ALGEBRAIC DYNAMICS OF SKEW-LINEAR SELF-MAPS

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Abstract. Let X be a variety defined over an algebraically closed field k of characteristic 0, let $N \in \mathbb{N}$, let $g : X \dashrightarrow X$ be a dominant rational self-map, and let $A : \mathbb{A}^N \longrightarrow \mathbb{A}^N$ be a linear transformation defined over $k(X)$, i.e., for a Zariski open dense subset $U \subset X$, we have that for $x \in U(k)$, the specialization $A(x)$ is an N-by-N matrix with entries in k. We let $f : X \times \mathbb{A}^N \dashrightarrow X \times \mathbb{A}^N$ be the rational endomorphism given by $(x, y) \mapsto (g(x), A(x)y)$. We prove that if the determinant of A is nonzero and if there exists $x \in X(k)$ such that its orbit $\mathcal{O}_g(x)$ is Zariski dense in X, then either there exists a point $z \in (X \times \mathbb{A}^N)(k)$ such that its orbit $\mathcal{O}_f(z)$ is Zariski dense in $X \times \mathbb{A}^N$ or there exists a nonconstant rational function $\psi \in k(X \times \mathbb{A}^N)$ such that $\psi \circ f = \psi$. Our result provides additional evidence to a conjecture of Medvedev and Scanlon.

1. Introduction

1.1. **Notation.** We let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Throughout our paper, we let k be an algebraically closed field of characteristic 0. Also, unless otherwise noted, all our subvarieties are assumed to be closed. In general, for a set S contained in an algebraic variety X, we denote by $\overline{S}$ its Zariski closure.

For a variety X defined over k and endowed with a rational self-map Φ, for any subvariety $V \subseteq X$, we define $\Phi(V)$ be the Zariski closure of the set $\Phi(V \setminus I(\Phi))$, where $I(\Phi)$ is the indeterminacy locus of Φ; in other words, $\Phi(V)$ is the strict transform of V under Φ. Also, we denote by $\mathcal{O}_\Phi(\alpha)$ the orbit of any point $\alpha \in X(K)$ under Φ, i.e., the set of all $\Phi^n(\alpha)$ for $n \in \mathbb{N}_0$ (as always in algebraic dynamics, we denote by Φ^n the n-th compositional power of the map Φ, where Φ^0 is the identity map, by convention). We say that α is periodic if there exists $n \in \mathbb{N}$ such that $\Phi^n(\alpha) = \alpha$; furthermore, the smallest positive integer n such that $\Phi^n(\alpha) = \alpha$ will be called the *period* of α. We say that α is preperiodic if there exists $m \in \mathbb{N}_0$ such that $\Phi^m(\alpha)$ is periodic. More generally, for an irreducible subvariety $V \subset X$, we say that V is periodic if $\Phi^n(V) = V$ for some $n \in \mathbb{N}$; if $\Phi(V) = V$ (i.e., $\overline{\Phi(V \setminus I(\Phi))} = V$), we say that V is invariant under the action of Φ (or simpler, invariant by Φ).

We will also encounter the following setup in our paper. Given a variety X defined over k and given $N \in \mathbb{N}$, we consider some N-by-N matrix A whose entries are rational functions on X; when the determinant of A is nonzero, then we write $A \in \mathrm{GL}_N(k(X))$. For any N-by-N matrix $A \in M_{N,N}(k(X))$ there exists an open, Zariski dense subset $U \subset X$ such that for each $x \in U$, the matrix $A(x)$ obtained by evaluating each entry of A at x is well-defined. We call *skew-linear self-map* a rational self-map $f : X \times \mathbb{A}^N \dashrightarrow X \times \mathbb{A}^N$ of the form $f(x, y) = (g(x), A(x)y)$, where $g : X \dashrightarrow X$ is a given rational self-map, while $A \in M_{N,N}(k(X))$.

1.2. **Zariski dense orbits.** The following conjecture was proposed by Medvedev and Scanlon [MS14, Conjecture 5.10] (and independently, by Amerik and Campana [AC08]); see also Zhang's [Zha06, Conjecture 4.1.6] regarding Zariski dense orbits for polarizable endomorphisms which motivated the aforementioned conjecture.

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Conjecture 1.1. Let X be a quasiprojective variety over k and $f : X \dashrightarrow X$ be a dominant rational self-map for which there exists no nonconstant rational function $\psi \in k(X)$ such that $\psi \circ f = \psi$. Then there exists a point $x \in X(k)$ whose orbit is Zariski dense in X.

The condition from Conjecture 1.1 that there is no nonconstant rational function $\psi \in k(X)$ such that $\psi \circ f = \psi$ is also refered as saying that f does not fix a nonconstant fibration. It is immediate to see that such a condition is absolutely necessary in order to hope for the conclusion in Conjecture 1.1 to hold; the difficulty in Conjecture 1.1 is to prove that such a condition is indeed sufficient for the existence of a Zariski dense orbit when the ground field is countable (note that the case when k is uncountable was established first in [AC08]).

In order to state our results, we introduce first the following definition.

Definition 1.2. Let X be any projective variety over k and $f : X \dashrightarrow X$ be a dominant rational self-map. We say that the pair (X, f) is *good* if Conjecture 1.1 holds for every pair which is birationally equivalent to (X, f), i.e., Conjecture 1.1 holds for any dynamical system (Y, g) for which there exists a birational map $\psi : X \dashrightarrow Y$ such that $\psi \circ f = g \circ \psi$.

Remark 1.3. It is immediate to see that if (X, f) and (Y, g) are birationally equivalent, then f fixes a nonconstant fibration if and only if g fixes a nonconstant fibration. Furthermore, if there is a point with a Zariski dense orbit under f in each (nontrivial) open subset of X, then for any pair (Y, g), which is birationally equivalent to (X, f), there exists a point in Y with a Zariski dense orbit under g.

Conjecture 1.1 predicts that each dynamical pair (X, f) is good; furthermore, in each of the important instances when Conjecture 1.1 holds for (X, f), then we actually know that the pair (X, f) is good (for more details, see Section 1.3).

We prove the following result for skew-linear self-maps.

Theorem 1.4. *Let $g : X \dashrightarrow X$ be a dominant rational map defined over k, let $N \in \mathbb{N}$, and let $f : X \times \mathbb{A}_k^N \dashrightarrow X \times \mathbb{A}_k^N$ be a dominant rational map defined by $(x, y) \mapsto (g(x), A(x)y)$ where $A \in \mathrm{GL}_N(k(X))$. If the pair (X, g) is good, then the pair $(X \times \mathbb{A}_k^N, f)$ is good.*

In Section 1.3 we discuss various cases when Conjecture 1.1 is known to hold; our Theorem 1.4 provides extensions of each one of those results since in the cases when Conjecture 1.1 is known to hold for a dynamical pair (X, f), then actually (X, f) is a good pair.

Very importantly, we note that the study of the dynamics of pairs $(X \times \mathbb{A}_k^N, f)$ where $f(x, y) = (g(x), A(x)y)$ for some endomorphism $g : X \longrightarrow X$ and some $A \in \mathrm{GL}_N(k(X))$ is quite subtle. Even in the special case when $X = \mathbb{G}_m^\ell$, $g : \mathbb{G}_m^\ell \longrightarrow \mathbb{G}_m^\ell$ is an algebraic group endomorphism and $A \in \mathbb{G}_N(k)$ is a constant matrix, it is a delicate question to get a complete characterization for which g, A and $x \in (\mathbb{G}_m^\ell \times \mathbb{A}^N)(k)$ we have that $\mathcal{O}_f(x)$ is Zariski dense. This last question is completely solved in [GH] using purely diophantine tools, thus very different techniques from the ones employed in our present paper.

1.3. **A brief history of previous results for the conjecture on the existence of Zariski dense orbits.** We work with the notation as in Conjecture 1.1.

The special case of Conjecture 1.1 when k is an uncountable field was proved in [AC08, Theorem 4.1] (which is stated more general, in the setting of Kähler manifolds); also, when k is uncountable, but in the special case f is an automorphism, Conjecture 1.1 was independently proven in [BRS10, Theorem 1.2]. Furthermore, if k is uncountable, Conjecture 1.1 holds even when k has positive characteristic (see [BGR17, Corollary 6.1]). If k is countable, Conjecture 1.1 has only been proved in a few special cases, using various techniques ranging from number theory, to p-adic dynamics, to higher dimensional algebraic geometry.

First, we note that Conjecture 1.1 holds if X has strictly positive Kodaira dimension and f is birational, as proven in [BGRS17, Theorem 1.2].

For varieties of negative Kodaira dimension, we note that Medvedev and Scanlon [MS14, Theorem 7.16] proved Conjecture 1.1 for endomorphisms f of $X = \mathbb{A}^m$ of the form $f(x_1, \dots, x_m) =$

$(f_1(x_1), \dots, f_m(x_m))$, where $f_1, \dots, f_m \in k[x]$. Combining techniques from model theory, number theory and polynomial decomposition theory, they obtain a complete description of all invariant subvarieties, which is the key to Conjecture 1.1 since orbit closures are invariant.

In the case when X is an abelian variety and $f\colon X \to X$ is a dominant self-map, Conjecture 1.1 was proved in [GS17]. The proof uses the explicit description of endomorphisms of an abelian variety and relies on the Mordell-Lang conjecture, due to Faltings [Fal94]. The strategy from [GS17] was then extended in [GS] to prove Conjecture 1.1 for all regular self-maps of any semiabelian variety.

Using methods from valuation theory (among several other tools), the second author proved in [Xie, Theorem 1.1] another important special case of Conjecture 1.1 for all polynomial endomorphisms f of $\mathbb{A}^2$. Previously, the same author established in [Xie15] the validity of Conjecture 1.1 for all birational automorphisms of surfaces (see also [BGT15] for an independent proof in the case of automorphisms of surfaces).

Finally, we observe that Conjecture 1.1 may be viewed as a dynamical analogue of a theorem of Rosenlicht (see [BGR17] for a comprehensive discussion on this theme). More precisely, the following result was proven by Rosenlicht [Ros56, Theorem 2].

Theorem 1.5. ([Ros56, Theorem 2]) *Consider the action of an algebraic group G on an irreducible algebraic variety X defined over an algebraically closed field k of characteristic* 0*. There exists a G-invariant dense open subvariety $X_0 \subset X$ and a G-equivariant morphism $g : X_0 \longrightarrow Z$ (where G acts trivially on Z), with the following properties:*

(i) *for each $x \in X_0(k)$, the orbit $G \cdot x$ equals the fiber $g^{-1}(g(x))$; and*
(ii) *$g^*k(Z) = k(X)^G := \{\psi \in k(X)\colon \psi \circ h = \psi$ for each $h \in G\}$.*

In particular, if there is no nonconstant fibration fixed by G, then for each $x \in X_0(k)$, we have $G \cdot x = X_0$ is Zariski dense in X.

Theorem 1.5 yields that Conjecture 1.1 holds for each automorphism $f : X \longrightarrow X$ contained in an algebraic group G (acting on X). Indeed (see also [BGR17]) one can apply Theorem 1.5 to X and the algebraic group G_0 which is the Zariski closure of the cyclic group spanned by f inside G and thus get that if f does not fix a nonconstant fibration, then there is $x \in X(k)$ such that $G_0 \cdot x$ is dense in X, and therefore $\mathcal{O}_f(x)$ is Zariski dense in X as well.

1.4. **Invariant subvarieties.** As a by-product of our method, we obtain the following characterization of invariant subvarieties under skew-linear automorphisms of $\mathbb{A}^1 \times \mathbb{A}^N$ of the form $(x, y) \mapsto (x + 1, A(x)y)$, where $A \in \mathrm{GL}_N(k[x])$.

Theorem 1.6. *Let $f : \mathbb{A}^1_k \times \mathbb{A}^N_k \to \mathbb{A}^1_k \times \mathbb{A}^N_k$ be an automorphism defined by $(x, y) \mapsto (x + 1, A(x)y)$ where $A(x)$ is a matrix in $\mathrm{GL}_N(k[x])$. Then there exists an automorphism h on $\mathbb{A}^1_k \times \mathbb{A}^N_k$ of the form $(x, y) \mapsto (x, T(x)y)$ where $T(x) \in \mathrm{GL}_N(k[x])$ such that for each subvariety V (not necessarily irreducible) of $\mathbb{A}^1_k \times \mathbb{A}^N_k$ invariant under f, we have $h^{-1}(V) = \mathbb{A}^1_k \times V_0 \subseteq \mathbb{A}^1_k \times \mathbb{A}^N_k$ where V_0 is a subvariety of $\mathbb{A}^N_k$.*

We also prove in Theorem 2.1 a more general version of the above result for invariant subvarieties under the action of a skew-linear self-map $f : X \times \mathbb{A}^N \dashrightarrow X \times \mathbb{A}^N$.

Remark 1.7. With the notation as in Theorem 1.6, we have that $h^{-1}(V)$ is invariant under $h^{-1} \circ f \circ h$; in other words, $h^{-1} \circ f \circ h = (x + 1, B(x)y)$ and $B(x)(V_0) = V_0$ for all $x \in \mathbb{A}^1_k$.

A skew-linear automorphism $\tilde{f} : \mathbb{A}^1 \times \mathbb{A}^N \longrightarrow \mathbb{A}^1 \times \mathbb{A}^N$ such as the automorphism $h^{-1} \circ f \circ h$ from Remark 1.7 will be called *straight*; more precisely, an automorphism of $\mathbb{A}^1 \times \mathbb{A}^N$ of the form $(x, y) \mapsto (x + 1, A(x)y)$ is straight if each invariant subvariety under its action is of the form $\mathbb{A}^1 \times V_0$ for some subvariety $V_0 \subseteq \mathbb{A}^N$ (see also Definition 3.5). Theorem 1.6 yields that any automorphism f of $\mathbb{A}^1 \times \mathbb{A}^N$ of the form $(x, y) \mapsto (x + 1, A(x)y)$ is conjugate to a straight automorphism (see Remark 1.7). In Section 3.2 we study more in-depth the straight automorphisms of $\mathbb{A}^1 \times \mathbb{A}^N$, which leads us to proving the following result.

Theorem 1.8. *Let $N \in \mathbb{N}$, let $A \in \mathrm{GL}_N(k[x])$, let $f : \mathbb{A}^1 \times \mathbb{A}^N \longrightarrow \mathbb{A}^1 \times \mathbb{A}^N$ be the automorphism given by $(x, y) \mapsto (x+1, A(x)y)$, and let V be a periodic subvariety of $\mathbb{A}^1 \times \mathbb{A}^N$ under the action of f. Then the period of V is uniformly bounded by a constant depending only on A (and independent of V).*

Actually, in Corollary 3.9 we prove a more precise version of Theorem 1.8 by showing that the period of any periodic subvariety V divides some positive integer intrinsically associated to A. We believe that Theorem 1.8 (and more generally, the results from Section 3) would be helpful in a further study of finding which points $x \in \mathbb{A}^1_k \times \mathbb{A}^N_k$ have a Zariski dense orbit under an automorphism f of the form $(x, y) \mapsto (x + 1, A(x)y)$.

Besides the intrinsic interest in the results of Section 3, they also provide a simpler proof of a special case of Theorem 2.1, thus helping the reader to understand the more general approach from Section 2.

1.5. **The plan for our paper.** In Section 2 we study the invariant subvarieties for skew-linear self-maps f of $X \times \mathbb{A}^N$ (for an arbitrary algebraic variety X) and subsequentely prove Theorems 2.1 and 1.4. In Section 3 we prove Theorem 1.6 (which is a more precise version of Theorem 2.1 when $X = \mathbb{A}^1$ and f is an automorphism) and then Theorem 1.8 (see Corollary 3.9). We conclude our paper with a more in-depth study of straight forms corresponding to skew-linear automorphisms of $\mathbb{A}^1 \times \mathbb{A}^2$; see Section 3.3.

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2. Zariski dense orbits

In this section, we let X be a variety defined over an algebraically closed field k of characteristic 0, endowed with a dominant self-map $g : X \dashrightarrow X$. We let $N \in \mathbb{N}$ and let $\pi : X \times \mathbb{A}^N_k \to X$ be the projection onto the first coordinate. We also let $A \in \mathrm{GL}_N(k(X))$ and (as in Theorem 1.4), we let $f : X \times \mathbb{A}^N \dashrightarrow X \times \mathbb{A}^N$ be the rational endomorphism given by $(x, y) \mapsto (g(x), A(x)y)$.

2.1. **Characterization of invariant subvarieties.** An important ingredient in our proof of Theorem 1.4 is a complete description of the subvarieties Y of $X \times \mathbb{A}^N$, which dominate X under the projection map π, and moreover, Y is invariant under the action of the skew-linear self-map f. So, we start by stating Theorem 2.1 which characterizes the (not necessarily irreducible) subvarieties of $X \times \mathbb{A}^N$, which are invariant under the rational self-map f; we state our result under the assumption that g fixes no nonconstant rational fibration, i.e., there is no nonconstant $\phi \in k(X)$ such that $\phi \circ g = \phi$. In Section 2.3 we explain that the general case can be reduced to Theorem 2.1.

Theorem 2.1. *Let $f : X \times \mathbb{A}^N_k \dashrightarrow X \times \mathbb{A}^N_k$ be a dominant rational map defined by $(x, y) \mapsto (g(x), A(x)y)$ where $A(x)$ is a matrix in $\mathrm{GL}_N(k(X))$, and let $\pi : X \times \mathbb{A}^N \longrightarrow X$ be the projection map. Suppose that there is no nonconstant rational function $\phi \in k(X)$ such that $\phi \circ g = \phi$. Then there exists:*

- *an integer $\ell \geq 1$;*
- *an irreducible variety Y endowed with a dominant rational map $g' : Y \dashrightarrow Y$ along with a generically finite map $\tau : Y \dashrightarrow X$ satisfying $\tau \circ g' = g^\ell \circ \tau$;*
- *a birational map h on $Y \times \mathbb{A}^N_k = Y \times_X X \times \mathbb{A}^N_k$ of the form $(x, y) \mapsto (x, T(x)y)$ where $T(x) \in \mathrm{GL}_N(k(Y))$,*

such that for any (not necessarily irreducible) subvariety $V \subset X \times \mathbb{A}^N_k$ with the properties that:

- *V is invariant under f, and*
- *each irreducible component of V dominates X under the induced projection map $\pi|_V : V \longrightarrow X$,*

we have $h^{-1}\left((\tau\times_X \mathrm{id})^{\#}(V)\right) = Y\times V_0 \subseteq Y\times \mathbb{A}^N_k$, *where* V_0 *is a subvariety of* $\mathbb{A}^N_k$ *and* $(\tau\times_X \mathrm{id})^{\#}(V)$ *is the corresponding strict transform*[1].

Theorem 2.1 is a generalization of Theorem 1.6 (though the latter result is slightly more precise, i.e., $\ell=1$ if $X=\mathbb{A}^1$ and $g(x)=x+1$). We prove Theorem 2.1 in Section 2.4.

2.2. **Invariant cycles.** Denote by $t_g := [k(X) : g^*(k(X))]\geq 1$ the topological degree of g.

For any irreducible subvariety W of $X\times\mathbb{A}^N_k$ which dominates X, denote by $f_\# W := d_W f(W)$ where d_W is the topological degree of $f|_W$ (and, as always, $f(W)$ is the Zariski closure of $f(W\setminus I(f))$). In our case, since W dominates X and the action of f on the fiber is linear, we have $d_W = t_g$.

Let V be an effective cycle of $X\times\mathbb{A}^N_k$ such that every irreducible component of V dominates X. Write $V=\sum_{i=1}^{\ell} a_i V_i$ where V_i are irreducible components of V and $a_i\geq 1$. Write $f_\# V := \sum_{i=1}^{\ell} a_i f_\# V_i = t_g\sum_{i=1}^{\ell} a_i f(V_i)$. We say that V is invariant under f if the support of V and $f_\#(V)$ are the same i.e. $f_\# V = t_g V$.

For any subvariety V of $X\times\mathbb{A}^N_k$ such that every irreducible component of V dominates X, we may view it as an effective cycle such that every irreducible component of V dominates X and all nonzero coefficients are equal to one. Then it is invariant under f if and only if as an effective cycle, it is invariant under f.

2.3. **Characterization of invariant subvarieties, general case.** In this section we explain that the case in which g fixes a nonconstant fibration can be reduced to Theorem 2.1. Indeed, first of all, we may suppose that X is projective (since g is a rational self-map). Then let

$$L = k(X)^g = \{\phi\in k(X):\ \phi\circ g=\phi\};$$

clearly, L is a subfield of $k(X)$ containing k. Let r be the transcendence degree of L over k; so, $1\leq r\leq \dim X$ since we assume that g fixes a nonconstant fibration.

Let R be a finitely generated k-subalgebra of L whose fraction field is L. Let B be an irreducible projective variety containing $\operatorname{Spec} R$ as a dense subset. The inclusion $R\hookrightarrow k(X)$ yields a dominant rational map $\psi: X \dashrightarrow B$. At the expense of replacing X by some suitable birational model, we may assume that X is smooth and that the map ψ is regular. By Stein factorization, we may further assume that the generic fiber of ψ is connected. By generic smoothness, we obtain that the generic fiber of ψ is smooth and thus geometrically irreducible.

Let η be the generic point of B. Let K be an algebraic closure of L. The geometric generic fiber of ψ is denoted by X_η over K. Then g induces a dominant rational self-map g_η on X_η and f induces a dominant rational self-map on $X_\eta\times\mathbb{A}^N$. Denote by I the set of invariant subvarieties of $X\times\mathbb{A}^N$ such that each of their irreducible components dominate X under the projection map $X\times\mathbb{A}^N\longrightarrow X$; we also let I_η be the set of invariant subvarieties of $X_\eta\times\mathbb{A}^N$ such that each of their irreducible components dominate X_η. For every invariant subvariety $V\in I$, we have that $V_\eta := V\times_X X_\eta$ is contained in I_η; the map $V\mapsto V_\eta$ is bijective.

By the construction of B, there is no nonconstant rational function $\phi\in K(X_\eta)$ satisfying $\phi\circ g=\phi$; therefore Theorem 2.1 applies for X_η. So, there exists an integer $\ell\geq 1$ and an irreducible variety Y_η endowed with a dominant rational self-map $g'_\eta : Y_\eta\to Y_\eta$ along with a generically finite map $\tau_\eta : Y_\eta \dashrightarrow X_\eta$ satisfying $\tau_\eta\circ g'_\eta = g_\eta^\ell\circ\tau_\eta$ such that there exists a birational map h_η on $Y_\eta\times\mathbb{A}^N = Y_\eta\times_{X_\eta} X_\eta\times\mathbb{A}^N$ of the form $(x,y)\mapsto (x,T(x)y)$ where $T(x)\in \mathrm{GL}_N(K(Y_\eta))$ with the property that for any subvariety $V_\eta\in I_\eta$, we have

$$h_\eta^{-1}((\tau_\eta\times_{X_\eta}\mathrm{id})^{\#}(V_\eta)) = Y_\eta\times V_0'\subseteq Y_\eta\times\mathbb{A}^N,$$

where V_0' is a subvariety of $\mathbb{A}^N_K$ and $(\tau_\eta\times_{X_\eta}\mathrm{id})^{\#}(V_\eta)$ is the strict transform. We note that X_η is in fact defined over L; furthermore, there exists a finite extension J over L such that Y_η,τ_η,h_η and V_0' are defined over J.

[1]Let $\phi: X_1\dashrightarrow X_2$ be any generically finite rational map between projective varieties. Let W any subvariety of X_2, we define the strictly transform $\phi^{\#}(W)$ of W to be the union of all irreducible components with the multiplicities of the Zariski closure of $\phi^{-1}|_{X_1\setminus I(\phi)}(W)$ on which ϕ are generically finite.

2.4. **Proof of Theorem 2.1.** We work with the notation as in Theorem 2.1.

Let $\mathcal{B}$ be the set of points $x \in X$ such that f is not a locally isomorphism on the fiber $\pi^{-1}(x)$. Then $\mathcal{B}$ is a proper closed subset of X.

Let I be the set of all effective invariant cycles V in $X \times \mathbb{A}_k^N$ for which every irreducible component of V dominates X under the projection map $\pi : X \times \mathbb{A}^N \longrightarrow X$. For any $x \in X$ and for any $V \in I$, we let

$$V_x := \pi^{-1}(x) \cap V \subseteq \mathbb{A}_k^N.$$

In the next result we show that over a Zariski dense subset of X, we have that each V_x is obtained through some linear transformation from a given V_{x_0}.

Proposition 2.2. *Let $V \in I$. There exists a Zariski open set U_V of X such that such that for any points $x_1, x_2 \in U_V(k)$, there exists $g \in \mathrm{GL}_N(k)$ such that $V_{x_2} = g(V_{x_1})$.*

Proof. After replacing X by some Zariski dense open subset, we may assume that there exists $d \geq 1$ such that $\deg V_x = d$ for all $x \in X(k)$. Let M_d be the variety parametrizing all effective cycles in $\mathbb{A}_k^N$ of degree d. Then f induces a rational map

$$F : X \times M_d \dashrightarrow X \times M_d \text{ given by } (x, W) \mapsto (g(x), A(x)(W)).$$

Let $\pi_1 : X \times M_d \to X$ be the projection onto the first coordinate.

At the expense of replacing X by some Zariski dense open subset, we may assume that the map s given by $x \mapsto (x, V_x)$ is a section from X to $X \times M_d$. For any point $W \in M_d(k)$, there is a morphism

$$\chi_W : X \times \mathrm{GL}_N \to X \times M_d \text{ given by } (x, g) \to (x, g(W)).$$

We note that $\chi_W(X \times \mathrm{GL}_N) = \chi_{g(W)}(X \times \mathrm{GL}_N)$ for any $g \in \mathrm{GL}_N(k)$.

The next Lemma yields (essentially) the conclusion in Proposition 2.2.

Lemma 2.3. *There exists $W \in M_d(k)$ and a Zariski dense open set U of X such that $s(U) \subseteq \chi_W(X \times \mathrm{GL}_N)$.*

Proof of Lemma 2.3. Let K be an algebraically closed uncountable field containing k. By [AC08] (see also [BGR17]), there exists a K-point $\alpha \in X(K)$ such that $g^n(\alpha) \notin \mathcal{B}$ for all $n \geq 0$ and its orbit is Zariski dense in X_K.

For all $n \in \mathbb{N}_0$, we have $s(g^n(\alpha)) \in \chi_{V_\alpha}(X_K \times \mathrm{GL}_N)$. Hence $s^{-1}\left(\chi_{V_\alpha}\right)(X_K \times \mathrm{GL}_N)$ is a Zariski dense constructible set in X_K and so, it contains a Zariski dense open set U_K in X_K. Since $X(k)$ is Zariski dense in X_K, there exists a point $\beta \in U_K \cap X(k)$. It follows that there exists $g \in \mathrm{GL}_N(k)$ such that $g(V_\alpha) = V_\beta$. Then we have $V_\beta \in M_d(k)$ and $\chi_{V_\alpha}(X_K \times \mathrm{GL}_N) = \chi_{V_\beta}(X_K \times \mathrm{GL}_N)$. Since s and $\chi_{V_\beta}(X_K \times \mathrm{GL}_N)$ are both defined over k, then $s^{-1}\left(\chi_{V_\beta}\right)(X_K \times \mathrm{GL}_N)$ is a Zariski dense constructible set in X_K which is defined over k. It follows that as a k-variety, $s^{-1}\left(\chi_{V_\beta}\right)(X \times \mathrm{GL}_N)$ is a Zariski dense constructible set in X. Then there exists a Zariski dense open set U of X such that $s(U) \subseteq \chi_{V_\beta}(X \times \mathrm{GL}_N)$, which concludes the proof of Lemma 2.3. $\square$

Now, let U be as in the conclusion of Lemma 2.3. Then for any $x_1, x_2 \in U$ there are $g_1, g_2 \in \mathrm{GL}_N(k)$ such that $V_{x_1} = g_1(W)$ and $V_{x_2} = g_2(W)$. Therefore $V_{x_2} = g_2 g_1^{-1}(V_{x_1})$, as desired in the conclusion of Proposition 2.2. $\square$

We observe that Proposition 2.2 applies to each $V \in I$ and so, we let U_V be the Zariski open subset of X satisfying the conclusion of Proposition 2.2 with respect to the variety V.

For any $V \in I$ and any points $\alpha, \beta \in U_V$, denote by $G_{\beta,\alpha}^V$ the set of $g \in \mathrm{GL}_N(k)$ such that $g(V_\beta) = V_\alpha$. By Proposition 2.2, the set $G_{\beta,\alpha}^V$ is nonempty, so let $g_{\beta,\alpha}^V$ be an element of $G_{\beta,\alpha}^V$. Then $G_{\beta,\alpha}^V = g_{\beta,\alpha}^V G_{\beta,\beta}^V$; we note that $G_{\beta,\beta}^V$ is an algebraic subgroup of $\mathrm{GL}_N(k)$.

The next result yields that the (a priori disjoint) sets $G_{\beta,\alpha}^V$ all contain some given sets $G_{\beta,\alpha}^S$ for a suitable invariant cycle $S \in I$.

Lemma 2.4. *There exists an effective invariant cycle $S \in I$ such that for any $V \in I$, there exists a Zariski dense open set $U \subseteq U_S \cap U_V$ with the property that for any two points $x_1, x_2 \in U(k)$ we have $G_{x_1,x_2}^S \subseteq G_{x_1,x_2}^V$.*

Proof. We note that, if $V_1, \ldots, V_s$ are invariant effective cycles in I, then $\sum_{i=1}^{s} n_i V_i$ (for arbitrary $n_i \in \mathbb{N}$) is also contained in I.

Let K be an algebraically closed field containing k such that the cardinality of K is strictly larger that the cardinality of I. Then there exists a point $\beta \in X(K)$ such that

$$\beta \in \bigcap_{V \in I} U_V(K). \tag{2.1}$$

For any $V \in I$, denote by $V_\beta := V \cap \pi^{-1}(\{\beta\})$ the fiber of V_K at the point $\beta \in X(K)$ from (2.1). Let

$$G_\beta^V := \{g' \in \mathrm{GL}_N \colon g'(V_\beta) = V_\beta\};$$

then G_β^V is an algebraic subgroup of GL_N. We also let

$$G_\beta := \bigcap_{V \in I} G_\beta^V;$$

then there exists a finite subset $\{V_1, \ldots, V_s\} \subseteq I$ such that

$$G_\beta := \bigcap_{i=1}^{s} G_\beta^{V_i}.$$

Let M be the maximum of the multiplicities of all irreducible components of $(V_1)_\beta, \ldots, (V_s)_\beta$ and let

$$S := \sum_{i=1}^{s} (M+1)^{i-1} V_i \in I.$$

Then for any $g' \in \mathrm{GL}_N(K)$, we have $g'(S_\beta) = S_\beta$ if and only if $g'((V_i)_\beta) = (V_i)_\beta$ for all $i = 1, \ldots, s$. In other words,

$$G_\beta^S = \bigcap_{i=1}^{s} G_\beta^{V_i} = G_\beta.$$

For any $V \in I$, denote by A_V the maximum of all multiplicities of all irreducible components of V. Now for any $V \in I$, let

$$M_V := \max\{A_V, A_S\} + 1 \text{ and } W := S + M_V \cdot V \in I,$$

and also let $U := U_W \cap U_V \cap U_S$ where the open sets U_W, U_V and U_S satisfy the conclusion of Proposition 2.2. For any $x_1, x_2 \in U(k)$, we claim that

$$G_{x_1,x_2}^S \subseteq G_{x_1,x_2}^V. \tag{2.2}$$

Since both G_{x_1,x_2}^S and G_{x_1,x_2}^V are defined over k and k is algebraically closed, we only need to show the inclusion (2.2) after base change K/k. So, we only need to show that $G_{x_1,x_2}^S(K) \subseteq G_{x_1,x_2}^V(K)$. Since $\beta \in U_W(K)$, for any $i = 1, 2$, there exists g_{β,x_i} satisfying

$$g_{\beta,x_i}(S_\beta + M_V \cdot V_\beta) = S_{x_i} + M_V \cdot V_{x_i}.$$

It follows that $g_{\beta,x_i}(S_\beta) = S_{x_i}$ and $g_{\beta,x_i}(V_\beta) = V_{x_i}$. Then we have

$$G_{x_1,x_2}^S(K) = g_{\beta,x_2} G_\beta^S(K) g_{\beta,x_1}^{-1} = g_{\beta,x_2} G_\beta(K) g_{\beta,x_1}^{-1}$$

and

$$G_{x_1,x_2}^V(K) = g_{\beta,x_2} G_\beta^V(K) g_{\beta,x_1}^{-1}.$$

Since $G_\beta \subseteq G_\beta^V$, we have $G_{x_1,x_2}^S(K) \subseteq G_{x_1,x_2}^V(K)$, as desired in Lemma 2.4. $\square$

Now we have all ingredients necessary to finish the proof of Theorem 2.1.

Proof of Theorem 2.1. Fix a point $\alpha \in U_S(k)$. Then $G_\alpha := G^S_{\alpha,\alpha}$ is an algebraic subgroup of GL_N. Let $\mathcal{G}$ be the subvariety of $U_S \times \mathrm{GL}_N$ of points $(x, g') \in U_S \times \mathrm{GL}_N$ such that $S_x = g'(S_\alpha)$. Lemma 2.2 yields that $\mathcal{G}$ is a G_α-torsor on U_S. Denote by

$$p : \mathcal{G} \subseteq U_S \times \mathrm{GL}_N \to U_S$$

the projection on the first coordinate. For any $x \in U_S$, let $G_x := G^S_{\alpha,x}$. We note that for any $x_1, x_2 \in U_S$, we have $G^S_{x_1,x_2} = G^S_{x_2} G^{-1}_{x_1}$. Note that for any $x \in (U_S \setminus \mathcal{B}) \cap g|^{-1}_{U_S \setminus \mathcal{B}}(U_S)$, we have $g' \in G_x$ and $A(x)g'(S_\alpha) = A(x)S_x = S_{g(x)}$. Then f induces a dominant rational map F on $\mathcal{G}$ defined by $(x, g') \mapsto (g(x), A(x)g')$.

Let G^0_α be the connected component of G_α; also let $\mu := G_\alpha/G^0_\alpha$, which is a finite group. Then the quotient $Y' := \mathcal{G}/G^0_\alpha$ is a μ-torsor on U_S. Observe that F induces a rational self-map f' on Y' such that $\pi' \circ f' = g \circ \pi'$ where $\pi' : Y' \to U_s$ is the projection to the base U_S. Let Y be an irreducible component of Y'. Then there exists $\ell \geq 1$ such that ${f'}^{\ell}(Y) = Y$. Let $g' := {f'}^{\ell}|_Y$ and $\tau := \pi'|_Y$. Then we have $\tau \circ g' = g^\ell \circ \tau$.

Now we consider the dominant rational map $f_Y := \mathrm{id} \times_X f$ on the base change $Y \times_X X \times \mathbb{A}^N$. Let $\mathcal{G}_Y := Y \times_{U_S} \mathcal{G}$. Then $\mathcal{G}_Y/G^0_\alpha = Y \times_{U_S} Y'$ has a section

$$T_0 : Y \to Y \times_{U_S} Y \subseteq \mathcal{G}_Y/G^0_\alpha \text{ sending } y \to (y, y).$$

The preimage $\mathcal{G}^0_Y$ of $T(Y)$ in $\mathcal{G}_Y$ is a connected component of $\mathcal{G}_Y$ which is a G^0_α torsor on Y. By [CO92], there exists a rational section

$$T : Y \to \mathcal{G}^0_Y \text{ satisfying } p_Y \circ T = \mathrm{id},$$

where p_Y is the projection from $\mathcal{G}_Y$ to Y and $T(\alpha) = 1 \in G_\alpha$ (i.e., $T(\alpha)$ is the identity element of G_α). We note that for any $x \in Y$, we have $T(x) \in G_{\tau(x)}$.

Let h be the rational map on $Y \times \mathbb{A}^N_k$ defined by $(x, y) \mapsto (x, T(x)y)$. Let $V \in I$ be an invariant subvariety of $X \times \mathbb{A}^N_k$. For any point $x \in Y$, denote by $(\tau \times_X \mathrm{id})^{-1}(V)_x$ the fiber of $(\tau \times_X \mathrm{id})^{-1}(V)$ at x. As a subvariety in A^N_k, we have $(\tau \times_X \mathrm{id})^{-1}(V)_x = V_{\tau(x)}$.

By Lemma 2.4, there exists a Zariski dense open set $U \subseteq U_S \cap U_V$ such that for any two points $x_1, x_2 \in U(k)$ we have $G^S_{x_1,x_2} \subseteq G^V_{x_1,x_2}$.

Pick a point $u_1 \in \tau^{-1}U(k)$. Let $V_0 := T(u_1)^{-1}((\tau \times_X \mathrm{id})^{-1}(V)_{u_1})$. For any $u_2 \in \tau^{-1}U(k)$, let $x_1 = \tau(u_1)$ and $x_2 = \tau(u_2)$. Since $T(u_i) \in G_{x_i}$ for $i = 1, 2$, we have $T(u_2)T(u_1)^{-1} \in G^S_{x_1,x_2} \subseteq G^V_{x_1,x_2}$. It follows that $T(u_2)T(u_1)^{-1}(V_{x_1}) = V_{x_2}$. We have

$$V_0 = T(u_1)^{-1}((\tau \times_X \mathrm{id})^{-1}(V)_{u_1}) = T(u_1)^{-1}(V_{x_1})$$

$$= T(u_2)^{-1}(T(u_2)T(u_1)^{-1}(V_{x_1})) = T(u_2)^{-1}(V_{x_2}) = T(u_2)^{-1}((\tau \times_X \mathrm{id})^{-1}(V)_{u_2}).$$

Then we get $h^{-1}(V) = Y \times V_0$, which concludes the proof of Theorem 2.1. $\square$

2.5. **Proof of Theorem 1.4.** We work under the hypotheses of Theorem 1.4.

Let $\mathcal{B}$ be the set of points $x \in X$ such that f is not a locally isomorphism on the fiber $\pi^{-1}(x)$. Then $\mathcal{B}$ is a proper closed subset of X.

If there exists a nonconstant rational function ψ on X invariant under g, then the nonconstant rational function $\psi \circ \pi$ on $X \times \mathbb{A}^N_k$ is invariant under f. So Theorem 1.4 holds. Now we may assume that there is no nonconstant rational function on X invariant under g. Then there exists a Zariski dense orbit in $X(k)$ under the action of g. Moreover, for any Zariski dense open set U of X, since the pair $(U, g|_U)$ is birationally equivalent to (X, g), then there exists a point $x_U \in U(k)$ with a Zariski dense orbit under the action of $g|_U$.

Let I be the set of all invariant subvarieties in $X \times \mathbb{A}^N_k$ for which every irreducible component of V dominates X under the projection map $X \times \mathbb{A}^N \longrightarrow X$.

Theorem 2.1 yields that (perhaps, at the expense of replacing f by a suitable iterate) there exists an irreducible variety Y endowed with a dominant rational self-map

$$g' : Y \dashrightarrow Y$$

and a generically finite map $\tau : Y \dashrightarrow X$ satisfying $\tau \circ g' = g \circ \tau$ such that there exists a birational map h on $Y \times \mathbb{A}_k^N = Y \times_X X \times \mathbb{A}_k^N$ of the form $(x, y) \mapsto (x, T(x)y)$ where $T(x) \in \mathrm{GL}_N(k(Y))$ such that for any subvariety $V \in I$, we have

$$h^{-1}((\tau \times_X \mathrm{id})^{\#}(V)) = Y \times V_0 \subseteq Y \times \mathbb{A}_k^N,$$

where V_0 is a subvariety of $\mathbb{A}_k^N$. Let $f' : Y \times \mathbb{A}^N \to Y \times \mathbb{A}^N$ be the rational map defined by

$$g' \times_{(X,g)} f : (x, y) \mapsto (g'(x), A(\tau(x))y).$$

We have $(\tau \times \mathrm{id}) \circ f' = f \circ (\tau \times \mathrm{id})$. Let

$$F := h^{-1} \circ f' \circ h : Y \times \mathbb{A}^N \to Y \times \mathbb{A}^N.$$

Then F is the map $(x, y) \mapsto (g'(x), B(x)y)$ where $B(x) := T^{-1}(g'(x))A(\tau(x))T(x)$. Let $\rho := (\tau \times \mathrm{id}) \circ h$. Then we have $\rho \circ F = f \circ \rho$. For any $V \in I$, we see that $\rho^{\#}(V)$ is invariant by F and it has the form $Y \times V_0$.

After replacing Y by some smaller open subset, we may assume that ρ is a regular morphism. Furthermore, we may assume that ρ is locally finite. Let

$$p : Y \times \mathbb{A}^N \to Y$$

be the projection to the first coordinate. Let $\mathcal{B}'$ be the set of points $x \in Y$ such that F is not locally an isomorphism on the fiber $p^{-1}(x)$. Then $\mathcal{B}'$ is a proper closed subset of Y. There exists a point $\alpha \in X(k)$, such that $\mathcal{O}_g(\alpha) \cap \mathcal{B} = \emptyset$; here we use the assumption about (X, g) being a good dynamical pair (so, in particular, there exists a point with a Zariski dense orbit contained in the complement of $\mathcal{B}$). At the expense of replacing α by some $g^n(\alpha)$, we may suppose that there exists a point $\beta \in Y$ such that $\tau(\beta) = \alpha$ and so, $\mathcal{O}_{g'}(\beta) \cap \mathcal{B}' = \emptyset$. Also, we may suppose that $T(\beta) = \mathrm{id}$.

For any $x \in X$ and $V \in I$, denote by $V_x := \pi^{-1}(x) \cap V \subseteq \mathbb{A}_k^N$. By Lemma 2.2, there exists a Zariski open set U_V of X such that such that for any points $x_1, x_2 \in U_V(k)$, there exists $g' \in \mathrm{GL}_N(k)$ such that $V_{x_2} = g'(V_{x_1})$. There exists $m \geq 0$, such that $g^m(\alpha) \in U_V$. There exists an open set U' containing α, such that $g^i(U') \cap \mathcal{B} = \emptyset$ for $i = 0, \ldots, m$ and moreover, $g^m(U') \subseteq U_V$. Then for any points $x_1, x_2 \in U'(k)$, there exists $g' \in \mathrm{GL}_N(k)$ such that

$$A(g^{m-1}(x_2)) \cdots A(x_2)V_{x_2} = g'A(g^{m-1}(x_1)) \cdots A(x_1)(V_{x_1});$$

it follows that

$$V_{x_2} = \left(A(g^{m-1}(x_2)) \cdots A(x_2)\right)^{-1} g'A(g^{m-1}(x_1)) \cdots A(x_1)(V_{x_1}).$$

So we may replace U_V by U' and therefore assume that $\alpha \in U_V$ for all $V \in I$.

For any $V \in I$, any points x_1, x_2 in U_V, denote by G_{x_1,x_2}^V the set of $g' \in \mathrm{GL}_N(k)$ such that $g'(V_{x_1}) = V_{x_2}$. Then there exists an element $g_{x_1,x_2}^V \in G_{x_1,x_2}^V$. We note that $G_\alpha^V := G_{\alpha,\alpha}^V$ is an algebraic subgroup of $\mathrm{GL}_N(k)$. Let $G_\alpha := \cap_{V \in I} G_\alpha^V$; this is an algebraic subgroup of GL_N. For any $V \in I$, we have $\rho^{\#}(V) = \rho^{-1}(V) = Y \times V_\alpha$. Hence $B(x) \in G_\alpha^V$ for all $x \in Y \setminus \mathcal{B}'$ and thus $B(x) \in G_\alpha$ for all $x \in Y \setminus \mathcal{B}'$.

Theorem 1.5 shows that either there exists a point $y \in \mathbb{A}^N(k)$ such that $G_\alpha \cdot y$ is Zariski dense in $\mathbb{A}^N$ or there exists a nonconstant rational function $\phi \in k(\mathbb{A}^N)$ such that $\phi \circ g' = \phi$ for all $g' \in G_\alpha$.

At first, we suppose that there exists a point $y \in \mathbb{A}^N(k)$ such that $G_\alpha \cdot y$ is Zariski dense in $\mathbb{A}^N$. Furthermore, Theorem 1.5 yields that *each* point in a dense open subset of $\mathbb{A}^N$ would have a Zariski dense orbit under the action of G_α. Now, let $\gamma := (\alpha, y) \in X \times \mathbb{A}^N$. Denote by Z the Zariski closure of $\mathcal{O}_f(\gamma)$. Since $\mathcal{O}_g(\alpha)$ is Zariski dense in X, then Z has at least one irreducible component which dominates X. Let V be the union of all irreducible components of Z which dominate X; then $V \in I$. There exists $m \geq 0$ such that $f^m(\alpha) \in V$ and so, $f^n(\alpha) \in V$ for all $n \geq m$.

Let γ' be the unique preimage of γ under ρ in the fiber $\pi^{-1}(\beta)$. Since we have assumed that $T(\alpha) = \mathrm{id}$, we have $\gamma' = (\beta, y)$. Then

$${f'}^m(\gamma') \in \rho^{-1}(V) = \rho^{\#}(V) = Y \times V_\alpha.$$

It follows that $B\left(g^{'(m-1)}(\beta)\right)\cdots B(g'(\beta))\cdot B(\beta)y \in V_\alpha$. Since

$$B\left(g^{'(m-1)}(\beta)\right)\cdots B(g'(\beta))\cdot B(\beta)\in G_\alpha \subseteq G_\alpha^V,$$

we have $y \in V_\alpha$. Then we have $G_\alpha \cdot y \subseteq V_\alpha$. Since $G_\alpha \cdot y$ is Zariski dense in $\mathbb{A}^N$, we have $V_\alpha = \mathbb{A}^N$. Then $\rho^{-1}(V) = Y \times A^N$. It follows that $V = X \times \mathbb{A}^N$. So $\mathcal{O}_f(\gamma)$ is Zariski dense in $X \times \mathbb{A}^N$.

Furthermore, we see that since any $\gamma = (\alpha, y)$ would have a Zariski dense orbit under f, where α is a point with a Zariski dense orbit under g avoiding $\mathcal{B}$ and therefore (since the pair (X, g) is good), α may be chosen in any open subset of X, while y is any point in a given open subset of $\mathbb{A}^N$, we have that there exist points with Zariski dense orbits under f in any nontrivial, open subsets of $X \times \mathbb{A}^N$. Hence, for any other dynamical pair (W, h), which is birationally equivalent to $(X \times \mathbb{A}^N, f)$, there exist k-points in W with a Zariski dense orbit under h (see Remark 1.3).

Now we assume that there exists a nonconstant rational function $\phi \in k(\mathbb{A}^N)$ such that $\phi \circ g' = \phi$ for all $g' \in G_\alpha$. Let χ be the rational function on $Y \times A^N$ defined by $(x, y) \mapsto \phi(y)$; it is invariant by f'. Let ψ be the rational function on $X \times \mathbb{A}^N$ defined by

$$\psi(x) = \prod_{x' \in \rho^{-1}(x)} \chi(x').$$

Then ψ is a nonconstant rational function on $X \times \mathbb{A}^N$ invariant under f; according to Remark 1.3, each dynamical pair (W, h), which is equivalent with $(X \times \mathbb{A}^N, f)$, also fixes some nonconstant fibration. This concludes the proof of Theorem 1.4.

3. A special class of automorphisms of the affine space

In this section we study in-depth the special case in Theorem 2.1 when $X = \mathbb{A}^1$ and $f : \mathbb{A}^1 \times \mathbb{A}^N \longrightarrow \mathbb{A}^1 \times \mathbb{A}^N$ is an automorphism given by $(x, y) \mapsto (x + 1, A(x)y)$ for some $A \in \mathrm{GL}_N(k[x])$. This leads to proving Theorem 1.6 and also to developing a theory of *straight models* (see Subsection 3.2) for linear transformations $A \in \mathrm{GL}_N(k[x])$, which we believe is of independent interest. In particular, we believe our results would be helpful for understanding better which points in $\mathbb{A}^1_k \times \mathbb{A}^N_k$ have Zariski dense orbits under an automorphism f as above.

3.1. Proof of Theorem 1.6.

We work under the hypotheses of Theorem 1.6. So, N is a positive integer, $A \in \mathrm{GL}_N(k[x])$ and $f : \mathbb{A}^1 \times \mathbb{A}^N \longrightarrow \mathbb{A}^1 \times \mathbb{A}^N$ is an automorphism given by $(x, y) \mapsto (x + 1, A(x)y)$.

For each $x \in \mathbb{A}^1(k)$, and each subvariety V invariant under f, we let

$$V_x := \pi^{-1}(x) \cap V \subseteq \mathbb{A}^N_k.$$

The next result is a more precise version of Proposition 2.2 in our setting.

Lemma 3.1. *For each $x \in \mathbb{A}^1(k)$, there exists $g_x \in \mathrm{GL}_N(k)$ such that $V_x = g_x(V_0)$.*

Proof. Let $d = \deg V_0$; then $\deg V_x = d$ for all $x \in \mathbb{A}^1(k)$. Let M_d be the variety parametrizing all subvarieties of $\mathbb{A}^N_k$ of degree d. Then f induces an automorphism

$$F : \mathbb{A}^1_k \times M_d \longrightarrow \mathbb{A}^1_k \times M_d \text{ defined by } (x, W) \mapsto (x + 1, A(x)(W)).$$

Denote by $\pi_1 : \mathbb{A}^1_k \times M_d \longrightarrow \mathbb{A}^1_k$ the projection to the first coordinate. There exists a section $s : \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k \times M_d$ defined by $x \mapsto (x, V_x)$ and there exists a morphism

$$\chi : \mathbb{A}^1_k \times \mathrm{GL}_N \longrightarrow \mathbb{A}^1 \times M_d \text{ given by } (x, g) \to (x, g(V_0)).$$

For all $n \in \mathbb{Z}$, we have that $s(n) \in \chi(\mathbb{A}^1_k \times \mathrm{GL}_N)$; therefore $s^{-1}\left(\chi(\mathbb{A}^1_k \times \mathrm{GL}_N)\right)$ is a Zariski dense constructible set in $\mathbb{A}^1_k$, thus it is a Zariski dense open subset of $\mathbb{A}^1_k$.

Observe that $s(\mathbb{A}^1_k)$ and $\chi(\mathbb{A}^1_k \times \mathrm{GL}_N)$ are invariant under F and so,

$$s(\mathbb{A}^1_k) \cap \chi(\mathbb{A}^1_k \times \mathrm{GL}_N) \text{ is also invariant under } F.$$

Thus $s^{-1}\left(\chi(\mathbb{A}^1_k \times \mathrm{GL}_N)\right)$ is invariant under $x \mapsto x+1$. Then $s^{-1}\left(\chi(\mathbb{A}^1_k \times \mathrm{GL}_N)\right) = \mathbb{A}^1_k$. Therefore for any $x \in \mathbb{A}^1(k)$, there exists $g_x \in \mathrm{GL}_N(k)$ such that $V_x = g_x(V_0)$. □

Let $\mathcal{G}^V := \{(x,g) \in \mathbb{A}^1_k \times \mathrm{GL}_N(k) \colon g(V_0) = V_x\}$. Then $\mathcal{G}^V$ is a subvariety of $\mathbb{A}^1_k \times \mathrm{GL}_N(k)$. Denote by $p_V : \mathcal{G}^V \longrightarrow \mathbb{A}^1_k$ the projection onto the first coordinate. For each $x \in \mathbb{A}^1(k)$, let $G^V_x := p_V^{-1}(x)$. We have $G^V_x = g_x G^V_0$.

Let I be the set of all invariant subvarieties in $\mathbb{A}^1_k \times \mathbb{A}^N_k$. Set $\mathcal{G} := \cap_{V \in I} \mathcal{G}^V$; it is a subvariety of $\mathbb{A}^1_k \times \mathrm{GL}_N(k)$. Denote by $p : \mathcal{G} \to \mathbb{A}^1_k$ the projection onto the first coordinate. Set $G_x := p^{-1}(x)$ for all $x \in \mathbb{A}^1(k)$. We have $G_x = g_x G_0$. Then $\mathcal{G}$ is a G_0-torsor on $\mathbb{A}^1_k$; in the next result we will show that $\mathcal{G}$ must be trivial.

Lemma 3.2. *Any G_0-torsor $\mathcal{G}$ on $\mathbb{A}^1_k$ is trivial.*

Proof. Let G^0_0 be the connected component of G_0, which is a normal subgroup of G_0. Consider the exact sequence

$$1 \to G^0_0 \to G_0 \to G_0/G^0_0 \to 1;$$

then we have the exact sequence

$$H^1_{\acute{e}t}(\mathbb{A}^1_k, G^0_0) \to H^1_{\acute{e}t}(\mathbb{A}^1_k, G_0) \to H^1_{\acute{e}t}(\mathbb{A}^1_k, G_0/G^0_0).$$

Since G_0/G^0_0 is finite and $\mathbb{A}^1_k$ is simply connected, then $H^1_{\acute{e}t}(\mathbb{A}^1_k, G_0/G^0_0) = 1$. So, we only need to show that $H^1_{\acute{e}t}(\mathbb{A}^1_k, G^0_0) = 1$.

Let R be the radical of G^0_0. Consider the exact sequence

$$1 \to R \to G^0_0 \to G^0_0/R \to 1.$$

We get the exact sequence

$$H^1_{\acute{e}t}(\mathbb{A}^1_k, R) \to H^1_{\acute{e}t}(\mathbb{A}^1_k, G^0_0) \to H^1_{\acute{e}t}(\mathbb{A}^1_k, G^0_0/R).$$

Since G^0_0/R is semisimple, by [CGP12] (see also [RR84]) we have $H^1_{\acute{e}t}(\mathbb{A}^1_k, G^0_0/R) = 1$. Thus we only need to show that $H^1_{\acute{e}t}(\mathbb{A}^1_k, R) = 1$. Since R is solvable, all is left to prove is that $H^1_{\acute{e}t}(\mathbb{A}^1_k, \mathbb{G}_m)$ and $H^1_{\acute{e}t}(\mathbb{A}^1_k, \mathbb{G}_a)$ are trivial. Obviously $H^1_{\acute{e}t}(\mathbb{A}^1_k, \mathbb{G}_m) = \mathrm{Pic}(\mathbb{A}^1_k)$ is trivial and $H^1_{\acute{e}t}(\mathbb{A}^1_k, \mathbb{G}_a) = H^1(\mathbb{A}^1, O_{\mathbb{A}^1}) = 1$, by [Har77, Theorem 3.5, Chapter III, p. 215]. This concludes our proof of Lemma 3.2. □

So, there exists a section $T : \mathbb{A}^1 \to \mathcal{G}$ satisfying $p \circ T = \mathrm{id}$ and $T(0) = 1 \in G_0$. Then $T \in \mathrm{GL}_N(k[x])$ and for all $x \in \mathbb{A}^1(k)$, we have $T(x) \in g_x G_0$. Let h be the automorphism on $\mathbb{A}^1_k \times \mathbb{A}^N_k$ defined by $(x,y) \mapsto (x, T(x)y)$. Let $V \in I$ be an invariant subvariety of $\mathbb{A}^1_k \times \mathbb{A}^N_k$ under the action of f. Then for any $x \in \mathbb{A}^1(k)$, we have

$$T(x)^{-1}(V_x) = T(x)^{-1}(g_x(V_0)) = V_0,$$

and so, we have $h^{-1}(V) = \mathbb{A}^1_k \times V_0$, which concludes the proof of Theorem 1.6.

3.2. **A straight model.** In this section we continue our study of the dynamical properties of automorphisms f of $\mathbb{A}^1 \times \mathbb{A}^N$ of the form $(x,y) \mapsto (x+1, A(x)y)$. We will prove Theorem 1.8 (see Corollary 3.9) which says that each periodic subvariety of $\mathbb{A}^1 \times \mathbb{A}^N$ under the action of f has its period uniformly bounded depending only on the matrix A.

For any $A \in \mathrm{GL}_N(k[x])$, denote by $f_A : \mathbb{A}^1_k \times \mathbb{A}^N_k \to \mathbb{A}^1_k \times \mathbb{A}^N_k$ the automorphism defined by $(x,y) \mapsto (x+1, A(x)y)$.

Definition 3.3. We say that A and A' are equivalent if f_A and $f_{A'}$ are conjugate by an automorphism of $\mathbb{A}^1_k \times \mathbb{A}^N_k$ given by $(x,y) \mapsto (x, T(x)y)$, i.e., if there exists an element $T \in \mathrm{GL}_N(k[x])$, such that $A'(x) = T(x+1)^{-1}A(x)T(x)$.

Denote by $\mathcal{P}^N$ the set of all subvarieties of $\mathbb{A}^N_k$ and denote by $\mathcal{P}^N_1$ the set of all subvarieties of $\mathbb{A}^1_k \times \mathbb{A}^N_k$ which dominate $\mathbb{A}^1_k$ under the projection map $\mathbb{A}^1 \times \mathbb{A}^N \longrightarrow \mathbb{A}^1$. We consider the map

$$r_0 : \mathcal{P}^N_1 \to \mathcal{P}^N \text{ given by } V \mapsto V \cap \pi^{-1}(0) \tag{3.1}$$

and also, consider the section $\sigma : \mathcal{P}^N \to \mathcal{P}^N_1$ given by $W \mapsto \mathbb{A}^1_k \times W$; then we have $r_0 \circ \sigma = \mathrm{id}$.

Let I_A be the set of all subvarieties $V \in \mathcal{P}_1^N$ which are invariant under f_A. Let I_A^0 be the set of all subvarieties $W \in \mathcal{P}^N$ such that $\sigma(W)$ is invariant under f_A. We have $\sigma(I_A^0) \subseteq I_A$ and $I_A^0 \subseteq r_0(I_A)$.

Lemma 3.4. *The map $r_0|_{I_A} : I_A \longrightarrow \mathcal{P}^N$ is injective.*

Proof. Let V_1, V_2 be two elements in I_A. Then $V_1 \cup V_2$ is also an element in I_A. If $r_0(V_1) = r_0(V_2)$, then $r_0(V_1) = r_0(V_1 \cup V_2)$. Lemma 3.1 yields that

$$\pi^{-1}(x) \cap V_1 = \pi^{-1}(x) \cap (V_1 \cup V_2)$$

for all $x \in \mathbb{A}^1(k)$. Then we have $V_1 = V_2$, as desired. $\square$

Definition 3.5. We say that A (or f_A) is *straight*, if $r_0(I_A) = I_A^0$.

Lemma 3.4, shows that A is straight if and only if $I_A = \sigma(I_A^0)$, i.e., all invariant subvarieties of $\mathbb{A}_k^1 \times \mathbb{A}_k^N$ are of the form $\mathbb{A}_k^1 \times W$.

For every $W \in \mathcal{P}^N$, denote by G_W the subgroup of GL_N which fixes W. Let

$$\mathbb{G}_A := \bigcap_{W \in r_0(I_A)} G_W;$$

this is an algebraic subgroup of GL_N. Let $A' \in \mathbb{G}_A(k[x])$ be an element equivalent to A, i.e., $A'(x) = T^{-1}(x+1)A(x)T(x)$ where $T \in \mathrm{GL}_N(k[x])$. Then we have

$$r_0(I'_A) = T(0)^{-1}(r_0(I_A)) \text{ and } \mathbb{G}_{A'} = T(0)^{-1}\mathbb{G}_A T(0).$$

So the conjugacy class of $\mathbb{G}_A$ in GL_N is an invariant in the equivalent class of A.

Remark 3.6. Theorem 1.6 (see also Remark 1.7) yields that for every $A \in \mathrm{GL}_N(k[x])$, there exists $A' \in \mathbb{G}_A(k[x])$ which is straight and moreover, A' and A are equivalent.

Proposition 3.7. *An element $A \in \mathrm{GL}_N(k[x])$ is straight if and only if $A(x) \in \mathbb{G}_A(k[x])$.*

Proof. First we suppose that A is straight. For any $W \in I_A^0 = r_0(I_A)$, we have that $\mathbb{A}_k^1 \times W$ is invariant under f_A. It follows that $A(x) \in G_W(k[x])$. Then

$$A(x) \in \bigcap_{W \in r_0(I_A)} G_W(k[x]) = \mathbb{G}_A(k[x]).$$

If $A(x) \in \mathbb{G}_A(k[x])$, then for each $V \in I_A$, we have that $W := r_0(V)$ is invariant under the action of $\mathbb{G}_A$. Then $\mathbb{A}_k^1 \times W$ is invariant under the action of f_A. So, $\mathbb{A}_k^1 \times W \in I_A$ and $r_0(\mathbb{A}_k^1 \times W) \in I_A = V$. Therefore $V = \mathbb{A}_k^1 \times W$, as claimed in the conclusion of Proposition 3.7. $\square$

The next result yields a good criterion for when a point $\alpha \in \mathbb{A}^1(k) \times \mathbb{A}^N(k)$ has a Zariski dense orbit under f.

Proposition 3.8. *Let $\alpha := (a, b) \in \mathbb{A}^1(k) \times \mathbb{A}^N(k)$ and let $A \in \mathrm{GL}_N(k[x])$ be a straight linear transformation. Then $\overline{\mathcal{O}_{f_A}(\alpha)} = \mathbb{A}_k^1 \times \overline{\mathbb{G}_A \cdot b}$.*

Proof. Since $\mathbb{A}_k^1 \times \overline{\mathbb{G}_A \cdot b}$ is f_A-invariant and $\alpha \in \mathbb{A}_k^1 \times \overline{\mathbb{G}_A \cdot b}$, then $\overline{\mathcal{O}_{f_A}(\alpha)} \subseteq \mathbb{A}_k^1 \times \overline{\mathbb{G}_A \cdot b}$. Using that $\overline{\mathcal{O}_{f_A}(\alpha)}$ is f_A-invariant, we get that there exists $W \in r_0(I_A)$ such that $\overline{\mathcal{O}_{f_A}(\alpha)} = \mathbb{A}_k^1 \times W$. By the definition of $\mathbb{G}_A$, we know that W is $\mathbb{G}_A$-invariant. Since $b \in W$, we have $\overline{\mathbb{G}_A \cdot b} \subseteq W$. Thus $\mathbb{A}_k^1 \times \overline{\mathbb{G}_A \cdot b} \subseteq \overline{\mathcal{O}_{f_A}(\alpha)}$, as desired. $\square$

We are ready now to prove that each periodic subvariety under the action of f_A has its period bounded depending only on A (see Theorem 1.8).

Corollary 3.9. *Let V be a periodic subvariety of f_A of period m. Then m divides the number of connected components of $\mathbb{G}_A$. In particular, the period of each periodic subvariety of $\mathbb{A}^1 \times \mathbb{A}^N$ under the action of f_A is uniformly bounded by a constant depending only on A.*

Proof. For each $i = 0, \dots, m-1$, let $W_i := r_0(f_A^i(V))$. We may assume that A is straight (see Remark 3.6). Then $f^i(V) = \mathbb{A}_k^1 \times W_i$. Since $\cup_{i=0}^{m-1} f_A^i(V)$ is invariant by f_A, then $\cup_{i=0}^{m-1} W_i$ is invariant by $\mathbb{G}_A$. Therefore $\mathbb{G}_A$ acts on the set $\{W_0, \dots, W_{m-1}\}$ transitively, thus proving that m must divide the number of components of $\mathbb{G}_A$. $\square$

3.3. **Straight forms when N is 2.** In this section, let $f : \mathbb{A}^1 \times \mathbb{A}^2 \longrightarrow \mathbb{A}^1 \times \mathbb{A}^2$ be an automorphism of the form $(x, y) \mapsto (x + 1, A(x)y)$.

We say that an invariant subvariety V of f is nontrivial, if V is not equal with $\mathbb{A}^1 \times \{0\}$ or with $\mathbb{A}^1 \times \mathbb{A}^2$.

Lemma 3.10. *If* $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ *where* $a_1, a_2 \in k^*$*, then* f_A *is straight.*

Proof. Let V be a nontrivial invariant subvariety of f. We need to show that $V = \mathbb{A}^1 \times r_0(V)$, where r_0 is defined in (3.1). We argue by contradiction; also, we may assume that all irreducible components of V have the same dimension. Thus there are only two cases to consider: $\dim r_0(V) = 0, 1$.

At first, we assume that $\dim r_0(V) = 1$. In this case, V is defined by a polynomial $P(x, y_1, y_2) \in k[x, y_1, y_2] \setminus k[x]$. There exists $q \in k^*$ such that $f_A^* P = qP$ i.e.

$$P(x + 1, a_1 y_1, a_2 y_2) = qP(x, y_1, y_2).$$

Write $P = \sum_I a_I(x)y^I$, where I is the multi-index and $a_I(x)$ is a polynomial in $k[x]$. We get $\sum_I a_I(x + 1)a^I y^I = \sum_I qa_I(x)y^I$. Then we have

$$a_I(x + 1) = a^{-I} qa_I(x).$$

Comparing the coefficient of the leading term, we have $a^{-I}q = 1$ if $a_I(x) \neq 0$. Thus $a_I(x) \in k$ for any I and so, $V = \mathbb{A}^1 \times r_0(V)$.

Now we assume $\dim r_0(V) = 0$.

Denote by $p_i : \mathbb{A}^1 \times \mathbb{A}^2 \to \mathbb{A}^1 \times \mathbb{A}^1$ the projection mapping (x, y_1, y_2) to (x, y_i) and let $f_i : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$ be the morphism $(x, y) \mapsto (x, a_i y)$. Then we have $p_i \circ f_A = f_i \circ p_i$ and $p_i(V)$ is an invariant subvariety of f_i of codimension 1 (for each $i = 1, 2$).

In this case, $p_i(V)$ is defined by a polynomial $P_i(x, y) \in k[x, y] \setminus k[x]$. There exists $q_i \in k^*$ such that $f_i^* P_i = q_i P_i$ i.e.

$$P_i(x + 1, a_i y) = q_i P(x, y_i).$$

Write $P_i = \sum_j a_j(x)y^j$, where $a_i(x)$ is a polynomial in $k[x]$. We get $\sum_j a_j(x + 1)a_i^j y^j = \sum_j q_i a_j(x)y^j$. Then we have

$$a_j(x + 1) = a_i^{-j} q_i a_j(x).$$

Comparing the coefficient of the leading term, we get $a_i^{-j} q_i = 1$ if $a_j(x) \neq 0$. Then $a_j(x) \in k$ for any j and so, $p_i(V) = \mathbb{A}^1 \times r_0(p_i(V))$. Furthermore, since $\dim r_0(V) = 0$, then also $\dim r_0(p_i(V)) = 0$.

Then we have $V \subseteq p_1^{-1}(p_1(V)) \cap p_2^{-1}(p_2(V)) = \mathbb{A}^1 \times (r_0(p_1(V)) \times r_0(p_2(V)))$. We note that $r_0(p_1(V)) \times r_0(p_2(V))$ is finite and $r_0(V) \subseteq r_0(p_1(V)) \times r_0(p_2(V))$. So $V = \mathbb{A}^1 \times r_0(V)$, as desired. □

Proposition 3.11. *Let f be an automorphism of $\mathbb{A}^1 \times \mathbb{A}^2$ of the form $(x, y) \mapsto (x + 1, A(x)y)$. If there exists a nontrivial invariant subvariety of f, then there exists* $B = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ *for some $a_1, a_2 \in k^*$ such that f is equivalent to f_B.*

Proof. Let V be a nontrivial invariant subvariety of f. We may assume that the dimension of all irreducible components of V are the same. By Theorem 1.6, we may suppose that $V = \mathbb{A}^1 \times V_0$ where V_0 is a subvariety of $\mathbb{A}^2$ which is invariant under $A(x)$ for all $x \in k$.

First, we observe that there exists $W_0 := \cup_{i=1}^s L_i \subseteq \mathbb{A}^2$ where L_i are distinct lines passing through the origin such that $W = \mathbb{A}^1 \times W_0$ is invariant under f. If $\dim V_0 = 0$, we may take W_0 be the union of lines passing through the origin and a point in V_0 (other than the origin). If $\dim V_0 = 1$, we consider the standard embeding $\mathbb{A}^2 \subseteq \mathbb{P}^2$ and then we may take W_0 be the union of lines passing through the origin and a point in the intersection of the Zariski closure of V_0 (in $\mathbb{P}^2$) and the line at infinity.

Now we may assume that V takes form $V = \mathbb{A}^1 \times (\bigcup_{i=1}^s L_i) = \bigcup_{i=1}^s \mathbb{A}^1 \times L_i$. Moreover, we may assume that $f(\mathbb{A}^1 \times L_i) = \mathbb{A}^1 \times L_{i+1}$ for $i = 1, \dots, s$ (where, by convention, we let $L_{s+1} := L_1$).

We have two cases: either $s = 1$ or $s \geq 2$.

Case $s = 1$**.** In suitable coordinates, we may assume that L_1 is defined by $y_2 = 0$. Then with respect to these coordinates, we may further assume that

$$A(x) = \begin{pmatrix} a_1(x) & b(x) \\ 0 & a_2(x) \end{pmatrix}$$

where $a_1(x), a_2(x), b(x) \in k[x]$. Because $\det A(x) = a_1(x)a_2(x)$ is a nonzero constant in $k[x]$, we have $a_1 := a_1(x)$ and $a_2 := a_2(x)$ are constants in k^*. We may assume that $b \neq 0$. Set $d := \deg b(x) \geq 0$. Denote by $k[x]_d$ the vector space of polynomials of degree at most d.

If $a_1 \neq a_2$, consider the linear map $T : k[x]_d \to k[x]_d$ defined by $T : P(x) \mapsto a_2 P(x+1) - a_1 P(x)$. Next we analyze the leading term of $T(P)$; we have $\deg(T(P)) = \deg(P)$. So T is injective and therefore, it must be surjective as well. Hence there exists $u \in k[x]_d$, such that $T(u(x)) = b(x)$. Let $U : \mathbb{A}^1 \times \mathbb{A}^2 \to \mathbb{A}^1 \times \mathbb{A}^2$ be the automorphism of the form

$$(x, y_1, y_2) \mapsto (x, y_1 + u(x)y_2, y_2);$$

then $U^{-1} \circ f \circ U = f_B$ where

$$B = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

If $a := a_1 = a_2$, consider the linear map $T : k[x]_{d+1} \to k[x]_d$ defined by $T : P(x) \mapsto aP(x+1) - aP(x)$. If $v(x) \in \ker(L)$, we have $v(x+1) = v(x)$. Then $v(x) \in k$. It follows that $\ker T = k$. Since $\dim k[x]_{d+1} = \dim k[x]_d + 1$, we obtain that T is surjective. Hence there exists $u(x) \in k[x]_{d+1}$ such that $T(u) = b(x)$. Let $U : \mathbb{A}^1 \times \mathbb{A}^2 \to \mathbb{A}^1 \times \mathbb{A}^2$ be the automorphism given by

$$(x, y_1, y_2) \mapsto (x, y_1 + u(x)y_2, y_2);$$

we obtain that $T^{-1} \circ f \circ T = f_B$ where

$$B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Case $s \geq 2$**.** Then, in suitable coordinates, we may assume that L_1 is defined by $y_2 = 0$ and L_2 is defined by $y_1 = 0$. So, with respect to these coordinates, we may assume that

$$A(x) = \begin{pmatrix} 0 & b(x) \\ c(x) & d(x) \end{pmatrix}$$

where $b(x), c(x), d(x) \in k[x]$. Because $\det A(x) = -b(x)c(x)$ is a nonzero constant in $k[x]$, we have $b := b(x)$ and $c := c(x)$ are constants in k^*. Then we have

$$f(\mathbb{A}^1 \times L_2) = \{(x, tb, td(x-1)) \colon x, t \in k\}.$$

We note that $f(\mathbb{A}^1 \times L_2) = \mathbb{A}^1 \times L_3$. Therefore $d(x-1)$ must be a constant; so, set $d := d(x) \in k$. Then we have $A(x) = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$. Let v be an eigenvector of A in $k^2 \setminus \{0\}$. Denote by L the line in $\mathbb{A}^2$ spaned by v. Then $\mathbb{A}^1 \times L$ is invariant by f. Then we reduced to the case $s = 1$ and conclude our proof. $\square$

Proposition 3.11 implies the following result immediately.

Corollary 3.12. *If there exists a nontrivial invariant subvariety of f, then there exist an invariant trivial subbundle of rank 1 in the vector bundle $\mathbb{A}^1 \times \mathbb{A}^2$ over $\mathbb{A}^1$.*

In other words, there exist $a(x), b(x) \in k[x]$ satisfying $\gcd(a(x), b(x)) = 1$ and $c \in k^*$ such that $f(x, a(x), b(x)) = (x + 1, ca(x + 1), cb(x + 1))$.

Proof. Assume that there exists a nontrivial invariant subvariety of f. By Proposition 3.11, we may assume that $f = f_B$ where $B = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ for some $a_1, a_2 \in k^*$. Then the subbundle of rank 1 defined by $y_1 = 0$ is invariant by f.

Now suppose that L is an invariant subbundle of rank 1 in the vector bundle $\mathbb{A}^1 \times \mathbb{A}^2$ over $\mathbb{A}^1$. Since $\mathrm{Pic}(\mathbb{A}^1) = \{0\}$, L is trivial. There exists a everywhere nonzero section s of L, i.e., there exist $a(x), b(x) \in k[x]$ satisfying $\gcd(a(x), b(x)) = 1$ such that $(x, a(x), b(x)) \in L$ for all $x \in \mathbb{A}^1$. Since L is invariant by f, the image $f(s)$ of s under f is also a nonzero section of L. Since $f(s)/s$ is a everywhere nonzero function on $\mathbb{A}^1$, it is constant. In other words $f(x, a(x), b(x)) = (x+1, ca(x+1), cb(x+1))$ for some $c \in k^*$. $\square$

We conclude by giving an example of an automorphism f which has no nontrivial invariant subvariety.

Proposition 3.13. *If* $A = \begin{pmatrix} 1 & 1 \\ x & x+1 \end{pmatrix}$*, then* f_A *has no nontrivial invariant subvariety. In particular,* f_A *is straight.*

Proof. If f_A has a nontrivial invariant subvariety, then by Corollary 3.12, there exist $a(x), b(x) \in k[x]$ satisfying $\gcd(a(x), b(x)) = 1$ and $c \in k^*$ such that

$$f(x, a(x), b(x)) = (x+1, ca(x+1), cb(x+1)).$$

It follows that $a(x) + b(x) = ca(x+1)$ and $xa(x) + (x+1)b(x) = cb(x+1)$. Then, combining these two equalities, we get:

$$b(x) = cb(x+1) - x(a(x) + b(x)) = cb(x+1) - cxa(x+1).$$

We have then

$$\deg b(x) \geq \deg(cb(x+1) - b(x)) = \deg(cxa(x+1)) = 1 + \deg a(x).$$

It follows that

$$\deg b(x) = \deg(a(x) + b(x)) = \deg(ca(x+1)) = \deg a(x) \leq \deg b(x) - 1.$$

Then we get a contradiction. $\square$

Remark 3.14. Let $A = \begin{pmatrix} 1 & 1 \\ x & x+1 \end{pmatrix}$. Proposition 3.13 yields that A is not equivalent to a constant matrix.

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