PERIODIC POINTS, LINEARIZING MAPS, AND THE DYNAMICAL MORDELL-LANG PROBLEM

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Abstract. Under suitable hypotheses, we prove a dynamical version of the Mordell-Lang conjecture for subvarieties of quasiprojective varieties X, endowed with the action of a morphism $\Phi : X \longrightarrow X$. We also prove a version of the Mordell-Lang conjecture that holds for any endomorphism of a semiabelian variety. We use an analytic method based on the technique of Skolem, Mahler, and Lech, along with results of Herman and Yoccoz from nonarchimedean dynamics.

1. Introduction

Let X be a quasiprojective variety over the complex numbers $\mathbb{C}$, let $\Phi : X \longrightarrow X$ be a morphism, and let V be a closed subvariety of X. For any integer $i \geq 0$, denote by Φ^i the i^{th} iterate $\Phi \circ \cdots \circ \Phi$; for any point $\alpha \in X(\mathbb{C})$, we let $\mathcal{O}_\Phi(\alpha) := \{\Phi^i(\alpha) : i \in \mathbb{N}\}$ be the Φ-orbit of α. If $\alpha \in X(\mathbb{C})$ has the property that there is some integer $\ell \geq 0$ such that $\Phi^\ell(\alpha) \in W(\mathbb{C})$, where W is a periodic subvariety of V, then there are infinitely many integers $n \geq 0$ such that $\Phi^n(\alpha) \in V$. More precisely, if $M \geq 1$ is the period of W (the smallest positive integer j for which $\Phi^j(W) = W$), then $\Phi^{kM+\ell}(\alpha) \in W(\mathbb{C}) \subseteq V(\mathbb{C})$ for integers $k \geq 0$. It is natural then to pose the following question.

Question 1.1. *If there are infinitely many integers $m \geq 0$ such that $\Phi^m(\alpha) \in V(\mathbb{C})$, are there necessarily integers $M \geq 1$ and $\ell \geq 0$ such that $\Phi^{kM+\ell}(\alpha) \in V(\mathbb{C})$ for all integers $k \geq 0$?*

Note that if $V(\mathbb{C})$ contains an infinite set of the form $\{\Phi^{kM+\ell}(\alpha)\}_{k\in\mathbb{N}}$ for some positive integers M and ℓ, then V contains a positive dimensional subvariety invariant under Φ^M (simply take the union of the positive dimensional components of the Zariski closure of $\{\Phi^{kM+\ell}(\alpha)\}_{k\in\mathbb{N}}$).

Denis [Den94] appears to have been the first to pose Question 1.1. He showed that the answer is "yes" under the additional hypothesis that the integers n for which $\Phi^n(\alpha) \in V(\mathbb{C})$ are sufficiently dense in the set of all positive integers; he also obtained results for automorphisms of projective space without using this additional hypothesis. Bell [Bel06] later solved the

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problem completely in the case of automorphisms of affine space, by showing that the set of all $n \in \mathbb{N}$ such that $\Phi^n(\alpha) \in V(\mathbb{C})$ is at most a finite union of arithmetic progressions. More recently, results were obtained in the case when $\Phi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ takes the form (f, g) for $f, g \in \mathbb{C}[t]$ ([GTZ08]) and the subvariety V is a line, and in the case when $\Phi : \mathbb{A}^g \longrightarrow \mathbb{A}^g$ has the form $(f, \dots, f)$ where $f \in K[t]$ (for a number field K) has no periodic critical points other than the point at infinity ([BGKT]).

The technique used in [BGKT] and [Bel06] is a modification of a method first used by Skolem [Sko34] (and later extended by Mahler [Mah35] and Lech [Lec53]) to treat linear recurrence sequences. The idea is to show that there is a positive integer M such that for each $i = 0, \dots, M-1$ there is an integer $j \equiv i \pmod{M}$ and a p-adic analytic function θ_j convergent on the closed unit disc in $\mathbb{C}_p$ such that $\theta_j(k) = \Phi^{kM+j}(\alpha)$ for all $k \in \mathbb{N}$. Given any polynomial F in the vanishing ideal of V, one thus obtains a p-adic analytic function $F \circ \theta_j$ that vanishes on all k for which $\Phi^{kM+j}(\alpha) \in V$. Since a power series cannot have infinitely many zeros in its domain of convergence unless it is identically zero, this implies that if there are infinitely many $n \equiv i \pmod{M}$ such that $\Phi^n(\alpha) \in V$, then $\Phi^{kM+j}(\alpha) \in V$ for *all* $k \in \mathbb{N}$.

In the case of [BGKT], the existence of the p-adic analytic function θ_j is proved by using linearizing maps developed by Rivera-Letelier [RL03]. One is able to show that the desired power series exists at a prime p provided that for any $\alpha_1, \dots, \alpha_g \in K$, there exists a nonnegative integer j such that $f^j(\alpha_i)$ is in a p-adically indifferent periodic residue class modulo p, for each $i = 1, \dots, g$. It seems plausible, and even likely, that this technique generalizes to the case of any map of the form $\Phi = (f_1, \dots, f_g)$ where $f_i \in \mathbb{C}[t]$. Thus, we make the following conjecture.

Conjecture 1.2. *Let $f_1, \dots, f_g \in \mathbb{C}[t]$ be polynomials, let Φ be their action coordinatewise on $\mathbb{A}^g$, let $\mathcal{O}_\Phi((x_1, \dots, x_g))$ denote the Φ-orbit of the point $(x_1, \dots, x_g) \in \mathbb{A}^g(\mathbb{C})$, and let V be a subvariety of $\mathbb{A}^g$. Then V intersects $\mathcal{O}_\Phi((x_1, \dots, x_g))$ in at most a finite union of orbits of the form $\mathcal{O}_{\Phi^M}(\Phi^\ell(x_1, \dots, x_g))$, for some nonnegative integers M and ℓ.*

Note that the orbits for which $M = 0$ are singletons, so that the conjecture allows not only infinite forward orbits but also finitely many extra points. We view our Conjecture 1.2 as a dynamical version of the classical Mordell-Lang conjecture, where subgroups of rank one are replaced by orbits under a morphism. Note that when each f_i is a monomial, Conjecture 1.2 reduces to a dynamical formulation of the classical Mordell-Lang conjecture for endomorphisms of the multiplicative group (see also our Theorem 1.8).

In this paper, we describe a more general framework for approaching Conjecture 1.2. Note that when Φ takes the form $(f_1, \dots, f_g)$ for $f_i \in \mathbb{C}[t]$, the Jacobian of Φ is always diagonalizable. Here we prove more general results about neighborhoods of fixed points with diagonalizable Jacobians. Our principal tool is work of Herman and Yoccoz [HY83] on linearizing maps for general diffeomorphisms in higher dimensions. Our first result is the following.

Theorem 1.3. *Let p be a prime number, let X be a quasiprojective variety defined over $\mathbb{C}_p$, and let $\Phi : X \longrightarrow X$ be a morphism defined over $\mathbb{C}_p$. Let $\alpha \in X(\mathbb{C}_p)$, and let V be a closed subvariety of X defined over $\mathbb{C}_p$. Assume the p-adic closure of the orbit $\mathcal{O}_\Phi(\alpha)$ contains a Φ-periodic point β of period dividing M such that β and all of its iterates are nonsingular, and such that the Jacobian of Φ^M at β is a nonzero homothety of p-adic absolute value less than one. Then $V(\mathbb{C}_p) \cap \mathcal{O}_\Phi(\alpha)$ is at most a finite union of orbits of the form $\mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$, for some nonnegative integers k and ℓ.*

As a special case of Theorem 1.3, we derive the main result of [GTb], which we state here as Theorem 1.5; this may be thought of as a special case of Conjecture 1.2. Before stating Theorem 1.5, we recall the definition of attracting periodic points for rational dynamics.

Definition 1.4. *If K is a field, and $\varphi \in K(t)$ is a rational function, then $z \in \mathbb{P}^1(\overline{K})$ is a* periodic point *for φ if there exists an integer $n \geq 1$ such that $\varphi^n(z) = z$. The smallest such integer n is the* period *of z, and $\lambda = (\varphi^n)'(z)$ is the* multiplier *of z. If $|\cdot|_v$ is an absolute value on K, and if $0 < |\lambda|_v < 1$, then z is called* attracting.

Theorem 1.5. *Let $g \geq 1$, let p be a prime number, let $\phi_1, \dots, \phi_g \in \mathbb{C}_p(t)$ be rational functions, and let $\Phi := (\phi_1, \dots, \phi_g)$ act coordinatewise on $\left(\mathbb{P}^1\right)^g$. Let $\alpha := (x_1, \dots, x_g) \in \left(\mathbb{P}^1\right)^g(\mathbb{C}_p)$, and let $V \subset \left(\mathbb{P}^1\right)^g$ be a subvariety defined over $\mathbb{C}_p$. Assume the p-adic closure of the orbit $\mathcal{O}_\Phi(\alpha)$ contains an attracting Φ-periodic point $\beta := (y_1, \dots, y_g)$ such that for some positive integer M, we have $\Phi^M(\beta) = \beta$ and $(\phi_1^M)'(y_1) = \cdots = (\phi_g^M)'(y_g)$. Then $V(\mathbb{C}_p) \cap \mathcal{O}_\Phi(\alpha)$ is at most a finite union of orbits of the form $\mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$, for some nonnegative integers k and ℓ.*

The following result generalizes the approach of [Bel06] and [BGKT]. Essentially, it shows that if an orbit gets close to some "indifferent" periodic point of X, then we can answer Question 1.1 in the affirmative (see [RL03] for the terminology of "indifferent" points in the context of rational dynamics).

Theorem 1.6. *Let X be a quasiprojective variety defined over a number field K, let $\Phi : X \longrightarrow X$ be a morphism defined over K, and let V be a closed subvariety of X defined over K. Let $\beta \in X(K)$ be a periodic point of period dividing M such that β and its iterates are all nonsingular points, and the Jacobian of Φ^M at β is a diagonalizable matrix whose eigenvalues $\lambda_1, \dots, \lambda_g$ satisfy*

$$\prod_{j=1}^{g} \lambda_j^{e_j} \neq \lambda_i, \tag{1.6.1}$$

for each $1 \leq i \leq g$, and any nonnegative integers $e_1, \dots, e_g$ such that $\sum_{j=1}^{g} e_j \geq 2$.

Then for all but finitely many primes p, there is a p-adic neighborhood $\mathcal{V}_p$ of β (depending only on p and β) such that if $\mathcal{O}_\Phi(\alpha) \cap \mathcal{V}_p$ is nonempty for $\alpha \in X(\mathbb{C}_p)$, then $V(\mathbb{C}_p) \cap \mathcal{O}_\Phi(\alpha)$ is at most a finite union of orbits of the form $\mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$, for some nonnegative integers k and ℓ.

There are a variety of other issues to be dealt with but this seems, at least, to represent evidence that Question 1.1 has a positive answer in general (see also Corollary 3.2). Typically, the Jacobian at a point should have distinct, nonzero, multiplicatively independent eigenvalues. Thus, Theorem 1.6 says that if some iterate of α belongs to a certain p-adic neighborhood $\mathcal{V}_p$ of a typical periodic point, then Question 1.1 has a positive answer for Φ and α. While many obstacles remain – most notably the issue of the size of the neighborhood $\mathcal{V}_p$ – we are hopeful that this approach will lead to a general answer for Question 1.1. We note that Fakhruddin ([Fak03]) has shown that periodic points are Zariski dense for a wide class of morphisms of varieties; moreover, his methods show that in many cases over number fields, there is a periodic point within any periodic residue class at a finite prime. Thus, it is reasonable to expect that some iterate of α will be p-adically close to some periodic point. In conclusion, we propose the following general conjecture.

Conjecture 1.7. *Let X be a quasiprojective variety defined over $\mathbb{C}$, let $\Phi : X \longrightarrow X$ be any morphism, and let $\alpha \in X(\mathbb{C})$. Then for each subvariety $V \subset X$, the intersection $V(\mathbb{C}) \cap \mathcal{O}_\Phi(\alpha)$ is a union of at most finitely many orbits of the form $\mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$, for some nonnegative integers k and ℓ.*

Finally, we prove Conjecture 1.7 for algebraic group endomorphisms of semiabelian varieties; in this case, one can show that the argument from the proof of Theorem 1.6 always applies. In [GTa], this is proved using deep results of Vojta [Voj99] and Faltings [Fal94] on integral points on semiabelian varieties. Here, we give a purely dynamical proof.

Theorem 1.8. *Let A be a semiabelian variety defined over a finitely generated subfield K of $\mathbb{C}$, and let $\Phi : A \longrightarrow A$ be an endomorphism defined over K. Then for every subvariety $V \subset A$ defined over K, and for every point $\alpha \in A(K)$, the intersection $V(K) \cap \mathcal{O}_\Phi(\alpha)$ is at most a finite union of orbits of the form $\mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$ for some $k, \ell \in \mathbb{N}$.*

Note that when X is a semiabelian variety and Φ is a multiplication-by-m map, this can likely be derived from Faltings' proof [Fal94] of the classical Mordell-Lang conjecture. Such a derivation seems less obvious, however, for more general endomorphisms.

The idea of the proof of Theorems 1.3 and 1.6 is fairly simple. In both cases, we begin by choosing an iterate $\Phi^\ell(\alpha)$ that is very close to β. Work of Herman and Yoccoz [HY83] give a p-adic function h in a neighborhood of β such that, for a suitable positive integer M, we have

$$\Phi^M \circ h = h \circ A,$$

for some linear function A. When A is a homothety, this means that iterates of $\Phi^\ell(\alpha)$ under Φ^M lie on an analytic line in $\mathbb{C}_p^g$. Composing with a polynomial in the vanishing ideal of the subvariety V gives a convergent p-adic power series in one variable; such a function is identically zero if it has infinitely many zeros. Thus, for each congruence class $i = 0, \dots, M-1$ modulo M, either there are finitely may $n \equiv i \pmod{M}$ such that $\Phi^n(\alpha) \in V$ or we have $\Phi^n(\alpha) \in V$ for all $n \geq \ell$ such that $n \equiv i \pmod{M}$. Under the conditions of Theorem 1.6, it is necessary to take p-adic logarithms of iterates in order to get a line in $\mathbb{C}^g$ but otherwise the proof is the same. Note that existence of the map h in Theorem 1.6 depends on Yu's [Yu90] results on linear forms in p-adic logarithms, which only apply over number fields. Under the conditions of Theorem 1.3, the map h exists even when the eigenvalues of A are transcendental.

We note that the Drinfeld module analog of Conjecture 1.2 has been proved using techniques similar to the ones employed in this paper (see [GT08]). This provided a positive answer to a conjecture proposed by Denis [Den92] in the case of Φ-submodules Γ of rank 1, where $\Phi := (\phi_1, \dots, \phi_g)$, each $\phi_i : \mathbb{F}_q[t] \longrightarrow \mathrm{End}_K(\mathbb{G}_a)$ is a Drinfeld module, and K is a function field of transcendence degree 1 over $\mathbb{F}_q$. However, Conjecture 1.2 seems much more difficult, since the proof [GT08] over Drinfeld modules makes use of the fact that the polynomials $\phi_i(a)$ are additive (for any $a \in \mathbb{F}_q[t]$); in particular, this allows us to find points of Γ which are arbitrarily close to 0 with respect to any valuation v of K at which the points of Γ are integral and Φ has good reduction. If we let $0 \neq P \in \mathbb{F}_q[t]$ such that $|P|_v < 1$, then 0 is an attracting fixed point for the dynamical system $\Phi(P)$, and each $\phi_i(P)$ has the same multiplier at 0; thus the main result of [GT08] is a special case of our Theorem 1.5.

We now briefly sketch the plan of our paper. In Section 2 we prove Theorem 1.3 and state a corollary of Theorem 1.5, while in Section 3 we prove Theorems 1.6 and 1.8.

Notation. We write $\mathbb{N}$ for the set of nonnegative integers. If K is a field, we write $\overline{K}$ for an algebraic closure of K. Given a prime number p, the field $\mathbb{C}_p$ will denote the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, the field of p-adic rationals. We denote by $|\cdot|_p$ the usual absolute value on $\mathbb{C}_p$; that is, we have $|p|_p = 1/p$. When we work in $\mathbb{C}_p^g$ with a fixed coordinate system, then, for $\vec{\alpha} = (\alpha_1 \dots, \alpha_g) \in \mathbb{C}_p^g$ and $r > 0$, we write $\mathbb{D}(\vec{\alpha}, r)$ for the open disk of radius r in $\mathbb{C}_p^g$ centered at α. More precisely, we have

$$\mathbb{D}(\vec{\alpha}, r) := \{(\beta_1, \dots, \beta_g) \in \mathbb{C}_p^g \ | \ \max_i |\alpha_i - \beta_i|_p < r\}.$$

Similarly, we let $\overline{\mathbb{D}}(\vec{\alpha}, r)$ be the *closed* disk of radius r centered at $\vec{\alpha}$. In the case where $g = 1$, we drop the vector notation and denote our discs as $\mathbb{D}(\alpha, r)$ and $\overline{\mathbb{D}}(\alpha, r)$. We say that a function F is *p-adic analytic* (or simply, analytic) on $\mathbb{D}(\alpha, r)$ (resp. $\overline{\mathbb{D}}(\alpha, r)$) if there is a power series $\sum_{n=0}^\infty a_n (z-\alpha)^n$, with coefficients in $\mathbb{C}_p$, convergent on all of $\mathbb{D}(\alpha, r)$ (resp. $\overline{\mathbb{D}}(\alpha, r)$) such that $F(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n$ for all $z \in \mathbb{D}(\alpha, r)$ (resp. $\overline{\mathbb{D}}(\alpha, r)$).

2. Attracting points

We will begin with a theorem of Herman and Yoccoz [HY83] on linearization of analytic maps near one of their fixed points. First we set up the notation. Let $\vec{0}$ be the zero vector in $\mathbb{C}_p^g$, and for $\vec{x} := (x_1, \dots, x_g)$ we let

$$f(\vec{x}) = \sum_{(i_1,\dots,i_g)\in\mathbb{N}^g} b_{i_1,\dots,i_g} x_1^{i_1} \cdots x_g^{i_g} \tag{2.0.1}$$

be a power series over $\mathbb{C}_p$ which fixes $\vec{0}$, and it has a positive radius of convergence; i.e., there is some $r > 0$ such that (2.0.1) converges on $\mathbb{D}(\vec{0}, r)$. Furthermore, we assume there exists $A \in \mathrm{GL}(g, \mathbb{C}_p)$ such that

$$f(\vec{x}) = A \cdot \vec{x} + \text{ higher order terms }.$$

In this case, f is a formal diffeomorphism in the terminology of [HY83]. More generally, for a formal power series ψ in $\mathbb{C}_p^g$ centered at $\vec{\alpha}$, we define $D\psi_{\vec{\alpha}}$ to be the linear part of the power series. Thus $Df_{\vec{0}} = A$. Note that this coincides with the usual definition of the D-operator from the theory of manifolds (that is, $D\psi_{\vec{\alpha}}$ is the usual Jacobian of ψ at $\vec{\alpha}$ – see [Jos02, I.1.5]).

Let $\lambda_1, \dots, \lambda_g$ be the eigenvalues of A. Suppose that there are constants $C, b > 0$ such that

$$|\lambda_1^{e_1} \cdots \lambda_g^{e_g} - \lambda_i|_p \geq C \left(\sum_{j=1}^{g} e_j \right)^{-b} \tag{2.0.2}$$

for any $1 \leq i \leq g$ and any tuple $(e_1, \dots, e_g) \in \mathbb{N}^g$ such that $\sum_{j=1}^{g} e_j \geq 2$ (this is condition (C) from page 413 of [HY83]). Note that (2.0.2) already implies that no $\lambda_i = 0$.

The following result is Theorem 1 of [HY83].

Theorem 2.1. *Let f and A be as above. There exists $r > 0$, and there exists a bijective, p-adic analytic function $h : \mathbb{D}(\vec{0}, r) \longrightarrow \mathbb{D}(\vec{0}, r)$ such that*

$$f(h(\vec{x})) = h(A\vec{x}), \tag{2.1.1}$$

for all $\vec{x} \in \mathbb{D}(\vec{0}, r)$, where $Dh_{\vec{0}} = \mathrm{Id}$.

Before continuing, we need to define $D\Phi$ more generally for Φ a morphism of a quasiprojective variety. If X is a quasiprojective variety defined over a field L, and $\Phi : X \longrightarrow X$ is a morphism, and both β and $\Phi(\beta)$ are nonsingular points in $X(L)$, then Φ induces an L-linear map

$$D\Phi_\beta : T_\beta \longrightarrow T_{\Phi(\beta)}$$

where T_β is the stalk of the tangent sheaf for X at β. Since β and $T(\beta)$ are nonsingular, both T_β and $T_{\Phi(\beta)}$ are vector spaces of dimension $\dim X$ over L (see [Har77, II.8]). Note that when $L = \mathbb{C}$ and β and $\Phi(\beta)$ are in a coordinate patch $\mathcal{U}$ on the complex manifold $X^{\text{nonsing}}(\mathbb{C})$, then $D\Phi_\beta$ can be written in coordinates using the partial derivatives of Φ with respect to these coordinates (i.e. as the Jacobian matrix of Φ expressed with respect to these coordinates).

We are ready to prove Theorem 1.3.

Proof. Let $\lambda \in \mathbb{C}_p$ such that $D\Phi^M_\beta = \lambda \cdot \mathrm{Id}$; according to our hypotheses, we have $0 < |\lambda|_p < 1$. Let $j \in \{0, \dots, M-1\}$ be fixed. Using the fact that $X(\mathbb{C}_p)$ is a p-adic analytic manifold in a neighborhood of each iterate $\Phi^j(\beta)$ we may find an analytic function $\mathcal{F}_j$ defined on a sufficiently small neighborhood $\mathcal{U}_j$ of $\vec{0} \in \mathbb{C}_p^g$ which maps $\mathcal{U}_j$ bijectively onto a neighborhood $\mathcal{V}_j$ of $\Phi^j(\beta)$. Then we write $\Psi_j := \mathcal{F}_j^{-1} \circ \Phi^M \circ \mathcal{F}_j$ as a function of the following form (note that $D(\Psi_j)_{\vec{0}} = \lambda \cdot \mathrm{Id}$):

$$\Psi_j(\vec{x}) := (\mathcal{F}_j^{-1} \circ \Phi^M \circ \mathcal{F}_j)(\vec{x}) = \lambda \cdot \vec{x} + \text{ higher order terms }.$$

Since $|\lambda^i - \lambda|_p = |\lambda|_p$ for $i \geq 2$, we see that (2.0.2) is satisfied; so we have a bijective analytic function $h_j : \mathbb{D}(\vec{0}, r_j) \longrightarrow \mathbb{D}(\vec{0}, r_j)$ (for some $r_j > 0$) such that

$$\Psi_j \circ h_j = h_j \circ \lambda \, \mathrm{Id}$$

and $D\left(h_j\right)_{\vec{0}} = \mathrm{Id}$, by Theorem 2.1. Let $r > 0$ such that for each $0 \leq j \leq M-1$, we have

$$(\Phi^j \circ \mathcal{F}_0)(\mathbb{D}(\vec{0}, r)) \subset \mathcal{F}_j(\mathbb{D}(\vec{0}, r_j)). \tag{2.1.2}$$

Let N_0 be the smallest positive integer such that $\Phi^{N_0}(\alpha) \in \mathcal{F}_0(\mathbb{D}(\vec{0}, r))$; then $\Phi^{N_j}(\alpha) \in \mathcal{F}_j(\mathbb{D}(\vec{0}, r_j))$, where $N_j := N_0 + j$ for each $j = 1, \dots, M-1$. Let $\vec{\alpha_j} \in \mathbb{D}(\vec{0}, r_j)$ satisfy $h_j(\vec{\alpha_j}) = \mathcal{F}_j^{-1}(\Phi^{N_j}(\alpha))$. Note that since $|\lambda|_p < 1$, we have

$$\left(\mathcal{F}_j^{-1} \circ \Phi^{kM}\right)\left(\Phi^{N_j}(\alpha)\right) = (\Psi_j^k \circ h_j)(\vec{\alpha_j}) = h_j(\lambda^k \cdot \vec{\alpha_j}). \tag{2.1.3}$$

Now, for each polynomial F in the vanishing ideal of V, we construct the function $\Theta_{F,j} : \overline{\mathbb{D}}(0,1) \longrightarrow \mathbb{C}_p$ given by

$$\Theta_{F,j}(z) := F\left(\left(\mathcal{F}_j \circ h_j\right)(z \cdot \vec{\alpha_j})\right).$$

The function $\Theta_{F,j}$ is analytic because each h_j is analytic on $\mathbb{D}(\vec{0}, r_j)$, and $\vec{\alpha_j} \in \mathbb{D}(\vec{0}, r_j)$.

For each $k \in \mathbb{N}$ such that $\Phi^{kM+N_j}(\alpha) \in V(\mathbb{C}_p)$, we have $\Theta_{F,j}(\lambda^k) = 0$ for each F. Since $\lim_{k\to\infty} \lambda^k = 0$, we conclude that if there are infinitely many k such that $\Phi^{N_0+j+Mk}(\alpha) \in V(\mathbb{C}_p)$, then $\Theta_{F,j}$ is identically equal to 0 (since the zeros of a p-adic analytic function cannot accumulate); hence $\Theta_{F,j}(\lambda^k) = 0$ for *all* $k \in \mathbb{N}$, which means that F vanishes on all points $\Phi^{N_0+j+Mk}(\alpha)$ for $k \in \mathbb{N}$. Applying this argument for all polynomials F in the vanishing ideal of V, we conclude that

$$\begin{aligned} &\text{either } \Phi^{N_0+j+Mk}(\alpha) \in V(\mathbb{C}_p) \text{ for all } k \in \mathbb{N} \\ &\text{or } V(\mathbb{C}_p) \cap \mathcal{O}_{\Phi^M}\left(\Phi^{N_0+j}(\alpha)\right) \text{ is finite.} \end{aligned} \tag{2.1.4}$$

Since

$$\mathcal{O}_\Phi(\alpha) = \{\Phi^i(\alpha) : 0 \le i \le N_0 - 1\} \bigcup \left(\bigcup_{j=0}^{M-1} \mathcal{O}_{\Phi^M}(\Phi^{N_0+j}(\alpha)) \right),$$

we conclude the proof of Theorem 1.3. □

The following result is an immediate corollary of Theorem 1.5.

Corollary 2.2. *Let $f \in \mathbb{C}_p[t]$ be defined by $f(t) = \sum_{i=1}^m a_i t^i$, where for each $i = 1, \dots, m$, we have $|a_i|_p \le 1$, while $0 < |a_1|_p < 1$. We consider Φ the coordinatewise action of f on $\mathbb{A}^g$ (where $g \ge 1$). Let $\alpha = (x_1, \dots, x_g) \in \mathbb{A}^g(\mathbb{C}_p)$ satisfy $|x_i|_p < 1$ for each $i = 1, \dots, g$. Then for any subvariety $V \subset \mathbb{A}^g$ defined over $\mathbb{C}_p$, the intersection $V(\mathbb{C}_p) \cap \mathcal{O}_\Phi(\alpha)$ is either finite, or it contains $\mathcal{O}_\Phi(\Phi^\ell(\alpha))$ for some $\ell \in \mathbb{N}$.*

Furthermore, both Theorem 1.3 and Theorem 1.5 (with its Corollary 2.2) can be made effective, as one can use Newton polygons to find the zeros of p-adic analytic functions.

3. Indifferent points

Using the same set-up as in the beginning of Section 2, we prove Theorem 1.6. But we first need to define an arbitrary power J^z of a *Jordan matrix*, whenever $z \in \mathbb{D}(0, p^{-1/(p-1)})$. By a Jordan matrix we mean a matrix which is in its Jordan form, i.e. it consists of its *Jordan blocks*. A Jordan block is either a multiple of the identity matrix, or it is an upper-triangular matrix whose entries on the diagonal are all equal, and the only nonzero entries outside the diagonal are the entries on the line above the diagonal which are all equal to one (see [Hun, VII.4] or [Lan02, XI.2]).

Proposition 3.1. *Let $J \in \mathrm{GL}(g, \mathbb{C}_p)$ be a Jordan matrix with the property that each eigenvalue λ_i of J satisfies $|\lambda_i|_p = 1$. Then there exists a positive integer d such that we may define $(J^d)^z$ for each $z \in \mathbb{D}(0, p^{-1/(p-1)})$ satisfying the following properties:*

(i) *for each $k \in \mathbb{N}$, the matrix $(J^d)^k$ is the usual (dk)-th power of J; and*
(ii) *the entries of $(J^d)^z$ are p-adic analytic functions of z.*

Proof. We let d be a positive integer such that $|\lambda_i^d - 1|_p < 1$ for each $i = 1, \dots, g$. We define the power $(J^d)^z$ for each $z \in \mathbb{D}(0, p^{-1/(p-1)})$ on each Jordan block. For each diagonal Jordan block, we define $(J^d)^z$ as the diagonal matrix whose eigenvalues are the corresponding $(\lambda_i^d)^z$ for each $z \in \mathbb{D}(0, p^{-1/(p-1)})$.

Assume now that J_0 is a Jordan block of dimension $m \le g$ corresponding to an eigenvalue λ. Then for each $z \in \mathbb{D}(0, p^{-1/(p-1)})$, we define $(J_0^d)^z$ be the following matrix

$$(\lambda^d)^z \cdot \begin{pmatrix} 1 & \frac{z}{1!\cdot\lambda} & \frac{z(z-1)}{2!\cdot\lambda^2} & \dots & \frac{z(z-1)\cdots(z-m+1)}{m!\cdot\lambda^{m-1}} \\ 0 & 1 & \frac{z}{1!\cdot\lambda} & \dots & \frac{z(z-1)\cdots(z-m+2)}{(m-1)!\cdot\lambda^{m-2}} \\ 0 & 0 & 1 & \dots & \frac{z(z-1)\cdots(z-m+3)}{(m-2)!\cdot\lambda^{m-3}} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

It is immediate to check that $(J_0^d)^z$ equals the usual power J_0^{dk} whenever $k \in \mathbb{N}$. Furthermore, since $|\lambda^d - 1|_p < 1$, we see that $(\lambda^d)^z = \exp_p((\log_p(\lambda^d))z)$ (where $\exp_p$ and $\log_p$ are the usual p-adic exponential and logarithm functions) is a p-adic analytic function of $z \in \mathbb{D}(0, p^{-1/(p-1)})$ (see [Rob00, Section 5.4.1]). Hence each entry of $(J_0^d)^z$ is a p-adic analytic function of $z \in \mathbb{D}(0, p^{-1/(p-1)})$. $\square$

Proof of Theorem 1.6. Let $B \in \mathrm{GL}(g, \overline{K})$ such that $B(D(\Phi^M)_\beta)B^{-1}$ is a diagonal matrix Λ; at the expense of replacing K by a finite extension, we may assume $B \in \mathrm{GL}(g, K)$. Then for all but finitely many primes p, the entries of both B and B^{-1} have p-adic absolute values at most equal to 1. Let λ_i (for $1 \le i \le g$) be the eigenvalues of $D(\Phi^M)_\beta$. According to our hypotheses, each λ_i is nonzero. Thus for all but finitely many primes p, each λ_i is a p-adic unit.

Fix a prime p satisfying the above conditions, and we fix an embedding of K into $\mathbb{C}_p$. Let $j \in \{0, \dots, M-1\}$ be fixed. Clearly, we have $D(\Phi^M)_{\Phi^j(\beta)} = D(\Phi^M)_\beta$. Since each $\Phi^j(\beta)$ is a nonsingular point, there exists a sufficiently small neighborhood $\mathcal{U}_j \subset \mathbb{C}_p^g$ of $\vec{0}$, and an analytic function $\mathcal{F}_j$ that maps $\mathcal{U}_j$ bijectively onto a small neighborhood of $\Phi^j(\beta) \in X(\mathbb{C}_p)$; let $\Psi_j := \mathcal{F}_j^{-1} \circ \Phi^M \circ \mathcal{F}_j$. Then

$$\Psi_j(\vec{x}) = (B^{-1}\Lambda B) \cdot \vec{x} + \text{ higher order terms }.$$

Using hypothesis (1.6.1) and [Yu90, Theorem 1], we conclude that (2.0.2) is satisfied by the eigenvalues λ_i. Using Theorem 2.1, we conclude that there exists a positive number $r_j > 0$ such that $\mathbb{D}(\vec{0}, r_j) \subset \mathcal{U}_j$, and there exists a bijective analytic function $h_j : \mathbb{D}(\vec{0}, r_j) \longrightarrow \mathbb{D}(\vec{0}, r_j)$ such that $\Psi_j \circ h_j = h_j \circ (B^{-1}\Lambda B)$.

Let r be a positive number such that for every $j = 0, \dots, M-1$ we have

$$\Phi^j(\mathcal{F}_0(\mathbb{D}(\vec{0}, r))) \subset \mathcal{F}_j(\mathbb{D}(\vec{0}, r_j)).$$

We let $\mathcal{V}_p := \mathcal{F}_0(\mathbb{D}(\vec{0}, r))$ be the corresponding p-adic neighborhood of β in $X(\mathbb{C}_p)$. Suppose that $\mathcal{O}_\Phi(\alpha) \cap \mathcal{V}_p$ is nonempty. Then there exists $N_0 \in \mathbb{N}$ such that $\Phi^{N_0}(\alpha) \in \mathcal{V}_p$, and so, $\Phi^{N_j}(\alpha) \in \mathcal{F}_j(\mathbb{D}(\vec{0}, r_j))$ where $N_j := N_0 + j$, for each $j = 0, \dots, M-1$. Let $\vec{\alpha_j} \in \mathbb{D}(\vec{0}, r_j)$ such that $h_j(\vec{\alpha_j}) = \mathcal{F}_j^{-1}(\Phi^{N_j}(\alpha))$. Then for each $k \in \mathbb{N}$, we have

$$\Phi^{kM+N_j}(\alpha) = (\mathcal{F}_j \circ h_j)\left(B^{-1}\Lambda^k B(\vec{\alpha_j})\right).$$

Note that $B^{-1}\Lambda^k B(\vec{\alpha_j})$ is in $\mathbb{D}(\vec{0}, r_j)$ for each $k \in \mathbb{N}$, since each entry of B, B^{-1}, and Λ is in $\overline{\mathbb{D}}(0,1)$,

Let d be a positive integer as in the conclusion of Proposition 3.1. Then the entries of the matrix $(\Lambda^d)^z$ are p-adic analytic functions of z in the disk $\mathbb{D}(0, p^{-1/(p-1)})$. Hence, the entries of the matrix $(\Lambda^d)^{2pz}$ are p-adic analytic functions of $z \in \overline{\mathbb{D}}(0,1)$ (since $|2pz|_p < p^{-1/(p-1)}$ for $|z|_p \le 1$). Therefore, for each fixed $\ell = 0, \dots, 2pd-1$, the entries of the matrix $\Lambda^\ell \cdot (\Lambda^d)^{2pz}$ are p-adic analytic functions of $z \in \overline{\mathbb{D}}(0,1)$.

Let F be any polynomial in the vanishing ideal of V. Then, for each $j = 0, \dots, M-1$ and for each $\ell = 0, \dots, 2pd-1$, the function

$$\Theta_{F,j,\ell} : \overline{\mathbb{D}}(0,1) \longrightarrow \mathbb{C}_p$$

defined by

$$\Theta_{F,j,\ell}(z) = F\left((\mathcal{F}_j \circ h_j)\left(B^{-1}\left(\Lambda^\ell \cdot (\Lambda^d)^{2pz}\right) B(\alpha_j)\right)\right)$$

is analytic. Furthermore, for each $k \in \mathbb{N}$ such that

$$\Phi^{N_0+j+M(2kpd+\ell)}(\alpha) \in V(\mathbb{C}_p)$$

we obtain that $\Theta_{F,j,\ell}(k) = 0$ for each F in the vanishing ideal of V. Because the zeros of a nonzero p-adic analytic function cannot accumulate (see [Rob00, Section 6.2.1]), we conclude that if there are infinitely many $k \in \mathbb{N}$ such that $\Theta_{F,j,\ell}(k) = 0$, then $\Theta_{F,j,\ell}$ is identically equal to zero, and thus F vanishes on all points $\Phi^{N_0+j+M(2kpd+\ell)}(\alpha)$ for $k \in \mathbb{N}$. Applying this argument to each F in the vanishing ideal of V, we conclude that

$$\text{either } \Phi^{N_0+j+M(2kpd+\ell)}(\alpha) \in V(\mathbb{C}_p) \text{ for all } k \in \mathbb{N}$$
$$\text{or } V(\mathbb{C}_p) \cap \mathcal{O}_{\Phi^{2Mpd}}\left(\Phi^{N_0+j+M\ell}(\alpha)\right) \text{ is finite.} \tag{3.1.1}$$

Since (3.1.1) holds for each $j = 0, \dots, M-1$ and for each $\ell = 0, \dots, 2pd-1$, this concludes the proof of Theorem 1.6. ☐

The following result follows from our Theorem 1.6.

Corollary 3.2. *Let $g \ge 1$, and let $f_1, f_2, \dots, f_g \in \bar{\mathbb{Q}}[t]$ such that for each $j = 1, \dots, g$, we have $f_j(t) = c_j t + \sum_{i=1}^{d_j} a_{i,j} t^i$ with $c_j \neq 0$. Assume $c_1, \dots, c_g$ are multiplicatively independent.*

Let Φ be the diagonal action of $(f_1, \dots, f_g)$ on the coordinates of $\mathbb{A}^g$, and let $V \subset \mathbb{A}^g$ be any subvariety defined over $\bar{\mathbb{Q}}$. Then for all but finitely many prime numbers p, there exists a neighborhood $\mathcal{U}_p$ of $0 \in \mathbb{C}_p$ such that if $\mathcal{O}_\Phi(\alpha) \cap \mathcal{U}_p^g$ is nonempty for some $\alpha \in \mathbb{A}^g(\mathbb{C}_p)$, then $V(\mathbb{C}_p) \cap \mathcal{O}_\Phi(\alpha)$ is at most a finite union of orbits of the form $\mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$ for some $k, \ell \in \mathbb{N}$.

Furthermore, using [BGT, Theorem 3.3] we obtain that we may take $\mathcal{U}_p = \mathbb{Z}_p$ and $\alpha \in \mathbb{A}^g(\mathbb{Z}_p)$ in the conclusion of Corollary 3.2.

We prove now Theorem 1.8.

Proof of Theorem 1.8. For each $n \geq 0$, we let $A_n := \Phi^n(A)$ (where $A_0 = A$). Clearly, $A_{n+1} \subset A_n$ for each $n \in \mathbb{N}$. Also, each A_n is connected because it is the image of a connected group through a morphism; hence, each A_n is a semiabelian variety itself. On the other hand, there is no infinite descending chain of semiabelian varieties; therefore there exists $N \in \mathbb{N}$ such that $A_n = A_N$ for each $n \geq N$, and so, Φ restricted to A_N is an isogeny. Clearly, it suffices to prove Theorem 1.8 after replacing α by $\Phi^N(\alpha)$. Thus, at the expense of replacing A by A_N, and V by $V \cap A_N$, we may assume that Φ is an isogeny.

We will proceed by induction on the dimension of V. The case $\dim V = 0$ is trivial. Using the inductive hypothesis, we may show that to prove our result for orbits of a point α it suffices to prove it for orbits of a multiple $m\alpha$ of α. Indeed, fix a positive integer m and suppose that Theorem 1.8 is true for orbits of $m\alpha$. Then, given any subvariety V, we know that the set of n for which $\Phi^n(m\alpha) \in mV$ forms a finite union $\mathcal{P}$ of arithmetic progressions. Thus, if we let W equal the inverse image of mV under the multiplication-by-m map we know that the set of n such that $\Phi^n(\alpha) \in W$ forms the same finite union $\mathcal{P}$ of arithmetic progressions, since Φ is a group endomorphism. We let $Z_1, \dots, Z_s$ be the positive dimensional irreducible components of the Zariski closure of $\{\Phi^n(\alpha)\}_{n\in\mathcal{P}}$ (if $\{\Phi^n(\alpha)\}_{n\in\mathcal{P}}$ is finite, then also $V(K) \cap \mathcal{O}_\Phi(\alpha)$ is finite, and so, we are done). Hence each Z_i is a Φ-periodic subvariety, and all but finitely many of the $\Phi^n(\alpha)$ for $n \in \mathcal{P}$ are contained in one of the $Z_i(K)$. Thus, we need only show that for each i, the set of n such that $\Phi^n(\alpha) \in V \cap Z_i$ forms a finite union of arithmetic progressions. Each Z_i is contained in one of irreducible components of W and thus $\dim Z_i \leq \dim V$. If Z_i is contained in V, then the set of n for which $\Phi^n(\alpha) \in Z_i = Z_i \cap V$ is a finite union of arithmetic progressions, since Z_i is Φ-periodic. If Z_i is not contained in V, then $Z_i \cap V$ has dimension less than $\dim Z_i \leq \dim V$ and the set of n such that $\Phi^n(\alpha) \in Z_i \cap V$ is a finite union of arithmetic progressions by the inductive hypothesis.

Let $L := D\Phi_0$; because Φ is an isogeny, we obtain that L is nonsingular. At the expense of replacing K by another finitely generated field, we may assume L is defined over K. We choose an embedding over $\mathbb{C}$ of $\iota : A \longrightarrow \mathbb{P}^N$ as an open subset of a projective variety (for some positive integer N). At the expense of enlarging K, we may assume the above embedding ι is defined over K. We write $\iota(A) = Z(\mathfrak{a}) \setminus Z(\mathfrak{b})$ for homogeneous ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $K[x_0, \dots, x_N]$, where $Z(\mathfrak{c})$ denotes the Zariski closed subset of $\mathbb{P}^N$ on which the ideal $\mathfrak{c}$ vanishes. We choose generators $F_1, \dots, F_m$ and $G_1, \dots, G_n$ for $\mathfrak{a}$ and $\mathfrak{b}$ respectively. We let $\oplus : A \times A \longrightarrow A$ denote the addition map and $\ominus : A \longrightarrow A$ the inversion map, written with respect to our chosen coordinates on $\mathbb{P}^N$.

Claim 3.3. *There exists a prime number p, and an embedding of K into $\mathbb{Q}_p$ such that*

(i) *there exists a* $\operatorname{Spec}(\mathbb{Z}_p)$*-scheme $\mathcal{A}$ whose generic fiber equals A.*

(ii) $\alpha \in \mathcal{A}(\mathbb{Z}_p)$.
(iii) *L is conjugate over $\mathbb{Z}_p$ to its Jordan canonical form Λ, and moreover each of its eigenvalues λ_i is a p-adic unit.*
(iv) *the maps Φ and $\ominus$ extend as endomorphisms of the $\mathbb{Z}_p$-scheme $\mathcal{A}$, while $\oplus$ extends to a morphism between $\mathcal{A} \times \mathcal{A}$ and $\mathcal{A}$.*

Proof of Claim 3.3. Let R be a finitely generated subring of K such that

(1) the coefficients of $F_1, \dots, F_m$, $G_1, \dots, G_n$, and of the polynomials defining Φ, $\oplus$, and $\ominus$ are all contained in R;
(2) $\alpha \in \mathbb{P}^N(R)$;
(3) L is conjugate over R to its Jordan canonical form Λ and all of the eigenvalues of Λ are in R.

Let $\mathcal{D}$ be the Zariski dense open subset of $\mathbb{P}^N_{\operatorname{Spec} R}$ defined by $G_i \neq 0$ for at least one $i \in \{1, \dots, n\}$. Then let $\mathcal{B}$ be the closed subset of $\mathcal{D}$ defined by the zeros of each F_j (for $j \in \{1, \dots, m\}$) and of the polynomials defining Φ. Because Φ is an endomorphism of A, we conclude that $\mathcal{B}$ does not intersect the generic fiber of $\mathcal{D} \longrightarrow \operatorname{Spec} R$. Therefore $\mathcal{B}$ is contained in a finite union of special fibers of $\mathbb{P}^N_{\operatorname{Spec}(R)} \longrightarrow \operatorname{Spec}(R)$. Let E_1 be the proper closed subset of $\operatorname{Spec} R$ corresponding to these special fibers. Thus, for any open affine subset $\operatorname{Spec} R'$ of $\operatorname{Spec} R \setminus E_1$, we obtain a $(\operatorname{Spec} R')$-scheme $\mathcal{A}_{R'}$ with generic fiber A such that Φ extends to an R'-morphism from $\mathcal{A}_{R'}$ to itself. By the same reasoning, there is a proper closed subset E_2 such that for any open affine $\operatorname{Spec} R'$ of $\operatorname{Spec} R \setminus E_2$, the maps $\oplus$ and $\ominus$ extend to morphisms

$$\oplus : \mathcal{A}_{R'} \times \mathcal{A}_{R'} \longrightarrow \mathcal{A}_{R'} \text{ and } \ominus : \mathcal{A}_{R'} \longrightarrow \mathcal{A}_{R'}.$$

Similarly, the Zariski closure of the point α in $\mathbb{P}^n(R)$ only meets $Z(\mathfrak{b})$ over primes contained in some closed subset E_3. Because each E_i is a closed subset of $\operatorname{Spec} R$, there exists a nonzero $f \in R$ which is contained in all prime ideals from E_i, for $i = 1, 2, 3$. Because R is a Noetherian integral domain, we conclude that also $R[\frac{1}{f}]$ is a Noetherian integral domain; thus Φ, $\oplus$, and $\ominus$ extend to maps of $\operatorname{Spec} R[\frac{1}{f}]$-schemes and α extends to a $\operatorname{Spec} R[\frac{1}{f}]$-point on $\mathcal{A}_{R[\frac{1}{f}]}$.

Since $R[\frac{1}{f}]$ is finitely generated as a ring, we may write $R[\frac{1}{f}] = \mathbb{Z}[u_1, \dots, u_e]$ for some nonzero elements u_i. By [Bel06, Lemma 3.1] (see also [Lec53]), there is a prime p such that the field of fractions of R embeds into $\mathbb{Q}_p$ in such a way that all of the u_i, all of the eigenvalues of Λ, and all of the reciprocals of the eigenvalues of Λ are sent to elements of $\mathbb{Z}_p$. Note that such a map gives an embedding of $R[\frac{1}{f}]$ into $\mathbb{Z}_p$. Extending the base of $\mathcal{A}_{R[\frac{1}{f}]}$, Φ, $\oplus$, and $\ominus$ from $R[\frac{1}{f}]$ to $\mathbb{Z}_p$ via this embedding yields the $\mathbb{Z}_p$-scheme $\mathcal{A}$ and the maps with the desired properties. □

Let p be a prime number for which the conclusion of Claim 3.3 holds. Then we have a $\mathbb{Z}_p$-scheme $\mathcal{A}$ whose generic fiber equals A such that $\alpha \in \mathcal{A}(\mathbb{Z}_p)$. Furthermore, letting $\mathcal{V}$ and $\mathcal{W}$ be the Zariski closures of $Z(\mathfrak{a})$ and respectively $Z(\mathfrak{b})$ in $\mathcal{A}$, we obtain that

$$\mathcal{A}(\mathbb{Z}_p) = \mathcal{V}(\mathbb{Z}_p) \cap \left(\mathbb{P}^N \setminus \mathcal{W}\right)(\mathbb{Z}_p)$$

is compact because it is the intersection of two compact subsets of $\mathbb{P}^N(\mathbb{Z}_p)$. Indeed, $\mathcal{V}(\mathbb{Z}_p)$ is compact because it is a closed subset of the compact set $\mathbb{P}^N(\mathbb{Z}_p)$. On the other hand, $\left(\mathbb{P}^N \setminus \mathcal{W}\right)(\mathbb{Z}_p)$ consists of finitely many residue classes of $\mathbb{P}^N(\mathbb{Z}_p)$ and thus, it is compact because $\mathbb{Z}_p$ is compact. The above finitely many residue classes correspond to points in $\left(\mathbb{P}^N \setminus Z(\bar{\mathfrak{b}})\right)(\mathbb{F}_p)$, where $\bar{\mathfrak{b}}$ is the ideal of $\mathbb{F}_p[x_0, \dots, x_N]$ generated by the reductions modulo p of each G_i.

Let d be a positive integer as in the conclusion of Proposition 3.1 corresponding to Λ. Let $(L^d)^z = B^{-1}(\Lambda^d)^z B$, where $L = B^{-1}\Lambda B$. Using Proposition 3.1, we conclude that $z \mapsto (L^d)^z$ is an analytic function whenever $z \in \mathbb{D}(0, p^{-1/(p-1)})$ (with values in $\mathrm{GL}(g, \mathbb{C}_p)$). Therefore, for each fixed vector $\vec{x_0} \in \mathbb{C}_p^g$,

(3.3.1)

the function $z \mapsto (L^d)^z \cdot \vec{x_0}$ is analytic whenever $z \in \mathbb{D}(0, p^{-1/(p-1)})$.

According to [Bou98, Proposition 3, p. 216] there exists a p-adic analytic map $\exp : \mathbb{C}_p^g \longrightarrow A(\mathbb{C}_p)$ which maps bijectively a sufficiently small neighborhood $\mathbb{D}(\vec{0}, r)$ of $\vec{0} \in \mathbb{C}_p^g$ onto a sufficiently small neighborhood of $0 \in A(\mathbb{C}_p)$. Because exp is a local isomorphism of the analytic groups $\mathbb{C}_p^g$ and $A(\mathbb{C}_p)$, we conclude that any endomorphism of A corresponds to an endomorphism of $(\mathbb{C}_p^g, +)$; thus there exists a linear map $\varphi : \mathbb{C}_p^g \longrightarrow \mathbb{C}_p^g$ such that

$$\Phi(\exp(z)) = \exp(\varphi(z)). \tag{3.3.2}$$

Computing the Jacobian at $\vec{0}$ in (3.3.2), we obtain that $D\varphi_{\vec{0}} = L$ because the embedding into $\mathbb{C}_p$ preserves the Jacobian L of Φ at 0. Therefore $\varphi(z) = L \cdot z$ for every $z \in \mathbb{C}_p^g$ (note that we know already that φ is a linear map).

After replacing α by $m \cdot \alpha$, for a positive integer m (as we may, by the remarks at the beginning of the proof), we may assume $\alpha \in \exp(\mathbb{D}(\vec{0}, r))$. To see this, we note that $\mathbb{D}(\vec{0}, r)$ is an additive subgroup because $|\cdot|_p$ is nonarchimedean, so its image in $A(\mathbb{Q}_p)$ is an open subgroup since exp is bijective and analytic on $D(\vec{0}, r)$. The fact that $\mathcal{A}(\mathbb{Z}_p)$ is compact means that any open subgroup of $\mathcal{A}(\mathbb{Z}_p)$ has finitely many cosets in $\mathcal{A}(\mathbb{Z}_p)$. Because $\alpha \in \mathcal{A}(\mathbb{Z}_p)$ there is a positive integer m such that $m\alpha \in \exp(\mathbb{D}(\vec{0}, r))$.

Let $\vec{\beta} \in \mathbb{D}(\vec{0}, r)$ such that $\exp(\vec{\beta}) = \alpha$. Since the coefficients of L are all p-adic integers, it follows that $L^n\vec{\beta}$ is in $\mathbb{D}(\vec{0}, r)$ for any $n \in \mathbb{N}$. Let $\vec{\beta_j} := L^j(\vec{\beta})$, for each $j = 0, \dots, 2pd - 1$.

The remainder of the argument now proceeds as in the proof of Theorem 1.6. Fix $j \in \{0, \dots, 2pd - 1\}$. Using (3.3.2) we obtain that for each $k \in \mathbb{N}$, we have

$$\Phi^{j+2pkd}(\alpha) = \exp\left((L^d)^{2pk} \cdot \vec{\beta_j}\right).$$

For each polynomial F in the vanishing ideal of V, we define the function $\Theta_{F,j}$ on $\overline{\mathbb{D}}(0,1)$ by

$$\Theta_{F,j}(z) = F\left(\exp\left((L^d)^{2pz} \cdot \vec{\beta_j}\right)\right).$$

Using (3.3.1) (along with the fact that $|2pz|_p < p^{-1/(p-1)}$ for any $|z|_p \le 1$), we conclude that $\Theta_{F,j}$ is analytic, and so, assuming that there are infinitely many $k \in \mathbb{N}$ such that $\Phi^{j+2pkd}(\alpha) \in V(\mathbb{C}_p)$, we see that $\Theta_{F,j}$ is identically equal to 0. This would mean that F vanishes on all points $\Phi^{j+2pkd}(\alpha)$ for $k \in \mathbb{N}$. We conclude that

$$\begin{aligned} &\text{either } \Phi^{j+2pkd}(\alpha) \in V(\mathbb{C}_p) \text{ for all } k \in \mathbb{N} \\ &\text{or } V(\mathbb{C}_p) \cap \mathcal{O}_{\Phi^{2pd}}(\Phi^j(\alpha)) \text{ is finite.} \end{aligned} \tag{3.3.3}$$

This concludes the proof of Theorem 1.8. ☐

As mentioned in Section 2, it should be possible to make effective Theorems 1.6 and 1.8. In particular, this should allow for the kinds of explicit computations that Flynn-Wetherell [FW99], Bruin [Bru03], and others performed in the context of rational points on curves of genus greater than one. It may also be possible to extend the proof of Theorem 1.8 to the case of any finite self-map of a nonsingular variety with trivial canonical bundle.

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