Lecture 16

The Model

$$Z = \mathbb{E}_{\substack{\text{massless} \\ \text{free field}}} F^{\Lambda}$$

where

$F_x \in C^3$, even function of $\nabla\phi$.

$$\left| \frac{\partial^p}{\partial(\nabla\phi)^p} \left(F_x - 1 \right) \right| \leq \epsilon \, e^{h^{-2} (\nabla\phi)_x^2}$$

F_x invariant under lattice symmetries that fix x. (Lattice symmetry)

Boundary conditions

Periodic or (Dimock 2009) ∞ vol massless free field

For ϵ small, h large

Theorem 1 The scaling limit is massless Gaussian with renormalised covariance.

Notation

Let $x = (x_1, x_2, \cdots, x_n) \in \Lambda^*$, $\hbar > 0$.

Write,

$$x! = n!$$

$$\hbar^x = \hbar^n$$

and, for $F \in C^\infty(\mathbb{R}^\Lambda)$

$$F_x(\phi) = \frac{\partial^n F(\phi)}{\partial\phi_{x_1} \cdots \partial\phi_{x_n}}$$

so that the Taylor expansion is

$$F(\phi+\zeta) \sim \sum_{x \in \Lambda^*} \frac{1}{x!} F_x(\phi)\, \zeta^x$$

$$\zeta^x = \prod_{i=1}^{n} \zeta_{x_i}$$

Test functions

We design a norm

$$\Phi = \{ g : \Lambda^* \to \mathbb{R} \}$$

$$\|g\|_{\Phi} = \sup_{x \in \Lambda} \sup_{\alpha \in A} h_j^{-x} \left| \nabla_j^{\alpha} g(x) \right|$$

$$h_j = h_0 L^{-j[\phi]}$$

$$\nabla_j = L^j \nabla$$

so that test function of norm one resemble products of fields: according to (scaling estimates)

$$\mathrm{Var}\left(h_j^{-1} \nabla_j^{\alpha} \zeta_j(x) \right) = O\left(L^{|\alpha| + [\phi]}\right)$$

(indep. of j)

Choose $A = \{$at most 2 derivatives ... with respect to each of $(x_1, x_2, \dots, x_n)\}$

Defn 2

$$\langle F, g \rangle_{\phi} = \sum_{x \in \Lambda^*} \frac{1}{x!} F_x(\phi)\, g_x$$

$$\| F \|_{T_\phi} = \sup \left\{ |\langle F, g \rangle_{\phi}| : \| g \|_{\Phi} = 1 \right\}$$

Remark 3 The norm is the result of replacing the product ζ^x of increments in $F(\phi + \zeta)$ by a test function of norm one

Analyticity We do not need F to be analytic so we add a condition to Φ that $g_x = 0$ if $x = (x_1, \ldots, x_n)$ has $n > p_N$. For $\nabla\phi$ models choose $p_N = 3$.

Proposition 4

$$\| F G \|_{T_\phi} \leq \| F \|_{T_\phi} \| G \|_{T_\phi}$$

Localisation of norms

This section can be omitted unless it comes up in questions about the use of $C^2(\Lambda)$.

$X \subset \Lambda$.

We say

$$F \in \mathcal{N}(X)$$

if $\quad F_x(\phi) = 0 \quad \forall\, x \notin X^*$

Define

$$\|g\|_{\Phi(X)} = \inf \{ \|g+f\|_{\Phi} : f_x = 0 \text{ for } x \in X^* \}$$

Then

$$|\langle F, g \rangle_\phi| \le \|F\|_{T_\phi} \|g\|_{\Phi(X)}$$

Proof $\forall f,\ f_x = 0$ for $x \in X^*$,

$$|\langle F, g \rangle_\phi| = |\langle F, g+f \rangle|$$

$$\le \|F\|_{T_\phi} \|g+f\|_{\Phi}$$

Take infimum over f. ∎

Weighted L_∞

In the hierarchical case we used

$$\|K\| = \sup_\phi \|K\|_{T_\phi}$$

but Euclidean models all require weighted L_∞ norms because

$$\tilde{K}(X) - J(X) = O(\nabla\phi)^4$$

grows as $|\nabla\phi| \nearrow$

Therefore we use

$$\|K_j(X)\|_{G_j} = \sup_\phi \|K_j(X)\|_{T_\phi} G_j^{-1}(X,\phi)$$

with weight G_j s.t.

$$\prod_{Y \subset\subset X} G_j(Y) \leq G_j(X)$$

$$\mathbb{E}_{j+1} G_j \leq G_{j+1} \qquad \text{(supermartingale)}$$

and $\|\phi\|_\Phi$ is dominated by G. See \cite{bry2009Brydges} for a detailed discussion

Loc

Monomials: let $\mathcal{S}$ be span of the monomials

$$\{1, \nabla\phi \cdot \nabla\phi\}$$

For $P \in \mathcal{S}$ and $X \subset \Lambda$

$$P(X) = \sum_{x \in X} P_x$$

$$\mathcal{S}(X) = \{P(X) : P \in \mathcal{S}\}$$

Polynomial test functions

Π is the space of $g : \Lambda^* \to \mathbb{R}$ s.t when restricted to X,

$$g|_{\Lambda^0} = 1$$

$$g|_{\Lambda^1} = \text{polynomials of degree} \leq d/2$$

$$g|_{\Lambda^2} = \text{polynomial degree } 0$$

Defn 5

$Loc_X : \mathcal{N} \to S(X)$ is the linear map characterized by

$$\langle F, g \rangle_0 = \langle P(X), g_0 \rangle$$

for all $g \in \Pi$

Proposition 6 This map exists. It is unique. It is bounded in T_0 norm.

Summary

RG map using

$$\| K_j \|_j \overset{\text{def}}{=} \sup_{X \in \mathcal{P}_{c,j}} \| K_j(X) \|_{G_j} A^{|X|_j}$$

given

$$(I_j, K_j) \quad \text{with} \quad \| \mathrm{Loc}_X K_j(X) \|_j := \text{negligible} \quad \cancel{(A \gg 1)}$$

We start with

$$(I_j, K_j) \longrightarrow (\tilde{I}, \tilde{K})$$

$$\tilde{K} = \tilde{K}_{main} + \tilde{\mathcal{L}} + \cancel{(A \gg 1)} \qquad \text{(Prop 14.6)}$$

and

$$\| \tilde{\mathcal{L}} \|_{j+1} \leq O\left(L^{d-4[\nabla\phi]}\right) \| K \|_j \qquad \text{(e: contracts)}$$

Next, by Prop 15.3,

$$(\tilde{I}, \tilde{K}) \longrightarrow (\tilde{I}, K')$$

$$K' = K_{main} + \tilde{\mathcal{L}} - J + \cancel{(A \gg 1)}$$

Choose

$$\mathcal{I}(X) = \text{Loc}_X \left(K_{main}(X) + \tilde{\mathcal{I}}(X) \right)$$

$$X \in \mathcal{S}_{j+1} \setminus \mathcal{B}_{j+1}$$

This achieves a map

$$(I_j, K_j) \longrightarrow (\tilde{I}, K')$$

with $\text{Loc}_X K'(X) = \cancel{A \gg 1}$ $\quad X \in \mathcal{S}_{j+1} \setminus \mathcal{B}_{j+1}$

(The norm still has $O(L^{-d})$ contraction in $(1-\text{Loc})\mathcal{L}$ part from (e: contracts))

Example 15.2:

$$(\tilde{I}, K') \longrightarrow (I_{j+1}, K_{j+1})$$

$$\text{Loc}_X K_{j+1}(X) = \cancel{A \gg 1} \quad \text{for } X \in \mathcal{S}_{j+1}$$

and since

$$\| K_{j+1} - (1-\text{Loc})\tilde{K}_{main} \|_{j+1} \leq O(L^{-d}) \ \| K_j \|_j$$

RG stays close to

$$(I_j, K) \longrightarrow (I_{j+1}, (1-\text{Loc})\tilde{K}_{main})$$

which is computable.

A point discussed in detail in
\cite{bry2009Bry}:

We want $I_j \longrightarrow$ Const. as $j \to \infty$ but this will only happen if we choose a "critical" κ as described below under the heading "$j = 0$".

i.e. V needs a "counterterm" $\frac{1-\kappa}{2}(\phi, -\Delta\phi)$ in order to be driven to zero.

This is why the scaling limit is massless Gaussian with renormalised covariance $\frac{1}{\kappa}\frac{1}{-\Delta}$

Need for the Lattice symmetry Hypothesis. This ensures that the Example 15-3 step just changes the constants a, b in

$$V = a(\nabla\phi)^2 + b$$

as opposed to adding terms like $\nabla_e \phi \, \nabla_{e'} \phi$ with $e \neq e'$

<u>$j=0$</u>

The Gaussian measure contains $e^{-\frac{1}{2}(\phi, -\Delta\phi)}$

For some not yet determined κ,

$$e^{-\frac{1}{2}(\phi,-\Delta\phi)} F^{\Lambda}$$

$$= e^{-\frac{\kappa}{2}(\phi,-\Delta\phi)} \; e^{-\frac{1-\kappa}{2}(\phi,-\Delta\phi)} \; F^{\Lambda}$$

$$= \;"\;\;"\;\;"\; \left(e^{-\frac{1-\kappa}{2}(\nabla\phi)^2}\right)^{\Lambda} F^{\Lambda}$$

$$= \;"\;\;"\;\;"\;\;"\;\;"\; (1+F-1)^{\Lambda}$$

$$= e^{-\frac{\kappa}{2}(\phi,-\Delta\phi)} (I_0 \circ K_0)^{\Lambda}$$

with $I_0 = e^{-\frac{1-\kappa}{2}(\nabla\phi)^2}$ and

$$K_0(X) = \left(e^{-\frac{1-\kappa}{2}(\nabla\phi)^2}(F-1)\right)^X$$

References

All proofs of this theorem using RG will have various details missing but I tried very hard in my Park City notes to be very detailed on everything I do cover.

Proofs based on convexity

\cite{NaddofSpencer1997}
\cite{Cotar2008}

Proofs based on RG

\cite{Dimock2008}
\cite{bry2009Brydges}

Many ideas in the last four lectures in
\cite{bry2009BrydgesSlade}