STATISTICAL MECHANICS AND THE RENORMALISATION GROUP

LECTURE NOTES FOR THE 2009 SUMMER SCHOOL IN PROBABILITY

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Part 1. Equilibrium Statistical Mechanics

Lecture 1. The Ideal Gas

During the 19th century chemists came to believe in the reality of indivisible units of matter, but this atomic hypothesis was far from universally accepted outside their science. A major question was whether thermodynamic concepts such as heat, temperature and entropy could be deduced from a "kinetic theory" of matter as an assembly of particles moving according to Newtonian mechanics. Ergodic theory began in the 1870's with Boltzmann's efforts to deduce from kinetic theory the probability law on phase space which correctly predicts the time averages of observables. In 1866 Maxwell independently postulated a Gaussian distribution for particle velocities also based on kinetic reasoning. In 1878 J.W. Gibbs considered a more general problem, namely to find the distribution of the states of the system in a subset of a much larger domain. His proposed solution to this problem is called the *grand canonical ensemble*. Due to fluctuation caused by the particles coming in and going out of the subset, neither the number of particles nor the total energy in the subset is conserved and in fact it is technically easier to work with probability laws that allow fluctuation in the energy and number of particles. The grand canonical ensemble is the starting point for this course.

[Pai82] reviews the 19th century controversies over the existence of atoms and discusses the origins of statistical physics.

Notation. For a set X,

$$X^n = \{(x_1, \dots, x_n) : x_i \in X\} \tag{1.1}$$

is the set of sequences in X with length n. By convention, X^0 is a set with only one element, written ().

$$X^* = \bigcup_{n \geq 0} X^n, \tag{1.2}$$

is the set of all sequences of arbitrary *finite* length. If $x \in X^*$, we write

$$x = (x_1, \dots, x_{N(x)}), \tag{1.3}$$

and we write $N = N(x)$.

If $X \subset \mathbb{R}^d$ we tacitly assume X is Lebesgue measurable and write $|X|$ for the Lebesgue measure of X. If X is a finite set, $|X|$ denotes the number of elements in X instead. Functions on $\mathbb{R}^d$ are always tacitly assumed Lebesgue measurable.

The indicator function is

$$\mathbb{1}_{x \in X} = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{else.} \end{cases} \tag{1.4}$$

Let $V : (\mathbb{R}^d)^* \to \mathbb{R} \cup \{+\infty\}$ be a function such that $V(x) = 0$ if $x \in X^0$ and

$$V(x) \geq -cN(x). \tag{1.5}$$

The last condition is called *stability.*

Example 1.1. $V(x) = -\sum_{i=1}^{N(x)} \phi(x_i)$ where ϕ is bounded $\mathbb{R}^d \to \mathbb{R}$; ϕ is called an *external field.*

Example 1.2. For some $v : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $V(x) = \sum_{1 \leq i < j \leq N(x)} v(x_i, x_j)$; v is called a *two body potential.* Conditions on v such that V is stable are discussed in the problems.

1.1. The Grand Canonical Ensemble.

Definition 1.3. Let $\Lambda \subset \mathbb{R}^d$, $|\Lambda| < \infty$. Let $z \geq 0$. For $E \subset \Lambda^*$

$$\mathbb{P}(E) = \frac{1}{Z} \sum_{n\geq 0} \frac{z^n}{n!} \int_{E\cap\Lambda^n} e^{-V} \, dx_1 \cdots dx_n \tag{1.6}$$

is called the *Grand Canonical Ensemble Gibbs measure*, or simply the *Gibbs measure*, where

$$Z = \sum_{n\geq 0} \frac{z^n}{n!} \int_{\Lambda^n} e^{-V} \, dx_1 \cdots dx_n \tag{1.7}$$

is the normalization factor such that $\mathbb{P}(\Lambda^*) = 1$.

We use the stability condition of the potential to check that $Z \neq \infty$ so that $\mathbb{P}(E)$ is well defined. In fact,

$$Z \leq \sum \frac{z^n}{n!} |\Lambda|^n e^{cn} = e^{z|\Lambda|e^c} < \infty.$$

Example 1.4. If $V(x) = -\sum_{i=1}^{N(x)} \phi(x_i)$, then

$$Z = Z(\phi) = \exp(z \int_\Lambda e^\phi \, dx). \tag{1.8}$$

Notation. For $F : \Lambda^* \to \mathbb{R}$, denote

$$\langle F \rangle = \mathbb{E}F = \int F \, d\mathbb{P} = \frac{1}{Z} \sum_{n\geq 0} \frac{z^n}{n!} \int_{\Lambda^n} e^{-V} F \, dx_1 \cdots dx_n. \tag{1.9}$$

Let $X \subset \Lambda$. A typical F is

$$N(X, x) = \#\text{ of particles of } x \text{ in } X = \sum_{i=1}^{N(x)} \mathbb{1}_{x_i \in X}. \tag{1.10}$$

We are working with the probability space $(\Lambda^*, \mathbb{P})$, where the σ-algebra $\mathcal{F}_\Lambda$ is generated by $(N_X, X \subset \Lambda)$, where

$$N_X = \{N(Y) : Y \subset X\}. \tag{1.11}$$

The Gibbs measure $\mathbb{P}_{V=0}$ defined setting $V = 0$ in Definition 1.3 is known as the *Ideal Gas*. We shall refer to the case $V = -\sum \phi(x_i)$ as the *Ideal Gas in External Field.*

The Gibbs measure $\mathbb{P}$ of Definition 1.3 with $V \neq 0$ can be written in terms of $\mathbb{P}_{V=0}$ as follows. For $E \subset \Lambda^*$,

$$\mathbb{P}(E) = \frac{1}{Z} \int_E e^{-V} \, d\mathbb{P}_{V=0}, \quad Z = \int e^{-V} \, d\mathbb{P}_{V=0}.$$

Lemma 1.5 (Ideal gas). *Let $V = 0$. Let $X_1, \ldots, X_n \subset \Lambda$, where $|\Lambda| < \infty$. Then*
a) $N(X_i) \sim$ Poisson$(z|X_i|)$;
b) if $|X_i \cap X_j| = 0$ for $i \neq j$, then $N(X_1), \ldots, N(X_n)$ are independent.

Proof. a) A Poisson(r) random variable Y has

$$\mathbb{E}e^{tY} = \sum_{n\geq 0} \frac{r^n}{n!} e^{-r} e^{tn} = e^{-r} e^{re^t} = e^{r(e^t-1)}. \tag{1.12}$$

Since the generating function characterises the distribution it is sufficient to prove that N_{X_i} also has this generating function, with $r = (z|X_i|)$. Denote $X = X_i$. By Example 1.4, with $\phi(x) = t\mathbb{1}_{x \in X}$,

$$\int e^{tN(x)}\, d\mathbb{P}_{V=0} = \frac{1}{Z}\sum_{n\geq 0}\frac{z^n}{n!}\int_{\Lambda^n} e^{tN(x)}\, dx_1\cdots dx_n = \frac{Z(\phi)}{Z(0)} = \exp(z\int_{\Lambda}(e^{\phi}-1)\, dx) \tag{1.13}$$
$$= \exp(z(|X|e^t + |\Lambda - X| - |X| - |\Lambda - X|)) = \exp(z|X|(e^t-1)).$$

This proves a). The statement b) comes from a similar calculation and is left to the reader (Problem 1.2). ■

Lemma 1.6. *If instead of zero potential $V = 0$ we consider $V = -\sum\phi(x_i)$, then a) $N_{X_i} \sim$* Poisson$(z\int_{X_i} e^{\phi}\, dx)$*, and b) also holds.*

Problems.

Problem 1.1. An $n\times n$ matrix A is said to be positive-definite if for all non-zero $\lambda = (\lambda_1,\ldots,\lambda_n)$ in $\mathbb{C}^n$,

$$\sum_{1\leq i,j\leq n}\lambda_i A_{ij}\bar{\lambda}_j > 0.$$

If the inequality is not strict then the matrix is said to be positive-semidefinite. A continuous function $f : \mathbb{R}\to\mathbb{R}$ is said to be positive-definite if for every $x\in\mathbb{R}^n$, the matrix $(f(x_i - x_j))_{1\leq i,j\leq n}$ is positive-semidefinite. Prove that:

(1) If f is positive-definite, then V given by a the two-body potential $v(x,y) = f(x-y)$ as in Example 1.2 satisfies the stability bound (1.5).
(2) If f is continuous and integrable so that the Fourier transform $\hat{f}(k) = \int f(x)e^{-ikx}\, dx$ exists, if $\hat{f}\geq 0$, then f is positive-definite. (This is the "easy" half of Bochner's theorem.) Concentrate on the case where $\hat{f}$ is also integrable, and then see exercise 8.4.30 in [Fol99] to remove this assumption.

This extends to $\mathbb{R}^d$. *Conjecture:* Every two-body potential $v(x,y) = f(x-y)$ such that V satisfies (1.5) has the form $f =$ non-negative function + positive-definite function.

Problem 1.2. Complete the proof of Lemma 1.5.

Problem 1.3. Prove the *Ideal Gas Law* which says, for $V = 0$,

$$p|\Lambda| = T\langle N(\Lambda)\rangle, \tag{1.14}$$

where by definition, $p/T = \log(Z)/|\Lambda|$. T is called the *temperature*; p is called the *pressure*.

Problem 1.4. Prove that $\langle N(\Lambda)\rangle$ is monotone in z.

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Lecture 2. Mean Field Theory

The next simplest case after the ideal gas is called mean field theory. It is important because it exhibits the phenomena of a "phase transition". In fact, it is a reasonable model for the transition in which liquid water becomes steam. As you know from every day experience, there is a very well defined temperature (100°C), at which the density of water has a jump: liquid water is much denser than steam.

Mean field theory should be formulated for the continuum models of last lecture, but in order to avoid a problem with stability, we will consider lattice systems instead. The topics of this lecture are: (1) how lattice systems are a special case of the continuum systems of lecture 1, (2) the limit of $\Lambda \nearrow \mathbb{R}^d$, (3) mean field theory for lattice systems, and (4) phase transitions.

[Min00] is a relatively friendly introduction to Gibbs measures. [Rue04] is a beautiful but harder book on Gibbs measures. The idea of regarding lattice systems as a special case of the continuum is explored in more detail in [RT09].

2.1. Notation. Paving $\mathbb{R}^d$ by blocks: Let $L \in \mathbb{N}$. For $x \in \mathbb{Z}^d$,

$$B(x) = \{y \in \mathbb{R}^d : \|y - Lx\|_\infty < L/2\}, \tag{2.1}$$

where

$$\|y\|_\infty = \max_{i=1,\dots,d} |y_i| \quad \text{for } y \in \mathbb{R}^d. \tag{2.2}$$

$B(x)$ is called a *block*. The set of all blocks is

$$\mathcal{B} = \mathcal{B}_L = \{B(x) : x \in \mathbb{Z}^d\}. \tag{2.3}$$

Let $\mathcal{P} = \mathcal{P}(\mathbb{R}^d)$ where, letting $\overline{B}$ denote the closure of B,

$$\mathcal{P}(\Lambda) = \{\text{all finite unions of } \overline{B}, B \in \mathcal{B}(\Lambda)\}. \tag{2.4}$$

A set $X \in \mathcal{P}$ is called a *polymer*. For $X \in \mathcal{P}$,

$$|X|_1 = |\mathcal{B}(X)| \tag{2.5}$$

is the number of $L = L^1$ blocks in X.

Choose $L = 1$ for this and the next lecture.

2.2. The random variables. After paving by blocks we are only interested in

$$N(B) = \# \text{ of particles in } B, \tag{2.6}$$

$$N = \sum_{B \in \mathcal{B}(\Lambda)} N(B) = \# \text{ of particles in } \Lambda, \tag{2.7}$$

$$\underline{N}_X = (N(B) : B \in \mathcal{B}(X)), \quad X \in \mathcal{P}, \tag{2.8}$$

and

$$\mathcal{F}_X = \sigma(\underline{N}_X). \tag{2.9}$$

$F \in m\mathcal{F}_X$ means that F is measurable with respect to $\mathcal{F}_X$.

2.3. The infinite volume limit. This refers to studying the joint distributions of $\underline{N}_X$, $X \in \mathcal{P}$, in a limit

$$\Lambda_1 \subset \Lambda_2 \subset \cdots, \quad \Lambda_i \in \mathcal{P}, \quad \bigcup_{i \geq 1} \Lambda_i = \mathbb{R}^d.$$

An *infinite volume limit* is a probability space $(\Omega_\infty, \mathbb{P}_\infty)$ carrying random variables $(N(B), B \in \mathcal{B})$ such that for some sequence Λ_i, for every $X \in \mathcal{P}$,

$$\lim_{i \to \infty} \mathbb{P}_{\Lambda_i}\{\underline{N}_X = \underline{n}\} = \mathbb{P}_\infty\{\underline{N}_X = \underline{n}\}. \tag{2.10}$$

Later in these lectures, when we encounter random variables which are not discrete (do not take values in $\mathbb{N}_0$), we will use the notion of weak convergence, which is equivalent to demanding that expectations of all bounded continuous functions of $\underline{N}_X$ converge to infinite volume expectations.

2.4. Mean field theory. For $\beta > 0$, mean field theory is defined by

$$V = \begin{cases} \infty & \text{if } N(B) > 1 \text{ for some } B \in \mathcal{B}(\Lambda) \\ -\frac{\beta}{|\Lambda|_1} \frac{N^2}{2} & \text{else.} \end{cases} \tag{2.11}$$

Let

$$\Omega = \{0, 1\}^{\mathcal{B}(\Lambda)}. \tag{2.12}$$

For $\underline{n} \in \Omega$,

$$z^{\underline{n}} = \prod_{B \in \mathcal{B}(\Lambda)} z^{n(B)}, \tag{2.13}$$

$$H(\underline{n}) = -\frac{\beta}{2|\Lambda|_1} \left(\sum_{B \in \mathcal{B}(\Lambda)} n(B) \right)^2. \tag{2.14}$$

Then, under the grand canonical ensemble,

$$\mathbb{P}\{\underline{N}_\Lambda = \underline{n}\} = \begin{cases} \frac{1}{Z} z^{\underline{n}} e^{-H(\underline{n})} & \underline{n} \in \Omega, \\ 0 & n \notin \Omega, \end{cases} \tag{2.15}$$

and

$$Z = \sum_{\underline{n} \in \Omega} z^{\underline{n}} e^{-H(\underline{n})}. \tag{2.16}$$

Proof. Since $V \in m\mathcal{F}_\Lambda$

$$\int_{\underline{N}_\Lambda = \underline{n}} e^{-V} \, d\mathbb{P}_{V=0} = \begin{cases} e^{-H(\underline{n})} \mathbb{P}_{V=0}\{\underline{N}_\Lambda = \underline{n}\} & \underline{n} \in \Omega \\ 0 & \underline{n} \notin \Omega \end{cases}$$

and for $n \in \Omega$,

$$e^{-H(\underline{n})} \mathbb{P}_{V=0}\{\underline{N}_\Lambda = \underline{n}\} = e^{-H(\underline{n})} \prod_{B \in \mathcal{B}(\Lambda)} \left(\frac{z^{N(B)}}{N(B)!} e^{-z} \right) = e^{-H(\underline{n})} z^{\underline{n}} e^{-z|\Lambda|_1}.$$

Dividing by the normalization and using

$$\mathbb{P}(E) = \frac{1}{Z} \int_E e^{-V} \, d\mathbb{P}_{V=0},$$

the result follows. ∎

The argument never used the specific form of V beyond $V \in m\mathcal{F}_\Lambda$, so by the same argument, a lattice model arises whenever, for the continuum model $V \in m\mathcal{F}_\Lambda$, and this is equivalent to

$$v(x,y) = v([x],[y]) \quad \text{a.e. Lebesgue} \tag{2.17}$$

in Example 1.2. $[x]$ is the point in $\mathbb{Z}^d$ closest to $x \in \mathbb{R}^d$, in the sense that $x \in B$ if and only if $B = B([x])$. $[x]$ is well-defined a.e. in $x \in \mathbb{R}^d$.

Proposition 2.1. *In the infinite volume limit, for every $X \in \mathcal{P}$ the probability law for $\underline{N}_X$ is a convex combination of* Bernoulli$(1 : ze^{\phi})$ *where ϕ is a constant in the set M_0 of global minima to*

$$S(\phi) = \frac{1}{2\beta}\phi^2 - \log(1 + ze^{\phi}). \tag{2.18}$$

In more detail, if $(\beta, z) \notin \{ze^{\beta/2} = 1\}$ or if $\beta \leq 4$ there is a unique global minimum ϕ and

$$\mathbb{P}\{\underline{N}_X = \underline{n}\} = \prod_{B\in\mathcal{B}(\Lambda)} \left(\frac{(ze^{\phi})^{n(B)}}{1 + ze^{\phi}}\right), \quad \underline{n} \in \Omega(X). \tag{2.19}$$

Otherwise $|M_0| = 2$ and

$$\mathbb{P}\{\underline{N}_X = \underline{n}\} = \frac{1}{2}\sum_{\phi\in M_0}\prod_{B\in\mathcal{B}(\Lambda)} \left(\frac{(ze^{\phi})^{n(B)}}{1 + ze^{\phi}}\right). \tag{2.20}$$

Recall:

$$Y \sim \text{Bernoulli}(1:t) \quad \text{means} \quad Y = \begin{cases} 1 & \text{with probability } \frac{t}{1+t} \\ 0 & \text{with probability } \frac{1}{1+t} \end{cases}$$

Discussion. Let $p \in [0,1]$. There exists a probability space $(\Omega_\infty^{(p)}, \mathbb{P}_\infty^{(p)})$ on which are defined random variables

$$(N(B), B \in \mathcal{B}(\mathbb{R}^d)), \quad N(B) : \Omega_\infty^{(p)} \to \mathbb{N} \tag{2.21}$$

and under the law $\mathbb{P}_\infty^{(p)}$ all these random variables are independent Bernoulli(p). By taking two copies, each carrying $1/2$ probability, we define a new probability space

$$(\Omega_\infty, \mathbb{P}_\infty), \quad \Omega_\infty = \Omega^{(p_1)} \cup \Omega_\infty^{(p_2)} \tag{2.22}$$

$$\mathbb{P}_\infty|_{\Omega_\infty^{(p_i)}} = \frac{1}{2}\mathbb{P}_\infty^{(p_i)} \quad (i = 1,2) \tag{2.23}$$

with an additional random variable

$$\rho = \begin{cases} p_1 & \text{on } \Omega_\infty^{p_1} \\ p_2 & \text{on } \Omega_\infty^{p_2} \end{cases} \tag{2.24}$$

Choose $p_i = \frac{ze^{\phi_i}}{1+ze^{\phi_i}}$, $\phi \in M_0$, $i = 1,2$ as in (2.20). Then $(\Omega_\infty, \mathbb{P}_\infty)$ is the infinite volume limit of mean field theory in case (2.20): For $X \in \mathcal{P}$, Proposition 2.1 says

$$\lim_{i\to\infty} \mathbb{P}_{\Lambda_i}\{\underline{N}_X = \underline{n}\} = \mathbb{P}_\infty\{\underline{N}_X = \underline{n}\} \tag{2.25}$$

However, ρ is not as new as it looks because we can create it from the random variables $(N(B), B \in \mathcal{B}(\mathbb{R}^d))$ by the construction

$$\rho = \lim_{X \nearrow} \frac{1}{|X|_1} \sum_{B \in \mathcal{B}(X)} N(B) \quad \text{a.s. } \mathbb{P}_\infty \tag{2.26}$$

Proof. Under $\mathbb{P}_\infty(\cdot|\rho)$ the random variables $N(B)$ are independent with expectation ρ so by the strong law of large numbers

$$\frac{1}{|X|_1} \sum_{B \in \mathcal{B}(X)} N(B) \to \mathbb{E}(N(B)|\rho) = \rho,$$

and a.s. $\mathbb{P}_\infty(\cdot|\rho)$ convergence implies a.s. $\mathbb{P}_\infty$ convergence. ∎

If we define $\mathcal{F}_X$ to be the σ-algebra generated by $\underline{N}_X$, then (2.26) implies ρ is $\mathcal{F}_{X^c}$ measurable for all X. In down to earth language, ρ does not depend on $\underline{N}_X$ because the $|X|_1 \to \infty$ limit in (2.26) washes out the contribution from $\underline{N}_X$. Thus ρ is $\mathcal{T}$-measurable where

$$\mathcal{T} = \bigcap_{X \subset \mathbb{R}^d} \mathcal{F}_{X^c}. \tag{2.27}$$

$\mathcal{T}$ is called the *tail σ-algebra* or the *algebra at ∞*. We say it is *non-trivial* because it contains sets which have probability $\neq 0$ or 1; equivalently, there are non-constant $\mathcal{T}$-measurable functions such as ρ.

In case (2.19) the infinite volume limit is $(\Omega_\infty^{(p)}, \mathbb{P}_\infty^{(p)})$, $p = ze^\phi/(1+ze^\phi)$, $\phi \in M_0$ is unique. In this case the only $\mathcal{T}$-measurable functions are constants, by the Hewitt-Savage 0-1 law.

Physically, ϕ is the density. In case (2.20) the system has two co-existing "phases", one has a higher density than the other, much like liquid water and gaseous water. The 1/2:1/2 mixure of the two is caused by me trying to keep it simple.

By only allowing a simplified form of V for mean field theory, I have only revealed the convex combination with coefficients 1/2 and 1/2. The infinite volume limit is normally set up in a more general way which includes in V an external field term that represents the interaction of particles inside Λ with a fixed configuration of particles outside Λ. By taking these more general infinite volume limits, one can achieve other convex combinations.

Problems.

Problem 2.1. For $v(x, y)$ as in (2.17) find $H(\underline{n})$ so that (2.15) holds. In other words, express

$$\sum_{1 \leq i < j \leq N(x)} v(x_i, x_j)$$

as an explicit function of the random variables $(N(B) : B \in \mathcal{B})$.

Problem 2.2. Ising models are usually expressed in terms of

$$\Omega_{\text{Ising}} = \{-1, 1\}^{\Lambda \cap \mathbb{Z}^d}, \tag{2.28}$$

e.g.

$$Z_{\text{Ising}} = \sum_{\sigma \in \Omega_{\text{Ising}}} e^{\beta \sum_{x,y \in \Lambda \cap \mathbb{Z}^d} \sigma_x \sigma_y}. \tag{2.29}$$

What Ising model is "the same as" our mean field theory in the case (2.20)? ($n = 0, 1 \leftrightarrow \sigma = -1, 1$)

Problem 2.3. Look up the de Finetti theorem in [Dur91] or any other good textbook, and explain what it has to do with mean field theory.

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Lecture 3. Laplace's Principle and Mean Field Theory

The main goal of this lecture is to prove Proposition 2.1. The secondary goal is to discuss the place of this result relative to the original goal of proving that the grand canonical ensembles constructed from potential that are more realistic than the mean field theory interaction also have 'liquid-to-gas' phase transitions. Very few continuum particle systems in the continuum are rigorously known to have such phase transitions.

We begin with a technical lemma (Lemma 3.1) which encapsulates a principle due to Laplace and then give the proof of Proposition 2.1. Notice the step marked with an exclamation point in this proof because we will re-use the same principle of expressing a two body interaction as a mixture of external fields.

Further Reference: [BF82]

Lemma 3.1 (Laplace). *Let S be a continuous function on $\mathbb{R}^n$ which has a unique global minimum at x_0. Furthermore, assume that $\int e^{-S}\,dx$ is finite and $\{x : S(x) \leq S(x_0)+1\}$ is compact. Then*

$$\lim_{t\to\infty} \frac{1}{(f=1)} \int e^{-tS} f\,dx = f(x_0) \tag{3.1}$$

for any bounded continuous $f : \mathbb{R}^n \to \mathbb{R}$.

The notation $(f = 1)$ stands for the appropriate normalization. The idea behind the lemma is that outside of the minimal value of S, the term e^{-tS} will decay fast, resulting in a delta function.

Proof. Consider

$$M_\epsilon = \{x : S(x) \leq S(x_0) + \epsilon\}$$

For $\epsilon > 0$, it contains $\{x : S(x) < S(x_0) + \epsilon\}$, which is open because S is continuous. Therefore,

$$\int_{M_\epsilon} e^{-S} dx \neq 0, \qquad\qquad \epsilon > 0$$

If U is an open set containing x_0, then $U^c \cap M_1$ is compact. So S has a minimum on $U^c \cap M_1$, which cannot equal x_0. Thus, there is $\epsilon > 0$ such that

$$S(x) \geq S(x_0) + \epsilon \qquad\qquad x \notin U.$$

We can, without loss of generality, assume that $S(x_0) = 0$. Let

$$I_t(E, f) = \int_E e^{-tS} f dx, \qquad\qquad E \subset \mathbb{R}^n$$

Then,

$$I_t(U^c, f) \leq \|f\|_\infty e^{(1-t)\epsilon} \int e^{-S} dx$$

$$\begin{aligned} I_t(\mathbb{R}^n, 1) &\geq I_t(M_{\epsilon/2}, 1) \\ &\geq e^{(1-t)\epsilon} \int_{M_{\epsilon/2}} e^{-S} dx. \end{aligned}$$

Therefore,

$$\frac{I_t(U^c, f)}{I_t(\mathbb{R}^n, 1)} \to 0 \qquad\qquad \text{as } t \to \infty \tag{a}$$

and

$$\text{(b)} \qquad \frac{I_t(U,1)}{I_t(\mathbb{R},1)} \to 1.$$

Let $\epsilon > 0$. Choose U such that $|f(x) - f(x_0)| < \epsilon$ for $x \in U$. Then using (a), we have

$$\begin{aligned}\frac{I_t(\mathbb{R},f)}{I_t(\mathbb{R},f)} &= \frac{I_t(U,f)}{I_t(\mathbb{R},1)} + o(t)\\ &\leq (f(x_0)+\epsilon)\frac{I_t(U,1)}{I_t(\mathbb{R},1)} + o(t)\end{aligned}$$

Using (b) this gives an upper bound in terms of $f(x_0)+\epsilon$. Likewise, a lower bound can be found in terms of $f(x_0)-\epsilon$. ∎

Proof of Proposition 2.1. Let $F = \mathbb{I}_{\underline{N}_X = \underline{n}}$ and $\alpha = \dfrac{\beta}{|\Lambda|_1}$. Then:

$$\begin{aligned}\langle F\rangle_{\mathrm{MFT},\Lambda} &= \frac{1}{(F=1)}\sum_{n\in\Omega} z^{\underline{n}} e^{-H} F\\ &= \frac{1}{(F=1)}\sum_{n\in\Omega} z^{\underline{n}} e^{\alpha\frac{N^2}{2}} F\\ \text{(!)} \qquad &= \frac{1}{(F=1)}\int\sum_{n\in\Omega} z^{\underline{n}} e^{\phi N} F e^{-\frac{\phi^2}{2\alpha}}\ d\phi.\end{aligned}$$

(!) is a direct consequence of a Laplace transform of a Gaussian:

$$\frac{1}{\sqrt{2\pi}}\int e^{-\frac{\phi^2}{2\alpha}} e^{\phi N} d\phi = e^{\frac{1}{2}\alpha N^2}$$

Define

$$\langle F\rangle_{\phi,\Lambda} = \frac{\sum z^{\underline{n}} e^{\phi N} F}{\sum z^{\underline{n}} e^{\phi N}}.$$

Since $F \in m\mathcal{F}_X$,

$$\langle F\rangle_{\phi,\Lambda} = \langle F\rangle_{\phi,X}.$$

This is because the Bernoulli random variables are independent, or more concretely, by explicitly expanding the numerator and the denominator in terms of X and $\Lambda\backslash X$ and factoring the terms reliant on $\Lambda\backslash X$. Since

$$\sum_{\underline{n}\in\Omega} z^{\underline{n}} e^{\phi N} = \sum (ze^{\phi})^{\underline{n}} = (1+ze^{\phi})^{|\Lambda|_1},$$

we have

$$\begin{aligned}\langle F\rangle_{\mathrm{MFT},\Lambda} &= \frac{1}{(F=1)}\int (1+ze^{\phi})^{|\Lambda|_1}\langle F\rangle_{\phi,X} e^{-\frac{\phi^2}{2\alpha}}\ d\phi\\ &= \frac{1}{(F=1)}\int e^{-|\Lambda|_1 S(\phi)}\langle F\rangle_{\phi,X}\ d\phi.\end{aligned}$$

Now we take the infinite volume limit as $|\Lambda|_1 \to \infty$.

If (z,β) are such that S has a unique global minimum ϕ then Lemma 3.1 and the choice F implies that, as $|\Lambda|_1 \nearrow \infty$

$$\langle F\rangle_{\mathrm{MFT},\Lambda} \to \langle F\rangle_{\phi,X} = \mathbb{P}_{\mathrm{Bernoulli}}\{\underline{N}_X = \underline{n}\}$$

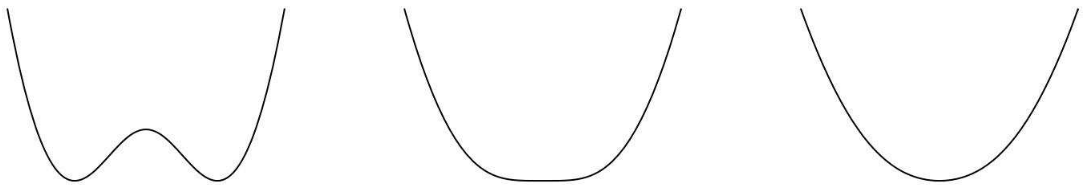

Figure 3.1. The plots from left to right represent the curves where $\beta > 4$, $\beta = 4$ and $\beta < 4$. It can be seen that for $\beta < 4$, the curve is convex.

Claim. *Analysis of $S(\phi)$ shows that $S(\phi)$ has a unique global minimum if $\beta \leq 4$ or if $ze^{\beta/2} \neq 1$.*

If $\beta > 4$ and $ze^{\beta/2} = 1$, Lemma 3.2 implies that there are two global minima related by symmetry. With the symmetry it is trivial to modify Lemma 3.1 to finish the case (2-phase).

The claim is not fully proved in these notes but see Figure 3.1 for an idea of the situation. ■

Lemma 3.2. *For $(\beta, z) \in \{ze^{\beta/2} = 1\}$*

$$S(\phi) = \frac{\eta^2}{2\beta} - \log(e^{-\eta/2} + e^{\eta/2}) + C_{\beta,2} \tag{3.2}$$

where $\eta = \phi - \beta/2$. There are two global minima $\phi = \beta/2 \pm \eta_c$ when $\beta > 4$, otherwise there is one global minimum.

Proof of Lemma 3.2. Let $\phi = \xi + \eta$, then

$$\log(1 + ze^{\phi}) = \log(1 + ze^{\xi}e^{\eta})$$

By choosing ξ so that $ze^{\xi} = 1$, then

$$\log(1 + ze^{\phi}) = \log(1 + e^{\eta}) = \log e^{\eta/2}(e^{-\eta/2} + e^{\eta/2}) = \eta/2 + \log(e^{-\eta/2} + e^{\eta/2})$$

Also, as $\dfrac{\phi^2}{2\beta} = \dfrac{\xi^2}{2\beta} + \dfrac{\xi\eta}{\beta} + \dfrac{\eta^2}{2\beta}$ then

$$S(\phi) = \frac{\xi^2}{2\beta} + \left(\frac{\xi}{\beta} - \frac{1}{2}\right)\eta + \frac{\eta^2}{2\beta} - \log(e^{-\eta/2} + e^{\eta/2})$$

If $\dfrac{\xi}{\beta} = \dfrac{1}{2}$, then we have the formula for $S(\phi)$ claimed in the Lemma. If $(\beta, z) \in \{ze^{\beta/2} = 1\}$, then we can simultaneously solve $\dfrac{\xi}{\beta} = \dfrac{1}{2}$ and $ze^{\xi} = 1$ as required. It is easy to check convexity iff $\beta \leq 4$. ■

3.1. Graphical interpretation.

The global minima are among the solutions to

$$\frac{\partial S}{\partial \phi} = 0 \tag{3.3}$$

which is

$$\frac{1}{\beta}\phi = \frac{ze^{\phi}}{1 + ze^{\phi}}. \tag{3.4}$$

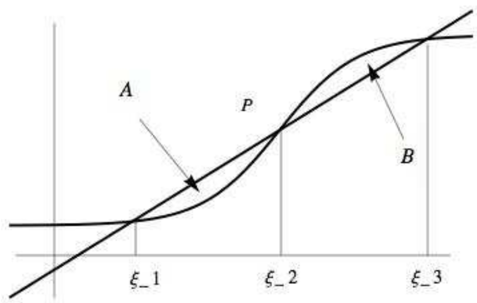

Figure 3.2. A plot of $\dfrac{\partial S}{\partial \phi}$ against ϕ for $\beta > 4$

Let ξ_2 be the maximum and ξ_1 and ξ_3 be local minima.

Then, using the notation in Figure 3.1, we have that

$$S(\xi_1) = S(\xi_2) - A, \quad S(\xi_3) = S(\xi_2) - B. \tag{3.5}$$

For two global minima the areas, A and B, must be equal. When P is the point of inflexion of $f(\phi)$ the two areas, A and B, are equal because f is odd about P. To fully prove the 1-phase case of Proposition 2.1 we have to show that the two areas A and B are not equal if ξ_2 is not a point of inflexion.

Discussion. Consider the grand canonical ensemble with V built from the 2-body potential

$$v(x,y) = \begin{cases} \infty & \text{if } |x-y| \leq 1 \\ \ell^{-d} f\left(\frac{\|x-y\|}{\ell}\right) & \text{else} \end{cases} \tag{3.6}$$

where $f \geq 0$ and $\int f dx = 1$. The

- The limit $\ell \to \infty$ is called the Kac limit [Kac59]. Intuitively one expects mean field theory in this limit because the range of the interaction is $O(\ell) \to \infty$ while the strength of the interaction is $O(\ell^{-3})$.
- Lebowitz-Penrose [LP66] proved in 1966 that the Kac limit of the infinite volume pressure is the mean field theory pressure for particles in the continuum with hardcore and attractive potential.
- Lebowitz-Mazel-Presutti [LMP99] proved in 1999 that the infinite volume limit of the grand canonical ensemble has a phase transition for ℓ sufficiently large, but not for this model. Instead they replaced the hard core by a less natural 4-body repulsion. It is a very interesting open problem to prove that the above models have phase transitions for ℓ sufficiently large. This formulates the idea that models that are close to mean field should have a phase transition because mean field theory does.
- This is interesting because at present hardly any continuum particle systems have been proved to exhibit phase transitions and the ones that have are very artificial. This open problem is the first step towards a natural class of models.

Problems.

Problem 3.1. Show that when $(\beta, z) \in \{ze^{\beta/2} = 1\}$ the probability of any configuration in the MFT model is invariant under ($n_B \longleftrightarrow 1 - n_B$ for all B).

Problem 3.2. Omit the step where we introduce the blocks $\mathcal{B}(\Lambda)$ and consider the grand canonical ensemble with

$$V(x) = -\frac{\beta}{|\Lambda|}\frac{N^2(x)}{2}.$$

Notice there is no hardcore condition. Apply the same idea,

$$e^{-\alpha\frac{N^2}{2}} = \frac{1}{\sqrt{2\pi}}\int e^{\phi N}e^{-\phi^2/(2\alpha)}d\phi$$

What is S in this case? What goes wrong and why did introducing the condition $V = \infty$ if any $N(B) > 1$ avoid this problem?

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Part 2. Lattice Gaussian Fields

Lecture 4. The Lattice Laplacian and Walks on the Lattice

For this lecture we put the particle systems away for now and work towards understanding two new systems called the massless and massive free fields on the lattice. For this we require some estimates on the lattice Laplacian and its resolvent. These are the topics of this lecture. In the next lecture we define the free fields.

Notation. Let $d \in \mathbb{N}$. Think of $\mathbb{Z}^d$ as a graph with edges

$$E = \mathrm{Edges}(\mathbb{Z}^d) = \Big\{ \{x, y\} : x, y \in \mathbb{Z}^d,\ \|x - y\|_2 = 1 \Big\}. \tag{4.1}$$

Here $\|\cdot\|_2$ is the Euclidean norm. We use the notation $xy = \{x, y\}$ for the edges.

For $\phi, \psi \colon \mathbb{Z}^d \to \mathbb{R}$, define

$$(\phi, \psi) = \sum_{x \in \mathbb{Z}^d} \phi(x)\psi(x). \tag{4.2}$$

We will only need this for the case when ϕ and ψ vanish outside a finite set.

Definition 4.1. For $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, the lattice Laplacian with Dirichlet boundary conditions outside Λ is the unique[1] symmetric $\Lambda \times \Lambda$ matrix $\Delta = \Delta_\Lambda$ such that

$$(\phi, -\Delta\phi) = \sum_{xy \in E} (\phi_x - \phi_y)^2 \tag{4.3}$$

for all $\phi \colon \mathbb{Z}^d \to \mathbb{R}$ such that $\phi = 0$ outside Λ.

Thus, $-\Delta$ is a linear operator $\mathbb{R}^\Lambda \to \mathbb{R}^\Lambda$. The eigenvalues of $-\Delta$ are positive because $(\phi, -\Delta\phi) > 0$ for $\phi \neq 0$. Therefore, $(\epsilon - \Delta)^{-1}$ exists for $\epsilon \geq 0$.[2] One can write the matrix elements of $-\Delta$ explicitly as follows:

$$-\Delta_{xy} = \begin{cases} 2d, & \text{if } x = y; \\ -1, & \text{if } xy \in E; \\ 0, & \text{otherwise,} \end{cases} \tag{4.4}$$

where $x, y \in \Lambda$.

Definition 4.2. Let $W_{ab}(\Lambda)$ denote the set of all sequences in Λ of the form

$$\omega = (\omega_0, \omega_1, \dots, \omega_n) \tag{4.5}$$

such that $\omega_0 = a$, $\omega_n = b$ and $(\omega_i, \omega_{i+1}) \in E$ for all $i = 0, \dots, n-1$. The number of elements in ω can be arbitrary and is denoted by $n = n(\omega)$.

Proposition 4.3. *For $\epsilon \geq 0$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, we have*[3]

$$(\epsilon - \Delta)^{-1}_{ab} = \sum_{\omega \in W_{ab}(\Lambda)} \left(\frac{1}{\epsilon + 2d} \right)^{n(\omega)+1}. \tag{4.6}$$

First, we give an idea of the proof. Write

$$\epsilon - \Delta = D - O, \tag{4.7}$$

where D is a diagonal matrix with entries $\epsilon + 2d$, and O is an off-diagonal matrix, and

$$O_{xy} = 1 \quad \text{iff} \quad xy \in \mathrm{Edges}(\Lambda). \tag{4.8}$$

---

[1] Look up polarisation to see that $(\phi, -\Delta\phi)$ determines $(\phi, -\Delta\phi')$ for $\phi' \neq \phi$.

[2] Here and below by $\epsilon - \Delta$ we denote the operator $\epsilon I - \Delta$.

[3] Here $(\epsilon - \Delta)^{-1}_{ab}$ denotes the ab'th element of the inverse matrix $(\epsilon - \Delta)^{-1}$.

Then the resolvent expression

$$(D-O)^{-1} = D^{-1} + D^{-1}OD^{-1} + D^{-1}OD^{-1}OD^{-1} + \dots \tag{4.9}$$

is the same as

$$(D-O)^{-1}_{ab} = \sum_{\omega \in W_{ab}(\Lambda)} (\epsilon + 2d)^{-n(\omega)-1} \tag{4.10}$$

because the matrix D^{-1} corresponds to the sum over sequences from $W_{ab}(\Lambda)$ of length zero, $D^{-1}OD^{-1}$ — over sequences of length 1, etc. Now we proceed with a proof.

Proof. Let

$$\begin{aligned} W_a(\Lambda) &= \bigcup_{b\in\Lambda} W_{ab}(\Lambda); \\ W_a^{(m)}(\Lambda) &= \{\omega \in W_a(\Lambda) \colon n(\omega) = m\}. \end{aligned} \tag{4.11}$$

The right hand side of (4.6) is absolutely convergent for $\epsilon > 0$ because

$$\begin{aligned} \sum_{\omega \in W_{ab}(\Lambda)} (\epsilon + 2d)^{-n(\omega)-1} &\leq \sum_{\omega \in W_a(\mathbb{Z}^d)} (\epsilon + 2d)^{-n(\omega)-1} \\ &= \sum_{n=0}^{\infty} (2d)^n (\epsilon + 2d)^{-n-1} = \frac{1}{\epsilon}. \end{aligned} \tag{4.12}$$

Once we know that $D^{-1} + D^{-1}OD^{-1} + D^1OD^{-1}OD^{-1} + \dots$ is convergent, multiplying by $D - O$ shows that it equals $(D-O)^{-1}$. By monotone convergence we can also conclude the case $\epsilon = 0$:

$$\sum_{w \in W_{ab}(\Lambda)} (2d)^{-n(\omega)-1} = \lim_{\epsilon \downarrow 0} \sum_{\omega \in W_{ab}(\Lambda)} (\epsilon + 2d)^{-n(\omega)-1} = \lim_{\epsilon\downarrow 0} (\epsilon - \Delta)^{-1}_{ab} = (-\Delta)^{-1}_{ab} \tag{4.13}$$

(here the operator $(-\Delta)$ is invertible because all its eigenvalues are positive). ■

Definition 4.4. Define for $k \in \mathbb{R}^d$,

$$f(k) := \sum_{x \in \mathbb{Z}^d \colon \|x\|_2 = 1} \left(e^{k\cdot x} - 1 \right). \tag{4.14}$$

Here $k \cdot x = \sum_{i=1}^d k^{(i)} x^{(i)}$.

Lemma 4.5. *For $k \in \mathbb{R}^d$ and $\epsilon > f(k)$ and all $a \in \mathbb{Z}^d$,*

$$e^{-k\cdot a} \sum_{\omega \in W_a(\mathbb{Z}^d)} (\epsilon + 2d)^{-n(\omega)-1} e^{k\cdot\omega_{n(\omega)}} = (\epsilon - f(k))^{-1}. \tag{4.15}$$

Proof. First, observe that

$$e^{-k\cdot a} e^{k\cdot\omega_{n(\omega)}} = e^{\sum_{i=0}^{n(\omega)-1} k\cdot(\omega_{i+1}-\omega_i)} = \prod_{i=1}^{n(\omega)-1} e^{k\cdot(\omega_{i+1}-\omega_i)}. \tag{4.16}$$

The left hand side of (4.15) becomes

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{\omega\in W_a^{(n)}(\mathbb{Z}^d)}\prod_{i=0}^{n(\omega)-1}\left(e^{k\cdot(\omega_{i+1}-\omega_i)}\frac{1}{\epsilon+2d}\right)\frac{1}{\epsilon+2d}\\ =\sum_{n=0}^{\infty}\left(\frac{1}{\epsilon+2d}\sum_{x\in\mathbb{Z}^d\colon \|x\|_2=1}e^{k\cdot x}\right)^n\frac{1}{\epsilon+2d}=(\epsilon-f(k))^{-1},\end{aligned}\tag{4.17}$$

this concluses the proof. ■

Corollary 4.6. *For $\epsilon>0$,*

$$\sum_{b\in\Lambda}(\epsilon-\Delta_\Lambda)^{-1}_{ab}\le\frac{1}{\epsilon},\tag{4.18}$$

and this increases to $\frac{1}{\epsilon}$ as Λ increases to the whole of $\mathbb{Z}^d$.

Proof. Set $k=0$ in Lemma 4.5 and use dominated convergence for controlling the limit $\Lambda\nearrow\mathbb{Z}^d$. ■

For $\lambda>0$ let $\eta=\sup\{f(k)\colon\|k\|_2=\lambda\}$.

Corollary 4.7. *For $\lambda>0$ and $\epsilon>\eta$,*

$$(\epsilon-\Delta_\Lambda)^{-1}_{ab}\le\frac{1}{\epsilon-\eta}e^{-\lambda\|b-a\|_2}.\tag{4.19}$$

Proof. By Proposition 4.3 and Lemma 4.5,

$$(\epsilon-\Delta_\Lambda)^{-1}_{ab}\le\sum_{\omega\in W_{ab}(\mathbb{Z}^d)}\left(\frac{1}{\epsilon+2d}\right)^{n(\omega)+1}\le\frac{1}{\epsilon-f(k)}e^{k\cdot(b-a)}.\tag{4.20}$$

Choose the direction of k such that $k\cdot(b-a)=-\lambda\|b-a\|_2$. After this one can replace the factor $(\epsilon-f(k))^{-1}$ by the upper bound $(\epsilon-\eta)^{-1}$. ■

Corollary 4.8. *For all $\epsilon>0$ and $a\in\mathbb{Z}^d$,*

$$\lim_{\Lambda\nearrow\mathbb{Z}^d}(\epsilon-\Delta_\Lambda)^{-1}_{aa}=(2\pi)^{-d}\int_{[-\pi,\pi]^d}\big(\epsilon-f(ik)\big)^{-1}dk.\tag{4.21}$$

The right hand side is bounded uniformly as $\epsilon\to0$ if $d\ge3$, otherwise it diverges as $\epsilon\to0$.

Proof. The main idea is the formula

$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}e^{ik\cdot x}dk=\delta_{x,0},\qquad x\in\mathbb{Z}^d.$$

Using this we have

$$\begin{aligned}(\epsilon-\Delta_\Lambda)^{-1}_{aa}&=\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\left(\sum_{b\in\Lambda}(\epsilon-\Delta_\Lambda)^{-1}_{ab}e^{ik\cdot(b-a)}\right)dk\\ &\to\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}(\epsilon-f(ik))^{-1}dk,\qquad\qquad \Lambda\nearrow\mathbb{Z}^d,\end{aligned}\tag{4.22}$$

by dominated convergence based on the bound (4.12).

This proves the Corollary apart from the claim about $\epsilon \downarrow 0$. For $\epsilon \downarrow 0$ note

$$f(ik) = \sum_{x \in \mathbb{Z}^d : \, \|x\|_2 = 1} \left(e^{ik \cdot x} - 1 \right) = \sum_{x \in \mathbb{Z}^d : \, \|x\|_2 = 1} \left(\cos\left(k \cdot x\right) - 1 \right). \tag{4.23}$$

This is real and non-positive, and equals zero in $[-\pi, \pi]^d$ iff $k = 0$. Near $k = 0$ we have the following expansion:

$$(\epsilon - f(ik))^{-1} = \frac{1}{\epsilon + \|k\|_2^2 + o\left(\|k\|_2^2\right)}. \tag{4.24}$$

This is integrable iff $d \geq 3$. The claim follows from monotone convergence. ■

Problems.

Problem 4.1. Adapt Lemma 4.5 and Corollary 4.7 to prove that, for $(A_{xy} : x, y \in \Lambda)$ any $\Lambda \times \Lambda$ matrix with the property that

$$\frac{1}{|A_{xx}|} \sum_{x \neq y} |A_{xy}| e^{\kappa \|x - y\|} \leq C < 1 \quad (x \in \Lambda), \tag{4.25}$$

the inverse A^{-1} exists, and uniformly in Λ, A^{-1}_{xy} decays exponentially in $\|x - y\|$.

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Lecture 5. Lattice Gaussian Fields

In this lecture the basic facts about Gaussian measures are introduced, but with a slant towards their role in theoretical physics where they serve as the underpinning for quantum field theory. Therefore the connection with graphs, Hermite polynomials, etc is included.

Let $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, and $\phi = (\phi_x, x \in \Lambda)$. Suppose that $A = (A_{xy} : x, y \in \Lambda)$ is symmetric with positive eigenvalues: $(\phi, A\phi) > 0$ if $\phi \neq 0$. A is said to be positive definite. Define a probability measure on $\mathbb{R}^\Lambda$ by

$$d\mu_C(\phi) = \frac{1}{N} e^{-\frac{1}{2}(\phi, A\phi)}\, d^\Lambda \phi, \quad C = A^{-1} \tag{5.1}$$

Then:

$$\int d\mu_C(\phi)\, e^{(f,\phi)} = e^{\frac{1}{2}(f, Cf)}, \quad f \in \mathbb{R}^\Lambda \tag{5.2}$$

$$\int d\mu_C\, \phi_a \phi_b = C_{ab} \tag{5.3}$$

$$N = (2\pi)^{|\Lambda|/2} (\det A)^{-1/2} \tag{5.4}$$

Lemma 5.1. *Given a $\Lambda \times \Lambda$ positive definite matrix C, there exists a unique probability measure such that* (5.2) *holds and it is $d\mu_C$.*

Proof. Existence: C is symmetric with positive eigenvalues. Therefore $A = C^{-1}$ exists and is symmetric and also has positive eigenvalues. Define $d\mu_C$ by (5.1). Uniqueness: The Laplace transform characterises the measure (a hard but well known theorem). ∎

Probability measures of the form (5.1) are said to be *Gaussian.* Here is a very important fact about these probability measures: If we are given a Gaussian probability measure and we integrate out some of the variables, the result is still Gaussian. This is what the next Lemma says and the proof is one of the problems for this lecture.

Lemma 5.2. *Let $d\mu_C$ be a Gaussian measure defined on $\mathbb{R}^\Lambda$ and let $\Lambda' \subset \Lambda$. Then there is a Gaussian measure $d\mu_{C'}$ defined on $\mathbb{R}^{\Lambda'}$ such that for any bounded function F defined on $\mathbb{R}^{\Lambda'}$, we have $\int_{\mathbb{R}^\Lambda} d\mu_C F = \int_{\mathbb{R}^{\Lambda'}} d\mu_{C'} F$.*

Definition 5.3. The massless free field is the case $A = -\Delta_\Lambda$. The free field with mass m is the case $A = m^2 - \Delta_\Lambda$.

Discussion. If $\overrightarrow{\phi} : \Lambda \to \mathbb{R}^d$ is vector-valued,

$$\frac{1}{2}(\overrightarrow{\phi}, -\Delta_\Lambda \overrightarrow{\phi}) = \frac{1}{2} \sum_{xy \in E} (\overrightarrow{\phi_x} - \overrightarrow{\phi_y})^2, \quad \overrightarrow{\phi_x} = 0 \text{ if } x \notin \Lambda. \tag{5.5}$$

is the energy of all the springs in a bedspring, and the frame is the Dirichlet boundary condition. Alternatively, this is a model for sound waves in a crystal.

Question. For a bedspring, does ϕ_0 remember the Dirichlet boundary condition as $\Lambda \nearrow \mathbb{Z}^d$? For F a function on $\mathbb{R}^\Lambda$ define the *expected value* of F, $\langle F \rangle = \int_{\mathbb{R}^\Lambda} d\mu_C F$, where $C = (-\Delta)^{-1}$. We only make this definition for F such that the right hand side is absolutely convergent. Now conside the special cases $F = \phi_0$ and $F = \phi_0^2$. As $\Lambda \nearrow \mathbb{Z}^d$, $\langle \phi_0 \rangle_\Lambda = 0$ but how about $\langle \phi_0^2 \rangle_\Lambda$?

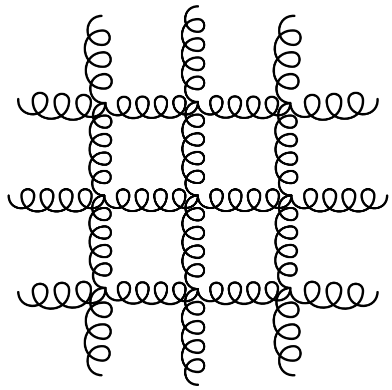


Figure 5.1. Bedspring

Example 5.4 (Mean Field Theory). If the domain Λ is paved with unit boxes and each box has zero or one particle, the the partition function is

$$Z = \sum_{\underline{n} \in \{0,1\}^{|\Lambda|}} z^{\underline{n}} e^{\frac{1}{2}\sum_{x,y\in\Lambda} n_x v_{xy} n_y}. \tag{5.6}$$

If v_{xy} is positive definite,

$$\begin{aligned} Z &= \sum_{\underline{n}} z^{\underline{n}} \int d\mu_v(\phi)\; e^{\sum_{x\in\Lambda}\phi_x n_x} \\ &= \int d\mu_v(\phi) \sum_{\underline{n}} z^{\underline{n}} e^{\sum \phi_x n_x} \\ &= \int d\mu_v(\phi) \prod_{x\in\Lambda}(1+ze^{\phi_x}) \\ &= \frac{1}{N}\int d^\Lambda\phi\; e^{-S(\phi)} \end{aligned} \tag{5.7}$$

where

$$S(\phi) = \frac{1}{2}(\phi, v^{-1}\phi) - \sum_{x\in\Lambda}\log(1+ze^{\phi_x}). \tag{5.8}$$

One possible choice is the lattice analogue of the Yukawa potential ($e^{-r}/(4\pi r)$ is the Green function for $1-\Delta$ on $\mathbb{R}^3$)

$$v_{xy} = \beta m^2(m^2-\Delta_\Lambda)^{-1}_{xy} \tag{5.9}$$

such that

$$S(\phi) = \frac{1}{2m^2\beta}\sum_{xy}(\phi_x-\phi_y)^2 + \frac{1}{2\beta}\sum_{x\in\Lambda}\phi_x^2 - \sum_{x\in\Lambda}\log(1+ze^{\phi}). \tag{5.10}$$

As $m \searrow 0$ the term $\frac{1}{2m^2\beta}\sum_{xy}(\phi_x-\phi_y)^2$ in $\exp[-S]$ concentrates the partition function on ϕ such that $\phi \simeq$ const. so that

$$\sum_{x\in\Lambda}\log(1+ze^{\phi}) \simeq |\Lambda|\log(1+ze^{\phi}), \tag{5.11}$$

$$\frac{1}{2\beta}\sum_{x\in\Lambda}\phi_x^2 \simeq \frac{1}{2\beta}|\Lambda|\phi^2. \tag{5.12}$$

Therefore, in this limit,

$$\frac{1}{|\Lambda|}\log Z \to \frac{1}{|\Lambda|}\log\frac{1}{N}\int_{\mathbb{R}} d\phi\, e^{-|\Lambda|\left(\frac{1}{2\beta}\phi^2-\log(1+ze^{\phi}\right)} \tag{5.13}$$

If we next take the limit $\Lambda \nearrow \mathbb{Z}^d$ we get $\inf_\phi\left(\frac{1}{2\beta}\phi^2-\log(1+ze^{\phi}\right)$, which is mean field theory. However we have committed a sin, the correct order of the limits is $\Lambda \nearrow \mathbb{Z}^d$ is followed by $m \searrow 0$, because one wants to be able to claim that mean field theory is asymptotic, uniformly in the volume. Uniformity in the volume is always the true challenge of statistical mechanics. This idea of transforming a two body potential to an integral over an external field was independently invented or exploited by [Str57], [Hub59], [Kac59], [Sie60].

Theorem 5.5 (Wick). *Let*

$$\Delta_C = \frac{1}{2}\sum_{x,y\in\Lambda} C_{xy}\frac{\partial}{\partial\phi_x}\frac{\partial}{\partial\phi_y}. \tag{5.14}$$

For P a polynomial,

$$\int d\mu_C\ P = e^{\frac{1}{2}\Delta_C}P|_{\phi=0}. \tag{5.15}$$

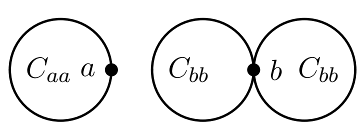
Proof. Homework (Problem 5.1). Hint: $\int d\mu_{tC}(\zeta)\ P(\zeta+\phi)$ and $e^{\frac{1}{2}\Delta_C}P$ are polynomials in ϕ with coefficients that depend on t that solve $\frac{\partial u(t,\phi)}{\partial t} - \frac{1}{2}\Delta_C u(t,\phi) = 0$. ∎

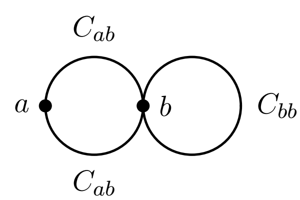
Example 5.6. Using the above theorem, we can easily prove one of the properties of Gaussian measures:

$$\int d\mu_C\ \phi_a\phi_b = e^{\frac{1}{2}\Delta_C}\phi_a\phi_b|_{\phi=0} = (1+\frac{\Delta_C}{2}+\cdots)\phi_a\phi_b|_{\phi=0} = C_{ab} \tag{5.16}$$

Example 5.7 (Feynman diagrams).

$$\int d\mu_C\ \frac{\phi_a^2}{2!}\ \frac{\phi_b^4}{4!} = \frac{1}{3!}(\frac{\Delta_C}{2})^3\frac{\phi_a^2}{2!}\frac{\phi_b^4}{4!}\Big|_{\phi=0} \tag{5.17}$$

$= \left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)$ 

$+ \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ 

Definition 5.8 (Wick polynomials). For polynomial P,

$$:P: \equiv :P:_C \equiv e^{-\frac{1}{2}\Delta_C}P. \tag{5.18}$$

Example 5.9. Using this notation,

$$:\phi_a^4: = \phi_a^4 - \frac{1}{2}(4)(3)C_{aa}\phi_a^2 + \frac{1}{2}\frac{1}{2}\frac{1}{2}C_{aa}^2 4! \tag{5.19}$$

$:\phi_x^p:$ is called the p[th] *Wick power*. That $\frac{\partial}{\partial\phi}:\phi^p: = p:\phi^{p-1}:$ follows from definition of "$:-:$".

Lemma 5.10. *If P, Q are monomials of different degrees,*

$$\int d\mu_C \, :P: \, :Q: = 0. \tag{5.20}$$

Remark 5.11. When $|\Lambda| = 1$, this proves that $:\phi^p:$ for $p = 0, 1, \dots$ are orthogonal polynomials on $\mathbb{R}$, so up to normalisation, they are Hermite polynomials.

Proof. The product rule for differentiation can be written $\frac{\partial}{\partial \phi} AB = \left(\frac{\partial}{\partial \phi_A} + \frac{\partial}{\partial \phi_B} \right) AB$ where $\frac{\partial}{\partial \phi_A}$ acts only on A and $\frac{\partial}{\partial \phi_B}$ acts only on B. Insert this decomposition into Δ_C. We suppress C and write the result as $\Delta = \Delta_{AA} + 2\Delta_{AB} + \Delta_{BB}$. For A, B polynomials,

$$e^{\frac{1}{2}\Delta} AB = e^{\frac{1}{2}\Delta_{AA} + \Delta_{AB} + \frac{1}{2}\Delta_{BB}} AB = e^{\Delta_{AB}} (e^{\frac{1}{2}\Delta_{AA}} A)(e^{\frac{1}{2}\Delta_{BB}} B).$$

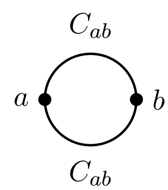
If $A = :P:$ then $e^{\frac{1}{2}\Delta_{AA}} :P: = P$, and so is it if $B = :Q:$, therefore,

$$e^{\frac{1}{2}\Delta} AB = e^{\Delta_{AB}} PQ = 0 \quad \text{at } \phi = 0$$

if P, Q have different degrees. ■

Example 5.12. Consider the following integral:

(5.21) $$\int d\mu_C \, \frac{:\phi_a^2:}{2!} \frac{:\phi_b^2:}{2!} = \frac{1}{2}$$



$$= \frac{1}{2} C_{ab}^2$$

Note that there are no self-loops!

Problems.

Problem 5.1. What is C' in Lemma 5.2.

Problem 5.2. Prove Lemma 5.2. Hint: Laplace transform and uniqueness.

Problem 5.3. Answer the Question above for $\mathbb{Z}^2$ by proving that for f continuous with compact support,

$$\langle f(\phi_0) \rangle_\Lambda \to 0 \quad \text{as } \Lambda \nearrow \mathbb{Z}^2. \tag{5.22}$$

Hint: use the previous problems.

Problem 5.4. Prove Wick's Theorem.

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Lecture 6. Fermionic Lattice Gaussian Fields

We have seen that Gaussian fields can be used to represent particle systems in a way that reveals the role of the Kac limit and mean field theory. Many other models also have Gaussian field representations. Today, we will see as further examples representations of self-avoiding "molecules". This also gives me a chance to briefly explain "Fermions", but my explanation is a device to get you used to the idea of "anticommuting variable" as a useful combinatorial tool by equating them with differential forms. The book by Berezin [Ber66] describes the standard setup which does not insist on any identification with differential forms. For more information on differential forms see any of [Arn89], [Spi65], [Fla89] [Rud76, Chapter 10].

6.1. Oriented Edges. Until now $A_{xy} = A_{yx}$, so $C_{xy} = C_{yx}$ and graphs have unoriented edges. To obtain oriented edges, let $A = (A_{xy})_{x,y\in\Lambda}$ be a not necessarily symmetric matrix such that

$$\mathrm{Re}(\phi, A\bar{\phi}) > 0, \quad \phi \neq 0, \phi \in \mathbb{C}^{\Lambda}. \tag{6.1}$$

Complex-valued ϕ can "see" the antisymmetric part of A, because

$$\begin{aligned}(\phi, A\bar{\phi}) &= (u + iv, A(u - iv))\\ &= (u, Au) + i(v, Au) - i(u, Av) + (v, Av).\end{aligned}$$

Recall that $(\phi, \psi) = \sum_x \phi_x\psi_x$ so there are no complex conjugates buried in the notation (ϕ, ψ). The terms $i(v, Au) - i(u, Av)$ vanish if $A_{xy} = A_{yx}$ and are a function only of the antisymmetric A'' in the decomposition $A = A' + A''$ with $A'_{xy} = (A_{xy} + A_{yx})/2$ and $A''_{xy} = (A_{xy} - A_{yx})/2$. For A a complex or real matrix such that $(\phi, A\bar{\phi})$ has positive real part for $\phi \neq 0$ define:

$$d\mu_C(\phi) = \frac{1}{N} e^{-(\phi,A\bar{\phi})}\, d^{2\Lambda}\phi, \tag{6.2}$$

where

$$C = A^{-1}, \quad N = \pi^{|\Lambda|}(\det A)^{-1}, \tag{6.3}$$

$$\phi_x = u_x + iv_x, \quad d^{2\Lambda}\phi = \prod_{x\in\Lambda} du_x dv_x. \tag{6.4}$$

Remark 6.1. If $A_{xy} = A_{yx}$, then $(\phi, A\bar{\phi}) = (u, Au) + (v, Av)$ which gives an easy way to prove that

$$N \propto (\det A)^{-1/2}(\det A)^{-1/2} = (\det A)^{-1}. \tag{6.5}$$

but this also holds when A is not symmetric.

Notation.

$$\frac{\partial}{\partial\phi} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial\bar{\phi}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right) \tag{6.6}$$

These are designed so that $\frac{\partial\phi}{\partial\phi} = 1$, $\frac{\partial\bar{\phi}}{\partial\phi} = 0$, etc.

Lemma 6.2. *If $F \in C^1$, then*

$$\int d\mu_C\; \bar{\phi}_a F = \int d\mu_C\; \sum_{x\in\Lambda} C_{ax}\frac{\partial F}{\partial\phi_x} \tag{6.7}$$

if both sides of this equation are integrable.

Proof. By using (6.6) it is easy to prove that integration by parts in the form $\int \frac{\partial A}{\partial \phi_a} B \, d^{2\Lambda}\phi = -\int A \frac{\partial B}{\partial \phi_a} \, d^{2\Lambda}\phi$ is valid provided the functions A, B tend to zero at infinity so that there are no boundary terms. We use this in the next lines

$$\int \bar{\phi}_a e^{-(\phi, A\bar{\phi})} F \; d^{2\Lambda}\phi = \int (CA\bar{\phi})_a e^{-(\phi, A\bar{\phi})} F \; d^{2\Lambda}\phi$$

$$= \sum_x C_{ax} \int \left(-\frac{\partial}{\partial \phi_x} e^{-(\phi, A\bar{\phi})} \right) F \; d^{2\Lambda}\phi = \sum_x C_{ax} \int e^{-(\phi, A\bar{\phi})} \frac{\partial F}{\partial \phi_x} \; d^{2\Lambda}\phi$$

■

As in Lecture 5, we have a *Wick's Theorem* with $\exp(\sum C_{xy} \frac{\partial}{\partial \bar{\phi}_x} \frac{\partial}{\partial \phi_y})$, but this Lemma is another form of Wick's Theorem.

Example 6.3.

$$\int d\mu_C \; \bar{\phi}_a \phi_b = C_{ab} \tag{6.8}$$

When Lemma 6.2 is applied to $\int d\mu_C \; \bar{\phi}_{a_1} \dots \bar{\phi}_{a_n} \phi_{b_1} \dots \phi_{b_n}$, the result is a sum over all ways to match each of $a_1, \dots, a_n$ with each of $b_1, \dots, b_n$. These are called perfect matchings and this observation prepares the way to quickly see why the next example is correct.

Example 6.4.

$$I_X = \int d\mu_C \prod_{\substack{x \in X \\ x \neq a,b}} (1 + {:}\phi_x \bar{\phi}_x{:}) \bar{\phi}_a \phi_b = \sum_{G \in \mathcal{G}_{ab}(X)} \prod_{(x,y) \in G} C_{xy} \tag{6.9}$$

where $G \in \mathcal{G}_{ab}(X)$ iff it is a graph on the set of vertices $X \cup \{a, b\}$ such that

(1) every $x \in X \setminus \{a, b\}$ has no edges or exactly one incoming and one outgoing edge and there are no loops.
(2) $x = a$ has one outgoing, $x = b$ one incoming edge.

That means, $G \in \mathcal{G}_{ab}(X)$ iff there exists a self-avoiding walk from a to b and an arbitary number of cycles, all disjoint. Case $a = b$ is also included, provided the terminology "self-avoiding walk from a to b" is understood to mean a cycle through $a = b$ or a loop at a.

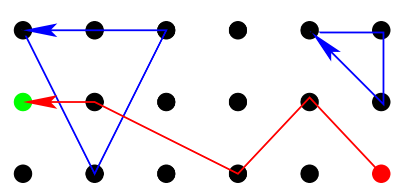

Figure 6.1. Self-avoiding walk and self-avoiding loops.

Proof. For $X \subset \Lambda$, define

$$J_X = \int d\mu_C \prod_{x \in X} (1 + {:}\phi_x \bar{\phi}_x{:}) = \sum_{G \in \mathcal{G}(X)} \prod_{(x,y) \in G} C_{xy} \tag{6.10}$$

where $G \in \mathcal{G}(X)$ iff every $x \in X$ has no edges or exactly one incoming and one outgoing edge and there are no loops. If X is empty we regard the sum as having a single term, the empty graph, for which the contribution on the right hand side is one, because the product under the sum is empty and empty products equal one.

Induction on $|X|$. The inductive hypothesis is that the (6.9) and (6.10) I_X hold if X is replaced by a strictly smaller subset. To initialise the induction: For $X = \varnothing$, the empty product $\prod_{x\in\varnothing}(1+{:}\phi_x\bar\phi_x{:}) = 1$, so $J_X = 1$ and so does the right hand side of (6.10); likewise $I_X = C_{ab}$ and therefore (6.9) also holds.

To prove (6.10): for $y \in X$, by algebra in the second equality and the inductive hypothesis in the third equality,

$$\begin{aligned} J_X &= \int d\mu_C \prod_{x\in X}(1+{:}\phi_x\bar\phi_x{:}) \\ &= \int d\mu_C \prod_{x\in X\setminus\{y\}}(1+{:}\phi_x\bar\phi_x{:}) + \int d\mu_C \prod_{x\in X\setminus\{y\}}(1+{:}\phi_x\bar\phi_x{:}){:}\phi_y\bar\phi_y{:} \\ &= \sum_{G\in\mathcal{G}(X\setminus\{y\})}\prod_{(x,y)\in G} C_{xy} + \sum_{G\in\mathcal{G}_{yy}(X\setminus\{y\})}\prod_{(x,y)\in G}{}' \, C_{xy} \end{aligned}$$

The prime on the product means that $(x,y) \neq (x,x)$. The first term is a sum over all cycles not passing though the vertex y and the second is the sum over all cycles that do contain y. Therefore they combine to give the right hand side of (6.10) and the inductive step is complete for (6.10).

To prove (6.9): By Lemma 6.2,

$$I_X = \sum_{x_1\in X\setminus\{a,b\}} C_{ax_1}\int d\mu_C \prod_{\substack{x\in X\setminus\{a,b\}\\ x\neq x_1}}(1+{:}\phi_x\bar\phi_x{:})\bar\phi_{x_1}\phi_b + C_{ab}\int d\mu_C \prod_{x\in X\setminus\{a,b\}}(1+{:}\phi_x\bar\phi_x{:}).$$

Apply inductive hypothesis to first term to find that it equals the contribution of all graphs in $\mathcal{G}_{a,b}$ which have a self-avoiding walk with two or more steps. The second term is the contribution for all graphs with a one step self-avoiding walk times $J_{X\setminus\{a,b\}}$. According to (6.10), this factor equals the contribution from cycles. ∎

6.2. Differential Forms = Fermions. The symbols

$$(du_x, dv_x : x \in \Lambda) \tag{6.11}$$

generate a finite dimensional algebra Ω^* over the ring of complex-valued function of $\phi_x = u_x + iv_x$, $x \in \Lambda$ via the *wedge product*:

$$du_x \wedge du_y = -du_y \wedge du_x \tag{6.12a}$$

$$du_x \wedge dv_y = -dv_y \wedge du_x \tag{6.12b}$$

$$dv_x \wedge dv_y = -dv_y \wedge dv_x \tag{6.12c}$$

This is a clever idea (Cartan) because we automatically get the Jacobian determinant (without the absolute value sign) when we make a change of variables as in

$$du \wedge dv = \left(\frac{\partial u}{\partial u'}du' + \frac{\partial u}{\partial v'}dv'\right) \wedge \left(\frac{\partial v}{\partial u'}du' + \frac{\partial v}{\partial v'}dv'\right) = \left(\frac{\partial u}{\partial u'}\frac{\partial v}{\partial v'} - \frac{\partial u}{\partial v'}\frac{\partial v}{\partial u'}\right) du' \wedge dv'.$$

This observation extends to higher dimensions. Because $\wedge$ looks like Λ, we will omit $\wedge$. The *degree* of a form is the degree as a polynomial in du_x, dv_y, $x, y \in \Lambda$. Ω^* is called the algebra of *differential forms*.

Example 6.5. Define

$$d\phi_x = du_x + idv_x, \quad d\bar{\phi}_x = du_x - idv_x, \tag{6.13}$$

$$d\bar{\phi}_x d\phi_x = (du_x - idv_x)(du_x + idv_x) = 2idu_x dv_x \tag{6.14}$$

Definition 6.6. The *volume form* on $\mathbb{C}^\Lambda = \mathbb{R}^{2\Lambda}$ is

$$\prod_{x\in\Lambda}(du_x dv_x) = (2i)^{-|\Lambda|}\prod_{x\in\Lambda}(d\bar{\phi}_x d\phi_x). \tag{6.15}$$

This is a top degree ($= 2|\Lambda|$) form. The particular way we have written it removes a sign ambiguity which would result if we did not carefully specify the order in which the du_x, dv_x must be written.

Definition 6.7. For $F \in \Omega^*$, let $f(u,v)\prod_{x\in\Lambda} du_x dv_x$ be the top degree part of F. Define

$$\int F = \int_{\mathbb{R}^{2\Lambda}} f(u,v)\, d^{2\Lambda}\phi. \tag{6.16}$$

(Recall that $d^{2\Lambda}\phi = \prod_{x\in\Lambda} du_x dv_x$ was defined before forms were introduced. It is the Lebesgue measure.)

Notice that $\int F = 0$ if F has zero top degree part.

Example 6.8. Let $N = |\Lambda|$.

$$\left(\sum_{x,y\in\Lambda} A_{xy} d\bar{\phi}_x d\phi_y\right)^N = \sum_{x_1,y_1}\cdots\sum_{x_N,y_N} A_{x_1,y_1}\cdots A_{x_N,y_N}\; d\bar{\phi}_{x_1} d\phi_{y_1}\cdots d\bar{\phi}_{x_N} d\phi_{y_N} \tag{6.17}$$

$$= N!\,(\det A)\prod_{x\in\Lambda} d\bar{\phi}_x d\phi_x \tag{6.18}$$

Example 6.9. Let

$$S = (\phi, A\bar{\phi}) + \frac{1}{2\pi i}\sum_{x,y\in\Lambda} A_{xy} d\phi_x d\bar{\phi}_y. \tag{6.19}$$

Define $e^{-S} \in \Omega^*$ by power series in the form part. Then:

$$\int e^{-S} = \int e^{-(\phi,A\bar{\phi})}\sum_{n\geq 0}\frac{1}{n!}\left(\frac{1}{2\pi i}\sum_{x,y\in\Lambda}(-A_{xy}) d\phi_x d\bar{\phi}_y\right)^n \tag{6.20}$$

$$\overset{\text{Ex. 6.8}}{=} (\det A^t)\pi^{-N}\int e^{-(\phi,A\bar{\phi})}\; d^{2\Lambda}\phi = 1 \tag{6.21}$$

This is self-normalization!

Define $\tau_x \in \Omega^*$ by

$$\tau_x = \phi_x\bar{\phi}_x + \frac{1}{2\pi i} d\phi_x d\bar{\phi}_x \tag{6.22}$$

We claim that for all $X \subset \Lambda$

$$\int e^{-S}\prod_{x\in X}(1+\tau_x) = 1. \tag{6.23}$$

Believing this for now:

Example 6.10 (SAW).

$$\int e^{-S} \prod_{\substack{x\in\Lambda \\ x\neq a,b}} (1+\tau_x)\bar{\phi}_a\phi_b = \sum_{\omega\in \mathrm{SAW}_{ab}(\Lambda)} \prod_{(x,y)\in\omega} C_{xy} \tag{6.24}$$

Sketch of proof.

$$\int e^{-S} \prod_{\substack{x\in\Lambda \\ x\neq a,b}} (1+\tau_x)\bar{\phi}_a\phi_b$$

is a sum of standard integrals in each of which Lemma 6.2 holds: it says that, under each of these integrals we can make the replacement

$$\bar{\phi}_a \to \sum_{x_1} C_{ax_1}\frac{\partial}{\partial\phi_{x_1}}$$

and then $\frac{\partial}{\partial\phi_{x_1}}$ differentiates everything in the integrand except $\exp(-\phi, A\bar{\phi}))d^{2\Lambda}\phi$. Therefore, we can reverse the expansion of the form integral into a sum of standard integrals and find that we have proved that

$$\int e^{-S} \prod_{\substack{x\in\Lambda \\ x\neq a,b}} (1+\tau_x)\bar{\phi}_a\phi_b = \int e^{-S}\sum_{x_1} C_{ax_1}\frac{\partial}{\partial\phi_{x_1}} \prod_{\substack{x\in\Lambda \\ x\neq a,b}} (1+\tau_x)\phi_b \tag{6.25}$$

$$\frac{\partial}{\partial\phi_{x_1}}(1+\tau_z) = \begin{cases} \bar{\phi}_{x_1} & \text{if } z = x_1, \\ 0 & \text{else.} \end{cases}$$

which is a sum of form integrals of the same form as our starting point so we can iterate and by induction get the sum over all self-avoiding walks ω of

$$\prod_{(x,y)\in\omega} C_{xy}\int e^{-S}\prod_{\substack{x\in\Lambda \\ x\notin\omega}}(1+\tau_x) = \prod_{(x,y)\in\omega} C_{xy}$$

by (6.23). ∎

In this example we see an interesting phenomenon. The integral $\int e^{-S}\prod_{\substack{x\in\Lambda \\ x\neq\omega}}(1+\tau_x)$ is a sum over all loops in $\Lambda\setminus\omega$. But supersymmetry (see below) leads to a huge cancellation so that this ends up being exactly one. Taking the view that one direction in the lattice represents time, a loop can be interpreted as the creation of a pair of particles "from the vacuum" followed later by pair annihilation. In this view, the vacuum is a very dynamic system in its own right because of all the creation /annihilation processes. Without supersymmetry the result is $exp(O(\text{volume of } \Lambda\setminus\omega))$. The exponent is the "energy of the vacuum". This leads to difficulties if one tries to include fields that represent gravitational forces because they are generated by energy and so the vacuum can generate large gravitational fields that we do not observe; The energy of the vacuum per unit volume is called the cosmological constant and supersymmetry implies that the cosmological constant is zero.

There is a precise sense in which τ_x is the time a continuous time random walk spends at site x. For more details see [BIS09].

6.3. Supersymmetry. The ideas in in this section are taken from [Wit92]. In this paper, Witten is interested in exact evaluations of the partition function for two dimensional Yang Mills theories on manifolds. His method uses extensions of the Duistermaat-Heckman Theorem, which is itself a far reaching generalisation of Lemma 6.12.

Define $i_X : \Omega^* \to \Omega^*$ by

(1) i_X is an antiderivation;
(2) $i_X(\text{zero form}) = 0$;
(3) $i_X d\phi_x = -2\pi i \phi_x$, $i_X d\bar{\phi}_x = 2\pi i \bar{\phi}_x$.

i_X lowers the degree. Recall that the exterior derivative d is also an antiderivation. Let

$$Q = d + i_X. \tag{6.26}$$

Q is called the *supersymmetry generator*. If $F \in \Omega^*$ and $QF = 0$ we say F is supersymmetric.

Example 6.11. τ_x is supersymmetric:

$$Q\tau_x = d\phi_x\bar{\phi}_x + \phi_x d\bar{\phi}_x + \frac{1}{2\pi i}\left((-2\pi i\phi_x)d\bar{\phi}_x - d\phi_x(2\pi i\bar{\phi}_x)\right) = 0 \tag{6.27}$$

Lemma 6.12 (Localisation). *Let $F \in \Omega^*$ be an even form (only even degree monomials) with smooth coefficients which together with derivatives decay integrably. If $QF = 0$ then*

$$\int F = F(\phi = 0, \bar{\phi} = 0, d\phi = 0, d\bar{\phi} = 0). \tag{6.28}$$

Note that this proves our claim (6.23).

Proof. By Problem 6.2

$$\sum_{x\in\Lambda} \tau_x = Q\omega, \quad \omega = \sum_{x\in\Lambda} \frac{1}{2\pi i}\phi_x d\bar{\phi}_x$$

$$\frac{d}{dt}\int F e^{-t\sum\tau_x} = -\int F(Q\omega)e^{-t\sum\tau_x} = -\int Q(F\omega e^{-t\sum\tau_x})$$
$$= -\int \underbrace{d(\cdots)}_{\text{Stoke's Theorem}} - \int \underbrace{i_X(\cdots)}_{\text{wrong degree}} = 0$$

Therefore

$$\int F = \lim_{t\to\infty}\int F e^{-t\sum\tau_x} = F(\phi = 0, \bar{\phi} = 0, d\phi = 0, d\bar{\phi} = 0).$$

The last step is a homework problem. ∎

Remark 6.13 (Origin of term supersymmetry).

$$Q^2 = (d + i_X)^2 = d^2 + d\circ i_X + i_X \circ d + i_X^2 \tag{6.29}$$

$d^2 = 0$ and i_X is also nilpotent, $i_X^2 = 0$, thus

$$Q^2 = d\circ i_X + i_X\circ d = \mathcal{L}_X, \tag{6.30}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to the vector field X that generates the $U(1)$ action

$$\phi \mapsto \phi e^{-2\pi i\theta}. \tag{6.31}$$

$Q^2 = \mathcal{L}_X$ says that Q is the square root of the $U(1)$ generator.

Problems.

Problem 6.1. Let f be a smooth function defined on $[0, \infty)$ with compact support. Let $\tau = \phi\bar{\phi} + \frac{1}{2\pi i} d\phi d\bar{\phi}$. This is a differential form of mixed degree on $\mathbb{R}^2$. Define a new differential form denoted $f(\tau)$ by the Taylor expansion of f about the point $\phi\bar{\phi}$, as in

$$f(\tau) = f(\phi\bar{\phi}) + f'(\phi\bar{\phi})\frac{1}{2\pi i} d\phi d\bar{\phi} \tag{6.32}$$

Prove, by direct calculation with polar coordinates, that

$$\int_{\mathbb{R}^2} f(\tau) = f(0)$$

Problem 6.2.

$$\tau_x = Q\left(\frac{1}{2\pi i}\phi_x d\bar{\phi}_x\right) \tag{6.33}$$

Problem 6.3. Justify the last step in the proof of Lemma 6.12.

Problem 6.4. Why are there no Wick powers in Example 6.10 whereas there are in Example 6.4?

Problem 6.5. Fix once and for all a square root $(2\pi i)^{-1/2}$ and define

$$\psi_x = (2\pi i)^{-1/2} d\phi_x \qquad \bar{\psi}_x = (2\pi i)^{-1/2} d\bar{\phi}_x. \tag{6.34}$$

Define differentiation with respect to ψ and $\bar{\psi}$ by specifying the derivatives on monomials in ψ and $\bar{\psi}$ and show that

$$Q = (2\pi i)^{1/2}\left(\psi_x \frac{\partial}{\partial \phi_x} + \bar{\psi}_x \frac{\partial}{\partial \bar{\phi}_x} - \phi_x \frac{\partial}{\partial \psi_x} + \bar{\phi}_x \frac{\partial}{\partial \bar{\psi}_x}\right) \tag{6.35}$$

Problem 6.6. Let A be a symmetric matrix. Define

$$B_x = \sum_y A_{xy}. \tag{6.36}$$

For $R \subset \Lambda$, let

$$B^R = \prod_{x \in R} B_x. \tag{6.37}$$

For a graph F, let

$$(-A)^F = \prod_{x,y \in E(F)} (-A_{xy}). \tag{6.38}$$

The matrix tree theorem says

$$\det A = \sum_{(F,R)} (-A)^F B^R, \tag{6.39}$$

where F is summed over all graphs on Λ which have no cycles and for each F, R is summed over all ways to choose one root in each connected component of F.

Prove the matrix tree theorem by starting with $\int e^{-S} = 1$. Write

$$\sum_{x,y} A_{xy} d\phi_x d\bar{\phi}_y = -\frac{1}{2}\sum_{x,y} A_{xy}(d\phi_x - d\phi_y)(d\bar{\phi}_x - d\bar{\phi}_y) + \sum_x B_x d\phi_x d\bar{\phi}_x. \tag{6.40}$$

Write $\phi_{xy} = \phi_x - \phi_y$, $d\phi_{xy} = d\phi_x - d\phi_y$ and expand $e^{\sum A d\phi d\bar{\phi}}$ in powers of $d\phi_{xy} d\bar{\phi}_{xy}$; likewise $e^{\sum B d\phi d\bar{\phi}}$ in terms of $d\phi_x d\bar{\phi}_x$. Argue that the terms in this expansion are naturally labelled by pairs $(F, R) = (\text{forest}, \text{root})$.

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Lecture 7. Infrared Bounds and Broken Symmetry

So far we have encountered different models and relations between them. All these models boil down to integrals of the form

$$Z = \int e^{-\alpha S(\phi)}\, d^\Lambda \phi \tag{7.1}$$

and the associated measure

$$\frac{1}{Z} e^{-\alpha S(\phi)}\, d^\Lambda \phi. \tag{7.2}$$

Then, there is the idea of mean field theory. When $\alpha \gg 1$, the measure concentrates onto the minima of $S(\phi)$. In our discussion, we have also encountered the enemy of this idea, which is that in the infinite volume limit, fluctuations around the minima may cause the model to forget which minimum was selected by the boundary condition. We have seen that the massless Gaussian in $\mathbb{Z}^2$ forgets the Dirichlet boundary condition at ∞, but in $\mathbb{Z}^d$, $d \geq 3$, this does not happen. Fluctuations around the minima are modeled by Gaussians because at the minimum, ϕ_0,

$$S(\phi) \approx S(\phi_0) + \frac{1}{2}(\phi - \phi_0) S''(\phi - \phi_0) \tag{7.3}$$

but $\approx$ involves non-Gaussian $O(\phi - \phi_0)^3$ corrections. Are we still able to use Gaussian intuition? In this lecture we see a proof of the existence of phase transitions that relies on 'Gaussian bounds' that captures this intuition.

Consider models of the form

$$Z = \int \prod_{x \in \Lambda} d\rho(\phi_x)\, e^{-\frac{1}{2}(\phi, -\Delta_\Lambda \phi)} \tag{7.4}$$

where $\phi : \Lambda \to \mathbb{R}^N$ is vector valued and $d\rho$ is $O(N)$ invariant (invariant under the action of the orthogonal group of N by N matrices). Suppose that Λ is a torus, i.e. it has periodic boundary conditions. Then,

$$(\phi, -\Delta_\Lambda \phi) = \sum_{xy \in \mathrm{Edges}(\Lambda)} \|x - y\|^2, \tag{7.5}$$

where $\mathrm{Edges}(\Lambda)$ represents the edges such that y is a nearest neighbor to x if $y = x + e \bmod(\mathrm{side}(\Lambda))$ and $\|e\| = 1$.

Example 7.1. Take $N = 1$ and

$$d\rho(\phi_x) = \delta(\phi_x + \sqrt{\beta}) + \delta(\phi_x - \sqrt{\beta}). \tag{7.6}$$

This gives the Ising model with the temperature given by β^{-1}.

Example 7.2. If $N > 1$, $d\rho$ is the surface Lebesgue measure on a sphere of radius $\sqrt{\beta}$. This is called the N-vector model or $O(N)$ model.

The joint distributions of $\phi = \{\phi_x : x \in \Lambda\}$ are $O(N)$ invariant, which means

$$\mathbb{P}_\Lambda\{\phi \in E\} = \mathbb{P}_\Lambda\{\phi \in RE\} \tag{7.7}$$

for $R \in O(N)$, and are translation invariant, which means

$$\mathbb{P}_\Lambda\{(\phi_x)_{x \in X} \in E\} = \mathbb{P}_\Lambda\{(\phi_x)_{x-a \in X} \in E\} \tag{7.8}$$

for $a \in \mathbb{Z}^d$.

Therefore any infinite volume limit $\mathbb{P}_\infty$ also has these properties. Let $\langle\cdot\rangle \equiv \langle\cdot\rangle_\infty$ be the expectation for the infinite volume limit.

Theorem 7.3 (Fröhlich-Simon-Spencer 1976 [FSS76]). *For $d \geq 3$, $\beta \gg 1$, there exists $c(\beta) > 0$ such that*
$$\lim_{y\to\infty} \langle \phi_x \cdot \phi_y\rangle = c(\beta).$$

Corollary 7.4. *The tail σ-algebra $\mathcal{T}$ is non-trivial.*

Proof. By the ergodic theorem,
$$Y := \lim_X \frac{1}{|X|}\sum_{x\in X}\phi_x \tag{7.9}$$
exists $\mathbb{P}_\infty$-a.s. and defines a tail measurable random variable $Y \in m\mathcal{T}$. It is not almost surely constant because
$$\mathrm{Var}(Y) = \langle Y^2\rangle_\infty - \langle Y\rangle_\infty^2 = \langle Y^2\rangle_\infty$$
by the $O(N)$ symmetry. This means that using the dominated convergence theorem gives
$$\mathrm{Var}(Y) = \lim_X \frac{1}{|X|^2}\sum_{x,y\in X}\langle\phi_x\cdot\phi_y\rangle_\infty = c(\beta) > 0. \tag{7.10}$$
Therefore $\{Y \in E\}$ is a non-trivial event in $\mathcal{T}$. ■

The *high temperature expansion* (not discussed in this course) proves that $\mathcal{T}$ is trivial for $\beta \ll 1$, so there exists β_c, a critical β, where the phase transition takes place.

Physically speaking, for $\beta > \beta_c$ there is long range order. This means that a boundary condition that selects a preferred direction for ϕ will be 'remembered' by ϕ_0 no matter how far away the boundary is. This is called *broken $O(N)$ symmetry.* For $\beta < \beta_c$, the boundary condition is not remembered; all correlations decay exponentially.

For the Ising model ($N = 1$), the hypothesis $d \geq 3$ is misleading in the sense that there is also a phase transition in $d = 2$. This is proved by the *Peierls argument*, which is not discussed in this course either.

7.1. Infrared bound. The difficult step in proving Theorem 7.3 is the following proposition, whose proof is deferred to later in this lecture.

Proposition 7.5 (Infrared bound). *For $f : \Lambda \to \mathbb{R}^N$ such that f is perpendicular to all constant fields,*
$$\langle (\phi, f)\,(\phi, f)\rangle_\Lambda \leq \left(f, (-\Delta_\Lambda)^{-1} f\right). \tag{7.11}$$

In previous lectures we were using the Laplacian with Dirichlet boundary conditions whose eigenvalues are positive and which is invertible. By (7.5), $(\phi, -\Delta_\Lambda\phi) = 0$ when ϕ is a constant field, so the Laplacian with periodic boundary conditions has zero eigenvalues and is not invertible. However the kernel of this Laplacian is exactly the subspace of constant fields and so it is invertible on the orthogonal complement of the constant fields and this is the reason for the hypothesis on f.

Proof of Theorem 7.3. Let $|\Lambda| \to \infty$. For f with compact support, $f \perp$ constant fields,
$$\langle (\phi, f)\,(\phi, f)\rangle_\infty \leq \int |\hat{f}(k)|^2 \frac{1}{\sum_{x:\|x\|=1}(1 - e^{ik\cdot x})} dk$$

where

$$\int dk = \int_{[-\pi,\pi]^d} dk \tag{7.12}$$

As f, defined by $f(x-y) = \langle \phi_x \cdot \phi_y \rangle_\infty$, is a positive-definite function, by Bochner's theorem [RS75, Theorem IX.9], there exists a positive measure $dw(k)$ such that

$$\langle \phi_x \cdot \phi_y \rangle_\infty = \int e^{ik\cdot(x-y)}\, dw(k). \tag{7.13}$$

In terms of dw,

$$\int |\hat{f}(k)|^2\, dw(k) \leq \int |\hat{f}(k)|^2 \frac{1}{\sum_{x:\|x\|=1}(1-e^{ik\cdot x})}\, dk. \tag{7.14}$$

The hypothesis that f is perpendicular to constant fields is the same as

$$\hat{f}(0) = 0 \tag{7.15}$$

which means that no admissible choice of f in (7.14) can detect whether $dw(k)$ has a point mass at $k=0$. However, we can choose test functions f in (7.14) such that $\hat{f}(k)$ is highly concentrated near specific points $k \neq 0$ and so deduce from (7.14) that (Problem 7.3)

$$dw(k) = c\delta(dk) + g(k)\, dk \tag{7.16}$$

where c is some constant, which could be zero, and $g(k) \geq 0$ with

$$g(k) \leq \frac{1}{\sum_{x:\|x\|=1}(1-e^{ik\cdot x})}.$$

For $d \geq 3$, this bound shows that g is integrable because, as in lecture 4,

$$\sum_{x:\|x\|=1}(1-e^{ik\cdot x}) = \|k\|^2 + o(\|k\|^2).$$

This implies that

$$\int g(k)\, dk \leq \text{const.} \tag{7.17}$$

The Riemann-Lebesgue Lemma implies that $g(x-y) \to 0$ as $y \to \infty$. Therefore,

$$\langle \phi_x \cdot \phi_y \rangle_\infty \to c \quad \text{as } y \to \infty.$$

To prove that $c > 0$, as $d\rho$ is surface measure on the sphere of radius $\sqrt{\beta}$, then

$$\langle \phi_x \cdot \phi_x \rangle_\infty = \beta.$$

Setting $x = y$ in (7.13) gives

$$\int dw(k) = \beta.$$

Integrating both sides of 7.16 gives

$$\beta = c + \int g(k)\, dk$$

but by (7.17), $\int g(k)\, dk$ is $O(\beta^0)$. This implies that as $\beta \to \infty$, then $c \to \infty$ which implies that $c > 0$ for $\beta \gg 1$. ■

Proposition 7.6. *Let*

$$Z(h) = \int \prod_{x\in\Lambda} d\rho(\phi_x - h_x)\, e^{-\frac{1}{2}(\phi,-\Delta_\Lambda\phi)} \tag{7.18}$$

where $h : \Lambda \to \mathbb{R}^N$. *Then,*

$$Z(h) \le Z(0) = Z. \tag{7.19}$$

The proposition yields the following corollary:

Corollary 7.7. *For f perpendicular to constant fields,*

(1)

$$\left\langle e^{-(f,\phi)}\right\rangle_\Lambda \le e^{\frac{1}{2}\left(f,(-\Delta_\Lambda)^{-1}f\right)}. \tag{7.20}$$

(2) *Proposition 7.5 holds.*

Proof of Corollary 7.7. We first prove that (1) implies (2). We replace f by tf, substract 1 from both sides and divide both sides by t^2. This gives

$$\frac{1}{t^2}\left\langle e^{-(\phi,tf)} - 1\right\rangle \le \frac{1}{t^2}\left(e^{\frac{1}{2}t^2\left(f,(-\Delta_\Lambda)^{-1}f\right)} - 1\right).$$

By $O(N)$ invariance, $\langle(\phi, f)\rangle = 0$. By the Taylor expansion in t and the limit $t \downarrow 0$, we obtain the infrared bound Proposition 7.5.

For the proof of (1), consider

$$\left\langle e^{-(\phi,f)}\right\rangle_\Lambda = \frac{1}{Z}\int \prod_{x\in\Lambda} d\rho(\phi_x)\, e^{-\frac{1}{2}(\phi,-\Delta\phi)}e^{-(\phi,f)}.$$

The idea is to evaluate the integral as if it were Gaussian. Thus we complete the square in the exponent by making a change of variables, $\phi_x = \phi'_x + h_x$. We choose h to eliminate terms which are linear in ϕ' and find that

$$\left\langle e^{-\langle\phi,f\rangle}\right\rangle_\Lambda = \frac{Z(-h)}{Z} e^{\frac{1}{2}\left(f,(-\Delta)^{-1}f\right)}.$$

But $Z(-h)/Z \le 1$ by Proposition 7.6. ■

7.2. Reflection Positivity. This is a separate and interesting topic which we need to prove Proposition 7.6. Suppose that Λ is a subset of $\mathbb{Z}^d$ which is invariant under a reflection about a hyperplane that divides Λ into Λ_+ on one side of the hyperplane and Λ_- on the other side. Thus

$$\Theta : \Lambda \to \Lambda \tag{7.21}$$

and $\Theta\Lambda_+ = \Lambda_-$ and $\Theta\Lambda_- = \Lambda_+$.

Example 7.8. The hyperplane is the point $x = 0$ in $\mathbb{Z}$. Then:

$$\Theta e^{\phi_1+\phi_2} = e^{\phi_{-1}+\phi_{-2}} \tag{7.22}$$

Definition 7.9. $\langle\cdot\rangle$ satisfies Osterwalder-Schrader positivity if

$$\langle(\Theta F)F\rangle \ge 0 \quad \text{for all } F \in \mathcal{F}_{\Lambda_+}. \tag{7.23}$$

Theorem 7.10. *Nearest neighbour ferromagnetic models are Osterwalder-Schrader.*

Proof. See [FSS76]. ■

Sketch of proof of Proposition 7.6. We have the following Cauchy-Schwarz inequality

$$\langle \Theta(F) G \rangle \leq \langle \Theta(F) F \rangle^{\frac{1}{2}} \langle \Theta(G) G \rangle^{\frac{1}{2}}$$

because $\langle \Theta(F) G \rangle$ defines an inner product (F, G) by using the Osterwalder-Schrader positivity. To understand the idea consider a periodic 2×2 lattice Λ. Then Λ has four points. To each point $x \in \Lambda$ there is a component h_x in $h = (h_x)_{x \in \Lambda}$ and these components are, in general, not equal to the same vector. The following pictorial representation describes a sequence of Cauchy-Schwarz inequalities applied to $Z(h)$ and in the pictures the different h_x are symbolised by the diamond, the heart, the club, and the spade. Each Cauchy inequality uses a reflection about a different hyperplane, but we have reflection positivity about all these hyperplanes because the torus Λ is translation invariant and invariant under rotation by $\pi/2$.

$$\begin{aligned} Z(h) = \begin{pmatrix} \diamondsuit & \heartsuit \\ \clubsuit & \spadesuit \end{pmatrix} &\leq \begin{pmatrix} \diamondsuit & \diamondsuit \\ \clubsuit & \clubsuit \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} \heartsuit & \heartsuit \\ \spadesuit & \spadesuit \end{pmatrix}^{\frac{1}{2}} \\ &\leq \begin{pmatrix} \diamondsuit & \diamondsuit \\ \diamondsuit & \diamondsuit \end{pmatrix}^{\frac{1}{4}} \begin{pmatrix} \clubsuit & \clubsuit \\ \clubsuit & \clubsuit \end{pmatrix}^{\frac{1}{4}} \begin{pmatrix} \heartsuit & \heartsuit \\ \heartsuit & \heartsuit \end{pmatrix}^{\frac{1}{4}} \begin{pmatrix} \spadesuit & \spadesuit \\ \spadesuit & \spadesuit \end{pmatrix}^{\frac{1}{4}} \end{aligned}$$

Therefore $Z(h) \leq \prod_{\text{constants}} Z(\text{constants})^{\frac{1}{4}}$. On the right hand side of this inequality we undo the translation by writing $\phi = \phi' - \text{const}$ and noting that

$$e^{-\frac{1}{2}(\phi, -\Delta\phi)} = e^{-\frac{1}{2}(\phi', -\Delta\phi')}$$

so that

$$\prod_{\text{constants}} Z(\text{constants})^{\frac{1}{4}} = \prod_{\text{constants}} Z(0)^{\frac{1}{4}} = Z.$$

■

Discussion. This is a very unstable method of proof.

(1) Add next-to-nearest-neighbour ferromagnetic interactions: This ruins Osterwalder-Schrader positivity and therefore the proof, but our intuition says it must strengthen trends towards order.
(2) The Fermions $d\phi$, $d\overline{\phi}$ ruin Osterwalder-Schrader positivity so we cannot prove the existence of collapsed phases of self-interacting walks by Osterwalder-Schrader positivity.

Proving the existence of phase transitions in systems with $O(N)$ symmetry, $N > 1$, is almost unimaginably hard by cluster expansions. Osterwalder-Schrader positivity is essentially the only reasonable technique we have (there are duality transformations for $N = 2$).

Open Problem. The quantum anti-ferromagnetic satisfies Osterwalder-Schrader positivity so we can prove there exists phase transitions. The quantum ferromagnetic does not satisfy Osterwalder-Schrader positivity. We can't prove there exists a phase transition.

Problems.

Problem 7.1. Why is the function f which is defined just below (7.12) positive-definite?

Problem 7.2. Justify (7.10).

Problem 7.3. Fill in the details in the passage from (7.14) to (7.16).

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Lecture 8. The Newman Central Limit Theorem

The result in this lecture is a model for the type of result that the rest of this course will be elaborating on. It is a very sophisticated central limit theorem that characterises the long distance structure of fluctuations in a class of statistical mechanical models (*ferromagnetic* models) which are not *critical*. The term critical will be defined later.

8.1. FKG systems.

Definition 8.1. We say that a function $F\colon \mathbb{R}^n \to \mathbb{R}$ is *increasing*, if $F(x) \leq F(y)$ for all $x, y \in \mathbb{R}^n$ such that $x_i \leq y_i$ for $i = 1, \dots, n$.

Definition 8.2. A finite set $X = \{X_1, \dots, X_n\}$ of random variables is *FKG* (Fortuin-Kasteleyn-Ginibre) if

$$\operatorname{Cov}(F(X), G(X)) \geq 0 \tag{8.1}$$

for all increasing functions F and G. An infinite set of random variables is FKG if every finite subset is FKG.

Note that all increasing functions of FKG random variables are themselves FKG random variables. Note also that for an FGK system $X = \{X_1, \dots, X_n\}$ we have $\operatorname{Cov}(X_i, X_j) \geq 0$ for all $i, j = 1, \dots, n$ because each X_k (viewed as a function $\mathbb{R}^n \to \mathbb{R}$) is increasing, $k = 1, \dots, n$.

Theorem 8.3 ([FKG71], in this form proved in [BR80]). *All ferromagnetic systems, that is, the systems of the form*

$$d\mu(x) = \frac{1}{Z}\, d^n x\, e^{F(x)}, \tag{8.2}$$

where $F(x)$ is such that

$$\frac{\partial^2 F}{\partial x_i \partial x_j} \geq 0, \qquad 1 \leq i \neq j \leq n, \tag{8.3}$$

are FKG.

Proof. [BR80, Theorem 1.1], see Problem 8.1. ■

Definition 8.4 (Block spins). For $x \in \mathbb{Z}^d$, $L \in \mathbb{N}$, set

$$\phi_L(x) = |B(x)|^{-1/2} \sum_{y \in B(x)} \left(\phi_y - \langle \phi_y \rangle\right), \tag{8.4}$$

where $B(x) \in \mathcal{B}_L$ is the block of size L centered on the point Lx.

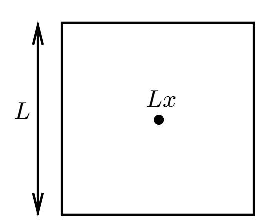

Figure 8.1. The block $B(x) \in \mathcal{B}_L$.

8.2. Formulation of Newman CLT. We assume that there is a system of random variables $\{\phi_x : x \in \mathbb{Z}^d\}$ indexed by the points of the lattice (a so-called *random field*) such that

(1) The probability law of $\{\phi_x\}$ is $\mathbb{Z}^d$ translation invariant;
(2) $\langle \phi_x^2 \rangle < \infty$ for some $x \in \mathbb{Z}^d$ (and hence for all $x \in \mathbb{Z}^d$);
(3) $\sum_{y \in \mathbb{Z}^d} \mathrm{Cov}(\phi_x, \phi_y) < \infty$ (this means that the model is *not critical*);
(4) The system $\{\phi_x : x \in \mathbb{Z}^d\}$ is FKG.

Theorem 8.5 ([New80])**.** *Under these assumptions we have*

$$\left\{\phi_L(x) : x \in \mathbb{Z}^d\right\} \Rightarrow \textit{i.i.d. Gaussian}, \qquad L \to \infty. \tag{8.5}$$

Without loss of generality in the folowing we assume that

$$\langle \phi_x \rangle = 0, \qquad \sum_{y \in \mathbb{Z}^d} \mathrm{Cov}(\phi_x, \phi_y) = 1. \tag{8.6}$$

For an arbitrary subset $X \subset \mathbb{Z}^d$ define

$$\phi(X) := |X|^{-1/2} \sum_{x \in X} \phi_x. \tag{8.7}$$

Denote

$$f_L(r) = \left\langle e^{ir\phi(B)} \right\rangle, \qquad B \in \mathcal{B}_L \tag{8.8}$$

(we can take any $B \in \mathcal{B}_L$ because $\{\phi_x\}$ is translation invariant).

8.3. Important properties of FKG systems.

Lemma 8.6. *If two random variables X and Y are FKG, and $f, g \in C^1$, then*

$$\mathrm{Cov}(f(X), g(Y)) \le \|f'\|_\infty \|g'\|_\infty \mathrm{Cov}(X, Y). \tag{8.9}$$

Proof. In the proof we assume that $f(s), g(s) \to 0$ as $s \to -\infty$ (Problem 8.2).
We have

$$\mathbb{E} f(X) = \int \mathbb{P}\{X > s\} f'(s) ds. \tag{8.10}$$

Indeed, insert $f(X) = \int_{s<X} f'(s) ds$ into the expectation $\mathbb{E} f(X)$ and switch $\mathbb{E}$ and $\int$.
Similarly, we obtain

$$\mathrm{Cov}(f(X), g(Y)) = \iint \big(\mathbb{P}\{X > s, Y > t\} - \mathbb{P}\{X > s\}\mathbb{P}\{Y > t\}\big) f'(s) g'(t)\, ds\, dt. \tag{8.11}$$

In (8.11) we no longer need the assumption $f(-\infty) = g(-\infty) = 0$.
Write

$$\mathbb{P}\{X > s, Y > y\} - \mathbb{P}\{X > s\}\mathbb{P}\{Y > t\} = \mathrm{Cov}(\mathbb{1}_{\{X>s\}} \mathbb{1}_{\{Y>t\}}) \ge 0, \tag{8.12}$$

because the indicator functions are increasing.
Now, using (8.12) we can take out $\|f'\|_\infty \|g'\|_\infty$ from (8.11) and write

$$\begin{aligned} &\mathrm{Cov}(f(X), g(Y)) \\ &\quad\le \|f'\|_\infty \|g'\|_\infty \iint \big(\mathbb{P}\{X > s, Y > t\} - \mathbb{P}\{X > s\}\mathbb{P}\{Y > t\}\big)\, ds\, dt \\ &\quad= \|f'\|_\infty \|g'\|_\infty \,\mathrm{Cov}(X, Y), \end{aligned} \tag{8.13}$$

The latter equality holds by choosing $f(s) = s$ and $g(t) = t$ in (8.11). This concludes the proof. ■

Proposition 8.7. *If* $\{X_j,\ j = 1, 2, \ldots, n\}$ *are FKG, then for all* $r_j \in \mathbb{R}$ *we have*

$$\left| \left\langle e^{i\sum_{j=1}^n r_j X_j} \right\rangle - \prod_{j=1}^n \left\langle e^{ir_j X_j} \right\rangle \right| \le \frac{1}{2} \sum_{1\le k\ne l\le n} \mathrm{Cov}(X_k, X_l)|r_k r_l|. \tag{8.14}$$

Remark 8.8. This Proposition implies that if $\mathrm{Cov}(X_k, X_l) = 0$ for all $1 \le k \ne l \le n$, then the variables $\{X_j\}$ are independent. This property of FKG systems is similar to that of the Gaussian systems.

Proof. By induction on n. Lemma 8.6 starts the induction at $n = 2$, and Lemma 8.6 also accomplishes the induction step. For details, see [New80]. ■

Lemma 8.9. *If* $g(r)$ *is* C^2 *at* $r = 0$ *(this means that* $g(r)$ *is doubly differentiable in some neighborhood of* 0 *and that* $g''(r)$ *is continuous at* $r = 0$*),* $g(0) = 1$ *and* $g'(0) = 0$*, then*

$$\lim_{n\to\infty} \left(g\left(\frac{r}{\sqrt{n}}\right)\right)^n = e^{g''(0)\frac{r^2}{2}}. \tag{8.15}$$

Proof. This can be proved using Taylor expansion (Problem 8.3). ■

8.4. Idea of the proof of Newman CLT. It suffices to prove that $f_L(r) \to e^{-\frac{1}{2}r^2}$ and that the variables $\left\{\phi(B(x))\colon x \in \mathbb{Z}^d\right\}$ become independent as $L \to \infty$.[4] We proceed by steps.

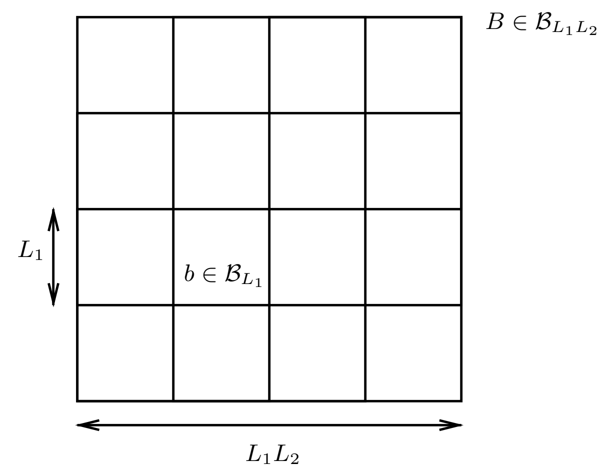

Figure 8.2.

Step 1. For $L_1 \gg 1$, all the pairs (x, y) such that $x \in b$, $y \in b' \ne b$ make negligible contribution to

$$\frac{1}{|B|} \sum_{x,y\in B} \mathrm{Cov}(\phi_x, \phi_y), \tag{8.16}$$

and this is uniform in L_2. Here $b, b' \in \mathcal{B}_{L_1}$ and $B \in \mathcal{B}_{L_1 L_2}$, see Figure 8.2.

[4]Here $B(x) \in \mathcal{B}_L$ is the block of size L centered on Lx, and $\phi(B(x))$ and $f_L(r)$ are defined in (8.7)–(8.8).

Step 2. Note that

$$\phi(B) = \sqrt{\frac{|b|}{|B|}} \sum_{b \in \mathcal{B}_{L_1}(B)} \phi(b). \tag{8.17}$$

Proposition 8.7 implies that

$$\left| \left\langle e^{ir\phi(B)} \right\rangle - \left(\left\langle e^{ir\frac{1}{\sqrt{n}}\phi(b)} \right\rangle \right)^n \right| \le \epsilon(L_1), \tag{8.18}$$

where $n = \frac{|B|}{|b|}$, and this estimate is uniform in n.

Now, as $L_2 \to \infty$ and $n \to \infty$, by Lemma 8.9 we get

$$\left(\left\langle e^{ir\frac{1}{\sqrt{n}}\phi(b)} \right\rangle \right)^n \to e^{-\frac{r^2}{2}\operatorname{Var}\phi(b)}. \tag{8.19}$$

Step 3. Combining Lemma 8.9 with Step 2 we get

$$\lim_{L_1\to\infty} \limsup_{L_2\to\infty} \left| f_{L_1 L_2}(r) - e^{-\frac{1}{2}r^2} \right| = 0. \tag{8.20}$$

Thus, we have a subsequence $L^{(k)}$, $k = 1, 2, \ldots$ such that

$$\lim_{k\to\infty} \left| f_{L^{(k)}}(r) - e^{-\frac{1}{2}r^2} \right| = 0. \tag{8.21}$$

Step 4. Finally, we prove

$$\lim_{L\to\infty} \left| f_L(r) - e^{-\frac{1}{2}r^2} \right| = 0. \tag{8.22}$$

8.5. The proof of Newman CLT. Here we prove Theorem 8.5 by steps that are indicated above.

8.5.1. *Step 1.*

Lemma 8.10. *For any block $B \in \mathcal{B}_L$ we have*

$$\lim_{L\to\infty} \sum_{B' \in \mathcal{B}_L(B^c)} \operatorname{Cov}(\phi(B), \phi(B')) = 0. \tag{8.23}$$

Here B^c denotes the complement $\mathbb{Z}^d \setminus B$.

Proof. Fix arbitrary $\epsilon > 0$ and let l be such that

$$\sum_{y\in\mathbb{Z}^d,\ \|x-y\|\ge l} \operatorname{Cov}(\phi_x, \phi_y) < \epsilon. \tag{8.24}$$

This can be done because the whole sum $\sum_{y\in\mathbb{Z}^d} \operatorname{Cov}(\phi_x, \phi_y)$ equals one, see (8.6).

Let

$$B^\circ := \{x \in B \colon \operatorname{dist}(x, B^c) \ge l\}, \tag{8.25}$$

see Figure 8.3.

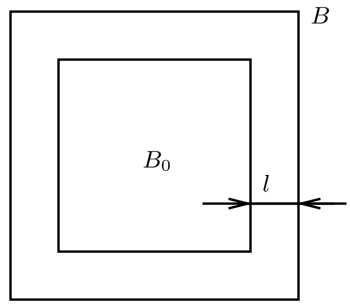

Figure 8.3.

We have

$$
\begin{aligned}
\sum_{B' \in \mathcal{B}(B^c)} \operatorname{Cov}\left(\phi(B), \phi(B')\right) &= \frac{1}{|B|} \sum_{x \in B} \sum_{y \notin B} \operatorname{Cov}(\phi_x, \phi_y) \\
&= \frac{1}{|B|} \sum_{x \in B} \sum_{y \notin B} \operatorname{Cov}(\phi_x, \phi_y) \mathbf{1}_{\|x-y\|<l} + \\
&\qquad + \frac{1}{|B|} \sum_{x \in B} \sum_{y \notin B} \operatorname{Cov}(\phi_x, \phi_y) \mathbf{1}_{\|x-y\| \geq l} \\
&\leq \frac{1}{|B|} \sum_{x \in B \setminus B^0} \sum_{y \in \mathbb{Z}^d} \operatorname{Cov}(\phi_x, \phi_y) + \\
&\qquad + \frac{1}{|B|} \sum_{x \in B} \sum_{y \notin B} \operatorname{Cov}(\phi_x, \phi_y) \mathbf{1}_{\|x-y\| \geq l} \\
&\leq \frac{|B \setminus B^0|}{|B|} + \frac{1}{|B|} \sum_{x \in B} \epsilon \\
&\leq 2\epsilon
\end{aligned} \tag{8.26}
$$

for all large L because $\lim_{L \to \infty} \frac{|B \setminus B^0|}{|B|} = 0$. ■

Lemma 8.11. $\lim_{L \to \infty} f_L''(0) = -1$.

Proof. This can be proved exactly as Lemma 8.10 (Problem 8.4). ■

8.5.2. *Step 2.*

Lemma 8.12.

$$
\lim_{L_1 \to \infty} \limsup_{L_2 \to \infty} \left| f_{L_1 L_2}(r) - \left(f_{L_1}\left(\frac{r}{L_2^{d/2}} \right) \right)^{L_2^d} \right| = 0. \tag{8.27}
$$

Proof. We have

$$
\phi(B) = L_2^{-d/2} \sum_{b \in \mathcal{B}_{L_1}(B)} \phi(b). \tag{8.28}
$$

By Proposition 8.7, for all L_2,

$$\begin{aligned}\left|f_{L_1L_2}(r) - f_{L_1}\left(\frac{r}{L_2^{d/2}}\right)^{L_2^d}\right| &\leq \frac{1}{2}L_2^{-d}\sum_{b\in\mathcal{B}_{L_1}(B)}\sum_{b'\in\mathcal{B}_{L_1}(b^c)}\mathrm{Cov}(\phi(b),\phi(b'))r^2 \\ &\leq \frac{1}{2}r^2\sum_{b'\in\mathcal{B}_{L_1}(b^c)}\mathrm{Cov}(\phi(b),\phi(b'))\end{aligned} \tag{8.29}$$

for any $b \in \mathcal{B}_{L_1}(B)$. By Lemma 8.10, the above sum tends to zero as $L_1 \to \infty$ uniformly in L_2. This concludes the proof. ■

8.5.3. *Step 3.*

Lemma 8.13.

$$\lim_{L_1\to\infty}\limsup_{L_2\to\infty}\left|f_{L_1L_2}(r) - e^{-\frac{1}{2}r^2}\right| = 0. \tag{8.30}$$

Proof. We have

$$\begin{aligned}&\left|f_{L_1L_2}(r) - e^{-\frac{1}{2}r^2}\right| \\ &\leq \left|f_{L_1L_2}(r) - \left(f_{L_1}\left(\frac{r}{L_2^{d/2}}\right)\right)^{L_2^d}\right| + \left|\left(f_{L_1}\left(\frac{r}{L_2^{d/2}}\right)\right)^{L_2^d} - e^{-\frac{1}{2}r^2}\right|\end{aligned} \tag{8.31}$$

First taking $\limsup_{L_2\to\infty}$, and then $\lim_{L_1\to\infty}$, we conclude that the first summand becomes zero by Lemma 8.12, and the second summand becomes zero by Lemma 8.9. ■

8.5.4. *Step 4.*

Lemma 8.14. *For $L_1, L \in \mathbb{N}$ define*

$$L_2 := \left\lfloor \frac{L}{L_1}\right\rfloor. \tag{8.32}$$

Then

$$\lim_{L\to\infty}(f_L(r) - f_{L_1L_2}(r)) = 0. \tag{8.33}$$

Proof. The number L_2 is defined such that

$$L_1L_2 < L < L_1L_2 + L_1. \tag{8.34}$$

Let

$$B \in \mathcal{B}_L, \qquad \widetilde{B} \in \mathcal{B}_{L_1L_2} \tag{8.35}$$

be the blocks centered on $x = 0$. Then

$$B = \widetilde{B} \cup X \tag{8.36}$$

for some $X \subset \mathbb{Z}^d$, and

$$\frac{|X|}{|\widetilde{B}|} \leq \frac{L^d - (L_1L_2)^d}{(L_1L_2)^d} \leq \frac{(L_1L_2 + L_1)^d - (L_1L_2)^d}{(L_1L_2)^d} = O\left(\frac{1}{L_2}\right). \tag{8.37}$$

This implies that X becomes negligible relative to $\widetilde{B}$.

Proposition 8.7 and Lemma 8.9 imply (Problem 8.5) that

$$\left\langle e^{ir\phi(B)}\right\rangle - \left\langle e^{ir\phi(\widetilde{B})}\right\rangle \to 0, \qquad L \to \infty. \tag{8.38}$$

This concludes the proof. ∎

Proof of Theorem 8.5. Writing

$$\left|f_L(r) - e^{-\frac{1}{2}r^2}\right| \le |f_L(r) - f_{L_1L_2}(r)| + \left|f_{L_1L_2}(r) - e^{-\frac{1}{2}r^2}\right| \tag{8.39}$$

and noting that the first summand goes to zero as $L \to \infty$ by Lemma 8.14, and the second summand is less than $\epsilon(L_1)$ (which in turn holds for all $\epsilon(L_1)$ because L_1 is arbitrary), we conclude that

$$\lim_{L\to\infty} f_L(r) = e^{-\frac{1}{2}r^2}. \tag{8.40}$$

This finally implies Theorem 8.5. ∎

Remark 8.15. There is a good book on limit theorems for FKG and related systems [BS07].

Problems.

Problem 8.1. Look up and be prepared to present the proof of Theorem 8.3 (a version of the FKG inequalities) in [BR80].

Problem 8.2. In the proof of Lemma 8.6 explain why the conditions $f(-\infty) = g(-\infty) = 0$ were dropped.

Problem 8.3. Prove Lemma 8.9.

Problem 8.4. Prove Lemma 8.11.

Problem 8.5. Complete the proof of Lemma 8.14.

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Part 3. The Hierarchical Renormalisation Group

Lecture 9. Scaling Limits and the Hierarchical Lattice

With Newman's theorem as motivation we introduce the idea of scaling limits. Scaling limit is a way to focus only on the long distance fluctuations of a statistical mechanical model. Many different models can have the same scaling limit. When two different models have the same scaling limit we say that they are in the same universality class. The grand goal of equilibrium statistical mechanics is to classify scaling limits. A starting point is to ask which models are in the universality class of the massless free field. The renormalization group is one way to answer this question. We will get used to the main ideas in the context of hierachical models.

9.1. White noise. White noise

$$W = \{W(X) : X \subset \mathbb{R}^d, |X| < \infty\} \tag{9.1}$$

is a collection of Gaussian random variables such that

$$\mathrm{Cov}(W(X), W(Y)) = |X \cap Y|, \tag{9.2}$$

$$W(\cup X_i) = \sum W(X_i) \quad \text{a.s.} \quad \text{if } \{X_i\} \text{ disjoint.} \tag{9.3}$$

For $X \subset \mathbb{R}^d$, $[\phi] > 0$, let

$$\phi(L, X) = L^{-d} \sum_{y \in LX \cap \mathbb{Z}^d} L^{[\phi]}(\phi_y - \langle \phi_y \rangle). \tag{9.4}$$

The conclusion of Newman's theorem can be restated as, for $X \in \mathcal{P}_{L=1}$,

$$\phi(L, X) \Longrightarrow W(X), \quad [\phi] = \frac{d}{2}. \tag{9.5}$$

We say that W is the scaling limit of ϕ. $[\phi]$ is called the dimension of ϕ. Choosing the "wrong" value for $[\phi]$ will give either no limit or a trivial limit concentrated on the zero field.

We say that two models are in the same universality class if they have the same scaling limit. Thus Newman's theorem is saying that all non-critical ferromagnetic models are in the same universality class, where non-critical means

$$\sum_y \mathrm{Cov}(\phi_x, \phi_y) < \infty. \tag{9.6}$$

The grand goal of equilibrium statistical mechanics is to classify the universality classes for models which are critical:

$$\sum_y \mathrm{Cov}(\phi_x, \phi_y) = \infty \tag{9.7}$$

Example 9.1. Recall that the infinite volume limit of the massless Gaussian on $\mathbb{Z}^d$ $(d > 2)$ has

$$\langle \phi_x \phi_y \rangle_\infty = \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle \phi_x \phi_y \rangle_\Lambda = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \frac{1}{\sum_{u \in \mathbb{Z}^d, \|u\|=1} (e^{ik.u} - 1)} e^{ik.(x-y)}\, dk. \tag{9.8}$$

Calculation (Problem 9.2) shows, for $[\phi] = \frac{d-2}{2}$,

$$\begin{aligned} \langle \phi(L, X)\phi(L, Y) \rangle &\overset{L \to \infty}{\longrightarrow} (2\pi)^{-d} \int_{\mathbb{R}^d} (1_X)\hat{}(k) \frac{1}{k^2} (1_Y)\hat{}(k)\, dk \\ &= c_d \int_X \int_Y \frac{1}{\|x - y\|^{d-2}}\, dx\, dy. \end{aligned} \tag{9.9}$$

Question. Is there a Gaussian field with covariance $\|x-y\|^{-(d-2)}$?

Let $[\phi] \in (0, \frac{d}{2})$.

Proposition 9.2. *There exists a probability space with*

$$\phi = \{\phi(X) : X \subset \mathbb{R}^d, \int_X \int_X \|x-y\|^{-2[\phi]}\, dx\, dy < \infty\}, \tag{9.10}$$

Gaussian random variables such that

$$\mathrm{Cov}(\phi(X), \phi(Y)) = \int_X \int_Y \|x-y\|^{-2[\phi]}\, dx\, dy, \tag{9.11}$$

$$\phi(X \cup Y) = \phi(X) + \phi(Y) \quad a.s. \quad if\ X, Y\ disjoint. \tag{9.12}$$

The case $[\phi] = \frac{d-2}{2}$ is called the massless continuum free field.

Proof. The next proposition constructs ϕ with these properties. ∎

Proposition 9.3. *Let $L > 1$. There exists a Gaussian random field*

$$\zeta = \{\zeta(x) : x \in \mathbb{R}^d\} \tag{9.13}$$

such that

1. *$\zeta \in C^\infty$ as a function of x,*
2. *$\mathrm{Cov}(\zeta(x), \zeta(y)) = 0$ for $\|x-y\| \geq L/2$,*
3. *for ζ_j independent scaled copies of ζ defined by*

$$\zeta_j(x) \overset{\mathcal{L}}{=} L^{-j[\phi]} \zeta(\frac{x}{L^j}), \tag{9.14}$$

the field $\phi = \{\phi(X) : X \subset \mathbb{R}^d\}$, given by the a.s. convergent sum

$$\phi(X) \overset{def.}{=} \sum_{j \in \mathbb{Z}} \int_X \zeta_j(x)\, dx, \tag{9.15}$$

which satisfies the conclusion of Proposition 9.2.

To prove this we use:

Lemma 9.4. *Let $u(x) = u(\|x\|) \in C_0(\mathbb{R}^d)$, $[\phi] \in (0, \frac{d}{2})$. There exists c such that for $\|x\| \neq 0$,*

$$\|x\|^{-2[\phi]} = \int_0^\infty \frac{dl}{l} l^{-2[\phi]} c u(\frac{x}{l}) \tag{9.16}$$

Proof. Let $l = \|x\| l'$. Then

$$\mathrm{RHS} = \|x\|^{-2[\phi]} \int_0^\infty \frac{dl'}{l'} l'^{-2[\phi]} c u(\frac{1}{l}) = \|x\|^{-2[\phi]}$$

by choice of c. ∎

Part of proof of Proposition 9.3. In Lemma 9.4, choose $u \in C_0^\infty$ and absorb c into u. We can also assume $\hat{u}(k) \geq 0$ because we can replace u by $u * u$ which is still C^∞ and of compact support. We can choose the support so that $u(x) = 0$ for $|x| \geq 1/2$. Let

$$C(x) = \int_1^L \frac{dl}{l} l^{-2[\phi]} u(x/l).$$

Then $\hat{C}(k) \geq 0$ and $C(x) = 0$ for $|x| \geq L/2$. The standard theory of Gaussian processes (Remark 9.5) shows that there exists $\zeta \in C^\infty$ with covariance $C(x-y)$. By Lemma 9.4,

$$\|x-y\|^{-2[\phi]} = \sum_{j\in\mathbb{Z}} L^{-2j[\phi]} C(\frac{x-y}{L^j}). \tag{9.17}$$

We construct a probability space carrying independent "increments" ζ_j with covariance $L^{-2j[\phi]}C(\frac{x-y}{L^j})$. Define $\phi(X)$ by (9.15). This series converges a.s. by Theorem 1.8.3 of [Dur91] and (Problem 9.1) $\phi(X)$ defined this way has the properties claimed in Proposition 9.2 because (9.17) makes the covariance match. ■

Remark 9.5. We have used the following statement:

> *For a C^∞ function $C(x)$ with $\hat{C}(k) \geq 0$ there exists a stationary zero-mean Gaussian process $\{\zeta(x)\}_{x\in\mathbb{R}^d}$ with covariance $C(x)$ and a.s. C^∞ sample paths.*

The following argument is still in progress and is not yet correct/complete. [5] To prove it, one could argue as follows.

Construct the process $\zeta(x)$ that has the covariance $C(x)$ (this is Kolmogorov's Construction Theorem applied to a Gaussian process [Wen81]). It can be easily shown that for all $i = 1,\ldots,d$ the process $\frac{1}{\epsilon}(\zeta(\epsilon e_i) - \zeta(0))$ (where e_i is the ith coordinate vector) is Cauchy in mean square as $\epsilon \to 0$. It follows that $\zeta(x)$ has mean square derivatives $\frac{\partial\zeta}{\partial x_i}(x)$, $i = 1,\ldots,d$. Similarly, $\zeta(x)$ has mean square partial derivatives of all orders and one can also compute covariances of these derivatives (see also [Wen81]). These can be realised as continuous functions by [Gar72],[IR78].

Thus, for all $m \geq 0$ we can start from the process

$$\left(\frac{\partial^d}{\partial x_1 \ldots \partial x_d}\right)^m \zeta(x)$$

and integrate it m times over $(-\infty, x)$ to get $\zeta(x)$. This implies that $\zeta(x)$ has differentiable sample paths up to certain order.

To complete the proof one should use Kolmogorov's Construction theorem again to pass from "finite-dimensional distributions" $\partial^\alpha\zeta(x)$ (where α are multiindices with bounded $|\alpha|$) to the joint distribution law of the process $\zeta(t)$ and all its derivatives of all orders. This shows that the sample paths of $\zeta(t)$ are C^∞.

The construction of Proposition 9.3 has created the scaling limit which labels the universality class of the lattice massless free field. What other models are in this universality class?

Theorem 9.6 (Aizenmann 1981 [Aiz82]**, Fröhlich 1981** [Frö82]**).** *In $d \geq 5$, the scaling limit of the nearest neighbor ferromagnetic Ising model, if it exists, is Gaussian.*

This is also true for the ϕ^4 field lattice field theory (which we have not yet defined). This result was proved by random walk representation related to Lecture 6. The Renormalisation Group (RG) is another way to prove this type of result. It is weaker in that it requires a small parameter and stronger in that it applies to a much wider class of models and also proves existence of scaling limit.

Since RG is complicated I want to first exhibit the idea for hierarchical models.

[5] If anyone knows a good reference please lets us know

9.2. Hierarchical models. These were invented by (Dyson, 1969), but not quite in the form I am about to describe, which is inspired by (Gallovatti et al 1978) and (Evans, 1989).

The d-dimensional hierarchical lattice Λ_∞ with parameter $L > 1$, $L \in \mathbb{N}$, is a countable Abelian group with the following properties:

(1) There is an ultrametric defined by a norm $|x + y| \leq \max(|x|, |y|)$.
(2) There is a map $L^{-1} : \Lambda_\infty \to \Lambda_\infty$ such that

$$|L^{-1}x| = \frac{|x|}{L} \qquad \text{if } L^{-1}x \neq 0. \tag{9.18}$$

(3) The ball $\{x : |x - y| \leq L^p\}$ has L^{dp} points.

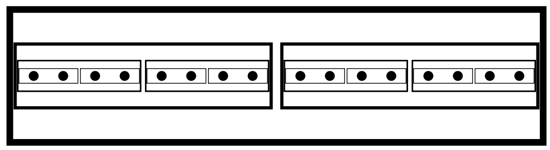

Figure 9.1. The balls of the hierarchical lattice with $L = 2, d = 1$

Example 9.7 ($L = 2, d = 1$).

$$\Lambda_\infty = \{\text{all finite binary sequences}\} \tag{9.19}$$

The group structure is $\bigoplus \mathbb{Z}_2$, so, for example, $100 - 11 = 111$. The map $2^{-1} : \Lambda_\infty \to \Lambda_\infty$ is right shift (collapse ball):

$$(x_n, x_{n-1} \cdots , x_2, x_1) \longmapsto (x_n, x_{n-1}, \cdots , x_2) \qquad x_n \neq 0 \tag{9.20}$$

The metric

$$|x| = \begin{cases} 2^n & x = (x_n, \cdots , x_1), x_n \neq 0, n \geq 1 \\ 0 & \end{cases} \tag{9.21}$$

satisfies

$$|2^{-1}x| = \frac{|x|}{2} \quad \text{if } |2^{-1}x| \neq 0, \tag{9.22}$$

$$|x + y| \leq \max(|x|, |y|). \tag{9.23}$$

Ultrametric means that no balls overlap: $B \cap B' \neq \emptyset \Rightarrow B \subset B'$ or $B' \subset B$. There are 2^p points in the ball $|x| \leq 2^p$.

9.3. The hierarchical free field. We construct the hierarchical Gaussian free field

$$\phi = \{\phi_x : x \in \Lambda_\infty\} \tag{9.24}$$

by creating the same structure as in Proposition 9.3. Let

$$\zeta = \{\zeta_x : x \in \Lambda_\infty\} \tag{9.25}$$

be a Gaussian random field such that

$$\mathrm{Cov}(\zeta_x, \zeta_y) = 0 \quad \text{if } |x - y| > L. \tag{9.26}$$

Then, for $[\phi] > 0$, define independent scaled copies

$$\zeta_j(x) \overset{\mathcal{L}}{=} L^{-j[\phi]}\zeta(L^{-j}x) \tag{9.27}$$

where

$$L^{-j} = (L^{-1})^j : \Lambda_\infty \to \Lambda_\infty. \tag{9.28}$$

Then we set

$$\phi(x) = \sum_{j\geq 1} \zeta_j(x). \tag{9.29}$$

Since $[\phi] > 0$, this series is a.s. convergent on a big probability space carrying all the increments ζ_j. This means that

$$\phi = \zeta_1 + \phi', \tag{9.30}$$

$$\phi' \stackrel{\mathcal{L}}{=} L^{-[\phi]}\phi(L^{-1}x), \tag{9.31}$$

and (Problem 9.4),

$$\phi'_x = \phi'_y \quad \text{a.s. for } |x-y| \leq L. \tag{9.32}$$

Since this is an ultrametric no balls overlap and balls are the same as blocks $B \in \mathcal{B}_L$.

Problems.

Problem 9.1. Prove that $\phi(X)$ defined by (9.15) has the properties claimed in Proposition 9.2.

Problem 9.2. Prove (9.9).

Problem 9.3. Construct a d-dimensional hierarchical lattice.

Problem 9.4. Prove (9.32).

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Lecture 10. The Hierarchical Gaussian Free Field

In this lecture, the hierarchical Gaussian free field is introduced. We define the renormalization group in the context of hierarchical models. We obtain some basic properties of this map and see an explanation for the role of the criterium $d \geq 5$ in the scaling limits.

10.1. The hierarchical free field. We recall the construction of hierarchical Gaussian free field

$$\phi = \{\phi_x : x \in \Lambda_\infty\} \tag{10.1}$$

by creating the same structure as in Proposition 9.3.

Let

$$\zeta = \{\zeta_x : x \in \Lambda_\infty\} \tag{10.2}$$

be a Gaussian random field such that

$$\mathrm{Cov}(\zeta_x, \zeta_y) = 0 \quad \text{if } |x-y| > L. \tag{10.3}$$

Then define independent scaled copies

$$\zeta_j(x) \overset{\mathcal{L}}{=} L^{-j[\phi]}\zeta(L^{-j}x), \tag{10.4}$$

where

$$L^{-j} = (L^{-1})^j : \Lambda_\infty \to \Lambda_\infty. \tag{10.5}$$

Definition 10.1. The hierarchical field is

$$\phi(x) = \sum_{j \geq 1} \zeta_j(x) \tag{10.6}$$

converging a.s. on a big probability space carrying all the increments ζ_j.

From this definition, it follows that

$$\phi = \zeta_1 + \phi', \quad \phi'_x \overset{\mathcal{L}}{=} L^{-[\phi]}\phi_{L^{-1}x}, \tag{10.7}$$

and (Problem 10.1),

$$\phi'_x = \phi'_y \quad \text{a.s.} \quad \text{for } |x-y| \leq L. \tag{10.8}$$

Since this is an ultrametric no balls overlap and balls are the same as blocks $B \in \mathcal{B}_L$.

10.2. Definition of the renormalization group. We define the following operations:

- *Integrating out* ζ: Define

$$\mathbb{E}_1(F) = \mathbb{E}(F|\zeta_2, \zeta_3, \dots). \tag{10.9}$$

- *Rescaling* ζ: For $F \in \sigma(\zeta_2, \zeta_3, \dots)$, define

$$\widehat{L^{-1}}F \tag{10.10}$$

  by replacing arguments $\zeta_{j+1}(x)$ with $L^{-[\phi]}\zeta_j(L^{-1}x)$, for $j \geq 1$.

  Remark 10.2. F and $\widehat{L^{-1}}F$ are equal in law.

- *RG Transformation*: For F, $\mathbb{E}|F| < \infty$, define

$$\mathrm{RG} : F \mapsto \widehat{L^{-1}} \circ \mathbb{E}_1(F). \tag{10.11}$$

Lemma 10.3.

$$\mathbb{E}F = \mathbb{E}(\mathrm{RG}(F)) \tag{10.12}$$

Proof. Let $\mathbb{E}_j := \mathbb{E}(F|\zeta_{j+1}, \zeta_{j+2}, \dots)$. Then:

$$\begin{aligned}
\mathbb{E}F &\overset{\text{Problem 10.2}}{=} \lim_{N\to\infty} \mathbb{E}_N \mathbb{E}_{N-1} \cdots \mathbb{E}_2 \mathbb{E}_1(F) \\
&\overset{\text{Remark 10.2}}{=} \lim_{N\to\infty} \mathbb{E}_{N-1} \cdots \mathbb{E}_1 \widehat{L^{-1}} \mathbb{E}_1(F) \\
&\overset{\text{Problem 10.2}}{=} \mathbb{E}(\mathrm{RG}(F))
\end{aligned}$$

■

Lemma 10.4. *For $P(\phi)$ a polynomial in ϕ,*

$$\mathbb{E}_1{:}P(\phi){:}_v = {:}P(\phi'){:}_{v'}, \tag{10.13}$$

where

$$v = \sum_{j\geq 1} C_j \quad : \textit{is the covariance of } \phi, \tag{10.14}$$

$$v' = \sum_{j\geq 2} C_j \quad : \textit{is the covariance of } \phi'. \tag{10.15}$$

Proof. Let

$$\Delta_C = \sum_{x,y} C(x,y) \frac{\partial}{\partial \phi'_x} \frac{\partial}{\partial \phi'_y},$$

$$\Delta_{C,\zeta} = \sum_{x,y} C(x,y) \frac{\partial}{\partial \zeta_x} \frac{\partial}{\partial \zeta_y}.$$

Then, from Lecture 4, for $Q = {:}\,P\,{:}_v$,

$$\begin{aligned}
\mathbb{E}_1 Q(\phi) = \mathbb{E}_1 Q(\phi' + \zeta) = e^{\frac{1}{2}\Delta_{C,\zeta}} Q(\phi'+\zeta)\big|_{\zeta=0} = e^{\frac{1}{2}\Delta_C} Q(\phi') \\
= e^{\frac{1}{2}\Delta_C} e^{-\frac{1}{2}\Delta_v} P(\phi') = e^{-\frac{1}{2}\Delta_{v-C}} P(\phi') = {:}P(\phi'){:}_{v'}
\end{aligned}$$

■

Lemma 10.5.

$$\mathrm{RG}{:}\phi_x^p{:}_v = L^{-p[\phi]} {:}\phi^p_{L^{-1}x}{:}_v \tag{10.16}$$

Proof. By Lemma 10.4,

$$\mathrm{RG}{:}\phi_x^p{:}_v = \widehat{L^{-1}}{:}\phi_x'^p{:}_{v'} = \widehat{L^{-1}} e^{-\frac{1}{2}\sum_{j\geq 2}\Delta_{C_j,\zeta_j}} \phi_x'^p = \widehat{L^{-1}} e^{-\frac{1}{2}\sum_{j\geq 2}\Delta_{C_j,\zeta_j}} \Big(\sum_{j\geq 2} \zeta_j\Big)^p$$

$\widehat{L^{-1}}$ replaces $\zeta_j(x)$ by $L^{-[\phi]}\zeta_{j-1}(L^{-1}x)$, thus

$$\mathrm{RG}{:}\phi_x^p{:}_v = L^{-p[\phi]}{:}\phi^p_{L^{-1}x}{:}.$$

■

10.3. Hierarchical models. Our models have had the form

$$Z = \int e^{-(\phi, -\Delta_\Lambda \phi)} F^\Lambda \, d^\Lambda \phi, \tag{10.17}$$

$$F^\Lambda = \prod_{x \in \Lambda} F_x \quad (\Lambda \subset \mathbb{Z}^d), \tag{10.18}$$

where F_x is a bounded function of ϕ_x. A close hierarchical analogue is

$$Z = \mathbb{E} F^\Lambda \quad (\Lambda \subset \Lambda_\infty). \tag{10.19}$$

Remark 10.6. It would be an even closer analogue if (10.17) had been the infinite volume Gaussian expectation of F^Λ. This can be understood as a different boundary condition at $\partial\Lambda$.

We intend to calculate Z by

$$Z = \lim_{n \to \infty} \mathbb{E}(\mathrm{RG})^n F^\Lambda. \tag{10.20}$$

Lemma 10.7.

$$\mathrm{RG}(F^\Lambda) = \prod_{x \in L^{-1}\Lambda} \mathrm{RG}(F^{B(x)}), \tag{10.21}$$

where

$$B(x) = \{y : L^{-1} y = x\}. \tag{10.22}$$

Proof.

$$\mathrm{RG}(F^\Lambda) = \widehat{L^{-1}} \mathbb{E}_1 \prod_{B \in \mathcal{B}_L(\Lambda)} F^B = \widehat{L^{-1}} \prod_{B \in \mathcal{B}_L(\Lambda)} \mathbb{E}_1 F^B = \prod_{x \in L^{-1}\Lambda} \underbrace{\widehat{L^{-1}} \mathbb{E}_1 F^{B(x)}}_{= \mathrm{RG}(F^{B(x)})}.$$

∎

Example 10.8.

$$F^\Lambda = e^{-V(\Lambda)}, \quad [\phi] = \frac{d-2}{2}, \tag{10.23}$$

$$V(\Lambda) = \sum_{x \in \Lambda} V_x, \quad V_x = g{:}\phi_x^4{:} + a{:}\phi_x^2{:} \tag{10.24}$$

Then, to order g, a, or equivalently, $V^2 = 0$,

$$\begin{aligned} \mathrm{RG}(F^{B(x)}) &= \mathrm{RG}(e^{-V(B(x))}) = \mathrm{RG}(1 - V(B(x))) \\ &= 1 - \sum_{y \in B} \left(g L^{-4[\phi]} {:}\phi_{L^{-1}y}^4{:} + a L^{-2[\phi]} {:}\phi_{L^{-1}y}^2{:} \right) = 1 - V'_x = e^{-V'_x}, \end{aligned} \tag{10.25}$$

where

$$V' = g'{:}\phi^4{:} + a'{:}\phi^2{:} \tag{10.26}$$

$$g' = |B| L^{-4[\phi]} g, \quad a' = |B| L^{-2[\phi]} a. \tag{10.27}$$

Putting in

$$[\phi] = \frac{d-2}{2}, \quad |B| = L^d, \tag{10.28}$$

we have

$$g' = L^{-d+4}g, \quad a' = L^2 a. \tag{10.29}$$

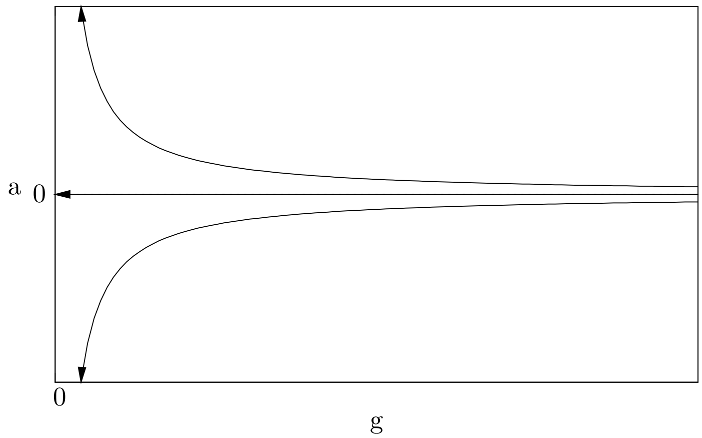

Figure 10.1. Approximate renormalization group trajectories ($d > 4$)

10.4. Correlation.

$$\langle \phi_a \phi_b \rangle = \frac{\mathbb{E}F(a,b)^{\Lambda}}{\mathbb{E}F^{\Lambda}}, \tag{10.30}$$

where, for $a \neq b$,

$$F_x(a,b) = e^{-V_x} \begin{cases} 1 & x \neq a, b \\ \phi_a & x = a \\ \phi_b & x = b \end{cases} \tag{10.31}$$

Apply RG to top and bottom of (10.30).

10.5. Problems.

Problem 10.1. Prove (10.8); c.f. [Dur91, Theorem 6.3].

Problem 10.2. Justify the limits in the proof of Lemma 10.3.

Problem 10.3. Find α such that

$$\mathrm{RG}(F(a,b)^{B(x)}) = \alpha \phi_x e^{-V_x} + O(g,a) \tag{10.32}$$

when $B(x)$ contains a but not b. If both $a, b \in B(x)$, what is

$$\mathrm{RG}^n(F(a,b)^{B(x)}) \tag{10.33}$$

to order V^0?

Bibliography

[Dur91] Richard Durrett. *Probability*. The Wadsworth & Brooks/Cole Statistics/Probability Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991. Theory and examples.

Lecture 11. The Renormalisation Group Step (1)

At the $O(V)$ level of the last lecture, the action of the renormalisation group is to replace

$$V = g{:}\phi^4{:} + a{:}\phi^2{:} \quad \text{by} \quad \tilde{V} = \tilde{g}{:}\phi^4{:} + \tilde{a}{:}\phi^2{:} \tag{11.1}$$

with $\tilde{g} = gL^{d-4[\phi]}$, $\tilde{a} = L^{d-2[\phi]}a$. To include all $O(V^2)$ corrections, we introduce an error term such that under the action of the renormalisation group

$$e^{-V} + K \to e^{-V'} + K'.$$

In this lecture, we introduce some of the main tools for controlling this K: which space K is in and how to measure its size. The ideas explained in this and the next lecture are based on pages 565–573 of [BI03].

11.1. The model. For $\Lambda \subset \Lambda_\infty$ a subset of the hierarchical lattice, denote

$$(e^{-V} + K)^\Lambda \stackrel{\text{def.}}{=} \prod_{x\in\Lambda}(e^{-V_x} + K_x) \tag{11.2}$$

where

$$V_x = g{:}\phi_x^4{:} + a{:}\phi_x^2{:} + b \tag{11.3}$$

with $|a| \leq \sqrt{g}$ and $K_x = K(\phi_x)$. Inductively assuming that K is even and such that $K(t) = O(t^6)$ as $t \to 0$, we are interested in the effect of the renormalisation group on

$$Z \stackrel{\text{def.}}{=} \mathbb{E}(e^{-V} + K)^\Lambda. \tag{11.4}$$

Initially, we could assume that $K = 0$, but after one renormalisation group step, we would need a K. Therefore, we choose a form of Z, which remains stable under the action of the renormalisation group.

$\mathbb{E}$ denotes the expectation for the hierarchical field. Then, for $L > 1$,

$$\phi_x \stackrel{\mathcal{L}}{=} L^{-[\phi]}\phi_{L^{-1}x} + \zeta \tag{11.5}$$

and, for $[\phi] > 0$, by the following remark we can assume that $\operatorname{Var}\zeta \leq 1$ for all L.

Remark 11.1. At present our construction of hierarchical ϕ seems to require a different probability model for each L, but if we assume $L \in \{3, 3^2, 3^3, \dots\}$ this can be avoided as follows. Construct the $L = 3$ probability space. On this space are defined $\{\zeta_j, j \geq 1\}$ and $\phi = \sum \zeta_j$. We write $\phi = (\zeta_1 + \zeta_2) + (\zeta_3 + \zeta_4)\dots$ and let $\xi = \zeta_1 + \zeta_2$. Then we have

$$\phi_x \stackrel{\mathcal{L}}{=} L^{-[\phi]}\phi_{L^{-1}x} + \xi, \quad L = 3^2. \tag{11.6}$$

More generally we obtain $L = 3^n$ by setting $\xi = \zeta_1 + \zeta_2 + \dots + \zeta_n$. Also

$$\operatorname{Var}(\xi_x) = \sum_{j=1}^{\log_3 L} 3^{-[\phi]j}\operatorname{Var}(\zeta_x). \tag{11.7}$$

This series is geometrically convergent for $L \to \infty$ so we can assume, by choice of $\operatorname{Var}(\zeta_x)$, that $\operatorname{Var}(\xi_x) \uparrow 1$ as $L \to \infty$, $L \in \{3^n : n \in \mathbb{N}\}$.

Assumption. We shall assume that $d - 4[\phi] < 0$.

Recall that d is the dimension of the space and $[\phi]$ represents the dimension of the field. As in the last lecture, the assumptions means that $g \to 0$ within the $O(V)$ calculations. When the coupling constants are contracted according to the $O(V)$ calculation, we say that they are *irrelevant*.

11.2. The T_ϕ norm. For F a function of finitely many $\{\phi_x, x \in \Lambda_\infty\}$, we define

$$\|F\|_{T_\phi} \stackrel{\text{def.}}{=} \|F\|_{T_\phi,\mathtt{h}} \stackrel{\text{def.}}{=} \sum_{x\in\Lambda_\infty^*} \frac{\mathtt{h}^n}{n!}\left|\frac{\partial^n F(\phi)}{\partial\phi_{x_1}\cdots\partial\phi_{x_n}}\right| \tag{11.8}$$

where $n = n(x)$ represents the length of the sequence x and $\mathtt{h} > 0$. (T_ϕ stands for tangent space.)

Example 11.2. If $F = F(\phi_x)$, then

$$\|F\|_{T_\phi} = \sum_{n=0}^{\infty} \frac{\mathtt{h}^n}{n!}\left|\frac{\partial^n F(\phi)}{\partial\phi_x^n}\right|. \tag{11.9}$$

This sum converges if F is analytic on a horizontal strip of width $\mathtt{h}$.

Lemma 11.3. (1)

$$\|F_1F_2\|_{T_\phi} \le \|F_1\|_{T_\phi}\|F_2\|_{T_\phi} \tag{11.10a}$$

and, therefore,

$$\left\|F^X\right\|_{T_\phi} \le \|F\|_{T_\phi}^X \tag{11.10b}$$

(2)

$$\sum_{x\in\Lambda_\infty^*} \frac{\mathtt{h}_1^n}{n!}\left\|\frac{\partial^n F(\phi)}{\partial\phi_{x_1}\ldots\partial\phi_{x_n}}\right\|_{T_\phi,\mathtt{h}_2} \le \|F\|_{T_\phi,\mathtt{h}_1+\mathtt{h}_2} \tag{11.11}$$

Proof. Problem 11.1. ■

Property (2) says that

$$\left\|\frac{\partial^n F(\phi)}{\partial\phi_{x_1}\ldots\partial\phi_{x_n}}\right\|_{T_\phi,\mathtt{h}_2} \le \frac{n!}{\mathtt{h}_1^n}\|F\|_{T_\phi,\mathtt{h}_1+\mathtt{h}_2} \tag{11.12}$$

which is a *Cauchy bound*. We will use that the derivatives are very small for large $\mathtt{h}_1$.

Example 11.4. Suppose $F = F(\phi)$. Here, we are interested in finding a bound for the T_ϕ norm for the renormalisation group step. This calculation shall be referred to later in the lecture. Recalling that $\mathrm{RG} = \widehat{L^{-1}}\mathbb{E}_1$,

$$(\mathbb{E}_1F)(\phi') = \int d\mu_C(\zeta)F(\phi'+\zeta), \tag{11.13}$$

$$(\mathrm{RG}(F))(\phi) = \int d\mu_C(\zeta)F(L^{-[\phi]}\phi+\zeta), \tag{11.14}$$

$$\frac{\partial}{\partial\phi}(\mathrm{RG}(F))(\phi) = L^{-[\phi]}\int d\mu_C(\zeta)F'(L^{-[\phi]}\phi+\zeta). \tag{11.15}$$

Applying the T_ϕ norm gives

$$\|\mathrm{RG}(F)\|_{T_\phi,\mathtt{h}} \le \int d\mu_C(\zeta)\sup_\zeta \|F\|_{T_{L^{-[\phi]}\phi+\zeta},L^{-[\phi]}\mathtt{h}} \le \sup_\phi \|F\|_{T_\phi,L^{-[\phi]}\mathtt{h}}. \tag{11.16}$$

We shall denote

$$\|F\|_{L^{-[\phi]}\mathtt{h}} \stackrel{\text{def.}}{=} \sup_\phi \|F\|_{T_\phi,L^{-[\phi]}\mathtt{h}}. \tag{11.17}$$

Lemma 11.5. *For $\mathtt{h} \leq g^{-1/4}$, there exists a constant C such that, for $|a| \leq \sqrt{g}$,*

$$\left\| e^{-g:\phi_x^4:-a:\phi_x^2:} \right\|_{T_\phi,\mathtt{h}} \leq e^{O(g\mathtt{h}^4)-\frac{1}{2}g\phi_x^4} \leq Ce^{-\frac{1}{2}g\phi_x^4}. \tag{11.18}$$

If $\mathtt{h} \leq cg^{-1/4}$ we have the same conclusion.

Proof. We give the proof for $e^{-g\phi^4}$. The complete case is Problem 11.2. The proof follows from using an approximation to the exponential:

$$\left\| \left(1 - \frac{g}{N}\phi_x^4\right)^N \right\|_{T_\phi,\mathtt{h}} \overset{\text{Lemma 11.3 (1)}}{\leq} \left\| 1 - \frac{g}{N}\phi_x^4 \right\|_{T_\phi,\mathtt{h}}^N$$

The definition of the T_ϕ norm gives

$$\left| (1 - \frac{g}{N}\phi_x^4) + \frac{g}{N}4|\phi_x|^3\mathtt{h} + \frac{g}{N}6|\phi_x|^2\mathtt{h}^2 + \cdots + \frac{g}{N}\mathtt{h}^4 \right|^N = \left| (1 + \frac{g}{N}\mathtt{h}^4(-t^4 + 4t^3 + 6t^2 + 4t + 1) \right|^N$$

by setting $t = |\phi_x|/\mathtt{h}$. Therefore, using $1 + x \leq e^x$,

$$\left\| \left(1 - \frac{g}{N}\phi_x^4\right)^N \right\|_{T_\phi,\mathtt{h}} \leq \left| 1 + \frac{g}{N}\mathtt{h}^4(-\frac{1}{2}t^4 + c) \right|^N \leq e^{-\frac{1}{2}g\phi_x^4}e^{cg\mathtt{h}^4}.$$

■

Notation. Let $\|F\|_{\mathtt{h}} = \sup_\phi \|F\|_{T_\phi,\mathtt{h}}$ and let $h = g^{-1/4}$ and $\tilde{h} = 2(L^{d-4[\phi]}g)^{-1/4}$.

11.3. RG Step I. Given (V, K) define $(\tilde{V}, \tilde{K})$ by

$$\mathrm{RG}(e^{-V} + K)^B = e^{-\tilde{V}_x} + \tilde{K}_x \tag{11.19}$$

where $x = L^{-1}B$ and $\tilde{V}_x = \mathrm{RG}(V(B))$. This equation defines $\tilde{K}$ because $\tilde{V}$ is already determined by $V(B)$. Define

$$\tilde{K}_{\mathrm{main},x} = \mathrm{RG}(e^{-V(B)}) - e^{-\mathrm{RG}(V(B))}. \tag{11.20}$$

Proposition 11.6. *There exists $c(L)$ such that as $L \to \infty$ with $g \leq c(L)$ and $\|K\|_h \leq c(L)$,*

$$\frac{\|\tilde{K} - \tilde{K}_{\mathrm{main},x}\|_{\tilde{h}}}{\|K\|_h} = O(L^{-d/2}). \tag{11.21}$$

Remark 11.7. $c(L)$ is determined in the proof such that

$$\lim_{L\to\infty} c(L) = 0$$

exponentially (faster than 2^{L^d}). We will make assumptions like $\|K\|_h \leq 1$, at very many places in the proof.

Part of proof of Proposition 11.6.

$$\tilde{K}_* - \tilde{K}_{\mathrm{main},x} = \sum_{y\in B} \widehat{L^{-1}} e^{-V(B\setminus\{y\})}K_y \tag{I}$$

$$+ \sum_{y\in B} \widehat{L^{-1}}(\mathbb{E}_1 - \mathrm{Id})(e^{-V(B\setminus\{y\})}K_y) \tag{II}$$

$$+ \sum_{Y\subset B, |Y|\geq 2} \widehat{L^{-1}}\mathbb{E}_1(e^{-V(B\setminus Y)}K(Y)) \tag{III}$$

which follows by doing a binomial expansion. Note that

$$L^{-[\phi]}\tilde{h} = L^{-[\phi]}2(L^{d-4[\phi]}g)^{-\frac{1}{4}} = 2L^{-\frac{d}{4}}g^{-\frac{1}{4}} \leq g^{-\frac{1}{4}} = h.$$

Term III. We prove that (III) is bounded in its T_ϕ norm as claimed in Proposition 11.6 by using Example 11.4 and Lemma 11.5 as follows:

$$\begin{aligned}\left\| \sum_{Y\subset B, |Y|\geq 2} \widehat{L^{-1}}\mathbb{E}_1(e^{-V(B\setminus Y)}K(Y)) \right\|_{T_\phi,\tilde{h}} &\leq \sum_{Y\subset B,|Y|\geq 2} \left\| e^{-V(B\setminus Y)}K(Y) \right\|_{\underbrace{L^{-[\phi]}\tilde{h}}_{\leq h}} \\ &\leq \sum_{Y\subset B,|Y|\geq 2} \left\| e^{-V(B\setminus Y)}K(Y) \right\|_h\end{aligned}$$

Then, we can separate K from the norm by Example 11.4. This gives

$$\begin{aligned}\left\| \sum_{Y\subset B, |Y|\geq 2} \widehat{L^{-1}}\mathbb{E}_1(e^{-V(B\setminus Y)}K(Y)) \right\|_{T_\phi,\tilde{h}} &\leq \left(\sum_{Y\subset B,|Y|\geq 2} \left\| e^{-V} \right\|_h^{B\setminus Y} \right) \|K\|_h^2 \\ &\leq \left(\sum_{Y\subset B} \left\| e^{-V} \right\|_h^{B\setminus Y} \right) \|K\|_h^2 .\end{aligned}$$

By reversing the binomial expansion and applying Lemma 11.5, this is the same as

$$\begin{aligned}\left\| \sum_{Y\subset B, |Y|\geq 2} \widehat{L^{-1}}\mathbb{E}_1(e^{-V(B\setminus Y)}K(Y)) \right\|_{T_\phi,\tilde{h}} &\leq \left(\left\| e^{-V} \right\|_h + 1 \right)^{|B|} \|K\|_h^2 \\ &\leq \left(c^{L^d} \|K\|_h \right) \|K\|_h .\end{aligned}$$

By choosing $c(L)$ to decrease sufficiently rapidly as $L \to \infty$ we arrange that $c^{L^d} \|K\|_h = o(L^{-d/2})$ as $L \to \infty$ and so the contribution of this term to

$$\frac{\|\tilde{K} - \tilde{K}_{\text{main},x}\|_{\tilde{h}}}{\|K\|_h} L^{d/2}$$

drops out. (I) and (II) will be bounded next lecture. ■

Problem 11.1. Prove Lemma 11.3.

Problem 11.2. Complete the proof of Lemma 11.5.

Bibliography

[BI03] David C. Brydges and John Z. Imbrie. Green's function for a hierarchical self-avoiding walk in four dimensions. *Commun. Math. Phys.*, 239(3):549–584, 2003.

Lecture 12. The Renormalisation Group Step (2)

Proof of Proposition 11.6 (cont'd). Recall the formulation of Proposition 11.6. That is, we want to prove that there exists $c(L)$ such that if $g \le c(L)$ and $\|K\|_h \le c(L)$ then

$$\frac{\|\tilde{K} - \tilde{K}_{\text{main}}\|_{\tilde{h}}}{\|K\|_h} = O(L^{-d/2}). \tag{12.1}$$

We wrote

$$\tilde{K}_x - \tilde{K}_{\text{main},x} = \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{12.2}$$

where

$$\mathrm{I} = \sum_{y \in B} \hat{L}^{-1} e^{-V(B\setminus\{y\})} K_y \tag{I}$$

$$\mathrm{II} = \sum_{y \in B} \hat{L}^{-1} (\mathbb{E}_1 - \mathrm{Id})(e^{-V(B\setminus\{y\})} K_y) \tag{II}$$

$$\mathrm{III} = \sum_{Y \subset B, |Y| \ge 2} \hat{L}^{-1} \mathbb{E}_1 (e^{-V(B\setminus Y)} K(Y)), \tag{III}$$

and B is the block such that $L^{-1}B = x$. In the previous lecture, we proved that there is a choice of $c(L)$ such that

$$\frac{\|\mathrm{III}\|_{T_\phi, \tilde{h}}}{\|K\|_h} = o(L^{-d/2}), \tag{12.4}$$

so our proof is complete for term III.

Term I. Since $K(\phi_y) = O(\phi_y^6)$, we can write

$$K(\phi_y) = \int_0^1 \frac{(1-t)^5}{5!} \left(\frac{d}{dt}\right)^6 K(t\phi_y)\, dt = \int_0^1 \frac{(1-t)^5}{5!} K^{(6)}(t\phi_y) \phi_y^6 \, dt \tag{12.5}$$

Note that

$$\hat{L}^{-1} e^{-V(B\setminus\{y\})} = e^{-(|B|-1)g\left(L^{-[\phi]}\phi_y\right)^4 + \cdots}, \tag{12.6}$$

where dots stand for terms containing ϕ_y^2 and $\phi_y^0 = 1$.

Preliminary calculation. Recall that

$$\tilde{h} = 2\tilde{g}^{-1/4}, \qquad \tilde{g} = L^{d-4[\phi]} g, \qquad L^{-[\phi]}\tilde{h} = 2|B|^{-1/4} h. \tag{12.7}$$

Thus,

$$\begin{aligned} &\left\| L^{-6[\phi]} \phi_y^6 e^{-(|B|-1) g L^{-4[\phi]} \phi_y^4} \right\|_{T_\phi, \tilde{h}} \\ &\le \left(L^{-[\phi]}\tilde{h}\right)^6 \left\| \frac{\phi_y}{\tilde{h}} \right\|_{\tilde{h}}^6 \left\| e^{-(1-|B|^{-1})\tilde{g}\phi_y^4} \right\|_{T_\phi, \tilde{h}} \\ &\le c \left(|B|^{-1/4} h\right)^6 \end{aligned} \tag{12.8}$$

because $\tilde{h} = 2\tilde{g}^{-1/4}$ so Lemma 11.5 applies.

Therefore,

$$\|I\|_{T_\phi, \tilde{h}} \le c|B| \left(|B|^{-1/4} h\right)^6 \sup_t \left\| K^{(6)}(t\phi_y) \right\|_{L^{-[\phi]}\tilde{h}}. \tag{12.9}$$

By Cauchy estimate from Lemma 11.3,

$$\begin{aligned}\|\mathrm{I}\|_{T_\phi,\tilde h} &\le c|B|\left(|B|^{-1/4}h\right)^6 \frac{1}{(h-L^{-[\phi]}\tilde h)^6}\,\|K\|_h \\ &\le c|B|\left(|B|^{-1/4}h\right)^6 \frac{1}{(h-2|B|^{-1/4}h)^6}\,\|K\|_h \\ &= O(|B|^{-1/2})\,\|K\|_h = L^{-d/2}\,\|K\|_h\,.\end{aligned}\tag{12.10}$$

Therefore,

$$\frac{\|\mathrm{I}\|_{T_\phi,h}}{\|K\|_h} = O(L^{-d/2}), \qquad L\to\infty.\tag{12.11}$$

The argument for II is given at the end of this lecture and it shows that

$$\frac{\|\mathrm{II}\|_{T_\phi,h}}{\|K\|_h} = O(\frac{1}{h^2}) = O(\sqrt{g}) = O(\sqrt{c(L)}),\tag{12.12}$$

so this can also be made $O(L^{-d/2})$, $L\to\infty$. ■

12.1. RG Step II. Proposition 11.6 required $K = O(\phi^6)$. The value $\tilde K$ will not obey this condition so we cannot use Proposition 11.6 for the next RG. Therefore, define (V',K'), where

$$V' = g'{:}\phi^4{:} + a'{:}\phi^2{:} + b'\tag{12.13}$$

so that

$$e^{-V'} + K' = e^{-\tilde V} + \tilde K, \quad K'(\phi_x) = O(\phi_x^6).\tag{12.14}$$

To see that a solution (V',K') exists, define V' by making Taylor expansion in $e^{-\tilde V}+\tilde K$ to order ϕ^4 and then let

$$K' = e^{-\tilde V} - e^{V'} + \tilde K.\tag{12.15}$$

Now K' is of order $O(\phi^6)$.

Lemma 12.1. *The solution* (V',K') *satisfies*

(1) $\|V'-\tilde V\|_{T_0,\mathtt{h}} \le c\,\|K\|_{T_0,\mathtt{h}}$, *where* T_0 *refers to the* T_ϕ *norm with* $\phi=0$*;*
(2) $\|K'\|_{\tilde h} \le c\|\tilde K\|_{\tilde h}$*;*
(3) $\|K'\|_{T_0,\mathtt{h}} \le c\|\tilde K\|_{T_0,\mathtt{h}}$,

where $\mathtt{h}\ge 1$.

Proof. See [BI03, p. 569]. ■

Now we can prove that K_{main} controls K.

Corollary 12.2. *For* L *large,* $g\le c(L)$*, if for some* z *such that* $z(d-4\,[\phi]) > -d/2$ *we have*

$$c\|\tilde K_{\mathrm{main}}\|_{\tilde h} \le \tilde g^z, \qquad \|K\|_h \le 2g^z,\tag{12.16}$$

then

$$\|K'\|_h \le 2\tilde g^z.\tag{12.17}$$

Proof. By Lemma 12.1 we have $\|K'\|_{\tilde h} \le c\|\tilde K\|_{\tilde h}$, and then we write

$$\begin{aligned}\|K'\|_{\tilde h} &\le c\|\tilde K - \tilde K_{\mathrm{main}}\|_{\tilde h} + c\|\tilde K_{\mathrm{main}}\|_{\tilde h} \\ &\le O(L^{-d/2})\,\|K\|_h + \tilde g^z \\ &\le O(L^{-d/2})2g^z + \tilde g^z \\ &\le O\left(L^{-d/2-(d-4[\phi])z}\right)2\tilde g^z + \tilde g^z \le 2\tilde g^z.\end{aligned} \tag{12.18}$$

The last inequality holds for all L large enough. This concludes the proof. ∎

Lemma 12.1 (1) says that

$$g' = L^{d-4[\phi]}g + O\left(\mathtt{h}^{-4}\|\tilde K\|_{T_0,\mathtt{h}}\right); \tag{12.19a}$$

$$a' = L^{d-2[\phi]}a + O\left(\mathtt{h}^{-2}\|\tilde K\|_{T_0,\mathtt{h}}\right). \tag{12.19b}$$

The next task is to prove that the corrections to linear terms are $o(g)$.

Notation. $h = g^{-1/4}$ (as before)

Proposition 12.3. *Let $p > 0$ and $\mathtt{h} = L^{[\phi]}$. There exists $c_p(L)$ such that if*

$$g \le c_p(L), \qquad \|K\|_{T_0,\mathtt{h}} \le c_p(L), \tag{12.20}$$

then

$$\frac{\|\tilde K - \tilde K_{\mathrm{main}}\|_{T_0,\mathtt{h}}}{\|K\|_{T_0,\mathtt{h}} \vee (h^{-p}\|K\|_h)} = O\big(L^{d-6[\phi]}\big). \tag{12.21}$$

Remark 12.4. In the lecture I wrote $O(L^{-d/2})$, but this is what the proof gives and it is better. By choosing $p = 12$ we have $h^{-p} = g^3$ which is so small that $h^{-p}\|K\|_h$ will drop out in our application of this result.

12.2. Domain. Let $\delta > 0$, $L \ge L_0(\delta)$,

$$g \le c(L); \tag{12.22a}$$

$$|a| \le g; \tag{12.22b}$$

$$\|K\|_{T_0,\mathtt{h}} \le g^{2-\delta}; \tag{12.22c}$$

$$\|K\|_h \le g^{1/2-\delta}. \tag{12.22d}$$

The last two inequalities are based on calculating $\|\tilde K_{\mathrm{main}}\|$. Then

$$g' = L^{d-4[\phi]}g + \epsilon_g, \qquad \epsilon_g \le g^{2-\delta}; \tag{12.23a}$$

$$a' = L^{d-2[\phi]}a + \epsilon_a, \qquad \epsilon_a \le g^{2-\delta}; \tag{12.23b}$$

$$b' = L^d b + \epsilon_b, \qquad \epsilon_b \le g^{2-\delta}, \tag{12.23c}$$

and K' obeys

$$\|K'\|_{T_0,\mathtt{h}} \le (g')^{2-\delta}; \tag{12.24a}$$

$$\|K'\|_{h'} \le (g')^{1/2-\delta}, \qquad h' := (g')^{-1/4}. \tag{12.24b}$$

Following [BS73] there exists a critical choice of a_c, b_c such that under the action of the renormalisation group the values g, a, b tend to zero.

Appendix 12.A. The bound on II in the proof of Proposition 11.6.

Notation.

$$\mathbb{E}^{(p-1)}F = \sum_{n=0}^{p-1} \frac{1}{n!}\left(\frac{\Delta_C}{2}\right)^n F. \tag{12.25}$$

Lemma 12.5.

$$\|\mathbb{E}_1 F - \mathbb{E}^{(p-1)}F\|_{T_\phi,\mathtt{h}_1} \leq \frac{(2p)!}{2^p p!}\left(\frac{C(0,0)}{\mathtt{h}_2^2}\right)^p \|F\|_{\mathtt{h}_1+\mathtt{h}_2} \tag{12.26}$$

Proof. We have

$$\begin{aligned}\mathbb{E}_1 F &= \mathbb{E}^{(p-1)}F + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}\left(\frac{d}{dt}\right)^p \mathbb{E}_t F \\ &= \mathbb{E}^{(p-1)}F + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}\mathbb{E}_t\left(\frac{\Delta_C}{2}\right)^p F,\end{aligned} \tag{12.27}$$

where $\mathbb{E}_t$ has covariance tC in place of C.

Therefore,

$$\begin{aligned}\|\mathbb{E}_1 F - \mathbb{E}^{(p-1)}F\|_{T_\phi,\mathtt{h}_1} &\leq \frac{1}{p!}\sup_t \left\|\mathbb{E}_t\left(\frac{\Delta_C}{2}\right)^p F\right\|_{T_\phi,\mathtt{h}_1} \\ &\leq \frac{1}{p!}\left\|\left(\frac{\Delta_C}{2}\right)^p F\right\|_{\mathtt{h}_1} \\ &\leq \frac{1}{p!}\frac{(2p)!}{2^p {\mathtt{h}_2}^{2p}}\left(C(0,0)\right)^p \|F\|_{\mathtt{h}_1+\mathtt{h}_2}\end{aligned} \tag{12.28}$$

by Lemma 11.3 and because $C(x,y) \leq C(0,0)$ by Cauchy-Schwarz and positive-definiteness. ∎

By taking $p=1$ we obtain a bound on $\mathbb{E}_1 F - F$ by $O(h^{-2})\|F\|_{2h}$ which is what is needed to bound term II in the proof of Proposition 11.6.

Appendix 12.B. Part of proof of Proposition 12.3. (1) If $F = F(\phi_x)$ and $F^{(n)}(0) = 0$ for $n = 0, 1, \ldots, p-1$, then

$$\|F\|_{T_0,\alpha\mathtt{h}} = \sum_{n\geq p} \frac{1}{n!}(\alpha\mathtt{h})^n |F^{(n)}(0)| \leq \alpha^p \|F\|_{T_0,\mathtt{h}}. \tag{12.29}$$

(2) Write

$$\tilde{K} - K_{\text{main}} = \sum_{y\in B} \hat{L}^{-1}\mathbb{E}_1\left(e^{-V(B\setminus\{y\})}K_y\right) + \sum_{y\subset B,\, |Y|\geq 2} \hat{L}^{-1}\mathbb{E}_1\left(e^{-V(B\setminus Y)}K^Y\right). \tag{12.30}$$

As in the proof of Proposition 11.6, the second term will turn out to be negligible so we consider the first term. Let $F = e^{-V(B\setminus\{y\})}K_y$, then $F = O(\phi^6)$. Thus, recalling $\mathtt{h} = L^{[\phi]}$,

$$
\begin{aligned}
\left\|\hat{L}^{-1}\mathbb{E}_1 F\right\|_{T_0,\mathtt{h}} &\leq \|\mathbb{E}_1 F\|_{T_0,L^{-[\phi]}\mathtt{h}}\,, \\
&\overset{\text{Lemma 12.5}}{\leq} \sum_{n=0}^{p-1}\frac{1}{n!}\left\|\left(\frac{\Delta_C}{2}\right)^n F\right\|_{T_0,L^{-[\phi]}\mathtt{h}} + O\left(\frac{1}{h-L^{-[\phi]}\mathtt{h}}\right)^{2p}\|F\|_h \\
&\leq c(p)\|F\|_{T_0,2L^{-[\phi]}\mathtt{h}} + O\left(\frac{1}{h-1}\right)^{2p}\|F\|_h \\
&\overset{(1)}{\leq} O\left(L^{-6[\phi]}\right)\|F\|_{T_0,\mathtt{h}} + O(h^{-2p})\|F\|_h \\
&\leq O\left(L^{-6[\phi]}\right)\|K\|_{T_0,\mathtt{h}} + O(h^{-2p})\|K\|_h.
\end{aligned}
\tag{12.31}
$$

By (2), the contribution to $\|\tilde{K}-K_{\mathrm{main}}\|_{T_0,\mathtt{h}}$ is, using $|B|$ to count terms in $\sum_{y\in B}$,

$$
|B|O\left(L^{-6[\phi]}\right)\|K\|_{T_0,\mathtt{h}} + O(h^{-2p})\|K\|_h \leq O\left(L^{d-6[\phi]}\right)\|K\|_{T_0,\mathtt{h}} \vee \left(h^{-2p+1}\|K\|_h\right), \tag{12.32}
$$

where we used $h \geq L^{6[\phi]-d}$, which is true by $h = g^{-1/4}$ and $g \leq c(L)$, and we can choose $c(L)$. Since this holds for all p, we can write p in place of $2p-1$ and we have

$$
\frac{\|\tilde{K}-K_{\mathrm{main}}\|_{T_0,\mathtt{h}}}{\|K\|_{T_0,\mathtt{h}} \vee (h^{-p}\|K\|_h)} = O\big(L^{d-6[\phi]}\big) \tag{12.33}
$$

as $L\to\infty$. ■

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Part 4. The Euclidean Renormalisation Group

Lecture 13. Scaling Estimates; Coordinates; Step I

In this and the remaining lectures we will see how the hierarchical model techniques can be lifted to the Euclidean $\mathbb{Z}^d$ case. First we will discuss the scheme in an abstract way and then describe how it is applied to the anharmonic lattice

$$Z = \int_{\mathbb{R}^\Lambda} \prod_{xy \in \mathrm{Edges}(\Lambda)} e^{-f(\phi_x - \phi_y)}\, d^\Lambda \phi \tag{13.1}$$

where f is "nearly" Gaussian,

$$f(\phi_x - \phi_y) \simeq \frac{1}{2}(\phi_x - \phi_y)^2. \tag{13.2}$$

Proposition 13.1 (Brydges-Guadagni-Mitter 2003 [BGM04]**).** *Let ϕ be the $\mathbb{Z}^d$ massless free field. Let $L \in \mathbb{N}$, $L \geq 2$. Let $d \geq 3$. There exist independent $\{\zeta_j : j \geq 1\}$, where $\zeta_j = \{\zeta_j(x) : j \geq 1\}$, such that*

(1) *ζ_j is Gaussian, its law is $\mathbb{Z}^d$ invariant,*
(2) *$\mathrm{Cov}(\zeta_j(x), \zeta_j(y)) = 0$ if $|x - y| \geq L^j/2$,*
(3) *$\phi = \sum_{j \geq 1} \zeta_j$.*

Furthermore the same is true for the massive $\mathbb{Z}^d$ free field for $d \geq 1$.

These "increments" ζ_j cannot be scalings of $\zeta = \zeta_1$ because a scaling would live on $L^{-j}\mathbb{Z}^d \neq \mathbb{Z}^d$. However, $C_j(x, y) := \mathrm{Cov}(\zeta_j(x), \zeta_j(y))$ obeys *scaling estimates*:

13.1. Scaling Estimates.

$$|(\nabla_x^\alpha \nabla_y^\alpha C_j)(x, y)|_{x=y} \leq C(\alpha) L^{-2(j-1)([\phi]+|\alpha|)} \tag{13.3}$$

where

$$\nabla_e f(x) = f(x + e) - f(x), \tag{13.4}$$

$$\alpha = (e_1, e_2, \cdots, e_n) \in (\text{unit vectors})^*. \tag{13.5}$$

Since $C(x, y) = C(x - y)$, $\nabla_x^\alpha \nabla_y^\alpha C(x, y)|_{x=y} = \nabla_x^{2\alpha} C(0, 0)$.

Remark 13.2. In the massive case C_j does more: It becomes essentially zero for $j \geq \log_L(\text{mass})^{-1}$. In the massless case, for $d = 1, 2$, there exists ζ_j such that $\nabla\phi = \sum \nabla\zeta_j$ (while ϕ itself does not exist).

13.2. Coordinates. In the hierarchical model, we had

$$(e^{-V} + K)^\Lambda = \sum_{X \subset \Lambda} e^{-V(\Lambda \setminus X)} K^X = \sum_{X \in \mathcal{P}_0(\Lambda)} e^{-V(\Lambda \setminus X)} K^X, \tag{13.6}$$

recalling that $\mathcal{P}_0$ is all unions of L^0 blocks, i.e. points in Λ. In particular, by definition, K^X factors over points: $K^X = \prod_{x \in X} K_x$. In the Euclidean model, it is not possible to define K on points. Instead, (13.6) at scale j is replaced by

$$\sum_{X \in \mathcal{P}_j(\Lambda)} e^{-V_j(\Lambda \setminus X)} K_j(X) \tag{13.7}$$

where

$$K_j = \{K_j(X) : X \in \mathcal{P}_j\} \tag{13.8}$$

is a collection of random variables defined on polymers of the current scale, such that:

(1) $K_j(X)$ depends on

$$\{\phi_x : x \in X^*\} \tag{13.9}$$

where X^* is a neighbourhood of X defined later.

(2) $K_j(X)$ factorises as

$$K_j(X) = \prod_{Y \in \mathcal{C}(X)} K_j(Y) \tag{13.10}$$

where $Y \in \mathcal{C}(X)$ means that Y is a connected component of X.

Figure 13.1. $Y \in \mathcal{P}_j$, $j \in \mathbb{N}$, is connected if any pair of points $a, b \in Y$ are such that there is a sequence $(a = x_1, x_2, \cdots, x_n = b)$ with $\|x_i - x_{i-1}\|_\infty = 1$ for $i = 2, 3, \ldots, n$ and $x_i \in Y$, $i = 1, \ldots, n$.

Definition 13.3. For $X \in \mathcal{P}_j$, F, G functions on $\mathcal{P}_j$,

$$(F \circ G)(X) \overset{\text{def.}}{=} \sum_{Y \in \mathcal{P}_j(X)} F(Y)G(X \setminus Y). \tag{13.11}$$

With this definition the "coordinates" (V_j, K_j) represent a random variable that depends on all the fields $\{\phi_x : x \in \Lambda\}$ by

$$(V_j, K_j) \to (e^{-V_j} \circ K_j)(\Lambda). \tag{13.12}$$

13.3. RG Step I. The Euclidean RG is a method to compute

$$\mathbb{E} e^{-V_0(\Lambda)}, \quad \text{where} \quad V_0(\Lambda) = \sum_{x \in \Lambda} V_{0,x} \tag{13.13}$$

via

$$\lim_{N \to \infty} \mathbb{E}_N \cdots \mathbb{E}_1 e^{-V_0(\Lambda)}. \tag{13.14}$$

In contrast to the renormalisation group on the hierarchical lattice, there is no rescaling: The RG step is only $F_j \mapsto \mathbb{E}_{j+1} F_j$. If $j = 0$,

$$e^{-V(\Lambda)} = \sum_{X \in \mathcal{P}_0(\Lambda)} e^{-V_0(\Lambda \setminus X)} K_0(X), \quad \text{where} \quad K_0(X) = \begin{cases} 1 & \text{if } X = \emptyset \\ 0 & \text{else.} \end{cases} \tag{13.15}$$

The RG step, given (V_j, K_j) by inductive assumption, is finding (V_{j+1}, K_{j+1}) such that

$$\mathbb{E}_{j+1}(e^{-V_j} \circ K_j)(\Lambda) = (e^{-V_{j+1}} \circ K_{j+1})(\Lambda) = \sum_{X \in \mathcal{P}_{j+1}(\Lambda)} K_{j+1}(X) e^{-V_{j+1}(\Lambda \setminus X)}. \tag{13.16}$$

We symbolise this condition as follows:

$$(V_j, K_j) \overset{\mathbb{E}_{j+1}}{\to} (V_{j+1}, K_{j+1})$$

It is a two and a half step process beginning as in hierarchical case with a "linear guess" that $V_{j+1} \simeq \tilde{V}$ where

$$\tilde{V}(X) = \mathbb{E}_{j+1} V_j(X). \tag{13.17}$$

Our first objective is a formula for $\tilde{K}$ such that $(V_j, K_j) \overset{\mathbb{E}_{j+1}}{\to} (\tilde{V}, \tilde{K})$.

Definition 13.4. For $X \in \mathcal{P}_j$, $\bar{X}$ is the smallest set in $\mathcal{P}_{j+1}$ that contains X.

Definition 13.5. For $U \in \mathcal{P}_{j+1}$, we say $X \in \bar{\mathcal{P}}_j(U)$ if $\bar{X} = U$.

Example 13.6. Let

$$I_x = e^{-V_x}, \quad \tilde{I}_x = e^{-\tilde{V}_x}, \quad \delta I_x = I_x - \tilde{I}_x. \tag{13.18}$$

Then

$$I^X = (\tilde{I} + \delta I)^X = \sum_{Y \subset X} \delta I^Y \tilde{I}^{X \setminus Y} = \left((\delta I) \circ \tilde{I}\right)(X), \tag{13.19}$$

where

$$\delta I(Y) = \delta I^Y, \quad \tilde{I}(Y) = \tilde{I}^Y. \tag{13.20}$$

Fact. *Properties of "$\circ$":*

$$A \circ B = B \circ A \tag{13.21}$$

$$A \circ (B \circ C) = (A \circ B) \circ C \tag{13.22}$$

Lemma 13.7. $(V_j, K_j) \overset{\mathbb{E}_{j+1}}{\to} (\tilde{V}, \tilde{K})$ *where for* $U \in \mathcal{P}_{j+1}$

$$\tilde{K}(U) = \sum_{X \in \bar{\mathcal{P}}_j(U)} \tilde{I}^{U \setminus X} \mathbb{E}_{j+1}(K_j \circ \delta I)(X) \tag{13.23}$$

and $\tilde{K}$ *satisfies the factorisation property* (13.10)*, as a function on* $\mathcal{P}_j$*; that is*

$$\tilde{K}(U) = \prod_{X \in \mathcal{C}(U)} \tilde{K}(X) \tag{13.24}$$

where $X \in \mathcal{C}(U)$ *means that* $X \in \mathcal{P}_{j+1}$ *is a connected component of* U *as a set in* $\mathcal{P}_{j+1}$*.*

Remark 13.8. This lemma does not depend on the choice $\tilde{V} = \mathbb{E}V$. It holds for any $\tilde{V}$ which is not a function of ζ_{j+1} so that

$$\mathbb{E}_{j+1}(\tilde{I}^X(-)) = \tilde{I}^X \mathbb{E}_{j+1}(-). \tag{13.25}$$

Proof.

$$\begin{aligned}
\mathbb{E}_{j+1}(I \circ K)(\Lambda) &= \mathbb{E}_{j+1}\left((\tilde{I} \circ \delta I) \circ K\right)(\Lambda) \\
&= \mathbb{E}_{j+1}\left(\tilde{I} \circ (\delta I \circ K)\right)(\Lambda) \\
&= \sum_{X \in \mathcal{P}_j(\Lambda)} \mathbb{E}_{j+1}\tilde{I}(\Lambda \backslash X)(\delta I \circ K)(X) \\
&= \sum_{X \in \mathcal{P}_j(\Lambda)} \tilde{I}^{\Lambda \backslash X}\mathbb{E}_{j+1}(\delta I \circ K)(X) \\
&= \sum_{U \in \mathcal{P}_{j+1}(\Lambda)} \sum_{X \in \bar{\mathcal{P}}_j(U)} \tilde{I}^{\Lambda \backslash \bar{X}}\tilde{I}^{\bar{X} \backslash X}\mathbb{E}_{j+1}(\delta I \circ K)(X) \\
&= \sum_{U \in \mathcal{P}_{j+1}(\Lambda)} \tilde{I}^{\Lambda \backslash U} \sum_{X \in \bar{\mathcal{P}}_j(U)} \tilde{I}^{U \backslash X}\mathbb{E}_{j+1}(\delta I \circ K)(X) \\
&= \sum_{U \in \mathcal{P}_{j+1}(\Lambda)} \tilde{I}^{\Lambda \backslash U}\tilde{K}(U) \\
&= (\tilde{I} \circ \tilde{K})(\Lambda)
\end{aligned}$$

Factorisation depends on the finite range property of ζ_{j+1} and (13.10). See Problem 13.2. ■

13.4. Problems.

Problem 13.1. Prove that if $u \in C_0^\infty(\mathbb{R}^d)$ then

$$\int_{L^{j-1}}^{L^j} \frac{dl}{l} l^{-2[\phi]} u(\frac{x-y}{l}) \tag{13.26}$$

obeys scaling estimates.

Problem 13.2. Prove (13.24).

Problem 13.3. Show that Lemma 13.7 returns the hierarchical formula for $\tilde{K}$ when connectedness is defined with the hierarchical metric and Λ_∞ replaces $\mathbb{Z}^d$.

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Lecture 14. Small Sets and the Negligible Part of Step I

Question. What does $\tilde{K}$ reduce to if the metric is hierarchical?

Example 14.1. Lemma 13.7 still applies because it made no assumption about the metric. Thus, $\tilde{K}(U)$ where $U \in \mathcal{P}_{j+1}$ factorizes as

$$\tilde{K}(U) = \prod_{B \in \mathcal{B}_{j+1}(U)} \tilde{K}(B) \tag{14.1}$$

because on the hierarchical lattice, blocks are connected components of U.

$$\begin{aligned}\tilde{K}(B) &= \sum_{X \in \bar{\mathcal{P}}_j(B)} \tilde{I}^{B \setminus X} \mathbb{E}_{j+1}(\delta I \circ K)(X) \\ &= \sum_{\substack{X_K, X_{\delta I} \in \mathcal{P}_j(B) \\ X_K \cap X_{\delta I} = \emptyset}} \mathbb{1}_{\overline{X_K \cup X_{\delta I}} = B} \tilde{I}^{B \setminus (X_K \cup X_{\delta I})} \mathbb{E}_{j+1} \delta I^{X_{\delta I}} K^{X_K}\end{aligned} \tag{14.2}$$

We work out the part that does not contain any K:

$$\begin{aligned}&\sum_{\substack{X_{\delta I} \in \mathcal{P}_j(B) \\ X_{\delta I} \neq \emptyset}} \mathbb{1}_{\overline{X_{\delta I}} = B} \tilde{I}^{B \setminus X_{\delta I}} \mathbb{E}_{j+1} \delta I^{X_{\delta I}} \\ &= \mathbb{E}_{j+1}(\tilde{I} + \delta I)^B - \tilde{I}^B = \mathbb{E}_{j+1} I^B - \tilde{I}^B = \mathbb{E}_{j+1} e^{V(B)} - e^{-\mathbb{E}_{j+1} V(B)} = K_{\text{main}}\end{aligned} \tag{14.3}$$

Remark 14.2.

$$\delta I^X = \prod_{b \in \mathcal{B}_j(X)} \delta I(b), \quad \delta I(b) = I(b) - \tilde{I}(b) \tag{14.4}$$

(No $\widehat{L^{-1}}$ to collapse b to a point.)

14.1. Small sets.

Definition 14.3.

$$\mathcal{S}_j = \{X \in \mathcal{P}_j : X \text{ connected}, |X|_j \leq 2^d\} \tag{14.5}$$

is the set of small sets, where, for $X \in \mathcal{P}_j$,

$$|X|_j = |\mathcal{B}_j(X)| \tag{14.6}$$

is the number of j blocks in X. For $B \in \mathcal{B}_j$, $X \in \mathcal{P}_j$,

$$B^* = \bigcup \{Y \in \mathcal{S} : Y \supset B\}, \tag{14.7}$$

$$X^* = \bigcup \{B^* : B \in \mathcal{B}_j(X)\} \tag{14.8}$$

are the small set neighbourhoods of B and X respectively. For $U \in \mathcal{P}_{j+1}$, we say

$$X \in \overline{\mathcal{S}}_j(U) \quad \text{if} \quad \{\bar{X} = U \ \text{ and } \ X \in \mathcal{S}_j\}. \tag{14.9}$$

The following geometric lemmas hold for $L \geq L_0(d)$.

Lemma 14.4. *There exists $c > 1$ such that if $X \notin \mathcal{S}$ and X is connected, then*

$$|X|_j \geq c |\bar{X}|_{j+1}. \tag{14.10}$$

Lemma 14.5. *There exists $c > 1$ such that*

$$|X|_j \geq c|\bar{X}|_{j+1} - c2^{d+1}n(X) \tag{14.11}$$

where $n(X)$ is the number of connected components of $X \in \mathcal{P}_j$.

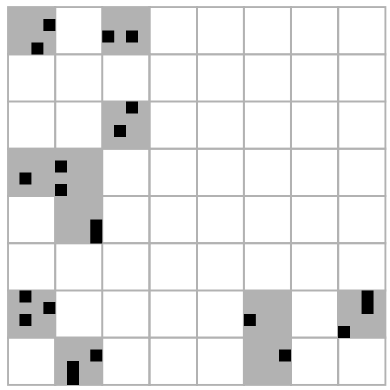

Figure 14.1. Illustration of geometric lemmas.

Let

$$\tilde{K}_{\text{main}}(U) = \sum_{X \in \bar{\mathcal{P}}_j(U)} \tilde{I}^{U \setminus X} \mathbb{E}_{j+1}(\delta I)^X \tag{14.12}$$

be the $X_K = \emptyset$ contribution to $\tilde{K}$. Let

$$* = \{(X_K, X_{\delta I}) \in \mathcal{P}_j^2(U) : \overline{X_K \cup X_{\delta I}} = U, X_K \cap X_{\delta I} = \emptyset, X_K \notin \mathcal{S}_j\}, \tag{14.13}$$

where $n(X_K)$ is the number of connected components of X_K. Let

$$R_*(U) = \sum_* \tilde{I}^{U \setminus (X_K \cup X_{\delta I})} \mathbb{E}_{j+1}(K(X_K)\delta I^{X_{\delta I}}) \tag{14.14}$$

be contribution to $\tilde{K}$ corresponding to the summands $*$. It will be negliglible in a sense to be made precise; to this end, we use properties of the norms which are only defined later.

Assumption (Norms). At each scale j, there are norms $\|\cdot\|_A \equiv \|\cdot\|$ for $A \geq 1$ (dependence on the scale j is suppressed in the notation), so that for all functions F, G on $\mathcal{P}_j$,

$$\|F(X)G(Y)\| \leq \|F(X)\| \, \|G(Y)\|, \quad \text{for all } X, Y \in \mathcal{P}_j \text{ disjoint.} \tag{14.15}$$

In the following, $\alpha > 1$ is a constant (not depending on the scale). We assume inductively that there is a constant $\epsilon_{\delta I}$ such that for δI and $\tilde{I}$ as in (13.18) and (13.17),

$$\|\mathbb{E}_{j+1}(\delta I)^X F(Y)\| \leq \alpha^{|X|_j + |Y|_j} \epsilon_{\delta I}^{|X|_j} \|F(Y)\|, \quad \text{for all } X, Y \in \mathcal{P}_j, \tag{14.16}$$

$$\|\tilde{I}^X\| \leq \alpha^{|X|_j}, \quad \text{for all } X \in \mathcal{P}_j, \tag{14.17}$$

and that there is a constant ϵ_K such that

$$\|K(X)\| \leq \epsilon_K^{n(X)} A^{-|X|_j}, \quad \text{for all } X \in \mathcal{P}_j. \tag{14.18}$$

Lemma 14.6. *There exists $\delta > 0$ and $c(A)$ such that*

$$\lim_{\substack{A\to\infty \\ \epsilon_K,\epsilon_{\delta I}\leq c(A)}} \frac{1}{\epsilon_K}\|R_*(U)\| A^{(1+\delta)|U|_{j+1}} = 0, \tag{14.19}$$

for L fixed.

In other words, for A sufficiently large, $\epsilon_K \leq c(A)$, $\epsilon_{\delta I} \leq c(A)$,

$$\|R_*(U)\| \leq 10^{-100}\epsilon_K A^{-(1+\delta)|U|_{j+1}}. \tag{14.20}$$

Proof. Note that $* = *_1 \cup *_2$ (disjoint) where

$$*_1 = \{(X_K, X_{\delta I}) \in * : n(X_K) \geq 2\},$$
$$*_2 = \{(X_K, X_{\delta I}) \in * : n(X_K) = 1\}.$$

Preliminary calculation: Lemma 14.5 implies

$$(\sharp)\quad A^{-|X_K|_j - |X_{\delta I}|_j} \leq A^{-c|\overline{X_K\cup X_{\delta I}}|_{j+1}} A^{c2^{d+1}(n(X_K)+n(X_{\delta I}))} \leq A^{-c|U|_{j+1}} A^{c2^{d+1}(n(X_K)+|X_{\delta I}|_j)}$$

because $n(X_{\delta I}) \leq |X_{\delta I}|_j$ and $\overline{X_K \cup X_{\delta I}} = U$. Then, by (14.15), (14.16),

$$\|\mathbb{E}_{j+1}(K(X_K)(\delta I)^{X_{\delta I}})\| \leq (\alpha^{|X_K|_j + |X_{\delta I}|_j}\epsilon_{\delta I}^{|X_{\delta I}|_j})(\epsilon_K^{n(X_K)} A^{-|X_K|_j})$$

and by (14.17)

$$\|\tilde{I}^{U\setminus(X_K\cup X_{\delta I})}\| \leq \alpha^{|U|_j - |X_K|_j - |X_{\delta I}|_j}.$$

Thus, inserting $1 = A^{|X_{\delta I}|_j}A^{-|X_{\delta I}|_j}$,

$$\begin{aligned}
\|R_{*_1}(U)\| &\leq \alpha^{|U|_j}\sum_{*}(\epsilon_{\delta I}A)^{|X_{\delta I}|_j}\epsilon_K^{n(X_K)}A^{-|X_K|_j}A^{-|X_{\delta I}|_j} \\
&\overset{(\sharp)}{\leq} \alpha^{|U|_j}A^{-c|U|_{j+1}}\sum_{*}\underbrace{(A^{c2^{d+1}+1}\epsilon_{\delta I})}_{\leq 1 \text{ by choice of } c(A)}{}^{|X_{\delta I}|_j}(A^{c2^{d+1}}\epsilon_K)^{n(X_K)} \\
&\leq \alpha^{|U|_j}A^{-c|U|_{j+1}}(A^{c2^{d+1}}\epsilon_K)^2\sum_{*}1 \\
&\leq \alpha^{|U|_j}A^{-c|U|_{j+1}}(A^{c2^{d+2}}\epsilon_K)\epsilon_K 3^{|U|_j} \\
&= \underbrace{(3\alpha)^{|U|_j}A^{-(c-1-\delta)|U|_{j+1}}}_{\substack{=((3\alpha)^{L^d}A^{-(c-1-\delta)})^{|U|_{j+1}} \\ \to 0 \text{ as } A\to\infty,\ L \text{ fixed}}}\ \underbrace{(A^{c2^{d+2}}\epsilon_K)}_{\leq 1 \text{ by } \epsilon_K < c(A)}\ (\epsilon_K A^{-(1+\delta)|U|_{j+1}}).
\end{aligned}$$

This shows that the contribution corresponding to $*_1$ satisfies the conclusion. The corresponding calculation for $*_2$ is Problem 14.1. ∎

14.2. Problems.

Problem 14.1. Use Lemma 14.4 to prove that the contribution corresponding to $*_2$ satisfies the conclusion of Lemma 14.6.

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Lecture 15. Cancellations on Small Sets and Step II

In the hierarchical model, we used a representation for the interaction

$$e^{-V_j} + K_j, \quad \text{with } K_j = O(\phi^6). \tag{15.1}$$

The RG action $(V_j, K_j) \to (V_{j+1}, K_{j+1})$ was constructed in two stages:

(1) $(V_j, K_j) \to (\tilde{V}, \tilde{K})$
(2) $(\tilde{V}, \tilde{K}) \to (V_{j+1}, K_{j+1})$

This lecture will focus on the Euclidean analogue of the second step.

15.1. Main, contractive and negligible parts of step I. Define

$$\tilde{L}(U) = \sum_{X \in \overline{\mathcal{S}}_j(U)} \tilde{I}^{U \setminus X} \mathbb{E}_{j+1} K(X). \tag{15.2}$$

Putting this definition together with our work in the last lecture, we find that the action of RG in (V, K) coordinates,

$$\mathbb{E}_{j+1}(I_j \circ K_j)(\Lambda) = \tilde{I} \circ \tilde{K}(\Lambda), \tag{15.3}$$

is given by

$$\tilde{I} = e^{-\tilde{V}}, \quad \tilde{K} = \tilde{K}_{\text{main}} + \tilde{L} + R_*, \tag{15.4}$$

where

$$\tilde{V} = \mathbb{E}_{j+1} V \text{ and } I_j = e^{-V_j}, \tag{15.5}$$

and

$$\tilde{K}_{\text{main}}(U) = \sum_{X \in \overline{\mathcal{P}}_j(U)} \tilde{I}^{U \setminus X} \mathbb{E}_{j+1} (\delta I)^X \tag{15.6}$$

is a function only of V and R_* is a negligible contribution of $\tilde{K}$ (Lemma 14.6).

Following the hierarchical 'yellow brick road', we would like to construct a domain for (V, K), where the RG is bounded in norm. The main step to the wizard will be to prove that $\tilde{L}$ is contractive, which is done in an algebraic fashion.

Remark 15.1. In this lecture we are not rescaling the norms. The norms will be introduced in the next lecture and shall be rescaled there.

15.2. Cancellations on blocks and small sets. Note that (15.2), the sum over $X \in \overline{\mathcal{S}}_j(U)$, the small sets at level j, has $O(L^d)$ terms. This sets the stage for an $O(L^d)$ expansion in the norm. The same issue appeared in the hierarchical case, where the remedy was to impose the inductive assumption $K_j = O(\phi^6)$, because this gave the compensating $L^{-6[\phi]}$ of (I) in Proposition 11.6. Then, in order to have $K = O(\phi^6)$ at the next scale, we have solved

$$e^{-\tilde{V}} + \tilde{K} = e^{-V_{j+1}} + K_{j+1}. \tag{15.7}$$

Example 15.2 (Cancellations on blocks). The analogous procedure for the Euclidean case is to adjust $\tilde{V}$ to V' in such a way that

$$e^{-\tilde{V}} \circ \tilde{K} = e^{-V'} \circ K' \tag{15.8}$$

with

$$K'(B) = O(\phi^6), \quad \text{for } B \in \mathcal{B}_{j+1}. \tag{15.9}$$

In more detail:

$$\begin{aligned} e^{-\tilde{V}} \circ \tilde{K} &= (e^{-V'} + e^{-\tilde{V}} - e^{-V'}) \circ \tilde{K} \\ &= (I' \circ \delta\tilde{I}) \circ \tilde{K} \\ &= I' \circ (\delta\tilde{I} \circ \tilde{K}) \end{aligned} \tag{15.10}$$

Thus,

$$\begin{aligned} K'(B) &= (\delta\tilde{I} \circ \tilde{K})(B) \\ &= \delta\tilde{I}(B) + \tilde{K}(B) \\ &= e^{-\tilde{V}(B)} - e^{-V'(B)} + \tilde{K}(B) \end{aligned} \tag{15.11}$$

and we can adjust $V'(B)$ so that $K'(B) = O(\phi^6)$, for $\phi =$ constant on B. In other words, this procedure only works for blocks.

It will thus not solve the problem of transferring from scale j to $j+1$ all the conditions

$$K_j(X) = O(\phi^6), \quad X \in S_K. \tag{15.12}$$

How to do this is the key problem to be surmounted in the Euclidean case. The solution I am about to describe is contained in my ongoing work with Gordon Slade [BS], but it evolved from a more primitive idea in [BY90].

Let $J = \{J(X) : X \in \mathcal{P}_{j+1}\}$ be so that

$$J(X) = 0 \quad \text{if } X \notin \mathcal{S}_{j+1}, \tag{15.13a}$$

$$\sum_{X \supset B} \frac{1}{|X|_{j+1}} \tilde{I}^{-X} J(X) = 0 \quad \text{for all } B \in \mathcal{B}_{j+1}. \tag{15.13b}$$

Let ϵ be so that

$$\|J(X)\| \leq \epsilon A^{-|X|_{j+1}} \tag{15.14}$$

and

$$\|\tilde{K}(X) - J(X)\| \leq \epsilon A^{-(1+\delta)|X|_{j+1}}. \tag{15.15}$$

Proposition 15.3. *There exists a constant $c(A)$ and K' such that $\tilde{I} \circ \tilde{K} = \tilde{I} \circ K'$, such that K' factors over connected components, and satisfies*

$$\lim_{A \to \infty, \epsilon \leq c(A)} \epsilon^{-1} A^{(1+\delta)|X|_{j+1}} \|K'(X) - (\tilde{K}(X) - J(X))\| = 0. \tag{15.16}$$

This solves the problem of arranging for $K'(X) = O(\phi^6)$ for all $X \in \mathcal{S}_{j+1} \backslash \mathcal{B}_{j+1}$ because we can choose $\{J(X) : X \in \mathcal{S}_{j+1} \backslash \mathcal{B}_{j+1}\}$ so that

$$\tilde{K}(X) - J(X) = O(\phi^6), \quad \text{for } X \in \mathcal{S}_{j+1} \setminus \mathcal{B}_{j+1} \tag{15.17}$$

(on $\phi =$ constant). The relation in (15.13b) then determines $J(B)$ for $B \in \mathcal{B}_{j+1}$. Therefore, we will not have the desired $K'(B) = O(\phi^6)$ for $B \in \mathcal{B}_{j+1}$, but this is the problem we know how to solve by adjusting $\tilde{I}(B)$ as in Example 15.2.

Proof of Proposition 15.3. i) Construction of K': Given $W \in \mathcal{P}_{j+1}$, let $\mathcal{I}(W)$ be the set of triples $(X, \overrightarrow{U}, U_M)$ where

(1) $X \in \mathcal{P}_{j+1}(W)$,
(2) $\overrightarrow{U} = \{U(B) \in \mathcal{S}_{j+1} : B \in \mathcal{B}_{j+1}(X), U(B) \supset B\}$,

(3) $U_M \in \mathcal{P}_{j+1}(W)$,
(4) $\{U(B) : B \in \mathcal{B}_{j+1}(X)\}$ and U_M are strictly disjoint,
(5) $X^* \cup U_M = W$,
(6) triples with $|X|_{j+1} = 1, U_M = \emptyset$ are omitted.

The conditions are not needed right away. They describe the constraints arising in the sums described as follows: Using $\tilde{K} = J + M$ where $M = \tilde{K} - J$, write

$$\begin{aligned}
\tilde{K} \circ \tilde{I}(\Lambda) &= \sum_{\tilde{U} \in \mathcal{P}_{j+1}(\Lambda)} \left(\prod_{U \in \mathcal{C}(\tilde{U})} (J(U) + M(U)) \right) \tilde{I}^{\Lambda \setminus \tilde{U}} \\
&= \sum_{U_J, U_M} \left(\prod_{U \in \mathcal{C}(U_J)} J(U) \right) \left(\prod_{V \in \mathcal{C}(U_M)} M(V) \right) \tilde{I}^{\Lambda \setminus (U_J \cup U_M)},
\end{aligned}$$

where the sum is over $U_J, U_M \in \mathcal{P}_{j+1}(\Lambda)$ that are strictly disjoint (no path contained in the union connects the two sets). Insert, for $U \in \mathcal{C}(U_J)$, the trivial identity

$$J(U) = \sum_{B \in \mathcal{B}_{j+1}(U)} \frac{1}{|U|_{j+1}} J(U).$$

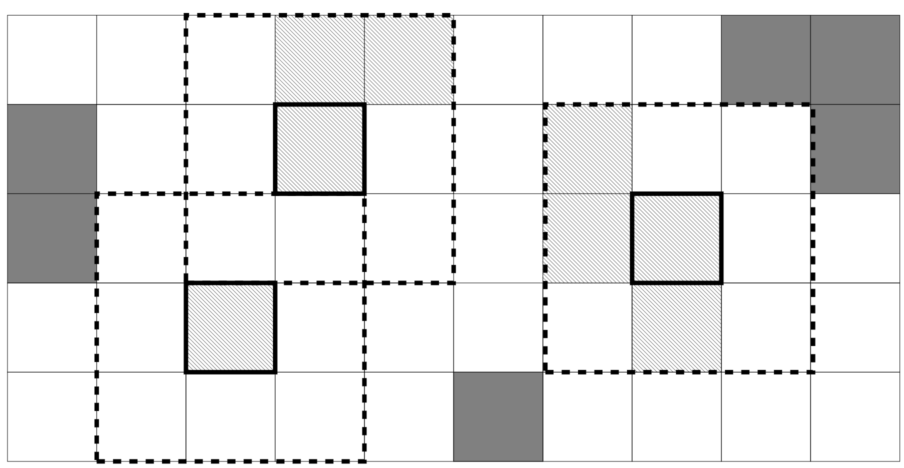

Figure 15.1. U_J (lightly shaded) and U_M (dark shaded) are strictly disjoint (no path contained in the union connects them). The connected components $U(B)$ of U_J are indexed by singled out blocks $B \in U(B)$ (framed in black). X is the union of these singled out blocks. Its small set neighbourhood X^* is the collection of blocks contained inside the dashed frames. Since the sets $U(B)$ can be taken small (J vanishes on sets that are not small), they are contained completely inside X^*. Finally, W is the union of X^* and U_M.

This creates a sum over pairs $\{(B,U(B)) : U(B) \in \mathcal{C}(U_J)\}$ (see Figure 15.1). Let $X \in \mathcal{P}_{j+1}$ be the union of these blocks $B \in \mathcal{B}_{j+1}$. Let $W = X^* \cup U_M$. Then,

$$\tilde{K} \circ \tilde{I}(\Lambda) = \sum_{W \in \mathcal{P}_{j+1}(\Lambda)} \left(\sum_{(X,\vec{U},U_M) \in \mathcal{I}(W)} \left(\prod_{B \in \mathcal{B}_{j+1}(X)} \frac{1}{|U(B)|_{j+1}} J(U(B)) \right) \left(\prod_{V \in \mathcal{C}(U_M)} M(V) \right) \tilde{I}^{W \setminus (U' \cup U_M)} \right) \tilde{I}^{\Lambda \setminus W} \quad \text{with } U' = \bigcup_{B \in \mathcal{B}_{j+1}(X)} U(B).$$

Let $K'(W)$ be the factor in the huge parenthesis, i.e.

$$K'(W) = \sum_{(X,\vec{U},U_M) \in \mathcal{I}(W)} \left(\prod_{B \in \mathcal{B}(X)} \frac{1}{|U(B)|_{j+1}} J(U(B)) \right) \left(\prod_{V \in \mathcal{C}(U_M)} M(V) \right) \tilde{I}^{W \setminus (U' \cup U_M)}$$

so that

$$\tilde{K} \circ \tilde{I} = K' \circ \tilde{I}$$

as claimed. In $K'(W)$, consider the terms where $U_M = \emptyset$, $X = B \in \mathcal{B}_{j+1}$. They are

$$\tilde{I}^W \sum_{U(B) \supset B} \frac{1}{|U(B)|_{j+1}} J(U(B)) \tilde{I}^{-U(B)} = 0$$

by (15.13b). Therefore, condition (6) holds.

ii) For the bound on $\|K'(W) - (\tilde{K}(W) - J(W))\|$, looking at the formula for K' we see that the contribution to K' when $X = \emptyset$ cancels with $\tilde{K} - J$ because $M = \tilde{K} - K$ for U_M with one component. Therefore $K'(W) - \tilde{K}(W) - I$ is second order in ϵ. These higher order terms are bounded using the same ideas as were used in the proof of Lemma 14.6. See [SS09] for the proof of a similar result. ■

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Lecture 16. Gradient Perturbations of the Massless Free Field

16.1. The model. In this lecture we consider a model with the following partition function:

$$Z = \mathbb{E}_{\text{massless free field}} F^{\Lambda}, \tag{16.1}$$

where

(1) $F_x \in C^3$ and F is an even function of $(\nabla\phi)_x$, where

$$(\nabla\phi)_x = \left\{ \nabla_e \phi(x) := \phi(x+e) - \phi(x) \colon e \text{ is a unit vector in } \mathbb{Z}^d \right\}; \tag{16.2}$$

(2) For $p = 0, 1, 2, 3$ and some positive constants ϵ and h,

$$\left| \frac{\partial^p}{\partial(\nabla\phi)^p}(F_x - 1) \right| \le \epsilon \cdot e^{h^{-2}(\nabla\phi)_x^2}, \tag{16.3}$$

where

$$(\nabla\phi)_x^2 = \sum_{e \in \mathbb{Z}^d \colon \|e\|=1} (\phi(x+e) - \phi(x))^2; \tag{16.4}$$

(3) F_x is invariant under the lattice symmetries of $\mathbb{Z}^d$ that fix $x \in \mathbb{Z}^d$ and under translations of $\mathbb{Z}^d$.

For example, one can take F like this:

$$F_x = \exp\left(-\epsilon \sum_{e \in \mathbb{Z}^d \colon \|e\|=1} (\phi(x+e) - \phi(x))^4 \right). \tag{16.5}$$

The boundary conditions of the model can be taken to be periodic or (as in [Dim08]) infinite volume massless free field.

Theorem 16.1. *For ϵ small, h large, the scaling limit of the model is massless Gaussian with renormalised covariance $\kappa^{-1}(-\Delta)^{-1}$ for some $0 < \kappa \ne 1$.*

16.2. Notation. Let $x = (x_1, \dots, x_n) \in \Lambda^*$, and $h > 0$. Write

$$x! = n!, \tag{16.6}$$

$$h^x = h^n, \tag{16.7}$$

and, for $F \in C^\infty(\mathbb{R}^\Lambda)$,

$$F_x(\phi) = \frac{\partial^n F(\phi)}{\partial\phi_{x_1} \dots \partial\phi_{x_n}}. \tag{16.8}$$

In this notation, the Taylor expansion is

$$F(\phi + \zeta) \sim \sum_{x \in \Lambda^*} \frac{1}{x!} F_x(\phi) \zeta^x, \quad \text{with } \zeta^x := \prod_{i=1}^n \zeta_{x_i}. \tag{16.9}$$

We write "$\sim$" in (16.9) because we do not know whether the Taylor series converges to F.

16.3. Test functions. We design a norm on the space

$$\Phi = \{g \colon \Lambda^* \to \mathbb{R}\}\,, \tag{16.10}$$

$$\|g\|_\Phi := \sup_{x\in\Lambda^*}\sup_{\alpha\in A} h_j^{-x}|(\nabla_j^\alpha g)(x)|. \tag{16.11}$$

Here $h_j = h_0 L^{-j[\phi]}$ is the scale at jth scaling level, and

$$\nabla_{j,e} = L^j \nabla_e, \quad \text{for } e \in \mathbb{Z}^d,\ \|e\| = 1. \tag{16.12}$$

We choose

$$A = \{\alpha \colon \text{at most two derivatives with respect to each of } (x_1,\ldots,x_n)\}\,. \tag{16.13}$$

We have defined test functions of scaling level j.

Remark 16.2. Test functions of norm one resemble products of fields. Indeed, according to the scaling estimates (13.3), we have

$$\mathrm{Var}(h_j^{-1}\nabla_j^\alpha \zeta_j(x)) = O\left(L^{|\alpha|+[\phi]}\right), \tag{16.14}$$

and this estimate is independent of j. Here ζ_j are independent Gaussian random variables of which our field ϕ can be constructed (Proposition 9.3).

Definition 16.3. For $F \in C^\infty(\mathbb{R}^\Lambda)$, $g \in \Phi$,

$$\langle F, g\rangle_\phi \overset{\text{def.}}{=} \sum_{x\in\Lambda^*} \frac{1}{x!} F_x(\phi) g_x, \tag{16.15}$$

$$\|F\|_{T_\phi} \overset{\text{def.}}{=} \sup\left\{ |\langle F, g\rangle_\phi| \colon\ \|g\|_\Phi = 1\right\}. \tag{16.16}$$

Remark 16.4. This norm is the result of replacing the product ζ^x in Taylor expansion (16.9) by a test function of norm one.

Remark 16.5 (Analyticity). We do not need F to be analytic, so we add a condition to Φ that $g_x = 0$ if $x = (x_1, \cdots, x_n)$ has $n > P_\mathcal{N}$. For $\nabla\phi$ models choose $P_\mathcal{N} = 3$.

Proposition 16.6.

$$\|FG\|_{T_\phi} \le \|F\|_{T_\phi} \|G\|_{T_\phi}\,. \tag{16.17}$$

Proof. Exercise. ■

16.4. Localization of norms. Let $X \subset \Lambda$ be a subset. We say that

$$F \in \mathcal{N}(X) \quad \text{iff} \quad F_x(\phi) = 0 \quad \text{for all } x \notin X^*. \tag{16.18}$$

Define

$$\|g\|_{\Phi(X)} = \inf\left\{\|g+f\|_\Phi : f_x = 0 \text{ for } x \in X^*\right\}. \tag{16.19}$$

Then, for $F \in \mathcal{N}(X)$, $g \in \Phi$,

$$|\langle F, g\rangle_\phi| \le \|F\|_{T_\phi} \|g\|_{\Phi(X)}\,. \tag{16.20}$$

Proof. For all f such that $f_x = 0$ for $x \in X^*$,

$$|\langle F, g\rangle_\phi| = |\langle F, g+f\rangle_\phi| \le \|F\|_{T_\phi} \|g+f\|_\Phi\,. \tag{16.21}$$

Now take the infimum over f. ■

16.5. Weighted L_∞. In the hierarchical case we used

$$\|K\| = \sup_\phi \|K\|_{T_\phi}, \tag{16.22}$$

but Euclidean models all require weighted L_∞ norms because

$$\tilde{K}(X) - J(X) = O(\nabla\phi)^4, \tag{16.23}$$

grows as $|\nabla\phi|$ grows. Therefore, we use

$$\|K_j(X)\|_{G_j} := \sup_\phi \|K_j(X)\|_{T_\phi} G_j^{-1}(X,\phi) \tag{16.24}$$

with weight G_j such that

$$\prod_{Y\in\mathcal{C}(X)} G_j(Y) \le G_j(X), \tag{16.25}$$

and

$$\mathbb{E}_{j+1}G_j \le G_{j+1} \quad \text{(supermartingale property)}, \tag{16.26}$$

and $\|\phi\|_\Phi$ is dominated by G. See [SS09] for a detailed discussion.

16.6. Loc. Let $\mathcal{S}$ be the span of the monomials

$$\{1, \nabla\phi, \nabla\phi\cdot\nabla\phi\}.$$

For a polynomial $P\in\mathcal{S}$, $x\in\Lambda$, let P_x be P evaluated at the fields at x. For $X\subset\Lambda$, let

$$P(X) \overset{\text{def.}}{=} \sum_{x\in X} P_x, \quad \mathcal{S}(X) \overset{\text{def.}}{=} \{P(X) \colon P\in\mathcal{S}\}. \tag{16.27}$$

Now we consider the space of polynomial test functions Π, that is, the space of functions $g\colon \Lambda^*\to\mathbb{R}$ that when restricted to X, satisfy

$$g\big|_{\Lambda^0} = \mathrm{c}\cdot 1, \tag{16.28a}$$

$$g\big|_{\Lambda^1} = \text{polynomial of degree} \le d/2, \tag{16.28b}$$

$$g\big|_{\Lambda^2} = \text{polynomial of degree } 0. \tag{16.28c}$$

Definition 16.7. $\mathrm{Loc}_X\colon \mathcal{N}\to\mathcal{S}(X)$ is the linear map characterised by

$$\langle F, g\rangle_0 = \langle P(X), g\rangle_0, \quad \text{for all } g\in\Pi, \tag{16.29}$$

where $P(X) = \mathrm{Loc}_X F$.

Proposition 16.8. *The map exists. It is unique. It is bounded in T_0 norm.*

16.7. Summary. We use the norms (depending on A)

$$\|K_j\|_j \overset{\text{def.}}{=} \sup_{X\in\mathcal{P}_{c,j}} \|K_j(X)\|_{G_j} A^{|X|_j}. \tag{16.30}$$

Step 1. Given

$$(I_j, K_j) \quad \text{with} \quad \|\mathrm{Loc}_X K_j(X)\|_j = (\text{negligible if } A\gg 1), \tag{16.31}$$

we start with

$$(I_j, K_j) \to (\tilde{I}, \tilde{K}), \tag{16.32}$$

where (Lemma 14.6)

$$\tilde{K} = \tilde{K}_{\mathrm{main}} + \tilde{\mathcal{L}} + (\text{negligible if } A \gg 1) \tag{16.33}$$

and

$$\|\tilde{\mathcal{L}}\|_{j+1} \leq O(L^{d-4[\nabla\phi]})\|K\|_j. \tag{16.34}$$

Step 2a. Next, by Proposition 15.3,

$$(\tilde{I}, \tilde{K}) \to (\tilde{I}, K'),$$

where

$$K' = \tilde{K}_{\mathrm{main}} + \tilde{\mathcal{L}} - J + (\text{negligible if } A \gg 1). \tag{16.35}$$

and $J(X)$ can be chosen arbitrarily for all small sets X except blocks. We choose

$$J(X) = \mathrm{Loc}_X \left(\tilde{K}_{\mathrm{main}}(X) + \tilde{\mathcal{L}}(X)\right), \quad \text{for } X \in \mathcal{S}_{j+1} \setminus \mathcal{B}_{j+1}. \tag{16.36}$$

This choice of J achieves a map

$$(I_j, K_j) \to (\tilde{I}, K'),$$

with (because $\mathrm{Loc}_X \mathrm{Loc}_X = \mathrm{Loc}_X$)

$$\mathrm{Loc}_X K'(X) = (\text{negligible if } A \gg 1), \quad \text{for all } X \in \mathcal{S}_{j+1} \setminus \mathcal{B}_{j+1}.$$

(The norm still has $O(L^{-d})$ contraction in the $(1 - \mathrm{Loc})\tilde{\mathcal{L}}$ part from (16.34).)

Step 2b. In Example 15.2 we worked out how to get rid of the blocks $B \in \mathcal{B}_{j+1}$. Thus,

$$(\tilde{I}, K') \to (I_{j+1}, K_{j+1}),$$

with

$$\mathrm{Loc}_X K_{j+1}(X) = (\text{negligible if } A \gg 1), \quad \text{for all } X \in \mathcal{S}_{j+1}.$$

Final Step. Since

$$\left\| K_{j+1} - (1 - \mathrm{Loc})\tilde{K}_{\mathrm{main}} \right\|_{j+1} \leq O\left(L^{-d}\right) \|K_j\|_j, \tag{16.37}$$

we conclude that RG stays close to the map

$$(I_j, K_j) \to \left(I_{j+1}, (1 - \mathrm{Loc})\tilde{K}_{\mathrm{main}}\right), \tag{16.38}$$

which is computable.

16.7.1. *Tuning.* A point discussed in detail in [SS09]: We want $I_j \to c$ as $j \to \infty$, but this will only happen if we choose a "critical" κ (in Theorem 16.1) as described below (the case $j = 0$). That is, V needs a "counterterm" $\frac{1}{2}(1 - \kappa)(\phi, -\Delta\phi)$ in order to be driven to zero. This is why in Theorem 16.1 the scaling limit is massless Gaussian with renormalised covariance $\kappa^{-1}(-\Delta)^{-1}$.

The need for the lattice symmetry of F can also be justified. Namely, this hypothesis ensures that the Example 15.2 step just changes the constants a, b in $V = a(\nabla\phi)^2 + b$ as opposed to adding terms of the form $\nabla_e\phi \cdot \nabla_{e'}\phi$ with $e \neq e'$.

16.7.2. *The case $j = 0$.* The Gaussian measure contains a factor $e^{-\frac{1}{2}(\phi,-\Delta\phi)}$. For some not yet determined κ, we have

$$
\begin{aligned}
e^{-\frac{1}{2}(\phi,-\Delta\phi)}F^{\Lambda} &= e^{-\frac{1}{2}\kappa(\phi,-\Delta\phi)}e^{-\frac{1}{2}(1-\kappa)(\phi,-\Delta\phi)}F^{\Lambda} \\
&= e^{-\frac{1}{2}\kappa(\phi,-\Delta\phi)}\left(e^{-\frac{1}{2}(1-\kappa)(\nabla\phi)^2}\right)^{\Lambda}F^{\Lambda} \\
&= e^{-\frac{1}{2}\kappa(\phi,-\Delta\phi)}\left(e^{-\frac{1}{2}(1-\kappa)(\nabla\phi)^2}\right)^{\Lambda}(1+F-1)^{\Lambda} \\
&= e^{-\frac{1}{2}\kappa(\phi,-\Delta\phi)}(I_0 \circ K_0)(\Lambda)
\end{aligned}
$$

with

$$
I_0 := e^{-\frac{1}{2}(1-\kappa)(\nabla\phi)^2}
$$

and

$$
K_0(X) := \left(e^{-\frac{1}{2}(1-\kappa)(\nabla\phi)^2}(F-1)\right)^X .
$$

16.8. References. All proofs of Theorem 16.1 using RG will have various details missing, but I tried very hard in the Park City notes to be very detailed in everything I do cover. Proofs based on convexity are [NS97, CD08]. Proofs based on RG are [Dim08, SS09]. Many ideas in the last four lectures are in [BS].

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