Proposition 6.1 follows from our assumption that $\deg(\mathbf{c}) \geq 1$. Indeed, if $|\lambda|_v$ is large, then $|\mathbf{c}(\lambda)|_v$ is large and thus $|\mathbf{f}^n(\mathbf{c}(\lambda))|_v \to \infty$ as $n \to \infty$. Furthermore, for nonarchimedean place v if $|\lambda|_v$ is sufficiently large, then (assuming v is nonarchimedean)

$$|\mathbf{f}^n(\mathbf{c}(\lambda))|_v = |\mathbf{c}(\lambda)|_v^{d^n} = |q_m\lambda^m|_v^{d^n}. \tag{6.2}$$

So, now we are left with the case that $\mathbf{f}$ is not a constant family, i.e. $r \geq 1$.

Lemma 6.3. *Assume $r \geq 1$, i.e.* **f** *is not a constant family of polynomials. Then $M_{\mathbf{c},v}$ is a bounded subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$.*

Proof. First we rewrite as before

$$f_\lambda(x) = P(x) + \sum_{j=1}^{r} Q_j(x) \cdot \lambda^{m_j},$$

with $P(x)$ in normal form of degree d, and each polynomial Q_j of degree $e_j \leq d-2$; also, $0 < m_1 < \cdots < m_r$. We know $m = \deg(\mathbf{c}) \geq m_r$.

Since $q_m\lambda^m$ is the leading monomial in $\mathbf{c}$, there exists a positive real number C_1 depending only on v, coefficients of $c_i(\lambda), i = 0, \ldots, d-2$ and on $\mathbf{c}$ such that if $|\lambda|_v > C_1$, then $|\mathbf{c}(\lambda)|_v > \frac{|q_m|_v}{2} \cdot |\lambda|_v^m$.

Let $\alpha := \max_{i=1}^r \frac{m_i}{d-e_i}$; then $\alpha \leq m_r/2$ since $e_i \leq d-2$ for all i. There exist positive real numbers C_2 and C_3 (depending only on v, and on the coefficients of $c_i(\lambda)$) such that if $|\lambda|_v > C_2$, and if $|x|_v > C_3|\lambda|_v^\alpha$, then

$$|f_\lambda(x)|_v > \frac{|x|_v^d}{2} > |x|_v,$$

and thus $|f_\lambda^n(x)|_v \to \infty$ as $n \to \infty$.

On the other hand, since $m \geq m_r \geq 2\alpha > \alpha$, we conclude that if $|\lambda|_v > (2C_3/|q_m|_v)^{1/(m-\alpha)}$ then

$$\frac{|q_m|_v}{2} \cdot |\lambda|_v^m > C_3|\lambda|_v^\alpha.$$

We let $C_4 := \max\left\{C_1, C_2, (2C_3/|q_m|_v)^{1/(m-\alpha)}, |q_m|_v^{-1/m}\right\}$. So, if $|\lambda|_v > C_4$ then

$$|\mathbf{c}(\lambda)|_v > \frac{|q_m|_v}{2} \cdot |\lambda|_v^m > C_3|\lambda|_v^\alpha,$$

and thus $|f_\lambda^n(\mathbf{c}(\lambda))|_v \to \infty$ as $n \to \infty$. We conclude that if $\lambda \in M_{\mathbf{c},v}$, then $|\lambda|_v \leq C_4$, as desired. □

Remark 6.4. It is possible to make the constants in the above proof explicit. Moreover, for a nonarchimedean place v the estimate of the absolute values can be precise. For example, if v is nonarchimedean, we can ensure that if $|\lambda|_v > C_4$, then

$$|f_\lambda^n(\mathbf{c}(\lambda))|_v = |q_m\lambda^m|_v^{d^n} \text{ for all } n \geq 1. \tag{6.5}$$

Theorem 6.6. *The logarithmic capacity of $M_{\mathbf{c},v}$ is $\gamma(M_{\mathbf{c},v}) = |q_m|_v^{-1/m}$.*

The strategy for the proof of Theorem 6.6 is to construct a continuous subharmonic function $G_{c,v} : \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \to \mathbb{R}$ satisfying Lemma 4.1 (3). Analogous to the family $f_\lambda(x) = x^d + \lambda$ treated in [2] we let

$$G_{\mathbf{c},v}(\lambda) := \lim_{n\to\infty} \frac{1}{\deg(g_{\mathbf{c},n})} \log^+[g_{\mathbf{c},n}(T)]_\lambda. \tag{6.7}$$

Then by a similar reasoning as in the proof of [2, Prop. 3.7], it can be shown that the limit exists for all $\lambda \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. In fact, by the definition of canonical local height, for $\lambda \in \mathbb{C}_v$ we have

$$\begin{aligned} G_{\mathbf{c},v}(\lambda) &= \lim_{n\to\infty} \frac{1}{md^n} \log^+ |f_\lambda^n(\mathbf{c}(\lambda))|_v \quad \text{since } \deg(g_{\mathbf{c},n}) = md^n \text{ by Lemma 5.2,} \\ &= \frac{1}{m} \cdot \widehat{\mathrm{h}}_{f_\lambda,v}(\mathbf{c}(\lambda)) \quad \text{by the definition of canonical local height.} \end{aligned}$$

As a consequence of the above computation, we have the following.

Proposition 6.8 (c.f. [29, Theorem II.0.1] and [30, Theorem III.0.1 and Corollary III.0.3])**.**

$$\widehat{\mathrm{h}}_{f_\lambda,v}(\mathbf{c}(\lambda)) = \deg(\mathbf{c}) G_{\mathbf{c},v}(\lambda).$$

Remark 6.9. The above formula holds in the more general case of Question 1.3; for example, one may work with a rational function $\mathbf{c} \in \mathbb{C}(\lambda)$.

Note that $G_{\mathbf{c},v}(\lambda) \geq 0$ for all $\lambda \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. Moreover, we see easily that $\lambda \in M_{\mathbf{c},v}$ if and only if $G_{\mathbf{c},v}(\lambda) = 0$.

Lemma 6.10. *$G_{\mathbf{c},v}$ is the Green's function for $M_{\mathbf{c},v}$ relative to ∞.*

The proof is essentially the same as the proof of [2, Prop. 3.7], we simply give a sketch of the idea.

Proof of Lemma 6.10. We deal with the case that v is nonarchimedean (the case when v is archimedean follows similarly). So, using the same argument as in the proof of [11, Prop. 1.2], we observe that as a function of λ,

$$\text{the function } \frac{\log^+ [g_{\mathbf{c},n}(T)]_\lambda}{\deg(g_{\mathbf{c},n})} \text{ converges uniformly on compact subsets of } \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}.$$

So,

$$\text{the function } \frac{\log^+ [g_{\mathbf{c},n}(T)]_\lambda}{\deg(g_{\mathbf{c},n})} \text{ is a continuous subharmonic function on } \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v},$$

which converges to $G_{\mathbf{c},v}$ uniformly; hence it follows from [4, Prop. 8.26(c)] that $G_{\mathbf{c},v}$ is continuous and subharmonic on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. Furthermore, as remarked above, $G_{\mathbf{c},v}$ is zero on $M_{\mathbf{c},v}$.

Arguing as in the proof of Lemma 6.3 (see (6.2) and (6.5)), if $|\lambda|_v > C_4$ then for $n \geq 1$ we have

$$|g_{\mathbf{c},n}(\lambda)|_v = |f_\lambda^n(\mathbf{c}(\lambda))|_v = |q_m \lambda^m|_v^{d^n}.$$

Hence, for $|\lambda|_v > C_4$ we have

$$\begin{aligned} G_{\mathbf{c},v}(\lambda) &= \lim_{n\to\infty} \frac{1}{md^n} \log |g_{\mathbf{c},n}(\lambda)|_v \\ &= \log|\lambda|_v + \frac{\log|q_m|_v}{m}. \end{aligned}$$

It follows from Lemma 4.1 (3), that $G_{\mathbf{c},v}$ is indeed the Green's function of $M_{\mathbf{c},v}$. ☐

Now we are ready to prove Theorem 6.6.

Proof of Theorem 6.6. As in the proof of Lemma 6.10, we have

$$G_{\mathbf{c},v}(\lambda) = \log|\lambda|_v + \frac{\log|q_m|_v}{m} + o(1)$$

for $|\lambda|_v$ sufficiently large. By Lemma 4.1 (3), we find that $V(M_{\mathbf{c},v}) = \frac{\log|q_m|_v}{m}$. Hence, the logarithmic capacity of $M_{\mathbf{c},v}$ is

$$\gamma(M_{\mathbf{c},v}) = e^{-V(M_{\mathbf{c},v})} = \frac{1}{|q_m|_v^{1/m}}$$

as desired. □

Let $\mathbb{M}_{\mathbf{c}} = \prod_{v\in\Omega} M_{\mathbf{c},v}$ be the generalized adelic Mandelbrot set associated to c. As a corollary to Theorem 6.6 we see that $\mathbb{M}_{\mathbf{c}}$ satisfies the hypothesis of Theorem 4.3.

Corollary 6.11. *For all but finitely many nonarchimedean places v, we have that $M_{\mathbf{c},v}$ is the closed unit disk $\mathcal{D}(0;1)$ in $\mathbb{C}_v$; furthermore $\gamma(\mathbb{M}_{\mathbf{c}}) = 1$.*

Proof. For each place v where all coefficients of $c_i(\lambda), i = 0, \dots, d-2$ and of $\mathbf{c}(\lambda)$ are v-adic integral, and moreover $|q_m|_v = 1$, we have that $M_{\mathbf{c},v} = \mathcal{D}(0,1)$. Indeed, $\mathcal{D}(0,1) \subset M_{\mathbf{c},v}$ since then $f_\lambda^n(\mathbf{c}(\lambda))$ is always a v-adic integer. For the converse implication we note that each coefficient of $g_{\mathbf{c},n}(\lambda)$ is a v-adic integer, while the leading coefficient is a v-adic unit for all $n \geq 1$; thus $|g_{\mathbf{c},n}(\lambda)|_v = |\lambda|_v^{md^n} \to \infty$ if $|\lambda|_v > 1$. Note that $q_m \neq 0$ and so, the second assertion in Corollary 6.11 follows immediately by the product formula in K. □

Using Proposition 6.8 and the decomposition of the global canonical height as a sum of local canonical heights we obtain the following result.

Corollary 6.12. *Let $\lambda \in \overline{K}$, let S be the set of $\mathrm{Gal}(\overline{K}/K)$-conjugates of λ, and let $h_{\mathbb{M}_{\mathbf{c}}}$ be defined as in (4.2). Then $\deg(\mathbf{c}) \cdot h_{\mathbb{M}_{\mathbf{c}}}(\lambda) = \widehat{\mathrm{h}}_{f_\lambda}(\mathbf{c}(\lambda))$.*

Remark 6.13. Let $h(\lambda)$ denote a Weil height function corresponding to the divisor ∞ of the parameter space which is the projective line in our case. Then, it follows from [12, Theorem 4.1] that

$$\lim_{h(\lambda)\to\infty} \frac{\widehat{\mathrm{h}}_{f_\lambda}(\mathbf{c}(\lambda))}{h(\lambda)} = \widehat{\mathrm{h}}_{\mathbf{f}}(\mathbf{c})$$

where $\widehat{\mathrm{h}}_{\mathbf{f}}(\mathbf{c})$ is the canonical height associated to the polynomial map $\mathbf{f}$ over the function field $\mathbb{C}(\lambda)$. Corollary 6.12 gives a precise relationship between the canonical height function on the special fiber, height of the parameter λ and $\widehat{\mathrm{h}}_{\mathbf{f}}(\mathbf{c})$ which is equal to $\deg(\mathbf{c})$ in this case.

7. Explicit formula for the Green function

In this Section we work under the assumption that $|\cdot|_v = |\cdot|$ is archimedean; we also denote $\mathbb{C}_v$ by simply $\mathbb{C}$ in this case. We show that in this setting we have an alternative way of representing the Green's function $G_c := G_{\mathbf{c},v}$ for the Mandelbrot set $M_c := M_{\mathbf{c},v}$. We continue to work under the same hypothesis on $\mathbf{c}(\lambda)$; in particular we assume that (5.1) holds. Furthermore, if $r = 0$ (i.e., $\mathbf{f}$ is a constant family of polynomials), then $m = \deg(\mathbf{c}) \geq 1$.

Since the degree in x of $f_\lambda(x)$ is d, there exists a unique function ϕ_λ which is an analytic homeomorphism on the set U_{R_λ} for some $R_\lambda \geq 1$ (where for any positive real number R, we denote by U_R the open set $\{z \in \mathbb{C} : |z| > R\}$) satisfying the following conditions:

(1) ϕ_λ has derivative equal to 1 at ∞, or more precisely, the analytic function $\psi_\lambda(z) := 1/\phi_\lambda(1/z)$ has derivative equal to 1 at $z = 0$; and
(2) $\phi_\lambda(f_\lambda(z)) = (\phi_\lambda(z))^d$ for $|z| > R_\lambda$.

We can make (1) above more precise by giving the power series expansion:

$$\phi_\lambda(z) = z + \sum_{n=1}^{\infty} \frac{A_{\lambda,n}}{z^n}. \tag{7.1}$$

From (7.1) we immediately conclude that $|\phi_\lambda(z)| = |z| + O_\lambda(1)$, and thus

$$\log|\phi_\lambda(z)| = \log|z| + O_\lambda(1) \quad \text{for } |z| \text{ large enough.} \tag{7.2}$$

So, using that $\phi_\lambda(f_\lambda(z)) = \phi_\lambda(z)^d$, we conclude that if $|z| > R_\lambda$, then

$$\lim_{n\to\infty} \frac{\log^+|f_\lambda^n(z)|}{d^n} = \lim_{n\to\infty} \frac{\log|\phi_\lambda(f_\lambda^n(z))|}{d^n} = \log|\phi_\lambda(z)|. \tag{7.3}$$

Hence (7.3) yields that the Green function G^λ for the (filled Julia set of the) polynomial f_λ equals

$$G^\lambda(z) := \lim_{n\to\infty} \frac{\log|f_\lambda^n(z)|}{d^n} = \log|\phi_\lambda(z)|, \text{ if } |z| > R_\lambda.$$

For more details on the Green function associated to any polynomial, see [13]. Now, we know by [13, Ch. III.4] that the function $\log|\phi_\lambda(z)|$ can be extended to a well-defined harmonic function on the entire basin of attraction A_∞^λ of the point at ∞ for the polynomial map f_λ. The set A_∞^λ is the complement of the filled Julia set of f_λ; more precisely, it is the set of all $z \in \mathbb{C}$ such that the orbit of z under f_λ is unbounded. Thus, on A_∞^λ we have that

$$G^\lambda(z) := \log|\phi_\lambda(z)| \tag{7.4}$$

is the Green function for (the filled Julia set of) the polynomial f_λ. Also by [13, Ch. III.4] we know that

$$R_\lambda := \max_{f_\lambda'(x)=0} e^{G^\lambda(x)} \geq 1.$$

In Proposition 7.6 we will show that if $|\lambda|$ is sufficiently large, then $\mathbf{c}(\lambda)$ is in the domain of analyticity for ϕ_λ. In particular, using (7.2) this would yield

$$G_c(\lambda) = \lim_{n\to\infty} \frac{\log^+|f_\lambda^n(\mathbf{c}(\lambda))|}{md^n} = \frac{\log|\phi_\lambda(\mathbf{c}(\lambda))|}{m} = \frac{G^\lambda(\mathbf{c}(\lambda))}{m}, \tag{7.5}$$

for $|\lambda|$ sufficiently large.

The proof of the next proposition is similar to the proof of [2, Lemma 3.2].

Proposition 7.6. *There exists a positive constant C_0 such that if $|\lambda| > C_0$, then $\mathbf{c}(\lambda)$ belongs to the analyticity domain of ϕ_λ.*

Proof. If $\mathbf{f}$ is a constant family of polynomials, then the conclusion is immediate since R_λ is constant (independent of λ) and thus for $|\lambda|$ sufficiently large, clearly $|\mathbf{c}(\lambda)| > R_\lambda$. So, from now on assume $\mathbf{f}$ is not a constant family of polynomials, which in particular yields that $r \geq 1$ and $0 < m_1 < \cdots < m_r$.

First we recall that

$$R_\lambda = e^{G^\lambda(x_0)} := \max_{f_\lambda'(x)=0} e^{G^\lambda(x)}.$$

Next we show that $R_\lambda \to \infty$ as $|\lambda| \to \infty$, which will be used later in our proof.

Lemma 7.7. *As* $|\lambda| \to \infty$*, we have* $R_\lambda \to \infty$.

Proof. We recall that

$$f_\lambda(x) = P(x) + \sum_{i=1}^{r} \lambda^{m_i} \cdot Q_i(x),$$

where $P(x)$ is a polynomial in normal form of degree d, and $0 < m_1 < \cdots < m_r$ are positive integers, while the Q_i's are nonzero polynomials of degrees $e_i \le d-2$. We have two cases.

Case 1. Each $Q_i(x)$ is a constant polynomial. Then the critical points of f_λ are independent of λ, i.e., $x_0 = O(1)$. We let $x_1 \in \mathbb{C}$ such that $f_\lambda(x_1) = x_0$. Since each Q_i is a nonzero constant polynomial, we immediately conclude that $|x_1| \gg |\lambda|^{m_r/d}$. On the other hand, since $U_{2R_\lambda} \subset \phi_\lambda^{-1}(U_{R_\lambda})$ (by [11, Corollary 3.3]) we conclude that $|x_1| \le 2R_\lambda$, and so, $R_\lambda \gg |\lambda|^{m_r/d}$. Indeed, if $|x_1| > 2R_\lambda$, then there exists $z_1 \in U_{R_\lambda}$ such that $\phi_\lambda^{-1}(z_1) = x_1$. Using the fact that ϕ_λ is a conjugacy map at ∞ for f_λ we would obtain that

$$x_0 = f_\lambda(x_1) = f_\lambda(\phi_\lambda^{-1}(z_1)) = \phi_\lambda^{-1}(z_1^d) \in U_{R_\lambda},$$

which contradicts the fact that x_0 is not in the analyticity domain of ϕ_λ.

Case 2. There exists $i = 1, \dots, r$ such that $Q_i(x)$ is not a constant polynomial. Then the critical points of f_λ vary with λ. In particular, there exists a critical point x_λ of maximum absolute value such that $|x_\lambda| \gg |\lambda|^{m_j/(d-e_j)}$ (for some $j = 1, \dots, r$), where for each $i = 1, \dots, r$ we have $e_i = \deg(Q_i) \le d-2$. Now, x_λ is not in the domain of analyticity of ϕ_λ and thus $|x_\lambda| \le R_\lambda$, which again shows that $R_\lambda \to \infty$ as $|\lambda| \to \infty$. □

Using that $R_\lambda \to \infty$, we will finish our proof. First we note that

$$|\phi_\lambda(f_\lambda(x_0))| = e^{G^\lambda(f_\lambda(x_0))} = e^{dG^\lambda(x_0)} = R_\lambda^d. \tag{7.8}$$

Note that $\phi_\lambda(z)$ is analytic on U_{R_λ}, while $\log|\phi_\lambda(z)|$ is continuous for $|z| \ge R_\lambda$. Moreover, whenever it is defined, $G^\lambda(f_\lambda(z)) = dG^\lambda(z)$; so, using also (7.4), we obtain (7.8).

Now, for $|\lambda|$ sufficiently large we have that $R_\lambda^d/2 > R_\lambda$ (since $R_\lambda \to \infty$ according to Lemma 7.7). So, $U_{R_\lambda^d} \subset \phi_\lambda\left(U_{R_\lambda^d/2}\right)$ (again using [11, Corollary 3.3]) and thus

$$|f_\lambda(x_0)| \ge \frac{R_\lambda^d}{2}. \tag{7.9}$$

Case 1. $\deg(Q_i) = 0$ for each i. Then $x_0 = O(1)$ as noticed in Lemma 7.7 and thus, using (7.9) we obtain that $|\lambda|^{m_r} \gg R_\lambda^d$. Since $\deg(\mathbf{c}) = m \ge m_r$, we obtain

$$|\mathbf{c}(\lambda)| \ge |q_m| \cdot |\lambda|^m - |O(\lambda^{m-1})| \gg R_\lambda^d > R_\lambda,$$

if $|\lambda|$ is sufficiently large.

Case 2. If not all of the Q_i's are constant polynomials, then we still know that

$$|x_0| \ll |\lambda|^{\max_{i=1}^r m_i/(d-e_i)} \ll |\lambda|^{m_r/2},$$

because $e_i \le d-2$ for each i. Therefore

$$R_\lambda^d \ll |f_\lambda(x_0)| \ll |\lambda|^{dm_r/2}. \tag{7.10}$$

On the other hand, $|\mathbf{c}(\lambda)| \sim |\lambda|^m$ and $m \ge m_r$, which yields that

$$|\mathbf{c}(\lambda)| \gg |\lambda|^m \gg R_\lambda^2 \gg R_\lambda,$$

by (7.10). This concludes the proof of Proposition 7.6. □

Therefore for large $|\lambda|$, the point $\mathbf{c}(\lambda)$ is in the domain of analyticity for ϕ_λ, which allows us to conclude that equation (7.5) holds.

We know (see [13]) that for each $\lambda \in \mathbb{C}$ and for each $z \in \mathbb{C}$ sufficiently large in absolute value, we have:

$$\phi_\lambda(z) = z \prod_{n=0}^{\infty} \left(\frac{f_\lambda^{n+1}(z)}{f_\lambda^n(z)^d} \right)^{\frac{1}{d^{n+1}}}, \tag{7.11}$$

and thus

$$\phi_\lambda(z) = z \prod_{n=0}^{\infty} \left(1 + \frac{Q_0\left(f_\lambda^n(z)\right) + \sum_{i=1}^r Q_i\left(f_\lambda^n(z)\right) \cdot \lambda^{m_i}}{f_\lambda^n(z)^d} \right)^{\frac{1}{d^{n+1}}}, \tag{7.12}$$

where $Q_0(x) := P(x) - x^d$ is a polynomial of degree at most equal to $d-2$. We showed in Proposition 7.6 that $\phi_\lambda(\mathbf{c}(\lambda))$ is well-defined; furthermore the function $\phi_\lambda(\mathbf{c}(\lambda))/\mathbf{c}(\lambda)$ can be expressed near ∞ as the above infinite product. Indeed, for each $n \in \mathbb{N}$, the order of magnitude of the numerator in the n-th fraction from the product appearing in (7.12) when we substitute $z = \mathbf{c}(\lambda)$ is at most

$$|\lambda|^{m+(d-2)md^n} \le |\lambda|^{m(d-1)d^n},$$

while the order of magnitude of the denominator is $|\lambda|^{md^{n+1}}$. This guarantees the convergence of the product from (7.12) corresponding to $\phi_\lambda(\mathbf{c}(\lambda))/\mathbf{c}(\lambda)$. We conclude that

(7.13) $\phi_\lambda(\mathbf{c}(\lambda))$ is an analytic function of λ (for large λ), and moreover

$$\phi_\lambda(\mathbf{c}(\lambda)) = q_m\lambda^m + O\left(\lambda^{m-1}\right). \tag{7.14}$$

8. Proof of Theorem 2.3: algebraic case

We work under the hypothesis of Theorem 2.3, and we continue with the notation from the previous Sections. Furthermore we prove Theorem 2.3 under the extra assumptions that

(8.1) $\mathbf{a}, \mathbf{b} \in \overline{\mathbb{Q}}[\lambda]$ and also, for each $i = 0, \dots, d-2$, $c_i \in \overline{\mathbb{Q}}[\lambda]$.

Recall that $f_\lambda(x) = x^d + \sum_{i=0}^{d-2} c_i(\lambda)x^i$ where we require that $c_i \in \overline{\mathbb{Q}}[\lambda]$ for $i = 0, \dots, d-2$. Let $\mathbf{a}, \mathbf{b} \in \overline{\mathbb{Q}}[\lambda]$ satisfying the hypothesis (i)-(ii) of Theorem 2.3. Let K be the number field generated by the coefficients of $c_i(\lambda)$ for $i = 0, \dots, d-2$, and of $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$. Let Ω_K be the set of all inequivalent absolute values on K.

Next, assume there exist infinitely many λ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ. At the expense of replacing $\mathbf{a}(\lambda)$ by $f_\lambda^k(\mathbf{a}(\lambda))$ and replacing $\mathbf{b}(\lambda)$ by $f_\lambda^\ell(\mathbf{b}(\lambda))$ we may assume that the polynomials

(8.2)
$\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ have the same leading coefficient, and the same degree $m \ge m_r$.

Let $h_{\mathbb{M}_\mathbf{a}}(z)$ ($h_{\mathbb{M}_\mathbf{b}}(z)$) be the height of $z \in \overline{K}$ relative to the adelic generalized Mandelbrot set $\mathbb{M}_\mathbf{a} := \prod_{v\in\Omega_K} M_{\mathbf{a},v}$ ($\mathbb{M}_\mathbf{b}$) as defined in Section 6. Note that if $\lambda \in \overline{K}$ is a parameter such that $\mathbf{a}(\lambda)$ (and $\mathbf{b}(\lambda)$) is preperiodic for f_λ, then $h_{\mathbb{M}_\mathbf{a}}(\lambda) = 0$ by Corollary 6.12. So, we may apply the equidistribution result from [4, Theorem 7.52] (see our Theorem 4.3) and conclude that $M_{\mathbf{a},v} = M_{\mathbf{b},v}$ for each place $v \in \Omega_K$. Indeed, we know that there exists an infinite sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of distinct numbers

$\lambda \in \overline{K}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ. So, for each $n \in \mathbb{N}$, we may take S_n be the union of the sets of Galois conjugates for λ_m for all $1 \le m \le n$. Clearly $\#S_n \to \infty$ as $n \to \infty$, and also each S_n is $\mathrm{Gal}(\overline{K}/K)$-invariant. Finally, $h_{\mathbb{M}_{\mathbf{a}}}(S_n) = h_{\mathbb{M}_{\mathbf{b}}}(S_n) = 0$ for all $n \in \mathbb{N}$, and thus Theorem 4.3 applies in this case. We obtain that $\mu_{\mathbb{M}_{\mathbf{a}}} = \mu_{\mathbb{M}_{\mathbf{b}}}$ and since they are both supported on $\mathbb{M}_{\mathbf{a}}$ (resp. $\mathbb{M}_{\mathbf{b}}$), we also get that $\mathbb{M}_{\mathbf{a}} = \mathbb{M}_{\mathbf{b}}$. The following Lemma applies in the generality of Theorem 2.3, and it will finish our proof (note that since K is a number field, it has at least one archimedean valuation).

Lemma 8.3. *Let* $\mathbf{f}$, $\mathbf{a}$ *and* $\mathbf{b}$ *be as in Theorem 2.3; in particular assume they are all defined over* $\mathbb{C}$. *Let* $|\cdot|$ *be the usual archimedean absolute value on* $\mathbb{C}$ *and let* M_a *and* M_b *be the corresponding complex Mandelbrot sets. If* $M_a = M_b$, *then* $\mathbf{a} = \mathbf{b}$.

Proof. Since $M_a = M_b$ then also the corresponding Green's functions are the same, i.e. (using (7.5) and (8.2))

$$|\phi_\lambda(\mathbf{a}(\lambda))| = |\phi_\lambda(\mathbf{b}(\lambda))| \text{ for all } |\lambda| \text{ sufficiently large.}$$

On the other hand, for $|z|$ large, the function $h(z) := \phi_z(\mathbf{a}(z))/\phi_z(\mathbf{b}(z))$ is an analytic function of constant absolute value (note that the denominator does not vanish since ϕ_λ is a homeomorphism for a neighborhood of ∞). By the Open Mapping Theorem, we conclude that $h(z) := u$ is a constant (for some $u \in \mathbb{C}$ of absolute value equal to 1); i.e.,

$$\phi_\lambda(\mathbf{a}(\lambda)) = u \cdot \phi_\lambda(\mathbf{b}(\lambda)). \tag{8.4}$$

Using (7.13) and (7.14) (also note that $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ have the same leading coefficient), we conclude that $u = 1$. Using that ϕ_λ is a homeomorphism on a neighborhood of the infinity, we conclude that $\mathbf{a}(\lambda) = \mathbf{b}(\lambda)$ for λ sufficiently large in absolute value, and thus for *all* λ, as desired (note that $\mathbf{a}$ and $\mathbf{b}$ are polynomials). □

Remark 8.5. Our proof (similar to the proof from [2]) only uses in an essential way the information that $M_a = M_b$, i.e., that the Mandelbrot sets over the complex numbers corresponding to $\mathbf{a}$ and $\mathbf{b}$ are equal, even though we know that $M_{\mathbf{a},v} = M_{\mathbf{b},v}$ for *all* places v.

9. Proof of Theorem 2.3: the converse implication

Now we prove the converse implication in Theorem 2.3 in the general case, i.e. for polynomials $c_0, \dots, c_{d-2}$, $\mathbf{a}$, and $\mathbf{b}$ with arbitrary complex coefficients. Again at the expense of replacing $\mathbf{a}(\lambda)$ by $f_\lambda^k(\mathbf{a}(\lambda))$ and replacing $\mathbf{b}(\lambda)$ by $f_\lambda^\ell(\mathbf{b}(\lambda))$ we may assume $\mathbf{a}(\lambda) = \mathbf{b}(\lambda)$. The following result will finish the converse statement in Theorem 2.3.

Proposition 9.1. *Let* $\mathbf{c} \in \mathbb{C}[\lambda]$ *of degree* $m \ge m_r$. *Let* $\mathrm{Prep}(\mathbf{c})$ *be the set consisting of all* $\lambda \in \mathbb{C}$ *such that* $\mathbf{c}(\lambda)$ *is preperiodic under* f_λ, *and let* M_c *be the set of all* $\lambda \in \mathbb{C}$ *such that the orbit of* $\mathbf{c}(\lambda)$ *under the action of* f_λ *is bounded with respect to the usual archimedean metric on* $\mathbb{C}$. *Then the closure in* $\mathbb{C}$ *of the set* $\mathrm{Prep}(\mathbf{c})$ *contains* ∂M_c. *In particular,* $\mathrm{Prep}(\mathbf{c})$ *is infinite.*

Proof. We first claim that the equation $f_z(\mathbf{c}(z)) = \mathbf{c}(z)$ has only finitely many solutions. Indeed, according to Lemma 5.2, the degree in z of $f_z(\mathbf{c}(z)) - \mathbf{c}(z)$ is dm, which means that there are at most dm solutions $z \in \mathbb{C}$ for the equation $f_z(\mathbf{c}(z)) = \mathbf{c}(z)$.

Let $x_0 \in \partial M_c$ which is *not* a solution z to $f_z(\mathbf{c}(z)) = \mathbf{c}(z)$; we will show that x_0 is contained in the closure in $\mathbb{C}$ of $\mathrm{Prep}(\mathbf{c})$. Since we already know that if $f_z(\mathbf{c}(z)) = \mathbf{c}(z)$ then $z \in \mathrm{Prep}(\mathbf{c})$, we will be done once we prove that each open neighborhood U of x_0 contains at least one point from $\mathrm{Prep}(\mathbf{c})$.

Now, let U be an open neighborhood of x_0 and let $h_i : U \longrightarrow \mathbb{P}^1(\mathbb{C})$ for $i = 1, 2, 3$ be three analytic functions with values taken in the compact Riemann sphere, given by:

$$h_1(z) := \infty;\ h_2(z) := \mathbf{c}(z) \text{ and } h_3(z) := g_{\mathbf{c},1}(z) = f_z(\mathbf{c}(z)).$$

Furthermore, since x_0 is not a solution for the equation $h_2(z) = h_3(z)$, then we may assume (at the expense of replacing U with a smaller neighborhood of x_0) that the closures of $h_2(U)$ and $h_3(U)$ are disjoint. Therefore the closures of $h_1(U)$, $h_2(U)$ and $h_3(U)$ in $\mathbb{P}^1(\mathbb{C})$ are all disjoint.

As before, we let $\{g_{\mathbf{c},n}\}_{n\ge 2}$ be the set of polynomials $g_{\mathbf{c},n}(z) := f_z^n(\mathbf{c}(z))$. Since $x_0 \in \partial M_c$, then the family of analytic maps $\{g_{\mathbf{c},n}\}_{n\ge 2}$ is not normal on U. Therefore, by Montel's Theorem (see [5, Theorem 3.3.6]), there exists $n \ge 2$ and there exists $z \in U$ such that $g_{\mathbf{c},n}(z) = \mathbf{c}(z)$ or $g_{\mathbf{c},n}(z) = f_z(\mathbf{c}(z))$ (clearly it cannot happen that $g_{\mathbf{c},n}(z) = \infty$). Either way we obtain that $z \in \mathrm{Prep}(\mathbf{c})$, as desired.

Since $\gamma(M_c) > 0$, we know that M_c is an uncountable subset of $\mathbb{C}$, and thus its boundary is infinite; hence also $\mathrm{Prep}(\mathbf{c})$ is infinite. □

10. Proof of Theorem 2.3: general case

In this Section we finish the proof of Theorem 2.3. With the same notation as in Theorem 2.3, we replace $\mathbf{a}$ and $\mathbf{b}$ by $f_\lambda^k(\mathbf{a}(\lambda))$ and respectively $f_\lambda^\ell(\mathbf{b}(\lambda))$; thus, $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are polynomials with the same degree and same leading coefficient. We assume there exist infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ; we will prove that $\mathbf{a} = \mathbf{b}$.

Let K denote the field generated over $\overline{\mathbb{Q}}$ by adjoining the coefficients of each c_i (for $i = 1, \dots, d-2$), and adjoining the coefficients of $\mathbf{a}$ and of $\mathbf{b}$. According to Corollary 5.3, if there exists $\lambda \in \mathbb{C}$ such that $\mathbf{a}(\lambda)$ (or $\mathbf{b}(\lambda)$) is preperiodic for f_λ, then $\lambda \in \overline{K}$ where $\overline{K}$ denotes the algebraic closure of K in $\mathbb{C}$. Let Ω_K be the set of inequivalent absolute values of K corresponding to the divisors of a projective $\overline{\mathbb{Q}}$-variety $\mathcal{V}$ regular in codimension 1; then the places in Ω_K satisfy a product formula.

As in Section 8, we let $h_{\mathbb{M}_{\mathbf{a}}}(z)$ ($h_{\mathbb{M}_{\mathbf{b}}}(z)$) be the height of $z \in \overline{K}$ relative to the adelic generalized Mandelbrot set $\mathbb{M}_{\mathbf{a}} = \prod_{v\in\Omega_K} \mathbf{M}_{\mathbf{a},v}$ ($\mathbb{M}_{\mathbf{b}}$) as defined in Section 6. Note that if $\lambda \in \overline{K}$ is a parameter such that $\mathbf{a}(\lambda)$ is preperiodic for f_λ, then $h_{\mathbb{M}_{\mathbf{a}}}(\lambda) = 0$ ($h_{\mathbb{M}_{\mathbf{b}}}(\lambda) = 0$ respectively) by Corollary 6.12 again. So, arguing as in Section 8, we may apply the equidistribution result from [4, Theorem 7.52] (Theorem 4.3) and conclude that $\mathbf{M}_{\mathbf{a},v} = \mathbf{M}_{\mathbf{b},v}$ for each place $v \in \Omega_K$.

As observed in our proof from Section 8 (see Remark 8.5), in order to finish the proof of Theorem 2.3 it suffices to prove that $M_a = M_b$, where M_a and M_b are the complex Mandelbrot sets corresponding to $\mathbf{a}$ and respectively, $\mathbf{b}$. By *complex* Mandelbrot sets M_a (M_b) we mean the Mandelbrot sets corresponding to $\mathbf{a}$ ($\mathbf{b}$) constructed with respect to the usual archimedean metric on $\mathbb{C}$.

As before, we denote by $\mathrm{Prep}(\mathbf{a})$ and $\mathrm{Prep}(\mathbf{b})$ the sets of all $\lambda \in \mathbb{C}$ such that $\mathbf{a}(\lambda)$ (respectively $\mathbf{b}(\lambda)$) is preperiodic for f_λ. As proved in Corollary 5.3 we know that both $\mathrm{Prep}(\mathbf{a})$ and $\mathrm{Prep}(\mathbf{b})$ are subsets of $\overline{K}$. In order to prove that $M_a = M_b$ it suffices to prove that $\mathrm{Prep}(\mathbf{a})$ differs from $\mathrm{Prep}(\mathbf{b})$ in at most finitely many points.

To ease the notation, we denote the symmetric difference of $\mathrm{Prep}(\mathbf{a})$ and $\mathrm{Prep}(\mathbf{b})$ by the following:

$$\mathrm{PrepDiff}(\mathbf{a},\mathbf{b}) := (\mathrm{Prep}(\mathbf{a}) \setminus \mathrm{Prep}(\mathbf{b})) \cup (\mathrm{Prep}(\mathbf{b}) \setminus \mathrm{Prep}(\mathbf{a})).$$

Proposition 10.1. *If the set* $\mathrm{PrepDiff}(\mathbf{a},\mathbf{b})$ *is finite, then* $M_a = M_b$*.*

Proof. Since M_a contains all points $\lambda \in \mathbb{C}$ such that $\lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(a)|}{d^n} = 0$, the Maximum Modulus Principle yields that the complement of M_a in $\mathbb{C}$ is connected; i.e., M_a is a full subset of $\mathbb{C}$ (see also [2]). So both M_a and M_b are full subsets of $\mathbb{C}$ containing the sets $\mathrm{Prep}(\mathbf{a})$ and $\mathrm{Prep}(\mathbf{b})$ respectively whose closures contain the boundary of M_a and respectively, of M_b (according to Proposition 9.1). As $\mathrm{Prep}(\mathbf{a})$ and $\mathrm{Prep}(\mathbf{b})$ differ by at most finitely many elements, we conclude that $M_a = M_b$. □

In order to prove that $\mathrm{Prep}(\mathbf{a})$ and $\mathrm{Prep}(\mathbf{b})$ differ by at most finitely many elements, we observe first that if $\lambda \in \mathrm{Prep}(\mathbf{a})$, then $\widehat{\mathrm{h}}_{f_\lambda}(\mathbf{a}(\lambda)) = 0$ and thus $\lambda^\sigma \in M_{\mathbf{a},v}$ for all v and all $\sigma \in \mathrm{Gal}(\overline{K}/K)$ (see (3.3); note that $\mathbf{a}(\lambda)^\sigma = \mathbf{a}(\lambda^\sigma)$ since $\mathbf{a} \in K[x]$). Similarly, if $\lambda \in \mathrm{Prep}(\mathbf{b})$ then $\lambda^\sigma \in M_{\mathbf{b},v}$ for each place $v \in \Omega_K$ and each Galois morphism σ. We would like to use the reverse implication, i.e., characterize the elements $\mathrm{Prep}(\mathbf{a})$ as the set of all $\lambda \in \overline{K}$ such that $\lambda^\sigma \in M_{\mathbf{a},v}$ for each place v and for each Galois morphism σ. This is true if f_λ is not isotrivial over $\overline{\mathbb{Q}}$ by Benedetto's result [6]. In this case, $\mathrm{Prep}(\mathbf{a})$ ($\mathrm{Prep}(\mathbf{b})$) is exactly the set of $\lambda \in \overline{K}$ such that $h_{\mathbb{M}_{\mathbf{a}}}(\lambda) = 0$ ($h_{\mathbb{M}_{\mathbf{a}}}(\lambda) = 0$ respectively). However, notice that if $f_\lambda \in \overline{\mathbb{Q}}[x]$ then

$$\lambda^\sigma \in M_{\mathbf{a},v} \text{ for all } v \in \Omega_K \text{ and } \sigma \in \mathrm{Gal}(\overline{K}/K) \text{ if and only if } \mathbf{a}(\lambda) \in \overline{\mathbb{Q}}.$$

We see that in this case $\mathrm{Prep}(\mathbf{a})$ is strictly smaller than the set of $\lambda \in \overline{K}$ such that $h_{\mathbb{M}_{\mathbf{a}}}(\lambda) = 0$. So, we will prove that $\mathrm{Prep}(\mathbf{a})$ and $\mathrm{Prep}(\mathbf{b})$ differ by at most finitely many elements by splitting our analysis into two cases depending on whether there exist infinitely many $\lambda \in \mathbb{C}$ such that f_λ is conjugate to a polynomial with coefficients in $\overline{\mathbb{Q}}$. The following easy result is key for our argument.

Lemma 10.2. *For any* $\lambda \in \mathbb{C}$*, the polynomial* $f_\lambda(x)$ *is conjugate to a polynomial with coefficients in* $\overline{\mathbb{Q}}$ *if and only if* $c_i(\lambda) \in \overline{\mathbb{Q}}$ *for each* $i = 1, \dots, d-2$*.*

Proof. One direction is obvious. Now, assume f_λ is conjugate to a polynomial with coefficients in $\overline{\mathbb{Q}}$. Let $\delta(x) := ax + b$ be a linear polynomial such that $\delta^{-1} \circ f_\lambda \circ \delta \in \overline{\mathbb{Q}}[x]$. Since f_λ is in normal form, we note that $a, b \in \overline{\mathbb{Q}}$ for otherwise the leading coefficient or the next-to-leading coefficient is not algebraic. Now, it is clear that each $c_i(\lambda) \in \overline{\mathbb{Q}}$ as desired. □

Let S be the set of all $\lambda \in \mathbb{C}$ such that f_λ is conjugate to a polynomial in $\overline{\mathbb{Q}}[x]$. Using Lemma 10.2, $S \subset \overline{K}$ since each polynomial c_i has coefficients in K, and $\overline{\mathbb{Q}} \subset K$. Also, S is $\mathrm{Gal}(\overline{K}/K)$-invariant since each coefficient of each c_i is in K.

Proposition 10.3. *We have*

$$\mathrm{PrepDiff}(\mathbf{a},\mathbf{b}) \subset S.$$

Proof. Let $\lambda \in \overline{K} \setminus S$. Since f_λ is not conjugate to a polynomial in $\overline{\mathbb{Q}}$, using Benedetto's result (see also (3.3)) we obtain that $\mathbf{a}(\lambda)$ is preperiodic for f_λ if and only if for each $v \in \Omega_K$ and $\sigma \in \mathrm{Gal}(\overline{K}/K)$, the local canonical height of $\mathbf{a}(\lambda)^\sigma =$

$\mathbf{a}(\lambda^\sigma)$ computed with respect to f_λ^σ equals 0. Since each coefficient of $c_i(\lambda)$ is defined over K, we get that $f_\lambda^\sigma = f_{\lambda^\sigma}$. Therefore, for each $\lambda \in \overline{K} \setminus S$, we see that $\mathbf{a}(\lambda)$ (or $\mathbf{b}(\lambda)$) is preperiodic for f_λ if and only if for all $v \in \Omega_K$ and all $\sigma \in \mathrm{Gal}(\overline{K}/K)$, we have $\lambda^\sigma \in M_{\mathbf{a},v}$ (respectively $\lambda^\sigma \in M_{\mathbf{b},v}$). Using the fact that $M_{\mathbf{a},v} = M_{\mathbf{b},v}$ for all $v \in \Omega_K$, we conclude that if $\lambda \in \overline{K} \setminus S$, then $\lambda \in \mathrm{Prep}(a)$ if and only if $\lambda \in \mathrm{Prep}(b)$. Hence, $\mathrm{PrepDiff}(\mathbf{a}, \mathbf{b}) \subset S$ as desired. ☐

Next, we have the following observation.

Lemma 10.4. *If $\lambda \in S$ and $\mathbf{a}(\lambda) \notin \overline{\mathbb{Q}}$, then $\mathbf{a}(\lambda)$ is not preperiodic for f_λ.*

Proof. The assertion is immediate since for $\lambda \in S$ we have $f_\lambda \in \overline{\mathbb{Q}}[x]$ by the definition of S (see also Lemma 10.2); hence the set of preperiodic points of f_λ is contained in $\overline{\mathbb{Q}}$. By assumption $\mathbf{a}(\lambda) \notin \overline{\mathbb{Q}}$, therefore $\mathbf{a}(\lambda)$ is not preperiodic for f_λ. ☐

Proposition 10.5. $\mathrm{PrepDiff}(\mathbf{a}, \mathbf{b})$ *is a finite set.*

Proof. If S is a finite set, then the assertion follows from Proposition 10.3. So, in the remaining part of the proof, we assume that S is an infinite set. By Lemma 10.2 we know that there exist infinitely many $\lambda \in \overline{K}$ such that $c_i(\lambda) \in \overline{\mathbb{Q}}$ for each $i = 0, \dots, d-2$. The following Lemma will be key for our proof.

Lemma 10.6. *Let $L_1 \subset L_2$ be algebraically closed fields of characteristic 0, and let $f_1, \dots, f_n \in L_2[x]$. If there exist infinitely many $z \in L_2$ such that $f_i(z) \in L_1$ for each $i = 1, \dots, n$, then there exists $h \in L_2[x]$, and there exist $g_1, \dots, g_n \in L_1[x]$ such that $f_i = g_i \circ h$ for each $i = 1, \dots, n$.*

Proof. Let $C \subset \mathbb{A}^n$ be the Zariski closure of the set

$$\{(f_1(z), \cdots, f_n(z)) : z \in L_2\}. \tag{10.7}$$

Then C is a rational curve which (by our hypothesis) contains infinitely many points over L_1. Therefore C is defined over L_1, and thus it has a rational parametrization over L_1. Let

$$(g_1, \dots, g_n) : \mathbb{A}^1 \longrightarrow C$$

be a birational morphism defined over L_1; we denote by $\psi : C \longrightarrow \mathbb{A}^1$ its inverse (for more details see [28, Chapter 1]). Since the closure of C in $\mathbb{P}^n$ (by considering the usual embedding of $\mathbb{A}^n \subset \mathbb{P}^n$) has only one point at infinity (due to the parametrization (10.7) of C), we conclude that (perhaps after a change of coordinates) we may assume each g_i is also a polynomial; more precisely, $g_i \in L_1[x]$. We let $h : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the rational map (defined over L_2) given by the composition

$$h := \psi \circ (f_1, \dots, f_n).$$

Therefore, for each $i = 1, \dots, n$, we have $f_i = g_i \circ h$, and since both f_i and g_i are polynomials, we conclude that also h is a polynomial, as desired. ☐

As an immediate consequence of Lemma 10.6, we have the following result.

Corollary 10.8. *Let $L_1 \subset L_2$ be algebraically closed fields of characteristic 0, and let $f_1, \dots, f_n \in L_2[x]$. If there exist infinitely many $z \in L_2$ such that $f_i(z) \in L_1$ for $i = 1, \dots, n$, then for any $i, j \in \{1, \dots, n\}$ and any $z \in L_2$ we have $f_i(z) \in L_1$ if and only if $f_j(z) \in L_1$.*

There are two possibilities: either there exist infinitely many $\lambda \in S$ such that $\mathbf{a}(\lambda) \in \overline{\mathbb{Q}}$, or not.

Lemma 10.9. *If there exist infinitely many* $\lambda \in S$ *such that* $\mathbf{a}(\lambda) \in \overline{\mathbb{Q}}$*, then* $\mathbf{a} = \mathbf{b}$*. In particular,* $\mathrm{Prep}(\mathbf{a}) = \mathrm{Prep}(\mathbf{b})$*.*

Proof. Using Corollary 10.8 we obtain that actually for *all* $\lambda \in S$ we have that $\mathbf{a}(\lambda) \in \overline{\mathbb{Q}}$. So, in this case *each* λ^σ belongs to each $M_{\mathbf{a},v}$ for each place v of the function field $K/\overline{\mathbb{Q}}$ and for each $\sigma \in \mathrm{Gal}(\overline{K}/K)$ (note that for such $\lambda \in S$ we have that both $f_\lambda \in \overline{\mathbb{Q}}[x]$ and $\mathbf{a}(\lambda) \in \overline{\mathbb{Q}}$, and also note that S is $\mathrm{Gal}(\overline{K}/K)$-invariant). Since $M_{\mathbf{a},v} = M_{\mathbf{b},v}$ for each place v, we conclude that $\lambda^\sigma \in M_{\mathbf{b},v}$ for each $\lambda \in S$, for each $v \in \Omega_K$ and for each $\sigma \in \mathrm{Gal}(\overline{K}/K)$. Since $f_\lambda \in \overline{\mathbb{Q}}[x]$, we conclude that $\mathbf{b}(\lambda) \in \overline{\mathbb{Q}}$ as well. Indeed, otherwise $|\mathbf{b}(\lambda)^\sigma|_v > 1$ for some place v and some Galois morphism σ, and thus $|f_\lambda^n(\mathbf{b}(\lambda^\sigma))|_v \to \infty$ as $n \to \infty$, contradicting the fact that $\lambda^\sigma \in M_{\mathbf{b},v}$. Hence both $\mathbf{a}(\lambda) \in \overline{\mathbb{Q}}$ and $\mathbf{b}(\lambda) \in \overline{\mathbb{Q}}$ for $\lambda \in S$.

Therefore, applying Lemma 10.6 to the polynomials $c_0, \dots, c_{d-2}$, $\mathbf{a}$ and $\mathbf{b}$, we conclude that there exist polynomials polynomials $c_0', \dots, c_{d-2}', \mathbf{a}', \mathbf{b}' \in \overline{\mathbb{Q}}[x]$ and $h \in \overline{K}[x]$ such that

$$c_i = c_i' \circ h \text{ for each } i = 0, \dots, d-2, \text{ and} \tag{10.10}$$

$$\mathbf{a} = \mathbf{a}' \circ h \text{ and } \mathbf{b} = \mathbf{b}' \circ h. \tag{10.11}$$

We let $\delta := h(\lambda)$, and define the family of polynomials

$$f_\delta'(x) := x^d + \sum_{i=0}^{d-2} c_i'(\delta) x^i.$$

So, we reduced the problem to the case studied in Section 8 for the family of polynomials $f_\delta' \in \overline{\mathbb{Q}}[x]$ and to the starting points $\mathbf{a}', \mathbf{b}' \in \overline{\mathbb{Q}}[\delta]$. Note that using hypothesis (i)-(ii) from Theorem 2.3, and also relations (10.10) and (10.11), $\mathbf{a}'(\delta)$ and $\mathbf{b}'(\delta)$ have the same leading coefficient and the same degree which is larger than the degrees of the c_i''s. So, since we know there exist infinitely many $\delta \in \mathbb{C}$ such that $\mathbf{a}'(\delta)$ and $\mathbf{b}'(\delta)$ are both preperiodic for f_δ' we conclude that $\mathbf{a}' = \mathbf{b}'$, as proved in Section 8. Hence $\mathbf{a} = \mathbf{b}$ and thus $\mathrm{Prep}(\mathbf{a}) = \mathrm{Prep}(\mathbf{b})$. □

Lemma 10.12. *If there exist finitely many* $\lambda \in S$ *such that* $\mathbf{a}(\lambda) \in \overline{\mathbb{Q}}$*, then* $\mathrm{PrepDiff}(\mathbf{a}, \mathbf{b})$ *is finite.*

Proof. First, note that there must be at most finitely many $\lambda \in S$ such that $\mathbf{b}(\lambda) \in \overline{\mathbb{Q}}$. Otherwise, arguing as in the proof of Lemma 10.9, we would obtain that for all the infinitely many $\lambda \in S$, both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are in $\overline{\mathbb{Q}}$, which violates the Lemma's hypothesis. So, let T be the finite subset of S containing all λ such that either $\mathbf{a}(\lambda) \in \overline{\mathbb{Q}}$ or $\mathbf{b}(\lambda) \in \overline{\mathbb{Q}}$.

Let $\lambda \in (\overline{K} \setminus T) \cap \mathrm{Prep}(\mathbf{a})$. If $\lambda \in S$ then by Lemma 10.4 we know that $\lambda \notin \mathrm{Prep}(\mathbf{a})$, a contradiction. Therefore, $\lambda \notin S$, so by Proposition 10.3, we have $\lambda \notin \mathrm{PrepDiff}(\mathbf{a}, \mathbf{b})$. Similarly, if $\lambda \in (\overline{K} \setminus T) \cap \mathrm{Prep}(\mathbf{b})$, then $\lambda \notin \mathrm{PrepDiff}(\mathbf{a}, \mathbf{b})$. Thus, $\mathrm{PrepDiff}(\mathbf{a}, \mathbf{b})$ is contained in the finite set T.

□

Lemmas 10.9 and 10.12 finish the proof of Proposition 10.5. □

Therefore Proposition 10.5 yields that $\mathrm{Prep}(\mathbf{a})$ and $\mathrm{Prep}(\mathbf{b})$ differ by at most finitely many elements. Then it follows from Proposition 10.1 that the corresponding complex Mandelbrot sets M_a and M_b are equal, and so we conclude our proof of Theorem 2.3 using Lemma 8.3.

11. Connections to the dynamical Manin-Mumford conjecture

We first prove Corollary 2.7 and then we present further connections between our Question 1.3 and the Dynamical Manin-Mumford Conjecture formulated by Ghioca, Tucker, and Zhang in [17].

Proof of Corollary 2.7. At the expense of replacing f by a conjugate $\delta^{-1} \circ f \circ \delta$, and replacing $\mathbf{a}$ (resp. $\mathbf{b}$) by $\delta^{-1} \circ \mathbf{a}$ (resp. $\delta^{-1} \circ \mathbf{b}$) we may assume f is in normal form. By the hypothesis of Corollary 2.7 we know that there are infinitely many $\lambda_n \in \overline{\mathbb{Q}}$ such that

$$\lim_{n\to\infty} \widehat{\mathrm{h}}_f(\mathbf{a}(\lambda_n)) + \widehat{\mathrm{h}}_f(\mathbf{b}(\lambda_n)) = 0.$$

We let $\mathbf{f} := f_\lambda := f$ be the constant family of polynomials f indexed by $\lambda \in \overline{\mathbb{Q}}$. As before, we let K be the field generated by coefficients of f, $\mathbf{a}$ and $\mathbf{b}$ and let $h_{\mathbb{M}_\mathbf{a}}(z)$ ($h_{\mathbb{M}_\mathbf{b}}(z)$) be the height of $z \in \overline{K}$ relative to the adelic generalized Mandelbrot set $\mathbb{M}_\mathbf{a} := \prod_{v\in\Omega_K} M_{\mathbf{a},v}$ ($\mathbb{M}_\mathbf{b}$) as defined in Section 6. So, we may apply the equidistribution result from [4, Theorem 7.52] (see our Theorem 4.3) and conclude that $M_{\mathbf{a},v} = M_{\mathbf{b},v}$ for each place $v \in \Omega_K$. Indeed, for each $n \in \mathbb{N}$, we may take S_n be the set of Galois conjugates of λ_n. Clearly $\#S_n \to \infty$ as $n \to \infty$ (since the points λ_n are distinct and their heights are bounded because the heights of $\mathbf{a}(\lambda_n)$ and $\mathbf{b}(\lambda_n)$ are bounded). Finally, $\lim_{n\to\infty} h_{\mathbb{M}_\mathbf{a}}(S_n) = \lim_{n\to\infty} h_{\mathbb{M}_\mathbf{b}}(S_n) = 0$ (by Corollary 6.12), and thus Theorem 4.3 applies in this case.

Using that $M_{\mathbf{a},v} = M_{\mathbf{b},v}$ for an archimedean place v, the same argument as in the proof of Theorem 2.3 yields that $\mathbf{a} = \mathbf{b}$, as desired. □

Next we discuss the connection between our Question 1.3 and the Dynamical Manin-Mumford Conjecture [17, Conjecture 1.4]. First we recall that for a projective variety X and an endomorphism Φ of X, we say that Φ is *polarizable* if there exists an integer $d > 1$ and there exists an ample line bundle $\mathcal{L}$ on X such that $\Phi^*(\mathcal{L}) = \mathcal{L}^{\otimes d}$.

Conjecture 11.1 (Ghioca, Tucker, Zhang)**.** *Let X be a projective variety, let $\varphi : X \longrightarrow X$ be a polarizable endomorphism defined over $\mathbb{C}$, and let Y be a subvariety of X which has no component included into the singular part of X. Then Y is preperiodic under φ if and only if there exists a Zariski dense subset of smooth points $x \in Y \cap \mathrm{Prep}_\varphi(X)$ such that the tangent subspace of Y at x is preperiodic under the induced action of φ on the Grassmanian $\mathrm{Gr}_{\dim(Y)}(T_{X,x})$. (Here we denote by $T_{X,x}$ the tangent space of X at the point x.)*

In [17], Ghioca, Tucker, and Zhang prove that Conjecture 11.1 holds whenever Φ is a polarizable algebraic group endomorphism of the abelian variety X, and also when $X = \mathbb{P}^1 \times \mathbb{P}^1$, Y is a line, and $\Phi(x, y) = (f(x), g(y))$ for any rational maps f and g. We claim that a positive answer to Question 1.3 yields the following special case of Conjecture 11.1 which is not covered by the results from [17]. Note that we do not need the condition on preperiodicity of tangent spaces in the Grassmanian, only an infinite family of preperiodic points; hence what one would obtain here is really a special case of Zhang's original dynamical Manin-Mumford conjecture (which did not require the extra hypothesis on tangent spaces).

Proposition 11.2. *If Question 1.3 holds, then for any endomorphism Φ of $\mathbb{P}^1\times\mathbb{P}^1$ given by $\Phi(x,y) := (f(x), f(y))$ for some rational map $f\in\mathbb{C}(x)$ of degree at least 2, a curve $Y\subset\mathbb{P}^1\times\mathbb{P}^1$ will contain infinitely many preperiodic points if and only if Y is preperiodic under Φ. In particular, Question 1.3 implies Conjecture 11.1 for such Y and Φ.*

Proof. Let $Y\subset\mathbb{P}^1\times\mathbb{P}^1$ be a curve containing infinitely many points (x,y) such that both x and y are preperiodic for f. Furthermore, we may assume Y projects dominantly on each coordinate of $\mathbb{P}^1\times\mathbb{P}^1$ since otherwise it is immediate to conclude that Y contains infinitely many preperiodic points for Φ if and only if $Y=\{c\}\times\mathbb{P}^1$, or $Y=\mathbb{P}^1\times\{c\}$, where c is a preperiodic point for f.

We let $\mathbf{f}=f_\lambda := f$ be the constant family of rational functions (equal to f) indexed by all points $\lambda\in Y$, and let K be the function field of Y. Let $(\mathbf{a},\mathbf{b})\in\mathbb{P}^1(K)\times\mathbb{P}^1(K)$ be a generic point for Y. By our assumption, there exist infinitely many $\lambda\in Y$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for $f_\lambda=f$. Since Y projects dominantly on each coordinate of $\mathbb{P}^1\times\mathbb{P}^1$, we get that neither $\mathbf{a}$ nor $\mathbf{b}$ is preperiodic under the action of f (otherwise, $\mathbf{a}$ or $\mathbf{b}$ would be constant). So, assuming Question 1.3 holds, we obtain that the curve $Y(\mathbb{C})=\{(\mathbf{a}(\lambda),\mathbf{b}(\lambda)) : \lambda\in Y\}\subset\mathbb{P}^1_K\times_K\mathbb{P}^1_K$ lies on a preperiodic proper subvariety Z of $\mathbb{P}^1\times\mathbb{P}^1$ defined over a finite extension of K. More precisely, we get that $Z=Y\otimes_{\mathbb{C}}K$ and so, Y must be itself preperiodic under the action of (f,f) on $\mathbb{P}^1\times\mathbb{P}^1$.

Conversely, suppose that Y is preperiodic under Φ. Then some iterate of Y contains a dense set of periodic points, by work of Fakhruddin [15], so Y contains an infinite set of preperiodic points. □

Remark 11.3. (a) In the proof of Proposition 11.2 we did not use the full strength of the hypothesis from Conjecture 11.1. Instead we used the weaker hypothesis of [33, Conjecture 2.5] or [34, Conjecture 1.2.1, Conjecture 4.1.7] (which was the original formulation of the Dynamical Manin-Mumford Conjecture). This is not surprising since for curves contained in $\mathbb{P}^1\times\mathbb{P}^1$ the *only* counterexamples to the original formulation of the Dynamical Manin-Mumford Conjecture are expected to occur when $\Phi := (f,g)$ for two *distinct* Lattès maps.
(b) Finally, we note that a positive answer to Conjecture 11.1 does not yield a positive answer to Question 1.3. Instead, Question 1.3 goes in a different direction which is likely to shed more light on the Dynamical Manin-Mumford Conjecture, especially in the case Y is a curve in Conjecture 11.1.

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