Math 405: 6a: Intro to Initial Value Problems

References: [LeVeque 2007]

Advanced reference: [Hairer, Norsett, Wanner]

Time-stepping methods for ODEs (and PDEs)

Consider ODE: $u_t = f(u)$ (or more generally $u_t = f(u(t), t)$ or $u_t = f(u(t), t) + g(t)$).

Initial Value Problem (IVP): $u(0) = \eta$ specified (in general at $t = T_0$).

Simplest method: forward Euler (a forward difference in time).

$$\frac{u^{n+1} - u^n}{k} = f(u^n)$$

Here we have a grid in time: $t_n = nk, k > 0$ a fixed **time step**. Often "Δt" used as well.

```
 |-------|-------|-------|-------|-------|-----
t0=0    t1=k    t2=2k   ...
```

Forward Euler is a "one-step method" and an "explicit method":

$$u^{n+1} = u^n + kf(u^n).$$

Example: van der Pol equation

$y'' + C(y^2 - 1)y' + y = 0$ and IC. Convert to system. Thus software has routines for systems of IVPs: Matlab ode45/ode15s, Fortran odepack, scipy.integerate.ode, etc.

Example: The Method of Lines (MOL)

Can think of $U(t)$ as a vector, so heat equation $u_t = u_{xx}$ with zero Dirichlet boundary conditions and some initial condition could be discretized in space (with our favourite matrix) to give:

```
    |        |            |-2    1                 |  |        |
    | u (t)  |            |                        |  | u (t)  |
    |  1     |            | 1   -2    1            |  |  1     |
    |        |            |                        |  |        |
d   |  .     |            |     .    .   .         |  |  .     |
--  |  .     |  =  1/h^2  |      .    .   .        |  |  .     |
dt  |  .     |            |       .    .   .       |  |  .     |
    |        |            |                        |  |        |
    |        |            |         1   -2    1    |  |        |
    | u (t)  |            |                        |  | u (t)  |
    |  m     |            |              1   -2    |  |  m     |
```

So we've gone from PDE to semidiscrete system of ODEs. Initial conditions of the PDE become initial conditions (a vector) for the ODE system. Now we can discretize in time (somewhat) independently.

Note: not all PDE finite differences methods are MOL.

Convergence for Initial Value Problems (IVP)

Focus on a fixed (but arbitrary) time $t = T$. With fixed time-step k it will take $N = T/k$ steps to get there. Convergence means:

$$\lim_{k \to 0, Nk=T} u^n = u(T)$$

Defn: a method is said to be *convergent* if we obtain convergence in above sense for any ODE problem with f Lipschitz continuous in u, for every fixed time $T > 0$ where the ODE has a unique solution, using any *reasonable starting values*.

Reasonable starting value? The method will need one or more starting value and these need to be consistent ($\lim_{k \to 0}$) with the initial conditions.

(We might also speak of methods which are convergent within a restricted class of problem.)

As in previous lectures, the method will first need to be *consistent* (solving the correct problem).

LTE, one-step error, and consistency for ODEs

Still Taylor series analysis. Two basic contradictory definitions of LTE in the literature (oops, see LeVeque Sec 5.5).

1. Apply the Taylor analysis to $\frac{u^{n+1}-u^n}{k} = f(u^n)$, the form of the "difference equation" that "directly models the derivatives" [LeVeque pg121].

   For forward Euler, this gives LTE of $O(k)$.

   This tends to be more convenient for PDEs: for now we will concentrate on this definition.

2. Apply the Taylor analysis to the form $u^{n+1} = u^n + kf(u^n)$. This is sometimes called the *one-step error* (the error we make assuming previous steps are exact). Under this, LTE for forward Euler is $O(k^2)$. But we apply $O(1/k)$ steps for an estimated/expected global error of $O(k)$.

ODE convergence for one-step methods

One-step methods: each step uses just one previous solution value. Forward Euler is an example. (Possibly temporary values are also computed but not directly carried forward).

For proving convergence of forward Euler, we can do something analogous to what we saw earlier for the BVP case. It bounds the growth of the local truncation error independent of time-step k. This gives a version of the "fundamental theorem": consistency + stability implies convergence. (see LeVeque 2007 or other books for details).

It would be easier to have stability analysis which is "uncoupled" from the LTE analysis... For example:

Stability for ODEs: Zero stability

Many specific types of stability (zero-stab, abs-stab, L-stab, A-stab, $L(\alpha)$-stab, G-stab, etc...)

Zero-stability: apply the method to $u' = 0$ and $u(0) = \text{const}$, check if the solution grows or not.

Surprisingly, for one-step methods this generalizes the convergence proof above (!) [we won't prove this.]

Another nice result: for one-step methods, any consistent method is zero-stable and thus convergent! [due to Dalquist, proof omitted.]

BUT zero-stability theory applies for $\lim_{k \to 0}$, and we seek something of more practical use.

Stability for ODEs: Absolute stability

Dahlquist test problem: $u' = \lambda u$, $u(0) = 1$.

Here λ is a complex number. Exact solution is $u(t) = \exp(\lambda t)$. Real part of λ negative gives decay in the exact solution.

We apply the numerical method to this test problem and:

> If soln grows without bound, we say *numerical method* is **unstable**.
> Otherwise, we say the *numerical method* is **(absolutely) stable**.

Note these ideas hold for finite k. Why do we expect Dahlquist test equation to tell us anything about other problems? Think of λ as eigenvalues of your RHS (eigendecomposition of linearized RHS), or as a Lipschitz constant for the RHS.

Some methods and stability regions

Region of absolute stability: region of the complex plane where choice of $k\lambda$ gives absolute stable.

1. Forward Euler: $O(k)$
   as above, put in the Dahlquist test problem. [sketch]
2. Improved Euler method: $O(k^2)$ [sketch]
   $v^{n+1} = v^n + \frac{k}{2}\big(f(v^n) + f(v^n + kf(v^n))\big)$
3. "*The* Fourth-order Runge-Kutta Method" $O(k^4)$
   $a = kf(t_n, v^n)$,
   $b = kf(t_n + k/2, v^n + a/2)$,
   $c = kf(t_n + k/2, v^n + b/2)$,
   $d = kf(t_n + k, v^n + c)$,
   $v^{n+1} = v^n + \frac{1}{6}(a + 2b + 2c + d)$.
   If you could take just one formula to a desert island, this is it. [sketch]
4. Backward Euler: $\frac{u^{n+1} - u^n}{k} = f(u^{n+1})$. $\quad O(k)$ [sketch]
5. Trapezoidal Rule: $\frac{u^{n+1} - u^n}{k} = \frac{1}{2}\left(f(u^n) + f(u^{n+1})\right)$.
   Accuracy: $O(k^2)$, but somewhat oscillatory (be careful). [sketch]

These last two are *implicit*, more on that later. The stability region including the left-hand plane is called "A-stability".

Dahlquist in 1979:

> I didn't like all these "strong", "perfect", "absolute", "generalized", "super", "hyper", "complete" and so on in mathematical definitions, I wanted something neutral; ... I chose the term "A-stable".

Growth factor

See above examples, note for forward Euler, we computed $u^n = (1 + z)^n$ with $z = k\lambda$. Here $G(z) = 1 + z$ is known as the *Growth Factor*. The growth factor should approximate e^z.

L-stability: asymptotic stability

Like A-stability, but with additional property: $G(z) \to 0$ as $\mathrm{Re}(z) \to -\infty$. This matches the asymptotic behaviour of the exact solution for large negative z.

Caution: some sources swap meaning of A-stability and L-stability (!!).