THE ORBIT INTERSECTION PROBLEM FOR LINEAR SPACES AND SEMIABELIAN VARIETIES

D. GHIOCA AND K. D. NGUYEN

Abstract. Let $f_1, f_2 : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ be affine maps $f_i(\mathbf{x}) := A_i\mathbf{x} + \mathbf{y}_i$ (where each A_i is an N-by-N matrix and $\mathbf{y}_i \in \mathbb{C}^N$), and let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{A}^N(\mathbb{C})$ such that $\mathbf{x}_i$ is not preperiodic under the action of f_i for $i = 1, 2$. If none of the eigenvalues of the matrices A_i is a root of unity, then we prove that the set $\{(n_1, n_2) \in \mathbb{N}_0^2 \colon f_1^{n_1}(\mathbf{x}_1) = f_2^{n_2}(\mathbf{x}_2)\}$ is a finite union of sets of the form $\{(m_1k + \ell_1, m_2k + \ell_2) \colon k \in \mathbb{N}_0\}$ where $m_1, m_2, \ell_1, \ell_2 \in \mathbb{N}_0$. Using this result, we prove that for any two self-maps $\Phi_i(x) := \Phi_{i,0}(x) + y_i$ on a semiabelian variety X defined over $\mathbb{C}$ (where $\Phi_{i,0} \in \mathrm{End}(X)$ and $y_i \in X(\mathbb{C})$), if none of the eigenvalues of the induced linear action $D\Phi_{i,0}$ on the tangent space at $0 \in X$ is a root of unity (for $i = 1, 2$), then for any two non-preperiodic points x_1, x_2, the set $\{(n_1, n_2) \in \mathbb{N}_0^2 \colon \Phi_1^{n_1}(x_1) = \Phi_2^{n_2}(x_2)\}$ is a finite union of sets of the form $\{(m_1k + \ell_1, m_2k + \ell_2) \colon k \in \mathbb{N}_0\}$ where $m_1, m_2, \ell_1, \ell_2 \in \mathbb{N}_0$. We give examples to show that the above condition on eigenvalues is necessary and introduce certain geometric properties that imply such a condition. Our method involves an analysis of certain systems of polynomial-exponential equations and the p-adic exponential map for semiabelian varieties.

1. Introduction

Throughout this paper, let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and let K be an algebraically closed field of characteristic 0. An arithmetic progression is a set of the form $\{mk + \ell :\ k \in \mathbb{N}_0\}$ for some $m, \ell \in \mathbb{N}_0$; note that when $m = 0$, this set is a singleton. For a map f from a set X to itself and for $m \in \mathbb{N}$, we let f^m denote the m-fold iterate $f \circ \ldots \circ f$, and let f^0 denote the identity map on X. If $x \in X$, we define the orbit $\mathcal{O}_f(x) := \{f^n(x) \colon n \in \mathbb{N}_0\}$. We say that x is preperiodic (or more precisely f-preperiodic) if its orbit $\mathcal{O}_f(x)$ is finite.

In algebraic dynamics, one studies the system $\{\Phi^n :\ n \in \mathbb{N}_0\}$ when X is a (quasi-projective) variety over K and Φ is a K-morphism. Motivated by the classical Mordell-Lang conjecture proved by Faltings [Fal94] and Vojta [Voj96], the Dynamical Mordell-Lang Conjecture predicts that for a given $x \in X(K)$ and a closed subvariety V of X, the set $\{n \in \mathbb{N} \colon \Phi^n(x) \in V(K)\}$ is a finite union of arithmetic progressions (see [GT09, Conjecture 1.7] along with the earlier work of Denis [Den94] and Bell [Bel06]). Considering X a semiabelian variety and Φ the translation by a point $x \in X(K)$, one recovers the cyclic case in the classical Mordell-Lang conjecture from the above stated Dynamical Mordell-Lang Conjecture. So, it is natural to seek a generalization of the Dynamical Mordell-Lang Conjecture (see [GTZ11]) to a statement which would contain as a special case the full statement of the classical Mordell-Lang conjecture. Therefore one can study the

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general *dynamical Mordell-Lang problem* by considering commuting K-morphisms $f_1, \dots, f_r$ from X to itself and asking whether the set

$$S(X, f_1, \dots, f_r, x, V) := \{(n_1, \dots, n_r) \in \mathbb{N}_0^r \colon f_1^{n_1} \circ \dots \circ f_r^{n_r}(x) \in V(K)\}$$

is a finite union of translates of subsemigroups of $\mathbb{N}_0^r$. This more general problem turns out to be rather delicate even when the underlying variety X is an algebraic group and each f_i is an endomorphism. Indeed, a recent theorem of Scanlon-Yasufuku [SY13] establishes that for any system of polynomial-exponential equations, its set of solutions is equal to a set of the form $S(X, f_1, \dots, f_r, x, V)$ where X is an algebraic torus, V is an algebraic subgroup, and $f_1, \dots, f_r$ are some commuting endomorphisms of X.

The Dynamical Mordell-Lang Conjecture has sparked significant interest and there have been many partial results; we refer the readers to [BGT16] for a survey of recent work. On the other hand, there are very few results known for the more general dynamical Mordell-Lang problem. In fact, not much is known even when we restrict to the following special case called *the orbit intersection problem*:

Question 1.1. *Let X be a variety over a K, let $r \geq 2$. For $1 \leq i \leq r$, let Φ_i be a K-morphism from X to itself, and let $\alpha_i \in X(K)$ that is not Φ_i-preperiodic. When can we conclude that the set $S := \{(n_1, \dots, n_r) \in \mathbb{N}_0^r \colon \Phi_1^{n_1}(\alpha_1) = \ldots = \Phi_r^{n_r}(\alpha_r)\}$ is a finite union of sets of the form $\{(n_1k + \ell_1, \dots, n_rk + \ell_r) \colon k \in \mathbb{N}_0\}$ for some $n_1, \dots, n_r, \ell_1, \dots, \ell_r \in \mathbb{N}_0$?*

Remark 1.2. For $1 \leq i \leq r$, let f_i be the self-map of X^r induced by the map Φ_i on the i-th factor and the identity map on all the other factors. Let Δ be the diagonal of X^r. The set S in Question 1.1 is exactly the set of $(n_1, \dots, n_r) \in \mathbb{N}_0^r$ such that $(f_1^{n_1} \circ \ldots \circ f_r^{n_r})(\alpha_1, \dots, \alpha_r) \in \Delta$. This explains why Question 1.1 is a special case of the general dynamical Mordell-Lang problem with the further requirement that S is a finite union of translates of subsemigroups of $\mathbb{N}_0^r$ *whose rank is at most* 1. When some α_i is preperiodic, it is trivial to describe the set S. This justifies our assumption on the α_i's.

Remark 1.3. Motivated by the examples in [GTZ11, Section 6], one may ask whether the following condition is sufficient for Question 1.1: there do not exist $m \in \mathbb{N}$ and a positive dimensional closed subvariety Y of X such that Φ_i^m restricts to an automorphism on Y for some $i \in \{1, \dots, r\}$. This condition is indeed sufficient when X is a semi-abelian variety (see Theorem 1.4 and Proposition 4.4); also this condition is often necessary as shown by various examples such as the following one. If $X = \mathbb{A}^2$, $m = 2$, and $\Phi_1 : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ is given by $\Phi_1(x, y) = (x + y^2, y^3)$ while $\Phi_2 : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ is given by $\Phi_2(x, y) = (x^2 + y, y^2)$, then both Φ_1 and Φ_2 restrict to endomorphisms of $V := \mathbb{A}^1 \times \{1\}$, and moreover $(\Phi_1)|_V$ is actually an automorphism. Then the set of all pairs (n_1, n_2) such that $\Phi_1^{n_1}(0, 1) = \Phi_2^{n_2}(0, 1)$ is infinite, but it is not a finite union of cosets of subsemigroups of $\mathbb{N}_0 \times \mathbb{N}_0$.

Question 1.1 in the case $X = \mathbb{P}_K^1$ and each Φ_i is a polynomial of degree larger than 1 has been settled by Tucker, Zieve, and the first author [GTZ08, GTZ12]. They also obtain various results for the general Mordell-Lang problem when X is a semiabelian variety and the self-maps are endormophisms satisfying certain technical conditions [GTZ11]. The case when $X = \mathbb{P}_K^1$ endowed by the action of certain generic rational functions is also established in an ongoing joint work of Zieve and the second author.

The goal of this paper is to answer Question 1.1 when X is a semiabelian variety and when $X = \mathbb{A}^n_K$ and the self-maps are affine transformations.

Theorem 1.4. *Let X be a semiabelian variety over K and $r \geq 2$. For $1 \leq i \leq r$, let $\Phi_i : X \to X$ be a K-morphism and let $\alpha_i \in X(K)$ that is not Φ_i-preperiodic. Let $\Phi_{i,0}$ be a K-endomorphism of X and $\alpha_{i,0} \in X(K)$ such that $\Phi_i(x) = \Phi_{i,0}(x) + \alpha_{i,0}$ and let $D\Phi_{i,0}$ be the linear transformation of the tangent space at the identity of X induced by $\Phi_{i,0}$. If none of the eigenvalues of $D\Phi_{i,0}$ is a root of unity for every i, then the set*

$$S := \{(n_1, \dots, n_r) \in \mathbb{N}_0^r \colon \Phi_1^{n_1}(\alpha_1) = \dots = \Phi_r^{n_r}(\alpha_r)\}$$

is a finite union of sets of the form $\{(n_1 k + \ell_1, \dots, n_r k + \ell_r) \colon k \in \mathbb{N}_0\}$ for some $n_1, \dots, n_r, \ell_1, \dots, \ell_r \in \mathbb{N}_0$.

We note that any self-map of a semiabelian variety is indeed a composition of a translation with an algebraic group endomorphism (see [NW14, Theorem 5.1.37]). The structure for self-maps on semiabelian varieties X is similar to the structure of affine self-maps on $\mathbb{A}^N$, and this allows us to reduce Theorem 1.4 (using the p-adic exponential map on X) to proving Question 1.1 for affine endomorphisms of $\mathbb{A}^N$ (see Theorem 1.6).

Example 1.5. We present an example to illustrate that the conclusion of Theorem 1.4 would fail without the assumption on the eigenvalues of the linear maps $D\Phi_{i,0}$'s. Consider the case $X = \mathbb{G}_m$, $\Phi_1(x) = 2x$, $\Phi_2(x) = x^2$, $\alpha_1 = 1$, and $\alpha_2 = 2$; then $\{(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0 \colon \Phi_1^m(\alpha_1) = \Phi_2^n(\alpha_2)\} = \{(2^n, n) \colon n \in \mathbb{N}_0\}$.

In Section 4, we present the proof of Theorem 1.4 and give some geometric conditions that imply the condition on the linear transformations $D\Phi_{i,0}$ in Theorem 1.4. For instance, let Φ_0 be an endomorphism of a semiabelian variety X defined over K, then none of the eigenvalues of $D\Phi_0$ is a root of unity *if and only if* Φ_0 does not preserve a non-constant fibration (see Proposition 4.3). Here we say that Φ_0 preserves a non-constant fibration if there exists a non-constant *rational* map $f : X \to \mathbb{P}^1_K$ such that $f \circ \Phi_0 = f$.

In [GTZ11, Theorem 1.3 (a)], a special case of Theorem 1.4 was obtained, i.e. when each $\Phi_i = \Phi_{i,0}$ is a group endomorphism and moreover, the Jacobians at $0 \in X$ of each Φ_i is *diagonalizable.* The hypothesis from [GTZ11] about the diagonalizability of the Jacobians of Φ_i greatly simplifies the problem since it allows one to reduce the problem to classical unit equations in diophantine geometry. In the absence of the diagonalizability condition, we have to use a much more refined analysis of the pairs $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$ such that $a_m = b_n$ for two arbitrary linear recurrence sequences. The result from [GTZ11] dealt only with the much easier case when the characteristic polynomials for these two linear recurrence sequences have *non-repeated* roots. As it was noted in [GTZ11, Section 6], if one of the maps Φ_i is an automorphism of X (or induces an automorphism of a positive dimensional subvariety of X), then the set S may no longer be a finite union of cosets of subsemigroups of $\mathbb{N}_0^r$. Essentially, the problem with one of the endomorphisms being actually an automorphism is the following: assuming X, α_i and Φ_i are defined over a number field, then the points in $\mathcal{O}_{\Phi_i}(\alpha_i)$ are not sufficiently sparse with respect to a Weil height on X and this increases the probability that $\mathcal{O}_{\Phi_i}(\alpha_i)$ intersects the other orbits.

The most important ingredient in the proof of Theorem 1.4 is the following result which also answers Question 1.1 when $X = \mathbb{A}^n_K$ and the maps Φ_i's are affine transformations.

Theorem 1.6. *Let $r, N \in \mathbb{N}$ with $r \geq 2$. For $i \in \{1, \dots, r\}$, let $f_i : K^N \longrightarrow K^N$ be an affine map which means there exist an $N \times N$-matrix $A_i \in M_N(K)$ and a vector $\mathbf{x}_i \in K^N$ such that $f(\mathbf{x}) = A_i\mathbf{x} + \mathbf{x}_i$ for every $\mathbf{x} \in K^N$. For $i \in \{1, \dots, r\}$, let $\mathbf{p}_i \in K^N$ that is not f_i-preperiodic. If none of the eigenvalues of A_i is a root of unity for each $i \in \{1, \dots, r\}$, then the set*

$$S := \{(n_1, \dots, n_r) \in \mathbb{N}_0^r \colon f_1^{n_1}(\mathbf{p}_1) = \ldots = f_r^{n_r}(\mathbf{p}_r)\}$$

is a finite union of sets of the form $\{(n_1k + \ell_1, \dots, n_rk + \ell_r) : k \in \mathbb{N}_0\}$ for some $n_1, \dots, n_r, \ell_1, \dots, \ell_r \in \mathbb{N}_0$.

The conclusion of this theorem would fail without the assumption on the eigenvalues of the matrices A_i. For example, let $N = 1$, $r = 2$, $A_1(x) = x+1$, $A_2(x) = 2x$, $\mathbf{p}_1 = 0$, and $\mathbf{p}_2 = 1$, then $S = \{(2^n, n) : n \in \mathbb{N}_0\}$. Also, Theorem 1.6 fails if one does not assume the maps f_i are affine, as shown by the following example. Let $r = 2$, $n = 1$, $f_1(x) = 2x$, $f_2(x) = x^2$, $\mathbf{p}_1 = 1$ and $\mathbf{p}_2 = 2$; then $S = \{(2^n, n) \colon n \in \mathbb{N}_0\}$.

The organization of this paper is as follows. In Section 3, we present the proof of Theorem 1.6 which requires a careful analysis of a certain system of polynomial-exponential equations in two variables. Some results on polynomial-exponential equations are given in the next section following Schmidt's exposition [Sch03]. In Section 4, Theorem 1.4 is reduced to Theorem 1.6 thanks to the use of the p-adic exponential map for an appropriate choice of the prime p.

We conclude this section with a brief discussion of the dynamical Mordell-Lang problem over fields of positive characteristic. We note right from the start that Question 1.1 fails even in the simplest examples of affine maps defined over $\mathbb{F}_p(t)$. Indeed, let $\Phi_i : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be affine maps given by $\Phi_1(x) = tx - t + 1$ and $\Phi_2(x) = (t+1)x$. It is immediate to see that $\Phi_1^m(2) = t^m + 1$ while $\Phi_2^n(1) = (t+1)^n$. Then the set

$$S = \{(m, n) \in \mathbb{N}_0^2 \colon \Phi_1^m(2) = \Phi_2^n(1)\} = \{(p^n, p^n) \colon n \in \mathbb{N}_0\}.$$

The above example stems from similar examples disproving a naive formulation of the Dynamical Mordell-Lang Conjecture in positive characteristic. A variant of the Dynamical Mordell-Lang Conjecture has been proposed by Scanlon and the first author [BGT16, Chapter 13]; however there are very few partial results since even the case of $\mathbb{G}_{\mathrm{m}}^n$ seems to be closely related to very difficult problems in diophantine geometry. We refer the readers to the discussion in [BGT16, Section 13.3] for more details. A deep theorem of Adamczewski and Bell [AB12, Theorem 1.4] implies that if K is a field of characteristic $p > 0$, then the set S in Theorem 1.6 is p-automatic.

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2. Some diophantine equations involving linear recurrence sequences

2.1. **Some classical results.** A large part of this subsection follows the notation from Schmidt's article [Sch03]. All the sequences considered in this section are sequences of complex numbers. A tuple $(a_1, \dots, a_k)$ of non-zero numbers is called non-degenerate if $\dfrac{a_i}{a_j}$ is not a root of unity for $1 \leq i < j \leq k$. A linear recurrence sequence is called non-degenerate if the tuple of (non-zero) characteristic roots is non-degenerate. We begin with the following well-known result:

Theorem 2.1 (Skolem-Mahler-Lech). *Let $\{u_n :\ n \in \mathbb{N}_0\}$ be a linear recurrence sequence. Then the set $Z := \{n :\ u_n = 0\}$ is a finite union of arithmetic progressions. Furthermore, if u_n is non-degenerate then Z is finite.*

We now consider non-degenerate linear recurrence sequences that are *not* of the form $P(n)\alpha^n$ where α is a root of unity. It is convenient to write such a sequence u as:

$$u_n = \sum_{i=0}^{q} P_i(n)\alpha_i^n \tag{1}$$

with the following convention [Sch03, Section 11]. If some root of the characteristic polynomial is a root of unity, let this root be α_0, and $\alpha_1, \ldots, \alpha_q$ the other roots. If no root of the characteristic polynomial is a root of unity, let these roots be $\alpha_1, \ldots, \alpha_q$, and set $\alpha_0 = 1$, $P_0 = 0$. Let v be another sequence written as

$$v_n = \sum_{i=0}^{q'} Q_i(n)\beta_i^n \tag{2}$$

with the same convention. The two sequences u and v are said to be *related* if $q = q'$ and after a suitable reordering of $\beta_1, \ldots, \beta_q$ we have:

$$\alpha_i^a = \beta_i^b \text{ for every } i \in \{1, \ldots, q\}$$

for certain non-zero integers a and b.

The next result follows from Schmidt's reformulation of a theorem by Laurent whose proof uses the celebrated Subspace Theorem:

Theorem 2.2 (Laurent). *Let u and v be non-degenerate linear recurrence sequences given by (1), (2), and under the convention described above. Consider the equation:*

$$u_m = v_n \textit{ for } (m, n) \in \mathbb{N}_0^2$$

and let Z be the set of solutions. We have the following:

(a) *If u and v are not related then Z is finite.*
(b) *$P_0(m)\alpha_0^m = Q_0(n)\beta_0^n$ for all but finitely many $(m, n) \in Z$.*

Proof. This follows from [Sch03, Theorem 11.2]. □

2.2. **Some consequences.**

Proposition 2.3. *Let $k \in \mathbb{N}$, let $a, b_1, \ldots, b_k \in \mathbb{C}^*$ none of which is a root of unity. Let $P(x), Q_1(x), \ldots, Q_k(x) \in \mathbb{C}[x] \setminus \{0\}$ and let $c \in \mathbb{C}$. Assume that $(b_1, \ldots, b_k)$ is non-degenerate. If $k \geq 2$ or $c \neq 0$, then the set*

$$Z := \left\{(m, n) \in \mathbb{N}_0^2 :\ P(m)a^m = c + \sum_{i=1}^{k} Q_i(n)b_i^n\right\}$$

is finite.

Proof. When $k \geq 2$, the two linear recurrence sequences $u_m = P(m)a^m = 0 \cdot 1^m + P(m)a^m$ and $v_n = c \cdot 1^n + \sum_{i=1}^{k} Q_i(n)b_i^n$ are not related. Hence Z is finite by part (a) of Theorem 2.2. If Z is infinite, we have $c = 0$ by part (b) of Theorem 2.2. □

If p is a prime, let $\mathbb{C}_p$ denote the completion of the algebraic closure of $\mathbb{Q}_p$. It is well-known that $\mathbb{C}_p$ is algebraically closed. We have:

Lemma 2.4. *Let $\gamma \in \mathbb{C}^*$ that is not a root of unity and let F be a finitely generated subfield of $\mathbb{C}$ containing γ. Then there exists a field $\mathcal{F}$ that is either $\mathbb{C}$ or $\mathbb{C}_p$ together with its usual absolute value $|\cdot|_0$ and an embedding $\sigma : F \to \mathcal{F}$ such that $|\sigma(\gamma)|_0 > 1$.*

Proof. It suffices to prove that there exist $\mathcal{F}$ that is $\mathbb{C}$ or $\mathbb{C}_p$ and an embedding $\sigma : \mathbb{Q}(\gamma) \to \mathcal{F}$ satisfying $|\sigma(\gamma)|_0 > 1$. Then it is possible to extend σ to F since $\mathcal{F}$ is algebraically closed and has infinite (in fact, uncountable) transcendence degree over $\mathbb{Q}$.

When γ is algebraic, since $\gamma \in \mathbb{C}^*$ is not a root of unity, a result of Kronecker (see, for instance, [BG06, Theorem 1.5.9]) gives that there is an absolute value $|\cdot|_v$ of the number field $\mathbb{Q}(\gamma)$ such that $|\gamma|_v > 1$. This absolute value $|\cdot|_v$ gives rise to the desired embedding into $\mathbb{C}$ if v is archimedean, or into $\mathbb{C}_p$ if $|\cdot|_v$ restricts to the p-adic absolute value of $\mathbb{Q}$. When γ is transcendental, we simply map γ to any transcendental number outside the unit disk. □

Proposition 2.5. *Let $\alpha, \beta_1, \beta_2 \in \mathbb{C}^*$ none of which is a root of unity. Let $P_1(x)$, $P_2(x)$, $Q_1(x)$, and $Q_2(x)$ be non-zero polynomials with complex coefficients. Let Z be the set of $(m,n) \in \mathbb{N}_0^2$ satisfying both $P_1(m)\alpha^m = Q_1(n)\beta_1^n$ and $P_2(m)\alpha^m = Q_2(n)\beta_2^n$. If Z is infinite then $\dfrac{\beta_1}{\beta_2}$ is a root of unity and $\deg(P_2) - \deg(P_1) = \deg(Q_2) - \deg(Q_1)$.*

Proof. For every fixed m (respectively n), there are only finitely many n (respectively m) such that $(m,n) \in Z$. Hence in every infinite subset of Z, m and n must be unbounded.

Fix any $\epsilon \in \mathbb{C}^*$ such that $P_1(x)+\epsilon P_2(x)$ is not the zero polynomial. If $(m,n) \in Z$, we have $(P_1(m)+\epsilon P_2(m))\alpha^m = Q_1(n)\beta_1^n + \epsilon Q_2(n)\beta_2^n$. By Proposition 2.3, we have that $\dfrac{\beta_1}{\beta_2}$ is a root of unity.

For $(m,n) \in Z$, we have $|P_1(m)/P_2(m)| = |Q_1(n)/Q_2(n)|$. By taking $(m,n) \in Z$ when both of m and n are large, we have that $\deg(P_2) = \deg(P_1)$, $\deg(P_2) > \deg(P_1)$, $\deg(P_2) < \deg(P_1)$ respectively if and only if $\deg(Q_2) = \deg(Q_1)$, $\deg(Q_2) > \deg(Q_1)$, $\deg(Q_2) < \deg(Q_1)$. For the rest of the proof, assume that $\deg(P_2) \neq \deg(P_1)$ and $\deg(Q_2) \neq \deg(Q_1)$. We have $\delta := \dfrac{\deg(Q_1) - \deg(Q_2)}{\deg(P_1) - \deg(P_2)} > 0$ and we need to prove that $\delta = 1$.

Assume that $\delta > 1$. From $|P_1(m)/P_2(m)| = |Q_1(n)/Q_2(n)|$ for $(m,n) \in Z$, we have $C_1 n^\delta < m < C_2 n^\delta$ for some positive constants C_1 and C_2; this is expressed succinctly as $m = \Theta(n^\delta)$. Let F be the field generated by α, β_1, β_2, and the coefficients of P_1, P_2, Q_1, Q_2. By Lemma 2.4, we can embed F into a field $\mathcal{F}$ which is $\mathbb{C}$ or $\mathbb{C}_p$ together with its usual absolute value $|\cdot|_0$ such that $|\alpha|_0 > 1$. Since $m = \Theta(n^\delta)$ and $\delta > 1$, we have $|Q_1(n)\beta_1^n|_0 = o(|P_1(m)\alpha^m|_0)$, contradiction.

The case $\delta < 1$ can be dealt with by similar arguments. We have $n = \Theta(m^{1/\delta})$. We now embed F into $\mathcal{F}$ such that $|\beta_1|_0 > 1$ to obtain $|P_1(m)\alpha^m|_0 = o(|Q_1(n)\beta_1^n|_0)$, contradiction. This finishes the proof. □

3. Proof of Theorem 1.6

Since we may restrict to a finitely generated subfield of K over which all the objects in the statement of Theorem 1.6 are defined, and we may embed this subfield into $\mathbb{C}$, for the rest of this section, we assume K is a subfield of $\mathbb{C}$.

3.1. **Some reductions.** First, we explain why it suffices to prove Theorem 1.6 when $r = 2$. Suppose that Theorem 1.6 is proved for $r = 2$, and assume $r \geq 3$. The set S' of pairs (m, n) satisfying $f_{r-1}^m(\mathbf{p}_{r-1}) = f_r^n(\mathbf{p}_r)$ is a finite union of sets of the form $\{(t_{r-1}k + \ell_{r-1}, t_r k + \ell_r) : \ k \in \mathbb{N}_0\}$ for some $t_{r-1}, t_r, \ell_{r-1}, \ell_r \in \mathbb{N}_0$. Fix one such set and the corresponding $t_{r-1}, t_r, \ell_{r-1}, \ell_r$. By ignoring finitely many pairs (m, n) in S', we may assume that t_{r-1} and t_r are positive. We are now looking for tuples $(n_1, \ldots, n_{r-2}, k) \in \mathbb{N}_0^{r-1}$ such that:

$$f_1^{n_1}(\mathbf{p}_1) = \ldots = f_{r-2}^{n_{r-2}}(\mathbf{p}_{r-2}) = (f_{r-1}^{t_{r-1}})^k(f_{r-1}^{\ell_{r-1}}(\mathbf{p}_{r-1})).$$

The map $f_{r-1}^{t_{r-1}}$ is associated to the matrix $A_{r-1}^{t_{r-1}}$ whose eigenvalues are not root of unity. So we have reduced to $r-1$ maps $f_1, \ldots, f_{r-2}, f_{r-1}^{t_{r-1}}$ at the starting points $\mathbf{p}_1, \ldots, \mathbf{p}_{r-2}, f_{r-1}^{t_{r-1}}(\mathbf{p}_{r-1})$ that satisfy the hypothesis of Theorem 1.6.

We now focus on the case $r = 2$. Let $\tilde{\mathbf{x}}_1$ be a fixed point of f_1, equivalently $A_1\tilde{\mathbf{x}}_1 + \mathbf{x}_1 = \tilde{\mathbf{x}}_1$. This is possible since $A_1 - I_N$ is invertible. Define $\psi(\mathbf{x}) = \mathbf{x} + \tilde{\mathbf{x}}_1$ so that $\psi^{-1} \circ f_1 \circ \psi(\mathbf{x}) = A_1\mathbf{x}$. Hence $f_1^n(\mathbf{x} + \tilde{\mathbf{x}}_1) = A_1^n\mathbf{x} + \tilde{\mathbf{x}}_1$. Similarly, let $\tilde{\mathbf{x}}_2$ be a fixed point of f_2, then we have $f_2^n(\mathbf{x} + \tilde{\mathbf{x}}_2) = A_2^n\mathbf{x} + \tilde{\mathbf{x}}_2$. Therefore we reduce to the problem of studying the set of pairs $(n_1, n_2) \in \mathbb{N}_0^2$ satisfying $A_1^{n_1}\mathbf{u} = A_2^{n_2}\mathbf{v} + \mathbf{w}$ where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are given vectors such that $\mathbf{u}$ (respectively $\mathbf{v}$) is not preperiodic under the map $\mathbf{x} \mapsto A_1\mathbf{x}$ (respectively $\mathbf{x} \mapsto A_2\mathbf{x}$).

Let P and Q be in $\mathrm{GL}_N(K)$ such that $A_1 = P^{-1}J_1P$ and $A_2 = Q^{-1}J_2Q$ where J_1 and J_2 are respectively the Jordan form of A_1 and A_2. The equation $A_1^{n_1}\mathbf{u} = A_2^{n_2}\mathbf{v} + \mathbf{w}$ is equivalent to $J_1^{n_1}P\mathbf{u} = PQ^{-1}J_2^{n_2}Q\mathbf{v} + P\mathbf{w}$. Replacing $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ by $(P\mathbf{u}, Q\mathbf{v}, P\mathbf{w})$, we reduce to proving the following (after a slight change of notation):

Theorem 3.1. *Let $A, B \in M_N(K)$ be in Jordan form and let $C \in \mathrm{GL}_N(K)$. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in K^N$ such that $\mathbf{u}$ and $\mathbf{v}$ are respectively not preperiodic under the maps $\mathbf{x} \mapsto A\mathbf{x}$ and $\mathbf{x} \mapsto B\mathbf{x}$. If neither A nor B have an eigenvalue which is a root of unity, then the set $S := \{(m, n) \in \mathbb{N}_0^2 : \ A^m\mathbf{u} = CB^n\mathbf{v} + \mathbf{w}\}$ is a finite union of sets of the form $\{(m_0k + \ell_1, n_0k + \ell_2) : \ k \in \mathbb{N}_0\}$ for some $m_0, n_0, \ell_1, \ell_2 \in \mathbb{N}_0$.*

3.2. **Proof of Theorem 3.1.** From now on, we assume the notation of Theorem 3.1. We start with the following easy result:

Lemma 3.2. *Let $P \in M_N(K)$, $\mathbf{p} \in K^N$, and V a closed subvariety of $\mathbb{A}_K^N$. The set $\{n \in \mathbb{N}_0 : \ P^n\mathbf{v} \in V(K)\}$ is a finite union of arithmetic progressions.*

Proof. Let $f_1, \ldots, f_k$ be polynomials defining V. Then each $\{f_i(P^n\mathbf{v}) : \ n \in \mathbb{N}_0\}$ is a linear recurrence sequence and we can apply Theorem 2.1. One can also get this result as an immediate consequence of [Bel06, Theorem 1.3]. ☐

We may assume that the tuple of non-zero eigenvalues of A and the tuple of non-zero eigenvalues of B are non-degenerate. This is possible since we can replace the data $(A, B, C, \mathbf{u}, \mathbf{v}, \mathbf{w})$ by $(A^M, B^M, C, A^{r_1}\mathbf{u}, B^{r_2}\mathbf{v}, \mathbf{w})$ for some $M \in \mathbb{N}$ and for all $0 \leq r_1, r_2 \leq M - 1$ and establish the conclusion of Theorem 3.1 for the set of pairs $(m, n) \in S$ satisfying $m \equiv r_1 \bmod M$ and $n \equiv r_2 \bmod M$.

For $\lambda \in \mathbb{C}$ and $s \in \mathbb{N}$, let $J_{\lambda,s}$ be the Jordan matrix of size s and eigenvalue λ. We have the formula:

$$J_{\lambda,s}^n = \begin{bmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \dots & \binom{n}{s-1}\lambda^{n-s+1} \\ 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} & \dots & \binom{n}{s-2}\lambda^{n-s+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda^n \end{bmatrix}$$

For convenience, we follow the convention that assigns any negative number to be a degree of the zero polynomial.

The key observation is that if $\lambda \neq 0$, then there are polynomials $P_{i,j}$ of degree $j-i$ for $1 \leq i, j \leq s$ such that the (i,j)-entry of $J_{\lambda,s}^n$ is $P_{i,j}(n)\lambda^n$ for every $n \in \mathbb{N}$. If $\lambda = 0$ *and* $n \geq s$, then $J_{\lambda,s}^n = 0_{s,s}$ (the zero matrix) so that we can still express the (i,j)-th entry of $J_{\lambda,s}^n$ as $\lambda^n \cdot P_{i,j}(n)$ where $P_{i,j}$ is any chosen polynomial of degree $j-i$.

So, if $\mathbf{a} = (a_1, \dots, a_s)^T$ is a fixed column vector in K^s and $n \geq s$, then there are polynomials $P_1, \dots, P_s$ such that:

$$J_{\lambda,s}^n \mathbf{a} = (P_1(n)\lambda^n, \dots, P_s(n)\lambda^n)^T$$

for every $n \in \mathbb{N}$. Moreover, if $\mathbf{a} \neq \mathbf{0}$ and $d := \max\{j : \ a_j \neq 0\}$ then by a direct calculation, we have $\deg(P_1) = d-1 \leq s-1$, $\deg(P_2) = d-2, \dots, \deg(P_s) = d-s \leq 0$.

We assume that the set S is infinite; otherwise there is nothing to prove. Since $\mathbf{u}$ and $\mathbf{v}$ are not preperiodic, for every fixed m (respectively n), there is at most one value of n (respectively m) such that $(m,n) \in S$. Hence m and n must be unbounded in every infinite subset of S. Hence it suffices to prove that the set

$$S_{\geq N} := \{(m,n) \in S \colon m, n \geq N\}$$

is a finite union of cosets of subsemigroups of $\mathbb{N}_0 \times \mathbb{N}_0$ of rank at most equal to 1.

Let p be the number of Jordan blocks in A, let J_{α_i,m_i} for $1 \leq i \leq p$, $\alpha_i \in \mathbb{C}$, $m_i \in \mathbb{N}$, and $\sum_i m_i = N$ be the Jordan blocks of A. Let q be the number of Jordan blocks in A, let J_{β_j,n_j} for $1 \leq j \leq q$, $\beta_j \in \mathbb{C}$, $n_j \in \mathbb{N}$, and $\sum_j n_j = N$ be the Jordan blocks of B. Note that the α_i's and β_j's are not root of unity. By a previous observation, there exist polynomials $P_{i,k}$ for $1 \leq i \leq p$ and $1 \leq k \leq m_i$ such that for $m \geq N$, we have

$$\begin{aligned} A^m\mathbf{u} =& (P_{1,1}(m)\alpha_1^m, \dots, P_{1,m_1}(m)\alpha_1^m, P_{2,1}(m)\alpha_2^m, \dots, P_{2,m_2}(m)\alpha_2^m, \\ & \dots, P_{p,1}(m)\alpha_p^m, \dots, P_{p,m_p}(m)\alpha_p^m)^T. \end{aligned} \tag{3}$$

Moreover, for $1 \leq i \leq p$, let $d_i = \deg(P_{i,1})$, then we have $d_i \leq m_i - 1$ and $\deg(P_{i,k}) = d_i - k + 1$ for $1 \leq k \leq m_i$. Similarly, there exist polynomials $Q_{j,\ell}$ for $1 \leq j \leq q$ and $1 \leq \ell \leq n_j$ with $e_j := \deg(Q_{j,1}) \leq n_j - 1$, $\deg(Q_{j,\ell}) = e_j - \ell + 1$ such that for $n \geq N$, we have

$$\begin{aligned} B^n\mathbf{v} =& (Q_{1,1}(n)\beta_1^n, \dots, Q_{1,n_1}(n)\beta_1^n, Q_{2,1}(n)\beta_2^n, \dots, Q_{2,n_2}(n)\beta_2^n, \\ & \dots, Q_{q,1}(n)\beta_q^n, \dots, Q_{q,n_q}(n)\beta_q^n)^T. \end{aligned} \tag{4}$$

Since $\mathbf{u}$ is not preperiodic under the map $\mathbf{x} \mapsto A\mathbf{x}$, there is at most one value of m such that $A^m\mathbf{u}$ is zero. Hence the set $\mathcal{I} := \{i : \ \alpha_i \neq 0 \text{ and } d_i \geq 0\}$ is non-empty. Similarly, the set $\mathcal{J} := \{j : \ \beta_j \neq 0 \text{ and } e_j \geq 0\}$ is non-empty. We have the following result.

Proposition 3.3. *The following hold:*

(a) *Let $i \in \{1,\dots,p\}$ and $k \in \{1,\dots,m_i\}$ be such that $\alpha_i \neq 0$ and $\deg(P_{i,k}) \geq 0$. Then there exist $j^* \in \{1,\dots,q\}$, $\ell^* \in \{1,\dots,n_{j^*}\}$, and a polynomial $Q(x)$ such that $\beta_{j^*} \neq 0$, $\deg(Q) = \deg(Q_{j^*,\ell^*}) \geq 0$, and $P_{i,k}(m)\alpha_i^m = Q(n)\beta_{j^*}^n$ for every $(m,n) \in S_{\geq N}$.*

(b) *Let $j \in \{1,\dots,q\}$ and $\ell \in \{1,\dots,n_j\}$ be such that $\beta_j \neq 0$ and $\deg(Q_{j,\ell}) \geq 0$. Then there exist $i^* \in \{1,\dots,p\}$, $k^* \in \{1,\dots,m_{i^*}\}$, and a polynomial $P(x)$ such that $\alpha_{i^*} \neq 0$, $\deg(P) = \deg(P_{i^*,k^*}) \geq 0$, and $P(m)\alpha_{i^*}^m = Q_{j,\ell}(n)\beta_j^n$ for every $(m,n) \in S_{\geq N}$.*

Proof. For part (a), fix $i \in \{1,\dots,p\}$ and $k \in \{1,\dots,m_i\}$ such that $\alpha_i \neq 0$ and $\deg(P_{i,k}) \geq 0$. Write $\mu = m_1 + \ldots + m_{i-1} + 1$ so that $P_{i,1}(m)\alpha_i^m$ is the μ-th entry of $A^m\mathbf{u}$. Write $\mathbf{w} = (w_1,\dots,w_N)^T$ and express the μ-th row of the matrix C as:

$$(c_{1,1},\dots,c_{1,n_1},\dots,c_{q,1},\dots,c_{q,n_q})^T.$$

For $(m,n) \in S_{\geq N}$, from $A^m\mathbf{u} = CB^n\mathbf{v} + \mathbf{w}$, (3), and (4), we have:

$$P_{i,1}(m)\alpha_i^m - w_\mu = \sum_{j=1}^{q}\left(\sum_{\ell=1}^{n_j} c_{j,\ell}Q_{j,\ell}(n)\right)\beta_j^n = \sum_{j=1}^{q} Q_j(n)\beta_j^n \tag{5}$$

where $Q_j(x) := \displaystyle\sum_{\ell=1}^{n_j} c_{j,\ell}Q_{j,\ell}(x)$.

Recall our assumption that the non-zero elements in $\{\beta_1,\dots,\beta_q\}$ form a non-degenerate tuple. By Proposition 2.3, $w_\mu = 0$ and there is a unique $j^* \in \{1,\dots,q\}$ such that $\beta_{j^*} \neq 0$, $Q(x) := Q_{j^*}(x) \neq 0$, and for $j \in \{1,\dots,q\} \setminus \{j^*\}$, we have $Q_j(n)\beta_j^n \equiv 0$ (this means either $\beta_j = 0$ or $Q_j(x)$ is the zero polynomial) so that equation (5) becomes:

$$P_{i,1}(m)\alpha_i^m = Q(n)\beta_{j^*}^n. \tag{6}$$

Since $Q = \displaystyle\sum_{\ell=1}^{n_{j^*}} c_{j^*,\ell}Q_{j^*,\ell}$, if we let ℓ^* be minimal such that $c_{j^*,\ell^*} \neq 0$ then $\deg(Q_{j^*,\ell^*}) = \deg(Q) \geq 0$. This finishes the proof of part (a).

The proof of part (b) is completely similar, this time we consider the equation $B^n\mathbf{v} = C^{-1}A^m\mathbf{u} - C^{-1}\mathbf{w}$ and compare the rows corresponding to the entry $Q_{j,\ell}(n)\beta_j^n$ in $B^n\mathbf{v}$. $\square$

Let $\tilde{i} \in \mathcal{I}$ be such that $d_{\tilde{i}} = \max\{d_i : \ i \in \mathcal{I}\}$. By part (a) of Proposition 3.3, there exist j^*, ℓ^*, and a polynomial $Q(x)$ such that $e^* := \deg(Q) = \deg(Q_{j^*,\ell^*}) \geq 0$ and

$$P_{\tilde{i},1}(m)\alpha_{\tilde{i}}^m = Q(n)\beta_{j^*}^n \tag{7}$$

for every $(m,n) \in S_{\geq N}$. We have:

Proposition 3.4. *The following hold:*

(a) *$d_{\tilde{i}} = e^*$, in other words $\deg(P_{\tilde{i},1}) = \deg(Q)$.*

(b) *There exist $\omega \in K^*$ such that $\alpha_{\tilde{i}}^m = \omega\beta_{j^*}^n$ for every $(m,n) \in S_{\geq N}$.*

Proof. First, we prove $d_{\tilde{i}} \le e^*$ as follows. For the entry $P_{\tilde{i},d_{\tilde{i}}+1}(m)\alpha_{\tilde{i}}^m$ in $A^m\mathbf{u}$, we have that $\deg(P_{\tilde{i},d_{\tilde{i}}+1}) = 0$. Applying part (a) of Proposition 3.3 to the pair $(\tilde{i}, d_{\tilde{i}}+1)$, we obtain j_*, ℓ_*, and a polynomial $R(x)$ such that $\beta_{j_*} \neq 0$, $\deg(R(x)) = \deg(Q_{j_*,\ell_*}) \ge 0$, and

$$P_{\tilde{i},d_{\tilde{i}}+1}(m)\alpha_{\tilde{i}}^m = R(n)\beta_{j_*}^n \tag{8}$$

for every $(m,n) \in S_{\ge N}$. Applying Proposition 2.5 to the pair of equations (7) and (8), we have that $d_{\tilde{i}} = e^* - \deg(R) \le e^*$. We also have $\dfrac{\beta_{j^*}}{\beta_{j_*}}$ is a root of unity, hence $j^* = j_*$ since the tuple of non-zero eigenvalues of B is non-degenerate.

The inequality $e^* \le d_{\tilde{i}}$ can be proved by similar arguments, as follows. We consider the entry $Q_{j^*,\ell^*+e^*}(n)\beta_{j^*}^n$ in $B^n\mathbf{v}$ for which $\deg(Q_{j^*,\ell^*+e^*}) = 0$. Applying part (b) of Proposition 3.3 to the pair (j^*, ℓ^*+e^*), we obtain i^*, k^*, and a polynomial $U(x)$ such that $\alpha_{i^*} \neq 0$, $\deg(U(x)) = \deg(P_{i^*,k^*}) \ge 0$, and

$$U(m)\alpha_{i^*}^m = Q_{j^*,\ell^*+e^*}(n)\beta_{j^*}^n \tag{9}$$

for every $(m,n) \in S_{\ge N}$. Applying Proposition 2.5 to the pair of equations (7) and (9), we have that $d_{\tilde{i}} - \deg(U) = e^*$. This finishes the proof of part (a).

For part (b), from $d_{\tilde{i}} = e^* - \deg(R)$ and part (a), we have that $\deg(R) = 0$. Hence both polynomials $P_{\tilde{i},d_{\tilde{i}}+1}$ and R are non-zero constants. Since $j_* = j^*$, equation (8) finishes the proof. ∎

It is possible to use the arguments in Proposition 3.4 to prove that $d_{\tilde{i}} = \max\{e_j : j \in \mathcal{J}\}$; however we will not use this fact. We can now easily complete the proof of Theorem 3.1. Part (b) of Proposition 3.4 shows that the set $S_{\ge N}$ *is contained in* one single set of the form $\{(m_0k+\ell_1, n_0k+\ell_2) : \ k \in \mathbb{N}_0\}$ with $\ell_1, \ell_2 \in \mathbb{N}_0$ and $m_0, n_0 \in \mathbb{N}$. In fact we can choose (m_0, n_0) to be the minimal pair of positive integers such that $\alpha_{\tilde{i}}^{m_0} = \beta_{j^*}^{n_0}$ and choose (ℓ_1, ℓ_2) to be the minimal pair of non-negative integers such that $\alpha_{\tilde{i}}^{\ell_1} = \omega\beta_{j^*}^{\ell_2}$. Both of these pairs exist by part (b) of Proposition 3.4 and our assumption that $S_{\ge N}$ is infinite. It remains to study the set of $k \in \mathbb{N}_0$ satisfying:

$$(A^{m_0})^k A^{\ell_1}\mathbf{u} = C(B^{n_0})^k B^{\ell_2}\mathbf{v} + \mathbf{w},\ m_0k+\ell_1 \ge N,\ n_0k+\ell_2 \ge N. \tag{10}$$

We now consider K^{2N} with the coordinates $(\mathbf{x},\mathbf{y})$ (where $\mathbf{x},\mathbf{y} \in K^N$), the linear map $L: \ K^{2N} \to K^{2N}$ given by $L(\mathbf{x},\mathbf{y}) = (A^{m_0}\mathbf{x}, B^{n_0}\mathbf{y})$, the starting point $(A^{\ell_1}\mathbf{u}, B^{\ell_2}\mathbf{v})$, and the subvariety defined by $\mathbf{x} = C\mathbf{y} + \mathbf{w}$. Applying Corollary 3.2 to the current data, we have that the set of $k \in \mathbb{N}$ satisfying (10) is a finite union of arithmetic progressions. This finishes the proof of Theorem 3.1.

4. Proof of Theorem 1.4 and further remarks

4.1. **Some reduction.** By using similar arguments as in Subsection 3.1, we reduce to the case $r = 2$. In other words, after a slight change of notation, we reduce to proving the following:

Theorem 4.1. *Let X be a semiabelian variety over K. Let Φ and Ψ be K-morphisms from X to itself. Let Φ_0 and Ψ_0 be K-endomorphisms of X and $\alpha_0, \beta_0 \in X(K)$ such that $\Phi(x) = \Phi_0(x) + \alpha_0$ and $\Psi(x) = \Psi_0(x) + \beta_0$. Let*

$\alpha, \beta \in X(K)$ such that α is not Φ-preperiodic and β is not Ψ-preperiodic. If none of the eigenvalues of $D\Phi_0$ and $D\Psi_0$ is a root of unity, then the set

$$S := \{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 \colon \Phi^m(\alpha) = \Psi^n(\beta)\}$$

is a finite union of sets of the form $\{(m_0k + \ell_1, n_0k + \ell_2) \colon k \in \mathbb{N}_0\}$ for some $m_0, n_0, \ell_1, \ell_2 \in \mathbb{N}_0$.

4.2. **Proof of Theorem 4.1.** Assume the notation of Theorem 4.1 throughout this subsection. We start with a further reduction.

Lemma 4.2. *Let k be a positive integer. It suffices to prove Theorem 4.1 for the maps $\tilde{\Phi}$ and $\tilde{\Psi}$ and starting points $\tilde{\alpha} = k\alpha$ and $\tilde{\beta} = k\beta$, where $\tilde{\Phi}(x) = \Phi_0(x) + k\alpha_0$ and $\tilde{\Psi}(x) = \Psi_0(x) + k\beta_0$.*

Proof. Assume Theorem 4.1 holds for endomorphisms $\tilde{\Phi}$ and $\tilde{\Psi}$ and starting points $\tilde{\alpha}$ and $\tilde{\beta}$. Then $\mathcal{O}_{\tilde{\Phi}}(\tilde{\alpha}) = k \cdot \mathcal{O}_{\Phi}(\alpha)$ and $\mathcal{O}_{\tilde{\Psi}}(\tilde{\beta}) = k \cdot \mathcal{O}_{\Psi}(\beta)$, where for any set T of points of X, we define

$$k \cdot T := \{k \cdot x \colon x \in T\}.$$

So, we know that the set $S := \{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 \colon \tilde{\Phi}^m(\tilde{\alpha}) = \tilde{\Psi}^n(\tilde{\beta})\}$ is a finite union of sets of the form $\{(m_0\ell + r_1, n_0\ell + r_2) \colon \ell \in \mathbb{N}_0\}$ for given $m_0, n_0, r_1, r_2 \in \mathbb{N}_0$. So, it suffices to prove that for each such $m_0, n_0, r_1, r_2 \in \mathbb{N}_0$, the set of $\ell \in \mathbb{N}_0$ such that $\Phi^{m_0\ell + r_1}(\alpha) = \Psi^{n_0\ell + r_2}(\beta)$ is a finite union of sets of the form $\{\ell_0 s + r_0\}_{s \in \mathbb{N}_0}$. Indeed, this last statement is a consequence of [BGT10, Theorem 4.1]. ☐

Let R be a finitely generated $\mathbb{Z}$-subalgebra of K over which X (along with its points $\alpha, \alpha_0, \beta, \beta_0$), and also Φ and Ψ are defined. By [BGT10, Proposition 4.4] (see also [GT09, Proposition 3.3] and [BGT16, Chapter 4]), there exist a prime number p and an embedding of R into $\mathbb{Z}_p$ such that

(i) X has a smooth semiabelian model $\mathcal{X}$ over $\mathbb{Z}_p$;
(ii) Φ and Ψ extend to endomorphism of $\mathcal{X}$;
(iii) $\alpha, \alpha_0, \beta, \beta_0$ extend to points in $\mathcal{X}(\mathbb{Z}_p)$.

Let f_0 and g_0 denote the linear maps induced on the tangent space at 0 by Φ_0, respectively Ψ_0. By (ii) above and [BGT10, Proposition 2.2], if one chooses coordinates for this tangent space via generators for the completed local ring at 0, then the entries of the N-by-N matrices A and B corresponding to f_0 and g_0 will be in $\mathbb{Z}_p$ (where $N = \dim(X)$). Fix one such set of coordinates and let $|\cdot|_p$ denote the corresponding p-adic metric on the tangent space at 0. We let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$.

According to [Bou98, Proposition 3, p. 216] there exists a p-adic analytic map exp which induces an analytic isomorphism between a sufficiently small neighborhood $\mathcal{U}$ of $\mathbb{C}_p^N$ and a corresponding p-adic neighborhood of the origin $0 \in \mathcal{X}(\mathbb{C}_p)$. Furthermore (at the expense of possibly replacing $\mathcal{U}$ by a smaller set), we may assume that the neighborhood $\mathcal{U}$ is a sufficiently small open ball, i.e., there exists a (sufficiently small) positive real number ϵ such that $\mathcal{U}$ consists of all $(z_1, \ldots, z_N) \in \mathbb{C}_p^N$ satisfying $|z_i|_p < \epsilon$. Because $\exp(\mathcal{U}) \cap \mathcal{X}(\mathbb{Z}_p)$ is an open subgroup of the compact group $\mathcal{X}(\mathbb{Z}_p)$ (see [GT09, p. 1402]), we conclude that $\exp(\mathcal{U}) \cap \mathcal{X}(\mathbb{Z}_p)$ has finite index $k \in \mathbb{N}$ in $\mathcal{X}(\mathbb{Z}_p)$. Using Lemma 4.2 (at the expense of replacing α, α_0, β, β_0 by $k\alpha$, $k\alpha_0$, $k\beta$ and respectively $k\beta_0$), we may assume that $\alpha, \alpha_0, \beta, \beta_0 \in \exp(\mathcal{U})$. Therefore, there exists $u, u_0, v, v_0 \in \mathcal{U}$ such that $\exp(u) = \alpha$, $\exp(u_0) = \alpha_0$, $\exp(v) = \beta$ and $\exp(v_0) = \beta$. Also, we let $f, g : \mathbb{A}^N \longrightarrow \mathbb{A}^N$ be the affine transformations given by $f(x) = f_0(x) + u_0$ and respectively $g(x) = g_0(x) + v_0$ for each $x \in \mathbb{A}^N$. Since the entries of the matrices A and B (corresponding to the linear transformations f_0 and g_0) have entries which are p-adic integers, we conclude that $\mathcal{O}_f(u), \mathcal{O}_g(v) \subset \mathcal{U}$. So, because exp is a local isomorphism, while $\Phi^m(\alpha) = \exp(f^m(u))$ and $\Psi^n(\beta) = \exp(g^n(v))$ for each $m, n \in \mathbb{N}_0$, the desired conclusion follows from Theorem 1.6.

4.3. **Further remarks.** Let X be a semiabelian variety over an algebraically closed field K of characteristic 0. We conclude this paper by introducing some geometric conditions that imply the condition that none of the eigenvalues of $D\Phi_0$ is a root of unity as required in the statement of Theorem 1.4 (or Theorem 4.1).

Recall (see, for example, [MS14, Section 7]) that a dominant K-endomorphism Φ_0 of X is said to preserve a non-constant fibration if there is a non-constant rational map $f \in K(X)$ such that $f \circ \Phi_0 = f$. We have the following:

Proposition 4.3. *Let Φ_0 be a dominant K-endomorphism of X. Then Φ_0 preserves a non-constant fibration if and only if at least one of the eigenvalues of $D\Phi_0$ is a root of unity.*

Proof. First we note that by [GS, Lemma 4.1], for any positive integer ℓ, we know that Φ_0 preserves a non-constant fibration if and only if Φ^ℓ preserves a non-constant fibration.

Assume now that $D\Phi_0$ has an eigenvalue which is a root of unity, say of order $\ell \in \mathbb{N}$. Therefore it suffices to prove that $\Phi_1 := \Phi_0^\ell$ preserves a non-constant fibration.

Let $f \in \mathbb{Z}[z]$ be the minimal (monic) polynomial for $D\Phi_1$; alternatively, $f(z)$ is the minimal monic polynomial with integer coefficients such that $f(\Phi_1) = 0 \in \mathrm{End}(X)$. We know that $f(1) = 0$; hence there exists $g \in \mathbb{Z}[z]$ such that $f(z) = (z-1) \cdot g(z)$. In particular, we know that $g(\Phi_1)$ is not the trivial endomorphism of X. We let $Y := g(\Phi_1)(X)$; then Y is a nontrivial semiabelian subvariety of X. We also let $\pi : X \longrightarrow Y$ be the map $x \mapsto g(\Phi_1)(x)$ and note that $\pi \circ \Phi_1 = \pi$ on X. Thus Φ_1 preserves a non-constant fibration, contradiction.

Assume now that $D\Phi_0$ has no eigenvalue which is a root of unity; we will show that Φ_0 does not preserve a nonconstant fibration. Again, at the expense of replacing Φ_0 by an iterate Φ_1, we may assume that all eigenvalues λ_i of $D\Phi_1$ have the property that if λ_i/λ_j is a root of unity, then $\lambda_i = \lambda_j$. Also, note that since Φ_0 and therefore its iterate Φ_1 is a dominant morphism, then each eigenvalue λ_i of $D\Phi_1$ is nonzero and not equal to a root of unity according to our hypothesis.

Arguing identically as in the proof of Theorem 1.4, we can find a prime number p and a suitable model $\mathcal{X}$ of X over $\mathbb{Z}_p$ such that each entry of $A := D\Phi_1$ is a p-adic integer, and moreover the p-adic exponential map exp induces a local isomorphism between a sufficiently small ball $\mathcal{B}$ in $\mathbb{C}_p^N$ and a corresponding small p-adic neighborhood $\mathcal{U}$ of the origin of $\mathcal{X}$.

We can write $A = B^{-1}JB$, where B is an invertible matrix and J is in Jordan form. Because B is invertible, we can choose $\mathbf{v} \in \mathcal{B}$ such that $B\mathbf{v}$ has all its entries nonzero. Then arguing identically as in Section 3, we get that the entries of $J^nB\mathbf{v}$ are of the form

$$(P_{1,1}(n)\lambda_1^n, \cdots, P_{1,m_1}(n)\lambda_1^n, P_{2,1}(n)\lambda_2^n, \cdots, P_{2,m_2}(n)\lambda_2^n, \cdots\cdots, P_{r,m_r}(n)\lambda_r^n)^T,$$

where each $P_{i,1}$ is a nonzero polynomial, and moreover $\deg(P_{i,j}) = \deg(P_{i,1}) - j + 1$ for each $i = 1, \ldots, r$ and for each $j = 1, \ldots, m_i$. Then for each $\mathbf{w} \in \mathbb{C}_p^N$ and for each

$a \in \mathbb{C}_p$, we have that $\left\{\mathbf{w}^T \cdot J^n B\mathbf{v} + a\right\}_{n\geq 1}$ is a linear recurrence sequence with non-degenerate characteristic roots, unless $\mathbf{w}$ is the zero-vector. Therefore, given any proper linear subspace $W \subset \mathbb{C}_p^N$, there are finitely many vectors $A^n\mathbf{v} = B^{-1}J^nB\mathbf{v}$ contained in the same coset $\mathbf{a} + W$ (for any given $\mathbf{a} \in \mathbb{C}_p^N$).

Let now $x = \exp(\mathbf{v}) \in \mathcal{U}$. We claim that $\mathcal{O}_{\Phi_1}(x)$ is Zariski dense in X. Indeed, otherwise there exists a coset $\beta + H$ of a proper algebraic subgroup H of X containing infinitely many points from the orbit $\mathcal{O}_{\Phi_1}(x)$. This last statement follows by noting that $\mathcal{O}_{\Phi_1}(x)$ is contained in a finitely generated subgroup of X (because there exists a monic nonzero polynomial $f \in \mathbb{Z}[z]$ such that $f(\Phi_1) = 0 \in \mathrm{End}(X)$) and then using the classical Mordell-Lang theorem (see [Voj96]). Because all entries of $D\Phi_1$ are p-adic integers, then we know that $\mathcal{O}_{\Phi_1}(x) \subset \mathcal{U}$. Since H is a proper algebraic subgroup of X, then $\exp^{-1}(H \cap \mathcal{U}) = H_0 \cap \mathcal{B}$ for some proper linear subgroup $H_0 \subset \mathbb{C}_p^N$. But then there are infinitely many vectors $A^n\mathbf{v}$ contained in a coset of H_0, which is a contradiction.

Now, since $\mathcal{O}_{\Phi_1}(x)$ is Zariski dense in X, we immediately get that Φ_1 and thus Φ_0 cannot preserve a non-constant fibration (see also [MS14, Section 7]). This concludes the proof of Proposition 4.3. □

Another condition has been mentioned in Remark 1.3.

Proposition 4.4. *Let Φ be a self-map of X over K with $\Phi(x) = \Phi_0(x) + \alpha_0$ where Φ_0 is a K-endomorphism and $\alpha_0 \in X(K)$. Assume there does not exist $m \in \mathbb{N}$ and a positive dimensional closed subvariety Y of X such that Φ^m restricts to an automorphism on Y. Then none of the eigenvalues of $D\Phi_0$ is a root of unity.*

Proof. We argue by contradiction and therefore, we assume $D\Phi_0$ has an eigenvalue which is a root of unity. At the expense of replacing Φ by an iterate Φ^m we may assume that all eigenvalues of $D\Phi_0$ are either equal to 1, or they are not roots of unity. We let (similar to the proof of Proposition 4.3) $f \in \mathbb{Z}[z]$ be the minimal monic polynomial such that $f(\Phi_0) = 0$. Then $f(t) = (t-1)^r \cdot f_1(t)$ for some monic polynomial $f_1 \in \mathbb{Z}[t]$ such that $f_1(1) \neq 0$, and some $r \in \mathbb{N}$. We let $Y := f_1(\Phi_0)(X)$ and $Z := \left(\Phi_0 - \mathrm{id}\,|_X\right)^r(X)$, where for any subvariety $V \subseteq X$, we denote by $\mathrm{id}\,|_V$ the identity map on V. Then Y and Z are semiabelian subvarieties of X. As proven in [GS, Lemma 6.1], we have that $X = Y + Z$ and $Y \cap Z$ is finite. Strictly speaking, [GS, Lemma 6.1] is written for endomorphisms of abelian varieties, but the proof goes verbatim to semiabelian varieties since no property applicable only to abelian varieties (such as the Poincaré's Reducibility Theorem—see [GS, Fact 3.2]) is used; essentially, all one uses is that the polynomials $(z-1)^r$ and $f_1(z)$ are coprime. So, Φ_0 restricts to an endomorphism τ of Z with the property that

$$(\tau - \mathrm{id}\,|_Z) : Z \longrightarrow Z$$

is an isogeny. Note that we allow the possibility that Z is the trivial algebraic subgroup of X; in this case, it is still true that $\tau - \mathrm{id}\,|_Z$ is surjective. We let $\beta_0 \in Y$ and $\gamma_0 \in Z$ such that $\alpha_0 = \beta_0 + \gamma_0$. Hence there exists $\gamma_1 \in Z$ such that

$$\Phi_0(\gamma_1) + \gamma_0 = \gamma_1. \tag{11}$$

In the case Z is the trivial subgroup, then clearly $\gamma_0 = \gamma_1 = 0$.

We claim that Φ restricts to an automorphism on the positive dimensional subvariety $V := \gamma_1 + Y$ (note that $r \geq 1$ and thus $\dim(Y) \geq 1$). First we note that Φ_0 restricts to an automorphism σ on Y; indeed, Φ_0 induces an endomorphism σ

of Y by definition, and then since $(\sigma - \mathrm{id}|_Y)^r = 0$, we get that $\sigma : Y \longrightarrow Y$ is an automorphism. Then for each $y \in Y$ we have that

$$\Phi(y+\gamma_1) = \Phi_0(y) + \Phi_0(\gamma_1) + \beta_0 + \gamma_0 = \sigma(y) + \beta_0 + \gamma_1.$$

Because σ is an automorphism of Y, while $\beta_0 \in Y$, we conclude that

$$y \mapsto \sigma(y) + \beta_0$$

is an automorphism of Y. This concludes the proof of Proposition 4.4. □

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Dragos Ghioca, Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada

E-mail address: dghioca@math.ubc.ca

Khoa D. Nguyen, Department of Mathematics, University of British Columbia, And Pacific Institute for The Mathematical Sciences, Vancouver, BC V6T 1Z2, Canada

E-mail address: dknguyen@math.ubc.ca

URL: www.math.ubc.ca/~dknguyen