Lecture 10

In this lecture I introduce the hierarchical Gaussian free field and define the renormalisation group map in the context of hierarchical models. We obtain some basic properties of this map and see an explanation for the role of the criterion $d \geq 5$ in scaling limits.

The hierarchical free field

We construct the hierarchical gaussian free field

$$\phi = \{\phi_x : x \in \Lambda_\infty\}$$

by creating the same structure as in Proposition 9.3

Let

$$\zeta = \{\zeta_x : x \in \Lambda_\infty\}$$

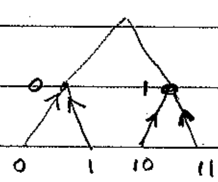
be gaussian such that

$$\mathrm{Cov}(\zeta_x, \zeta_y) = 0$$

if $|x-y| > L$.

Then define independent scaled copies

$$\zeta_j(x) \overset{\mathcal{D}}{=} L^{-j[\phi]} \zeta(L^{-j} x)$$



action of L^{-1}

where

$$L^{-j} = (L^{-1})^j : \Lambda_\infty \to \Lambda_\infty$$

<u>Definition 1</u> The hierarchical field is

$$\phi(x) = \sum_{j \geq 1} \zeta_j(x)$$

a.s. convergence on a big probability space carrying all the increments ζ_j.

From this definition it follows that

$$\phi = \zeta_1 + \phi'$$

$$\phi' \overset{\mathcal{D}}{=} L^{-[\phi]} \phi(L^{-1} x)$$

and, (problem 1),

$$\phi'_x = \phi'_y \quad \text{a.s.}$$

$$\text{for } |x-y| \leq L \tag{e: phiprime}$$

Since this is an ultrametric no balls overlap and balls are the same as blocks $B \in \mathcal{B}_L$.

Integrate out ζ_1

Define $\mathbb{E}_1$:

$$\mathbb{E}_1(F) = \mathbb{E}(F \mid \zeta_2, \zeta_3, \dots)$$

Rescale

$$F \in \sigma(\zeta_2, \zeta_3, \dots)$$

define $\hat{L}^{-1} F$ by replacing arguments

$$\zeta_{j+1}(x) \quad \text{replaced by} \quad L^{-[\phi]} \zeta_j (L^{-1} x), \qquad j \geq 1$$

Remark 2

They are equal in law.

RG Transformation

For F, $\mathbb{E}|F| < \infty$,

$$F \overset{RG}{\longmapsto} \hat{L}^{-1} \circ \mathbb{E}_1(F)$$

Lemma 3

$$\mathbb{E} F = \mathbb{E}\left(RG(F) \right)$$

Proof $\quad \mathbb{E}_j(F) \stackrel{\text{def}}{=} \mathbb{E}(F \mid \zeta_{j+1}, \zeta_{j+2}, \dots)$.

$$\mathbb{E} F = \lim_{N \to \infty} \mathbb{E}_N \mathbb{E}_{N-1} \cdots \mathbb{E}_2 \mathbb{E}_1(F)$$

$$\stackrel{\text{Remark 2}}{=} \lim_{N \to \infty} \mathbb{E}_{N-1} \cdots \cdots \mathbb{E}_1 \hat{\mathcal{L}}^{-1} \mathbb{E}_1(F)$$

$$= \mathbb{E}\left(RG(F) \right)$$

Problem 2: justify the limits

∎

Lemma 4

For $P(\phi)$ polynomial in ϕ,

$$\mathbb{E}_1 :P(\phi):_v = :P(\phi'):_{v'},$$

Proof Let

$$\Delta_C = \sum_{x,y} C(x,y) \, \partial/\partial\phi'_x \, \frac{\partial}{\partial\phi'_y}$$

and let $\Delta_{C,\zeta}$ be the same operator with $\partial/\partial\phi$ replaced by $\partial/\partial\zeta$. Then, from Lecture 4, for $Q = :P:_v$

$$\mathbb{E}_1 Q(\phi) = \mathbb{E}_1 Q(\phi'+\zeta)$$

$$= e^{\frac{1}{2}\Delta_{C_1,\zeta}} Q(\phi'+\zeta)\Big|_{\zeta=0}$$

$$= e^{\frac{1}{2}\Delta_C} Q(\phi')$$

$$= e^{\frac{1}{2}\Delta_C} e^{-\frac{1}{2}\Delta_v} P(\phi')$$

$$= e^{-\frac{1}{2}\Delta_{v-C}} P(\phi')$$

$$= :P(\phi'):_{v'}$$

∎

Lemma 5

$$RG :\phi_x^p:_v = L^{-p[\phi]} :\phi_{L^{-1}x}^p:_v$$

Proof

By Lemma 4

$$RG :\phi_x^p:_v = \hat{L}^{-1} :\phi_x^{\prime p}:_{v'}$$

$$= \hat{L}^{-1} e^{-\frac{1}{2}\sum_{j\geq 2}\Delta_j} \left(\sum_{j\geq 2} \zeta_j(x)\right)^p$$

$$\Delta_j = C_j(x,x)\frac{\partial^2}{\partial \zeta_j(x)^2}$$

$\hat{L}^{-1}$ replaces $\zeta_j(x)$ by $L^{-[\phi]}\zeta_{j-1}(L^{-1}x)$

$$= L^{-p[\phi]} :\phi_{L^{-1}x}^p:$$

Hierarchical Models

Our models have had the form

$$Z = \int e^{-(\phi, -\Delta_\Lambda \phi)} F^\Lambda \, d^\Lambda \phi \qquad \text{(e:model)}$$

$$F^\Lambda = \prod_{x \in \Lambda} F_x$$

where F_x is a bounded function of ϕ_x so a close hierarchical analogue is

$$Z = \mathbb{E} F^\Lambda, \qquad \Lambda \subset \Lambda_\infty$$

Remark 6

It would be an even closer analogue if (e:model) had been the ∞ vol. Gaussian expectation of F^Λ. This can be understood as a different boundary condition at $\partial\Lambda$.

Calculate Z by

$$Z = \lim_{n \to \infty} \mathbb{E} (RG)^n F^\Lambda$$

Lemma 7

$$RG(F^\Lambda) = \prod_{x \in L^{-1}\Lambda} RG(F^{B(x)})$$

where

$$B(x) = \{ y : L^{-1} y = x \}$$

Proof

$$RG(F^\Lambda) = \hat{L}^{-1} \mathbb{E}_1 \prod_{B \in \mathcal{B}_L(\Lambda)} F^B$$

$$= \hat{L}^{-1} \prod_{B \in \mathcal{B}_L(\Lambda)} \mathbb{E}_1 F^B$$

because $Cov(\zeta_1(x), \zeta_1(y)) = 0$ for $|x-y| > L$

$$= \prod_{x \in L^{-1}\Lambda} \underbrace{\hat{L}^{-1} \mathbb{E}_1 F^{B(x)}}$$

$$= RG(F^{B(x)})$$

∎

Example 8

$$F^{\Lambda} = e^{-V(\Lambda)}, \qquad [\phi] = \frac{d-2}{2}$$

$$V(\Lambda) = \sum_{x \in \Lambda} V_x$$

$$V_x = g :\phi_x^4: + a :\phi_x^2:$$

Then, to order g, a, or equivalently, $V^2 = 0$,

$$RG\left(F^{B(x)}\right) = RG\, e^{-V(B(x))}$$

$$= RG\left(1 - V(B(x))\right)$$

$$= 1 - \sum_{y \in B}\left(g L^{-4[\phi]} :\phi^4_{L^{-1}y}: \right.$$

$$\left. + a L^{-2[\phi]} :\phi^2_{L^{-1}y}: \right)$$

$$= 1 - V'_x$$

$$= e^{-V'_x}$$

where

$$V' = g' :\phi^4: + a' :\phi^2:$$

$$g' = |B| \, L^{-4[\phi]} g$$

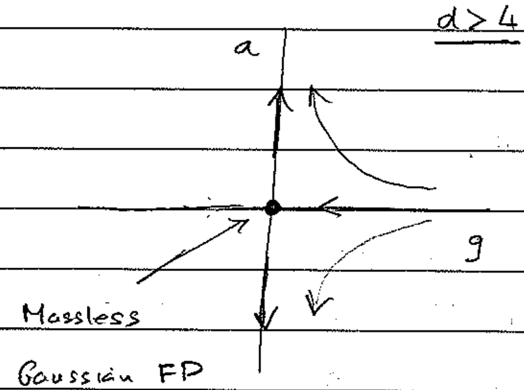
$$a' = |B| \, L^{-2[\phi]} a$$

Putting in

$$[\phi] = \frac{d-2}{2}$$

$$|B| = L^d$$

we have



$$g' = L^{-d+4} g$$

$$a' = L^2 a$$

Correlation

$$\langle \phi_a \phi_b \rangle = \frac{E\, F(a,b)^\Lambda}{E\, F^\Lambda}$$

where, for $a \neq b$,

$$F_x(a,b) = e^{-V_x} \begin{cases} 1 & x \neq a, b \\ \phi_a & x = a \\ \phi_b & x = b \end{cases}$$

Apply RG top and bottom.

Problems

1. Prove (e:phiprime).
   c.f. Durrett Thm 6-3

2. Find α s.t

$$RG\left(F(a,b)^{B(x)}\right) = \alpha\,\phi_x\,e^{-V_x'} + O(g,a)$$

when $B(x)$ contains a but not b.
If both $a, b \in B(x)$, what is

$$(RG)^n\left(F(a,b)^{B(x)}\right)$$

to order V^0 ?