

PERIODIC SUBVARIETIES OF SEMIABELIAN VARIETIES AND ANNIHILATORS OF IRREDUCIBLE REPRESENTATIONS

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ABSTRACT. Let G be a semiabelian variety defined over a field of characteristic 0, endowed with an endomorphism Φ . We prove there is no proper subvariety $Y \subset G$ which intersects the orbit of each periodic point of G under the action of Φ . As an application, we are able to give a topological characterization of the annihilator ideals of irreducible representations in certain skew polynomial algebras.

1. INTRODUCTION

1.1. A sample of our results. In our paper we prove various connections between purely algebraic properties of rings and arithmetic dynamics on semiabelian varieties. In order to state one of our main results, we first need to introduce the terminology of *primitive ideals*, of *skew polynomial rings*, and also of *rational ideals*.

Given a simple left R -module M , the annihilator, $\text{Ann}_R(M)$, of M is the set of $x \in R$ such that $xM = (0)$. This is a two-sided ideal and any ideal of this form is called a *primitive ideal*. To be pedantic, it is called a *left primitive ideal*, with right primitivity of ideals being defined analogously. Bergman [Ber64] gave an example of a ring in which (0) is right primitive but not left primitive, but his example is highly pathological and in practice, for well behaved classes of algebras, one has that the notions of left and right primitivity coincide and hence we omit the “left” when talking about primitive ideals, since it is well known that the two notions coincide in the context we consider in this paper. As is customary, we say that a ring R is *primitive* if (0) is a primitive ideal. Also, we recall that given a ring R and an automorphism σ of R , one can form the *skew polynomial ring* $R[t; \sigma]$, which is, as a set, $R[t]$, but with “twisted” multiplication given by $t \cdot r = \sigma(r)t$ for $r \in R$. One can similarly define the skew Laurent polynomial ring $R[t^{\pm 1}; \sigma]$.

Finally, in order to introduce the notion of rational ideals (see also Section 1.2), we note that given a left noetherian ring R and a prime ideal P , Goldie (see [Row88a, Chapter 3]) shows that one can localize $S = R/P$ at the set of elements that are not left or right zero divisors and one will obtain a quotient ring $Q(S)$, which is a simple Artinian ring. One can intuitively think of R/P as being a “noncommutative field of fractions” of R and the Artin-Wedderburn theorem says that $Q(S)$ is isomorphic to a matrix ring over a division ring. In the case that k is a field and R is a k -algebra,

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we say that a prime ideal P of R is *rational* if the centre of $Q(R/P)$ is an algebraic extension of the base field.

In Section 3, we prove the following result.

Theorem 1.1. *Let k be a field of characteristic zero, let $R = k[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ and let σ be a k -algebra automorphism of R . Then the following results hold:*

- (a) *a prime ideal P of $S := R[t; \sigma]$ is primitive if and only if P is locally closed in $\text{Spec}(S)$;*
- (b) *a prime ideal P of $T := R[t^{\pm 1}; \sigma]$ is primitive if and only if P is rational.*

We note that the full Dixmier-Moeglin equivalence does not hold in either Theorem 1.1 (a) or (b). In the case of part (a), we show there exist examples of rational prime ideals that are not primitive or locally closed (and these examples can be done in positive characteristic in addition to characteristic zero); in the case of part (b), we show there exist examples of primitive and rational ideals that are not locally closed. See §4 for further details.

1.2. The motivation for our problem from a ring theoretical point of view. One of the interesting developments that has come out of the evolution of noncommutative projective geometry is the relationship between ring theoretic properties of algebras produced from geometric data and dynamical phenomena involving algebraic varieties equipped with an automorphism. In fact, these connections were in part what gave impetus to the first collaboration of the authors (along with Tom Tucker) [BGT10], where one of the consequences of the authors' work was to prove a conjecture of Keeler, Rogalski, and Stafford [KRS05] concerning when algebras associated to the naïve blowup of an integral projective scheme at a closed point are noetherian. In fact, many basic ring theoretic questions about twisted homogeneous coordinate rings and skew polynomial rings often lead to deep and highly non-trivial dynamical questions. A recent example of this can be seen by the work of Brown, Carvalho, and Matczuk [BCM17], who studied the property of when skew polynomial rings of automorphism type have the property that the injective hulls of simple modules are locally Artinian. In their paper, in addition to characterizing when this property holds for many skew polynomial rings, they use an argument supplied by Goodearl to fill a gap that occurs in a paper of Jordan [Jor93] about skew polynomial algebras. Jordan looked at a special example of skew rings in which $R = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ and σ is the \mathbb{C} -algebra automorphism given by $\sigma(x) = y$ and $\sigma(y) = x^{-1}y$. Jordan's goal was to show that the corresponding skew polynomial algebra has the property that every irreducible representation has a non-trivial annihilator.

Given an associative ring R , a fundamental problem is to classify the irreducible representations (simple R -modules) of R . Understanding these representations yields a great deal of information about the ring itself and this representation theoretic approach to understanding rings has become an essential part of the study of representations of finite groups (and their group rings) and can be seen in the various local-global principles that arise in commutative algebra. Unfortunately, it is generally an intractable problem to classify the simple modules of a general ring and in practice one often settles for a coarser understanding, proposed as an alternative by Dixmier, to instead

understand annihilators of simple modules, which leads naturally to the aforementioned problem of characterizing primitive ideals in the prime spectrum.

We note that if M is a simple R -module with annihilator P , then M can be viewed as an R/P -module in a natural way and it becomes a faithful R/P -module; that is, $\text{Ann}_{R/P}(M) = (0)$. In this case, Jacobson's density theorem (see [Row88a, Chapter 2]) says that R/P embeds as a dense subring of an endomorphism ring of a Δ -vector space, where $\Delta = \text{End}_{R/P}(M)$. For commutative rings R , a simple module is isomorphic to a cyclic module of the form R/P with P a maximal ideal. In this case, P is the annihilator so primitive ideals are precisely the maximal ideals of R , which are exactly the closed points in $\text{Spec}(R)$. In general, the primitive ideals form a subset of the prime ideals of R , where a two-sided ideal I of R is prime if whenever $JL \subseteq I$, with J, L two-sided ideals of R , we have either $J \subseteq I$ or $L \subseteq I$. The problem of determining the subset of primitive ideals of $\text{Spec}(R)$ is an important part of understanding the representation theory of R . In the case of enveloping algebras of finite-dimensional complex Lie algebras, Dixmier [Dix77] and Moeglin [Moe80], gave a concrete description of the primitive spectrum in terms of the related properties of an ideal being rational and also being locally closed. Dixmier and Moeglin proved that for $P \in \text{Spec}(U)$ with U the enveloping algebra of a finite-dimensional complex Lie algebra, we have

$$P \text{ primitive} \iff P \text{ rational} \iff P \text{ locally closed in } \text{Spec}(U).$$

Our Theorem 1.1 is inspired in part by the results of Dixmier and Moeglin and proves analogues of their results for prime ideals in rings of the form $R[t; \sigma]$ or $R[t^{\pm 1}; \sigma]$, where $R = k[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$.

1.3. Connections to algebraic geometry and our results regarding arithmetic dynamics on semiabelian varieties. When one gets into rings “coming from geometry”, the problem of classifying primitive ideals often takes on a dynamical flavour. One of the simplest classes of such rings are twisted polynomial rings $R[t; \sigma]$ with R a finitely generated k -algebra and $\sigma : R \rightarrow R$ a k -algebra automorphism of R . In this case, the primitive ideals were characterized by Leroy and Matczuk [LM96]. Given a ring R and $\sigma \in \text{Aut}(R)$, we say that an element a of R is σ -special if for every σ -stable ideal I of R we have $a\sigma(a) \cdots \sigma^n(a) \in I$ for some n and there is no $m \geq 1$ such that $a\sigma(a) \cdots \sigma^m(a) = 0$. Leroy and Matczuk [LM96] show that if R is a ring and σ is an automorphism of R then $R[t; \sigma]$ is primitive if and only if it has a σ -special element and σ has infinite order. When one looks at the case when R is a finitely generated k -algebra that is an integral domain and σ is a k -algebra automorphism, one has a corresponding automorphism τ of the irreducible affine variety $X = \text{Spec}(R)$. The σ -special property can then be translated as follows: R is σ -special if and only if there is a proper closed subset Y of X such that every τ -periodic subvariety of X has an irreducible component contained in Y . On the other hand, the property of (0) being locally closed in the skew polynomial algebra is equivalent to the union of all proper periodic subvarieties being a proper Zariski closed set (see our Lemma 3.2). It is natural to ask whether primitive ideals are precisely those prime ideals that are locally closed in the Zariski topology for algebras of the form $R[t; \sigma]$. In terms of arithmetic dynamics, if one has a quasiprojective variety X equipped with an endomorphism Φ , this is asking about the equivalence of the two properties:

- (I) there is a proper subvariety $Y \subset X$ that contains all the proper, irreducible, periodic subvarieties of X (under the action of Φ).
- (II) there is a proper subvariety $Y \subset X$ that contains some iterate of each proper, irreducible, periodic subvariety of X .

Clearly, property (I) implies property (II); the difficult part is to prove that (II) also implies (I).

In Section 2 we prove the following results for semiabelian varieties (which are themselves extensions of abelian varieties by algebraic tori in the category of algebraic groups; for more details regarding semiabelian varieties, see [NW14]).

Theorem 1.2. *Let G be a semiabelian variety defined over an algebraically closed field K of characteristic 0, i.e., there exists a short exact sequence of K -algebraic groups*

$$1 \longrightarrow \mathbb{G}_m^N \longrightarrow G \longrightarrow A \longrightarrow 1,$$

for some nonnegative integer N and some abelian variety A . Let Φ be a dominant group endomorphism of G . Then there is no proper subvariety of G which intersects the orbit of each torsion point that is periodic under the action of Φ .

Theorem 1.2 yields that neither property (I) nor property (II) holds for a dominant group endomorphism of a semiabelian variety. Also, we note that it suffices in Theorem 1.2 to restrict to orbits of *torsion* points of G that are periodic under the action of Φ and still derive that no proper subvariety of G may intersect all these orbits. In particular, Theorem 1.2 allows us to derive the equivalence of properties (I) and (II) for any regular self-map on a semiabelian variety, as established in the next result.

Theorem 1.3. *Let G be a semiabelian variety defined over an algebraically closed field K of characteristic 0 and let $\Phi : G \rightarrow G$ be a dominant, regular self-map. Then one of the following two statements must hold:*

- (A) *there is a proper subvariety $X \subset G$ that contains all the proper, irreducible, periodic subvarieties of G (under the action of Φ).*
- (B) *there is no proper subvariety $Y \subset G$ that contains some iterate of each proper, irreducible, periodic subvariety of G .*

In particular, Theorem 1.3 in the special case $G = \mathbb{G}_m^n$ is used in deriving the conclusion in Theorem 1.1.

1.4. Further connections with algebraic dynamics. Besides the motivation from algebra (explained in Subsection 1.3), Theorems 1.2 and 1.3 are also motivated by various questions in algebraic geometry and arithmetic dynamics, as we will explain below.

If Φ is a polarizable endomorphism of a projective variety X defined over a field of characteristic 0 (i.e., there exists an ample line bundle \mathcal{L} on X such that $\Phi^*\mathcal{L}$ is linearly equivalent with $\mathcal{L}^{\otimes d}$ for some $d > 1$), then Fakhruddin [Fak03, Theorem 5.1] proved that the periodic points are dense. Furthermore, a more careful analysis of the proof of [Fak03, Proposition 5.5] yields the existence

of a set S of periodic points of X with the property that choosing a point from the orbit of each point in S would always yield a Zariski dense set in X ; thus the conclusion in Theorem 1.2 holds for polarizable dynamical systems (X, Φ) .

Next we discuss the connections between our results and the Dynamical Manin-Mumford conjecture. The original form of the Dynamical Manin-Mumford conjecture, formulated by Zhang in early 1990s (see [Zha06]) predicts that for a polarizable dynamical system (X, Φ) defined over a field of characteristic 0, if $V \subset X$ is a subvariety which contains a Zariski dense set of preperiodic points, then V must be preperiodic itself. Later (see [GTZ08]), this conjecture was amended; however, it is expected that a variant of the Dynamical Manin-Mumford Conjecture holds for more general dynamical systems that are not necessarily polarizable (see [DF17, GNY19, GNY18] for similar results). So, assume that for the endomorphism Φ of a quasiprojective variety X , the aforementioned Dynamical Manin-Mumford Conjecture holds; we claim that this yields that a variant of the conclusions from Theorems 1.2 and 1.3 must hold. Indeed, assume there is a proper subvariety $Y \subset X$ which contains some iterate of each periodic point of X . Without loss of generality, we may assume Y is the Zariski closure of a set of periodic points; then the Dynamical Manin-Mumford Conjecture yields that Y must be preperiodic and therefore, its orbit under Φ is a closed, proper, subvariety Z of X . Then our assumption on Y yields that Z must contain *all* periodic points of X .

Our Theorems 1.2 and 1.3 (and especially, their proofs) are also motivated by the special case of the Medvedev-Scanlon conjecture for regular self-maps on semiabelian varieties. The Medvedev-Scanlon conjecture from [MS14] is based on a much earlier conjecture of Zhang from the early 1990s stated in [Zha06]; the Medvedev-Scanlon conjecture predicts that for a dominant rational self-map Φ of a quasiprojective variety X defined over an algebraically closed field K of characteristic 0, either there exists a point $x \in X(K)$ with a Zariski dense orbit, or there exists a non-constant rational function $f : X \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$. This conjecture is known to hold when K is uncountable (see [AC08, BGR17]), but it is very difficult in the case of countable fields (for various results in this case, see [MS14, BGRS17, GH18, GS17, GS, GX18]). Similar to the proof of the Medvedev-Scanlon conjecture when X is a semiabelian variety (see [GS], which extends the results of [GS17]), in our arguments deriving Theorem 1.3 we reduce to the special case Φ is a composition of a translation with a unipotent endomorphism (this last case bears resemblance also to the study of wild automorphisms for abelian varieties from [RRZ07]). In Lemma 3.3 we prove that given a dynamical system (X, Φ) , if there exists a proper subvariety $Y \subset X$ containing an iterate of each proper, irreducible, periodic subvariety of X , then there must exist an algebraic point of X with a Zariski dense orbit under Φ . Our results allow us to simplify a characterization of Jordan for the primitivity of skew Laurent polynomial algebras over affine commutative domains (see Proposition 3.4).

2. PROOFS OF THEOREMS 1.2 AND 1.3

We start by proving some useful reductions for the proof of Theorem 1.2; similar statements hold also for Theorem 1.3. The first statement is a simple, but useful observation, which we will use throughout the entire proof of Theorem 1.2. We also note that throughout this section, our semiabelian variety G is always assumed to be defined over an algebraically closed field K of characteristic 0. Before stating the next result, we recall that for any self-map Φ on a set X , a point $x \in X$ is called *preperiodic* if there exist positive integers $m < n$ such that $\Phi^m(x) = \Phi^n(x)$.

Lemma 2.1. *Let G be a semiabelian variety endowed with a group endomorphism Φ .*

- (a) *Then the orbit of each periodic, torsion point of G contains only torsion points.*
- (b) *Each torsion point is preperiodic under the action of Φ .*
- (c) *Then $Y \subset G$ intersects the orbit of each periodic, torsion point if and only if Y intersects the orbit of each torsion point.*

Proof. The proof is immediate, by noting that Φ commutes with the multiplication-by- m map (always denoted by $[m]$) on G , for any integer m . In particular, letting $G[m]$ be the (finite) set of torsion points of G killed by $[m]$, we see that $\Phi(G[m]) \subseteq G[m]$ (which justifies both (a) and (b)). Finally, part (c) follows from part (b). \square

Our next observation is also simple and useful.

Lemma 2.2. *The conclusion of Theorem 1.2 is unchanged if we replace the dynamical system (G, Φ) by the dynamical system (G, Φ^ℓ) for some positive integer ℓ .*

Proof. First of all, it is clear that the two dynamical systems share the same set of periodic points; therefore, the conclusion of Theorem 1.2 for (G, Φ) yields the same conclusion for (G, Φ^ℓ) . Conversely, if there were some proper subvariety $Y \subset G$ containing some iterate of each periodic point of G under the action of Φ , then $\cup_{i=0}^{\ell-1} \Phi^i(Y)$ is also a proper subvariety of G , which would then contain some iterate of each periodic point under the action of Φ^ℓ . \square

The next result shows that we can always replace the dynamical system (G, Φ) by an isogenous copy of it.

Lemma 2.3. *Theorem 1.2 is invariant if we replace G by an isogenous semiabelian variety G' .*

Proof. Let $\tau : G \rightarrow G'$ be an isogeny, and assume Theorem 1.2 holds for any dominant group endomorphism of G' ; we show next that it must hold also for any dominant group endomorphism of G .

Let Φ be a group endomorphism of G , and let m be a positive integer such that $\ker(\tau) \subset G[m]$. We let $\Phi_m := \Phi \circ [m]$ and note that $\Phi'_m := \tau \circ \Phi_m \circ \tau^{-1}$ is a well-defined, dominant group endomorphism of G' . Since $\tau \circ \Phi_m = \Phi'_m \circ \tau$, we obtain that Theorem 1.2 holds for the dynamical system (G, Φ_m) if and only if it holds for the dynamical system (G', Φ'_m) ; note that if there were a proper subvariety $Y \subset G$ intersecting the orbit of each torsion point of G (see Lemma 2.1), then $\tau(Y)$ is a proper

subvariety of G' intersecting the orbit of each torsion point of G' (and also the converse holds since τ is a finite map). Therefore, it remains to prove that if Theorem 1.2 holds for (G, Φ_m) , then it must hold also for (G, Φ) .

Assume that Theorem 1.2 fails for the dynamical system (G, Φ) ; then there exists some proper subvariety $Y \subset G$ intersecting the orbit of each torsion point of G (see Lemma 2.1 (c)). Without loss of generality we may assume Y is the Zariski closure of a subset of torsion points (see Lemma 2.1 (a)). Thus, by Laurent's theorem [Lau84], we get that Y is a finite union of torsion translates of proper algebraic subgroups of G . In particular, this yields that $Y_m := \bigcup_{k=1}^{\infty} [m^k](Y)$ (i.e., the union of the image of Y under the multiplication-by- m^k morphisms, as we let k vary) is also a proper, Zariski closed subset of G . Furthermore, by construction (and also using the hypothesis regarding Y), we get that Y_m intersects the orbit of each torsion point of G under the action of Φ_m . This contradicts the fact that conclusion of Theorem 1.2 must hold for the dynamical system (G, Φ_m) , thus proving the desired reduction from Lemma 2.3. \square

The next result shows in particular that proving Theorem 1.2 reduces to deriving the desired conclusion in the case of abelian varieties and respectively, in the case of algebraic tori (note that each semiabelian variety is an extension of an abelian variety by a torus).

Lemma 2.4. *Let G be a semiabelian variety endowed with a dominant group endomorphism $\bar{\Phi}$ and assume there exists a short exact sequence of semiabelian varieties:*

$$(2.4.1) \quad 1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1$$

such that $\Phi|_{G_1}$ is an endomorphism of G_1 , while $\bar{\Phi}$ is the corresponding induced endomorphism of G_2 . If the conclusion in Theorem 1.2 holds for the dynamical systems $(G_1, \Phi|_{G_1})$ and $(G_2, \bar{\Phi})$, then the same conclusion from Theorem 1.2 holds for the dynamical system (G, Φ) .

Proof. Let $Y \subseteq G$ be a subvariety which contains a point from the orbit of each periodic torsion point of G under the action of Φ . For each point $x \in G$, we let $\bar{x} \in G_2$ be its image under the morphism $G \rightarrow G_2$ from (2.4.1).

Let $x \in G$ be a torsion point that is periodic under the action of Φ ; let $N(x) \in \mathbb{N}$ be the length of the period of x . Then for any torsion point $y \in G_1$ that is periodic under the action of $\Phi|_{G_1}$, we have that also $x + y$ is both torsion and periodic under the action of Φ (since Φ is a group endomorphism). Our hypothesis regarding Y yields that for each such point $y \in G_1$, there exists some positive integer n_y such that $\Phi^{n_y}(x) + \Phi^{n_y}(y) \in Y$. Our hypothesis regarding the dynamical system $(G_1, \Phi|_{G_1})$ yields that the set $\{\Phi^{n_y}(y)\}_y$ is Zariski dense in G_1 . Thus, there exists some positive integer $N_0(x) \leq N(x)$ with the property that the set

$$\{\Phi^{n_y}(y) : n_y \equiv N_0(x) \pmod{N(x)}\}$$

is Zariski dense in G_1 . So, $\Phi^{N_0(x)}(x) + G_1 \subset Y$ for each torsion point x that is periodic under the action of Φ . Using the fact that the points $\{\overline{\Phi^{N_0(x)}(x)}\}_x$ are Zariski dense in G_2 (according to the hypothesis applied to the dynamical system $(G_2, \bar{\Phi})$), we conclude that $Y = G$, as desired. \square

A special case of Lemma 2.4 yields the following statement.

Corollary 2.5. *If the conclusion in our Theorem 1.2 holds for the semiabelian varieties G_1 and G_2 equipped with dominant group endomorphisms Φ_1 , respectively Φ_2 , then the same conclusion holds for the dynamical system $(G_1 \times G_2, \Phi)$, where Φ is the endomorphism of $G := G_1 \times G_2$ defined by $\Phi(x_1, x_2) = (\Phi_1(x_1), \Phi_2(x_2))$.*

We continue by proving Theorem 1.2 for powers of simple semiabelian varieties (i.e., G is either isomorphic to some power of \mathbb{G}_m , or to some power of a simple abelian variety); this special case is instrumental in deriving the general case in Theorem 1.2.

Theorem 2.1. *Theorem 1.2 holds when $G = A^m$ for a simple semiabelian variety A (i.e., there is no proper semiabelian subvariety of A).*

We split our proof of Theorem 2.1 based on whether A is isomorphic to the multiplicative group \mathbb{G}_m , or A is a simple abelian variety.

Proof of Theorem 2.1 when $A \xrightarrow{\sim} \mathbb{G}_m$. Suppose there exists a proper subvariety Y of G (which we assume to be the m -th cartesian power of \mathbb{G}_m) intersecting the orbit of each torsion point of G that is periodic under the action of Φ . In particular, we may assume $Y \subset G$ is the Zariski closure of a set of torsion points and thus, by Laurent's theorem [Lau84], Y is a finite union of torsion translates of algebraic subgroups of G .

We argue by induction on m ; the case $m = 1$ is obvious since there exist infinitely many torsion points of \mathbb{G}_m (which are the usual roots of unity in K).

Because Y is a finite union of torsion translates of algebraic subgroups of A^m , then there must exist finitely many m -tuples $(\alpha_{i,1}, \dots, \alpha_{i,m}) \in \mathbb{Z}^m \setminus \{(0, \dots, 0)\}$ (for some $i = 1, \dots, k$) such that each point $(x_1, \dots, x_m) \in G$ of Y satisfies an equation of the form:

$$(2.5.1) \quad \prod_{j=1}^m x_j^{\alpha_{i,j}} = 1,$$

for some $i = 1, \dots, k$. Indeed, each algebraic subgroup U of the m -th cartesian power of \mathbb{G}_m is given by finitely many equations of the form (2.5.1); then all points contained in a torsion translate $\zeta \cdot U$ (for a torsion point ζ of order ℓ) satisfy equations of the form (2.5.1) in which the exponents $\alpha_{i,j}$ are replaced by $\ell \cdot \alpha_{i,j}$.

We let M_Φ be the m -by- m matrix with integer entries which corresponds to the endomorphism Φ of G . Since Φ is dominant, all eigenvalues of M_Φ are nonzero. We let L be a finite Galois extension of \mathbb{Q} containing all the eigenvalues of M_Φ and also, containing the entries of all the eigenvectors of M_Φ . Then there exists an infinite set S of primes p satisfying the following properties:

- (i) p splits completely in L/\mathbb{Q} ; and
- (ii) fixing some prime \mathfrak{p} of L lying above p , we have that each eigenvalue of M_Φ is a \mathfrak{p} -adic unit.

Fix some $p \in S$. We let $\epsilon_p \in \overline{\mathbb{Q}}$ be a root of unity of order precisely p . Our hypotheses (i)–(ii) yield the existence of a nonzero vector $v := (c_1, \dots, c_m) \in \mathbb{Z}^m$ and also the existence of an integer b coprime with p such that

$$(2.5.2) \quad M_\Phi \cdot v \equiv b \cdot v \pmod{p}.$$

Note that the congruence equation (2.5.2) first holds modulo \mathfrak{p} , but then since M_Φ and also the vector v have all their entries integral, then we obtain a congruence modulo p . We let $\vec{x}_p := (\epsilon_p^{c_1}, \dots, \epsilon_p^{c_m}) \in G(\overline{\mathbb{Q}})$. Then (2.5.2) yields that

$$(2.5.3) \quad \Phi(\vec{x}_p) = \vec{x}_p^b,$$

where for each vector $\vec{v} := (v_1, \dots, v_m) \in \overline{\mathbb{Q}}^m$, we let $\vec{v}^b := (v_1^b, \dots, v_m^b)$. Thus $\Phi^j(x_p) = \vec{x}_p^{bj}$; since $\gcd(b, p) = 1$, we get that \vec{x}_p is periodic under the action of Φ . Furthermore, our assumption on Y yields the existence of some $j \in \mathbb{N}$ and also the existence of some $i_p \in \{1, \dots, k\}$ such that

$$\prod_{\ell=1}^m \epsilon_p^{\alpha_{i_p, \ell} c_\ell b^j} = 1;$$

hence (because b is coprime with p , while ϵ_p has order p), we have

$$(2.5.4) \quad c_1 \alpha_{i_p, 1} + \dots + c_m \alpha_{i_p, m} \equiv 0 \pmod{p}.$$

By the pigeonhole principle, there exists some $\tilde{i} \in \{1, \dots, k\}$ such that $i_p = \tilde{i}$ for infinitely many primes $p \in S$. We let $w := (\alpha_{\tilde{i}, 1}, \dots, \alpha_{\tilde{i}, m}) \in \mathbb{Z}^m$. Then, using (2.5.4) and (2.5.2), we have

$$(2.5.5) \quad w \cdot M_\Phi^n v \equiv 0 \pmod{p},$$

for all nonnegative integers n (where $w_1 \cdot w_2$ represents the dot product of the two vectors w_1 and w_2). Note that in equation (2.5.5), the vector v changes with the prime p , but the vector w is unchanged.

Consider the L -vector space U spanned by all vectors $(M_\Phi^t)^n w$ for $n \geq 0$ (where M^t is the transpose of the matrix M). If $\dim U = m$, then equation (2.5.5) cannot hold for infinitely many primes p since for large primes p , there is no nonzero vector v orthogonal modulo p to each element in U .

Thus it must be that $\dim U < m$. In this case, we let A_1 be the algebraic subgroup of G corresponding to the orthogonal complement of the linear subspace U of the tangent space of G at the origin; more precisely, A_1 is the algebraic subgroup of G consisting of all $x := (x_1, \dots, x_m)$ such that for each n , we have $[w](\Phi^n(x)) = 0$, where $[w] : G \rightarrow \mathbb{G}_m$ is given by the following formula (note that $w = (\alpha_{\tilde{i}, 1}, \dots, \alpha_{\tilde{i}, m})$):

$$(z_1, \dots, z_m) \mapsto \prod_{j=1}^m z_j^{\alpha_{\tilde{i}, j}}.$$

Clearly, A_1 is fixed by Φ ; furthermore, since $\dim(U) < m$, we get that A_1 is a proper algebraic subgroup of A^m (also note that not all entries of w are equal to 0). Furthermore, an iterate of Φ restricts to an endomorphism of A_1^0 , which is the connected component of A_1 ; note that A_1^0 is isomorphic to the m_1 -st cartesian power of \mathbb{G}_m (for some $0 < m_1 < m$). Since the conclusion of

Theorem 1.2 is unchanged if we replace Φ by an iterate of it (see Lemma 2.2), without loss of generality, we assume Φ induces an endomorphism of A_1^0 . Thus we have a short exact sequence of algebraic groups:

$$1 \longrightarrow A_1^0 \longrightarrow G \longrightarrow A_2 \longrightarrow 1,$$

where A_2 is isomorphic to $\mathbb{G}_m^{m_2}$ for some integer $0 < m_2 < m$. Moreover, Φ induces also an endomorphism $\bar{\Phi}$ of A_2 (because Φ restricts to an endomorphism of A_1^0). Applying the inductive hypothesis to the dynamical systems $(A_1^0, \Phi|_{A_1^0})$ and $(A_2, \bar{\Phi})$ combined with Lemma 2.4 yields the desired conclusion in Theorem 1.2 in the case G is the cartesian power of the multiplicative group. \square

Proof of Theorem 2.1 when A is an abelian variety. We know that A is an abelian variety of dimension $g \geq 1$. We start by proving some easy facts regarding endomorphisms of abelian varieties.

Lemma 2.6. *Let Ψ be a dominant group endomorphism of A^ℓ (for some positive integer ℓ). Then for all but finitely many primes p , we have that for each $n \in \mathbb{N}$, the endomorphism Ψ^n of A^ℓ induces a bijection on $A[p]^\ell$ (where $A[p]$ is the p -torsion subset of A).*

Proof of Lemma 2.6. We let $g \in \mathbb{Z}[t]$ be a monic polynomial of minimal degree, which kills the endomorphism Ψ of A^ℓ , i.e., $g(\Psi) = 0$; for more details, see [GS17, Fact 3.3] and [GS, Fact 2.6]. Since Ψ is a dominant endomorphism, $c := g(0)$ is a nonzero integer. So, there exists a polynomial $h \in \mathbb{Z}[t]$ such that $h(\Psi) \cdot \Psi = [c]_\ell$, where $[c]_\ell$ is the coordinatewise multiplication-by- c morphism on A^ℓ . Clearly, for each prime p that does not divide c , the endomorphism $[c]_\ell$ of A^ℓ induces a bijection on $A[p]^\ell$. In particular, $(h(\Psi) \cdot \Psi)|_{A[p]^\ell}$ is a bijection, which forces $\Psi|_{A[p]^\ell}$ be a bijection on $A[p]^\ell$; then Ψ^n , for any positive integer n , induces a bijection of $A[p]^\ell$, as claimed. \square

Lemma 2.7. *Let p be a prime number and let Ψ_1, \dots, Ψ_ℓ be group endomorphisms of A with the property that there exists some $i_1 \in \{1, \dots, \ell\}$ such that Ψ_{i_1} induces a bijection on $A[p]$. Then there are exactly $p^{2g(\ell-1)}$ points $(x_1, \dots, x_\ell) \in A[p]^\ell$ such that*

$$(2.7.1) \quad \Psi_1(x_1) + \dots + \Psi_\ell(x_\ell) = 0.$$

Proof of Lemma 2.7. For each $j \in \{1, \dots, \ell\} \setminus \{i_1\}$, we let x_j be an arbitrary point in $A[p]$; this yields precisely $p^{2g(\ell-1)}$ such $(\ell-1)$ -tuples (since $\#A[p] = p^{2g}$ because the multiplication-by- p map on an abelian variety of dimension g has degree p^{2g}). For each such $(\ell-1)$ -tuple, there is exactly one solution $x_{i_1} \in A[p]$ such that equation (2.7.1) is satisfied, as claimed. \square

We continue our proof arguing again by induction on m , similar to the case G is a power of the multiplicative group; also, as before, the case $m = 1$ is obvious since there exist infinitely many torsion points and A has no proper algebraic subgroups because it is a simple abelian variety. We let $D := \text{End}(A)$. Furthermore, as before, we assume there exists a proper subvariety Y of G intersecting the orbit of each torsion point of G . Moreover, we may assume $Y \subset G$ is the Zariski closure of a set of torsion points and thus, by Laurent's theorem [Lau84], Y is a finite union of torsion translates of algebraic subgroups of G . Thus there must exist finitely many m -tuples

$(\alpha_{i,1}, \dots, \alpha_{i,m}) \in D^m \setminus \{(0, \dots, 0)\}$ (for some $i = 1, \dots, k$) such that each point $(x_1, \dots, x_m) \in A^m$ of Y satisfies an equation of the form:

$$(2.7.2) \quad [\alpha_{i,1}](x_1) + \dots + [\alpha_{i,m}](x_m) = 0,$$

for some $i = 1, \dots, k$ (where $[\alpha]$ is the endomorphism of A represented by the element $\alpha \in D$). Also, we fix $M_\Phi \in M_{m,m}(D)$ representing the endomorphism $\Phi \in \text{End}(A^m)$.

Let $f \in \mathbb{Z}[t]$ be the minimal, monic polynomial of the endomorphism Φ ; i.e., $f(\Phi) = 0$. Next we prove that we may assume that f is irreducible.

Claim 2.8. If $f(t)$ is not irreducible, then the conclusion in Theorem 2.1 follows from the inductive hypothesis.

Proof of Claim 2.8. Assume $f(t) = g(t) \cdot h(t)$ for some non-constant polynomials $g, h \in \mathbb{Z}[t]$. We let $A_1 := \ker(g(\Phi))$ and also, let $A_2 := g(\Phi)(A^m)$. Then A_2 is an abelian variety (isogenous to A^r for some integer $r < m$ since A is a simple abelian variety), while A_1 is an algebraic subgroup of A^m (not necessarily connected); however, the connected component A_1^0 of A_1 is isogenous to A^{m-r} . Then we have a short exact sequence of algebraic groups:

$$1 \longrightarrow A_1 \longrightarrow A^m \longrightarrow A_2 \longrightarrow 1;$$

furthermore, Φ restricts to a group endomorphism of both A_1 and of A_2 . Moreover, a suitable iterate Φ^ℓ (for some $\ell \in \mathbb{N}$) induces an endomorphism of A_1^0 as well. Then, using Lemma 2.2, and also applying the inductive hypothesis to each dynamical system (A_1^0, Φ^ℓ) and (A_2, Φ) , coupled with Lemma 2.4 yields the desired conclusion for the dynamical system (A^m, Φ) , as claimed. \square

From now on, we may assume that the polynomial $f(t)$ is irreducible; in particular, the gcd of $f(t)$ and $f'(t)$ is 1 and so this means that all the roots of $f(t)$ are distinct (and nonzero, since Φ is a dominant group endomorphism). Let L be the splitting field for the polynomial $f(t)$; so, L is a finite Galois extension of \mathbb{Q} containing all the roots of f . Then there exists an infinite set S of primes p , which split completely in L/\mathbb{Q} . Furthermore, at the expense of excluding finitely many primes p from the infinite set S , we may even assume that:

- (A) the prime p and the endomorphism Φ of A^m satisfy the conclusion of Lemma 2.6; and
- (B) for each $i = 1, \dots, k$, there exists an endomorphism $[\alpha_{i,j(i)}]$ (for some $j(i) \in \{1, \dots, m\}$) which induces a bijection on $A[p]$.

Note that Lemma 2.6 yields that only finitely many primes p do not satisfy condition (A) above. Similarly, since for each $i = 1, \dots, k$, we know that there is some $j(i) \in \{1, \dots, m\}$ such that $\alpha_{i,j(i)} \neq 0$, then Lemma 2.6 yields that for all but finitely many primes p , the endomorphism $[\alpha_{i,j(i)}]$ of A induces a bijection of $A[p]$.

Let $p \in S$. Since f has distinct, nonzero roots and p splits completely in L/\mathbb{Q} , we get that the reduction of $f(t)$ modulo p divides the polynomial $t^{p-1} - 1$ in $\mathbb{F}_p[t]$; thus there exists some polynomials $g_p, h_p \in \mathbb{Z}[t]$ such that

$$(2.8.1) \quad t^{p-1} - 1 = f(t) \cdot g_p(t) + p \cdot h_p(t).$$

Hence

$$\Phi^{p-1}(x) = x + ph_p(\Phi)(x) = x + (h_p(\Phi))(px)$$

for each $x \in G = A^m$. Equation (2.8.1) yields that for each $x \in G[p]$, we have

$$(2.8.2) \quad \Phi^{p-1}(x) = x.$$

Let $x \in G[p]$. By our assumption, there exists some positive integer $n(x)$ such that $\Phi^{n(x)}(x) \in Y$; using (2.8.2), we see that we may assume $n(x) \in \{1, \dots, p-1\}$. We let $x = (x_1, \dots, x_m) \in A[p]^m$; so by (2.7.2), there exists some $i(x) \in \{1, \dots, k\}$ such that

$$(2.8.3) \quad [\overrightarrow{\alpha_{i(x)}}] \cdot \Phi^{n(x)}(x) = 0,$$

where for any $\delta_1, \dots, \delta_m \in D$ we define the morphism $[\delta_1, \dots, \delta_m] : A^m \rightarrow A$ as follows:

$$[\delta_1, \dots, \delta_m] \cdot x := [\delta_1](x_1) + \dots + [\delta_m](x_m).$$

Lemma 2.9. *Let $p \in S$, let $i \in \{1, \dots, k\}$ and let $n \in \mathbb{N}$. Then there are $p^{2g(m-1)}$ points $x \in A[p]^m$ such that $[\overrightarrow{\alpha_i}] \cdot \Phi^n(x) = 0$.*

Proof of Lemma 2.9. By our choice of the prime p satisfying condition (A) above, we get that Φ^n induces a bijection of $A[p]^m$. Then condition (B), along with Lemma 2.7 finishes the proof of Lemma 2.9. \square

Lemma 2.9 yields that there are at most $k(p-1)p^{2g(m-1)} < kp^{2gm-1}$ tuples $x := (x_1, \dots, x_m) \in A[p]^m$, which satisfy the equation $[\overrightarrow{\alpha_i}] \cdot \Phi^n(x) = 0$ for some $i = 1, \dots, k$ and for some $n = 1, \dots, p-1$. On the other hand, our hypothesis on the points of Y yields that for each point $x \in A[p]^m$ there exists some $i(x) \in \{1, \dots, k\}$ and some $n(x) \in \{1, \dots, p-1\}$ satisfying equation (2.8.3). This yields an inequality $p^{2gm} < kp^{2gm-1}$, which is false when $p > k$. This contradiction concludes our proof of Theorem 2.1 when G is a power of a simple abelian variety A . \square

Using the proof of Theorem 1.2 in the case G is a power of a simple abelian variety (along with Lemmas 2.3 and 2.4), we can derive the same conclusion when G is any abelian variety.

Proof of Theorem 1.2 when G is an abelian variety. Poincaré's theorem (see, for example, [GS17, Fact 3.2]) gives that G is isogenous with a direct product of simple abelian varieties; more precisely, G is isogenous to a product of the form $\prod_{i=1}^{\ell} A_i^{m_i}$, where each A_i is a simple abelian variety. Combining Lemma 2.3 and Corollary 2.5, along with the proof of Theorem 1.2 for powers of simple abelian varieties yields the conclusion in Theorem 1.2 for any abelian variety. \square

Combining the fact that we have proven the conclusion of Theorem 1.2 for abelian varieties and also for tori, we can complete the proof of the general case in Theorem 1.2.

Proof of Theorem 1.2. First we note that we proved that when G is either an abelian variety or an algebraic torus, then there is no proper subvariety intersecting the orbit of each torsion point of

G . For the general case of a semiabelian variety G , we know there exists a short exact sequence of algebraic groups:

$$(2.9.1) \quad 1 \longrightarrow \mathbb{G}_m^N \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where A is an abelian variety and \mathbb{G}_m^N is the toric part of G . Furthermore, Φ induces a group endomorphism $\bar{\Phi}$ of A and also, $\Phi|_{\mathbb{G}_m^N}$ is a group endomorphism of \mathbb{G}_m^N (since there exist no non-trivial morphisms between an abelian variety and an algebraic torus; see also [GS, Fact 2.3]). Then Lemma 2.4 yields the desired conclusion. \square

Theorem 1.2 is an important ingredient in proving Theorem 1.3.

Proof of Theorem 1.3. We first note that each regular self-map on a semiabelian variety is a composition of a translation with an algebraic group endomorphism (see [GS, Fact 2.1]).

We start by proving some useful reductions, similar to the ones established in the beginning of our proof of Theorem 1.2.

Claim 2.10. It suffices to prove Theorem 1.3 for an isogenous dynamical system, i.e., if there exists an isogeny $\tau : G \longrightarrow G'$ of semiabelian varieties endowed with dominant, regular self-maps $\Phi : G \longrightarrow G$ and $\Phi' : G' \longrightarrow G'$ such that $\Phi' \circ \tau = \tau \circ \Phi$, then the conclusion of Theorem 1.3 for (G, Φ) yields the conclusion of Theorem 1.3 for (G', Φ') .

Proof of Claim 2.10. First we show that knowing conclusion (A) in Theorem 1.3 holds for the proper subvariety $X \subset G$ (under the action of Φ) yields that also conclusion (A) holds for the proper subvariety $X' := \tau(X) \subset G'$ (under the action of Φ'). Indeed, given any proper, irreducible, periodic subvariety $Z' \subset G'$ (under the action of Φ'), we let $Z \subset G$ be an irreducible component of $\tau^{-1}(Z')$, which is periodic under the action of Φ . The fact that conclusion (A) in Theorem 1.3 holds for the dynamical system (G, Φ) yields that $Z \subseteq X$; so, $Z' = \tau(Z) \subseteq \tau(X) = X'$, as claimed.

Next we show that if conclusion (B) holds for (G, Φ) , then it must also hold for (G', Φ') . Indeed, assume on the contrary that there exists some proper subvariety $Y' \subset G'$ which contains an iterate of each periodic, irreducible, proper subvariety of G' (under the action of Φ'). We claim that $Y := \tau^{-1}(Y')$, which is a proper subvariety of G , must contain an iterate of each irreducible, proper, periodic subvariety of G (under the action of Φ), therefore contradicting conclusion (B) for the dynamical system (G, Φ) . Let $Z \subset G$ be a proper, irreducible, periodic subvariety of G and let $Z' := \tau(Z) \subset G'$; clearly, Z' is a proper, irreducible, periodic subvariety of G' . Then there exists some $m \in \mathbb{N}$ such that $(\Phi')^m(Z') \subseteq Y'$; so,

$$(\Phi')^m(\tau(Z)) = \tau(\Phi^m(Z)) \subseteq Y',$$

which yields that $\Phi^m(Z) \subseteq \tau^{-1}(Y') = Y$, as claimed.

This finishes our proof of Claim 2.10. \square

Claim 2.11. With the notation as in Theorem 1.3, let $\tau : G \longrightarrow G$ be an automorphism and let $\Phi' := \tau \circ \Phi \circ \tau^{-1}$. Then conclusion (A) holds for (G, Φ) if and only if it holds for (G, Φ') and also, conclusion (B) holds for (G, Φ) if and only if it holds for (G, Φ') .

Proof of Claim 2.11. This is an immediate consequence of Claim 2.10. \square

Claim 2.12. Assume conclusion (B) from Theorem 1.3 holds for the dynamical system (G_1, Φ_1) , where G_1 is a semiabelian varieties endowed with a dominant, regular self-map Φ_1 . Then for any semiabelian variety G_2 endowed with a dominant, regular self-map Φ_2 , we have that also conclusion (B) from Theorem 1.3 holds for the semiabelian variety $G_1 \times G_2$ endowed with the dominant, regular self-map Φ given by the rule $(x_1, x_2) \mapsto (\Phi_1(x_1), \Phi_2(x_2))$.

Proof of Claim 2.12. Let $Y \subset G_1 \times G_2$ be a subvariety containing some iterate of each proper, irreducible, periodic subvariety of $G_1 \times G_2$ (under the action of Φ). We know that there is no proper subvariety of G_1 which contains some iterate of each proper, irreducible, periodic subvariety of G_1 , under the action of Φ_1 . Then for any proper, irreducible, periodic subvariety $V_1 \subset G_1$ (under the action of Φ_1), we have that $V_1 \times G_2$ is a proper, irreducible, periodic subvariety of G (under the action of Φ); thus Y must contain a variety of the form $\Phi_1^i(V_1) \times G_2$ for some positive integer i . Our hypothesis regarding the dynamical system (G_1, Φ_1) yields that Y must project dominantly onto G_1 , and moreover $Y = G_1 \times G_2$, as claimed. \square

We know that Φ is the composition of a translation (by some point $y \in G(K)$) and a dominant group endomorphism $\Phi_0 : G \rightarrow G$. The next reduction is crucial for our proof.

Claim 2.13. It suffices to prove the conclusion under the assumption that $\Phi_0 - \text{id}$ is a nilpotent endomorphism of G (where id will always represent the identity morphism).

Proof of Claim 2.13. Let $f \in \mathbb{Z}[t]$ be the minimal, monic polynomial satisfied by Φ_0 , i.e., $f(\Phi_0) = 0$. Since Φ (and therefore also Φ_0) is dominant, the roots of $f(t)$ are nonzero. We let r be the order of 1 as a root of $f(t)$ and also, we let $g \in \mathbb{Z}[t]$ such that $f(t) = (t - 1)^r \cdot g(t)$. Then we let $G_1 := (\Phi_0 - \text{id})^r(G)$ and $G_2 := (g(\Phi_0))(G)$.

Because the polynomials $(t - 1)^r$ and $g(t)$ are coprime, coupled with the fact that the group of K -rational points of a semiabelian variety is divisible, we get that $G_1 + G_2 = G$; also, $G_1 \cap G_2$ is finite (these facts are proven in [GS17, Lemma 6.1] and [GS, (4.0.2)]). Thus G is isogenous with the direct product $G_1 \times G_2$; more precisely, we consider the isogeny $\tau : G_1 \times G_2 \rightarrow G$ given by $\tau(x, y) = x + y$.

We let $y_1 \in G_1(K)$ and $y_2 \in G_2(K)$ such that $y = y_1 + y_2$; then we let $\Phi_1 : G_1 \rightarrow G_1$ be the composition of $(\Phi_0)|_{G_1}$ with the translation by y_1 , while $\Phi_2 : G_2 \rightarrow G_2$ is given by the composition of $(\Phi_0)|_{G_2}$ with the translation by y_2 . We get that for each $x_1 \in G_1$ and $x_2 \in G_2$ we have:

$$\Phi(\tau(x_1, x_2)) = \tau(\Phi_1(x_1), \Phi_2(x_2));$$

in other words, the dynamical system (G, Φ) is isogenous to the dynamical system $(G_1 \times G_2, \Phi_1 \times \Phi_2)$. Thus, by Claim 2.10, it suffices to prove the conclusion of Theorem 1.3 for $(G_1 \times G_2, \Phi_1 \times \Phi_2)$.

We immediately observe that conclusion (B) from Theorem 1.3 always holds for (G_1, Φ_1) , since in this case, our problem reduces to Theorem 1.2. Indeed, we observe that $\Phi_0 - \text{id}$ is dominant on G_1 (note that the minimal polynomial satisfied by Φ_0 as an endomorphism of G_1 is $g(t)$, which has

no root equal to 1). Thus there exists some $y_3 \in G_1(K)$ such that $\Phi_0(y_3) - y_3 = -y_1$ (note that each dominant group endomorphism is actually surjective) and this yields (after denoting by τ_z the translation-by- z map on G_1 for any point $z \in G_1(K)$) that

$$(\tau_{-y_3} \circ \Phi_1 \circ \tau_{y_3})(x) = \Phi_0(x) \text{ for each } x \in G_1;$$

therefore $\tau_{-y_3} \circ \Phi_1 \circ \tau_{y_3}$ is a group endomorphism of G_1 . Using Claim 2.11, we see that it is no loss of generality to assume that $y_1 = 0$, i.e., $\Phi_1 : G_1 \rightarrow G_1$ is a dominant group endomorphism, and then, as proven in Theorem 1.2, we know that alternative (B) holds in this case.

Using Claim 2.12 (and also Claim 2.10), we are left to prove Theorem 1.3 for the dynamical system (G_2, Φ_2) , in which case we do know (as claimed) that $\Phi_2 - \text{id}$ is a nilpotent endomorphism of G_2 . This finishes our proof of Claim 2.13. \square

From now on, we work with the extra hypothesis on the dynamical system (G, Φ) established by Claim 2.13 and thus assume that $\Phi = \tau_y \circ \Phi_0$ (where τ_y is the translation-by- y map on G) and furthermore $\Psi := \Phi_0 - \text{id}$ is a nilpotent endomorphism of G . We prove that Theorem 1.3 must hold under these assumptions.

Let W be an irreducible, proper subvariety of G that is of maximal dimension among all irreducible, periodic subvarieties under the action of Φ ; clearly, if there is no such W , i.e., there is no proper subvariety of G , periodic under the action of Φ , then alternative (A) holds trivially.

We note that the conclusion in Theorem 1.3 is preserved if we replace Φ by an iterate Φ^N ; the argument is identical with the proof of Lemma 2.2.

With the above observation, we may assume that W is fixed by Φ .

Claim 2.14. For each $x \in \ker(\Psi)$, we have that $x + W$ is also fixed by Φ .

Proof of Claim 2.14. Let $w \in W$; we know that $\Phi(w) \in W$ and furthermore, since $x \in \ker(\Phi_0 - \text{id})$, $\Phi_0(x) + \Phi(w) \in x + W$. But $\Phi_0(x) + \Phi(w) = \Phi(x + w)$ and therefore, $\Phi(x + w) \in x + W$ for each $w \in W$, thus proving that $x + W$ is indeed fixed by Φ . \square

Claim 2.14 yields that the Zariski closure Z of $\ker(\Psi) + W$ is fixed by Φ . Since W has maximal dimension among the periodic, proper, subvarieties of G , we must have that either $Z = G$, or that W is an irreducible component of Z (and that they both have the same dimension). We split our analysis into these two cases.

Case 1. Assume $Z = G$.

In this case, we get that there is no proper subvariety $Y \subset G$ which contains an iterate of each periodic, proper, irreducible subvariety of G , because each $x + W$ (for $x \in \ker(\Psi)$) is fixed by Φ and their union is Zariski dense in G , according to our assumption.

Case 2. W is an irreducible component of Z (of the same dimension).

In this case, we let $W_1 := Z$, which is a proper subvariety of G , fixed under the action of Φ ; as observed, W_1 may no longer be irreducible.

Claim 2.15. For each $x \in \ker(\Psi^2)$, $x + W_1$ is also fixed by Φ .

Proof of Claim 2.15. Indeed, for each $w \in W_1$ and each $x \in \ker(\Psi^2)$, we have that

$$\Phi(x + w) = \Phi(w) + \Phi_0(x) = x + \Psi(x) + \Phi(w) \in x + W_1,$$

because $\Phi(w) \in W_1$ (since W and therefore, also W_1 is fixed by Φ), while $\Psi(x) \in \ker(\Psi)$ and $W_1 = \ker(\Psi) + W_1$ since $W_1 = Z$ is the Zariski closure of $W + \ker(\Psi)$. Hence $x + W_1$ is fixed by Φ , as claimed. \square

Claim 2.15 yields that for each $x \in \ker(\Psi^2)$, the subvariety $x + W$ is fixed by Φ^{N_1} , where N_1 is a positive integer depending solely on the number of irreducible components of $W_1 = Z$. Without loss of generality, we replace again Φ by its iterate Φ^{N_1} such that each $x + W$ is now fixed by Φ (for each $x \in \ker(\Psi^2)$).

We argue as before and if $\ker(\Psi^2) + W_1 = G$, conclusion (B) holds in Theorem 1.3 because the union of all irreducible, proper subvarieties of G , which are fixed by Φ , is actually Zariski dense in G . Hence it remains to analyze the case $W_2 := \ker(\Psi^2) + W_1 = \ker(\Psi^2) + W$ is a subvariety of G of the same dimension as W . Then we argue as in the proofs of Claims 2.14 and 2.15 and inductively derive that unless conclusion (B) holds in Theorem 1.3, then we must have that $\ker(\Psi^j) + W$ is a proper subvariety of G for any $j \in \mathbb{N}$. However, Ψ is nilpotent and so we conclude (in finitely many steps) that conclusion (B) must hold because $\ker(\Psi^r) = G$ for sufficiently large integers r .

This finishes our proof of Theorem 1.3. \square

3. PROOF OF THEOREM 1.1

In order to prove Theorem 1.1, we need a result about invariant subvarieties of \mathbb{G}_m^d ; it would be interesting (by itself) to know whether an analogue of this result holds in a more general setting. Before stating our lemma, we recall that an endomorphism Φ of some quasiprojective variety X , which is not necessarily irreducible, preserves a non-constant fibration if there exists a rational function $f : X \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$ and such that f is defined and non-constant on a dense open subset of some irreducible component of X ; this notion plays a crucial role in the Medvedev-Scanlon conjecture from [MS14] regarding Zariski dense orbits.

Lemma 3.1. *Let $n \in \mathbb{N}$ and let $\phi : \mathbb{G}_m^n \rightarrow \mathbb{G}_m^n$ be the composition of a translation with an algebraic group automorphism. If $X \subseteq \mathbb{G}_m^n$ is an irreducible subvariety defined over an algebraically closed field k of characteristic 0 that is invariant under ϕ and moreover, if $\phi|_X$ does not preserve a non-constant fibration, then X is a translate of some subtorus of \mathbb{G}_m^n .*

Proof. Let $Y \subseteq \mathbb{G}_m^n$ be an irreducible subvariety of minimal dimension with respect to having the following properties:

- (i) Y is invariant under some power of ϕ ;
- (ii) Y is a translate of a subtorus of \mathbb{G}_m^n ;
- (iii) $Y \supseteq X$.

Such a Y exists by hypothesis. We claim that $Y = X$. To see this, we have $\mathcal{O}(Y) = k[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$, where $d = \dim(Y)$. If $\mathcal{O}(Y) \neq \mathcal{O}(X)$ then there is a non-trivial linear combination

$$(3.1.1) \quad \sum_{i=1}^r c_i m_i$$

that vanishes identically on X , where $c_1, \dots, c_r \in k^*$ and m_1, \dots, m_r are distinct monomials in $x_1^{\pm 1}, \dots, x_d^{\pm 1}$. We may assume that r is minimal with respect to such a dependence holding on X and since the m_i are units in $\mathcal{O}(X)$, we see that $r \geq 2$.

Then let H denote the subgroup of $\mathcal{O}(X)$ generated by the images of $x_1^{\pm 1}, \dots, x_d^{\pm 1}$. By replacing ϕ by a power, we may assume that X and Y are both ϕ -invariant. For every $t \geq 1$ we have

$$(3.1.2) \quad \sum_{i=1}^r c_i (\phi^*)^t(m_i)$$

vanishes on X , and we have $(\phi^*)^t(m_i) \in k^*H$ for every t . In particular, by the theory of S -unit equations for function fields (see, for example, [Zan93]), there exist $s < t$ such that

$$(3.1.3) \quad (\phi^*)^t(m_i) = \lambda_i (\phi^*)^s(m_i)$$

on X for $i = 1, \dots, r$ for some $\lambda_1, \dots, \lambda_r \in k^*$. Indeed, first of all, we can reduce to the case that no proper subsum of the sum appearing in (3.1.1) equals 0 (otherwise we replace the sum from (3.1.1) with that particular subsum); furthermore, at the expense of restricting to an infinite set of positive integers t , we may assume that no proper subsum of the sum appearing in (3.1.2) vanishes. Then the first theorem stated on [Zan93, p. 87] (which is itself a restatement of the main result of [Mas84]) yields that the primitive solutions $x_1, \dots, x_r \in k^* \cdot H$ to the equation

$$(3.1.4) \quad c_1 x_1 + \dots + c_r x_r = 0$$

(i.e., solutions for which no proper subsum of the sum appearing in (3.1.4) equals 0) have bounded height. Therefore, modulo constants from k^* , there are only finitely many solutions to (3.1.4) and thus also to (3.1.2), which delivers the desired claim from (3.1.3).

Since ϕ is an automorphism, then (3.1.3) gives us that $(\phi^*)^{t-s}(m_i) = \lambda_i m_i$ on X for $i = 1, \dots, r$. In particular the fact that

$$\sum_{i=1}^r c_i m_i$$

vanishes identically on X gives that

$$\sum_{i=1}^r c_i (\phi^*)^{t-s}(m_i)$$

vanishes identically on X and so $\sum_{i=1}^r c_i \lambda_i m_i = 0$ on X . In particular, we have

$$\sum_{i=2}^r c_i (\lambda_i - \lambda_1) m_i = 0$$

on X . By minimality of r in our relation, we see that $\lambda_1 = \dots = \lambda_r$. Then since $r \geq 2$ we have

$$(\phi^*)^{t-s}(m_1/m_2) = m_1/m_2.$$

Letting $h = m_1/m_2$ we then see that either h is constant on X or $\phi^{t-s}|_X$ preserves a non-constant fibration. But the latter possibility would then give that $\phi|_X$ preserves a non-constant fibration (see [BGRS17, Lemma 2.1]), which is a contradiction. It follows that there is some $\lambda \in k^*$ such that $h = \lambda$ on X . Then we can write $h = x_1^{i_1} \cdots x_d^{i_d}$ with $i_1, \dots, i_d \in \mathbb{Z}$, not all zero since $m_1 \neq m_2$. We may also assume that the integers i_1, \dots, i_d are coprime since otherwise we could replace h by a root of it in H and it would still be identically equal to some nonzero value of k on X since k is algebraically closed. Then we have a surjective map

$$\mathbb{Z}^d \rightarrow \mathbb{Z}$$

given by $(a_1, \dots, a_d) \mapsto \sum a_j i_j$ and since \mathbb{Z} is projective as a module over itself the sequence splits and since all projectives are free, we see that $\mathcal{O}(Y)^*/k^* \cong \langle h \rangle \oplus \langle h_1, \dots, h_{d-1} \rangle$, where h_1, \dots, h_{d-1} are monomials in $x_1^{\pm 1}, \dots, x_d^{\pm 1}$. In particular, $\mathcal{O}(Y)/(h - \lambda) \cong \mathcal{O}(\mathbb{G}_m^{d-1})$. Let $Y' = Y \cap V(h - \lambda)$. Then Y' contains X , it is ϕ^{t-s} -invariant and irreducible, and also it is a translate of an algebraic subtorus, contradicting the minimality of Y . It follows that $Y = X$ and the result follows. \square

We note that a subvariety not preserving a non-constant fibration is related to the property of the ideal (0) of the associated skew polynomial ring being rational—this connection is made precise in Lemma 3.5. Finally, we need a result relating being locally closed to σ -invariant ideals.

We first recall some basic facts about skew polynomial rings. Given a ring R with an automorphism σ , an ideal I is σ -invariant if $\sigma(I) \subseteq I$; a σ -invariant ideal I is σ -prime if whenever $JL \subseteq I$ with J, L σ -invariant ideals, then we must have $J \subseteq I$ or $L \subseteq I$. Then given a prime ideal P of either $R[t; \sigma]$ or $R[t^{\pm 1}; \sigma]$, where in the former case we assume $t \notin P$, we obtain a σ -prime ideal $P \cap R$ of R and conversely if I is a σ -prime ideal of R then $IR[t; \sigma]$ is a prime ideal of $R[t; \sigma]$ that does not contain t and similarly $IR[t^{\pm 1}; \sigma]$ is a prime ideal of $R[t^{\pm 1}; \sigma]$. In general this map from σ -prime ideals of R to prime ideals of $R[t; \sigma]$ is injective and not surjective. We note that a σ -prime ideal is a *semiprime* ideal of R and not necessarily prime; that is, it is an intersection of prime ideals of R . In the case when R is Noetherian, the σ -prime ideals are precisely the ideals of the form $\bigcap_{i=1}^t P_i$, where P_1, \dots, P_t is a finite σ -orbit of prime ideals. More details and proofs can be found in [BG02, Chapter II].

For Theorem 1.1 we are interested in three different properties of prime ideals: being primitive, being rational, and being locally closed in the prime spectrum. We recall that given a field k , a k -algebra R satisfies the strong Nullstellensatz if every prime ideal is an intersection of primitive ideals and if whenever M is a simple R -module, the endomorphism ring $\text{End}_R(M)$ is a finite-dimensional k -algebra. If R is a finitely generated commutative k -algebra then $R[t; \sigma]$ and $R[t^{\pm 1}; \sigma]$ satisfy the strong Nullstellensatz [McC82, Theorem 4.6]. For a Noetherian algebra S satisfying the strong Nullstellensatz we always have the following implications for $P \in \text{Spec}(S)$:

$$P \text{ locally closed} \implies P \text{ primitive} \implies P \text{ rational.}$$

Lemma 3.2. *Let k be a field and let R be a finitely generated commutative k -algebra that is an integral domain and let σ be a k -algebra automorphism of R . If R is not a field then (0) is a locally*

closed prime ideal of $R[t; \sigma]$ if and only if the intersection of the nonzero σ -prime ideals of R is nonzero.

Proof. If the intersection of the nonzero σ -prime ideals of R is zero, then the intersection of the nonzero prime ideals of $R[t; \sigma]$ is zero and so (0) is not locally closed, by the remarks in the penultimate paragraph above. It suffices to show that if the intersection of the nonzero σ -prime ideals of R is nonzero then (0) is locally closed. Henceforth we assume that the intersection of the nonzero σ -prime ideals of R is nonzero. There are three types of nonzero prime ideals in $\text{Spec}(R[t; \sigma])$:

- (i) primes Q such that $Q \cap R \neq (0)$ and $t \notin Q$;
- (ii) primes Q such that $t \in Q$;
- (iii) nonzero primes Q such that $t \notin Q$ and $Q \cap R = (0)$.

Since a finite intersection of nonzero ideals in a prime ring is nonzero, it suffices to show that the intersection of the nonzero primes of each type is nonzero. Note that if $t \notin Q$ then $Q \cap R$ is σ -invariant, since for $q \in Q$ we have $tq = \sigma(q)t \in Q$ and since $tR[t; \sigma] = R[t; \sigma]t$ we then have

$$\sigma(q)R[t; \sigma]t \subseteq Q,$$

and so $\sigma(q) \in Q$ since Q is prime and $t \notin Q$. Then the intersection of prime ideals of type (i) is nonzero since $Q \cap R$ is a σ -prime ideal when $t \notin Q$ and we are assuming that the intersection of nonzero σ -prime ideals is nonzero.

For primes of type (ii), we have t is in the intersection.

We note that if $t \notin Q$ then no power of t is in Q , because $tR[t; \sigma] = R[t; \sigma]t$ and so if $t^m \in Q$ then $(t)^m \subseteq Q$ and since Q is prime we then have $(t) \subseteq Q$, which we have assumed is not the case. Thus primes of type (iii) cannot contain an element of the form st^m with $m \geq 0$ and $s \in R \setminus \{0\}$, and hence they survive in the localization $K[t^{\pm 1}; \sigma]$, where K is the field of fractions of R . But this algebra is simple unless σ has finite order [Jor93, §6] and so we see that nonzero primes of type (iii) do not exist unless σ has finite order. Since R is a finitely generated commutative k -algebra and a domain, (0) is an intersection of the maximal ideals [Eis95, Theorem 4.19] and since σ has finite order we see from the penultimate paragraph before this lemma that the finite intersection of the orbit of each maximal ideal is a nonzero σ -prime ideal and the intersection of all such ideals is equal to zero. Thus, under our standing hypotheses, there are no primes of type (iii). The result follows. \square

We are now ready to prove Theorem 1.1 (a).

Proof of Theorem 1.1 (a). By the Irving-Small reduction techniques for relating locally closed and primitive ideals (see [Row88b, Lemma 8.4.28]) we may assume that k is algebraically closed. Let $S = R[t; \sigma]$. We divide the proof into three cases. The first case is when $t \notin P$ and $(P \cap R)S$ is strictly contained in P . Let $I = P \cap R$ and observe that I is σ -prime and that σ induces a non-trivial automorphism of R/I . Then S/P is a non-trivial homomorphic image of $(R/I)[t; \sigma]$. Let \bar{P} denote the image of P in $(R/I)[t; \sigma]$. Then $\bar{P} \cap (R/I) = (0)$ and so we see that \bar{P} survives in the localization

$Q(R/I)[t; \sigma]$ and we let P_0 denote the ideal generated by \bar{P} in this localization. Now P_0 necessarily contains a monic polynomial, since the collection of leading coefficients of elements of P_0 forms a nonzero σ -invariant ideal L of $Q(R/I)$ and since I is a σ -prime ideal of R , L is necessarily all of $Q(R/I)$. In particular, $Q(R/I)[t; \sigma]/P_0$ is a finite $Q(R/I)$ -module and thus it satisfies a polynomial identity. It follows that $(R/I)[t; \sigma]/\bar{P}$ satisfies a polynomial identity. We note that P is primitive if and only if \bar{P} is primitive and since in a polynomial identity algebra primitive ideals are precisely those ideals that are maximal we see that P is primitive if and only if it is maximal in this case. In particular, P is vacuously locally closed. This completes the first case.

The second case is when $t \in P$. In this case S/P is a homomorphic image of R and so P is primitive if and only if it is maximal, and this holds if and only if P is locally closed.

Finally, we may assume that $I = P \cap R$ has the property that $IS = P$ and $t \notin P$. Since $tS = St$ and P is a prime ideal, no power of t is in P . Let $G = \text{Spec}(R)$. Let X be the zero set of I . Then X is σ -invariant and σ acts transitively on the irreducible components of X . We let X_1, \dots, X_d denote the irreducible components of X and let $J = I(X_1)$. Then J is prime and is σ^m -invariant for some m . By the Leroy-Matczuk theorem, $(R/I)[t; \sigma]$ is primitive if and only if $(R/J)[t^m; \sigma^m]$ is primitive (see [LM96, Corollary 2.2]) and by [Let89, Theorem 2.3], to show that (0) is locally closed in $(R/I)[t; \sigma]$, it suffices to show that (0) is locally closed in $(R/J)[t^m; \sigma^m]$. Suppose first that (0) is primitive. Then since $(R/J)[t^m; \sigma^m]$ satisfies the strong Nullstellensatz, by the discussion before Lemma 3.2, (0) is rational. Since (0) is rational, $X_1 = \text{Spec}(R/J) \cong \mathbb{G}_m^d$ for some $d \geq 0$ by Lemma 3.5 (below) and Lemma 3.1 (also, note that each regular self-map of a power of the multiplicative group is a composition of a translation by a group endomorphism; see [GS, Fact 2.1]). By the Leroy-Matczuk theorem, as explained in Subsection 1.3, (0) is primitive if and only if $\tau := \sigma^m$ has infinite order on X_1 and there is some hypersurface $Y \subseteq X_1$ such that every proper τ -periodic subvariety of X_1 has the property that some irreducible component is contained in Y . In particular, by Theorem 1.3, this occurs if and only if the union of the proper τ -periodic subvarieties of X_1 is not Zariski dense, which is the same as saying that the intersection of the nonzero τ -prime ideals of R/J is nonzero. Then by Lemma 3.2, we see that this gives that (0) is locally closed, unless X_1 is a point. But if X_1 is a point, we have $(R/J)[t^m; \sigma^m]$ is isomorphic to $k[x]$ and (0) is neither primitive nor locally closed in this case.

Conversely, if (0) is locally closed then (0) is primitive since $R[t; \sigma]$ satisfies the strong Nullstellensatz [Row88b, Proposition 8.4.18]. \square

In the case where we have a skew Laurent polynomial ring $S = R[t^{\pm 1}; \sigma]$ the criterion for primitivity is due to Jordan [Jor93, Theorem 7.3]: in this case, (0) is a primitive ideal if and only if either R has a σ -special element or there is a maximal ideal Q with the property that $\bigcap_{n \in \mathbb{Z}} \sigma^n(Q) = (0)$. The property of the existence of Q such that $\bigcap_{n \in \mathbb{Z}} \sigma^n(Q) = (0)$ is, conjecturally, equivalent to σ not preserving a non-constant fibration. (This in fact would be implied by the Medvedev-Scanlon conjecture [MS14].) Furthermore, the following easy lemma provides an interesting reduction in Jordan's criterion.

Lemma 3.3. *Let X be a quasiprojective variety defined over an algebraically closed field k of characteristic 0, endowed with an automorphism σ . Assume there exists a proper subvariety $Y \subset X$ which contains an iterate of each proper, irreducible, periodic subvariety of X , under the action of σ . Then there exists a point $x \in X(k)$ whose orbit under σ is Zariski dense in X .*

Proof. As proven by Amerik [Ame11], there exists a point $x \in X(k)$ whose entire orbit under σ avoids the proper, Zariski-closed subset $Y \subset X$. We claim that the orbit of x under σ must be Zariski dense in X . Indeed, otherwise, its Zariski closure must be a finite union of proper, irreducible, periodic subvarieties of X ; then let Z be any one of these subvarieties. By hypothesis, there exists $i \geq 0$ such that $\sigma^i(Z) \subset Y$. On the other hand, by construction, there exists some $m \geq 0$ such that $\sigma^m(x) \in Z$. Thus $\sigma^{m+i}(x) \in Y$, which contradicts our choice for x .

This concludes our proof of Lemma 3.3. □

Lemma 3.3 implies that if there is a σ -special element then there is a maximal ideal Q with the property that $\bigcap_{n \in \mathbb{Z}} \sigma^n(Q) = (0)$ and so we can simplify Jordan's criterion for finitely generated noetherian algebras R . We record this in the following statement.

Proposition 3.4. *Let k be an algebraically closed field and let R be a finitely generated integral domain with an automorphism σ . Then $R[t^{\pm 1}; \sigma]$ is primitive if and only if there is a closed k -point $x \in X := \text{Spec}(R)$ whose orbit under the map induced by σ is Zariski dense in X .*

Proof. This follows immediately from Jordan's theorem [Jor93, Theorem 7.3] along with Lemma 3.3. □

Lemma 3.5. *Let k be an algebraically closed field and let R be a finitely generated commutative k -algebra and let σ be a k -algebra automorphism of R . Let P be a prime ideal of $S = R[t^{\pm 1}; \sigma]$. Then P is a rational ideal of S if and only if either P is maximal or $P \cap R$ has infinite codimension in R and $P = (R \cap P)S$ and the automorphism of $X = \text{Spec}(R/(P \cap R))$ induced by σ does not preserve a non-constant fibration.*

Proof. Let P be a prime ideal of S , so $Q := P \cap R$ is a semiprime σ -invariant ideal of R . In proving the lemma, we may assume without loss of generality that $Q = (0)$, so R is σ -prime and $(0) = Q_1 \cap \dots \cap Q_s$ with the minimal primes Q_1, \dots, Q_s of R permuted transitively by σ .

\implies : Suppose first that P is rational. $P \neq (0)$ then by a result of Irving [Irv79, Theorem 4.3], σ has finite order on R and so $R[t^{\pm 1}; \sigma]$ satisfies a polynomial identity. Since the only rational ideals of polynomial identity algebras are maximal ideals, by Kaplansky's theorem [Row88b, Theorem 6.1.25], we see that P is maximal in this case.

If $R/P \cap R$ is finite-dimensional, then for some $m \geq 1$, σ^m induces the identity on $R/P \cap R$. Thus R/P satisfies a polynomial identity, and again P is maximal.

Thus we are left to deal with the case where $P = (0)$, with R infinite-dimensional and σ -prime. Let $X = \text{Spec}(R)$. Then X is the union of the irreducible subvarieties X_1, \dots, X_s , where $X_i = \text{Spec}(R/Q_i)$. Moreover, there is some $\tau = \sigma^m$ such that Q_1, \dots, Q_s are τ -stable. Furthermore, (0) is

a rational prime ideal of $(R/Q)[t^{\pm 1}; \sigma]$ if and only if (0) is a rational prime ideal of $(R/Q_1)[t^{\pm m}; \tau]$ (see [Let89, Corollary 1.2]) and the automorphism of X induced by σ preserves a non-constant fibration if and only if the automorphism of X_1 induced by τ preserves a non-constant fibration (see [BGRS17, Lemma 2.1]). If the automorphism of X_1 induced by τ preserves a non-constant fibration, there is a non-constant rational map $f : X_1 \rightarrow \mathbb{P}^1$ such that the automorphism of X_1 induced by τ preserves the fibres of f . In particular if we take the embedding $f^* : k(\mathbb{P}^1) \rightarrow k(X_1)$, then by construction τ is the identity on the image of f^* . Thus τ preserves the function field and so $f^*(k(\mathbb{P}^1))$ is a central subfield of the division ring of quotients of $(R/Q_1)[t^{\pm m}; \tau]$ and so (0) is not a rational prime of $(R/Q_1)[t^{\pm m}; \tau]$, and therefore, by [Let89, Corollary 1.2], the ideal (0) of S is not rational. This is a contradiction, and completes the proof of this implication.

\Leftarrow : Suppose now that P is not rational. Then P is not maximal, by the Nullstellensatz. If R is finite-dimensional there is nothing further to prove. So suppose that R is infinite-dimensional. Since $P_1 := Q_1 R[t^{\pm m}; \tau]$ is not rational, the Goldie ring of quotients of $(R/Q_1)[t^{\pm m}; \tau]$ has a central element z that is not in k . Let K denote the field of fractions of R/Q_1 . Then since $K((t^m; \tau))$ is a division ring, we have $Q((R/Q_1)[t^{\pm m}; \tau])$ embeds in this skew power series ring and thus if we let z denote an element of $Q((R/Q_1)[t^{\pm m}; \tau])$ that is not algebraic over k then we have a Laurent power series expansion

$$z = \sum_{i \geq -M} \alpha_i t^{mi},$$

with $\alpha_i \in K$ and $\alpha_{-M} \neq 0$. Now z commutes with t^m and hence

$$\sum_{i \geq -M} \tau(\alpha_i) t^{mi+m} = t^m z = z t^m = \sum_{i \geq -M} \alpha_i t^{mi+m},$$

and so $\tau(\alpha_i) = \alpha_i$ for all $i \geq -M$. In particular, if some $\alpha_i \in K \setminus k$ then α_i is a rational function on $\text{Spec}(R/Q_1)$ and $\tau(\alpha_i) = \alpha_i$ and thus the automorphism induced by τ preserves a non-constant fibration. If, on the other hand, $\alpha_i \in k$ for all i then the fact that z is central and using the fact that $az = za$ for all $a \in R/Q_1$ shows that $\tau^j(a) = a$ for all $a \in R/Q_1$ for which $\alpha_j \neq 0$. Thus either τ has finite order on R/Q_1 or $\alpha_j = 0$ for $j \neq 0$. But the latter case gives that $z \in k$, a contradiction. Thus τ has finite order on R/Q_1 , but in this case the automorphism induced by τ preserves a non-constant fibration. \square

Proof of Theorem 1.1 (b). By the Irving-Small reduction techniques (see [Row88b, Lemma 8.4.29]) we may replace k by its algebraic closure and assume that k is algebraically closed. If P is a prime ideal of $R[t^{\pm 1}; \sigma]$ then, by Lemma 3.5, since $R[t^{\pm 1}; \sigma]$ satisfies the strong Nullstellensatz, primitive ideals are rational. Thus it suffices to prove rational ideals are primitive. Let P be a rational ideal of $R[t^{\pm 1}; \sigma]$. Then by [GS, Fact 2.1] and Lemma 3.5, either P is maximal, in which case it is primitive and there is nothing to prove, or $P = (R \cap P)R[t^{\pm 1}; \sigma]$. In the latter case, we let Q_1, \dots, Q_s denote the minimal prime ideals above $P \cap R$ in R . Then σ permutes these primes and acts transitively on this set of primes. Thus there is some d such that $\tau = \sigma^d$ has the property that $\tau(Q_1) = Q_1$. Now $R/(P \cap R)[t^{\pm 1}; \sigma]$ is primitive if and only if $R/Q_1[t^{\pm d}; \tau]$ is primitive (see [LM96, Corollary 2.2]) and $Q_1 R[t^d; \tau]$ is a rational prime ideal of $R[t^{\pm d}, \tau]$ by [Let89, Corollary 1.2]. Thus it suffices to prove

that $(R/Q_1)[t^{\pm d}; \tau]$ is primitive. Let $X = \text{Spec}(R/Q_1)$. Then the map induced by τ on X does not preserve a non-constant fibration since $Q_1 R[t^{\pm d}; \tau]$ is a rational prime ideal of $R[t^{\pm d}; \tau]$. Thus by Lemma 3.1, $X \cong \mathbb{G}_m^d$ for some d and so $R/Q_1 \cong k[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$. Moreover, by [GS], since the map induced by τ on X does not preserve a non-constant fibration, we then have that there is some $x \in X(k)$ that has a Zariski dense orbit. In particular there is a maximal ideal above Q_1 whose bi-infinite τ orbit has intersection equal to Q_1 . Then by Proposition 3.4 we see that $(R/Q_1)[t^{\pm d}, \tau]$ is primitive which gives that $R[t^{\pm 1}; \sigma]/P$ is primitive and so P is primitive. \square

4. EXAMPLES

We make a few concluding remarks concerning the remaining implications in the Dixmier-Moeglin equivalence in Theorem 1.1 (a) and (b). Let k be a field and let $R = k[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ and let σ be a k -algebra automorphism of R . Observe that Theorem 1.1 (a) gives, when k is of characteristic zero, that a prime ideal of $S := R[t; \sigma]$ is primitive if and only if P is locally closed in $\text{Spec}(S)$, while the Nullstellensatz gives that primitive ideals are rational. Thus the only remaining question in this case is whether rational ideals are primitive. In fact, both in characteristic zero and in positive characteristic, this is not the case. We are indebted to the referee for the positive characteristic example in this section.

In the case of characteristic zero, if $R = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ and $\sigma \in \text{Aut}_{\mathbb{C}}(R)$ given by $\sigma(y) = x$, $\sigma(x) = yx^{-1}$, then $S = R[t; \sigma]$ is not primitive but $T = R[t^{\pm 1}; \sigma]$ is [BCM17, §7.4].¹ It follows that (0) is not a primitive ideal of $R[t; \sigma]$ but it is rational. In the case of positive characteristic one can use work of Roseblade as follows. Let k be the algebraic closure of a finite field, let A be a finitely generated free abelian group of rank at least two, and let $\langle t \rangle$ be the infinite cyclic group and let $G = A \rtimes \langle t \rangle$ with the action of $\langle t \rangle$ on A being chosen so that $\mathbb{Q}\langle t^n \rangle$ acts irreducibly on $\mathbb{Q} \otimes A$ for all positive integers n . By a result of Roseblade [Ros73, Theorem A], the simple kG -modules are finite-dimensional. Writing R for kA and $S := R[t; \sigma] \subseteq kG$, we see that the same property holds for the simple S -modules, since a simple S -module is either annihilated by t or is t -divisible and hence acquires a natural kG -module structure. Hence S is not primitive and so (0) is not locally closed by the Nullstellensatz. But we note that (0) is a rational ideal of kG and hence of S . To see this, by Lemma 3.5 it suffices to show that the automorphism $\tau : (k^*)^2 \rightarrow (k^*)^2$ given by $\tau(a, b) = (ba^{-1}, a)$ does not preserve a non-constant fibration. If there is a non-constant rational function $f(x, y)$ such that $f \circ \tau = f$, then we have $f(yx^{-1}, x) = f(x, y)$. We write $f(x, y) = p(x, y)/q(x, y)$ with $p(x, y), q(x, y) \in k[x^{\pm 1}, y]$ having gcd 1. Then $f \circ \tau = f$ gives that $p(yx^{-1}, x)q(x, y) = p(x, y)q(yx^{-1}, x)$. Since $\text{gcd}(p, q) = 1$ in $k[x^{\pm 1}, y]$, we see that there is $C \in k^*$ and $a \in \mathbb{Z}$ such that $p(yx^{-1}, x) = Cx^a p(x, y)$ and $q(yx^{-1}, x) = Cx^a q(x, y)$. Now we write $p(x, y) = \sum c_{i,j} x^i y^j$. Then we have

$$\sum C c_{i,j} y^i x^{j-i+a} = \sum c_{i,j} c_{i,j} x^i y^j.$$

¹The arXiv version available at the time of writing this paper does not show that S is primitive, but the authors have since added a proof to their accepted manuscript.

In particular, if $X = \{(i, j) \in \mathbb{Z}^2 : c_{i,j} \neq 0\}$ then if $(i, j) \in X$ we have $(j - i + a, i) \in X$. In particular, if

$$B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

then if $(i, j) \in X$, we have $(i', j') \in X$ where

$$\begin{bmatrix} i' \\ j' \end{bmatrix} := B \begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}.$$

In particular, since X is finite, if $(i, j) \in X$ and $v = [i, j]^T$ and $w = [a, 0]^T$ then there exist positive integers m and n with $m < n$ such that

$$B^n v + B^{n-1} w + B^{n-2} w + \cdots + w = B^m v + B^{m-1} w + \cdots + w.$$

Multiplying by $(B - I)$ then gives that $B^n(B - I)v + B^n w = B^m(B - I)v + B^m w$, or equivalently $(B^n - B^m)((B - I)v + w) = 0$. Since B is invertible and has no roots of unity as eigenvalues, $B^n - B^m$ is invertible and so we then see $(B - I)v + w = 0$ and then by the invertibility of $(B - I)$ gives that v is uniquely determined by w . Thus $|X| = 1$ and so $p(x, y)$ is a monomial. Similarly, $q(x, y)$ is a monomial and so $f(x, y) = p(x, y)/q(x, y)$ is of the form $C'x^b y^c$ for some integers b, c and some $C' \in k^*$. But by assumption, $f \circ \tau = f$ and so $C'x^{-b+c}y^b = C'x^b y^c$ and so solving this gives $b = c = 0$ and so f is constant, a contradiction.

In the setting of Theorem 1.1 (b), we have primitivity is equivalent to rationality and the Nullstellensatz gives that locally closed ideals are primitive and so the remaining question is whether primitive ideals are necessarily locally closed. An example of Lorenz [Lor77] shows that there exists a group $G = \mathbb{Z} \times \mathbb{Z}$ such that kG is primitive but such that (0) is not locally closed whenever k is not an algebraic extension of a finite field. This group algebra is isomorphic to an algebra of the form $k[x^{\pm 1}, y^{\pm 1}][t^{\pm 1}; \sigma]$, and so Lorenz' example shows that Theorem 1.1 (b) cannot be strengthened in general to give the full Dixmier-Moeglin equivalence.

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