Numerical Methods for Differential Equations: Homework 5

Due by 2pm Tuesday 29th November. ***Please submit your hardcopy at the start of lecture.*** *The answers you submit should be attractive, brief, complete, and should include program listings and plots where appropriate. The use of "publish" in Matlab/Octave is one possible approach.*

Problem 1: Leap frog/midpoint rule in time Consider the multistep scheme constructed from 2nd-order centered differences in time for the ODE problem $u_t = f(u)$.

Pperform an absolute stability analysis (Dalquist test, etc) for this time-stepping scheme. What is different about the region of absolute stability, compared to other methods we have looked at? Hint: you may need to numerically experiment with the complex roots of a quadratic to determine what is included in the stability region.

Problem 2: von Neumann analysis Consider the Leap Frog method for the advection problem $u_t + au_x = 0$. That is, 2nd-order centred differences in both space and time.

Perform a von Neumann stability analysis. Do the algebra in terms of the "*Courant number* $\nu = \frac{ka}{h}$" where k is the time-step and h is the spatial step. What restrictions on ν do there seem to be? What does this mean for k?

Hint: you might want to look at [1] or another book on von Neumann analysis for details.

Problem 3: Leap frog numerical experiment Write a code for 1D advection using the leap frog scheme from the previous exercise to solve $u_t + u_x = 0$.

Use the domain $0 \leq x \leq 25$ with periodic boundary conditions and $T_f = 17$. For an initial condition, take:

$$u(x, 0) = \eta(x) = \exp(-20(x-2)^2) + \exp(-(x-5)^2)$$

(Hint: you will need a second starting value: use the exact value of $u(x, k)$: an appropriate choice is the initial condition with x shifted to $x - k$—or maybe its $x + k$; check).

Use $h = 0.05$ and $\nu = 1.1$. What happens?

Use $h = 0.05$ and $\nu = 0.8$. What happens?

FYI: von Neumann analysis can also give us inside into the "dispersive" properties of a scheme, telling us whether waves of different wavenumbers move at different speeds. You may have seen the dispersion relation and "group velocity versus phase velocity" for continuous problems: this is analogous.

Problem 4: Laplacian Eigenvalues Consider the second derivative operator u_{xx}. Consider two problems: one with periodic boundary conditions $u(0) = u(\pi) = 0$ and the other with zero-Dirichlet boundary conditions.

By hand, what are the *eigenfunctions* and corresponding eigenvalues of these operator/BCs? Hint: you are looking for functions $v(x)$ whose second derivative is a constant multiple of $v(x)$. They must also satisfy the BCs. You will probably have some free parameters: be precise about what values they can take.

Problem 5: Laplacian Eigenvalues Now consider the finite difference matrices to approximate these problems: matrix A with periodic BCs and matrix B with zero-dirichlet BCs. Using "eigs" in Matlab/Octave or another tool, compute the 20 *smallest-magnitude* eigenvalues and associated eigenvectors for your two matrices. Use $h = \pi/100$.

Do the results agree with your by-hand analysis of the continuous problem? If not, check your work!

For the remainder of the problem, consider the zero-dirichlet matrix.

Do a convergence study in h, hopefully showing the eigenvalues of the discrete problem converge to those of the continuous problem. What is the convergence rate?

Do a convergence study in h, hopefully showing the eigenvectors of the discrete problem converge to the eigenfunctions. Measure error in the vector max norm (after projecting the eigenfunction on the grid). What is the convergence rate for the eigenfunction corresponding to the eigenvalue λ_3 (the third-smallest eigenvalue)?

References

[1] R. J. LeVeque. *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems.* SIAM, 2007.