Density of orbits of endomorphisms of commutative linear algebraic groups

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Abstract. We prove a conjecture of Medvedev and Scanlon for endomorphisms of connected commutative linear algebraic groups G defined over an algebraically closed field $\Bbbk$ of characteristic 0. That is, if $\Phi\colon G \longrightarrow G$ is a dominant endomorphism, we prove that one of the following holds: either there exists a non-constant rational function $f \in \Bbbk(G)$ preserved by Φ (i.e., $f \circ \Phi = f$), or there exists a point $x \in G(\Bbbk)$ whose Φ-orbit is Zariski dense in G.

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1. Introduction

Throughout our paper, we work over an algebraically closed field $\Bbbk$ of characteristic 0. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any self-map Φ on a set X, and any $n \in \mathbb{N}_0$, we denote by Φ^n the n-th compositional power, where Φ^0 is the identity map. For any $x \in X$, we denote by $\mathcal{O}_\Phi(x)$ its forward orbit under Φ, i.e., the set of all iterates $\Phi^n(x)$ for $n \in \mathbb{N}_0$. An *endomorphism* of an algebraic group G is defined as a self-morphism of G in the category of algebraic groups.

Our main result is the following.

Theorem 1.1. *Let G be a connected commutative linear algebraic group defined over an algebraically closed field $\Bbbk$ of characteristic 0, and $\Phi\colon G \longrightarrow G$ a dominant endomorphism. Then either there exists a point $x \in G(\Bbbk)$ such that $\mathcal{O}_\Phi(x)$ is Zariski dense in G, or there exists a non-constant rational function $f \in \Bbbk(G)$ such that $f \circ \Phi = f$.*

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Theorem 1.1 answers affirmatively the following conjecture raised by Medvedev and Scanlon in [MS14] for the case of endomorphisms of $\mathbb{G}_a^k \times \mathbb{G}_m^\ell$. Note that any connected commutative linear algebraic group splits over an algebraically closed field $\Bbbk$ of characteristic 0 as a direct product of its largest unipotent subgroup (which is in our case a vector group, i.e., the additive group $\mathbb{G}_a^k$ of a finite-dimensional $\Bbbk$-vector space) with an algebraic torus $\mathbb{G}_m^\ell$.

Conjecture 1.2 (cf. [MS14, Conjecture 7.14])**.** *Let X be a quasi-projective variety defined over an algebraically closed field $\Bbbk$ of characteristic 0, and $\varphi\colon X \dashrightarrow X$ a dominant rational self-map. Then either there exists a point $x \in X(\Bbbk)$ whose orbit under φ is Zariski dense in X, or φ preserves a non-constant rational function $f \in \Bbbk(X)$, i.e., $f \circ \varphi = f$.*

With the notation as in Conjecture 1.2, it is immediate to see that if φ preserves a non-constant rational function, then there is no Zariski dense orbit. So, the real difficulty in Conjecture 1.2 lies in proving that there exists a Zariski dense orbit for a dominant rational self-map φ of X, which preserves no non-constant rational function.

The origin of [MS14, Conjecture 7.14] lies in a much older conjecture formulated by Zhang in the early 1990s (and published in [Zha10, Conjecture 4.1.6]). Zhang asked that for each polarizable endomorphism φ of a projective variety X defined over $\overline{\mathbb{Q}}$ there must exist a $\overline{\mathbb{Q}}$-point with Zariski dense orbit under φ. Medvedev and Scanlon [MS14] conjectured that as long as φ does not preserve a non-constant rational function, then a Zariski dense orbit must exist; the hypothesis concerning polarizability of φ already implies that no non-constant rational function is preserved by φ. We describe below the known partial results towards Conjecture 1.2.

(i) In [AC08], Amerik and Campana proved Conjecture 1.2 for all uncountable algebraically closed fields $\Bbbk$ (see also [BRS10] for a proof of the special case of this result when φ is an automorphism). In fact, Conjecture 1.2 is true even in positive characteristic, as long as the base field $\Bbbk$ is uncountable (see [BGR17, Corollary 6.1]); on the other hand, when the transcendence degree of $\Bbbk$ over $\mathbb{F}_p$ is smaller than the dimension of X, there are counterexamples to the corresponding variant of Conjecture 1.2 in characteristic p (as shown in [BGR17, Example 6.2]).

(ii) In [MS14], Medvedev and Scanlon proved their conjecture in the special case $X = \mathbb{A}_{\Bbbk}^n$ and φ is given by the coordinatewise action of n one-variable polynomials $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$; their result was established over an arbitrary field $\Bbbk$ of characteristic 0 which is not necessarily algebraically closed.

(iii) Conjecture 1.2 is known for all projective varieties of positive Kodaira dimension; see for example [BGRS17, Proposition 2.3].

(iv) In [Xie15], Conjecture 1.2 was proven for all birational automorphisms of surfaces (see also [BGT15] for an independent proof of the case of automorphisms). Later, Xie [Xie17] established the validity of Conjecture 1.2 for all polynomial endomorphisms of $\mathbb{A}^2_{\Bbbk}$.

(v) In [BGRS17], the conjecture was proven for all smooth minimal 3-folds of Kodaira dimension 0 with sufficiently large Picard number, contingent on certain conjectures in the Minimal Model Program.

(vi) In [GS17], Conjecture 1.2 was proven for all abelian varieties; later this result was extended to dominant regular self-maps for all semi-abelian varieties (see [GS]).

(vii) In [GX], it was proven that if Conjecture 1.2 holds for the dynamical system (X, φ), then it also holds for the dynamical system $(X \times \mathbb{A}^k_{\Bbbk}, \psi)$, where $\psi\colon X \times \mathbb{A}^k_{\Bbbk} \dashrightarrow X \times \mathbb{A}^k_{\Bbbk}$ is given by $(x, y) \mapsto (\varphi(x), A(x)y)$ and $A \in \mathrm{GL}_k(\Bbbk(X))$.

We note that combining the results of [GS] (which, in particular, proves Conjecture 1.2 when $X = \mathbb{G}_m^\ell$) with the results of [GX], one recovers our Theorem 1.1. However, our proof of Theorem 1.1 avoids the more complicated arguments from algebraic geometry which were used in the proofs from [GX] and instead we use mainly number-theoretic tools, employing in a crucial way a theorem of Laurent [Lau84] regarding polynomial-exponential equations. So, with this new tool which we bring to the study of the Medvedev–Scanlon conjecture, we are able to construct explicitly points with Zariski dense orbits (which is not available in [GX]). Besides the intrinsic interest in our new approach, as part of our proof, we also obtain in Theorem 2.1 a more precise result of when a linear transformation has a Zariski dense orbit.

2. Proof of main results

We start by proving the following more precise version of the special case in Theorem 1.1 when G is a connected commutative unipotent algebraic group over $\Bbbk$, i.e., $G = \mathbb{G}_a^k$ for some $k \in \mathbb{N}$.

Theorem 2.1. *Let $\Phi\colon \mathbb{G}_a^k \longrightarrow \mathbb{G}_a^k$ be a dominant endomorphism defined over an algebraically closed field $\Bbbk$ of characteristic 0. Then the following are equivalent:*

(i) Φ preserves a non-constant rational function.

(ii) There is no $\alpha \in \mathbb{G}_a^k(\Bbbk)$ whose orbit under Φ is Zariski dense in $\mathbb{G}_a^k$.

(iii) The matrix A representing the action of Φ on $\mathbb{G}_a^k$ is either diagonalizable with multiplicatively dependent eigenvalues, or it has at most $k-2$ multiplicatively independent eigenvalues.

Proof. Clearly, (i) $\Longrightarrow$ (ii). We will prove that (iii) $\Longrightarrow$ (i) and then that (ii) $\Longrightarrow$ (iii). First of all, using [GS17, Lemma 5.4], we may assume that A is in Jordan (canonical) form. Strictly speaking, [GS17, Lemma 5.4] proves that the Medvedev–Scanlon conjecture for abelian varieties is unaffected after replacing the given endomorphism by a conjugate of it through an automorphism; however, its proof goes verbatim for any endomorphism of any quasi-projective variety. Also, because the part (iii) above is unaffected after replacing A by its Jordan form, then from now on, we assume that A is a Jordan matrix.

Now, assuming (iii) holds, we shall show that (i) holds. Indeed, if A is diagonalizable, then since it has multiplicatively dependent eigenvalues $\lambda_1, \dots, \lambda_k$, i.e., there exist some integers $c_1, \dots, c_k$ not all equal to 0 such that $\prod_{i=1}^k \lambda_i^{c_i} = 1$, then Φ preserves the non-constant rational function

$$f\colon \mathbb{G}_a^k \longrightarrow \mathbb{P}^1_{\Bbbk} \text{ given by } f(x_1, \dots, x_k) = \prod_{i=1}^k x_i^{c_i},$$

as claimed. Now, assuming A is not diagonalizable and it has at most $k-2$ multiplicatively independent eigenvalues, we will derive (i). There are 3 easy cases to consider:

Case 1. A has $k-2$ Jordan blocks of dimension 1 and one Jordan block of dimension 2 and moreover, the corresponding $k-1$ eigenvalues $\lambda_1, \dots, \lambda_{k-1}$ are multiplicatively dependent, i.e., there exist some integers $c_1, \dots, c_{k-1}$ not all equal to 0 such that $\prod_{i=1}^{k-1} \lambda_i^{c_i} = 1$. Without loss of generality, we may assume that λ_1 corresponds to the unique Jordan block of dimension 2. Namely,

$$A = J_{\lambda_1,2} \bigoplus \operatorname{diag}(\lambda_2, \dots, \lambda_{k-1}).$$

Then we conclude that Φ preserves a non-constant rational function

$$f\colon \mathbb{G}_a^k \longrightarrow \mathbb{P}^1_{\Bbbk} \text{ given by } f(x_1, \dots, x_k) = \prod_{i=1}^{k-1} x_{i+1}^{c_i}.$$

Case 2. A has at least two Jordan blocks of dimension 2 each. Again, we may assume that the first two Jordan blocks of A are given by $J_{\lambda_i,2}$ with $i = 1, 2$ (it may happen that $\lambda_1 = \lambda_2$). Then we see that Φ preserves the non-constant rational function $\mathbb{G}_a^k \longrightarrow \mathbb{P}^1_{\Bbbk}$ given by

$$(x_1, \dots, x_k) \mapsto \frac{x_1}{\lambda_2 x_2} - \frac{x_3}{\lambda_1 x_4}.$$

(Note that $\lambda_1\lambda_2 \neq 0$ because the endomorphism Φ is dominant and hence none of its eigenvalues equals 0. This is also true in the following cases.)

Case 3. A has a Jordan block of dimension at least equal to 3 which is denoted by $J_{\lambda,m}$ with $3 \leq m \leq k$. Clearly, it suffices to prove that the endomorphism $\varphi\colon \mathbb{G}_a^m \longrightarrow \mathbb{G}_a^m$ (induced by the action of Φ restricted on the generalized eigenspace corresponding to the eigenvalue λ) preserves a non-constant rational function. Note that the action of φ is given by the Jordan matrix

$$J_{\lambda,m} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

We conclude that φ preserves the non-constant rational function $f\colon \mathbb{G}_a^m \longrightarrow \mathbb{P}^1_{\Bbbk}$ given by

$$f(x_1,\dots,x_m) = \frac{2x_{m-2}}{x_m} - \frac{x_{m-1}^2}{x_m^2} + \frac{x_{m-1}}{\lambda x_m}.$$

Therefore, it remains to prove that if (ii) holds, then (iii) must follow. In order to prove this, we show that if A is either diagonalizable with multiplicatively independent eigenvalues, or if A has $k-2$ Jordan blocks of dimension 1 and one Jordan block of dimension 2 and moreover, the $k-1$ eigenvalues corresponding to these $k-1$ Jordan blocks are all multiplicatively independent, then there exists a $\Bbbk$-point with a Zariski dense orbit. So, we have two more cases to analyze.

Case 4. A is diagonalizable with multiplicatively independent eigenvalues $\lambda_1,\dots,\lambda_k$. In this case, we shall prove that the orbit of $\alpha := (1,1,\dots,1)$ is Zariski dense in $\mathbb{G}_a^k$. Indeed, if there were a nonzero polynomial $F \in \Bbbk[x_1,\dots,x_k]$ vanishing on the points of the orbit of α under Φ, then we would have that $F(\lambda_1^n,\dots,\lambda_k^n) = F(\Phi^n(\alpha)) = 0$ for each $n \in \mathbb{N}_0$. Let

$$F(x_1,\dots,x_k) = \sum_{(i_1,\dots,i_k)} c_{i_1,\dots,i_k} \prod_{j=1}^{k} x_j^{i_j},$$

where the coefficients $c_{i_1,\dots,i_k}$'s are nonzero (and clearly, there are only finitely many of them appearing in the above sum). Then it follows that

$$\sum_{(i_1,\dots,i_k)} c_{i_1,\dots,i_k} \cdot \Lambda^n_{i_1,\dots,i_k} = 0 \text{ for each } n \in \mathbb{N}_0,$$

where $\Lambda_{i_1,\dots,i_k} := \prod_{j=1}^k \lambda_j^{i_j}$. On the other hand, since for $(i_1,\dots,i_k) \neq (j_1,\dots,j_k)$ we know that $\Lambda_{i_1,\dots,i_k}/\Lambda_{j_1,\dots,j_k}$ is not a root of unity (because the λ_i's are multiplicatively independent), $F(\Phi^n(\alpha))$ is a non-degenerate linear recurrence sequence (see [Ghi, Definition 3.1]). Hence [Sch03] (see also [Ghi, Proposition 3.2]) yields that as long as F is not identically equal to 0 (i.e., not all coefficients $c_{i_1,\dots,i_k}$ are equal to 0), then there are at most finitely many $n \in \mathbb{N}_0$ such that $F(\Phi^n(\alpha)) = 0$, which is a contradiction. So, indeed, $\mathcal{O}_\Phi(\alpha)$ is Zariski dense in $\mathbb{G}_a^k$.

Case 5. A has $k-2$ Jordan blocks of dimension 1 and one Jordan block of dimension 2 and moreover, the corresponding $k-1$ eigenvalues $\lambda_1,\dots,\lambda_{k-1}$ are multiplicatively independent. Without loss of generality, we may assume that

$$A = J_{\lambda_1,2} \bigoplus \operatorname{diag}(\lambda_2,\dots,\lambda_{k-1}) = \begin{pmatrix} \lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{k-1} \end{pmatrix},$$

and so,

$$A^n = J_{\lambda_1,2}^n \bigoplus \operatorname{diag}(\lambda_2^n,\dots,\lambda_{k-1}^n) = \begin{pmatrix} \lambda_1^n & n\lambda_1^{n-1} & 0 & \cdots & 0 \\ 0 & \lambda_1^n & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{k-1}^n \end{pmatrix}.$$

We shall prove again that the orbit of $\alpha = (1,\dots,1)$ under the action of Φ is Zariski dense in $\mathbb{G}_a^k$. Let $\Psi\colon \mathbb{G}_a^k \longrightarrow \mathbb{G}_a^k$ be the automorphism given by

$$(x_1, x_2, x_3, \dots, x_k) \mapsto (\lambda_1(x_1 - x_2), x_2, x_3, \dots, x_k)$$

(note that all λ_i's are nonzero because Φ is dominant). It suffices to prove that $\Psi(\mathcal{O}_\Phi(\alpha))$ is Zariski dense in $\mathbb{G}_a^k$. This is equivalent with proving that there is no nonzero polynomial $F \in \Bbbk[x_1,\dots,x_k]$ vanishing on

$$\Psi(\Phi^n(\alpha)) = (n\lambda_1^n, \lambda_1^n, \lambda_2^n, \dots, \lambda_{k-1}^n).$$

So, letting $F(x_1,\dots,x_k) \coloneqq \sum_{(i_1,\dots,i_k)} c_{i_1,\dots,i_k} x_1^{i_1}\cdots x_k^{i_k}$, we get that

$$F(\Psi(\Phi^n(\alpha))) = \sum_{(i_1,\dots,i_k)} c_{i_1,\dots,i_k} n^{i_1} \left(\lambda_1^{i_1+i_2}\cdot\lambda_2^{i_3}\cdots\lambda_{k-1}^{i_k}\right)^n = 0. \tag{2.1.1}$$

Letting $\Lambda_{j_1,\dots,j_{k-1}} \coloneqq \lambda_1^{j_1}\cdot\lambda_2^{j_2}\cdots\lambda_{k-1}^{j_{k-1}}$, we can rewrite (2.1.1) as

$$\sum_{(j_1,\dots,j_{k-1})} Q_{j_1,\dots,j_{k-1}}(n)\cdot \Lambda_{j_1,\dots,j_{k-1}}^n = 0, \tag{2.1.2}$$

where

$$Q_{j_1,\dots,j_{k-1}}(n) \coloneqq \sum_{\substack{i_1+i_2=j_1 \text{ and} \\ i_3=j_2,\dots,i_k=j_{k-1}}} c_{i_1,i_2,i_3,\dots,i_k} n^{i_1}.$$

As in the previous Case 4, the left-hand side of (2.1.2) represents the general term of a non-degenerate linear recurrence sequence (i.e., such that the quotient of any two of its distinct characteristic roots is not a root of unity). It follows from [Sch03] (see also [Ghi, Proposition 3.2]) that there are at most finitely many $n \in \mathbb{N}_0$ such that (2.1.2) holds, unless $F = 0$ (i.e., each coefficient $c_{i_1,\dots,i_k}$ equals 0). Therefore, $\Psi(\mathcal{O}_\Phi(\alpha))$ is indeed Zariski dense in $\mathbb{G}_a^k$ and hence so is $\mathcal{O}_\Phi(\alpha)$.

This concludes our proof of Theorem 2.1. □

Remark 2.2. We note that in Theorem 2.1 we actually proved a stronger statement as follows. If A is a Jordan matrix acting on $\mathbb{G}_a^k$ and either it has k multiplicatively independent eigenvalues, or it is not diagonalizable, but it still has $k-1$ multiplicatively independent eigenvalues, then there is no proper subvariety of $\mathbb{G}_a^k$ which contains infinitely many points from the orbit of $(1,\dots,1)$ under the action of A. So, not only that the orbit of $(1,\dots,1)$ is Zariski dense in $\mathbb{G}_a^k$, but any infinite subset of its orbit must also be Zariski dense in $\mathbb{G}_a^k$. This strengthening is similar to the one obtained in [BGT10, Corollary 1.4] for the action of any étale endomorphism of a quasi-projective variety (see also [BGT16] for the connections of these results to the dynamical Mordell–Lang conjecture).

The next result will be used in our proof of Theorem 1.1.

Proposition 2.3. *Let $A \in \mathbb{M}_{\ell,\ell}(\mathbb{Z})$ be a matrix with nonzero determinant, and let $\vec{p} \in \mathbb{M}_{\ell,1}(\mathbb{Z})$ be a nonzero vector. Let c_1 and c_2 be positive real numbers. If there exists an infinite set S of positive integers such that for each $n \in S$, we have that $A^n \cdot \vec{p}$ is a vector whose entries are all bounded in absolute value by $c_1 n + c_2$, then A has an eigenvalue which is a root of unity.*

Proof. Let $B \in \mathbb{M}_{\ell,\ell}(\overline{\mathbb{Q}})$ be an invertible matrix such that $J := BAB^{-1}$ is the Jordan canonical form of A. For each $n \in \mathbb{N}$, let $\vec{p_n} := A^n \cdot \vec{p}$ and $\vec{q_n} := B \cdot \vec{p_n}$. So, we know that each entry in $\vec{p_n}$ is an integer bounded in absolute value by $c_1 n + c_2$ for any $n \in S \subseteq \mathbb{N}$. Then, according to our hypotheses, there exist some positive constants c_3 and c_4 such that each entry in $\vec{q_n}$ is bounded in absolute value by $c_3 n + c_4$. Furthermore, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, denoting by $\vec{v}^\sigma$ the vector obtained by applying σ to each entry of the vector $\vec{v} \in \mathbb{M}_{\ell,1}(\overline{\mathbb{Q}})$, we have that each entry in $\vec{q_n}^\sigma$ is bounded by $c_5 n + c_6$ for some positive constants c_5 and c_6 which are independent of n and σ. Indeed, this claim follows from the observation that $\vec{q_n}^\sigma = B^\sigma \cdot \vec{p_n}$, because $\vec{p_n}$ has integer entries (since both $\vec{p}$ and A have integer entries) and moreover, the entries in $\vec{p_n}$ are all bounded in absolute value by $c_1 n + c_2$.

Denote by $\ell_1, \dots, \ell_m$ the dimensions of the Jordan blocks of J in the order as they appear in the matrix J (so, $\ell = \ell_1 + \cdots + \ell_m$). Let $\vec{q} := B \cdot \vec{p}$. Since $\vec{p} \neq \vec{0}$ and B is invertible, then $\vec{q}$ is not the zero vector either. Without loss of generality, we may assume that one of the first ℓ_1 entries in $\vec{q}$ is nonzero. Next, we will prove that the eigenvalue of J corresponding to its first Jordan block (of dimension ℓ_1) must have absolute value at most equal to 1. We state and prove our result from Lemma 2.4 in much higher generality than needed since it holds for any valued field $(L, |\cdot|)$ (our application will be for $L = \overline{\mathbb{Q}}$ equipped with the usual archimedean absolute value $|\cdot|$).

Lemma 2.4. *Let $(L, |\cdot|)$ be an arbitrary valued field, let $J_{\lambda_1,r} \in \mathbb{M}_{r,r}(L)$ be a Jordan block of dimension $r \geq 1$ corresponding to a nonzero eigenvalue λ_1, and let $\vec{v} \in \mathbb{M}_{r,1}(L)$ be a nonzero vector. If there exist positive constants c_5, c_6, and an infinite set S_1 of positive integers such that for each $n \in S_1$, we have that each entry in $J_{\lambda_1,r}^n \cdot \vec{v}$ is bounded in absolute value by $c_5 n + c_6$, then $|\lambda_1| \leq 1$.*

Proof of Lemma 2.4. Let s be the largest integer with the property that the s-th entry v_s in $\vec{v}$ is nonzero; so, $1 \leq s \leq r$. Then for each $n \in S_1$, we have that the s-th entry in $J_{\lambda_1,r}^n \cdot \vec{v}$ is $v_s \lambda_1^n$ and hence according to our hypothesis, we have

$$|v_s \lambda_1^n| \leq c_5 n + c_6. \tag{2.4.1}$$

Since $v_s \neq 0$ and equation (2.4.1) holds for each n in the infinite set S_1, we conclude that $|\lambda_1| \le 1$, as desired. Thus, the lemma follows.

So, our assumptions coupled with Lemma 2.4 yield that the eigenvalue λ_1 corresponding to the first Jordan block of the matrix J has absolute value at most equal to 1. Furthermore, as previously explained, for each $n \in S$ and for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have that each entry in

$$\vec{q_n}^{\sigma} = (B \cdot \vec{p_n})^{\sigma} = (BA^n \cdot \vec{p})^{\sigma} = (J^n \cdot \vec{q})^{\sigma} = (J^{\sigma})^n \cdot \vec{q}^{\sigma}$$

is bounded in absolute value by $c_5 n + c_6$. Thus, applying again Lemma 2.4, this time to the first Jordan block of the matrix J^{σ}, we obtain that $|\sigma(\lambda_1)| \le 1$.

Now, λ_1 is an algebraic integer (since it is the eigenvalue of a matrix with integer entries) and for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have that $|\sigma(\lambda_1)| \le 1$. Because the product of all the Galois conjugates of λ_1 must be a nonzero integer, we conclude that actually $|\sigma(\lambda_1)| = 1$ for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then a classical lemma from algebraic number theory yields that λ_1 must be a root of unity, as desired. □

Now we are ready to prove our main theorem stated in the introduction.

Proof of Theorem 1.1. Because G is a connected commutative linear algebraic group defined over an algebraically closed field $\Bbbk$ of characteristic 0, then G is isomorphic to $\mathbb{G}_a^k \times \mathbb{G}_m^{\ell}$ for some $k, \ell \in \mathbb{N}_0$. Since there are no nontrivial homomorphisms between $\mathbb{G}_a$ and $\mathbb{G}_m$, then Φ splits as $\Phi_1 \times \Phi_2$, where Φ_1 and Φ_2 are dominant endomorphisms of $\mathbb{G}_a^k$ and $\mathbb{G}_m^{\ell}$, respectively. So, our conclusion follows once we prove the following statement: if neither Φ_1 nor Φ_2 preserve any non-constant rational function, then there exists a point $\alpha \in (\mathbb{G}_a^k \times \mathbb{G}_m^{\ell})(\Bbbk)$ with a Zariski dense orbit under Φ.

Thus, we assume that Φ_1 and Φ_2 do not preserve any non-constant rational function. In particular, this means that the action of Φ_2 on the tangent space of the identity of $\mathbb{G}_m^{\ell}$ is given through a matrix A_2 whose eigenvalues are not roots of unity (since otherwise one may argue as in the proof of [GS17, Lemma 6.2] or [GS, Lemma 4.1] that Φ_2 preserves a non-constant fibration which is not the case). Also, our Theorem 2.1 yields that either the matrix A_1 (which represents Φ_1) is diagonalizable with multiplicatively independent eigenvalues, or the Jordan canonical form of A_1 has $k-2$ blocks of dimension 1 and one block of dimension 2 such that the $k-1$ eigenvalues are multiplicatively independent. Next, we will analyze in detail the second possibility for A_1 (when there is a Jordan block of dimension 2), since the former possibility (when A_1 is diagonalizable with multiplicatively independent eigenvalues) turns out to be a special case of the latter one.

Arguing as in the proof of Theorem 2.1, at the expense of replacing Φ_1 and therefore Φ by a conjugate through an automorphism, we may assume that A_1 is a Jordan matrix of the form

$$A_1 = J_{\lambda_1,2} \bigoplus \operatorname{diag}(\lambda_2, \ldots, \lambda_{k-1}) = \begin{pmatrix} \lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{k-1} \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_{k-1}$ are multiplicatively independent eigenvalues. We will prove that there exists a point $\alpha \in (\mathbb{G}_a^k \times \mathbb{G}_m^\ell)(\Bbbk)$ with a Zariski dense orbit. Suppose that we have proved it for the time being. Then restricting the action of Φ_1 (and thus of A_1) to the last $k-1$ coordinate axes of $\mathbb{G}_a^k$, we obtain a diagonal matrix with multiplicatively independent eigenvalues. Letting $\hat{\pi}_1$ be the projection of $\mathbb{G}_a^k$ to $\mathbb{G}_a^{k-1}$ with the first coordinate omitted, we obtain a point $\gamma := (\hat{\pi}_1 \times \mathrm{id}_{\mathbb{G}_m^\ell})(\alpha)$ whose orbit under the induced endomorphism of $\mathbb{G}_a^{k-1} \times \mathbb{G}_m^\ell$ is Zariski dense. This justifies our earlier claim that it suffices to consider the case of a non-diagonalizable linear action Φ_1 since the diagonal case reduces to this more general case.

Let $\alpha := (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell) \in (\mathbb{G}_a^k \times \mathbb{G}_m^\ell)(\overline{\mathbb{Q}})$ such that $\alpha_1 = \cdots = \alpha_k = 1$, while $\lambda_1, \ldots, \lambda_{k-1}, \beta_1, \ldots, \beta_\ell$ are all multiplicatively independent. We will prove that $\mathcal{O}_\Phi(\alpha)$ is Zariski dense in $\mathbb{G}_a^k \times \mathbb{G}_m^\ell$. Since $\lambda_1, \ldots, \lambda_{k-1}$ are multiplicatively independent elements of $\Bbbk$ (which is an algebraically closed field containing $\overline{\mathbb{Q}}$), without loss of generality, we may assume that each $\lambda_i \in \overline{\mathbb{Q}}$. This follows through a standard specialization argument as shown in [Mas89, Section 5] (see also [Zan12, p. 39]); one can actually prove that there are infinitely many specializations which would yield multiplicatively independent $\lambda_1, \ldots, \lambda_{k-1}, \beta_1, \ldots, \beta_\ell$. (Note that if the orbit of α under the action of a specialization of Φ has a Zariski dense orbit, then $\mathcal{O}_\Phi(\alpha)$ must itself be Zariski dense in $\mathbb{G}_a^k \times \mathbb{G}_m^\ell$.)

Now, suppose to the contrary that there is a hypersurface Y (not necessarily irreducible) of $\mathbb{G}_a^k \times \mathbb{G}_m^\ell$ containing $\mathcal{O}_\Phi(\alpha)$. Similar to the proof of Theorem 2.1 (see the Case 5), considering the birational automorphism $\Psi_1 \colon \mathbb{G}_a^k \dashrightarrow \mathbb{G}_a^k$ given by

$$(x_1, x_2, x_3, \ldots, x_k) \mapsto \left(\frac{\lambda_1 (x_1 - x_2)}{x_2}, x_2, x_3, \ldots, x_k \right),$$

which extends to a birational automorphism $\Psi := \Psi_1 \times \mathrm{id}_{\mathbb{G}_m^\ell}$ of $\mathbb{G}_a^k \times \mathbb{G}_m^\ell$, we see that $\Psi(Y)$ is a hypersurface of $\mathbb{G}_a^k \times \mathbb{G}_m^\ell$ containing $\Psi(\mathcal{O}_\Phi(\alpha))$. In particular, this yields that there exists some nonzero polynomial $F \in \overline{\mathbb{Q}}[x_1, \ldots, x_{k+\ell}]$ (since the entire orbit of α is defined over $\overline{\mathbb{Q}}$) vanishing at the following set of $\overline{\mathbb{Q}}$-points:

$$\begin{aligned} &\Psi(\mathcal{O}_\Phi(\alpha)) \\ &= \left\{ \left(n, \lambda_1^n, \lambda_2^n, \ldots, \lambda_{k-1}^n, \beta_{n,1}, \beta_{n,2}, \ldots, \beta_{n,\ell} \right) \in (\mathbb{G}_a^k \times \mathbb{G}_m^\ell)(\overline{\mathbb{Q}}) : n \in \mathbb{N}_0 \right\}, \end{aligned}$$

where $(\beta_{n,1},\dots,\beta_{n,\ell}) \coloneqq \Phi_2^n(\beta_1,\dots,\beta_\ell)$. So, letting $\left\{m_{i,j}^{(n)}\right\}_{1\le i,j\le \ell}$ be the entries of the matrix A_2^n, then the point $\Phi_2^n(\beta_1,\dots,\beta_\ell)\in \mathbb{G}_m^\ell(\overline{\mathbb{Q}})$ equals

$$\left(\prod_{j=1}^{\ell}\beta_j^{m_{1,j}^{(n)}},\dots,\prod_{j=1}^{\ell}\beta_j^{m_{\ell,j}^{(n)}}\right),$$

or alternatively, we can write it as $\beta^{A_2^n}$, where $\beta\coloneqq(\beta_1,\dots,\beta_\ell)\in\mathbb{G}_m^\ell(\overline{\mathbb{Q}})$. More generally, for a matrix $M\in\mathbb{M}_{\ell,\ell}(\mathbb{Z})$ and some $\gamma\coloneqq(\gamma_1,\dots,\gamma_\ell)\in\mathbb{G}_m^\ell(\overline{\mathbb{Q}})$, we let γ^M be $\varphi(\gamma)$, where $\varphi\colon\mathbb{G}_m^\ell\longrightarrow\mathbb{G}_m^\ell$ is the endomorphism corresponding to the matrix M (with respect to the action of φ on the tangent space of the identity of $\mathbb{G}_m^\ell$). Furthermore, for any $\vec{a}\coloneqq(a_1,\dots,a_\ell)\in\mathbb{Z}^\ell$, we let $\gamma^{\vec{a}}\in\mathbb{G}_m(\overline{\mathbb{Q}})$ be $\prod_{i=1}^\ell\gamma_i^{a_i}$.

We also write

$$F(x_1,\dots,x_{k+\ell})=\sum_{(i_1,\dots,i_{k+\ell})}c_{i_1,\dots,i_{k+\ell}}x_1^{i_1}\cdots x_{k+\ell}^{i_{k+\ell}},$$

where each coefficient $c_{i_1,\dots,i_{k+\ell}}$ is nonzero so that it is a finite sum. We denote $\overrightarrow{i_{2,\dots,k}}\coloneqq(i_2,\dots,i_k)\in\mathbb{Z}^{k-1}$, $\overrightarrow{i_{k+1,\dots,k+\ell}}\coloneqq(i_{k+1},\dots,i_{k+\ell})\in\mathbb{Z}^\ell$, and $\Lambda\coloneqq(\lambda_1,\dots,\lambda_{k-1})\in\mathbb{G}_m^{k-1}(\overline{\mathbb{Q}})$. Note that the λ_i's are nonzero since Φ_1 is a dominant endomorphism. Let $M\coloneqq(m_{r,s})\in\mathbb{M}_{\ell,\ell}(\mathbb{Z})$ be a matrix of integer variables and consider the polynomial-exponential equation

$$\sum_{(i_2,\dots,i_{k+\ell})}\left(\sum_{i_1}c_{i_1,\dots,i_{k+\ell}}n^{i_1}\right)\cdot\left(\Lambda^{\overrightarrow{i_{2,\dots,k}}}\right)^n\cdot\beta^{\overrightarrow{i_{k+1,\dots,k+\ell}}\cdot M}=0; \tag{2.4.2}$$

in particular, $\beta^{\overrightarrow{i_{k+1,\dots,k+\ell}}\cdot M}$ equals

$$\prod_{s=1}^{\ell}\beta_s^{\sum_{r=1}^\ell i_{k+r}m_{r,s}}=\prod_{r,s=1}^{\ell}\beta_s^{i_{k+r}m_{r,s}}.$$

With the notation as in (2.4.2), we let

$$Q_{\overrightarrow{i_{2,\dots,k+\ell}}}(n)\coloneqq\sum_{i_1}c_{i_1,\dots,i_{k+\ell}}n^{i_1}.$$

So, the polynomial-exponential equation (2.4.2) has ℓ^2+1 integer variables; denoting $\Lambda_{i_2,\dots,i_k}\coloneqq\Lambda^{\overrightarrow{i_{2,\dots,k}}}$, we have

$$\sum_{(i_2,\dots,i_{k+\ell})}Q_{\overrightarrow{i_{2,\dots,k+\ell}}}(n)\cdot\Lambda_{i_2,\dots,i_k}^n\cdot\prod_{r,s=1}^{\ell}\left(\beta_s^{i_{k+r}}\right)^{m_{r,s}}=0. \tag{2.4.3}$$

We are going to apply [Lau84, Théorème 6]. Note that each $n\in\mathbb{N}_0$ for which

$$F(\Psi(\Phi^n(\alpha)))=0$$

yields an integer solution $\left(n, \left(m_{i,j}^{(n)}\right)_{1\le i,j\le \ell}\right)$ of the equation (2.4.3). Now, for each $n \in \mathbb{N}_0$, we let $\mathscr{P}_n$ be a maximal compatible partition of the set of indices $(i_2, \dots, i_{k+\ell})$ in the sense of Laurent (see [Lau84, p. 320]) with the property that for each part I of the partition $\mathscr{P}_n$, we have that

$$\sum_{(i_2,\dots,i_{k+\ell})\in I} Q_{\overrightarrow{i_2,\dots,k+\ell}}(n)\cdot \Lambda^n_{i_2,\dots,i_k}\cdot \prod_{r,s=1}^{\ell}\left(\beta_s^{i_{k+r}}\right)^{m_{r,s}^{(n)}} = 0. \tag{2.4.4}$$

Since there are only finitely many partitions of the finite index set of all $(i_2, \dots, i_{k+\ell})$, we fix some partition $\mathscr{P}$ for which we assume that there exists an infinite set S of positive integers n such that $\mathscr{P} := \mathscr{P}_n$. Then we define $\mathscr{H}_{\mathscr{P}}$ as the subgroup of $\mathbb{Z}^{\ell^2+1}$ consisting of all $\left(n, \left(m_{i,j}^{(n)}\right)_{1\le i,j\le \ell}\right)$ such that for each part I of the partition $\mathscr{P}$ and for any two indices $\vec{i} := (i_2, \dots, i_{k+\ell})$ and $\vec{j} := (j_2, \dots, j_{k+\ell})$ contained in I, we have that

$$\Lambda^n_{i_2,\dots,i_k}\cdot \prod_{r,s=1}^{\ell}\left(\beta_s^{i_{k+r}}\right)^{m_{r,s}^{(n)}} = \Lambda^n_{j_2,\dots,j_k}\cdot \prod_{r,s=1}^{\ell}\left(\beta_s^{j_{k+r}}\right)^{m_{r,s}^{(n)}}. \tag{2.4.5}$$

Then by [Lau84, Théorème 6], we can write the solution $\left(n, \left(m_{i,j}^{(n)}\right)_{1\le i,j\le \ell}\right)$ as $\overrightarrow{N_0}(n) + \overrightarrow{N_1}(n)$, where $\overrightarrow{N_0} := \overrightarrow{N_0}(n)$, $\overrightarrow{N_1} := \overrightarrow{N_1}(n) \in \mathbb{Z}^{1+\ell^2}$ and moreover, $\overrightarrow{N_0} \in \mathscr{H}_{\mathscr{P}}$ while the absolute value of each entry in $\overrightarrow{N_1}$ is bounded above by $C_1 \log(U_n) + C_2$, where C_1 and C_2 are some positive constants independent of n, and

$$U_n := \max\left\{n, \max_{1\le i,j\le \ell}\left|m_{i,j}^{(n)}\right|\right\}.$$

A simple computation for $A_2^n = \left(m_{i,j}^{(n)}\right)_{1\le i,j\le \ell}$ yields that there exists a positive constant C_3 such that $U_n \le C_3^n$ for all $n \in \mathbb{N}$. We then conclude that each entry in $\overrightarrow{N_1}$ is bounded in absolute value by $C_4 n + C_5$, for some absolute constants C_4 and C_5 independent of n. Next, we will determine the subgroup $\mathscr{H}_{\mathscr{P}}$ of $\mathbb{Z}^{1+\ell^2}$.

We may first assume that at least one part I of the partition $\mathscr{P}$ satisfies the property that $\pi(I)$ has at least 2 elements, where $\pi\colon \mathbb{Z}^{k+\ell-1} \longrightarrow \mathbb{Z}^\ell$ is the projection on the last ℓ coordinates, i.e., $(i_2, \dots, i_{k+\ell}) \mapsto (i_{k+1}, \dots, i_{k+\ell})$. Indeed, if $\#(\pi(I)) = 1$ for each part I of $\mathscr{P}$, then equation (2.4.4) would actually yield that

$$\sum_{(i_2,\dots,i_{k+\ell})\in I} Q_{\overrightarrow{i_2,\dots,k+\ell}}(n)\cdot \Lambda^n_{i_2,\dots,i_k} = 0. \tag{2.4.6}$$

Thus, since (2.4.6) holds for each part I of $\mathscr{P}$, we would get that there exists a proper subvariety of $\mathbb{G}_a^k \times \mathbb{G}_m^\ell$ of the form $Z \times \mathbb{G}_m^\ell$ containing infinitely many points of $\Psi(\mathcal{O}_\Phi(\alpha))$. In particular, Z would be a proper subvariety of

$\mathbb{G}_a^k$ containing infinitely many points of $\Psi_1(\mathcal{O}_{\Phi_1}(1,\dots,1))$, which contradicts the proof of Theorem 2.1 (see also Remark 2.2). Therefore, we may indeed assume that there exists at least one part I of $\mathscr{P}$ such that $\pi(I)$ contains at least two distinct elements $(i_{k+1},\dots,i_{k+\ell})$ and $(j_{k+1},\dots,j_{k+\ell})$.

Let $\overrightarrow{N_0} := \left(n_0, \left(m_{0,i,j}^{(n)}\right)_{1\le i,j\le \ell}\right)$. Since $\overrightarrow{N_0} \in \mathscr{H}_{\mathscr{P}}$, by the definition of $\mathscr{H}_{\mathscr{P}}$, we apply (2.4.5) to $\overrightarrow{N_0}$ and to $(i_2,\dots,i_{k+\ell})$, $(j_2,\dots,j_{k+\ell}) \in I$ for which $(i_{k+1},\dots,i_{k+\ell}) \neq (j_{k+1},\dots,j_{k+\ell})$ and get that

$$\Lambda_{i_2,\dots,i_k}^{n_0} \cdot \prod_{r,s=1}^{\ell} \left(\beta_s^{i_{k+r}}\right)^{m_{0,r,s}^{(n)}} = \Lambda_{j_2,\dots,j_k}^{n_0} \cdot \prod_{r,s=1}^{\ell} \left(\beta_s^{j_{k+r}}\right)^{m_{0,r,s}^{(n)}}. \tag{2.4.7}$$

Using the fact that $\Lambda_{i_2,\dots,i_k} = \prod_{t=1}^{k-1} \lambda_t^{i_{t+1}}$ and that $\lambda_1,\dots,\lambda_{k-1},\beta_1,\dots,\beta_\ell$ are multiplicatively independent, equation (2.4.7) yields that

$$\sum_{r=1}^{\ell} i_{k+r} m_{0,r,s}^{(n)} = \sum_{r=1}^{\ell} j_{k+r} m_{0,r,s}^{(n)} \text{ for any } 1 \le s \le \ell. \tag{2.4.8}$$

Denote $M_n^0 := \left(m_{0,r,s}^{(n)}\right)_{1\le r,s\le \ell}$ and also let $\vec{p} := (i_{k+1}-j_{k+1},\dots,i_{k+\ell}-j_{k+\ell})^t \in \mathbb{M}_{\ell,1}(\mathbb{Z})$. Then we may write equation (2.4.8) as $\vec{p}^{\,t} \cdot M_n^0 = \vec{0}$.

Let $\overrightarrow{N_1} := \left(n_1, \left(m_{1,r,s}^{(n)}\right)_{1\le r,s\le \ell}\right)$ and denote $M_n^1 := \left(m_{1,r,s}^{(n)}\right)_{1\le r,s\le \ell}$. Then we have $A_2^n = M_n^0 + M_n^1$ for each $n \in S$, i.e., $m_{r,s}^{(n)} = m_{0,r,s}^{(n)} + m_{1,r,s}^{(n)}$ for each $1 \le r,s \le \ell$. Using that $\vec{p}^{\,t} \cdot M_n^0 = \vec{0}$, we obtain that for each $n \in S$ we have

$$\vec{p}^{\,t} \cdot A_2^n = \vec{p}^{\,t} \cdot M_n^1, \text{ or equivalently, } (A_2^t)^n \cdot \vec{p} = (M_n^1)^t \cdot \vec{p}, \tag{2.4.9}$$

where D^t always represents the transpose of the matrix D. Using the fact that each entry in $(M_n^1)^t$ is bounded in absolute value by $C_4 n + C_5$, we obtain that each entry of the vector

$$\overrightarrow{p_n} := (A_2^t)^n \cdot \vec{p} = (M_n^1)^t \cdot \vec{p} \tag{2.4.10}$$

is also bounded in absolute value by $C_6 n + C_7$ (again for some positive constants C_6 and C_7 independent of n). Note that $\vec{p} \neq \vec{0}$ and (2.4.10) holds for all n in the infinite set S of positive integers. It follows from Proposition 2.3 that one of the eigenvalues of A_2 must be a root of unity, which contradicts our assumption on A_2 at the beginning of the proof.

This concludes our proof of Theorem 1.1. $\square$

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