Math 405 10b: The Singular Value Decomposition

Ref: Trefethen and Bau text 2007, Chap 4, Chap 5.

The SVD is conceptually related to eigenvalue decomposition but is probably more useful in practice in numerical linear algebra.

Consider a $m \times n$ matrix A.

Interprete geometrically as mapping from unit sphere in R^n to hyperellipse in R^m... [Draw figure].

What is happening in the figure:

$$Av_j = \sigma_j u_j$$

where σ_j is the semi-axis length.

Collecting this for all j, gives the SVD:

$$AV = U\Sigma$$

$$A = U\Sigma V^T$$

where here:

- U: left singular vectors, $m \times m$, orthog matrix.
- V: right singular vectors, $n \times n$, orthog matrix.
- Σ: singular values on diagonal, in descending order (by convention).

Every matrix has a SVD. Eigenvalue computations are problematic: non-diagonalizable (think Jordan canonical form), complex eigenvalues from real-valued matrices, "illconditioning" for non-normal matrices, etc.

(note, unfort. that the left singular vectors are on the *right* of the figure: position from the equation not the graphic!)

Properties of matrices from the SVD

Matrix 2-norm: $||A||_2 = \sigma_1$.

Rank of A: number of non-zero singular values:

- SVD gives A as sum of rank-1 outer products: $A = \sum \sigma_j u_j v_j^T$
- Throw some σ away: get the best possible "low-rank" approx to A (in 2-norm and Frobinius norm). Leads to Principle Component Analysis" (PCA) for example.

Computing nullspace and range of A: use U and V.

Determinant: $|\det(A)| = \prod(\sigma_i)$.

Computation of the SVD

Similar to the QR Algorithm. See Youtube for old video.