Num Soln of DEs: 06b: more on Initial Value Problems

Runge-Kutta methods

We've seen several of these (forward Euler, improved Euler, RK4). They use temporary intermediate "stage values" to advance from U^n to U^{n+1}.

Matlab's code "ode45" uses Runge-Kutta methods.

Linear-Multistep Methods

Uses previous step values (e.g., U^{n-1} and U^n) to advance to U^{n+1}.

Example 1: unstable method, discuss zero-stability...

Example 2: BDF-2 method, look at absolute stability analysis—in fact, it is A-stable.

Matlab's code "ode15s" uses implicit linear multistep methods with variable order.

Implicit time-stepping methods

(see earlier in notes for Backward Euler and Trapezoidal Rule methods).

$u^{n+1} = u^n + kf(u^{n+1})$

u^{n+1} on RHS makes this more *expensive* than forward Euler.

Advantages? A-stability. Some implicit methods (including BE and TR) have no time-step restriction for (absolute) stability.

Implicit methods are often useful for stiff problems.

Stiffness?

The classical zero-stability/consistency/convergence theory for ODEs was established by Dahlquist in 1956.

Just a few years later it began to be widely appreciated that something was missing from this theory.

Key paper: [Dahlquist, 1963]

(Chemists Curtiss & Hirschfelder [1952], used the term "stiff", which may actually have originated with the statistician John Tukey (who also invented "FFT" and "bit").

Example

ODE $u' = -\sin(t)$ with IC $u(0) = 1$ has solution $u(t) = \cos(t)$.

Change ODE to

$$u' = -100(u(t) - \cos(t)) - \sin(t),$$

then this *still* has solution $u(t) = \cos(t)$.

But the numerics are much different: [demo_06_stiff.m], a convergence study showing forward Euler/backward Euler convergence on these problems.

Analysis: linearize around the soln: let $u(t) = \cos(t) + w(t)$ and we get an ODE for $w(t)$ of $w' = -100w$ which does indeed have a very different *time scale* than $\cos(t)$.

Definition of stiffness

- A stiff ODE is one with widely varying time scales.
- More precisely, an ODE with solution of interest $u(t)$ is stiff when there are time scales present in the equation that are much shorter than that of $u(t)$ itself.

Neither ideal.

- My favourite: a stiff *problem* is one were implicit methods work better. (I learned this from Raymond Spiteri but probably due to Gear.) C.W. Gear, 1982:

Stiffness ratio: smallest eigenvalue to largest eigenvalue. (Continuous diffusion is infinitely stiff.)

Non-linearity

For nonlinear problems (or nonlinear discretizations of linear problems), implicit methods require solving nonlinear equations at each time-step. Similarly, for nonlinear steady state problems.

For this, see Newton's method, covered earlier in the course.

IMEX methods

Implicit/Explicit methods. Best of both worlds? Treat some part of equation (often linear diffusion or hyperdiffusion) implicitly to avoid time-step restrictions. But treat to nonlinear terms explicitly (to avoid nonlinear system solves).

Example: [demo_kuramoto_sivashinsky.m]

$$u_t = -u_{xx} - u_{xxxx} - (u^2/2)_x$$