A QUESTION FOR ITERATED GALOIS GROUPS IN ARITHMETIC DYNAMICS

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ABSTRACT. We formulate a general question regarding the size of the iterated Galois groups associated to an algebraic dynamical system and then we discuss some special cases of our question. Our main result answers this question for certain split polynomial maps whose coordinates are unicritical polynomials.

1. Introduction and Statement of Results

1.1. **A general question.** For any self-map Φ on a variety X and for any integer $n \geq 0$, we let Φ^n be the n-th iterate of Φ (where Φ^0 is the identity map, by definition). For a point $x \in X$ with the property that each point $\Phi^n(x)$ avoids the indeterminacy locus of Φ, we denote by $\mathcal{O}_\Phi(x)$ the orbit of x under Φ, i.e., the set of all $\Phi^n(x)$ for $n \geq 0$; also, the strict forward orbit of the point x refers to the set of all $\Phi^n(x)$ for $n > 0$. We say that x is preperiodic if its orbit $\mathcal{O}_\Phi(x)$ is finite; furthermore, if $\Phi^n(x) = x$ for some positive integer n, then we say that x is periodic.

For a projective variety X endowed with an endomorphism Φ, we say that Φ is polarizable if there exists an ample line bundle $\mathcal{L}$ on X such that $\Phi^*\mathcal{L}$ is linearly equivalent to $\mathcal{L}^{\otimes d}$ for some integer $d > 1$. In particular, polarizable endomorphisms are dominant.

Question 1.1. *Let K be a number field or a function field of finite transcendence degree over a field of characteristic 0, let X be a smooth projective variety defined over K and let $\Phi : X \longrightarrow X$ be a polarizable endomorphism.*

For each positive integer n, we have that Φ^n induces an inclusion of the function field $K(X)$ into itself; we let $G_n(\Phi, X)$ be the Galois group of the Galois closure of $K(X)$ over itself with respect to this inclusion. We let $G_\infty := G_\infty(\Phi, X)$ be the inverse limit of these groups $G_n(\Phi, X)$.

For each point $x \in X(K)$, we let $G_n(\Phi, x)$ be the Galois group of $K(\Phi^{-n}(x))$ over $K(x)$. We let $G_\infty(x) := G_\infty(\Phi, x)$ be the inverse limit of the groups $G_n(\Phi, x)$. We have that there is a natural embedding of $G_\infty(x)$ inside G_∞.

Then is it true that at least one of the following statements must hold?

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(A) *The index* $[G_\infty(\Phi, X) : G_\infty(\Phi, x)]$ *is finite.*
(B) *The point* x *lies in the strict forward orbit of a point in the ramification locus of* Φ.
(C) *The point* x *lies on a proper subvariety* $Y \subset X$ *that is invariant under a non-identity self-map* $\Psi : X \longrightarrow X$ *with the property that* $\Phi^n \circ \Psi = \Psi \circ \Phi^n$ *for some some positive integer* n.

Jones [Jon13, Conjecture 3.11] proposes a similar conjecture for quadratic rational functions $\Phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ over number fields, but our Question 1.1 has not previously been posed for arbitrary function fields K (of characteristic 0) or for self-maps of higher dimensional varieties. Also, we are very grateful to the anonymous referee for pointing out that condition (C) in Question 1.1 covers some well-known "degenerate" cases, as follows. So, if $X = \mathbb{P}^1$ while $\Phi(y) = y^2 - 2$ and $x = 0$, then in this case $G_\infty(\Phi, X)$ is the affine general linear group over $\mathbb{Z}_2$, while $G_\infty(\Phi, x)$ is an infinite-index cyclic subgroup; hence condition (A) in Question 1.1 does not hold in this example (also, condition (B) does not hold in this example). However, for this example, the proper subvariety $Y = \{x\}$ is invariant under the degree-3 Chebyshev polynomial $T_3(y) = y^3 - 3y$, which commutes with Φ. Another well-known degenerate case is when the point x is in fact periodic under Φ, and here we have $Y = \{x\}$ and $\Psi = \Phi^m$, where m is the period of x under Φ.

We note that one needs to exclude the case of finite fields in Question 1.1 since generally, the Galois group $G_\infty = G_\infty(\Phi, X)$ is not abelian (even in the case $X = \mathbb{P}^1$ and Φ is a rational function), while $G_\infty(x)$ would have to be abelian if X were defined over a finite field $\mathbb{F}_q$ because the Galois group of any extension of finite fields is abelian. One could ask the same question from Question 1.1 under the assumption that K is a finitely generated infinite field (even when its characteristic is positive); however, there are potential complications when the map Φ is not separable. On the other hand, if Φ were separable, it is conceivable that one might expect the same conclusion as in Question 1.1.

Now, in Question 1.1, one can see that if either conclusion (B) or (C) holds, then this is likely to prevent the index $[G_\infty : G_\infty(x)]$ from being finite; this is similar to what happens even in the case of rational functions, i.e., $X = \mathbb{P}^1$ (see [BDG+20, Proposition 3.2]).

In this paper we discuss some special cases of Question 1.1 which generally fall outside the scope of the previous studies of the arboreal Galois representation associated to a dynamical system. In particular, we treat the case when K is a function field of transcendence degree greater than one (and $\Phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is a polynomial mapping), and also we discuss several cases when X is a higher dimensional variety.

1.2. **The case of polynomials defined over function fields of higher transcendence degree.** Here we explain our results towards Question 1.1 when $X = \mathbb{P}^1$ and Φ is a polynomial mapping (which we denote by f), while K is the function field of an arbitrary smooth projective variety defined over $\overline{\mathbb{Q}}$. We start by explaining in more detail the groups appearing in Question 1.1 for polynomial mappings f.

So, let K be any field, let $f \in K[x]$ with $d = \deg f \geq 2$ and let $\beta \in \mathbb{P}^1(\overline{K})$. For $n \in \mathbb{N}$, let $K_n(f, \beta) = K(f^{-n}(\beta))$ be the field obtained by adjoining the nth preimages of β under f to $K(\beta)$. (We declare that $K(\infty) = K$.) Set $K_\infty(f, \beta) = \bigcup_{n=1}^{\infty} K_n(f, \beta)$. For $n \in \mathbb{N} \cup \{\infty\}$, define $G_n(f, \beta) = \mathrm{Gal}(K_n(f, \beta)/K(\beta))$. In most of the paper, we will write $G_n(\beta)$ and $K_n(\beta)$, suppressing the dependence on f if there is no ambiguity.

The group $G_\infty(\beta)$ embeds into $\mathrm{Aut}(T^d_\infty)$, the automorphism group of an infinite d-ary rooted tree T^d_∞. Recently there has been much work on the problem of determining when the index $[\mathrm{Aut}(T^d_\infty) : G_\infty(\beta)]$ is finite (see [BFH+16, BIJ+17, BGJT19, BG19, BJ07, BJ09, JKMT16, Juu19, Jon07, JM14, Odo85, Odo88, Pin13a, Pin13b, Jon13, BDG+20]). By work of Odoni [Odo85], one expects that a generically chosen rational function has a surjective arboreal representation, i.e., that $[\mathrm{Aut}(T^d_\infty) : G_\infty(\beta)] = 1$.

In [BDG+20], the authors studied the family of polynomials $f(x) = x^d + c \in K[x]$, where K is the function field of a smooth irreducible projective curve defined over $\overline{\mathbb{Q}}$; note that up to a change of variables, the above polynomials $f(x)$ represent all polynomials with precisely one (finite) critical point. Since the field K contains a primitive d-th root of unity, then it is easy to show that for f in this family, $G_\infty(\beta)$ sits inside $[C_d]^\infty$, the infinite iterated wreath product of the cyclic group C_d (with d elements); actually, with the notation as in Question 1.1, we have that $G_\infty = [C_d]^\infty$. As $\mathrm{Aut}(T^d_n) \cong [S_d]^n$, this means that if $d \geq 3$, then $[\mathrm{Aut}(T^d_\infty) : [C_d]^\infty] = \infty$. Thus it is impossible for $G_\infty(\beta)$ to have finite index in $\mathrm{Aut}(T^d_\infty)$ within this family (except when $d = 2$). However, this simply means that, given the constraint on the size of $G_\infty(\beta)$, we should ask a different finite index question. So, the correct problem to study is when $G_\infty(\beta)$ has finite index in $[C_d]^\infty = G_\infty$, exactly as predicted by Question 1.1. In our current paper, we extend the results of [BDG+20] to the case where K is the function field of a higher-dimensional variety defined over $\overline{\mathbb{Q}}$.

So, let K be the function field of a smooth projective irreducible variety V over $\overline{\mathbb{Q}}$. We say that $f \in K[x]$ is isotrivial if f is defined over $\overline{\mathbb{Q}}$ up to a change of variables, that is, if $\varphi^{-1} \circ f \circ \varphi \in \overline{\mathbb{Q}}[x]$ for some $\varphi \in \overline{K}[x]$ of degree 1. In the special case of a unicritical polynomial $f(x) = x^d + c \in K[x]$, we have that f is isotrivial if and only if $c \in \overline{\mathbb{Q}}$. We say that β is postcritical for f if $f^n(\alpha) = \beta$ for some $n \geq 1$ and some critical point α of f.

The following result is an extension of [BDG+20, Theorem 1.1] to function fields of arbitrary finite transcendence degree and it represents a special case of our Question 1.1.

Theorem 1.2. *Let K be the function field of a smooth projective variety defined over $\overline{\mathbb{Q}}$. Let $q = p^r$ ($r \geq 1$) be a power of the prime number p, let $c \in K \setminus \overline{\mathbb{Q}}$, let $f(x) = x^q + c \in K[x]$ and let $\beta \in K$. Then the following are equivalent:*

(1) *The point β is neither periodic nor postcritical for f.*
(2) *The group $G_\infty(\beta)$ has finite index in G_∞.*

It is fairly easy to see that the conditions on β in Theorem 1.2 are necessary (see [BDG+20, Proposition 3.2]); so, the entire difficulty lies in showing that these conditions are sufficient. Also, we note that the isotrivial case (i.e., $c \in \overline{\mathbb{Q}}$) follows verbatim using the proof from [BDG+20, Section 10]. We require the degree d of our polynomials in Theorem 1.2 be powers of prime numbers so that the map $x \mapsto x^d$ is injective on $\overline{\mathbb{F}_p}$; this fact was crucial in the proof of [BDG+20, Proposition 6.3], which is then used in the proof of our Proposition 4.1.

As proven in [BDG+20], one of the key steps in the proof of Theorem 1.2 is an eventual stability result. As is usual in arithmetic dynamics, we say that the pair (f, β) is eventually stable over the field K if the number of irreducible K-factors of $f^n(x) - \beta$ is uniformly bounded for all n. The following result extends [BDG+20, Theorem 1.3] to function fields of arbitrary transcendence degree.

Theorem 1.3. *Let K be the function field of a smooth projective variety defined over $\overline{\mathbb{Q}}$. Let $q = p^r$ ($r \geq 1$) be a power of the prime number p. Let $f \in K[x]$ be a polynomial of the form $x^q + c$ where $c \notin \overline{\mathbb{Q}}$. Then for any non-periodic $\beta \in K$, the pair (f, β) is eventually stable over K.*

We also prove the following disjointness theorem for fields generated by inverse images of different points under different maps; our result is a generalization of [BDG+20, Theorem 1.4].

Theorem 1.4. *Let K be the function field of a smooth projective variety defined over $\overline{\mathbb{Q}}$. For $i = 1, \dots, m$ let $f_i(x) = x^q + c_i \in K[x]$, where $c_i \notin \overline{\mathbb{Q}}$, and let $\alpha_i \in K$. Suppose that there are no distinct i, j with the property that (α_i, α_j) lies on a curve in $\mathbb{A}^2$ that is periodic under the action of $(x, y) \mapsto (f_i(x), f_j(y))$. For each i, let M_i denote $K_\infty(f_i, \alpha_i)$. Then for each $i = 1, \dots, m$, we have that*

$$\left[M_i \cap \left(\prod_{j \neq i} M_j \right) : K \right] < \infty.$$

Theorem 1.4 coupled with Theorem 1.2 yields the following special case of Question 1.1.

Theorem 1.5. *Let K be the function field of a smooth projective variety defined over $\overline{\mathbb{Q}}$, let q be a power of the prime number p, and let m be a positive integer. For $i = 1, \dots, m$ let $f_i(x) = x^q + c_i \in K[x]$, where $c_i \notin \overline{\mathbb{Q}}$, and let $\alpha_i \in K$. We let $\underline{\alpha} := (\alpha_1, \dots, \alpha_m)$ and let $\Phi := (f_1, \dots, f_m)$ acting on $X := (\mathbb{P}^1)^m$. Then let $G_n(\Phi, \underline{\alpha})$ be the Galois group of $K(\Phi^{-n}(\underline{\alpha}))$ over K. We let $G_\infty(\underline{\alpha}) := G_\infty(\Phi, \underline{\alpha})$ be the inverse limit of the groups $G_n(\Phi, \underline{\alpha})$.*

Then at least one of the following must hold:

(A) *$[G_\infty(\Phi, X) : G_\infty(\Phi, \underline{\alpha})]$ is finite;*

(B) $\underline{\alpha}$ *lies in the strict forward orbit of a point in the ramification locus of* Φ*; or*

(C) $\underline{\alpha}$ *lies on a proper subvariety* $Y \subset X$ *that is invariant under a non-identity self-map* $\Psi : X \longrightarrow X$ *with the property that* $\Phi \circ \Psi = \Psi \circ \Phi$.

Finally, using a similar strategy as employed in [BG19], we obtain the following result regarding the arboreal Galois representation associated to cubic polynomials. Once again, we use the notation from Question 1.1 for G_∞ and $G_\infty(x)$ and since both groups lie naturally inside $\mathrm{Aut}(T^3_\infty)$, then in order to prove the finiteness of the index of $G_\infty(x)$ inside G_∞, it suffices to prove $[\mathrm{Aut}(T^3_\infty) : G_\infty(x)] < \infty$.

Theorem 1.6. *Let* K *be the function field of a smooth projective variety defined over* $\overline{\mathbb{Q}}$*. Let* $f \in K[x]$ *be a cubic polynomial. Assume that* f *is not isotrivial over* $\overline{\mathbb{Q}}$*, that* β *is not periodic or postcritical for* f*, that the pair* (f, β) *is eventually stable, and that* f *has distinct finite critical points* γ_1*, and* γ_2*, and* $f^n(\gamma_1) \neq f^n(\gamma_2)$ *for all* $n \geq 1$*. Then*

$$[\mathrm{Aut}(T^3_\infty) : G_\infty(\beta)] < \infty.$$

Our proofs rely mainly on specialization techniques in order to extend the results of [BDG+20] and [BG19] to the generality of the current article. Actually, considering the extension of our results from [BDG+20] to arbitrary function fields led us to formulate the general Question 1.1.

1.3. **The case of higher dimensional varieties.** Question 1.1 was also motivated by the case of the multiplication-by-m maps Φ (for some integer $m > 1$) on abelian varieties X. In that case, the conclusion of Question 1.1 is known due to Ribet's work [Rib79] (who generalizes results of Bachmakov [Bac70]); as long as the point $x \in X$ is not torsion, we know that the index $[G_\infty : G_\infty(x)]$ is finite. Actually, the first result in this direction was the case of the monomial map $\Phi(z) := z^m$ (for integer $m > 1$) acting on $\mathbb{P}^1$, in which case, the conclusion in Question 1.1 reduces to the classical theory of Kummer extensions.

Similarly, due to work of Pink [Pin16], one establishes also the conclusion of Question 1.1 in the special case of Drinfeld modules, i.e., if Φ is a separable additive polynomial (which is, therefore, an endomorphism of $\mathbb{G}_a$ defined over a field of characteristic p) of degree larger than 1 and whose derivative is a transcendental element over $\mathbb{F}_p$. This justifies our belief that as long as Φ is separable, then Question 1.1 should hold as well for finitely generated, infinite fields of positive characteristic.

In Section 6 we present additional evidence supporting our Question 1.1.

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2. Wreath products

In this section (which overlaps with [BDG+20, Section 2]), we give a brief introduction to wreath products, which arise naturally from the Galois theory of the preimage fields $K_n(\beta) = K(f^{-n}(\beta))$.

Let G be a permutation group acting on a set X, and let H be any group. Let H^X be the group of functions from X to H with multiplication defined pointwise, or equivalently the direct product of $|X|$ copies of H. The wreath product of G by H is the semidirect product $H^X \rtimes G$, where G acts on H^X by permuting coordinates: for $f \in H^X$ and $g \in G$ we have

$$f^g(x) = f(g^{-1}x)$$

for each $x \in X$. We will use the notation $G[H]$ for the wreath product, suppressing the set X in the notation. (Another common convention is $H \wr G$ or $H \wr_X G$ if we wish to call attention to X.)

Fix an integer $d \geq 2$. For $n \geq 1$, let T_n^d be the complete rooted d-ary tree of level n. It is easy to see that $\mathrm{Aut}(T_1^d) \cong S_d$, and standard to show that $\mathrm{Aut}(T_n^d)$ satisfies the recursive formula

$$\mathrm{Aut}(T_n^d) \cong \mathrm{Aut}(T_{n-1}^d)[S_d].$$

Therefore we may think of $\mathrm{Aut}(T_n^d)$ as the "nth iterated wreath product" of S_d, which we will denote $[S_d]^n$. In general, for $f \in K[x]$ of degree d and $\beta \in K$, the Galois group $G_n(\beta) = \mathrm{Gal}(K_n(\beta)/K)$ embeds into $[S_d]^n$ via the faithful action of $G_n(\beta)$ on the nth level of the tree of preimages of β (see for example [Odo85] or [BG19, Section 2]).

Assume now that $f(x) := x^d + c \in K[x]$, where K is a field of characteristic 0 that contains the d-th roots of unity. For $\beta \in K$ such that $\beta - c$ is not a dth power in K, we have $K_1(\beta) = K((\beta - c)^{1/d})$ and $G_1(\beta) \cong C_d$. For any $n \geq 2$, the extension $K_n(\beta)$ is a Kummer extension attained by adjoining to $K_{n-1}(\beta)$ the d-th roots of $z - c$ where z ranges over the roots of $f^{n-1}(x) = \beta$. Thus we have

$$\mathrm{Gal}(K_n(\beta)/K_{n-1}(\beta)) \subseteq \prod_{f^{n-1}(z)=\beta} \mathrm{Gal}(K_{n-1}(\beta)((z-c)^{1/d})/K_{n-1}(\beta)) \subseteq C_d^{d^{n-1}}.$$

This is clear if $f^{n-1}(x) - \beta$ has distinct roots in $\overline{K}$. If $f^{n-1}(x) - \beta$ has repeated roots, then $\mathrm{Gal}(K_n(\beta)/K_{n-1}(\beta))$ sits inside a direct product of a smaller number of copies of C_d, so the stated containments still hold.

Considering the Galois tower

$$K_n(\beta) \supseteq K_{n-1}(\beta) \supseteq K$$

we see that

$$G_n(\beta) \subseteq \mathrm{Gal}(K_n(\beta)/K_{n-1}(\beta)) \rtimes G_{n-1}(\beta) \cong G_{n-1}(\beta)[C_d],$$

where the implied permutation action of $G_{n-1}(\beta)$ is on the set of roots of $f^{n-1}(x) - \beta$. By induction, $G_n(\beta)$ embeds into $[C_d]^n$, the n-th iterated wreath product of C_d. Observe that $[C_d]^n$ sits as a subgroup of $\mathrm{Aut}(T_n^d) \cong$

$[S_d]^n$ via the obvious action on the tree. Taking inverse limits, $G_\infty(\beta)$ embeds into $[C_d]^\infty$, which sits as a subgroup of $\mathrm{Aut}(T_\infty)$.

We summarize our basic strategy for proving that $G_\infty(\beta)$ has finite or infinite index in $[C_d]^\infty$ as Proposition 2.1 (whose proof is identical with the one for [BDG+20, Proposition 2.1]).

Proposition 2.1. *Let $f = x^d + c \in K[x]$. Then $[[C_d]^\infty : G_\infty(\beta)] < \infty$ if and only if $\mathrm{Gal}(K_n(\beta)/K_{n-1}(\beta)) \cong C_d^{d^{n-1}}$ for all sufficiently large n.*

3. Heights

In this section we set up the notation regarding heights. Since later we will need to consider function fields over arbitrary fields (see the proof of Theorem 1.4, for example), we will introduce the heights associated to function fields in a general setting; for more details, see [BG06, Chapter 1]. First, we recall the notation of $\log^+$: for each real number z, we have $\log^+ |z| := \log \max\{1, |z|\}$.

So, K is the function field of a smooth projective (irreducible) variety V defined over a field k (of characteristic 0). As proven in [BG06, Section 1.4], there exists a set Ω_V of places of the function field K/k associated to the codimension 1 irreducible subvarieties of V; furthermore, there exist positive integers n_v (for each $v \in \Omega_V$) such that the product formula holds for the nonzero elements $z \in K$:

$$\prod_{v \in \Omega_V} |z|_v^{n_v} = 1. \tag{3.0.1}$$

Then we have the Weil height associated to the set of places in Ω_V; for simplicity, we omit the variety V from the notation for the Weil height and instead, we simply denote

$$h_{K/k}(z) = \sum_{v \in \Omega_V} n_v \cdot \log^+ |z|_v.$$

Naturally, the Weil height extends to all points in $\overline{K}$ (see [BG06, Section 1.5]).

We denote by $v(\cdot)$ the (exponential) valuation associated to each place in Ω_V. For $f \in K[x]$ with $\deg f = d \geq 2$, let $\widehat{h}_f(z)$ be the Call-Silverman canonical height of z relative to f [CS93], defined by

$$\widehat{h}_f(z) = \lim_{n \to \infty} \frac{h_{K/k}(f^n(z))}{d^n}.$$

The next two results are [BDG+20, Lemma 3.1] and a generalization of [BDG+20, Proposition 3.2] to the more general setting of arbitrary function fields; for the latter, the proof is identical to that of the corresponding statement from [BDG+20].

Lemma 3.1 (Capelli's lemma)**.** *Let K be any field and let $f, g \in K[x]$. Suppose $\alpha \in \overline{K}$ is any root of f. Then $f(g(x))$ is irreducible over K if and only if both $f(x)$ is irreducible over K and $g(x) - \alpha$ is irreducible over $K(\alpha)$.*

Proposition 3.2. *Let K be a function field of finite transcendence degree over a field of characteristic* 0*. Suppose $f(x) = x^d + c \in K[x]$ with $d \geq 2$, and let $\beta \in K$. If β is either periodic or postcritical for f, then $[[C_d]^\infty : G_\infty(\beta)] = \infty$.*

4. Eventual Stability

In this section, we show that if K is a function field over a finitely generated field of characteristic 0 and $f \in K[x]$ is a non-isotrivial unicritical polynomial of degree equal to a prime power, then f is eventually stable. Our results are an extension of the results obtained in [BDG+20, Section 6].

Proposition 4.1. *Let $q = p^r$ (for $r \geq 1$) be a power of the prime number p and let K be a function field of transcendence degree* 1 *over a finitely generated field k of characteristic* 0*. Let $f(x) = x^q + c \in K[x]$, where $c \notin k$. Then for any $\beta \in K$ that is not periodic under f, the pair (f, β) is eventually stable over K.*

Remark 4.2. We note that since in Theorem 1.3 we work under the assumption that the polynomial $x^q + c$ is not defined over $\overline{\mathbb{Q}}$, i.e., $c \notin \overline{\mathbb{Q}}$, then we can readily choose some finitely generated field k such that K is the function field of a curve C defined over k and $c \notin k$ (which is equivalent with $c \notin \overline{k}$ since k is algebraically closed in K). Indeed, letting $t_1, \dots, t_m \in K$ be algebraically independent over $\overline{\mathbb{Q}}$ such that K is algebraic over $\mathbb{Q}(t_1, \dots, t_m)$, then since $c \notin \overline{\mathbb{Q}}$, there exists some $i \in \{1, \dots, m\}$ such that $c \notin \overline{k_i}$, where $k_i := \mathbb{Q}(t_1, \dots, t_{i-1}, t_{i+1}, \dots t_m)$ (note that $\bigcap_i \overline{k_i} = \overline{\mathbb{Q}}$). Since $\operatorname{trdeg} K/k_i = 1$, this allows us in Theorem 1.3 to reduce to the case K/k is a function field of transcendence degree 1 and $c \notin \overline{k}$.

Proof of Proposition 4.1. Since K/k is a function field of transcendence degree 1 and $c \notin k$, then K must be a finite extension of $k(c)$ (and therefore, also a finite extension of $k(c, \beta)$). Thus, the pair (f, β) is eventually stable over K if and only if it is eventually stable over $k(c, \beta)$; hence, from now on, we may assume that $K = k(c, \beta)$. Furthermore, since for each n, we have that $f^{-n}(\beta)$ is contained in some algebraic extension of $\mathbb{Q}(c, \beta)$, then we may (and do) assume that K is a finite extension of $\mathbb{Q}(c, \beta)$; note that here we use in an essential way that k is a finitely generated field and therefore, $K \cap \overline{\mathbb{Q}(c, \beta)}$ must be a finite extension of $\mathbb{Q}(c, \beta)$.

Now, if β is not algebraic over $\mathbb{Q}(c)$, then $f^n(x) - \beta$ is easily seen to be irreducible over $\mathbb{Q}(c, \beta)$ for all n; for example, this follows immediately by looking at its Newton polygon at the place at infinity for the function field $\mathbb{Q}(c, \beta)/\mathbb{Q}(c)$. This would imply that $f^n(x) - \beta$ has finitely many factors over K; hence, from now on, we may assume that $\mathbb{Q}(c, \beta)$ is an algebraic extension of $\mathbb{Q}(c)$. But then we are back to the setting of [BDG+20, Proposition 6.3] (i.e., we have a function field of transcendence degree one over a number field) and the result follows immediately. This concludes our proof of Proposition 4.1. $\square$

5. Proof of Main Theorems

Proof of Theorem 1.3. We know that K is the function field of a projective smooth irreducible variety defined over $\overline{\mathbb{Q}}$; also, we know that the number c (where $f(x) = x^q + c$) is not contained in $\overline{\mathbb{Q}}$. As explained in Remark 4.2, there exists a finitely generated field k such that

(A) $\mathrm{trdeg}_k K = 1$; and
(B) $c \notin \overline{k}$.

Furthermore, at the expense of replacing both k and K by finite extensions, we may assume there exists a smooth projective curve C such that $c, \beta \in L := k(C)$ and moreover, $K = \overline{\mathbb{Q}}L$.

By Proposition 4.1, the pair (f, β) is eventually stable over L. It will suffice now to show that

$$\overline{\mathbb{Q}} \cap \left(\bigcup_{n=1}^{\infty} L(f^{-n}(\beta)) \right) \text{ is a finite extension of } \mathbb{Q}.$$

Since (f, β) is eventually stable over L, there exists an m such that $f^n(x) - \alpha_i$ is irreducible over $L(\alpha_i)$ for all α_i such that $f^m(\alpha_i) = \beta$ and all $n \geq m$ (see [BG19, Prop 4.2] for a proof of this fact). Applying [BDG+20, Proposition 7.7] (which extends verbatim to our setting if $s = 1$ in [BDG+20, Proposition 7.7]) and also applying [BDG+20, Proposition 8.1] (again in the special case $s = 1$, which does not require the results of [BDG+20, Section 5]), we see then that there is an integer n_1 such that for all $n > n_1$, the field $L_n(\beta) := L(f^{-n}(\beta))$ contains no nontrivial extensions of $L_{n-1}(\beta)$ that are unramified over $L_{n-1}(\beta)$. Let γ be any element of $\overline{\mathbb{Q}} \cap L_\ell(\beta)$ for some ℓ. Let N be minimal among all integers such that $\gamma \in L_N(\beta)$. Since $L_{N-1}(\beta)(\gamma)$ is unramified over $L_{N-1}(\beta)$, it follows that $N \leq n_1$. Hence $\overline{\mathbb{Q}} \cap \bigcup_{n=1}^{\infty} L(f^{-n}(\beta))$ is a finite extension of $\mathbb{Q}$, as desired. □

Proof of Theorem 1.2. As proven in [BDG+20, Proposition 3.2], we already know that the conditions are necessary. Therefore assume that β is not postcritical nor periodic for f. By Theorem 1.3, the pair (f, β) is eventually stable. Again using Lemma 3.1, there is some m such that for all $\alpha \in f^{-m}(\beta)$ and for all $n \geq 1$, $f^n(x) - \alpha$ is irreducible over $K_m(\beta)$. By [BDG+20, Proposition 7.7] and also using [BDG+20, Proposition 8.1] (which are both valid in our setting in the special case when $s = 1$, which does not require the results of [BDG+20, Section 5]), there exists n_0 such that for all $n \geq n_0$, we have

$$\mathrm{Gal}(K_n(\beta)/K_{n-1}(\beta)) \cong C_q^{q^{n-1}}.$$

By Proposition 2.1, we are done. □

Proof of Theorem 1.4. We note that Theorem 1.4 was proven in [BDG+20, Theorem 1.4] when K is a function field of transcendence degree 1 over a number field. So, from now on, assume K has transcendence degree $m > 1$ (over a number field). Then (at the expense of replacing K by a finite extension) there exists a finite tower of field extensions $L_0 \subset L_1 \subset \cdots \subset L_m = K$, where each L_i/L_{i-1} is a function field of (relative) transcendence degree 1 and furthermore, L_0 is a number field. Our strategy is to show that given any function field K/L of transcendence degree 1 (where L itself is a function field over another field F) with the property that the α_i's and the c_i's satisfy the hypotheses of Theorem 1.4, then there exists a specialization at a place γ for the function field K/L with the property that the corresponding $\alpha_i(\gamma)$'s and the $c_i(\gamma)$'s still satisfy the same hypotheses. So, after finitely many suitable specializations, we descend our problem to the case when all the points are contained in a function field of transcendence degree 1 over a number field and therefore, the desired conclusion is delivered by [BDG+20, Theorem 1.4].

First of all, we let $h_{K/L}$ be the Weil height for the function field $K = L(C)$ where C is a projective smooth curve defined over L. We assume the α_i's and the c_i's verify the hypotheses of Theorem 1.4. Also, we let $h_C : C(\overline{L}) \longrightarrow \mathbb{R}_{\geq 0}$ be the Weil height associated to a given ample divisor on C. For each polynomial $f \in K[x]$ of degree larger than one, we have the canonical height $\widehat{h}_f$ associated to the polynomial f. Also, since L itself is a function field over some other field F, then for each point z of $\overline{L}$, we let $h_{L/F}(z)$ be its Weil height computed with respect to the function field L/F.

For each $\gamma \in C(\overline{L})$ for which the coefficients of $f(x)$ are well-defined at γ, we let f_γ be the specialization of f at γ; in our proof, we will work with $f(x) := x^q + c$ for $c \in K$ and thus (viewing $c \in L(C)$), the specialization $f_\gamma(x) := x^q + c(\gamma)$ is constructed for points $\gamma \in C(\overline{L})$ such that $c(\gamma)$ is well-defined; clearly, the c_i's and the α_i's are well-defined at all but finitely many specializations for the function field K/L. Also, for all but finitely many specializations, we have that

(i) if for a pair of distinct indices $i \neq j$, we have that c_i/c_j is not a $(q-1)$-st root of unity, then also

$$c_i(\gamma)/c_j(\gamma) \text{ is not a } (q-1)\text{-st root of unity.} \tag{5.0.1}$$

Lemma 5.1. *There exists $M > 0$ such that whenever $h_C(\gamma) > M$, we have that each $c_i(\gamma) \notin \overline{\mathbb{Q}}$.*

Proof of Lemma 5.1. Clearly, it suffices to prove this statement for one c_i, which we will denote by c. If $c \in \overline{L}$, then c is invariant under specialization, so our hypothesis that $c \notin \overline{\mathbb{Q}}$ ensures that $c(\gamma) \notin \overline{\mathbb{Q}}$ for every specialization. On the other hand, if $c \notin \overline{L}$, then $h_{K/L}(c) > 0$ and using [CS93], we have that $\lim_{h_C(\gamma)\to\infty} h_{L/F}(c(\gamma))/h_C(\gamma) = h_{K/L}(c) > 0$, thus proving that for some $M > 0$, whenever $h_C(\gamma) > M$ then $h_{L/F}(c(\gamma)) > 0$, which yields that $c(\gamma) \notin \overline{\mathbb{Q}}$ (since those points would have height equal to 0 for the function field L/F). ∎

Next we show that for all specializations of sufficiently large height,

(ii) there are no distinct indices $i \neq j$ such that

(5.1.1)
$$(\alpha_i(\gamma), \alpha_j(\gamma)) \text{ lies on a periodic curve under the action of } (f_{i,\gamma}, f_{j,\gamma}).$$

We achieve this goal by combining Lemmas 5.2 and 5.3.

Lemma 5.2. *Let $K = L(C)$ be the function field of a curve (where L/F is itself a function field), let $c \in K$, let $f(x) := x^q + c$ and let $\beta_1, \beta_2 \in K$ such that there is no integer $n \geq 0$ with the property that $f^n(\beta_1) = \beta_2$. If either c or β_1 is not contained in $\overline{L}$, then there exists a real number $M_0 > 0$ such that for all points $\gamma \in C(\overline{L})$ satisfying $h_C(\gamma) > M_0$, we have that there exists no integer $n \geq 0$ such that $f_\gamma^n(\beta_1(\gamma)) = \beta_2(\gamma)$.*

Proof of Lemma 5.2. First we note that if β_1 is preperiodic, then its orbit under f is finite and therefore, away from finitely many points of C, the specialization of β_2 will avoid the corresponding specialization of a point in the orbit of β_1 under f. So, from now on, we assume β_1 is not preperiodic.

Since not both c and β_1 are contained in $\overline{L}$, then we get that $\widehat{h}_f(\beta_1) > 0$. Indeed, we know that β_1 is not preperiodic and therefore, according to [Ben05], either $\widehat{h}_f(\beta_1) > 0$ or the pair (f, β_1) is isotrivial for the function field K/L, which in our case is equivalent with both c and β_1 being contained in $\overline{L}$ (note that either $f(x) = x^q + c$ is defined over $\overline{L}$, or no conjugate of f under a linear transformation would be defined over $\overline{L}$). So, our hypothesis for Lemma 5.2 yields that $\widehat{h}_f(\beta_1) > 0$. Then [CS93] yields that

$$\lim_{h_C(\gamma)\to\infty} \frac{\widehat{h}_{f_\gamma}(\beta_1(\gamma))}{h_C(\gamma)} = \widehat{h}_f(\beta_1) > 0. \tag{5.2.1}$$

In particular, there exist positive real numbers C_0 and M_0 such that that if $h_C(\gamma) > M_0$, then

$$\widehat{h}_{f_\gamma}(\beta_1(\gamma)) > C_0 \cdot h_C(\gamma). \tag{5.2.2}$$

Again using [CS93] (see also [Sil07]), we get that there exist positive real numbers C_1 and C_2 such that

$$h_{L/F}(\beta_2(\gamma)) \leq C_1 \cdot h_C(\gamma) + C_2. \tag{5.2.3}$$

Using [Sil07] (see also [DGK+19, Proposition 2.4]), we get that there exist positive constants C_3 and C_4 such that for each $z \in \overline{L}$, we have

$$h_{L/F}(z) > \widehat{h}_{f_\gamma}(z) - C_3 \cdot h_C(\gamma) - C_4. \tag{5.2.4}$$

Now, let $\gamma \in C(\overline{L})$ such that $h_C(\gamma) > M_0$ and assume there exists some nonnegative integer n such that $f_\gamma^n(\beta_1(\gamma)) = \beta_2(\gamma)$. Using (5.2.4) and the definition of the canonical height, we get

(5.2.5)
$$h_{L/F}\left(f_\gamma^n(\beta_1(\gamma))\right) > \widehat{h}_{f_\gamma}\left(f_\gamma^n(\beta_1(\gamma))\right) - C_3 h_C(\gamma) - C_4 = q^n \widehat{h}_{f_\gamma}(\beta_1(\gamma)) - C_3 h_C(\gamma) - C_4.$$

Using inequality (5.2.2) in (5.2.5), we get

$$h_{L/F}\left(f_\gamma^n(\beta_1(\gamma))\right) > (q^n C_0 - C_3) \cdot h_C(\gamma) - C_4. \tag{5.2.6}$$

Now, combining inequalities (5.2.3) and (5.2.6) along with the equality $f_\gamma^n(\beta_1(\gamma)) = \beta_2(\gamma)$, we obtain that

$$(q^n C_0 - C_3 - C_1) \cdot h_C(\gamma) < C_2 + C_4, \tag{5.2.7}$$

which yields that if $h_C(\gamma) > M_0$, then

$$n < \log_q \left(\frac{\frac{C_2+C_4}{M_0} + C_1 + C_3}{C_0} \right).$$

On the other hand, for each of the finitely many nonnegative integers n satisfying the above inequality, there exist finitely many points $\gamma \in C(\overline{L})$ such that $f_\gamma^n(\beta_1(\gamma)) = \beta_2(\gamma)$. So, at the expense of replacing M_0 by a larger number, we obtain the conclusion from Lemma 5.2. □

Next we note that if we assume that c, β_1 and also β_2 are contained in $\overline{L}$, then for *every* specialization, we get that these elements are unchanged through the corresponding specialization. Therefore, if originally there was no integer n such that $f^n(\beta_1) = \beta_2$ (where $f(x) := x^q + c$), then this conclusion remains valid for *each* specialization. So, we are left to analyze the case when $c, \beta_1 \in \overline{L}$, while $\beta_2 \in K \setminus \overline{L}$.

Lemma 5.3. *Let $c, \beta_1 \in \overline{L}$, let $f(x) := x^q + c$ and let $\beta_2 \in K \setminus \overline{L}$. If $c \notin \overline{\mathbb{Q}}$, then the following statements hold:*

(a) *If β_1 is preperiodic under the action of f, then for all but finitely many specializations γ, there is no integer n such that $f^n(\beta_1) = \beta_2(\gamma)$.*
(b) *If β_1 is not preperiodic under the action of f, then there exist positive constants C_8 and C_9 such that for any nonnegative integer n and for any $\gamma \in C(\overline{L})$, if $f^n(\beta_1) = \beta_2(\gamma)$ then*

$$|h_C(\gamma) - C_8 \cdot q^n| \le C_9.$$

Proof of Lemma 5.3. The proof of part (a) is identical with the corresponding case (β_1 is preperiodic under the action of f) from Lemma 5.2. So, from now on, we assume β_1 is not preperiodic under the action of f.

Now, since $c \notin \overline{\mathbb{Q}}$, we have that the polynomial $f(x) = x^q + c$ is not isotrivial for the function field $L/\mathbb{Q}$ and so, [Ben05] yields that $C_5 := \widehat{h}_{f,L/\mathbb{Q}}(\beta_1) > 0$ (where $\widehat{h}_{f,L/\mathbb{Q}}$ is the canonical height of the polynomial $f \in \overline{L}[x]$ constructed with respect to the height for the function field $L/\mathbb{Q}$, which is a function field of finite transcendence degree, in general larger than one). As in the proof of Lemma 5.2, we have (see [Sil07]) that

$$\left| h_{L/\mathbb{Q}}(\beta_2(\gamma)) - C_1 \cdot h_C(\gamma) \right| \le C_2 \tag{5.3.1}$$

for some positive constants C_1 and C_2; note that $C_1 > 0$ since $\beta_2 \notin \overline{L}$. Also, there exists a positive constant C_7 (see [CS93]) such that for each $z \in \overline{L}$ we have

$$\left| h_{L/\mathbb{Q}}(z) - \widehat{h}_{f,L/\mathbb{Q}}(z) \right| \le C_7. \tag{5.3.2}$$

So, if $f^n(\beta_1) = \beta_2(\gamma)$ for some nonnegative integer n, then equations (5.3.1) and (5.3.2) (along with the fact that $\widehat{h}_{f,L/\mathbb{Q}}(f^n(\beta_1)) = q^n \cdot C_5$) yield that

$$|C_1 h_C(\gamma) - q^n C_5| \le C_2 + C_7. \tag{5.3.3}$$

Then taking $C_8 := \frac{C_5}{C_1}$ and $C_9 := \frac{C_2+C_7}{C_1}$, we see that inequality (5.3.3) provides the desired conclusion from part (b) of Lemma 5.3. ☐

Now we explain how to combine Lemmas 5.1, 5.2 and 5.3 to provide the condition (ii) (see (5.1.1)) for some suitable specialization at a point $\gamma \in C(\overline{L})$. First, we notice that since no $c_i \in \overline{\mathbb{Q}}$, then (in particular) $c_i \ne 0, -2$ (note that $c_i = 0$ yields a monomial function $f_i(x)$, while $c_i = -2$ and $d = 2$ yields the second Chebyshev polynomial); hence, in the language from [MS14] (see also [GN16, GN17, GNY19]), the polynomials $f_i(x) = x^q + c_i$ are disintegrated, or non-special (i.e., not conjugated to monomials or Chebyshev polynomials). Also, the hypothesis of Theorem 1.4 yields that no α_i can be periodic for the corresponding polynomial f_i. Now, [GNY19, Proposition 7.7] yields that if there exists a plane curve, projecting dominantly onto each coordinate, which is periodic under the action of $(x, y) \mapsto (f_i(x), f_j(y))$, then there must be some $(q-1)$-st root of unity ζ such that $c_j = \zeta \cdot c_i$. Furthermore, as proven in [MS14] as a result of a deep analysis of polynomial decompositions along with a powerful study of the model theory of an algebraically closed field with a distinguished automorphism ACFA_0 (see also [GN16, Proposition 2.5] and [GN17, Proposition 5.5]), assuming $c_j = \zeta c_i$, we have that each plane curve (projecting dominantly onto each coordinate) which is periodic under the action of $(x, y) \mapsto (f_i(x), f_j(y))$ must be of the form

$$y = \zeta \cdot f_i^n(x) \text{ for some } n \ge 0 \tag{5.3.4}$$

or

$$x = \zeta^{-1} \cdot f_i^n(y) \text{ for some } n \ge 0. \tag{5.3.5}$$

Using the fact that the only periodic curves under the action of $(x, y) \mapsto (f_i(x), f_j(x))$ are the ones described by equations (5.3.4) and (5.3.5) and that can only happen if $c_j = \zeta \cdot c_i$, then Lemmas 5.2 and 5.3 yield that any point $\gamma \in C(\overline{L})$ for which $f_C(\gamma)$ is sufficiently large (see Lemma 5.2) and, furthermore, which also satisfies the property

$$h_C(\gamma) \notin [C_8 q^n - C_9, C_8 q^n + C_9] \text{ for all } n \ge 0 \tag{5.3.6}$$

for some suitable positive constants C_8 and C_9 (see Lemma 5.3) would induce a specialization which satisfies the conditions (i)-(ii) (see (5.0.1) and (5.1.1)). Easily, we see that there exist infinitely many such suitable specializations.

Now, for such a suitable specialization at a point $\gamma \in C(\overline{L})$, we note that specializing at γ a field extension $K_1 \subset K_2$ (which are themselves finite field extensions of K) yields a finite field extension $L_1 \subset L_2$ (which are themselves finite extensions of L) and moreover,

$$[L_2 : L_1] \le [K_2 : K_1]. \tag{5.3.7}$$

So, following the proof of Theorem 1.4 for the specializations of f_i and of α_i at γ yields that there is a (minimal) integer n_2 such for all $n > n_2$, we have

$$\mathrm{Gal}(L_n(\boldsymbol{f}_\gamma, \boldsymbol{\alpha}(\gamma))/L_{n-1}(\boldsymbol{f}_\gamma, \boldsymbol{\alpha}(\gamma))) \cong C_q^{mq^{n-1}}, \tag{5.3.8}$$

where L_n is the specialization at γ of K_n (while $\boldsymbol{f}_\gamma$ and $\boldsymbol{\alpha}(\gamma)$ are the respective specializations of $\boldsymbol{f} = (f_1, \dots, f_m)$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$ at γ). Then combining (5.3.7) with (5.3.8) yields that

$$\mathrm{Gal}(K_n(\boldsymbol{f}, \boldsymbol{\alpha})/K_{n-1}(\boldsymbol{f}, \boldsymbol{\alpha})) \cong C_q^{mq^{n-1}},$$

and then the rest of the proof of Theorem 1.4 follows verbatim as in the proof of [BDG+20, Theorem 1.4]. □

Proof of Theorem 1.5. Assume that condition (B) does not hold. Then for each $i = 1, \dots, m$, we have that α_i is not a postcritical point (i.e., it is not in the strict forward orbit of 0 under the map f_i). Now, assume that also condition (C) does not hold; in particular, this means that no α_i is a periodic point for the corresponding map f_i. Using the fact that α_i is neither periodic nor postcritical, Theorem 1.2 yields that each Galois group $G_\infty(f_i, \alpha_i)$ has finite index inside the corresponding $G_\infty(f_i, \mathbb{P}^1)$.

Furthermore, since condition (C) does not hold, then there are no distinct i, j with the property that (α_i, α_j) lies on a plane curve that is periodic under the action of $(x, y) \mapsto (f_i(x), f_j(y))$. So, letting M_i be $K_\infty(f_i, \alpha_i)$, then for each $i = 1, \dots, m$, we have that

$$\left[M_i \cap \left(\prod_{j \neq i} M_j \right) : K \right] < \infty, \tag{5.3.9}$$

by Theorem 1.4. Using (5.3.9), we obtain that $G_\infty(\Phi, \underline{\alpha})$ has finite index in $\prod_{i=1}^m G_\infty(f_i, \alpha_i)$ and then combining this information with the fact that $G_\infty(f_i, \alpha_i)$ has finite index in $G_\infty(f_i, \mathbb{P}^1)$, we obtain the desired conclusion from Theorem 1.5. □

Proof of Theorem 1.6. At the expense of replacing $f(x)$ with a conjugate, we may assume that $f(x) = x^3 + bx + a$. Since $f^{-n}(\beta)$ is algebraic over $\mathbb{Q}(a, b, \beta)$ for all n, we may assume then that K is a finite extension of $\mathbb{Q}(a, b, \beta)$. In the case where K has transcendence degree 1 over $\mathbb{Q}$, then [BG19, Theorem 1.1] states $[\mathrm{Aut}(T_\infty^3) : G_\infty] < \infty$. If β is not algebraic over $\mathbb{Q}(a, b)$, then [BG19, Proposition 12.1] gives the even stronger result that $[\mathrm{Aut}(T_\infty) : G_\infty] = 1$.

Thus, we are left with treating the case where a and b are algebraically independent and β is algebraic over $\mathbb{Q}(a, b)$. We treat this case by specializing from K to a finite extension of $\mathbb{Q}(b)$, so that we may apply [BG19, Theorem 1.1]; in particular, we will work with specializations t such that $a_t = G(b)$ where G is a polynomial with positive integer coefficients. Note that when $f_t(x) := x^3 + bx + G(b)$, for G an even polynomial whose nonzero coefficients are positive integers (say, $G(b) = b^4 + 3b^2 + 5$), f_t must be eventually stable, since f can be further specialized (by letting $b := 3m$ for $m \in \mathbb{Z}$) to a polynomial of the form $x^3 + 3mx + n$ (where $m, n \in \mathbb{Z}$), and such polynomials are known to be eventually stable over any finite extension of $\mathbb{Q}$ by [JL17]. Furthermore, we then have $f_t^i((\gamma_1)_t) \neq f_t^i((\gamma_2)_t)$ for all i since sending b to $-3e^2$, for e a positive integer, yields critical points $\pm e$ and $f_t^i(-e) > f_t^i(e)$ for all i for a polynomial G as above (whose only nonzero terms have even degree and positive integer coefficients). Similarly, if $\deg G(b)$ is larger than two then neither $(\gamma_1)_t$ nor $(\gamma_2)_t$ can be preperiodic, since the heights of the iterates of each must go to infinity.

Now, by [CS93, Theorem 4.1], we have

$$\lim_{h(t)\to\infty} \frac{\widehat{h}_{f_t}(\beta)}{h(t)} = \widehat{h}_f(\beta), \tag{5.3.10}$$

where $\widehat{h}_f$ is the canonical height for the polynomial f with respect to the heights on the function field $\mathbb{Q}(a,b)/\mathbb{Q}(b)$, while $\widehat{h}_{f_t}$ is the canonical height of the specialization polynomial f_t with respect to the heights for the function field $\mathbb{Q}(b)/\mathbb{Q}$; also, $h(t)$ refers to the Weil height for the curve $\mathbb{P}^1_{\mathbb{Q}(b)}$ so that the height of the point when we specialize $a_t := G(b)$ is simply the degree of the polynomial G.

If β is not preperiodic, then since f is not isotrivial over $\mathbb{Q}(b)$, we have $\widehat{h}_f(\beta) > 0$ by [Ben05], so $\widehat{h}_{f_t}(\beta_t) > 0$ when $h(t)$ is large. If β is preperiodic, then there are at most finitely many specializations t such that β_t is periodic, since β itself is not periodic. Thus, in either event, β_t is not periodic for all t of sufficiently large height. We also have

$$\lim_{h(t)\to\infty} \frac{\widehat{h}_{f_t}(\gamma_i)}{h(t)} = \widehat{h}_f((\gamma_i)_t) > 0, \tag{5.3.11}$$

again by [CS93, Theorem 4.1] and [Ben05]. Thus, choosing a j such that $3^j\widehat{h}_f(\gamma_i) > \widehat{h}_f(\beta)$, for $i = 1, 2$, we see that for all specializations t of sufficiently large height, we have $\widehat{h}_{f_t}(f_t^N((\gamma_i)_t)) > \widehat{h}_{f_t}(\beta_t)$ for $i = 1, 2$ for all $N \geq j$. Since there are at most finitely many t such that $f_t^m((\gamma_i)_t) = \beta_t$ for $i = 1, 2$ and $m < j$, we see again that for all t of sufficiently large height, we have $f_t^n((\gamma_i)_t) \neq \beta_t$ for all n.

Since we may choose a specialization t such that $a_t = G(b)$, for G a polynomial of arbitrarily large degree, then there are specializations t with $h(t)$ arbitrarily large such that f_t has the desired form $x^3+bx+G(b)$ (where G is a polynomial whose nonzero terms have even degrees and positive integer coefficients). Hence, there is a specialization t such that f_t is eventually stable, β_t is not post-critical and not periodic, the critical points $(\gamma_1)_t, (\gamma_2)_t$ of f_t are not periodic, and we do not have $f_t^i((\gamma_1)_t) = f_t^i((\gamma_2)_t)$ for any positive integers i. Thus, the pair (f_t, β_t) satisfies the hypotheses of [BG19, Theorem 1.1]. Now, for all n the degree $[K(f^{-n}(\beta)) : K]$ is at least as large as $[k_t(f_t^{-n}(\beta_t)) : k_t]$, where k_t is the field of definition of the point t, so we must have $[\mathrm{Aut}(T^3_\infty) : G_\infty] < \infty$, as desired. $\square$

6. A generalization of our main question by taking pullbacks of higher-dimensional varieties

It makes sense to consider the following more general question, which provides a geometric extension to Question 1.1. So, given a polarizable endomorphism Φ of a smooth projective variety X defined over a field K of characteristic 0, we let $G_\infty := G_\infty(\Phi)$ be defined as in Question 1.1 as the inverse limit of the Galois groups for the Galois closures of $K(X)/(\Phi^n)^*K(X)$ (as n goes to infinity). Now, let $Z \subset X$ be a proper (closed, irreducible) subvariety and let $G_n(\Phi, Z)$ be the Galois group for Galois closure of the cover $\Phi^{-n}(Z) \longrightarrow Z$; then let $G_\infty(Z) := G_\infty(\Phi, Z)$ be the inverse limit of all these groups $G_n(\Phi, Z)$. Similar to Question 1.1, one expects that at least one of the following three possibilities must hold:

(A') $[G_\infty : G_\infty(Z)]$ is finite;
(B') $\Phi^{-n}(Z)$ does not intersect properly with the ramification locus of Φ, for some $n \geq 0$; or
(C') there exists a proper subvariety $Y \subset X$ containing Z, which is invariant under the action of a non-identity self-map $\Psi : X \longrightarrow X$ with the property that $\Psi \circ \Phi^n = \Phi^n \circ \Psi$ for some positive integer n.

In the case when Question 1.1 is known, then also the above question has positive answer. Indeed, if $\dim(X) = 1$, then the above question is precisely the problem investigated by Question 1.1. On the other hand, when X is an arbitrary abelian variety endowed with the multiplication-by-m map Φ, condition (B') above is vacuous since there is no ramification for Φ. So, in this case, we have that either the index $[G_\infty : G_\infty(Z)]$ is finite, or Z is contained in a proper algebraic subgroup of X, which is essentially the content of (C').

Furthermore, for any polarizable dynamical system (X, Φ), given any subvariety $Z \subset X$, it is sufficient to find one point x inside Z which does not satisfy conditions (B)-(C) from Question 1.1; then, according to Question 1.1, we have that $[G_\infty : G_\infty(x)] < \infty$. Indeed, for any point $x \in Z$, we have that $G_\infty(x)$ is the decomposition group of $G_\infty(Z)$ corresponding to the prime associated to the closed point x on Z (this fact can be established by analyzing the statement at each level n and then taking inverse limits).

So, as explained in the previous paragraph, as long as the closed subvariety Z contains a point x which verifies the conditions (B)-(C) from Question 1.1, then we would expect that also $[G_\infty : G_\infty(Z)] < \infty$. On the other hand, one expects to find such a point $x \in Z$ as long as Z does not satisfy conditions (B')-(C'). For example, condition (C') yields that Z is not preperiodic under the action of Φ and therefore, assuming the Dynamical Manin-Mumford Conjecture holds (see [Zha06]), there should be a Zariski dense set of non-preperiodic points inside Z. Actually, the stronger assumption from (C') which refers to the action on Z by any self-map Ψ commuting with Φ should yield - coupled with the Dynamical Manin-Mumford Conjecture - the existence of a point $x \in Z$ satisfying condition (C), as predicted by [GTZ11, YZ17, GNY18].

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