ON THE MEDVEDEV-SCANLON CONJECTURE FOR MINIMAL THREEFOLDS OF NON-NEGATIVE KODAIRA DIMENSION

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Abstract. Motivated by work of Zhang from the early '90s, Medvedev and Scanlon formulated the following conjecture. Let K be an algebraically closed field of characteristic 0 and let X be a quasiprojective variety defined over K endowed with a dominant rational self-map Φ. Then there exists a point $\alpha \in X(K)$ with Zariski dense orbit under Φ if and only if Φ preserves no nontrivial rational fibration, i.e., there exists no non-constant rational functions $f \in K(X)$ such that $\Phi^*(f) = f$. The Medvedev-Scanlon conjecture holds when K is uncountable. The case where K is countable (e.g., $K = \overline{\mathbb{Q}}$) is much more difficult; here the Medvedev-Scanlon conjecture has only been proved in a small number of special cases. In this paper we show that the Medvedev-Scanlon holds for all varieties of positive Kodaira dimension, and explore the case of Kodaira dimension 0. Our results are most complete in dimension 3.

Contents

1. Introduction

Consider a dominant rational self-map $\phi\colon X \dashrightarrow X$ of an irreducible variety X, defined over a field k. For an integer $n \geq 0$, we will denote by ϕ^n the n-th compositional power of ϕ. Given a point $\alpha \in X$, we define its orbit under ϕ (denoted $\mathcal{O}_\phi(\alpha)$) to be the set of all $\phi^n(\alpha)$ (as n ranges over the non-negative integers) whenever α is not in the indeterminacy locus for ϕ^n.

In this paper, we will prove the Medvedev-Scanlon conjecture for a large class of projective varieties X. This is a conjecture in arithmetic dynamics that predicts when there is a point in $X(\overline{\mathbb{Q}})$ with dense ϕ-orbit. Certainly, no such $\overline{\mathbb{Q}}$-point can exist if ϕ *preserves a rational fibration*, i.e. if there is a dominant rational map $\pi : X \dashrightarrow Y$ with $\dim Y > 0$ such that $\pi \circ \phi = \pi$. The Medvedev-Scanlon conjecture asserts that this necessary condition is also sufficient.

Conjecture 1.1 ([MS14, 7.14])**.** *Let X be an irreducible variety over an algebraically closed field K of characteristic 0 and $\phi\colon X \dashrightarrow X$ be a dominant rational self-map. If ϕ does not preserve a rational fibration, then there is a point $x \in X(K)$ with Zariski dense forward orbit under ϕ.*

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We note that the special case of Conjecture 1.1 when K is an uncountable field was earlier proved in [AC08, Theorem 4.1] (we also mention that Conjecture 1.1 was independently proven in [BRS10, Theorem 1.2] when ϕ is an automorphism and K is uncountable). Actually, the result from [AC08] (which is stated a bit more general, in the setting of Kähler manifolds), along with a much older conjecture of Zhang [Zha06, Conjecture 4.1.6] about Zariski dense orbits for polarizable endomorphisms motivated the above conjecture made by Medvedev and Scanlon in [MS14]. It is interesting to note that when K is uncountable, Conjecture 1.1 holds even when K has positive characteristic (see [BGR, Corollary 6.1]), and furthermore, over uncountable fields, it holds in a more general setting which allows one to replace ϕ by an arbitrary semigroup of dominant rational self-maps $X \dashrightarrow X$ (see [BGR, Theorem 1.2] which was itself motivated by finding a purely algebraic, characteristic-free proof of the result from [AC08]).

For the rest of the introduction we will assume that K is a countable algebraically closed field of characteristic 0 (e.g., $K = \bar{\mathbb{Q}}$). Here the Medvedev-Scanlon conjecture has only been proved in a few special cases, using subtle diophantine techniques.

In particular, Medvedev and Scanlon [MS14, Theorem 7.16] established Conjecture 1.1 for endomorphisms ϕ of $X = \mathbb{A}^m$ of the form $\phi(x_1, \ldots, x_m) = (f_1(x_1), \ldots, f_m(x_m))$, where $f_1, \ldots, f_m \in K[x]$. Their proof combines techniques from model theory, number theory and polynomial decomposition theory to obtain a complete description of all periodic subvarieties.

In the case where X is an abelian variety and $\phi \colon X \to X$ is dominant self-map, Conjecture 1.1 was proved in [GS]. The proof uses the explicit description of endomorphisms of an abelian variety and relies on the Mordell-Lang conjecture, due to Faltings [Fal94].

In the case where $\dim(X) \leq 2$ and $\phi \colon X \dashrightarrow X$ is a birational isomorphism, Conjecture 1.1 is established by Xie [Xie15]; we also note that recently, Xie [Xie, Theorem 1.1] proved Conjecture 1.1 for all polynomial endomorphisms of $\mathbb{A}^2$. We remark that in [Xie15, Theorem 1.4], the result for birational automorphisms ϕ of surfaces is stated under the additional assumption that the first dynamical degree of ϕ is greater than 1. However, the same proof goes through without this assumption. We will not use [Xie15, Theorem 1.4] in this paper, but we will appeal to the case of regular automorphisms of surfaces, which was settled earlier in [BGT15, Theorem 1.3]. These results are proved by p-adic techniques, in particular, the so-called p-adic arc lemma. For details on the p-adic arc lemma and its applications we refer the reader to [BGT16, Chapter 4].

Finally, we note that in [ABR11], also employing p-adic techniques, it is proven the validity of Conjecture 1.1 when ϕ satisfies some very special conditions. More precisely, it is shown that if ϕ has a fixed point x in $X(\bar{\mathbb{Q}})$ such that the induced map ϕ^* on the tangent space T_x has multiplicatively independent eigenvalues, then there is a point in $X(\bar{\mathbb{Q}})$ whose orbit under ϕ is Zariski dense in X. As noted in [ABR11], it is difficult to find interesting examples of self-maps ϕ satisfying these hypotheses; in other words, given any variety X endowed with a self-map $\phi : X \dashrightarrow X$ which does not preserve a fibration, one cannot expect that ϕ satisfies the hypothesis regarding the multiplicative independence of the eigenvalues at one of its fixed points.

In this paper we will explore Conjecture 1.1 in the case where $\phi \colon X \dashrightarrow X$ is a birational automorphism and $\dim(X) \geq 3$ by using techniques of higher-dimensional algebraic geometry. Our first main result settles the Medvedev-Scanlon conjecture for birational self-maps of varieties of positive Kodaira dimension.

Theorem 1.2. *If X is an irreducible projective variety of Kodaira dimension $\kappa(X) > 0$ defined over $\overline{\mathbb{Q}}$ and $\phi \colon X \dashrightarrow X$ is a birational self-map, then ϕ preserves a rational fibration. In particular, the Medvedev-Scanlon Conjecture 1.1 is vacuously true in this case.*

Our next result shows that if X is a smooth minimal model with $\kappa(X) = 0$, then assuming standard conjectures in the minimal model program, Conjecture 1.1 can be reduced to products of special kinds of varieties: Calabi-Yau, hyperkähler, and abelian varieties.

Recall that a smooth projective variety X over $\overline{\mathbb{Q}}$ is called *hyperkähler* if its complex analytification is simply connected and $H^0(\Omega_X^2)$ is spanned by a symplectic form. In dimension 2, hyperkähler varieties are nothing more than K3 surfaces.

We use the convention that a smooth projective variety of dimension ≥ 3 defined over $\overline{\mathbb{Q}}$ is *Calabi-Yau* if the complex analytification $X_{\mathbb{C}}$ is simply connected, $K_X \simeq \mathcal{O}_X$, and $H^p(\mathcal{O}_X) = 0$ for $0 < p < \dim X$. Since we are working over $\overline{\mathbb{Q}}$, by the symmetry of the Hodge diamond, this latter condition is equivalent to requiring $H^0(\Omega_X^p) = 0$ for $0 < p < \dim X$.

Our next result relies on the abundance conjecture, which is known for curves, surfaces, and threefolds. We state it in the form that we need, although the conjecture itself is more general.

Conjecture 1.3 (Abundance [KM98, Corollary 3.12])**.** *If X is a smooth projective minimal variety of Kodaira dimension 0, then K_X is numerically trivial.*

Remark 1.4 (For those unfamiliar with MMP). For readers more familiar with diophantine geometry and less familiar with the techniques of the minimal model program, we emphasize that assuming the abundance conjecture is akin to assuming the generalized Riemann hypothesis. Although the result is not yet known in higher dimension, it is largely expected that the conjecture is true.

Theorem 1.5. *Fix an integer $n \geq 1$. Assuming the Abundance Conjecture 1.3, the Medvedev-Scanlon Conjecture 1.1 holds for birational self-maps of smooth projective minimal n-folds over $\overline{\mathbb{Q}}$ of Kodaira dimension 0 if and only if it holds for those n-folds of the form $A \times \prod_i Y_i \times \prod_j Z_j$, where A is an abelian variety, the Y_i are Calabi-Yau, and the Z_j are hyperkähler.*

In the case of threefolds, we unconditionally reduce to the case of Calabi-Yau varieties.

Theorem 1.6. *The Medvedev-Scanlon Conjecture 1.1 holds for birational self-maps of smooth projective minimal threefolds over $\overline{\mathbb{Q}}$ of Kodaira dimension* 0 *if and only if it holds for smooth Calabi-Yau threefolds.*

Finally, we handle the case of Calabi-Yau threefolds, contingent on conjectures in the minimal model program. Via the intersection product, the second Chern class $c_2(X)$ defines a linear form on the nef cone $\mathrm{Nef}(X)$. Miyaoka [Miy87] shows that this linear form always assumes non-negative values on the nef cone. We separately consider the cases where $c_2(X)$ is strictly positive and where it is not.

Theorem 1.7. *Let X be a smooth projective Calabi-Yau threefold over $\overline{\mathbb{Q}}$. Then the Medvedev-Scanlon Conjecture 1.1 holds for all (regular) automorphisms $\phi \colon X \to X$ if either:*

(1) *$c_2(X)$ is positive on* $\mathrm{Nef}(X)$*, or*
(2) *there is a semi-ample divisor $D \neq 0$ on X such that $c_2(X) \cdot D = 0$.*

Here by "divisor" we mean that D is an integral point of $\mathrm{Nef}(X)$, i.e., D is the linear combination of classes of codimension 1 irreducible subvarieties of X with integer coefficients. Note also that here $c_2(X) \neq 0$. Indeed, otherwise there would exist a finite étale cover $A \to X$, where A is an abelian variety. Since we are assuming that X is simply connected, this cannot happen.

Remark 1.8 (Concerning the hypothesis in Theorem 1.7 (2)). If the hypothesis in Theorem 1.7 (1) fails, then as mentioned above, Miyaoka's theorem implies $Z := c_2(X)^{\perp} \cap \mathrm{Nef}(X)$ is a non-zero face of $\mathrm{Nef}(X)$. A priori, Z could be irrational. If Z contains a non-zero rational class D, then the semi-ampleness conjecture [LOP, Conjecture 2.1] implies that some scalar multiple mD is a semi-ample divisor, and so the hypothesis in (2) holds.

Thus, assuming the semi-ampleness conjecture, the only Calabi-Yau varieties X that Theorem 1.7 does not apply to are those for which Z is non-zero and contains no non-zero rational classes. If [Ogu01, Question-Conjecture 2.6] of Oguiso is true over $\overline{\mathbb{Q}}$, then this situation never occurs when the Picard number $\rho(X)$ is sufficiently large.

In light of Remark 1.8, we have the following result.

Corollary 1.9. *If the semi-ampleness conjecture* [LOP, Conjecture 2.1] *and* [Ogu01, Question-Conjecture 2.6] *are true over $\overline{\mathbb{Q}}$, then the Medvedev-Scanlon Conjecture 1.1 is true for all automorphisms of smooth minimal threefolds of non-negative Kodaira dimension and sufficiently large Picard number.*

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2. The case of positive Kodaira dimension: proof of Theorem 1.2

We begin with two useful lemmas.

Lemma 2.1. *In order to prove Conjecture 1.1 for the dynamical system (X, ϕ), it is sufficient to prove Conjecture 1.1 for an iterate (X, ϕ^m), for some $m \in \mathbb{N}$.*

Proof. It is clear that if ϕ^m has a Zariski dense orbit, then so does ϕ.

It remains to show that if ϕ does not preserve a nonconstant fibration, then neither does ϕ^m. Indeed, suppose there exists a nonconstant $f \in K(X)$ such that $(\phi^m)^*(f) = f$. Then ϕ preserves the symmetric function g_i in the rational functions $f, \phi^*(f), \ldots, (\phi^{m-1})^*(f)$, for each $i = 1, \ldots, m$. Since f is nonconstant, then at least one of $g_1, \ldots, g_m$ is non-constant. In other words, there exists a non-constant function g_i which is nonconstant and thus fixed by ϕ^*, as desired. □

Lemma 2.2. *Let $\phi : X \dashrightarrow X$ be a birational automorphism defined over a field k. Let K be an uncountable algebraically closed field containing k. Then the following conditions are equivalent:*

(1) $k(X)^\phi = k$,

(2) There exists a K-point $x \in X(K)$ such that the orbit $\{\phi^n(x) \,|\, n = 0, 1, 2, \ldots\}$ is dense in X.

(3) $K(X)^\phi = K$.

Proof. The implication (1) $\Longrightarrow$ (2) follows from [BGR, Theorem 1.2].

The remaining implications (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are obvious. □

Proof of Theorem 1.2. The theorem asserts that $k(X)^\phi \neq k$. By Lemma 2.2, we may also assume that $k = \mathbb{C}$ is the field of complex numbers.

Note that we may replace X by a birationally equivalent variety; this does not change $\mathbb{C}(X)$ or $\mathbb{C}(X)^\phi$. After resolving the singularities of X, we may also assume that X is smooth.

Next, consider the Iitaka fibration, i.e. the rational map $f : X \dashrightarrow \mathbb{P}^N$ defined by the complete linear system $|mK_X|$ for m sufficiently divisible. The image is a projective variety Y of dimension $\kappa(X)$. Since $\phi^* K_X^{\otimes m} \simeq K_X^{\otimes m}$, we have an induced action $\overline{\phi}$ on $\mathbb{P}^N$ such that $f \circ \phi = \overline{\phi} \circ f$. By a theorem of Deligne and Ueno [Uen75, Thm 14.10], the image of the m-th pluricanonical representation $\rho_m : \mathrm{Bir}(X) \to \mathrm{GL}(H^0(mK_X))$ is a finite group. Thus, after replacing ϕ by an iterate, we may assume that $\rho_m(\phi) = \mathrm{id}$; that is, $\overline{\phi}$ is the identity on $\mathbb{P}^N$. So, ϕ preserves a rational fibration, as claimed. □

3. The Beauville-Bogomolov decomposition theorem over $\overline{\mathbb{Q}}$

We now recall the Beauville-Bogomolov decomposition theorem. Suppose X is a smooth complex projective variety with numerically canonical divisor K_X. Beauville [Bea83, p. 9] defines $\pi : \widetilde{X} \to X$ to be a *minimal split cover* if it is a finite étale Galois cover, $\widetilde{X} \simeq A \times S$, where A is an abelian variety and S is simply connected, and there is no non-trivial element of the Galois group that simultaneously acts as translation on A and the identity on S. The main theorem together with

Proposition 3 of [Bea83] show that every such X has a minimal split covering and that it is unique up to non-unique isomorphism.

In the sequel we will need a variant of the Beauville-Bogomolov decomposition theorem [Bea83] over $\overline{\mathbb{Q}}$. For lack of a suitable reference, we will prove it below.

Proposition 3.1. *Let X be a smooth projective minimal variety over $\overline{\mathbb{Q}}$ with K_X numerically trivial. Then there exists a finite étale Galois cover $\widetilde{X} \to X$ defined over $\overline{\mathbb{Q}}$ such that*

(1) *$\widetilde{X} = A \times \prod_i Y_i \times \prod_j Z_j$, where A is an abelian variety, the Y_i are Calabi-Yau, and the Z_j are hyperkähler,*
(2) *no element of the Galois group acts simultaneously as translation on A and the identity on all of the Y_i and Z_j.*

Proof. The Beauville-Bogomolov decomposition theorem tells us that there is a finite group G and a G-torsor $T \to X_{\mathbb{C}}$ with $T = A \times \prod_i Y_i \times \prod_j Z_j$, where A is an abelian variety, the Y_i are Calabi-Yau varieties, and the Z_j are hyperkähler varieties. By a standard limit argument, there exists a finitely generated field extension $K/\overline{\mathbb{Q}}$ so that we can descend $T \to X_{\mathbb{C}}$ to a G-torsor $T' \to X_K$, the abelian variety A to an abelian variety A' over K, and the Y_i (resp. Z_j) to smooth proper K-schemes Y_i' (resp. Z_j'). Moreover, after possibly enlarging K, we can descend the isomorphism $T \simeq A \times \prod_i Y_i \times \prod_j Z_j$ to an isomorphism $T' \simeq A' \times \prod_i Y_i' \times \prod_j Z_j'$. Since $H^0(\Omega^p_{Y_i'}) \otimes_K \mathbb{C} = H^0(\Omega^p_{Y_i})$, we have $H^0(\Omega^p_{Y_i'}) = 0$ for $0 < p < \dim Y_i'$. By similar reasoning, we see $H^0(\Omega^2_{Z_j'})$ is 1-dimensional and that $K_{Y_i'} \simeq \mathcal{O}_{Y_i'}$; the latter statement can be proved by using the fact that a line bundle $\mathcal{L}$ on a projective variety is trivial if and only if $H^0(\mathcal{L})$ and $H^0(\mathcal{L}^\vee)$ are both non-zero. Choosing a generator $\omega_j \in H^0(\Omega^2_{Z_j'})$, we have an induced map $T_{Z_j'} \to \Omega^1_{Z_j'}$ and non-degeneracy of ω_j is equivalent to this map being an isomorphism. Since this is true after a field extension from K to $\mathbb{C}$, it is true over K.

Next, let V be a smooth $\overline{\mathbb{Q}}$-variety with function field K. After possibly shrinking V, we can extend $T' \to X_K$ to a G-torsor $T'' \to X_V$, extend A' to an abelian scheme $A'' \to V$, Y_i' and Z_j' to smooth proper V-schemes Y_i'' and Z_j'', and can assume $T'' \simeq A'' \times \prod_i Y_i'' \times \prod_j Z_j''$ over V. Let $\pi_i : Y_i'' \to V$ and $\psi_j : Z_j'' \to V$ be the structure maps. After suitably shrinking V, we can assume $(\pi_i)_* \Omega^p_{Y_i''/V} = 0$ for $0 < p < \dim Y_i''$, that $(\psi_j)_* \Omega^2_{Z_j''/V} \simeq \mathcal{O}_V$, and that there is a non-vanishing section ω_j of $(\psi_j)_* \Omega^2_{Z_j''/V}$ whose induced map $T_{Z_j''/V} \to \Omega^1_{Z_j''/V}$ is an isomorphism.

Finally, we show that for all maps $t : \operatorname{Spec} \mathbb{C} \to V$, the complex analytifications of the $(Y_i'')_t$ and $(Z_j'')_t$ are simply connected. First note that by the Beauville-Bogomolov decomposition theorem, these varieties have virtually abelian fundamental groups; specifically, if W denotes one of these varieties, then there is a finite Galois cover $A \times S \to W$ with A an abelian variety and S simply connected, so $\pi_1(W)$ contains $\pi_1(A) \simeq \mathbb{Z}^r$ as a finite index subgroup. Next, note that if the étale fundamental group $\pi_1^{et}(W)$ is trivial, then so is $\pi_1(W)$. Indeed, if $\pi_1^{et}(W) = 0$, then $r = 0$, so $\pi_1(W)$ is finite and therefore, $\pi_1(W) = \pi_1^{et}(W) = 0$. Thus, it suffices to prove that for every geometric point $\overline{v}$ of V, the étale fundamental groups $\pi_1^{et}((Y_i'')_{\overline{v}})$ and $\pi_1^{et}((Z_j'')_{\overline{v}})$ are trivial. Since the étale fundamental groups of the geometric generic fibers $(Y_i'')_{\overline{\eta}} = Y_i$ and $(Z_j'')_{\overline{\eta}} = Z_j$ are trivial, this follows immediately from specialization results of the étale fundamental group [StPrj, Proposition 0C0Q].

Now, choosing any $\overline{\mathbb{Q}}$-point $v \in V$ gives our desired G-torsor of $T_v'' \to X$. $\square$

4. Proof of Theorem 1.5

In this section we will prove Theorem 1.5. The key ingredients of the proof are supplied by Lemma 4.1 and Proposition 4.3 below.

Lemma 4.1. *Consider the commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, dashed, "\psi"] & Y, \end{tikzcd}$$

where $\pi\colon X \to Y$ is a dominant morphism of irreducible varieties, ϕ and ψ are birational isomorphisms of X and Y, respectively, and the entire diagram is defined over $\overline{\mathbb{Q}}$. Further suppose that $\dim(X) = \dim(Y)$ *and $\overline{\mathbb{Q}}(X)^\phi = \overline{\mathbb{Q}}$. Then*

(a) $\overline{\mathbb{Q}}(Y)^\psi = \overline{\mathbb{Q}}$.

In parts (b) and (c), assume further that $\pi\colon X \to Y$ is a G-torsor for some finite smooth group scheme G.

(b) If ϕ is regular at $x \in X$, then ψ is regular at $y := \pi(x) \in Y$.

(c) If the Medvedev-Scanlon conjecture holds for X, then there exists a point $y \in Y(\overline{\mathbb{Q}})$ whose ψ-orbit is dense in Y.

Proof. (a) follows from Lemma 2.2 using the embedding of $\mathbb{C}(Y)$ into $\mathbb{C}(X)$ through the induced map π^*.

(b) The composition $\pi \circ \phi\colon X \dashrightarrow Y$ is a G-invariant rational map which is regular at x. Hence, it descends to a rational map $Y \dashrightarrow Y$ which is regular at y. Clearly, this map coincides with ψ. In other words, ψ is regular at y, as claimed.

(c) Since the Medvedev-Scanlon Conjecture holds for ϕ, there exists a point $x \in X(\overline{\mathbb{Q}})$ such that the ϕ-orbit of x is dense in X. Using part (b) for each iterate of ϕ, we conclude that for each $n \in \mathbb{N}$ such that ϕ^n is defined at x, we have that ψ^n is defined at $y := \pi(x)$. Furthermore, since the orbit of x under ϕ is dense in X, we conclude that also the orbit of y under ψ is dense in Y. $\square$

Remark 4.2. Let X be a minimal threefold with K_X torsion. If ϕ is a birational automorphism of X, then as Lazić shows in [Laz13, p. 197] between Remarks 6.1 and 6.2, ϕ is a pseudo-automorphism, i.e. neither ϕ nor ϕ^{-1} contracts a divisor.

Proposition 4.3. *Let X be a smooth projective minimal variety over $\overline{\mathbb{Q}}$ with K_X numerically trivial, and let $\pi : \widetilde{X} \to X$ be a minimal split cover provided by Proposition 3.1. Then for every birational automorphism ϕ of X over $\overline{\mathbb{Q}}$, there exists a birational automorphism $\widetilde{\phi}$ of $\widetilde{X}$ over $\overline{\mathbb{Q}}$ such that $\pi \circ \widetilde{\phi} = \phi \circ \pi$.*

Proof. We know that $\pi\colon \widetilde{X} \to X$ is a G-torsor for some finite étale group scheme G, and that $\widetilde{X} = A \times S$ with S simply connected and A an abelian variety. Since X is smooth, ϕ is regular on an open subset $U \subseteq X$ with $X \smallsetminus U$ having codimension at least 2. Consider the Cartesian diagram

$$\begin{tikzcd} \widetilde{X} \times_X U \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "\pi"] \\ U \arrow[r, "\phi|_U"] & X \end{tikzcd}$$

Since $X \smallsetminus U$ has codimension at least 2, by [Ols12, Proposition 3.2], the G-torsor $\widetilde{X} \times_X U \to U$ extends uniquely to a G-torsor $\pi' : \widetilde{X}' \to X$. We therefore have a commutative diagram

$$\begin{tikzcd} \widetilde{X}' \arrow[r, dashed, "\widetilde{\phi}"] \arrow[d, "\pi'"] & \widetilde{X} \arrow[d, "\pi"] \\ X \arrow[r, dashed, "\phi"] & X \end{tikzcd}$$

So, to finish the proof, it suffices to show that $\widetilde{X}'$ is split, i.e. the product of an abelian variety and a simply connected variety; indeed, if $\widetilde{X}'$ is split, then by [Bea83, Proposition 3], there exists a map $\alpha\colon \widetilde{X}' \to \widetilde{X}$ such that $\pi' = \pi \circ \alpha$. Then by degree considerations, α must be an isomorphism and so $\widetilde{\phi} \circ \alpha$ is the birational map whose existence we asserted in the statement of the proposition.

The rest of the proof is devoted to showing that $\widetilde{X}'$ is split. Since π' is étale, we see $(\pi')^*K_X = K_{\widetilde{X}'}$ and so $K_{\widetilde{X}'}$ is numerically trivial. Thus, by Proposition 3.1, there is a minimal split cover $p'\colon Y' \to \widetilde{X}'$ defined over $\overline{\mathbb{Q}}$. After replacing p' by a further étale cover, we can assume that $\pi \circ p'$ is Galois with group Γ, although now p' is merely a split covering instead of a minimal split covering. Since Y' is split, we know $Y' = B' \times T'$ with T' simply connected and B' an abelian variety. Let H be the Galois group of Y' over $\widetilde{X}'$.

Mimicking the argument in the first paragraph of the proof, we obtain a diagram

$$\begin{tikzcd}
Y' \arrow[r, dashed, "\psi"] \arrow[d, "p'"] & Y \arrow[d, "p"] \\
\widetilde{X}' \arrow[r, dashed, "\widetilde{\phi}"] \arrow[d, "\pi'"] & \widetilde{X} \arrow[d, "\pi"] \\
X \arrow[r, dashed, "\phi"] & X
\end{tikzcd}$$

where p is an H-torsor and $\pi \circ p$ is a Γ-torsor. Indeed, by Remark 4.2, there exist open subsets U and V of X whose complements have codimension at least 2 and such that $\phi|_U\colon U \to V$ is an isomorphism. Pulling back $Y'|_U \to \widetilde{X}'|_U$ via the isomorphism $\widetilde{X}' \smallsetminus \widetilde{X}'|_U$, we obtain a Cartesian diagram

$$\begin{tikzcd}
Y' \arrow[d, "p'"] & \supseteq & Y'|_U \arrow[r, "\simeq"] \arrow[d] & Z \arrow[d] & & \\
\widetilde{X}' \arrow[d, "\pi'"] & \supseteq & \widetilde{X}'|_U \arrow[r, "\simeq"] \arrow[d] & \widetilde{X}|_V \arrow[d] & \subseteq & \widetilde{X} \arrow[d, "\pi"] \\
X & \supseteq & U \arrow[r, "\simeq"] & V & \subseteq & X
\end{tikzcd}$$

where the map $Z \to \widetilde{X}|_V$ is an H-torsor. Since π is étale, hence codimension preserving, $\widetilde{X} \smallsetminus \widetilde{X}|_V$ has codimension at least 2. By [Ols12, Proposition 3.2], the H-torsor $Z \to \widetilde{X}|_V$ extends uniquely to an H-torsor $p\colon Y \to \widetilde{X}$. Since $X \smallsetminus U$ has codimension at least 2, another application of [Ols12, Proposition 3.2] shows that $\pi \circ p$ is a Γ-torsor.

Since Y is a finite étale cover of $\widetilde{X} = A \times S$, we necessarily have $Y = B \times T$ with B an abelian variety and T simply connected. Moreover, since $\widetilde{X}$ is the minimal split covering of X, the proof of [Bea83, Proposition 3] tells us that the H-action on Y realizes H as the normal subgroup of elements in Γ acting simultaneously as translation on B and the identity on T. As a result, $\widetilde{X} = (B/H) \times T$, so p induces isomorphisms $T \simeq S$ and $B/H \simeq A$.

To finish the proof, it suffices to show that H acts on Y' as translation on B' and the identity on T'. Indeed, provided we can show this, we then know that $\widetilde{X}' = (B'/H) \times T'$, hence it is split as desired. To prove that H acts on Y' as stated, we compare it with the H-action on $Y = B \times T$, which we already know acts as translation on B and the identity on T. Since ψ is an H-equivariant map by construction, it induces an H-equivariant birational map $\overline{\psi} : B' \dashrightarrow B$ on Albanese varieties. Every rational map of abelian varieties is regular, so $\overline{\psi}$ is in fact an isomorphism. Moreover, after suitable choice of origin, it respects the group structure. Given $\gamma \in H$, we know it acts on B as translation t_z by some z, so γ acts on B as $\overline{\psi}^{-1} t_z \overline{\psi}$ which is translation by $\overline{\psi}^{-1}(z)$. Now, choosing a general point $b \in B'$, ψ induces a birational map on fibers $T' = Y'_b \dashrightarrow Y_{\psi(b)} = T$ that commutes with the H-action. Since each $\gamma \in H$ acts as the identity on T, we see that the automorphism $\gamma\colon T' \to T$ agrees with the identity map on a dense open. As a result, it is the identity map, which proves our desired claim. □

Proof of Theorem 1.5. Let X be a smooth projective minimal variety over $\overline{\mathbb{Q}}$ of Kodaira dimension 0, and let ϕ be a birational automorphism of X. The Abundance Conjecture 1.3 tells us that K_X is numerically trivial. Then by Proposition 3.1, there exists a minimal split cover $\pi : \widetilde{X} \to X$ defined over $\overline{\mathbb{Q}}$. By Proposition 4.3, ϕ lifts to a birational automorphism $\widetilde{\phi}$ of $\widetilde{X}$. By Lemma 4.1 (c), it is then enough to show that Medvedev-Scanlon holds for $\widetilde{\phi}$. □

5. Proof of Theorem 1.6

Our proof will rely on the following lemma.

Lemma 5.1. *Consider the commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "\psi"] & Y, \end{tikzcd}$$

where $\pi\colon X \to Y$ is a dominant morphism of irreducible varieties, ϕ is birational isomorphisms of X, ψ is an automorphism of Y, and the entire diagram is defined over $\overline{\mathbb{Q}}$. Suppose $\overline{\mathbb{Q}}(X)^{\phi} = \overline{\mathbb{Q}}$ (and hence, $\overline{\mathbb{Q}}(Y)^{\phi} = \overline{\mathbb{Q}}$; see Lemma 4.1(a)), and there exists a $y \in Y(\overline{\mathbb{Q}})$ whose ψ-orbit is dense in Y. Assume further that either (a) π is birational or (b) ϕ is a (regular) automorphism and $\dim(X) = \dim(Y) + 1$*. Then there exists an $x \in X(\overline{\mathbb{Q}})$ whose ϕ-orbit is dense in X.*

Proof. (a) Suppose π restricts to an isomorphism between dense open subsets X_0 of X and Y_0 of Y. After replacing y by an iterate, we may assume that $y \in Y_0$. We claim that the preimage $x \in X_0$ of y has a dense ϕ-orbit in X. Indeed, set $y_n := \psi^n(y) \in Y$. Then there is a sequence $i_1 \leqslant i_2 \leqslant \dots$ such that the points $y_{i_1}, y_{i_2}, \dots$, all lie in Y_0 are are dense in Y. Then $x_n := \phi^n(x)$ are well defined for $n = i_1, i_2, \dots$ and are dense in X. This proves the claim.

(b) By [BRS10, Theorem 1.2], X has only finitely many ϕ-invariant codimension 1 subvarieties. Denote their union by $H \subset X$. Once again, set $y_n := \psi^n(y) \in Y$. The union of the fibers $\pi^{-1}(y_n)$, as n ranges over the non-negative integers, is dense in X. Hence, one of these fibers is not contained in H. After replacing y by an iterate, we may assume that $\pi^{-1}(y) \not\subset H$. Choose a $\overline{\mathbb{Q}}$-point $x \in \pi^{-1}(y)$ which does not lie in H. We claim that the ϕ-orbit of x is dense in X. Indeed, denote Zariski closure of the orbit of x by Z. By our construction $\pi(Z)$ contains the ψ-orbit of y and thus is dense in Y. Hence, $\dim(Y) \leqslant \dim(Z) \leqslant \dim(X) = \dim(Y) + 1$. On the other hand, since $x \notin H$, Z cannot be a hypersurface in Y. Thus $\dim(Z) = \dim(X) = \dim(Y) + 1$, i.e., $Z = X$, as desired. □

We now proceed with the proof of Theorem 1.6. Since the abundance conjecture is known for threefolds [Kaw92], we can apply Theorem 1.5. Thus, the Medvedev-Scanlon Conjecture 1.1 holds for all smooth projective minimal threefolds of Kodaira dimension 0 if and only if it holds for products of Calabi-Yau varieties, hyperkähler varieties, and abelian varieties over $\overline{\mathbb{Q}}$. We are therefore reduced to three possibilities: (i) X is an abelian threefold, (ii) X is a product $E \times S$, where E is an elliptic curve and S is a K3 surface, or (iii) X is a smooth Calabi-Yau 3-fold. The Medvedev-Scanlon conjecture holds in case (i) by [GS]. The main result of this section, Proposition 5.3, asserts that Conjecture 1.1 also holds in case (ii). This will leave us with case (iii), thus completing the proof of Theorem 1.6.

Lemma 5.2. *Suppose $X = E \times S$, where E an elliptic curve and S is a smooth minimal surface with trivial Albanese and $\kappa(S) \geq 0$. Every birational isomorphism $\phi\colon X \dashrightarrow X$ is of the form $\phi = \phi_E \times \phi_S$ with ϕ_E an automorphism of E and ϕ_S an automorphism of S. In particular, every birational isomorphism of X is regular.*

Proof. The projection $\pi : X \to E$ is the Albanese map for X. Thus ϕ induces a birational automorphism ϕ_E of E such that $\pi \circ \phi = \phi_E \circ \pi$. Since E is a smooth curve, ϕ_E is an automorphism of E. Replacing ϕ by $\phi \circ (\phi_E^{-1}, \mathrm{id}_S)$, we see that to prove the lemma, we may assume $\phi_E = \mathrm{id}_E$.

Since X is smooth, the indeterminacy locus $I(\phi)$ of ϕ has codimension at least 2, and so $I(\phi) \cap X_t$ has codimension at least 1 for all $t \in E$. We therefore obtain a map $f\colon E \to \mathrm{Bir}(S)$ given by $t \mapsto \phi|_{X_t}$. Since $\kappa(S) \geq 0$, S is not ruled, so S is a unique smooth minimal surface in its birational class, and $\mathrm{Bir}(S) = \mathrm{Aut}(S)$, see for example [Bea96, Theorem V.19]. Our goal is to show that the resulting map $f\colon E \to \mathrm{Aut}(S)$ is constant. Choose a point $t_0 \in E$ and let $\sigma := f(t_0) \in \mathrm{Aut}(S)$. After composing ϕ with $(1, \sigma^{-1})\colon E \times S \to E \times S$, we may assume that $f(t_0) = 1 \in \mathrm{Aut}(S)$. Since E is irreducible, this implies that the image of f lies in $\mathrm{Aut}^0(S)$. Since S has trivial Albanese, by [Fuj78, Corollary 5.8], $\mathrm{Aut}^0(S)$ is an affine algebraic group. Thus, f must be a constant map, as claimed. We now define ϕ_S to be the image of this map. $\square$

Proposition 5.3. *Suppose $X = E \times S$, were E an elliptic curve and S is a surface with trivial Albanese and $\kappa(S) \geq 0$. Let $\phi\colon X \dashrightarrow X$ be a birational isomorphism such that $\overline{\mathbb{Q}}(X)^{\phi} = \overline{\mathbb{Q}}$. Then Conjecture 1.1 holds for (X, ϕ).*

Proof. Let $\pi\colon S \to S_{min}$ be the minimal model of S. By Lemma 5.2, ϕ descends to an automorphism $E \times S_{min} \to E \times S_{min}$ of the form (ϕ_E, ϕ_{min}), where ϕ_E is an automorphism of E and ϕ_{min} is an automorphism of S_{min}. Now consider the commutative diagram

$$\begin{tikzcd}
E \times S \arrow[r, dashed, "\phi"] \arrow[d, "\mathrm{id}\times\pi"'] & E \times S \arrow[d, "\mathrm{id}\times\pi"'] \\
E \times S_{min} \arrow[r, "\phi_E \times \phi_{min}"] \arrow[d, "\mathrm{pr}"'] & E \times S_{min} \arrow[d, "\mathrm{pr}"'] \\
S_{min} \arrow[r, "\phi_{min}"] & S_{min},
\end{tikzcd}$$

By [BGT15, Theorem 1.3] the Medvedev-Scanlon conjecture holds for the automorphism ϕ_{min} of the surface S_{min}. By Lemma 5.2(b), $E \times S_{min}$ has a $\overline{\mathbb{Q}}$-point with a dense (ϕ_E, ϕ_{min})-orbit. Applying Lemma 5.2(a), we conclude that the automorphism $E \times S$ has a $\overline{\mathbb{Q}}$-point with a dense ϕ-orbit, as desired. $\square$

6. Pseudo-automorphisms that preserve a line bundle

The following result will be used in the proof of Theorem 1.7 in the next section.

Proposition 6.1. *Suppose $\phi\colon X \dashrightarrow X$ is pseudo-automorphism of a smooth projective variety defined over a field k of characteristic 0, L is a line bundle such that $\phi^*(L) \simeq L$, and Y is the closure of the image of the natural rational map $i\colon X \dashrightarrow \mathbb{P}H^0(X, L)$ for large n. Here, as usual $H^0(X, L)$ denotes the finite-dimensional space of global sections of L, and $\mathbb{P}H^0(X, L)$ is the associated projective space. Then*

(a) ϕ induces a linear automorphism $\bar{\phi}$ of the projective space $\mathbb{P}H^0(X, L)$ preserving Y.

Moreover, assume $k(X)^{\phi} = k$. Then

(b) there is a dense $\bar{\phi}$-invariant subset U of Y such that the $\bar{\phi}$-orbit of y is dense in Y for every $y \in U$,

(c) Y is a rational variety over the algebraic closure $\overline{k}$.

Note that since ϕ is a pseudo-automorphism, it induces an automorphism $\phi^*\colon \operatorname{Pic}(X) \to \operatorname{Pic}(X)$.

Proof. (a) We begin with the following preliminary observation. Suppose L and L' are isomorphic line bundles on a complete variety X defined over k. We claim that there is a canonically defined linear isomorphism between the finite-dimensional projective spaces $\mathbb{P}H^0(X, L)$ and $\mathbb{P}H^0(X, L')$. To define this linear isomorphism, write $L = \mathcal{O}_X(D)$ and $L' = O_X(D')$, where D and D' are divisors on X. Since L and L' are isomorphic, these divisors are linearly equivalent. That is,

$$D' = D + (f), \tag{6.2}$$

where (f) denotes the divisor associated to a rational function $f \in k(X)$. Once f is chosen, we can define an isomorphism of vector spaces $H^0(X, L) \to H^0(X, L')$ given by $\alpha \mapsto f\alpha$. The rational function f in (6.2) is uniquely determined by L and L' up to a non-zero scalar factor. The isomorphism of projective spaces $\mathbb{P}H^0(X, L) \to \mathbb{P}H^0(X, L')$ thus defined depends only on L and L' and not on the choice of f. This proves the claim.

We now apply this claim in the setting of the proposition, with $L' := \phi^*(L)$. The line bundles L and L' are isomorphic by our assumption. On the other hand, ϕ induces an isomorphism

$$\phi^* : H^0(X, L) \to H^0(X, L')$$

via pull-back. Composing with the inverse of the linear isomorphism $\mathbb{P}H^0(X, L) \to \mathbb{P}H^0(X, L')$ constructed above, we obtain a desired automorphism $\bar{\phi}\colon \mathbb{P}H^0(X, L) \to \mathbb{P}H^0(X, L)$ such that the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d, dashed, "i"'] & X \arrow[d, dashed, "i"] \\ \mathbb{P}H^0(X, L) \arrow[r, "\bar{\phi}"] & \mathbb{P}H^0(X, L) \end{tikzcd}$$

commutes.

(b) Let Y be the closure of image of X in $\mathbb{P}(V)$ under i, where $V := H^0(X, L)$. Since $k(X)^\phi = k$, clearly $k(Y)^\phi = k$ as well.

Set G to be the subgroup of $\operatorname{PGL}(V)$ consisting of automorphisms of $\mathbb{P}(V)$ which preserve Y. Then $\bar{\phi} \in G$, and G is a closed subgroup of $\operatorname{PGL}(V)$ and hence, a linear algebraic group. Let G_0 be the Zariski closure of the subgroup generated by $\bar{\phi}$ inside G. Then G_0 is an abelian linear algebraic group. Moreover, for any $y \in Y$, the orbit of y under ϕ has the same closure in Y as the orbit of y under G_0. So, it suffices to show that there is a dense open subset $U \subset Y$ such that every $y \in U$ has a dense orbit under G_0. The last assertion is a consequence of Rosenlicht's theorem; see [Ro56, Theorem 2], *cf.* also [BGR, Theorem 1.1] and [BRS10, Proposition 7.4(1)]; in fact, we can take U to be a dense G_0-orbit in Y.

(c) Since U is a G_0-orbit, it is isomorphic to the homogeneous space G_0/H_0, for some subgroup $H_0 \subset G_0$. Since G_0 is abelian, H_0 is normal in G_0. Hence, as a variety, U is isomorphic to the abelian linear algebraic group G_0/H_0. Every abelian linear irreducible algebraic group over $\overline{k}$ is isomorphic to a direct product of copies of $\mathbb{G}_a$ and $\mathbb{G}_m$; we conclude that U is rational over $\overline{k}$ and hence, so is Y. □

7. Proof of Theorem 1.7

Let X be a minimal threefold with K_X torsion. Then by Remark 4.2, ϕ is a pseudo-automorphism, i.e. neither ϕ nor ϕ^{-1} contracts a divisor. As a result, ϕ induces an automorphism of the nef cone $\operatorname{Nef}(X)$. Every smooth minimal threefold Y with $c_2(Y) = 0$ has an étale cover by an abelian variety, so if X is a Calabi-Yau variety, hence simply connected, we must have $c_2(X) \neq 0$. As mentioned in the introduction, a theorem of Miyaoka [Miy87] then tells us that $c_2(X)$ is positive on the ample cone $\mathrm{Amp}(X)$ and non-negative on $\mathrm{Nef}(X)$. We first consider the case where $c_2(X)$ is strictly positive on the nef cone. This approach is based on arguments given in Chapter 4 of [Ki10].

Lemma 7.1. *Suppose $\ell : \mathbb{R}^n \to \mathbb{R}$ is a linear function and C is a closed cone in $\mathbb{R}^n$ such that $\ell(z) > 0$ for any $z \in C$ other than the origin. Then for any real number $M \geq 0$, the region $C_M := \{z \in C \mid \ell(z) \leq M\}$ is compact.*

Proof. Let S be the intersection of C with the unit sphere. Clearly S is compact. Define the function $f : S \to \mathbb{R}$ given as follows. For $p \in S$, let I_p be the intersection of the line through p and the origin with the strip $0 \leq \ell(z) \leq M$. Since ℓ is positive on C, I_p is an interval of finite length. Let $f(p)$ be the length of I_p. Since f is continuous and S is compact, f attains its maximal value r on S. Consequently, C_M is contained in the ball of radius r centered at the origin. Thus C_M is closed and bounded, hence compact. $\square$

Proof of Theorem 1.7. (1) Since $c_2(X)$ is strictly positive on $\mathrm{Nef}(X)$, Lemma 7.1 shows that for all $M \geq 0$, the region $\{D \in \mathrm{Nef}(X) \mid c_2(X) \cdot D \leq M\}$ is compact. As a result, $c_2(X)$ achieves a minimum positive value on $\mathrm{Pic}(X) \cap \mathrm{Amp}(X)$ and this value is achieved by only finitely many D_i. Taking the sum of these finitely many D_i, we obtain an ample class A which is fixed by ϕ^*. Let $\mathcal{M}$ be an ample line bundle representing the class of A. Since the Albanese of X is trivial, rational equivalence is the same as linear equivalence. Since $\phi^*A = A$ in $\mathrm{NS}(X) \otimes \mathbb{C}$, we have $\phi^*\mathcal{M} \simeq \mathcal{M} \otimes \mathcal{N}$ where $\mathcal{N}$ is a torsion line bundle. Replacing A by a scalar multiple, we may assume that $\phi^*(A)$ is isomorphic to A and that A is very ample. If ϕ preserves a rational fibration, we are done. Otherwise, with notation as in Proposition 6.1(b), there is a dense set of $y \in Y$ with dense orbit under $\bar{\phi}$. However, A is very ample, so $Y = X$ which gives the desired conclusion.

(2) We will now consider the case where there is a semi-ample divisor $D \neq 0$ on X such that $c_2(X) \cdot D = 0$. Let $\pi : X \to Y$ be the associated c_2-contraction. Oguiso shows ([Ogu01, Theorem 4.3]) that there are only finitely many c_2-contractions, and so after replacing ϕ by a further iterate, we can assume $\phi^*[D] = [D]$. By Proposition 6.1(a), ϕ descends to an automorphism $\overline{\phi}$ of Y. Since D is non-zero, Y is not a point. We now consider three cases.

Case 1: $\dim(Y) = 3$, i.e., D is big. Since contractions have connected fibers, π is birational. If X preserves a rational fibration, we are done. Otherwise, Proposition 6.1(c) tells us that Y is rational over $\overline{\mathbb{Q}}$, which is not possible since X has Kodaira dimension 0. So, the Medvedev-Scanlon Conjecture for ϕ holds in this case.

Case 2: $\dim(Y) = 2$. Here the Medvedev-Scanlon conjecture holds by Lemmas 2.2 and 5.1.

Case 3: $\dim(Y) = 1$. By Proposition 6.1(c), $Y \simeq \mathbb{P}^1$ (over $\overline{\mathbb{Q}}$).

Let $Z \subseteq \mathbb{P}^1$ be the locus of points t where the fiber X_t is singular. Then $\overline{\phi}(Z) = Z$. Since Z is a finite set, after replacing ϕ by a further iterate, we can assume $\overline{\phi}$ fixes Z point-wise. By [VZ01, Theorem 0.2], we know that Z contains at least 3 points. It follows that $\overline{\phi}$ is the identity since it fixes at least three points of $\mathbb{P}^1$. In other words, there exists a rational function on X which is invariant under some iterate of ϕ, a contradiction. $\square$

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