BOUNDING PERIODS OF SUBVARIETIES OF $(\mathbb{P}^1)^n$

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Abstract. Using methods of p-adic analysis, along with the powerful result of Medvedev-Scanlon [MS14] for the classification of periodic subvarieties of $(\mathbb{P}^1)^n$, we bound the length of the orbit of a periodic subvariety $Y \subset (\mathbb{P}^1)^n$ under the action of a dominant endomorphism.

1. Introduction

First of all, as a matter of convention, all the subvarieties appearing in our paper are closed and irreducible. Secondly, given a periodic subvariety $V \subseteq X$ (under the action of some self-map Φ of X), its *period* is the length of the orbit of V (under the action of Φ).

In [MS94], Morton and Silverman conjecture that there is a constant $C(N, d, D)$ such that for any morphism $f : \mathbb{P}^N \longrightarrow \mathbb{P}^N$ of degree d defined over a number field K with $[K : \mathbb{Q}] \leq D$, the number of preperiodic points of f over K is less than or equal to $C(N, d, D)$. This conjecture remains very much open, but in the case where f has good reduction at a prime $\mathfrak{p}$, a great deal has been proved about bounds depending on $\mathfrak{p}$, N, d, D (see [Zie96, Pez05, Hut09]).

In this paper, we study the more general problem of bounding length of the orbit of periodic subvarieties of any dimension; we prove the following results.

Theorem 1.1. *Let K be a finite extension of $\mathbb{Q}_p$. Let $f_1, \dots, f_n \in K(x)$ be rational functions of good reduction and let $\Phi : (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^n$ be the endomorphism given by*

$$(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n)). \tag{1.1.1}$$

Then there exists a constant C depending only on n, p, $[K : \mathbb{Q}_p]$ and the degree of the rational functions f_i such that if Y is any irreducible subvariety of $(\mathbb{P}^1)^n$ defined over K satisfying the following two properties:

- *Y is periodic under the action of Φ, and*
- *there is a nonsingular point $(\alpha_1, \dots, \alpha_n) \in Y(K)$,*

then its period is bounded by C.

We prove Theorem 1.1 as a consequence of the next result.

Theorem 1.2. *Let K be a finite extension of $\mathbb{Q}_p$. Let $f_1, \dots, f_n \in K(x)$ be rational functions of good reduction and let $\Phi : (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^n$ be the endomorphism given by*

$$(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n)). \tag{1.2.1}$$

Then there exists a constant C depending only on n, p, $[K : \mathbb{Q}_p]$, and the degrees of the rational functions f_i such that if Y is any irreducible subvariety of $(\mathbb{P}^1)^n$ defined over K satisfying the following two properties:

- *Y is periodic under the action of Φ, and*
- *there is a point $(\alpha_1, \dots, \alpha_n) \in Y(K)$ such that for each $i = 1, \dots, n$, we have that α_i is a nonpreperiodic point for f_i,*

then its period is bounded by C.

We note that any dominant (regular) endomorphism Φ of $(\mathbb{P}^1)^n$ is split, i.e., it is of the form

$$(x_1, x_2, \dots, x_n) \mapsto \left(f_1(x_{\sigma(1)}), f_2(x_{\sigma(2)}), \dots, f_n(x_{\sigma(n)})\right),$$

where the f_i's are rational functions, while σ is a permutation of $\{1, \dots, n\}$. So, at the expense of replacing Φ by an iterate Φ^k (for some positive integer k dividing $n!$), each dominant endomorphism of $(\mathbb{P}^1)^n$ is of the form (1.2.1).

In [BGT15], a similar result was obtained bounding the length of the orbit of any periodic subvariety (with a distinguished smooth p-adic point of good reduction) under the action of an étale endomorphism of an arbitrary variety (not only $(\mathbb{P}^1)^n$). However, the presence of ramification as it is in our case adds new complications. Also, we note that in [GN17], using results regarding the decomposition of polynomials, it was shown that the length of the orbit of a periodic subvariety of $\mathbb{A}^n$ under the coordinatewise action of n one-variable polynomials is uniformly bounded by a constant depending only on n and on the degrees of the polynomials. It is important to point out that the method from [GN17] cannot be extended to treat the case of rational functions; essentially, for rational functions, one lacks all the sharper results regarding their decomposition, as established in [MS14] for the case of polynomials.

In our Theorem 1.2, the constant C bounding the length of the orbit of a periodic subvariety of $(\mathbb{P}^1)^n$ depends on the degrees of the rational functions f_i and also on the good reduction data associated to our problem, which refers to the knowledge about a single point on Y. So, from this point of view, our result is less restrictive than the one obtained in [Hut], where a similar bound as the one from our Theorem 1.2 is established under the additional assumption that also the periodic subvariety Y (along with its iterates under Φ) have good reduction modulo p, i.e., reducing modulo p yields a subvariety $\bar{Y}$, which is an irreducible subvariety defined over $\overline{\mathbb{F}_p}$ of same degree as Y (and a similar property holds for each iterate of Y under Φ). Even though the result from [Hut] is valid for more general dynamical systems (i.e., endomorphisms of $\mathbb{P}^n$ as opposed to endomorphisms of $(\mathbb{P}^1)^n$), the additional hypothesis regarding the good behaviour of the periodic subvariety modulo p makes it more restrictive than our Theorem 1.2.

Theorem 1.2 is proved using a p-adic analytic parametrization of forward orbits under the action of a rational function (such as in [BGKT10]), combined with the description of periodic subvarieties of $(\mathbb{P}^1)^n$ (obtained by Medvedev-Scanlon [MS14]).

We start by describing in Section 2 the various useful results regarding the p-adic parametrization of orbits of points under the action of a rational function. We continue by proving in Section 3 the special case of Theorem 1.2 when the periodic subvariety is a curve (see Theorem 3.1). We conclude our paper by reducing the general Theorem 1.2 to the special Theorem 3.1, using the description of periodic subvarieties of $(\mathbb{P}^1)^n$ provided by Medvedev-Scanlon in [MS14]; then we derive the conclusion of Theorem 1.1 as a consequence of Theorem 1.2.

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2. Useful results

In this Section, we follow the setup and the results from [BGKT10, Section 2]; see also [BGT16, Chapter 6]. For a prime p, we let $\mathbb{Z}_p$ denote the ring of p-adic integers, $\mathbb{Q}_p$ will denote the field of p-adic rationals, and $\mathbb{C}_p$ will denote the completion of an algebraic closure of $\mathbb{Q}_p$ with respect to the p-adic absolute value $|\cdot|_p$. Given a point $y \in \mathbb{C}_p$ and a real number $r > 0$, write

$$D(y,r) = \{x \in \mathbb{C}_p : |x-y|_p < r\}, \quad \overline{D}(y,r) = \{x \in \mathbb{C}_p : |x-y|_p \le r\}$$

for the open and closed disks, respectively, of radius r about y in $\mathbb{C}_p$. We write $[y] \subseteq \mathbb{P}^1(\mathbb{C}_p)$ for the residue class of a point $y \in \mathbb{P}^1(\mathbb{C}_p)$. That is, $[y] = D(y,1)$ if $|y| \le 1$, or else $[y] = \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}(0,1)$ if $|y| > 1$.

Let K be a finite extension of $\mathbb{Q}_p$, let $\mathfrak{o}_K$ be its ring of p-adic integers, let π be a uniformizer for $\mathfrak{o}_K$ and let κ be its corresponding residue field. Let $f \in K(x)$ be a rational function of degree $d \ge 1$. We say that f has good reduction if writing the rational function f as

$$f([X_0 : X_1]) = [f_0(X_0,X_1) : f_1(X_0,X_1)]$$

for some coprime homogeneous polynomials $f_0, f_1 \in \mathfrak{o}_K[X_0, X_1]$ of degree d for which at least one coefficient is a π-adic unit, then reducing the coefficients of both f_0 and f_1 modulo π yields two coprime homogeneous polynomials in $\kappa[X_0, X_1]$ of same degree d. Note that our choice of coordinates here induces a choice of model $\mathbb{P}^1_{\mathfrak{o}_K}$ and a morphism $F : \mathbb{P}^1_{\mathfrak{o}_K} \longrightarrow \mathbb{P}^1_{\mathfrak{o}_K}$ of $\mathfrak{o}_K$-schemes such that $(F)_K = f$ when f has good reduction at π.

The action of a p-adic power series $f \in \mathfrak{o}_K[[z]]$ on $D(0,1)$ is either attracting (i.e., f contracts distances) or quasiperiodic (i.e., f is distance-preserving), depending on its linear coefficient. Rivera-Letelier gives a more precise description of this dichotomy in [RL03, Sections 3.1 and 3.2]. The following two Lemmas follow from [RL03, Propositions 3.3 and 3.16] and they were stated in [BGKT10, Lemmas 2.1 and 2.2] in the case $K = \mathbb{Q}_p$; the general case follows by an identical argument (for more details regarding the applications to arithmetic dynamics of the p-adic parametrization of orbits, see [BGT16]).

Lemma 2.1. *Let K be a finite extension of $\mathbb{Q}_p$, let $\mathfrak{o}_K$ be its ring of p-adic integers, and let π be a uniformizer for $\mathfrak{o}_K$. We let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in \mathfrak{o}_K[[z]]$ be a nonconstant power series with $|a_0|_p, |a_1|_p < 1$. Then there is a point $y \in \pi \cdot \mathfrak{o}_K$ such that $f(y) = y$, and $\lim_{n\to\infty} f^n(z) = y$ for all $z \in D(0,1)$. Write $\lambda = f'(y)$; then $|\lambda|_p < 1$, and:*

(i) *(Attracting). If $\lambda \neq 0$, then there is a radius $0 < r < 1$ and a power series $u \in K[[z]]$ mapping $\overline{D}(0,r)$ bijectively onto $\overline{D}(y,r)$ with $u(0) = y$, such that for all $z \in D(y,r)$ and $n \ge 0$,*

$$f^n(z) = u(\lambda^n u^{-1}(z)).$$

(ii) *(Superattracting). If $\lambda = 0$, then write f as*

$$f(z) = y + c_m(z-y)^m + c_{m+1}(z-y)^{m+1} + \cdots \in \mathfrak{o}_K[[z-y]]$$

with $m \geq 2$ and $c_m \neq 0$. If c_m has an $(m-1)$-st root in K (or equivalently, in $\mathfrak{o}_K$), then there are radii $0 < r, s < 1$ and a power series $u \in K[[z]]$ mapping $\overline{D}(0,s)$ bijectively onto $\overline{D}(y,r)$ with $u(0) = y$, such that for all $z \in D(y,r)$ and $n \geq 0$,

$$f^n(z) = u\Big((u^{-1}(z))^{m^n}\Big).$$

Proof. The proof is identical to the proof of [BGKT10, Lemma 2.1]; see also [SS] for a similar result in the superattracting case. □

The next result is a slight generalization of [BGKT10, Lemma 2.2].

Lemma 2.2 (Indifferent or quasiperiodic)**.** *Let K, $\mathfrak{o}_K$ and π be as in Lemma 2.1 and let κ be the residue field of K. Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in \mathfrak{o}_K[[z]]$ be a nonconstant power series with $|a_0|_p < 1$ but $|a_1|_p = 1$. Then for any nonperiodic $x \in \pi \cdot \mathfrak{o}_K$, there are: an integer $k \geq 1$ depending only on p, e and $[\kappa : \mathbb{F}_p]$, radii $0 < r < 1$ and $s \geq |k|_p$, and a power series $u \in K[[z]]$ mapping $\overline{D}(0,s)$ bijectively onto $\overline{D}(x,r)$ with $u(0) = x$, such that for all $z \in \overline{D}(x,r)$ and $n \geq 0$,*

$$f^{nk}(z) = u(nk + u^{-1}(z)).$$

Proof. The conclusion of Lemma 2.2 (including the fact that k is bounded solely in terms of p, e and $[\kappa : \mathbb{F}_p]$) follows by combining [BGT15, Proposition 2.1] with [Poo13]. □

3. The case of curves

We first prove Theorem 1.2 when Y is a curve in $(\mathbb{P}^1)^n$.

Theorem 3.1. *Let K be a finite extension of $\mathbb{Q}_p$. Let $f_1, \cdots, f_n \in K(x)$ be rational functions of good reduction and let $\Phi : (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^n$ be the endomorphism given by $(x_1, \cdots, x_n) \mapsto (f_1(x_1), \cdots, f_n(x_n))$. Then there exists a constant c depending only on p, $[K : \mathbb{Q}_p]$ and on the degrees of the f_i's, such that for any irreducible curve $Y \subset (\mathbb{P}^1)^n$ defined over K satisfying the following two properties:*

- *Y is periodic under the action of Φ, and*
- *there exists a point $(\alpha_1, \ldots, \alpha_n) \in Y(K)$ such that each α_i is not preperiodic under the action of f_i,*

then its period is bounded by c.

Proof. We let κ be the residue field of K and we let e be the ramification index for $K/\mathbb{Q}_p$. We let $\mathfrak{o}_K$ be the subring of p-adic integers in K, and we let π be a uniformizer for $\mathfrak{o}_K$.

At the expense of replacing K by a finite extension of degree bounded solely in terms of the degrees of the rational functions f_i, we may assume the following. For some $i = 1, \ldots, n$ if α_i modulo π is in the cycle of a periodic critical point $Q \in \mathbb{P}^1_K$ of f_i of period $\tilde{\ell}_i$, given a local coordinate $x_{i,Q}$ at Q, we write

$$f_i^{\tilde{\ell}_i}(x_{i,Q}) = c_{i,Q} x_{i,Q}^{m_{i,Q}} + O\left(x_{i,Q}^{m_{i,Q}+1}\right), \tag{3.1.1}$$

where $m_{i,Q} \geq 2$ and $c_{i,Q} \neq 0$; then K contains all the $(m_{i,Q} - 1)$-st roots of $c_{i,Q}$. Since there are at most finitely many superattracting periodic points and furthermore, the period of any periodic point of some f_i contained in $\mathbb{P}^1(K)$ is bounded solely in terms of p, the degree of the f_i and the degree of the field extension $K/\mathbb{Q}_p$ (see [Hut09, Zie96]), then we are guaranteed that a finite extension of K suffices. If there is no superattracting periodic point modulo p in the orbit of α_i under the action of f_i modulo π, then we do not need to extend K.

We proceed as in the proof of [BGKT10, Theorem 1.4] (see also [BGT16, Chapter 11]); our goal is to show that there exists a positive integer k depending only on the local data coming from K (i.e, its ramification index and also the size of its residue field) and also there exists a positive integer ℓ (which may depend on the α_i's) such that for each $i = 1, \dots, n$ and for each $0 \le j \le k-1$ there exist positive integers m_i and p-adic analytic power series $F_{i,j} \in K[[z_0, z_1, z_{m_i}]]$ such that (modulo linear transformations) we have p-adic analytic parametrizations:

$$f_i^{\ell+j+nk}(\alpha_i) = F_{i,j}\left(n, \pi^n, \pi^{m_i^n}\right) \text{ for each } n \ge 0. \tag{3.1.2}$$

The integers m_i from (3.1.2) are the same as the integers $m_{i,Q}$ from equation (3.1.1); if α_i is not superattracting, then actually $F_{i,j}$ is a p-adic analytic power series only in the variables z_0 and z_1, or alternatively, $m_i = 1$ in that case.

In order to derive the p-adic parametrization (3.1.2), we proceed as in the proof of [BGKT10, Theorem 1.4]; in particular, we split our analysis into various steps.

Step (i). First we note that there exist positive integers ℓ_0 and k_0 such that the residue class of $f_i^{\ell_0}(\alpha_i)$ (for $i = 1, \dots, n$) is fixed under the induced action on $\mathbb{P}^1_\kappa$ of the reduction of $f_i^{k_0}$ modulo π; here we use the fact that each f_i has good reduction modulo π. Furthermore, note that both integers k_0 and ℓ_0 depend only on p and on $[\kappa : \mathbb{F}_p]$; for more details, see [BGKT10, Step (ii), p. 1063]. By a $\mathrm{PGL}(2, \mathfrak{o}_K)$-change of coordinates for each $i = 1, 2$, we may assume that $f_i^{\ell_0}(\alpha_i) \in \pi\mathfrak{o}_K$, and therefore $f_i^{k_0}$ may be written as a nonconstant power series in $\mathfrak{o}_K[[z]]$ mapping $D(0,1)$ into itself.

So, $f_i^{\ell_0}(\alpha_i)$ lands in an indifferent, attracting, or superattracting residue class for the action of $f_i^{k_0}$. Now, if it lands in a superattracting residue class, then it means that the orbit of $f_i^{\ell_0}(\alpha_i)$ under the action of $f_i^{k_0}$ converges π-adically to a periodic critical point Q_i for f_i and therefore, Q_i must live in $\mathbb{P}^1_K$. This justifies why we can restrict in equation (3.1.1) only to periodic critical points contained in $\mathbb{P}^1_K$.

Step (ii). Let $i \in \{1, \dots, n\}$. If $|(f_i^{k_0})'(f_i^{\ell_0}(\alpha_i))|_p = 1$, we apply Lemma 2.2 to $f_i^{k_0}$ and the point $f_i^{\ell_0}(\alpha_i)$ to obtain radii r_i and s_i (along with a positive integer n_i depending only on p, e and $[\kappa : \mathbb{F}_p]$) and a power series u_i such that

- $s_i \ge |n_i|_p$;
- the power series $u_i \in K[[z]]$ maps $\overline{D}(0, s_i)$ bijectively onto $\overline{D}\left(f_i^{\ell_0}(\alpha_i), r_i\right)$ with $u_i(0) = f_i^{\ell_0}(\alpha_i)$; and
- for all $z \in \overline{D}\left(f_i^{\ell_0}(\alpha_i), r_i\right)$ and $n \ge 0$,
$$f_i^{k_0 n_i n}(z) = u_i(k_0 n_i n + u_i^{-1}(z)).$$

Let $\ell_{1,i} := \ell_0$ and also let $k_{3,i} = n_i k_0$ (note that $k_{3,i}$ depends only on p, e and $[\kappa : \mathbb{F}_p]$); then $f_i^{k_{3,i}+\ell_1}(\alpha_i) \in \overline{D}(f_i^{\ell_1}(\alpha_i), r_i)$. (Following the notation from [BGKT10], the jump from a subscript of 0 to 3 is because certain complications, to be addressed in Steps (iii)–(v), do not arise in the quasiperiodic case.) Note that $f_i^{\ell_{1,i}+nk_{3,i}}(\alpha_i)$ may be expressed as a power series in the integer $n \ge 0$; specifically, $f_i^{\ell_{1,i}+nk_{3,i}}(\alpha_i) = u_i(k_{3,i} \cdot n)$.

Step (iii). If $\left|\left(f_i^{k_0}\right)'\left(f_i^{\ell_0}(\alpha_i)\right)\right|_p < 1$, then Lemma 2.1 yields that there is a point $y_i \in D(0,1)$ fixed by $f_i^{k_0}$ along with radii r_i and s_i (where $s_i := r_i$ in the non-superattracting case), and an associated power series $u_i \in K[[z]]$. Set $k_{1,i} = k_0$ and $\ell_{1,i} = \ell_0 + n_i k_{1,i}$ for a suitable integer $n_i \geq 0$ so that

$$f_i^{\ell_{1,i}}(\alpha_i) \in \overline{D}(y_i, r_i). \tag{3.1.3}$$

Define $\lambda_i := (f_i^{k_0})'(y_i)$ to be the multiplier of the point y_i; then in this case, $|\lambda_i|_p < 1$. Define $\mu_i := u_i^{-1}(f_i^{\ell_{1,i}}(\alpha_i))$; note that $\mu_i \in \pi \cdot \mathfrak{o}_K$, because $s_i < 1$. In addition, $\mu_i \neq 0$, because u_i is bijective and $u_i(0) = y_i$ is fixed by $f_i^{k_0}$, while $u_i(\mu_i) = f_i^{\ell_{1,i}}(\alpha_i)$ is not fixed by $f_i^{k_0}$.

Step (iv). In this step, we consider only the case that $0 < |\lambda_i|_p < 1$ (i.e., attracting but not superattracting). We will express certain functions of n as power series in n and π^n and thus obtain a p-adic parametrization of the orbit of α_i under f_i as in (3.1.2), where the corresponding p-adic analytic function $F_{i,j}$ is actually a function of only 2 variables (i.e., $m_i = 1$ with the notation as in (3.1.2)).

Write $\lambda_i = \alpha_i \pi^{e_i}$, where $e_i \geq 1$ and $\alpha_i \in \mathfrak{o}_K^*$. If α_i is a root of unity, we can choose an integer $M_{1,i} \geq 1$ such that $\alpha_i^{M_{1,i}} = 1$. If α_i is not a root of unity, it is well known that there is an integer $M_{1,i} \geq 1$ such that $\alpha_i^{nM_{1,i}}$ can be written as a power series in n with coefficients in $\mathfrak{o}_K$. Furthermore, $M_{1,i}$ depends only on p, e and $[\kappa : \mathbb{F}_p]$, which can be seen also applying Lemma 2.2 to the function $z \mapsto \alpha_i z$; for more details, see [BGKT10, Step (iii)]. We set

$$k_{3,i} := M_{1,i} k_{1,i}; \tag{3.1.4}$$

also, we replace λ_i by $\lambda_i^{M_{1,i}}$ and replace e_i by $M_{1,i} \cdot e_i$. (The subscript again jumps to 3 in (3.1.4) because of the complications of Step (iv).) Thus, we can write

$$\lambda_i^n = (\pi^n)^{e_i} g_{1,i}(n) \qquad \text{for all integers } n \geq 0, \tag{3.1.5}$$

for some power series $g_{1,i}(z) \in \mathfrak{o}_K[[z]]$.

Step (v). In this step, we consider only the superattracting case, that $\lambda_i = 0$, and we will express certain functions of n as power series in n, π^n, and $\pi^{m_i^n}$, where $m_i \geq 2$ is a certain integer (see (3.1.1)).

We let first the integer $m_{i,0}$ be the order of the unique superattracting point of f_i in $D(0,1)$, as in Lemma 2.1(ii). Then we write $m_{i,0} := a_i \pi^{b_i}$, for some $a_i \in \mathfrak{o}_K^*$ and some integer $b_i \geq 0$. Then as in Step (iv), we can find a positive integer $M_{1,i}$ depending only on p, e and $[\kappa : \mathbb{F}_p]$ such that $a_i^{nM_{1,i}}$ can be written as a power series in n with coefficients in $\mathfrak{o}_K$. Set

$$k_{2,i} := M_{1,i} \cdot k_{1,i} \quad \text{and} \quad m_{i,1} := m_{i,0}^{M_{1,i}}.$$

Then $m_{i,1}^n$ can be written as a power series in n and π^n, with coefficients in $\mathfrak{o}_K$.

In addition, recall that $\mu_i = u_i^{-1}(f_i^{\ell_i}(\alpha_i))$ satisfies $0 < |\mu_i|_p < 1$; thus, we can write $\mu_i = \beta_i \pi^{e_i}$, where $e_i \geq 1$ and $\beta_i \in \mathfrak{o}_K^*$. If β_i is a root of unity with, say, $\beta_i^{M_{2,i}} = 1$ for some positive integer $M_{2,i}$ (depending only on p, e and $[\kappa : \mathbb{F}_p]$, since the size of the group of roots of unity contained in K depends only on p, e and $[\kappa : \mathbb{F}_p]$), choose a positive integer $M_{3,i}$ (again depending only on p, e and $[\kappa : \mathbb{F}_p]$) so that $M_{2,i} | \left(m_{i,1}^{2M_{3,i}} - m_{i,1}^{M_{3,i}}\right)$. Set

$$k_{3,i} := M_{3,i} k_{2,i} \text{ and } m_{i,2} := m_{i,1}^{M_{3,i}}$$

and note that $\beta_i^{m_{i,2}^n}$ is constant in n.

On the other hand, if β_i is not a root of unity, then as in Step (iv), there is an integer $M'_{2,i}$ depending only on p, e and $[\kappa : \mathbb{F}_p]$ such that $\beta_i^{nM'_{2,i}}$ can be written as a power series in n with coefficients in $\mathfrak{o}_K$. As above, choose a positive integer $M'_{3,i}$ (depending only on p, e and $[\kappa : \mathbb{F}_p]$) such that $M'_{2,i} | \big(m_{i,1}^{2M'_{3,i}} - m_{i,1}^{M'_{3,i}}\big)$, and set

$$k_{3,i} := M'_{3,i} k_{2,i} \text{ and also, } m_{i,2} := m_{i,1}^{M'_{3,i}}.$$

Then $m_{i,2}^n \equiv m_{i,2} \pmod{M'_{2,i}}$ for all positive integers n, and therefore

$$\beta_i^{m_{i,2}^n} = \beta_i^{m_{i,2}} \cdot \beta_i^{m_{i,2}^n - m_{i,2}} = \beta_i^{m_{i,2}} \cdot \left(\beta_i^{M'_{2,i}}\right)^{(m_{i,2}^n - m_{i,2})/M'_{2,i}}$$

can be written as a power series in $(m_{i,2}^n - m_{i,2})/M'_{2,i}$ with coefficients in $\mathfrak{o}_K$. Using the fact that $p \nmid M'_{2,i}$, and expressing $m_{i,2}^n = (m_{i,1}^n)^{M'_{3,i}}$ as a power series in n and π^n with coefficients in $\mathfrak{o}_K$, we conclude that $\beta_i^{m_{i,2}^n}$ can in fact be written as a power series in n and π^n, with coefficients in $\mathfrak{o}_K$.

Thus, whether or not β_j is a root of unity, we can write

$$\mu_i^{m_{i,2}^n} = \left(\pi^{m_{i,2}^n}\right)^{e_i} g_{1,i}(n, \pi^n) \qquad \text{for all integers } n \geq 0, \tag{3.1.6}$$

for some power series $g_{1,i}(z_0, z_1) \in \mathfrak{o}_K[[z_0, z_1]]$.

Step (vi). Let k be the least common multiple of $k_{3,1}$ and of $k_{3,2}$; then k is bounded by a constant depending only on p, e and $[\kappa : \mathbb{F}_p]$.

Note that replacing each $k_{3,i}$ by k does not change the conclusions of our previous Steps since the radii r_i, s_i, the power series u_i, the point μ_i and the integer $\ell_{1,i}$ are unaffected when we replace k by a multiple of it. Moreover, in the quasiperiodic case, $f_i^{k+\ell_{1,i}}(\alpha_i)$ still lies in $\overline{D}(f_i^{\ell_{1,i}}(\alpha_i), r_i)$.

In the attracting case, we replace λ_i by $\lambda_i^{k/k_{3,i}}$, also replace e_i by $\frac{k}{k_{3,i}} e_i$, and let

$$g_{2,i}(z) := \big(g_{1,i}(z)\big)^{k/k_{3,i}} \in \mathfrak{o}_K[[z]];$$

and in the superattracting case, we replace m_i by $m_i^{k/k_{3,i}}$ and let

$$g_{2,i}(z_0, z_1) := g_{1,i}\Big(\frac{k}{k_{3,i}} z_0, z_1^{k/k_{3,i}}\Big) \in \mathfrak{o}_K[[z_0, z_1]].$$

With this new notation, it follows from Steps (i)–(v) that for any integer $n \geq 0$,

1. $f_i^{\ell_{1,i}+nk}(\alpha_i) = u_i(nk)$, if $f_i^{\ell_{1,i}}(\alpha_i)$ lies in a quasiperiodic residue class;
2. $f_i^{\ell_{1,i}+nk}(\alpha_i) = u_i\big(\lambda_i^n \mu_i\big) = u_i\big((\pi^n)^{e_i} g_{2,i}(n)\mu_i\big)$, if $f_i^{\ell_{1,i}}(\alpha_i)$ lies in an attracting residue class; and
3. $f_i^{\ell_{1,i}+nk}(\alpha_i) = u_i\big(\mu_i^{m_i^n}\big) = u_i\Big(\big(\pi^{m_i^n}\big)^{e_i} g_{2,i}(n, \pi^n)\Big)$, if $f_i^{\ell_{1,i}}(\alpha_i)$ lies in a superattracting residue class,

where $\mu_i = u_i^{-1}(f_i^{\ell_{1,i}}(\alpha_i))$ is as in Step (iii). In particular, in all three cases, we have expressed $f_i^{\ell_{1,i}+nk}(\alpha_i)$ as a power series in n, π^n, and, if needed, $\pi^{m_i^n}$.

Let $L = \max\{\ell_{1,1}, \cdots, \ell_{1,n}\}$. For each $\ell = L, \ldots, L+k-1$ and each $i = 1, \cdots, n$, choose a linear fractional transformation $\eta_{i,\ell} \in \mathrm{PGL}(2, \mathfrak{o}_K)$ so that $\eta_{i,\ell} \circ f_i^{\ell}(\alpha_i) \in D(0,1)$. Then $\eta_{i,\ell} \circ f_i^{\ell-\ell_{1,i}}(D(0,1)) \subseteq D(0,1)$, because f_i has good reduction.

Finally, define $E_{i,\ell} = \eta_{i,\ell} \circ f_i^{\ell-\ell_{1,i}} \circ u_i$, so that $E_{i,\ell} \in K[[z]]$ maps $\overline{D}(0, s_i)$ into $D(0, 1)$.

Step (vii). In this step, we will write down power series $F_{i,\ell}$ for $f_i^{\ell+nk}$ in terms of n, π^n, and $\pi^{m_i^n}$. We will also produce bounds $B_{i,\ell}$ to be used in applying [BGKT10, Lemma 3.1]. For each $i = 1, \cdots, n$, we consider the three cases that $f_i^\ell(\alpha_i)$ lies in a quasiperiodic, attracting (but not superattracting), or superattracting residue class for the function f_i^k.

In the quasiperiodic case, for each $\ell = L, \ldots, L + k - 1$, define the power series

$$F_{i,\ell}(z_0) = E_{i,\ell}(kz_0) \in K[[z_0]],$$

so that $F_{i,\ell}(n) = \eta_{i,\ell} \circ f_i^{\ell+nk}(\alpha_i)$ for all $n \geq 0$. All coefficients of $F_{i,\ell}$ have absolute value at most 1, because $|k|_p \leq s_i$ and $E_{i,\ell}$ maps $\overline{D}(0, s_i)$ into $D(0, 1)$. Hence, we set our bound $B_{i,\ell}$ to be $B_{i,\ell} := 0$.

Second, in the attracting (but not superattracting) case, for each $\ell = L, \ldots, L + k - 1$, define the power series

$$F_{i,\ell}(z_0, z_1) = E_{i,\ell}\big(z_1^{e_i} g_{2,i}(z_0)\mu_i\big) \in K[[z_0, z_1]],$$

where $E_{i,\ell}$ and $g_{2,i}$ are as in Step (vi), so that $F_{i,\ell}(n, \pi^n) = \eta_{i,\ell} \circ f_i^{\ell+nk}(\alpha_i)$ for all $n \geq 0$.

Still in the attracting (but not superattracting) case, because $E_{i,\ell}$ maps $\overline{D}(0, s_i)$ into $D(0, 1)$, there is some $B_{i,\ell} > 0$ such that for every $j \geq 0$, the coefficient of z^j in $E_{i,\ell}(z)$ has absolute value at most $p^{jB_{i,\ell}}$. Recalling also that $g_{2,i} \in \mathfrak{o}_K[[z]]$ and $|\mu_i|_p < 1$, it follows that if we write $F_{i,\ell}(z_0, z_1) = \sum_{j=0}^\infty h_j(z_0)z_1^j$ (where each $h_j \in K[[z]]$), then for each $j \geq 0$, all coefficients of h_j have absolute value at most $p^{jB_{i,\ell}}$.

Third, in the superattracting case, for each $\ell = L, \ldots, L+k-1$, define the power series

$$F_{i,\ell}(z_0, z_1, z_{m_i}) = E_{i,\ell}\big(g_{2,i}(z_0, z_1)z_{m_i}^{e_i}\big) \in K[[z_0, z_1, z_{m_i}]],$$

where $E_{i,\ell}$ and $g_{2,i}$ are as in Step (v), so that $F_{i,\ell}(n, \pi^n, \pi^{m_i^n}) = \eta_{i,\ell} \circ f_i^{\ell+nk}(\alpha_i)$ for all $n \geq 0$.

Still in the superattracting case, because $E_{i,\ell}$ maps $\overline{D}(0, s_i)$ into $D(0, 1)$, there is some $B_{i,\ell} > 0$ such that for every $j \geq 0$, the coefficient of z^j in $E_{i,\ell}(z)$ has absolute value at most $p^{jB_{i,\ell}}$. Hence, if we write $F_{i,\ell}(z_0, z_1, z_{m_i}) = \sum_{j_1,j_2\geq 0} h_{j_1,j_2}(z_0)z_1^{j_1} z_{m_i}^{j_2}$ (where each $h_{j_1,j_2} \in K[[z]]$), then as before, since $g_{2,i} \in \mathfrak{o}_K[[z_0, z_1]]$, all coefficients of h_{j_1,j_2} have absolute value at most $p^{j_2B_{i,\ell}} \leq p^{B_{i,\ell}(j_1+j_2)}$. Finally, set

$$B := \max_{L\leq\ell\leq L+k-1} (B_{1,\ell} + \cdots + B_{n,\ell})\,.$$

Step (viii). For each $\ell = L, \ldots, L + k - 1$, let $\{\mathcal{H}_{\ell,i}\}_i$ be a finite set of polynomials in $\mathfrak{o}_K[t_1, \ldots, t_n]$ generating the vanishing ideal of the curve $Y \subset (\mathbb{P}^1)^n$ dehomogenized with respect to the coordinates determined by $(\eta_{1,\ell}, \cdots, \eta_{n,\ell})$. We let $m := \max\{m_1, \ldots, m_n\}$ and then we define

$$G_{\ell,i}(z_0, z_1, z_2, \ldots, z_m) = \mathcal{H}_{\ell,i}(F_{1,\ell}, \ldots, F_{n,\ell}) \in K[[z_0, z_1, z_2, \ldots, z_m]]. \tag{3.1.7}$$

We note that we may not need all variables $z_0, \ldots, z_m$ in equation (3.1.7); however, in all cases, by construction, $G_{\ell,i}(n, \pi^n, \pi^{2^n}, \ldots, \pi^{m^n})$ is defined for all integers $n \geq 0$, and moreover,

$$G_{\ell,i}(n, \pi^n, \pi^{2^n}, \ldots, \pi^{m^n}) = 0 \text{ for all } i, \text{ if and only if } \Phi^{\ell+nk}(\alpha) \in Y. \tag{3.1.8}$$

Now, if for some $\ell = L, \cdots, L+k-1$ we have that $G_{\ell,i} = 0$ for all i, then we get that $\Phi^{\ell+nk}(\alpha) \in Y$ for all $n \geq 0$ and since α is not preperiodic, while Y is a curve, we conclude that Y must be fixed under the action of Φ^k, as desired (note that k is bounded solely in terms of the degrees of the f_i's and also on p, e and $[\kappa : \mathbb{F}_p]$).

So, from now on, we assume that for each $\ell = L, \cdots, L+k-1$ there exists some i for which $G_{\ell,i} \neq 0$. Then for each such nonzero $G_{\ell,i}$, we write

$$G_{\ell,i}(z_0, \pi^n, \pi^{2^n}, \ldots, \pi^{m^n}) = \sum_{w \in \mathbb{N}^m} g_w(z_0)\pi^{f_w(n)},$$

where for each $w := (w_1, \ldots, w_m) \in \mathbb{N}^m$, we have that

$$f_w(n) := w_1 n + w_2 2^n + \cdots + w_m m^n.$$

Then we let v be the smallest element of $\mathbb{N}^m$ with respect to the usual lexicographic order on $\mathbb{N}^m$ such that $g_v \neq 0$. Also, for each element $w := (w_1, \ldots, w_m) \in \mathbb{N}^m$, we define $|w| := w_1 + \cdots + w_m$; for more details, see [BGKT10, Section 3].

By our choice of the bound B in Step (vi), and because all coefficients of $\mathcal{H}_{\ell,i}$ lie in $\mathfrak{o}_K$, all coefficients of g_w have absolute value at most $p^{B|w|}$, for every $w \in \mathbb{N}^m$. Since $G_{\ell,i}\left(n, \pi^n, \pi^{2^n}, \cdots, \pi^{m^n}\right)$ is defined at every $n \geq 0$, then g_v must converge on $\overline{D}(0,1)$; therefore, we may choose a radius $0 < s_\ell \leq 1$ for g_v satisfying the hypotheses of [BGKT10, Lemma 3.1]. More precisely, g_v has only finitely many zeros in $\overline{D}(0,1)$ and so, we let $0 < s_\ell \leq 1$ be the minimum distance between any two distinct such zeros.

Let s be the minimum of all the s_ℓ as we vary $\ell \in \{L, \cdots, L+k-1\}$. The set $\mathfrak{o}_K$ may be covered by finitely many disks $D(\gamma, s)$ (for some points $\gamma \in \mathfrak{o}_K$). We let $N := k \cdot p^M$ for some sufficiently large integer M such that $|p^M|_p < s$.

For each nonzero $G_{\ell,i}$ we may apply [BGKT10, Lemma 3.1] (with the bound B from Step (vi) and radius s from the previous paragraph) and conclude that for each $j \in \{1, \ldots, N\}$, the set

$$\{n \in \mathbb{N} \colon \Phi^{j+nN}(\alpha) \in Y\} \text{ has natural density } 0. \tag{3.1.9}$$

In order to derive (3.1.9), we use (3.1.8) and [BGKT10, Lemma 3.1 and Corollary 1.5]. However, since Y is periodic, the set

$$\{n \in \mathbb{N} \colon \Phi^n(\alpha) \in Y\} \text{ contains an infinite arithmetic progression,}$$

which contradicts (3.1.9) by Szemerédi's theorem. In conclusion, it must be that for some ℓ, we have that $G_{\ell,i} = 0$ for all i and therefore Y must be fixed by Φ^k.

This concludes the proof of Theorem 3.1. □

4. Proof of our main results

We first prove Theorem 1.2. Our strategy is to prove separately its conclusion assuming that either

(1) all rational functions f_i are *disintegrated*, using the terminology from [MS14] (or *non-special*, using the terminology from [GN16, GN17]), i.e., for each $i = 1, \ldots, n$, we have that $\deg(f_i) \geq 2$ and also, $f_i(x)$ is not conjugated (through a fractional linear transformation) to $x^{\pm \deg(f_i)}$, or to $\pm C_{\deg(f_i)}$ (where, for each $m \geq 2$, $\pm C_m(x)$ is the m-th Chebyshev polynomial, which satisfies $C_m\left(x + \frac{1}{x}\right) = x^m + \frac{1}{x^m}$), or to a Lattés map (i.e., the quotient of an endomorphism of an elliptic curve); or

(2) all rational functions f_i are special, i.e., for each $i = 1, \dots, n$, we have that $\deg(f_i) \geq 2$ and also, $f_i(x)$ is conjugated either to $x^{\pm \deg(f_i)}$, or to $\pm C_{\deg(f_i)}$, or to a Lattés map; or
(3) each rational function f_i has degree equal to 1.

In each of the above cases (1)-(3), we show that the period of a periodic subvariety (satisfying the hypotheses of Theorem 1.2) is uniformly bounded. Case (1) follows from combining our Theorem 3.1 along with the description from [MS14] of the periodic subvarieties of $(\mathbb{P}^1)^n$ under the coordinatewise action of n disintegrated rational functions (for more details, see Theorem 4.1). Case (2) above follows using classical results regarding algebraic subgroups of product of tori and of elliptic curves (for more details, see Theorem 4.4). Finally, case (3) follows using a direct analysis since for linear maps, one can find explicit formulas for the points in any of their orbits (for more details, see Theorem 4.3). Then we finish the proof of Theorem 1.2 by employing one more time the results of [MS14] which allows us to prove the conclusion of Theorem 1.2 after splitting the action of the f_i's in the above 3 cases and using Theorems 4.1, 4.3 and 4.4.

Theorem 4.1. *Theorem 1.2 holds under the additional assumption that each rational function f_i has degree greater than* 1*, and moreover, no $f_i(x)$ is conjugated to either $x^{\pm \deg(f_i)}$, or to $\pm C_{\deg(f_i)}$, or to a Lattés map.*

Before proceeding to the proof of Theorem 4.1, we will make some general reductions, which may be useful beyond the present paper.

So, working under the hypotheses of Theorem 4.1, i.e., that each f_i is a disintegrated rational function (of degree larger than 1) then, according to [MS14, Proposition 2.21], we know that each irreducible periodic subvariety $Y \subset (\mathbb{P}^1)^n$ is an irreducible component of the intersection of finitely many hypersurfaces $Y_{i,j}$ of $(\mathbb{P}^1)^n$, which are of the form $Y_{i,j} := \pi_{i,j}^{-1}(C_{i,j})$, where $C_{i,j}$ is an irreducible periodic curve defined over K under the action of (f_i, f_j) on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\pi_{i,j} : (\mathbb{P}^1)^n \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is the projection on the (i, j)-th coordinate axes; note that $C_{i,j}$ is also defined over K since the projection of the irreducible K-subscheme Y of $(\mathbb{P}^1)^n$ on any two coordinates would still be an irreducible K-subscheme (of $(\mathbb{P}^1)^2$). In particular, we claim that Y is a product of irreducible periodic curves defined over K, which is proved below in Lemma 4.2. Before stating formally our result, we introduce some useful notation: given a subset $I \subset \{1, \dots, n\}$, we denote by $(\mathbb{P}^1)^{|I|}$ the product of all the axes of $(\mathbb{P}^1)^n$ corresponding to the indices from the subset I.

Lemma 4.2. *Let K be a field of characteristic* 0*, let $f_1, \dots, f_n \in K(x)$ be disintegrated rational functions of degree larger than* 1 *and let Φ be their coordinatewise action on $(\mathbb{P}^1)^n$. Let $Y \subset (\mathbb{P}^1)^n$ be a periodic subvariety under the action of Φ, also defined over K. There exists a partition of $\{1, \dots, n\}$ into $r = \dim Y$ subsets $I_1, \dots, I_r$ and there exist irreducible curves $Y_j \subset (\mathbb{P}^1)^{|I_j|}$ defined over K, which are periodic under the induced action of the rational functions f_i for $i \in I_j$ on the coordinate axes corresponding to the subset I_j such that $Y = Y_1 \times \cdots \times Y_r$.*

Proof. Without loss of generality, we may assume Y projects dominantly onto each coordinate axis; otherwise $Y = \{\zeta\} \times Y_0$ for some periodic point $\zeta \in (\mathbb{P}^1)^{|I_0|}$ (for a suitable $I_0 \subset \{1, \dots, n\}$) and some periodic subvariety $Y_0 \subset (\mathbb{P}^1)^{|\{1,\dots,n\} \setminus I_0|}$ and it suffices to prove the desired conclusion for the periodic subvariety Y_0.

Next, we introduce a relation on the set $\{1, \dots, n\}$ by saying that $i \sim j$ if the projection of Y on the two coordinate axes i-th and j-th yields a curve. Clearly, this relation is symmetric and it is easy to see that it is transitive as well (since we get that $\pi_{i,j}(Y)$ is a curve if and only if the coordinate rational functions x_i and x_j, seen as elements of $K(Y)$ are algebraically dependent). Also, this relation is reflexive in a trivial way because the projection of Y on any single coordinate axis must be a curve, since it is dominant; thus $\sim$ is an equivalence relation.

So, we partition $\{1, \dots, n\}$ into the equivalence classes $I_1, \dots, I_r$ corresponding to the relation $\sim$. By construction, the projection of Y on $(\mathbb{P}^1)^{|I_j|}$ must be an irreducible curve Y_j and we claim that $Y = Y_1 \times \cdots \times Y_r$. Indeed, we know from [Med07, MS14] that Y is an irreducible component of the intersection of various hypersurfaces of the form $\pi_{i,j}^{-1}(C_{i,j})$; this yields that Y is an irreducible component of the intersection

$$\bigcap_{j=1}^{r} \left(Y_j \times (\mathbb{P}^1)^{|\{1,\dots,n\}\setminus I_j|}\right). \tag{4.2.1}$$

However, the intersection from equation (4.2.1) is precisely the product of the curves Y_j, which is irreducible; hence $Y = Y_1 \times \cdots \times Y_r$, as claimed. Furthermore, the curves Y_j are defined over K (since Y is defined over K and each Y_j is the projection of Y on $(\mathbb{P}^1)^{|I_j|}$) and also, each Y_j must be periodic under the induced action of the rational functions f_i for $i \in I_j$ on the coordinate axes of $(\mathbb{P}^1)^{|I_j|}$.

This concludes our proof of Lemma 4.2. ☐

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. So, Lemma 4.2 yields that any given periodic subvariety $Y \subset (\mathbb{P}^1)^n$ is of the form

$$Y = Y_1 \times \cdots \times Y_r$$

for some periodic curves $Y_j \subset (\mathbb{P}^1)^{|I_j|}$ defined over K (for some suitable partition $\{I_j\}_{j=1}^r$ of $\{1, \dots, n\}$, as in Lemma 4.2).

By Theorem 3.1, we know that the period of each curve Y_j under the induced action of the f_i's is bounded independently of the Y_j's (and only depending on the degrees of the f_i's and also on p and $[K : \mathbb{Q}_p]$); note that we know that the projection of $(\alpha_1, \dots, \alpha_n)$ on each $(\mathbb{P}^1)^{|I_j|}$ is a point of Y_j whose coordinates are nonpreperiodic. Therefore, Y has its period bounded only on p, $[K : \mathbb{Q}_p]$ and on the degrees of the f_i's. This concludes the proof of Theorem 4.1. ☐

Theorem 4.3. *Theorem 1.2 holds under the assumption that each rational function f_i has degree equal to* 1.

Proof. We use Poonen's [Poo13] result on p-adic interpolation of iterates. Since each f_i is a degree one map with good reduction at π, there is an M depending only on p, e, and $[\kappa : \mathbb{F}_p]$ such that $f_i^M(x) \equiv x \pmod{p^c}$, where c is the smallest integer greater than $1/(p-1)$. Now, let $\gamma = (\gamma_1 \dots, \gamma_n)$ be any point in $Y(\overline{K})$, and let R be the ring of p-adic integers in $K(\gamma)$. Then taking the local power series expansion for f_i^M at γ_i gives a power series $g_i \in R[x]$, convergent on R, such that $g_i(z) = f_i^M(z)$ for all $z \in R$. Since $f_i^M(x) \equiv x \pmod{p^c}$, we have $g_i(x) \equiv x \pmod{p^c}$. By [Poo13], for each i, there is therefore a power series $\theta_i \in R[x]$, convergent on R, such that $\theta_i(k) = (f_i^M)^k(\gamma_i)$ for all positive integers k.

Now, let H be any polynomial in the vanishing ideal of Y, dehomogenized at γ. Since there are infinitely many k such that $\Phi^{kM}(\gamma) \in Y$ (because Y is periodic under Φ), there are infinitely many k such that $H(\theta_1(k), \dots, \theta_n(k)) = 0$. Since a convergent p-adic power series has finitely many zeros unless the series is identically zero, it follows that $H(\theta_1(k), \dots, \theta_n(k)) = 0$ for all k, so $\Phi^{Mk}(\gamma) \in Y$ for all k. Since this is true for all $\gamma \in Y(\overline{K})$, we must have that Y has period dividing M.

As an aside, we could argue also directly, using the fact that each f_i is a linear map to find explicit formulas for $f_i^n(\alpha_i)$ and thus construct explicit p-adic analytic functions as above, thus proving the desired conclusion in Theorem 4.3. □

Theorem 4.4. *Theorem 1.2 holds under the assumption that each rational function f_i has degree larger than 1 and is conjugate either to $x^{\pm \deg(f_i)}$, or to $\pm C_{\deg(f_i)}$, or to a Lattés map.*

Proof. At the expense of replacing each f_i by a conjugate (which does not change the conclusion of our result since it replaces the action of Φ by the action of a conjugate of it through an automorphism of $(\mathbb{P}^1)^n$), we may assume each $f_i(x)$ is either a Lattés map, or equal to $\pm C_{\deg(f_i)}$, or equal to $x^{\pm \deg(f_i)}$.

In this case there exist elliptic curves $E_1, \dots, E_k$ (for some $0 \le k \le n$) and there exists an endomorphism Ψ of $S := \mathbb{G}_m^{n-k} \times \prod_{i=1}^k E_i$ along with a finite morphism $\eta : S \longrightarrow (\mathbb{P}^1)^n$ such that

$$\eta \circ \Psi = \Phi \circ \eta. \tag{4.4.1}$$

Indeed, for each Lattés map f_i (without loss of generality, we assume $f_{n-k+1}, \dots, f_n$ are all the Lattés maps), we know there exists an elliptic curve E_i along with some endomorphism g_i and also there exists some morphism $\eta_i : E_i \longrightarrow \mathbb{P}^1$ of degree 2 (identifying each point P of E_i with $-P$) such that $\eta_i \circ g_i = f_i \circ \eta_i$ for each $i = n-k+1, \cdots, n$. Furthermore, assuming $f_1, \dots, f_\ell$ (for some $0 \le \ell \le n-k$) are all the rational functions conjugated to $\pm C_{\deg(f_i)}$, then for each $i = 1, \dots, \ell$, we let $\eta_i : \mathbb{G}_m \longrightarrow \mathbb{P}^1$ be defined by $\eta_i(x) = x + \frac{1}{x}$; also, we let $g_i(x) = \pm x^{\deg(f_i)}$ for each such i. Finally, for each $i = \ell+1, \dots, n-k$, we let $g_i = f_i$ (which is thus equal to $x^{\pm \deg(f_i)}$) and also let $\eta_i(x) = x$ for each such $i = \ell+1, \dots, n-k$. Then (4.4.1) holds with $\eta := (\eta_1, \dots, \eta_n)$ and also with $\Psi := (g_1, \dots, g_n) : S \longrightarrow S$ (acting coordinatewise).

Now, $Y \subset (\mathbb{P}^1)^n$ is periodic under the action of Φ if and only if an irreducible component of $Z := \eta^{-1}(Y)$ is periodic under the action of Ψ; moreover, the two periodic varieties would then have the same length for their corresponding orbits. Hence, it suffices to bound the period for any periodic subvariety of S defined over some given local field. Note that if Y is defined over K, then Z is defined over another local field L whose degree over K is bounded by 2^n since for each $i = 1, \dots, \ell$ and also for each $i = n-k+1, \dots, n$, the degree of η_i is 2 (while $\deg(\eta_i) = 1$ if $i = \ell+1, \dots, n-k$). Therefore, replacing K by L simply increases the size of the residue field of the new local field and also increases its ramification index (over $\mathbb{Q}_p$), but both the size of the residue field of K and the ramification index of K increase by a factor which depends only on n.

Furthermore, at the expense of conjugating each $g_i(x)$ (for $1 \le i \le \ell$) by a suitable linear transformation $x \mapsto \zeta_i x$, we may actually assume each $g_i(x) = x^{\deg(f_i)}$ (i.e., modulo a conjugation by a linear scaling map given by a root of unity, we may assume the sign infront of each monomial from the definition of $g_i(x)$ is positive). In order to replace the original maps by the new maps, we might need to replace K by a finite extension, which depends only on the degrees of the original rational functions f_i.

We observe that S is a split semiabelian variety (whose abelian part is a product of elliptic curves). Thus the periodic subvarieties of S (under the coordinatewise action of group homomorphisms on each of its 1-dimensional factors) are torsion translates of algebraic subgroups (a more general result holds inside any semiabelian variety, as proven in [Hin88, Lemme 10]). Moreover, noting that there are no nontrivial morphisms between a torus and an elliptic curve, then each periodic subvariety Z of S is the zero locus of finitely many equations of the form

$$\prod_{i=1}^{n-k} x_i^{c_i} = \zeta \tag{4.4.2}$$

for some root of unity $\zeta \in K$ and some integers c_i, or

$$\sum_{i=1}^{k} \psi_i(y_i) = Q, \tag{4.4.3}$$

where E is one of the elliptic curves E_j (for some $j = 1, \dots, k$) and $\psi_i : E_i \longrightarrow E$ are group endomorphisms, while Q is a torsion point of $E(K)$. Clearly, if E_i and E are not isogenuous, then ψ_i is the trivial map.

Furthermore, the algebraic subgroup of $\mathbb{G}_m^{n-k}$ defined by the equations

$$\prod_{i=1}^{n-k} x_i^{c_i} = 1 \tag{4.4.4}$$

(obtained by replacing each ζ by 1 in (4.4.2)) must be periodic under the coordinatewise action of the g_i's. Then we note that for each equation (4.4.4), letting I be the set consisting of all indices i such that $c_i \neq 0$, we have that $\deg(g_i)$ is the same for each $i \in I$, or otherwise the hypersurface given by the equation (4.4.4) is not periodic under the coordinatewise action of the g_i's for $i \in I$. So, each hypersurface of the form (4.4.4) is invariant under the coordinatewise action of $g_1^2, \dots, g_{n-k}^2$ on $\mathbb{G}_m^{n-k}$ since each g_i is a monomial $x^{\pm \deg(g_i)}$ and therefore $g_i^2(x) = x^{\deg(f_i)^2}$; in particular, the algebraic subgroup which is the zero locus of all equations (4.4.4) is fixed by the coordinatewise action of the g_i^2's.

Similarly, the algebraic subgroup of $E_1 \times \cdots \times E_k$ given by the equations

$$\sum_{i=1}^{k} \psi_i(y_i) = 0, \tag{4.4.5}$$

(obtained by replacing each Q by 0 in (4.4.3)) must be periodic under the coordinatewise action of $g_{n-k+1}, \dots, g_n$. We claim that its period is bounded by the number of roots of unity contained in the endomorphism ring of E. Indeed, whenever ψ_i is nontrivial, then E_i and E have isomorphic endomorphism rings and moroever, the endomorphism g_i of E_i descends to an endomorphism of E. Note that if ψ_i and ψ_j are nontrivial and the endomorphisms g_{n-k+i} and g_{n-k+j} of E_i, respectively of E_j, correspond to elements ω_i and ω_j in the endomorphism ring R of E whose quotient ω_i/ω_j is not a root of unity, then the algebraic subgroup given by equation (4.4.5) is not periodic under the action of $(g_{n-k+1}, \cdots, g_n)$. Therefore, the period of the algebraic group given by equation (4.4.5) is absolutely bounded because there are at most 6 roots of unity in an order of an imaginary quadratic number ring (such as the ring R).

Using the fact that Z is the zero locus of finitely many equations of the form (4.4.2) and (4.4.3), while the algebraic groups given by equations (4.4.4) and (4.4.5) have the size of their orbit under Ψ bounded by 6, we obtain that the length of the orbit of Z under Ψ is bounded in terms of the orders of the roots of unity ζ appearing in equations of the form (4.4.2) and also in terms of the orders of the torsion points Q appearing in equations of the form (4.4.3). So, we conclude that the period under the action of Ψ of any periodic subvariety of S (defined over the local field K) is bounded solely in terms of the size of the group of roots of unity contained in K and also in terms of the size of the torsion subgroups of $E_i(K)$, for $i = 1, \dots, k$. Since any local field contains finitely many roots of unity (the bound depending solely on the size of its residue field and also depending on its ramification index over $\mathbb{Q}_p$), and also, any elliptic curve has finitely many torsion points over the local field K, with an upper bound for the size of its torsion depending solely on the size of the residue field of K and on the ramification index for $K/\mathbb{Q}_p$ (see [Sil86, Chapter VII]), then we obtain the desired conclusion in Theorem 4.4. □

Finally, we can prove Theorem 1.2

Proof of Theorem 1.2. As proven in [Med07] (see also [MS14, Theorem 2.30] for the case when each f_i is a polynomial), each periodic subvariety Y of the dynamical system $((\mathbb{P}^1)^n, \Phi)$ may be written (after a suitable re-ordering of the coordinate axes, and thus of the rational functions acting on them) as $Y_1 \times Y_2 \times Y_3$, where $Y_i \subset (\mathbb{P}^1)^{n_i}$ for some integers n_i satisfying $n_1 + n_2 + n_3 = n$ and moreover, the following holds:

- the rational functions $f_1, \cdots, f_{n_1}$ are all disintegrated;
- each rational function $f_i(x)$ from the list $f_{n_1+1}, \cdots, f_{n_1+n_2}$ is conjugate either to $x^{\pm \deg(f_i)}$, or to $\pm C_{\deg(f_i)}$, or to a Lattés function (and its degree is greater than 1);
- the rational functions $f_{n_1+n_2+1}, \cdots, f_{n_1+n_2+n_3}$ have degree equal to 1;
- Y_1 is periodic under the action of $\Phi_1 := (f_1, \cdots, f_{n_1})$ on $(\mathbb{P}^1)^{n_1}$, while Y_2 is periodic under the action of $\Phi_2 := (f_{n_1+1}, \cdots, f_{n_1+n_2})$ and Y_3 is periodic under the action of $\Phi_3 := (f_{n_1+n_2+1}, \cdots, f_{n_1+n_2+n_3})$.

Therefore, the uniform bound on the length of the orbit of Y under the the action of Φ follows using Theorems 4.1, 4.3 and 4.4, which provide uniform bounds for the periods of each of the periodic subvarieties Y_j (for $1 \le j \le 3$). □

We conclude this section by proving Theorem 1.1.

Proof of Theorem 1.1. We argue by induction on n; the case $n = 1$ follows from [Zie96]. So, we assume now that $n \ge 2$ and that Theorem 1.1 holds for endomorphisms of $(\mathbb{P}^1)^{n-1}$. In particular, we may assume that Y projects dominantly onto each coordinate axes, since otherwise $Y = Y_0 \times \{\gamma\}$, where $\gamma \in \mathbb{P}^1_K$ is a periodic point (of bounded period, according to [Zie96]) and then the inductive hypothesis yields the desired conclusion.

Let π be a given uniformizer for the subring of π-adic integers $\mathfrak{o}_K$ contained in K; also, let κ be the residue field of $\mathfrak{o}_K$ and let e be the ramification index for $K/\mathbb{Q}_p$.

Let $\mathcal{Y}$ be a model for Y over $\mathfrak{o}_K$ such that $\mathcal{Y}$ is closed in $(\mathbb{P}^1_{\mathfrak{o}_K})^n$. Then $\mathcal{Y}$ is a projective $\mathfrak{o}_K$-scheme, so every point in $Y(K)$ extends to a unique point in $\mathcal{Y}(\mathfrak{o}_K)$. We let α denote the point in $\mathcal{Y}(\mathfrak{o}_K)$ to which $(\alpha_1, \dots, \alpha_n)$ extends.

Let $\bar{\alpha}$ be the reduction of α modulo π, i.e., it is the intersection of α with the special fiber of $\mathcal{Y}$. Since the intersection of α with the generic fiber Y of $\mathcal{Y}$ is a smooth point x, then [BGT15, Proposition 2.2] yields the existence of a Zariski dense, uncountable set $U \subset \mathcal{Y}(\mathfrak{o}_K)$ consisting of sections β whose intersection with the special fiber of $\mathcal{Y}$ equals $\bar{\alpha}$. The fact that U is uncountable follows from the p-adic implicit function theorem used in the proof of [BGT15, Proposition 2.2], which yields the existence of a p-adic submanifold of $\mathcal{Y}$ whose points all have the same reduction modulo π. More precisely, as shown in [BGT15, Proposition 2.2], letting $d := \dim(Y)$, at the expense of relabelling the coordinate axes, we know that for a sufficiently small p-adic neighborhood $\mathcal{U}$ of $(\alpha_{n-d+1}, \dots, \alpha_n)$ (where $\alpha := (\alpha_1, \dots, \alpha_n)$), we have p-adic analytic functions $\mathcal{F}_1, \dots, \mathcal{F}_n$ on $\mathcal{U}$ such that for each $\gamma \in \mathcal{U}$, the point $(\mathcal{F}_1(\gamma), \dots, \mathcal{F}_n(\gamma))$ is on $\mathcal{Y}$.

Each one of the points $\beta \in U$ is of the form $(\beta_1, \dots, \beta_n)$ and if for some point β, we have that each β_i is not preperiodic for the action of f_i, then the result follows from Theorem 1.2. So, assume for each $\beta \in U$, there exists some $i \in \{1, \dots, n\}$ such that β_i is preperiodic for f_i. Therefore there must exists some $i_0 \in \{1, \dots, n\}$ such that there exists an uncountable set $U_0 \subset \mathcal{Y}(\mathfrak{o}_K)$ consisting of points β with the property that the corresponding β_{i_0} is preperiodic for the action of f_{i_0}. Furthermore, we may assume that there exists no submanifold $\mathcal{U}_0$ of $\mathcal{U}$ of dimension less than d such that for each $\beta \in U_0$, there exists some $\gamma \in \mathcal{U}_0$ such that $\beta = (\mathcal{F}_1(\gamma), \cdots, \mathcal{F}_n(\gamma))$.

For the sake of simplifying the notation, we identify each β_{i_0} with its intersection with the generic fiber of $\mathbb{P}^1_{\mathfrak{o}_K}$ and thefore, view each β_{i_0} as a point in $\mathbb{P}^1(K)$ whose reduction modulo π is $\overline{\alpha_{i_0}}$. There are two cases.

Case 1. The set $\{\beta_{i_0} : \beta \in U_0\}$ is uncountable.

Then, according to our assumption, we obtain that f_{i_0} has an uncountable set of preperiodic points, which automatically yields that f_{i_0} itself must be preperiodic; more precisely, $\deg f_{i_0} = 1$ (i.e, f_{i_0} is an automorphism) and $f_{i_0}^{\ell}$ is the identity, for some positive integer ℓ. So, at the expense of replacing $\mathcal{Y}$ by $\mathcal{Y}^{\ell}$, we may assume $\mathcal{Y}$ is fixed under the action of f_{i_0} on the i_0-th coordinate axis. Therefore, the length of the orbit of $\mathcal{Y}$ equals the length of the orbit of $\mathcal{Y}_0$ under the coordinatewise action of the rational functions f_i for $i \neq i_0$, where $\mathcal{Y}_0$ is the projection of $\mathcal{Y}$ on the remaining $(n-1)$ coordinate axes, other than the i_0-th coordinate axis. In this case, we are done by the inductive hypothesis; also, note that ℓ depends solely on p, e and $[\kappa : \mathbb{F}_p]$. Indeed, the linear map $f_{i_0}(x)$ is conjugated over a quadratic extension L of K to one of these two maps $x \mapsto x+1$ or $x \mapsto ax$; furthermore, since f_{i_0} has finite order, then $f_{i_0}(x)$ must be conjugated to a map $x \mapsto ax$, where a is a root of unity contained in L. Because the number of roots of unity in a local field L is bounded solely in terms of p and of the degree $[L : \mathbb{Q}_p]$, our claim follows.

Case 2. The set $\{\beta_{i_0} : \beta \in U_0\}$ is countable.

Let $\mathcal{U}_0 \subset \mathcal{U}$ be the set consisting of all points $\gamma \in \mathcal{U}$ such that $(\mathcal{F}_1(\gamma), \dots, \mathcal{F}_n(\gamma)) \in U_0$. Then we know that the set $\{\mathcal{F}_{i_0}(\gamma) : \gamma \in \mathcal{U}_0\}$ is countable; furthermore, by our assumption, we know that $\mathcal{U}_0$ is an uncountable set, which is not contained in a submanifold of $\mathcal{U}$ of dimension less than d. Therefore, $\mathcal{F}_{i_0}$ must be constant, and thus the projection of $\mathcal{Y}$ on the i_0-th coordinate axis must be constant, contrary to our hypothesis from the beginning of our proof. This concludes the proof of Theorem 1.1. □

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