# SIEGEL'S THEOREM FOR DRINFELD MODULES

D. GHIOCA AND T. J. TUCKER

Abstract. We prove a Siegel type statement for finitely generated $\phi$-submodules of $\mathbb{G}_a$ under the action of a Drinfeld module $\phi$. This provides a positive answer to a question we asked in a previous paper. We also prove an analog for Drinfeld modules of a theorem of Silverman for nonconstant rational maps of $\mathbb{P}^1$ over a number field.

## 1. Introduction

In 1929, Siegel ([Sie29]) proved that if $C$ is an irreducible affine curve defined over a number field $K$ and $C$ has at least three points at infinity, then there are at most finitely many $K$-rational points on $C$ that have integral coordinates. The proof of this famous theorem uses diophantine approximation along with the fact that certain groups of rational points are finitely generated; when $C$ has genus greater than 0, the group in question is the Mordell-Weil group of the Jacobian of $C$, while when $C$ has genus 0, the group in question is the group of $S$-units in a finite extension of $K$.

Motivated by the analogy between rank 2 Drinfeld modules and elliptic curves, the authors conjectured in [GT06] a Siegel type statement for finitely generated $\phi$-submodules $\Gamma$ of $\mathbb{G}_a$ (where $\phi$ is a Drinfeld module of arbitrary rank). For a finite set of places $S$ of a function field $K$, we defined a notion of $S$-integrality and asked whether or not it is possible that there are infinitely many $\gamma \in \Gamma$ which are $S$-integral with respect to a fixed point $\alpha \in \overline{K}$. We also proved in [GT06] a first instance of our conjecture in the case where $\Gamma$ is a cyclic submodule and $\alpha$ is a torsion point for $\phi$. Our goal in this paper is to prove our Siegel conjecture for every finitely generated $\phi$-submodule of $\mathbb{G}_a(K)$, where $\phi$ is a Drinfeld module defined over the field $K$ (see our Theorem 2.4). We will also establish an analog (also in the context of Drinfeld modules) of a theorem of Silverman for nonconstant morphisms of $\mathbb{P}^1$ of degree greater than 1 over a number field (see our Theorem 2.5).

We note that recently there has been significant progress on establishing additional links between classical diophantine results over number fields and similar statements for Drinfeld modules. Denis [Den92a] formulated analogs for Drinfeld modules of the Manin-Mumford and the Mordell-Lang

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[1]dghioca@math.mcmaster.ca; ttucker@math.rochester.edu

conjectures. The Denis-Manin-Mumford conjecture was proved by Scanlon in [Sca02], while a first instance of the Denis-Mordell-Lang conjecture was established in [Ghi05] by the first author (see also [Ghi06b] for an extension of the result from [Ghi05]). The authors proved in [GT07] several other cases of the Denis-Mordell-Lang conjecture. In addition, the first author proved in [Ghi06a] an equidistribution statement for torsion points of a Drinfeld module that is similar to the equidistribution statement established by Szpiro-Ullmo-Zhang [SUZ97] (which was later extended by Zhang [Zha98] to a full proof of the famous Bogomolov conjecture). Breuer [Bre05] proved a special case of the André-Oort conjecture for Drinfeld modules, while special cases of this conjecture in the classical case of a number field were proved by Edixhoven-Yafaev [EY03] and Yafaev [Yaf06]. Bosser [Bos99] proved a lower bound for linear forms in logarithms at an infinite place associated to a Drinfeld module (similar to the classical result obtained by Baker [Bak75] for usual logarithms, or by David [Dav95] for elliptic logarithms). Bosser's result was used by the authors in [GT06] to establish certain equidistribution and integrality statements for Drinfeld modules. Moreover, Bosser's result is believed to be true also for linear forms in logarithms at finite places for a Drinfeld module (as was communicated to us by Bosser). Assuming this last statement, we prove in this paper the natural analog of Siegel's theorem for finitely generated $\phi$-submodules. We believe that our present paper provides additional evidence that the Drinfeld modules represent a good arithmetic analog in characteristic $p$ for abelian varieties in characteristic 0.

The basic outline of this paper can be summarized quite briefly. In Section 2 we give the basic definitions and notation, and then state our main results. In Section 3 we prove these main results: Theorems 2.4 and 2.5.

## 2. Notation

**Notation.** $\mathbb{N}$ stands for the non-negative integers: $\{0, 1, \dots\}$, while $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ stands for the positive integers.

2.1. **Drinfeld modules.** We begin by defining a Drinfeld module. Let $p$ be a prime and let $q$ be a power of $p$. Let $A := \mathbb{F}_q[t]$, let $K$ be a finite field extension of $\mathbb{F}_q(t)$, and let $\overline{K}$ be an algebraic closure of $K$. We let $\tau$ be the Frobenius on $\mathbb{F}_q$, and we extend its action on $\overline{K}$. Let $K\{\tau\}$ be the ring of polynomials in $\tau$ with coefficients from $K$ (the addition is the usual addition, while the multiplication is the composition of functions).

A Drinfeld module is a morphism $\phi : A \to K\{\tau\}$ for which the coefficient of $\tau^0$ in $\phi(a) =: \phi_a$ is $a$ for every $a \in A$, and there exists $a \in A$ such that $\phi_a \neq a\tau^0$. The definition given here represents what Goss [Gos96] calls a Drinfeld module of "generic characteristic".

We note that usually, in the definition of a Drinfeld module, $A$ is the ring of functions defined on a projective nonsingular curve $C$, regular away from a closed point $\eta \in C$. For our definition of a Drinfeld module, $C = \mathbb{P}^1_{\mathbb{F}_q}$ and $\eta$ is the usual point at infinity on $\mathbb{P}^1$. On the other hand, every ring of regular functions $A$ as above contains $\mathbb{F}_q[t]$ as a subring, where $t$ is a nonconstant function in $A$.

For every field extension $K \subset L$, the Drinfeld module $\phi$ induces an action on $\mathbb{G}_a(L)$ by $a * x := \phi_a(x)$, for each $a \in A$. We call *$\phi$-submodules* subgroups of $\mathbb{G}_a(\overline{K})$ which are invariant under the action of $\phi$. We define the *rank* of a $\phi$-submodule $\Gamma$ be

$$\dim_{\mathrm{Frac}(A)} \Gamma \otimes_A \mathrm{Frac}(A).$$

As shown in [Poo95], $\mathbb{G}_a(K)$ is a direct sum of a finite torsion $\phi$-submodule with a free $\phi$-submodule of rank $\aleph_0$.

A point $\alpha$ is *torsion* for the Drinfeld module action if and only if there exists $Q \in A \setminus \{0\}$ such that $\phi_Q(\alpha) = 0$. The monic polynomial $Q$ of minimal degree which satisfies $\phi_Q(\alpha) = 0$ is called the *order* of $\alpha$. Since each polynomial $\phi_Q$ is separable, the torsion submodule $\phi_{\mathrm{tor}}$ lies in the separable closure $K^{\mathrm{sep}}$ of $K$.

2.2. **Valuations and Weil heights.** Let $M_{\mathbb{F}_q(t)}$ be the set of places on $\mathbb{F}_q(t)$. We denote by $v_\infty$ the place in $M_{\mathbb{F}_q(t)}$ such that $v_\infty(\frac{f}{g}) = \deg(g) - \deg(f)$ for every nonzero $f, g \in A = \mathbb{F}_q[t]$. We let $M_K$ be the set of valuations on $K$. Then $M_K$ is a set of valuations which satisfies a product formula (see [Ser97, Chapter 2]). Thus

- for each nonzero $x \in K$, there are finitely many $v \in M_K$ such that $|x|_v \neq 1$; and
- for each nonzero $x \in K$, we have $\prod_{v \in M_K} |x|_v = 1$.

We may use these valuations to define a Weil height for each $x \in K$ as

$$h(x) = \sum_{v \in M_K} \max \log(|x|_v, 1). \tag{2.0.1}$$

**Convention.** Without loss of generality we may assume that the normalization for all the valuations of $K$ is made so that for each $v \in M_K$, we have $\log |x|_v \in \mathbb{Z}$.

**Definition 2.1.** *Each place in $M_K$ which lies over $v_\infty$ is called an infinite place. Each place in $M_K$ which does not lie over $v_\infty$ is called a finite place.*

2.3. **Canonical heights.** Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of *rank* $d$ (i.e. the degree of $\phi_t$ as a polynomial in $\tau$ equals $d$). The canonical height of $\beta \in K$ relative to $\phi$ (see [Den92b]) is defined as

$$\widehat{h}(\beta) = \lim_{n\to\infty} \frac{h(\phi_{t^n}(\beta))}{q^{nd}}.$$

Denis [Den92b] showed that a point is torsion if and only if its canonical height equals 0.

For every $v \in M_K$, we let the local canonical height of $\beta \in K$ at $v$ be

$$\widehat{h}_v(\beta) = \lim_{n\to\infty} \frac{\log\max(|\phi_{t^n}(\beta)|_v, 1)}{q^{nd}}. \tag{2.1.1}$$

Furthermore, for every $a \in \mathbb{F}_q[t]$, we have $\widehat{h}_v(\phi_a(x)) = \deg(\phi_a) \cdot \widehat{h}_v(x)$ (see [Poo95]). It is clear that $\widehat{h}_v$ satisfies the triangle inequality, and also that $\sum_{v \in M_K} \widehat{h}_v(\beta) = \widehat{h}(\beta)$.

2.4. **Completions and filled Julia sets.** By abuse of notation, we let $\infty \in M_K$ denote any place extending the place $v_\infty$. We let $K_\infty$ be the completion of $K$ with respect to $|\cdot|_\infty$. We let $\overline{K_\infty}$ be an algebraic closure of $K_\infty$. We let $\mathbb{C}_\infty$ be the completion of $\overline{K_\infty}$. Then $\mathbb{C}_\infty$ is a complete, algebraically closed field. Note that $\mathbb{C}_\infty$ depends on our choice for $\infty \in M_K$ extending $v_\infty$. However, each time we will work with only one such place $\infty$, and so, there will be no possibility of confusion.

Next, we define the *$v$-adic filled Julia set* $J_{\phi,v}$ corresponding to the Drinfeld module $\phi$ and to each place $v$ of $M_K$. Let $\mathbb{C}_v$ be the completion of an algebraic closure of $K_v$. Then $|\cdot|_v$ extends to a unique absolute value on all of $\mathbb{C}_v$. The set $J_{\phi,v}$ consists of all $x \in \mathbb{C}_v$ for which $\{|\phi_Q(x)|_v\}_{Q \in A}$ is bounded. It is immediate to see that $x \in J_{\phi,v}$ if and only if $\{|\phi_{t^n}(x)|_v\}_{n \geq 1}$ is bounded.

One final note on absolute values: as noted above, the place $v \in M_K$ extends to a unique absolute value $|\cdot|_v$ on all of $\mathbb{C}_v$. We fix an embedding of $i : \overline{K} \longrightarrow \mathbb{C}_v$. For $x \in \overline{K}$, we denote $|i(x)|_v$ simply as $|x|_v$, by abuse of notation.

2.5. **The coefficients of $\phi_t$.** Each Drinfeld module is isomorphic to a Drinfeld module for which all the coefficients of $\phi_t$ are integral at all the places in $M_K$ which do not lie over $v_\infty$. Indeed, we let $B \in \mathbb{F}_q[t]$ be a product of all (the finitely many) irreducible polynomials $P \in \mathbb{F}_q[t]$ with the property that there exists a place $v \in M_K$ which lies over the place $(P) \in M_{\mathbb{F}_q(t)}$, and there exists a coefficient of $\phi_t$ which is not integral at $v$. Let $\gamma$ be a sufficiently large power of $B$. Then $\psi : A \to K\{\tau\}$ defined by $\psi_Q := \gamma^{-1}\phi_Q\gamma$ (for each $Q \in A$) is a Drinfeld module isomorphic to $\phi$, and all the coefficients of $\psi_t$ are integral away from the places lying above $v_\infty$. Hence, from now on, we assume that all the coefficients of $\phi_t$ are integral away from the places lying over $v_\infty$. It follows that for every $Q \in A$, all coefficients of $\phi_Q$ are integral away from the places lying over $v_\infty$.

2.6. **Integrality and reduction.**

**Definition 2.2.** *For a finite set of places $S \subset M_K$ and $\alpha \in \overline{K}$, we say that $\beta \in \overline{K}$ is $S$-integral with respect to $\alpha$ if for every place $v \notin S$, and for every morphisms $\sigma, \tau : \overline{K} \to \overline{K}$ (which restrict to the identity on $K$) the following are true:*

- *if $|\alpha^\tau|_v \leq 1$, then $|\alpha^\tau - \beta^\sigma|_v \geq 1$.*
- *if $|\alpha^\tau|_v > 1$, then $|\beta^\sigma|_v \leq 1$.*

We note that if $\beta$ is $S$-integral with respect to $\alpha$, then it is also $S'$-integral with respect to $\alpha$, where $S'$ is a finite set of places containing $S$. Moreover, the fact that $\beta$ is $S$-integral with respect to $\alpha$, is preserved if we replace

$K$ by a finite extension. Therefore, in our results we will always assume $\alpha, \beta \in K$. For more details about the definition of $S$-integrality, we refer the reader to [BIR05].

**Definition 2.3.** *The Drinfeld module $\phi$ has good reduction at a place $v$ if for each nonzero $a \in A$, all coefficients of $\phi_a$ are $v$-adic integers and the leading coefficient of $\phi_a$ is a $v$-adic unit. If $\phi$ does not have good reduction at $v$, then we say that $\phi$ has bad reduction at $v$.*

It is immediate to see that $\phi$ has good reduction at $v$ if and only if all coefficients of $\phi_t$ are $v$-adic integers, while the leading coefficient of $\phi_t$ is a $v$-adic unit.

We can now state our Siegel type result for Drinfeld modules.

**Theorem 2.4.** *With the above notation, assume in addition $K$ has only one infinite place. Let $\Gamma$ be a finitely generated $\phi$-submodule of $\mathbb{G}_a(K)$, let $\alpha \in K$, and let $S$ be a finite set of places in $M_K$. Then there are finitely many $\gamma \in \Gamma$ such that $\gamma$ is $S$-integral with respect to $\alpha$.*

As mentioned in Section 1, we proved in [GT06] that Theorem 2.4 holds when $\Gamma$ is a cyclic $\phi$-module generated by a nontorsion point $\beta \in K$ and $\alpha \in \phi_{\mathrm{tor}}(K)$ (see Theorem 1.1 and Proposition 5.6 of [GT06]). Moreover, in [GT06] we did not have in our results the extra hypothesis from Theorem 2.4 that there exists only one infinite place in $M_K$. Even though we believe Theorem 2.4 is true without this hypothesis, our method for proving Theorem 2.4 requires this technical hypothesis. On the other hand, we are able to prove the following analog for Drinfeld modules of a theorem of Silverman (see [Sil93]) for nonconstant morphisms of $\mathbb{P}^1$ of degree greater than 1 over a number field, without the hypothesis of having only one infinite place in $M_K$.

**Theorem 2.5.** *With the above notation, let $\beta \in K$ be a nontorsion point, and let $\alpha \in K$ be an arbitrary point. Then there are finitely many $Q \in A$ such that $\phi_Q(\beta)$ is $S$-integral for $\alpha$.*

As explained before, in [GT06] we proved Theorem 2.5 in the case $\alpha$ is a torsion point in $K$.

## 3. Proofs of our main results

We continue with the notation from Section 2. In our argument, we will be using the following key fact.

**Fact 3.1.** *Assume $\infty \in M_K$ is an infinite place. Let $\gamma_1, \dots, \gamma_r, \alpha \in K$. Then there exist (negative) constants $C_0$ and $C_1$ (depending only on $\phi$, $\gamma_1, \dots, \gamma_r, \alpha$) such that for any polynomials $P_1, \dots, P_r \in A$ (not all constants), either $\phi_{P_1}(\gamma_1) + \cdots + \phi_{P_r}(\gamma_r) = \alpha$ or*

$$\log |\phi_{P_1}(\gamma_1) + \cdots + \phi_{P_r}(\gamma_r) - \alpha|_\infty \geq C_0 + C_1 \max_{1 \leq i \leq r} (\deg(P_i) \log \deg(P_i)).$$

Fact 3.1 follows easily from the lower bounds for linear forms in logarithms established by Bosser (see Théorème 1.1 in [Bos99]). Essentially, it is the same proof as our proof of Proposition 3.7 of [GT06] (see in particular the derivation of the inequality (3.7.2) in [GT06]). For the sake of completeness, we will provide below a sketch of a proof of Fact 3.1.

*Proof of Fact 3.1.* We denote by $\exp_\infty$ the exponential map associated to the place $\infty$ (see [Gos96]). We also let $\mathcal{L}$ be the corresponding lattice for $\exp_\infty$, i.e. $\mathcal{L} := \ker(\exp_\infty)$. Finally, let $\omega_1, \dots, \omega_d$ be an $A$-basis for $\mathcal{L}$ of "successive minima" (see Lemma (4.2) of [Tag93]). This means that for every $Q_1, \dots, Q_d \in A$, we have

$$|Q_1\omega_1 + \cdots + Q_d\omega_d|_\infty = \max_{i=1}^{d} |Q_i\omega_i|_\infty. \tag{3.1.1}$$

Let $u_0 \in \mathbb{C}_\infty$ such that $\exp_\infty(u_0) = \alpha$. We also let $u_1, \dots, u_r \in \mathbb{C}_\infty$ such that for each $i$, we have $\exp_\infty(u_i) = \gamma_i$. We will find constants $C_0$ and $C_1$ satisfying the inequality from Fact 3.1, which depend only on $\phi$ and $u_0, u_1, \dots, u_r$.

There exists a positive constant $C_2$ such that $\exp_\infty$ induces an isomorphism from the ball $B := \{z \in \mathbb{C}_\infty : |z|_\infty < C_2\}$ to itself (see Lemma 3.6 of [GT06]). If we assume there exist no constants $C_0$ and $C_1$ as in the conclusion of Fact 3.1, then there exist polynomials $P_1, \dots, P_r$, not all constants, such that

$$\sum_{i=1}^{r} \phi_{P_i}(\gamma_i) \neq \alpha \tag{3.1.2}$$

and $|\sum_{i=1}^r \phi_{P_i}(\gamma_i) - \alpha|_\infty < C_2$. Thus we can find $y \in B$ such that $|y|_\infty = |\sum_{i=1}^r \phi_{P_i}(\gamma_i) - \alpha|_\infty$ and

$$\exp_\infty(y) = \sum_{i=1}^{r} \phi_{P_i}(\gamma_i) - \alpha. \tag{3.1.3}$$

Moreover, because $\exp_\infty$ is an isomorphism on the metric space $B$, then for every $y' \in \mathbb{C}_\infty$ such that $\exp_\infty(y') = \sum_{i=1}^r \phi_{P_i}(\gamma_i) - \alpha$, we have $|y'|_\infty \geq |y|_\infty$. But we know that

$$\exp_\infty\left(\sum_{i=1}^{r} P_i u_i - u_0\right) = \sum_{i=1}^{r} \phi_{P_i}(\gamma_i) - \alpha. \tag{3.1.4}$$

Therefore $|\sum_{i=1}^r P_i u_i - u_0|_\infty \geq |y|_\infty$. On the other hand, using (3.1.3) and (3.1.4), we conclude that there exist polynomials $Q_1, \dots, Q_d$ such that

$$\sum_{i=1}^{r} P_i u_i - u_0 = y + \sum_{i=1}^{d} Q_i\omega_i.$$

Hence $|\sum_{i=1}^{d} Q_i\omega_i|_\infty \le |\sum_{i=1}^{r} P_iu_i - u_0|_\infty$. Using (3.1.1), we obtain

$$\begin{aligned}\left|\sum_{i=1}^{d} Q_i\omega_i\right|_\infty = \max_{i=1}^{d} |Q_i\omega_i|_\infty &\le \left|\sum_{i=1}^{r} P_iu_i - u_0\right|_\infty \\ &\le \max\left(|u_0|_\infty, \max_{i=1}^{r} |P_iu_i|_\infty\right) \\ &\le C_3 \cdot \max_{i=1}^{r} |P_i|_\infty,\end{aligned} \tag{3.1.5}$$

where $C_3$ is a constant depending only on $u_0, u_1, \dots, u_r$. We take logarithms of both sides in (3.1.5) and obtain

$$\begin{aligned}\max_{i=1}^{d} \deg Q_i &\le \max_{i=1}^{r} \deg P_i + \log C_3 - \min_{i=1}^{d} \log |\omega_i|_\infty \\ &\le \max_{i=1}^{r} \deg P_i + C_4,\end{aligned} \tag{3.1.6}$$

where $C_4$ depends only on $\phi$ and $u_0, u_1, \dots, u_r$ (the dependence on the $\omega_i$ is actually a dependence on $\phi$, because the $\omega_i$ are a fixed basis of "successive minima" for $\phi$ at $\infty$). Using (3.1.6) and Proposition 3.2 of [GT06] (which is a translation of the bounds for linear forms in logarithms for Drinfeld modules established in [Bos99]), we conclude that there exist (negative) constants $C_0$, $C_1$, $C_5$ and $C_6$ (depending only on $\phi$, $\gamma_1, \dots, \gamma_r$ and $\alpha$) such that

(3.1.7)

$$\begin{aligned}\log\left|\sum_{i=1}^{r} \phi_{P_i}(\gamma_i) - \alpha\right|_\infty &= \log |y|_\infty \\ &= \log\left|\sum_{i=1}^{r} P_iu_i - u_0 - \sum_{i=1}^{d} Q_i\omega_i\right|_\infty \\ &\ge C_5 + C_6\left(\max_{i=1}^{r} \deg P_i + C_4\right) \log \max_{i=1}^{r} (\deg P_i + C_4) \\ &\ge C_0 + C_1\left(\max_{i=1}^{r} \deg P_i\right) \log \max_{i=1}^{r} (\deg P_i),\end{aligned}$$

as desired. □

In our proofs for Theorems 2.5 and 2.4 we will also use the following statement, which is believed to be true, based on communication with V. Bosser. Therefore we assume its validity without proof.

**Statement 3.2.** *Assume $v$ does not lie above $v_\infty$. Let $\gamma_1, \dots, \gamma_r, \alpha \in K$. Then there exist positive constants $C_1, C_2, C_3$ (depending only on $v$, $\phi$, $\gamma_1, \dots, \gamma_r$ and $\alpha$) such that for any $P_1, \dots, P_r \in \mathbb{F}_q[t]$, either $\phi_{P_1}(\gamma_1) + \dots + \phi_{P_r}(\gamma_r) = \alpha$ or*

$$\log |\phi_{P_1}(\gamma_1) + \dots + \phi_{P_r}(\gamma_r) - \alpha|_v \ge -C_1 - C_2 \max_{1\le i\le r} (\deg(P_i))^{C_3}.$$

Statement 3.2 follows after one establishes a lower bound for linear forms in logarithms at finite places $v$. In a private communication, V. Bosser told us that it is clear to him that his proof ([Bos99]) can be adapted to work also at finite places with minor modifications.

We sketch here how Statement 3.2 would follow from a lower bound for linear forms in logarithms at finite places. Let $v$ be a finite place and let $\exp_v$ be the formal exponential map associated to $v$. The existence of $\exp_v$ and its convergence on a sufficiently small ball $B_v := \{x \in \mathbb{C}_v : |x|_v < C_v\}$ is proved along the same lines as the existence and the convergence of the usual exponential map at infinite places for $\phi$ (see Section 4.6 of [Gos96]). In addition,

$$|\exp_v(x)|_v = |x|_v \tag{3.2.1}$$

for every $x \in B_v$. Moreover, at the expense of replacing $C_v$ with a smaller positive constant, we may assume that for each $F \in A$, and for each $x \in B_v$, we have (see Lemma 4.2 in [GT06])

$$|\phi_F(x)|_v = |Fx|_v. \tag{3.2.2}$$

Assume we know the existence of the following lower bound for (nonzero) linear forms in logarithms at a finite place $v$.

**Statement 3.3.** *Let $u_1, \dots, u_r \in B_v$ such that for each $i$, $\exp_v(u_i) \in \overline{K}$. Then there exist positive constants $C_4$, $C_5$, and $C_6$ (depending on $u_1, \dots, u_r$) such that for every $F_1, \dots, F_r \in A$, either $\sum_{i=1}^r F_i u_i = 0$, or*

$$\log\left|\sum_{i=1}^r F_i u_i\right|_v \geq -C_4 - C_5\left(\max_{i=1}^r \deg F_i\right)^{C_6}.$$

As mentioned before, Bosser proved Statement 3.3 in the case $v$ is an infinite place (in his result, $C_6 = 1 + \epsilon$ and $C_4 = C_\epsilon$ for every $\epsilon > 0$).

We will now derive Statement 3.2 assuming Statement 3.3 holds.

*Proof.* (*That Statement 3.3 implies Statement 3.2.*) Clearly, it suffices to prove Statement 3.2 in the case $\alpha = 0$. So, let $\gamma_1, \dots, \gamma_r \in K$, and assume by contradiction that there exists an infinite sequence $\{F_{n,i}\}_{\substack{n\in\mathbb{N}^*\\ 1\le i\le r}}$ such that for each $n$, we have

$$-\infty < \log\left|\sum_{i=1}^r \phi_{F_{n,i}}(\gamma_i)\right|_v < \log C_v. \tag{3.3.1}$$

For each $n \geq 1$, we let $\mathcal{F}_n := (F_{n,1}, \dots, F_{n,r}) \in A^r$. We view $A^r$ as an $r$-dimensional $A$-lattice inside the $r$-dimensional $\mathrm{Frac}(A)$-vector space $\mathrm{Frac}(A)^r$. In addition, we may assume that for $n \neq m$, we have $\mathcal{F}_n \neq \mathcal{F}_m$.

Using basic linear algebra, because the sequence $\{F_{n,i}\}_{\substack{n\in\mathbb{N}^*\\ 1\le i\le r}}$ is infinite, we can find $n_0 \geq 1$ such that for every $n > n_0$, there exist $H_n, G_{n,1}, \dots, G_{n,n_0} \in$

$A$ (not all equal to 0) such that

$$H_n \cdot \mathcal{F}_n = \sum_{j=1}^{n_0} G_{n,j} \cdot \mathcal{F}_j. \tag{3.3.2}$$

Essentially, (3.3.2) says that $\mathcal{F}_1, \dots, \mathcal{F}_{n_0}$ span the linear subspace of $\mathrm{Frac}(A)^r$ generated by all $\mathcal{F}_n$. Moreover, we can choose the $H_n$ in (3.3.2) in such a way that $\deg H_n$ is bounded independently of $n$ (e.g. by a suitable determinant of some linearly independent subset of the first $n_0$ of the $\mathcal{F}_j$). Furthemore, there exists a constant $C_7$ such that for all $n > n_0$, we have

$$\max_{j=1}^{n_0} \deg G_{n,j} < C_7 + \max_{i=1}^{r} \deg F_{n,i}. \tag{3.3.3}$$

Because $\left|\sum_{i=1}^r \phi_{F_{n,i}}(\gamma_i)\right|_v < C_v$, equation (3.2.2) yields

$$\left|\phi_{H_n}\left(\sum_{i=1}^r \phi_{F_{n,i}}(\gamma_i)\right)\right|_v = |H_n|_v \cdot \left|\sum_{i=1}^r \phi_{F_{n,i}}(\gamma_i)\right|_v. \tag{3.3.4}$$

Using (3.3.2), (3.3.4), and the fact that $|H_n|_v \leq 1$, we obtain

$$\begin{aligned} \left|\sum_{i=1}^r \phi_{F_{n,i}}(\gamma_i)\right|_v &\geq \left|\phi_{H_n}\left(\sum_{i=1}^r \phi_{F_{n,i}}(\gamma_i)\right)\right|_v \\ &= \left|\sum_{j=1}^{n_0} \phi_{G_{n,j}}\left(\sum_{i=1}^r \phi_{F_{j,i}}(\gamma_i)\right)\right|_v. \end{aligned} \tag{3.3.5}$$

Since $\left|\sum_{i=1}^r \phi_{F_{j,i}}(\gamma_i)\right|_v < C_v$ for all $1 \leq j \leq n_0$, there exist $u_1, \dots, u_{n_0} \in B_v$ such that for every $1 \leq j \leq n_0$, we have

$$\exp_v(u_j) = \sum_{i=1}^r \phi_{F_{j,i}}(\gamma_i).$$

Then Statement 3.3 implies that there exist constants $C_4, C_5, C_6, C_8, C_9$ (depending on $u_1, \dots, u_{n_0}$), such that

$$\begin{aligned} \log\left|\sum_{j=1}^{n_0} \phi_{G_{n,j}}\left(\sum_{i=1}^r \phi_{F_{j,i}}(\gamma_i)\right)\right|_v &= \log\left|\sum_{j=1}^{n_0} G_{n,j} u_j\right|_v \\ &\geq -C_4 - C_5\left(\max_{j=1}^{n_0} \deg G_{n,j}\right)^{C_6} \\ &\geq -C_8 - C_9\left(\max_{i=1}^{r} \deg F_{n,i}\right)^{C_6}, \end{aligned} \tag{3.3.6}$$

where in the first equality we used (3.2.1), while in the last inequality we used (3.3.3). Equations (3.3.5) and (3.3.6) show that Statement 3.2 follows from Statement 3.3, as desired. □

Next we prove Theorem 2.5 which will be a *warm-up* for our proof of Theorem 2.4. For its proof, we will only need the following weaker (but also still conjectural) form of Statement 3.2 (i.e., we only need Statement 3.3 be true for non-homogeneous 1-forms of logarithms).

**Statement 3.4.** *Assume $v$ does not lie over $v_\infty$. Let $\gamma, \alpha \in K$. Then there exist positive constants $C_1$, $C_2$ and $C_3$ (depending only on $v$, $\phi$, $\gamma$ and $\alpha$) such that for each polynomial $P \in \mathbb{F}_q[t]$, either $\phi_P(\gamma) = \alpha$ or*

$$\log |\phi_P(\gamma) - \alpha|_v \geq -C_1 - C_2 \deg(P)^{C_3}.$$

*Proof of Theorem 2.5.* The following Lemma is the key to our proof.

**Lemma 3.5.** *For each $v \in M_K$, we have $\widehat{h}_v(\beta) = \lim_{\deg Q \to \infty} \frac{\log |\phi_Q(\beta) - \alpha|_v}{q^{d \deg Q}}$.*

*Proof of Lemma 3.5.* Let $v \in M_K$. If $\beta \notin J_{\phi,v}$, then $|\phi_Q(\beta)|_v \to \infty$, as $\deg Q \to \infty$. Hence, if $\deg Q$ is sufficiently large, then $|\phi_Q(\beta) - \alpha|_v = |\phi_Q(\beta)|_v = \max\{|\phi_Q(\beta)|_v, 1\}$, which yields the conclusion of Lemma 3.5.

Thus, from now on, we assume $\beta \in J_{\phi,v}$. Hence $\widehat{h}_v(\beta) = 0$, and we need to show that

$$\lim_{\deg Q \to \infty} \frac{\log |\phi_Q(\beta) - \alpha|_v}{q^{d \deg Q}} = 0. \tag{3.5.1}$$

Also note that since $\beta \in J_{\phi,v}$, then $|\phi_Q(\beta) - \alpha|_v$ is bounded, and so, $\limsup_{\deg Q \to \infty} \frac{\log |\phi_Q(\beta) - \alpha|_v}{q^{d \deg Q}} \leq 0$. Thus, in order to prove (3.5.1), it suffices to show that

$$\liminf_{\deg Q \to \infty} \frac{\log |\phi_Q(\beta) - \alpha|_v}{q^{d \deg Q}} \geq 0. \tag{3.5.2}$$

If v is an infinite place, then Fact 3.1 implies that for every polynomial $Q$ such that $\phi_Q(\beta) \neq \alpha$, we have $\log |\phi_Q(\beta) - \alpha|_\infty \geq C_0 + C_1 \deg(Q) \log \deg(Q)$ (for some constants $C_0, C_1 < 0$). Then taking the limit as $\deg Q \to \infty$, we obtain (3.5.2), as desired.

Similarly, if $v$ is a finite place, then (3.5.2) follows from Statement 3.4. ☐

Theorem 2.5 follows easily using the result of Lemma 3.5. We assume there exist infinitely many polynomials $Q_n$ such that $\phi_{Q_n}(\beta)$ is $S$-integral with respect to $\alpha$. We consider the sum

$$\Sigma := \sum_{v \in M_K} \lim_{n \to \infty} \frac{\log |\phi_{Q_n}(\beta) - \alpha|_v}{q^{d \deg Q_n}}.$$

Using Lemma 3.5, we obtain that $\Sigma = \widehat{h}(\beta) > 0$ (because $\beta \notin \phi_{\text{tor}}$).

Let $\mathcal{T}$ be a finite set of places consisting of all the places in $S$ along with all places $v \in M_K$ which satisfy at least one of the following conditions:

1. $|\beta|_v > 1$.
2. $|\alpha|_v > 1$.
3. $v$ is a place of bad reduction for $\phi$.

Therefore by our choice for $\mathcal{T}$ (see 1. and 3.), for every $v \notin \mathcal{T}$, we have $|\phi_{Q_n}(\beta)|_v \le 1$. Thus, using also 2., we have $|\phi_{Q_n}(\beta) - \alpha|_v \le 1$. On the other hand, $\phi_{Q_n}(\beta)$ is also $\mathcal{T}$-integral with respect to $\alpha$. Hence, because of 2., then for all $v \notin \mathcal{T}$, we have $|\phi_{Q_n}(\beta) - \alpha|_v \ge 1$. We conclude that for every $v \notin \mathcal{T}$, and for every $n$, we have $|\phi_{Q_n}(\beta) - \alpha|_v = 1$. This allows us to interchange the summation and the limit in the definition of $\Sigma$ (because then $\Sigma$ is a finite sum over all $v \in \mathcal{T}$). We obtain

$$\Sigma = \lim_{n\to\infty} \frac{1}{q^{d \deg Q_n}} \sum_{v \in M_K} \log|\phi_{Q_n}(\beta) - \alpha|_v = 0,$$

by the product formula applied to each $\phi_{Q_n}(\beta) - \alpha$. On the other hand, we already showed that $\Sigma = \widehat{h}(\beta) > 0$. This contradicts our assumption that there are infinitely many polynomials $Q$ such that $\phi_Q(\beta)$ is $S$-integral with respect to $\alpha$, and concludes the proof of Theorem 2.5. ☐

Before proceeding to the proof of Theorem 2.4, we prove several facts about local heights. In Lemma 3.10 we will use the technical assumption of having only one infinite place in $K$.

From now on, let $\phi_t = \sum_{i=0}^{d} a_i \tau^i$. As explained in Section 2, we may assume each $a_i$ is integral away from $v_\infty$. Also, from now on, we work under the assumption that there exists a *unique* place $\infty \in M_K$ lying above $v_\infty$.

**Fact 3.6.** *For every place $v$ of $K$, there exists $M_v > 0$ such that for each $x \in K$, if $|x|_v > M_v$, then for every nonzero $Q \in A$, we have $|\phi_Q(x)|_v > M_v$. Moreover, if $|x|_v > M_v$, then $\widehat{h}_v(x) = \log|x|_v + \frac{\log|a_d|_v}{q^d - 1} > 0$.*

Fact 3.6 is proved in Lemma 4.4 of [GT06]. In particular, Fact 3.6 shows that for each $v \in M_K$ and for each $x \in K$, we have $\widehat{h}_v(x) \in \mathbb{Q}$. Indeed, for every $x \in K$ of positive local canonical height at $v$, there exists a polynomial $P$ such that $|\phi_P(x)|_v > M_v$. Then $\widehat{h}_v(x) = \frac{\widehat{h}_v(\phi_P(x))}{q^{d \deg P}}$. By Fact 3.6, we already know that $\widehat{h}_v(\phi_P(x)) \in \mathbb{Q}$. Thus also $\widehat{h}_v(x) \in \mathbb{Q}$.

**Fact 3.7.** *Let $v \in M_K \setminus \{\infty\}$. There exists a positive constant $N_v$, and there exists a nonzero polynomial $Q_v$, such that for each $x \in K$, the following statements are true*

(i) *if $|x|_v \le N_v$, then for each $Q \in A$, we have $|\phi_Q(x)|_v \le |x|_v \le N_v$.*
(ii) *either $|\phi_{Q_v}(x)|_v \le N_v$, or $|\phi_{Q_v}(x)|_v > M_v$.*

*Proof of Fact 3.7.* This was proved in [Ghi07b]. It is easy to see that

$$N_v := \min_{1 \le i \le d} |a_i|_v^{-\frac{1}{q^i - 1}}$$

satisfies condition $(i)$, but the proof of $(ii)$ is much more complicated. In [Ghi07b], the first author proved that there exists a positive integer $d_v$ such that for every $x \in K$, there exists a polynomial $Q$ of degree at most $d_v$ such that either $|\phi_Q(x)|_v > M_v$, or $|\phi_Q(x)|_v \le N_v$ (see Remark 5.12 which is valid for every place which does not lie over $v_\infty$). Using Fact 3.6 and $(i)$, we conclude that the polynomial $Q_v := \prod_{\deg P \le d_v} P$ satisfies property $(ii)$. □

Using Facts 3.6 and 3.7 we prove the following important result valid for finite places.

**Lemma 3.8.** *Let $v \in M_K \setminus \{\infty\}$. Then there exists a positive integer $D_v$ such that for every $x \in K$, we have $D_v \cdot \widehat{h}_v(x) \in \mathbb{N}$. If in addition we assume $v$ is a place of good reduction for $\phi$, then we may take $D_v = 1$.*

*Proof of Lemma 3.8.* Let $x \in K$. If $\widehat{h}_v(x) = 0$, then we have nothing to show. Therefore, assume from now on that $\widehat{h}_v(x) > 0$. Using $(ii)$ of Fact 3.7, there exists a polynomial $Q_v$ (depending only on $v$, and not on $x$) such that $|\phi_{Q_v}(x)|_v > M_v$ (clearly, the other option from $(ii)$ of Lemma 3.7 is not available because we assumed that $\widehat{h}_v(x) > 0$). Moreover, using the definition of the local height, and also Fact 3.6,

$$\widehat{h}_v(x) = \frac{\widehat{h}_v(\phi_{Q_v}(x))}{q^{d \deg Q_v}} = \frac{\log|\phi_{Q_v}(x)|_v + \frac{\log|a_d|_v}{q^d - 1}}{q^{d \deg Q_v}}. \tag{3.8.1}$$

Because both $\log|\phi_{Q_v}(x)|_v$ and $\log|a_d|_v$ are integer numbers, (3.8.1) yields the conclusion of Lemma 3.8 (we may take $D_v = q^{d \deg Q_v}(q^d - 1)$).

The second part of Lemma 3.8 follows immediately from Lemma 4.13 of [Ghi07b]. Indeed, if $v$ is a place of good reduction for $\phi$, then $|x|_v > 1$ (because we assumed $\widehat{h}_v(x) > 0$). But then, $\widehat{h}_v(x) = \log|x|_v$ (here we use the fact that $v$ is a place of good reduction, and thus $a_d$ is a $v$-adic unit). Hence $\widehat{h}_v(x) \in \mathbb{N}$, and we may take $D_v = 1$. □

The following result is an immediate corollary of Fact 3.8.

**Corollary 3.9.** *There exists a positive integer $D$ such that for every $v \in M_K \setminus \{\infty\}$, and for every $x \in K$, we have $D \cdot \widehat{h}_v(x) \in \mathbb{N}$.*

Next we prove a similar result as in Lemma 3.8 which is valid for the *only* infinite place of $K$.

**Lemma 3.10.** *There exists a positive integer $D_\infty$ such that for every $x \in K$, either $\widehat{h}_v(x) > 0$ for some $v \in M_K \setminus \{\infty\}$, or $D_\infty \cdot \widehat{h}_\infty(x) \in \mathbb{N}$.*

Before proceeding to its proof, we observe that we cannot remove the assumption that $\widehat{h}_v(x) = 0$ for every finite place $v$, in order to obtain the existence of $D_\infty$ in the statement of Lemma 3.10. Indeed, we know that in $K$ there are points of arbitrarily small (but positive) local height at $\infty$ (see Example 6.1 from [Ghi07b]). Therefore, there exists *no* positive integer $D_\infty$ which would clear all the possible denominators for the local heights at $\infty$ of those points. However, it turns out (as we will show in the proof of Lemma 3.10) that for such points $x$ of *very* small local height at $\infty$, there exists some other place $v$ for which $\widehat{h}_v(x) > 0$.

*Proof of Lemma 3.10.* Let $x \in K$. If $x \in \phi_{\mathrm{tor}}$, then we have nothing to prove (every positive integer $D_\infty$ would work because $\widehat{h}_\infty(x) = 0$). Thus, we assume $x$ is a nontorsion point. If $\widehat{h}_v(x) > 0$ for some place $v$ which does not lie over $v_\infty$, then again we are done. So, assume from now on that $\widehat{h}_v(x) = 0$ for every finite place $v$.

By proceeding as in the proof of Lemma 3.8, it suffices to show that there exists a polynomial $Q_\infty$ of degree bounded independently of $x$ such that $|\phi_{Q_\infty}(x)|_\infty > M_\infty$ (with the notation as in Fact 3.6). This is proved in Theorem 4.4 of [Ghi07a]. The first author showed in [Ghi07a] that there exists a positive integer $L$ (depending only on the number of places of bad reduction of $\phi$) such that for every nontorsion point $x$, there exists a place $v \in M_K$, and there exists a polynomial $Q$ of degree less than $L$ such that $|\phi_Q(x)|_v > M_v$. Because we assumed that $\widehat{h}_v(x) = 0$ for every $v \neq \infty$, then the above statement yields the existence of $D_\infty$. □

We will prove Theorem 2.4 by showing that a certain lim sup is positive. This will contradict the existence of infinitely many $S$-integral points in a finitely generated $\phi$-submodule. Our first step will be a result about the lim inf of the sequences which will appear in the proof of Theorem 2.4.

**Lemma 3.11.** *Suppose that* $\Gamma$ *is a torsion-free* $\phi$*-submodule of* $\mathbb{G}_a(K)$ *generated by elements* $\gamma_1, \dots, \gamma_r$*. For each* $i \in \{1, \dots, r\}$ *let* $(P_{n,i})_{n \in \mathbb{N}^*} \subset \mathbb{F}_q[t]$ *be a sequence of polynomials such that for each* $m \neq n$*, the* $r$*-tuples* $(P_{n,i})_{1 \le i \le r}$ *and* $(P_{m,i})_{1 \le i \le r}$ *are distinct. Then for every* $v \in M_K$*, we have*

$$\liminf_{n \to \infty} \frac{\log |\sum_{i=1}^r \phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \ge 0. \tag{3.11.1}$$

*Proof.* Suppose that for some $\epsilon > 0$, there exists a sequence $(n_k)_{k \ge 1} \subset \mathbb{N}^*$ such that $\sum_{i=1}^r \phi_{P_{n_k,i}}(\gamma_i) \neq \alpha$ and

$$\frac{\log |\sum_{i=1}^r \phi_{P_{n_k,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n_k,i}}} < -\epsilon, \tag{3.11.2}$$

for every $k \ge 1$. But taking the lower bound from Fact 3.1 or Statement 3.2 (depending on whether $v$ is the infinite place or not) and dividing through by $\sum_{i=1}^r q^{d \deg P_{n_k,i}}$, we see that this is impossible. □

The following proposition is the key technical result required to prove Theorem 2.4. This proposition plays the same role that Lemma 3.5 plays in the proof of Theorem 2.5, or that Corollary 3.13 plays in the proof of Theorem 1.1 from [GT06]. Note that is does not provide an exact formula for the canonical height of a point, however; it merely shows that a certain limit is positive. This will suffice for our purposes since we only need that a certain sum of limits be positive in order to prove Theorem 2.4.

**Proposition 3.12.** *Let* $\Gamma$ *be a torsion-free* $\phi$*-submodule of* $\mathbb{G}_a(K)$ *generated by elements* $\gamma_1, \dots, \gamma_r$*. For each* $i \in \{1, \dots, r\}$ *let* $(P_{n,i})_{n \in \mathbb{N}^*} \subset \mathbb{F}_q[t]$ *be a* *sequence of polynomials such that for each $m \neq n$, the $r$-tuples $(P_{n,i})_{1\le i\le r}$ and $(P_{m,i})_{1\le i\le r}$ are distinct. Then there exists a place $v \in M_K$ such that*

$$\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^{r} \phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^{r} q^{d\deg P_{n,i}}} > 0. \tag{3.12.1}$$

*Proof.* Using the triangle inequality for the $v$-adic norm, and the fact that

$$\lim_{n\to\infty} \sum_{i=1}^{r} q^{d\deg P_{n,i}} = +\infty,$$

we conclude that proving that (3.12.1) holds is equivalent to proving that for some place $v$, we have

$$\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^{r} \phi_{P_{n,i}}(\gamma_i)|_v}{\sum_{i=1}^{r} q^{d\deg P_{n,i}}} > 0. \tag{3.12.2}$$

We also observe that it suffices to prove Proposition 3.12 for a subsequence $(n_k)_{k\ge 1} \subset \mathbb{N}^*$.

We prove (3.12.2) by induction on $r$. If $r = 1$, then by Corollary 3.13 of [GT06] (see also our Lemma 3.5),

$$\limsup_{\deg P\to\infty} \frac{\log |\phi_P(\gamma_1)|_v}{q^{d\deg P}} = \widehat{h}_v(\gamma_1) \tag{3.12.3}$$

and because $\gamma_1 \notin \phi_{\mathrm{tor}}$, there exists a place $v$ such that $\widehat{h}_v(\gamma_1) > 0$, thus proving (3.12.2) for $r = 1$. Therefore, we assume (3.12.2) is established for all $\phi$-submodules $\Gamma$ of rank less than $r$ and we will prove it for $\phi$-submodules of rank $r$.

In the course of our argument for proving (3.12.2), we will replace several times a given sequence with a subsequence of itself (note that passing to a subsequence can only make the lim sup smaller). For the sake of not clustering the notation, we will drop the extra indices which would be introduced by dealing with the subsequence.

Let $S_0$ be the set of places $v \in M_K$ for which there exists some $\gamma \in \Gamma$ such that $\widehat{h}_v(\gamma) > 0$. The following easy fact will be used later in our argument.

**Fact 3.13.** *The set $S_0$ is finite.*

*Proof of Fact 3.13.* We claim that $S_0$ equals the *finite* set $S_0'$ of places $v \in M_K$ for which there exists $i \in \{1,\dots,r\}$ such that $\widehat{h}_v(\gamma_i) > 0$. Indeed, let $v \in M_K \setminus S_0'$. Then for each $i \in \{1,\dots,r\}$ we have $\widehat{h}_v(\gamma_i) = 0$. Moreover, for each $i \in \{1,\dots,r\}$ and for each $Q_i \in \mathbb{F}_q[t]$, we have

$$\widehat{h}_v(\phi_{Q_i}(\gamma_i)) = \deg(\phi_{Q_i}) \cdot \widehat{h}_v(\gamma_i) = 0. \tag{3.13.1}$$

Using (3.13.1) and the triangle inequality for the local canonical height, we obtain that

$$\widehat{h}_v\left(\sum_{i=1}^{r} \phi_{Q_i}(\gamma_i)\right) = 0.$$

This shows that indeed $S_0 = S_0'$, and concludes the proof of Fact 3.13. $\square$

If the sequence $(n_k)_{k\geq 1} \subset \mathbb{N}^*$ has the property that for some $j \in \{1, \dots, r\}$, we have

$$\lim_{k\to\infty} \frac{q^{d \deg P_{n_k,j}}}{\sum_{i=1}^r q^{d\deg P_{n_k,i}}} = 0, \tag{3.13.2}$$

then the inductive hypothesis will yield the desired conclusion. Indeed, by the induction hypothesis, and also using (3.13.2), there exists $v \in S_0$ such that

$$\limsup_{k\to\infty} \frac{\log |\sum_{i\neq j} \phi_{P_{n_k,i}}(\gamma_i)|_v}{\sum_{i=1}^r q^{d\deg P_{n_k,i}}} > 0. \tag{3.13.3}$$

If $\widehat{h}_v(\gamma_j) = 0$, then $\left|\phi_{P_{n_k,j}}(\gamma_j)\right|_v$ is bounded as $k \to \infty$. Thus, for large enough $k$,

$$\left|\sum_{i=1}^r \phi_{P_{n_k,i}}(\gamma_i)\right|_v = \left|\sum_{i\neq j} \phi_{P_{n_k,i}}(\gamma_i)\right|_v$$

and so, (3.13.3) shows that (3.12.2) holds.

Now, if $\widehat{h}_v(\gamma_j) > 0$, then we proved in Lemma 4.4 of [GT06] that

$$\log |\phi_P(\gamma_j)|_v - q^{d\deg P}\widehat{h}_v(\gamma_j) \tag{3.13.4}$$

is uniformly bounded as $\deg P \to \infty$ (note that this follows easily from simple arguments involving geometric series and coefficients of polynomials). Therefore, using (3.13.2), we obtain

$$\lim_{k\to\infty} \frac{\log \left|\phi_{P_{n_k,j}}(\gamma_j)\right|_v}{\sum_{i=1}^r q^{d\deg P_{n_k,i}}} = 0. \tag{3.13.5}$$

Using (3.13.3) and (3.13.5), we conclude that for large enough $k$,

$$\left|\sum_{i=1}^r \phi_{P_{n_k,i}}(\gamma_i)\right|_v = \left|\sum_{i\neq j} \phi_{P_{n_k,i}}(\gamma_i)\right|_v$$

and so,

$$\limsup_{k\to\infty} \frac{\log \left|\sum_{i=1}^r \phi_{P_{n_k,i}}(\gamma_i)\right|_v}{\sum_{i=1}^r q^{d\deg P_{n_k,i}}} > 0, \tag{3.13.6}$$

as desired. Therefore, we may assume from now on that there exists $B \geq 1$ such that for every $n$,

$$\frac{\max_{1\leq i\leq r} q^{d\deg P_{n,i}}}{\min_{1\leq i\leq r} q^{d\deg P_{n,i}}} \leq B \text{ or equivalently,} \tag{3.13.7}$$

$$\max_{1\leq i\leq r} \deg P_{n,i} - \min_{1\leq i\leq r} \deg P_{n,i} \leq \frac{\log_q B}{d}. \tag{3.13.8}$$

We will prove that (3.12.2) holds for some place $v$ by doing an analysis at each place $v \in S_0$. We know that $|S_0| \geq 1$ because all $\gamma_i$ are nontorsion.

Our strategy is to show that in case (3.12.2) does not hold, then we can find $\delta_1, \dots, \delta_r \in \Gamma$, and we can find a sequence $(n_k)_{k\geq 1} \subset \mathbb{N}^*$, and a sequence of polynomials $(R_{k,i})_{\substack{k\in\mathbb{N}^* \\ 1\leq i\leq r}}$ such that

$$\sum_{i=1}^{r} \phi_{P_{n_k,i}}(\gamma_i) = \sum_{i=1}^{r} \phi_{R_{k,i}}(\delta_i) \text{ and} \tag{3.13.9}$$

$$\sum_{i=1}^{r} \widehat{h}_v(\delta_i) < \sum_{i=1}^{r} \widehat{h}_v(\gamma_i) \text{ and} \tag{3.13.10}$$

$$0 < \liminf_{k\to\infty} \frac{\sum_{i=1}^{r} q^{d\deg P_{n_k,i}}}{\sum_{i=1}^{r} q^{d\deg R_{k,i}}} \leq \limsup_{k\to\infty} \frac{\sum_{i=1}^{r} q^{d\deg P_{n_k,i}}}{\sum_{i=1}^{r} q^{d\deg R_{k,i}}} < +\infty. \tag{3.13.11}$$

Equation (3.13.9) will enable us to replace the $\gamma_i$ by the $\delta_i$ and proceed with our analysis of the latter. Inequality (3.13.10) combined with Corollary 3.9 and Lemma 3.10 will show that for each such $v$, in a finite number of steps we either construct a sequence $\delta_i$ as above for which all $\widehat{h}_v(\delta_i) = 0$, *or* (3.12.2) holds for $\delta_1, \dots, \delta_r$ and the corresponding polynomials $R_{k,i}$, i.e.

$$\limsup_{k\to\infty} \frac{\log|\sum_{i=1}^{r} \phi_{R_{k,i}}(\delta_i)|_v}{\sum_{i=1}^{r} q^{d\deg R_{k,i}}} > 0. \tag{3.13.12}$$

Equation (3.13.11) shows that (3.12.2) is equivalent to (3.13.12) (see also (3.13.9)).

We start with $v \in S_0 \setminus \{\infty\}$. As proved in Lemma 4.4 of [GT06], for each $i \in \{1, \dots, r\}$ such that $\widehat{h}_v(\gamma_i) > 0$, there exists a positive integer $d_i$ such that for every polynomial $Q_i$ of degree at least $d_i$, we have

$$\log|\phi_{Q_i}(\gamma_i)|_v = q^{d\deg Q_i}\widehat{h}_v(\gamma_i) - \frac{\log|a_d|_v}{q^d - 1}. \tag{3.13.13}$$

We know that for each $i$, we have $\lim_{n\to\infty} \deg P_{n,i} = +\infty$ because of (3.13.8). Hence, for each $n$ sufficiently large, and for each $i \in \{1, \dots, r\}$ such that $\widehat{h}_v(\gamma_i) > 0$, we have

$$\log|\phi_{P_{n,i}}(\gamma_i)|_v = q^{d\deg P_{n,i}}\widehat{h}_v(\gamma_i) - \frac{\log|a_d|_v}{q^d - 1}. \tag{3.13.14}$$

We now split the problem into two cases.

***Case 1.*** There exists an infinite subsequence $(n_k)_{k\geq 1}$ such that for every $k$, we have

$$\left|\sum_{i=1}^{r} \phi_{P_{n_k,i}}(\gamma_i)\right|_v = \max_{1\leq i\leq r} \left|\phi_{P_{n_k,i}}(\gamma_i)\right|_v. \tag{3.13.15}$$

For the sake of not clustering the notation, we drop the index $k$ from (3.13.15) (note that we need to prove (3.12.2) only for a *subsequence*). At the expense of replacing again $\mathbb{N}^*$ by a subsequence, we may also assume that for some *fixed* $j \in \{1, \dots, r\}$, we have

$$\left|\sum_{i=1}^{r} \phi_{P_{n,i}}(\gamma_i)\right|_v = \max_{i=1}^{r} \left|\phi_{P_{n,i}}(\gamma_i)\right| = \left|\phi_{P_{n,j}}(\gamma_j)\right|_v, \tag{3.13.16}$$

for all $n \in \mathbb{N}^*$. Because we know that there exists $i \in \{1, \dots, r\}$ such that $\widehat{h}_v(\gamma_i) > 0$, then for such $i$, we know $|\phi_{P_{n,i}}(\gamma_i)|_v$ is unbounded (as $n \to \infty$). Therefore, using (3.13.16), we conclude that also $|\phi_{P_{n,j}}(\gamma_j)|_v$ is unbounded (as $n \to \infty$). In particular, this means that $\widehat{h}_v(\gamma_j) > 0$.

Then using (3.13.14) for $\gamma_j$, we obtain that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^r \phi_{P_{n,i}}(\gamma_i)|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&= \limsup_{n\to\infty} \frac{\log |\phi_{P_{n,j}}(\gamma_j)|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&= \limsup_{n\to\infty} \frac{q^{d \deg P_{n,j}} \widehat{h}_v(\gamma_j) - \frac{\log |a_d|_v}{q^d - 1}}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&= \lim_{n\to\infty} \frac{q^{d \deg P_{n,j}} \widehat{h}_v(\gamma_j) - \frac{\log |a_d|_v}{q^d - 1}}{q^{d \deg P_{n,j}}} \cdot \limsup_{n\to\infty} \frac{q^{d \deg P_{n,j}}}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&> 0,
\end{aligned} \tag{3.13.17}$$

since

$$\lim_{n\to\infty} \frac{q^{d \deg P_{n,j}} \widehat{h}_v(\gamma_j) - \frac{\log |a_d|_v}{q^d - 1}}{q^{d \deg P_{n,j}}} = \widehat{h}_v(\gamma_j) > 0 \text{ and}$$

$$\limsup_{n\to\infty} \frac{q^{d \deg P_{n,j}}}{\sum_{i=1}^r q^{d \deg P_{n,i}}} > 0 \quad \text{because of (3.13.8).}$$

***Case 2.*** For all but finitely many $n$, we have

$$\left|\sum_{i=1}^{r} \phi_{P_{n,i}}(\gamma_i)\right|_v < \max_{1 \le i \le r} \left|\phi_{P_{n,i}}(\gamma_i)\right|_v. \tag{3.13.18}$$

Using the pigeonhole principle, there exists an infinite sequence $(n_k)_{k\ge 1} \subset \mathbb{N}^*$, and there exist $j_1, \dots, j_s \in \{1, \dots, r\}$ (where $s \ge 2$) such that for each $k$, we have

$$|\phi_{P_{n_k,j_1}}(\gamma_{j_1})|_v = \cdots = |\phi_{P_{n_k,j_s}}(\gamma_{j_s})|_v > \max_{i \in \{1,\dots,r\} \setminus \{j_1,\dots,j_s\}} |\phi_{P_{n_k,i}}(\gamma_i)|_v. \tag{3.13.19}$$

Again, as we did before, we drop the index $k$ from the above subsequence of $\mathbb{N}^*$. Using (3.13.19) and the fact that there exists $i \in \{1, \dots, r\}$ such that $\widehat{h}_v(\gamma_i) > 0$, we conclude that for all $1 \le i \le s$, we have $\widehat{h}_v(\gamma_{j_i}) > 0$. Hence, using (3.13.14) in (3.13.19), we obtain that for sufficiently large $n$, we have

$$q^{d \deg P_{n,j_1}} \widehat{h}_v(\gamma_{j_1}) = \cdots = q^{d \deg P_{n,j_s}} \widehat{h}_v(\gamma_{j_s}). \tag{3.13.20}$$

Without loss of generality, we may assume $\widehat{h}_v(\gamma_{j_1}) \geq \widehat{h}_v(\gamma_{j_i})$ for all $i \in \{2, \dots, s\}$. Then (3.13.20) yields that $\deg P_{n,j_i} \geq \deg P_{n,j_1}$ for $i > 1$. We divide (with quotient and remainder) each $P_{n,j_i}$ (for $i > 1$) by $P_{n,j_1}$ and for each such $j_i$, we obtain

$$P_{n,j_i} = P_{n,j_1} \cdot C_{n,j_i} + R_{n,j_i}, \tag{3.13.21}$$

where $\deg R_{n,j_i} < \deg P_{n,j_1} \leq \deg P_{n,j_i}$. Using (3.13.8), we conclude that $\deg C_{n,j_i}$ is uniformly bounded as $n \to \infty$. This means that, at the expense of passing to another subsequence of $\mathbb{N}^*$, we may assume that there exist polynomials $C_{j_i}$ such that

$$C_{n,j_i} = C_{j_i} \text{ for all } n.$$

We let $R_{n,i} := P_{n,i}$ for each $n$ and for each $i \in \{1, \dots, r\} \setminus \{j_2, \dots, j_s\}$.

Let $\delta_i$ for $i \in \{1, \dots, r\}$ be defined as follows:

$$\delta_i := \gamma_i \text{ if } i \neq j_1; \text{ and}$$

$$\delta_{j_1} := \gamma_{j_1} + \sum_{i=2}^{s} \phi_{C_{j_i}}(\gamma_{j_i}).$$

Then for each $n$, using (3.13.21) and the definition of the $\delta_i$ and $R_{n,i}$, we obtain

$$\sum_{i=1}^{r} \phi_{P_{n,i}}(\gamma_i) = \sum_{i=1}^{r} \phi_{R_{n,i}}(\delta_i). \tag{3.13.22}$$

Using (3.13.8) and the definition of the $R_{n,i}$ (in particular, the fact that $R_{n,j_1} = P_{n,j_1}$ and $\deg R_{n,j_i} < \deg P_{n,j_1}$ for $2 \leq i \leq s$), it is immediate to see that

$$0 < \liminf_{n\to\infty} \frac{\sum_{i=1}^{r} q^{d \deg P_{n,i}}}{\sum_{i=1}^{r} q^{d \deg R_{n,i}}} \leq \limsup_{n\to\infty} \frac{\sum_{i=1}^{r} q^{d \deg P_{n,i}}}{\sum_{i=1}^{r} q^{d \deg R_{n,i}}} < +\infty. \tag{3.13.23}$$

Moreover, because of (3.13.22) and (3.13.23), we get that

$$\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^{r} \phi_{P_{n,i}}(\gamma_i)|_v}{\sum_{i=1}^{r} q^{d \deg P_{n,i}}} > 0 \tag{3.13.24}$$

if and only if

$$\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^{r} \phi_{R_{n,i}}(\delta_i)|_v}{\sum_{i=1}^{r} q^{d \deg R_{n,i}}} > 0. \tag{3.13.25}$$

We claim that if $\widehat{h}_v(\delta_{j_1}) \geq \widehat{h}_v(\gamma_{j_1})$, then (3.13.25) holds (and so, also (3.13.24) holds). Indeed, in that case, for large enough $n$, we have

$$\begin{aligned} \log|\phi_{R_{n,j_1}}(\delta_{j_1})|_v &= q^{d\deg R_{n,j_1}}\widehat{h}_v(\delta_{j_1}) - \frac{\log|a_d|_v}{q^d-1} \\ &\geq q^{d\deg P_{n,j_1}}\widehat{h}_v(\gamma_{j_1}) - \frac{\log|a_d|_v}{q^d-1} \\ &= \log|\phi_{P_{n,j_1}}(\gamma_{j_1})|_v \\ &> \max_{i=2}^{s}\log|\phi_{R_{n,j_i}}(\gamma_{j_i})|_v, \end{aligned} \tag{3.13.26}$$

where in the last inequality from (3.13.26) we used (3.13.20) and (3.13.14), and that for each $i \in \{2,\dots,s\}$ we have $\deg R_{n,j_i} < \deg P_{n,j_i}$. Moreover, using (3.13.26) and (3.13.19), together with the definition of the $R_{n,i}$ and the $\delta_i$, we conclude that for large enough $n$, we have

$$\begin{aligned} \log\left|\sum_{i=1}^{r}\phi_{R_{n,i}}(\delta_i)\right|_v &= \log\left|\phi_{R_{n,j_1}}(\delta_{j_1})\right|_v \\ &= q^{d\deg P_{n,j_1}}\widehat{h}_v(\gamma_{j_1}) - \frac{\log|a_d|_v}{q^d-1}. \end{aligned} \tag{3.13.27}$$

Because $R_{n,j_1} = P_{n,j_1}$, equations (3.13.8) and (3.13.23) show that

$$\limsup_{n\to\infty}\frac{q^{d\deg R_{n,j_1}}}{\sum_{i=1}^{r}q^{d\deg R_{n,i}}} > 0. \tag{3.13.28}$$

Equations (3.13.27) and (3.13.28) show that we are now in ***Case 1*** for the sequence $(R_{n,i})_{\substack{n\in\mathbb{N}^*\\1\leq i\leq r}}$. Hence

$$\limsup_{n\to\infty}\frac{\log|\sum_{i=1}^{r}\phi_{R_{n,i}}(\delta_i)|_v}{\sum_{i=1}^{r}q^{d\deg R_{n,i}}} > 0, \tag{3.13.29}$$

as desired.

Assume from now on that $\widehat{h}_v(\delta_{j_1}) < \widehat{h}_v(\gamma_{j_1})$. Because $v \in M_K \setminus \{\infty\}$, using Corollary 3.9 and also using that if $i \neq j_1$, then $\delta_i = \gamma_i$, we conclude

$$\sum_{i=1}^{r}\widehat{h}_v(\gamma_i) - \sum_{i=1}^{r}\widehat{h}_v(\delta_i) \geq \frac{1}{D}.$$

Our goal is to prove (3.13.24) by proving (3.13.25). Because we replace some of the polynomials $P_{n,i}$ with other polynomials $R_{n,i}$, it may very well be that (3.13.8) is no longer satisfied for the polynomials $R_{n,i}$. Note that in this case, using induction and arguing as in equations (3.13.2) through (3.13.6), we see that

$$\limsup_{n\to\infty}\frac{\log|\sum_{j=1}^{r}\phi_{R_{n,j}}(\delta_j)|_w}{\sum_{j=1}^{r}q^{d\deg R_{n,j}}} > 0,$$

for some place $w$. This would yield that (see (3.13.22) and (3.13.23))

$$\limsup_{n\to\infty} \frac{\log|\sum_{j=1}^{r}\phi_{P_{n,j}}(\gamma_j)|_w}{\sum_{j=1}^{r} q^{d\deg P_{n,j}}} > 0,$$

as desired. Hence, we may assume again that (3.13.8) holds.

We continue the above analysis this time with the $\gamma_i$ replaced by $\delta_i$. Either we prove (3.13.25) (and so, implicitly, (3.13.24)), or we replace the $\delta_i$ by other elements in $\Gamma$, say $\beta_i$ and we decrease even further the sum of their local heights at $v$:

$$\sum_{i=1}^{r}\widehat{h}_v(\delta_i) - \sum_{i=1}^{r}\widehat{h}_v(\beta_i) \geq \frac{1}{D}.$$

The above process cannot go on infinitely often because the sum of the local heights $\sum_{i=1}^{r}\widehat{h}_v(\gamma_i)$ is decreased each time by at least $\frac{1}{D}$. Our process ends when we cannot replace anymore the eventual $\zeta_i$ by new $\beta_i$. Thus, at the final step, we have $\zeta_1,\dots,\zeta_r$ for which we cannot further decrease their sum of local canonical heights at $v$. This happens either because all $\zeta_i$ have local canonical height equal to 0, or because we already found a sequence of polynomials $T_{n,i}$ for which

$$\limsup_{n\to\infty} \frac{\log|\sum_{i=1}^{r}\phi_{T_{n,i}}(\zeta_i)|_v}{\sum_{i=1}^{r} q^{d\deg T_{n,i}}} > 0. \tag{3.13.30}$$

Since

$$\sum_{i=1}^{r}\phi_{P_{n,i}}(\gamma_i) = \sum_{i=1}^{r}\phi_{T_{n,i}}(\zeta_i), \tag{3.13.31}$$

this would imply that (3.12.2) holds, which would complete the proof. Hence, we may assume that we have found a sequence $(\zeta_i)_{1\leq i\leq r}$ with canonical local heights equal to 0. As before, we let the $(T_{n,i})_{\substack{n\in\mathbb{N}^*\\ 1\leq i\leq r}}$ be the corresponding sequence of polynomials for the $\zeta_i$, which replace the polynomials $P_{n,i}$.

Next we apply the above process to another $w\in S_0\setminus\{\infty\}$ for which there exists at least one $\zeta_i$ such that $\widehat{h}_w(\zeta_i) > 0$. Note that when we apply the above process to the $\zeta_1,\dots,\zeta_r$ at the place $w$, we might replace (at certain steps of our process) the $\zeta_i$ by

$$\sum_{j}\phi_{C_j}(\zeta_j)\in\Gamma. \tag{3.13.32}$$

Because for the places $v\in S_0$ for which we already completed the above process, $\widehat{h}_v(\zeta_i) = 0$ for all $i$, then by the triangle inequality for the local height, we also have

$$\widehat{h}_v\left(\sum_{j}\phi_{C_j}(\zeta_j)\right) = 0.$$

If we went through all $v \in S_0 \setminus \{\infty\}$, and if the above process did not yield that (3.13.24) holds for some $v \in S \setminus \{\infty\}$, then we are left with $\zeta_1, \dots, \zeta_r \in \Gamma$ such that for all $i$ and all $v \neq \infty$, we have $\widehat{h}_v(\zeta_i) = 0$. Note that since $\widehat{h}_v(\zeta_i) = 0$ for each $v \neq \infty$ and each $i \in \{1, \dots, r\}$, then by the triangle inequality for local heights, for all polynomials $Q_1, \dots, Q_r$, we have

$$\widehat{h}_v\left(\sum_{i=1}^{r} \phi_{Q_i}(\zeta_i)\right) = 0 \text{ for } v \neq \infty. \tag{3.13.33}$$

Lemma 3.10 and (3.13.33) show that for all polynomials $Q_i$,

$$D_\infty \cdot \widehat{h}_\infty\left(\sum_{i=1}^{r} \phi_{Q_i}(\zeta_i)\right) \in \mathbb{N}. \tag{3.13.34}$$

We repeat the above process, this time for $v = \infty$. As before, we conclude that either

$$\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^{r} \phi_{T_{n,i}}(\zeta_i)|_\infty}{\sum_{i=1}^{r} q^{d \deg T_{n,i}}} > 0 \tag{3.13.35}$$

or we are able to replace the $\zeta_i$ by some other elements $\beta_i$ (which are of the form (3.13.32)) such that

$$\sum_{i=1}^{r} \widehat{h}_\infty(\beta_i) < \sum_{i=1}^{r} \widehat{h}_\infty(\zeta_i).$$

Using (3.13.34), we conclude that

$$\sum_{i=1}^{r} \widehat{h}_\infty(\zeta_i) - \sum_{i=1}^{r} \widehat{h}_\infty(\beta_i) \geq \frac{1}{D_\infty}. \tag{3.13.36}$$

Therefore, after a finite number of steps this process of replacing the $\zeta_i$ must end, and it cannot end with all the new $\beta_i$ having local canonical height 0, because this would mean that all $\beta_i$ are torsion (we already knew that for $v \neq \infty$, we have $\widehat{h}_v(\zeta_i) = 0$, and so, by (3.13.33), $\widehat{h}_v(\beta_i) = 0$). Because the $\beta_i$ are nontrivial "linear" combinations (in the $\phi$-module $\Gamma$) of the $\gamma_i$ which span a torsion-free $\phi$-module, we conclude that indeed, the $\beta_i$ cannot be torsion points. Hence, our process ends with proving (3.13.35) which proves (3.13.24), and so, it concludes the proof of our Proposition 3.12. $\square$

*Remark* 3.14. If there is more than one infinite place in $K$, then we cannot derive Lemma 3.10, and in particular, we cannot derive (3.13.36). The idea is that in this case, for each nontorsion $\zeta$ which has its local canonical height equal to 0 at finite places, we only know that there exists *some* infinite place where its local canonical height has *bounded* denominator. However, we do not know if that place is the one which we analyze at that particular moment in our process from the proof of Proposition 3.12. Hence, we would not necessarily be able to derive (3.13.36).

Now we are ready to prove Theorem 2.4.

*Proof of Theorem 2.4.* Let $(\gamma_i)_i$ be a finite set of generators of $\Gamma$ as a module over $A = \mathbb{F}_q[t]$. At the expense of replacing $S$ with a larger finite set of places of $K$, we may assume $S$ contains all the places $v \in M_K$ which satisfy at least one of the following properties:

1. $\widehat{h}_v(\gamma_i) > 0$ for some $1 \le i \le r$.
2. $|\gamma_i|_v > 1$ for some $1 \le i \le r$.
3. $|\alpha|_v > 1$.
4. $\phi$ has bad reduction at $v$.

Expanding the set $S$ leads only to (possible) extension of the set of $S$-integral points in $\Gamma$ with respect to $\alpha$. Clearly, for every $\gamma \in \Gamma$, and for every $v \notin S$ we have $|\gamma|_v \le 1$. Therefore, using 3., we obtain

(3.14.1)
$$\gamma \in \Gamma \text{ is } S\text{-integral with respect to } \alpha \Longleftrightarrow |\gamma - \alpha|_v = 1 \text{ for all } v \in M_K \setminus S.$$

Moreover, using 1. from above, we conclude that for every $\gamma \in \Gamma$, and for every $v \notin S$, we have $\widehat{h}_v(\gamma) = 0$ (see the proof of Fact 3.13).

Next we observe that it suffices to prove Theorem 2.4 under the assumption that $\Gamma$ is a free $\phi$-submodule. Indeed, because $A = \mathbb{F}_q[t]$ is a principal ideal domain, $\Gamma$ is a direct sum of its finite torsion submodule $\Gamma_{\mathrm{tor}}$ and a free $\phi$-submodule $\Gamma_1$ of rank $r$, say. Therefore,

$$\Gamma = \bigcup_{\gamma \in \Gamma_{\mathrm{tor}}} \gamma + \Gamma_1.$$

If we show that for every $\gamma_0 \in \Gamma_{\mathrm{tor}}$ there are finitely many $\gamma_1 \in \Gamma_1$ such that $\gamma_1$ is $S$-integral with respect to $\alpha - \gamma_0$, then we obtain the conclusion of Theorem 2.4 for $\Gamma$ and $\alpha$ (see (3.14.1)).

Thus from now on, we assume $\Gamma$ is a free $\phi$-submodule of rank $r$. Let $\gamma_1, \dots, \gamma_r$ be a basis for $\Gamma$ as an $\mathbb{F}_q[t]$-module. We reason by contradiction. Let

$$\sum_{i=1}^{r} \phi_{P_{n,i}}(\gamma_i) \in \Gamma$$

be an infinite sequence of elements $S$-integral with respect to $\alpha$. Because of the $S$-integrality assumption (along with the assumptions on $S$), we conclude that for every $v \notin S$, and for every $n$ we have

$$\frac{\log |\sum_{i=1}^r \phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} = 0.$$

Thus, using the product formula, we see that

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{v \in S} \frac{\log |\sum_{i=1}^r \phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&= \limsup_{n\to\infty} \sum_{v \in M_K} \frac{\log |\sum_{i=1}^r \phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&= 0.
\end{aligned}$$

On the other hand, by Proposition 3.12, there is some place $w \in S$ and some number $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^r \phi_{P_{n,i}}(\gamma_i) - \alpha|_w}{\sum_{i=1}^r q^{d \deg P_{n,i}}} = \delta > 0.$$

So, using Lemma 3.11, we see that

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{v\in S} \frac{\log |\sum_{i=1}^r \phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&\geq \sum_{\substack{v\in S\\ v\neq w}} \liminf_{n\to\infty} \frac{\log |\sum_{i=1}^r \phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} + \limsup_{n\to\infty} \frac{\log |\sum_{i=1}^r \phi_{P_{n,i}}(\gamma_i) - \alpha|_w}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&\geq 0 + \delta \\
&> 0.
\end{aligned}$$

Thus, we have a contradiction which shows that there cannot be infinitely many elements of $\Gamma$ which are $S$-integral for $\alpha$. $\square$

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Dragos Ghioca, Department of Mathematics, McMaster University, 1280 Main Street West, Hamilton, Ontario, Canada L8S 4K1,
*E-mail address*: dghioca@math.mcmaster.ca

Thomas Tucker, Department of Mathematics, Hylan Building, University of Rochester, Rochester, NY 14627,
*E-mail address*: ttucker@math.rochester.edu