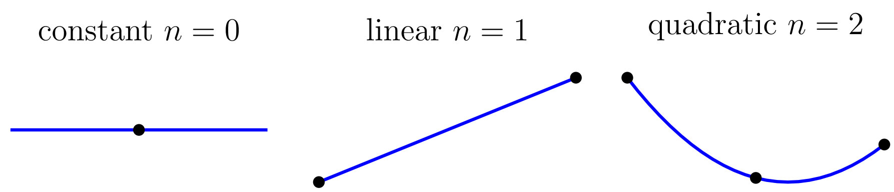
Math 405: Numerical Methods for Differential Equations 2016 W1
Topic 3: Lagrange Interpolation

This lecture adapted from chapter 6 of the numerical analysis textbook by Süli and Mayers.

Notation: $\Pi_n = \{$real polynomials of degree $\leq n\}$

Setup: given data f_i at distinct x_i, $i = 0, 1, \ldots, n$, with $x_0 < x_1 < \cdots < x_n$, can we find a polynomial p_n such that $p_n(x_i) = f_i$? Such a polynomial is said to **interpolate** the data.

E.g.:



Theorem. $\exists p_n \in \Pi_n$ such that $p_n(x_i) = f_i$ for $i = 0, 1, \ldots, n$.

Proof. (Constructive!) Consider, for $k = 0, 1, \ldots, n$, the "cardinal polynomial"

$$L_{n,k}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} \in \Pi_n. \tag{1}$$

Then

$$L_{n,k}(x_i) = 0 \;\text{ for }\; i = 0, \ldots, k-1, k+1, \ldots, n \;\text{ and }\; L_{n,k}(x_k) = 1.$$

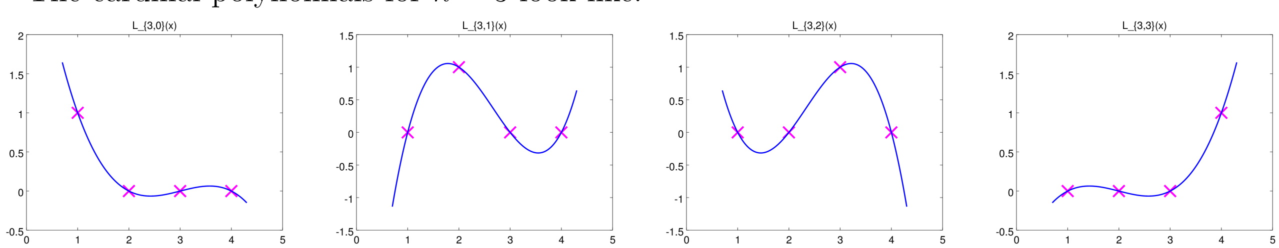
So now define

$$p_n(x) = \sum_{k=0}^{n} f_k L_{n,k}(x) \in \Pi_n \tag{2}$$

$\Longrightarrow$

$$p_n(x_i) = \sum_{k=0}^{n} f_k L_{n,k}(x_i) = f_i \;\text{ for }\; i = 0, 1, \ldots, n. \qquad \square$$

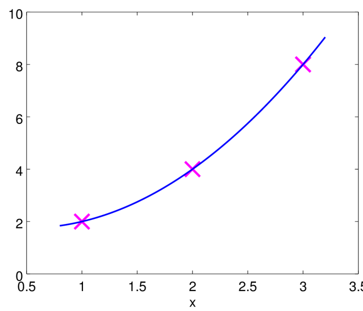
The polynomial (2) is the **Lagrange interpolating polynomial**.
The cardinal polynomials for $n = 3$ look like:

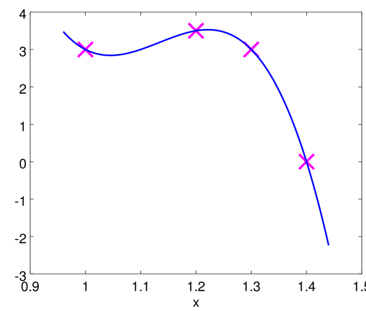


Theorem. The interpolating polynomial of degree $\leq n$ (through $n+1$ points) is unique.

Proof. "One root too many". Consider two interpolating polynomials p_n, $q_n \in \Pi_n$. Difference $d_n = p_n - q_n \in \Pi_n$ satisfies $d_n(x_k) = 0$ for $k = 0, 1, \ldots, n$, i.e., d_n is a polynomial of degree at most n but has at least $n+1$ distinct roots. Fundam. Thm. Algebra $\implies$ $d_n \equiv 0 \implies p_n = q_n$. □

Demos: See `demo_03_lagrange.m` and `demo_03_lagrange_construct.m`. Here is the output of `lagrange([1 2 3], [2 4 8])` and `lagrange([1 1.2 1.3 1.4], [3 3.5 3 0])`:





Data from an underlying smooth function: Suppose that $f(x)$ has at least $n+1$ smooth derivatives in the interval (x_0, x_n). Let $f_k = f(x_k)$ for $k = 0, 1, \ldots, n$, and let p_n be the Lagrange interpolating polynomial for the data (x_k, f_k), $k = 0, 1, \ldots, n$.

Error: how large can the error $f(x) - p_n(x)$ be on the interval $[x_0, x_n]$?

Theorem. For every $x \in [x_0, x_n]$ there exists $\xi = \xi(x) \in (x_0, x_n)$ such that

$$e(x) \stackrel{\text{def}}{=} f(x) - p_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)}(\xi)}{(n+1)!},$$

where $f^{(n+1)}$ is the $(n+1)$-st derivative of f.

Proof. Trivial for $x = x_k$, $k = 0, 1, \ldots, n$ as $e(x) = 0$ by construction. So suppose $x \neq x_k$. Let

$$\phi(t) \stackrel{\text{def}}{=} e(t) - \frac{e(x)}{\pi(x)}\pi(t),$$

where

$$\begin{aligned} \pi(t) &\stackrel{\text{def}}{=} (t - x_0)(t - x_1)\cdots(t - x_n) \\ &= t^{n+1} + \cdots \in \Pi_{n+1}. \end{aligned}$$

Now note that ϕ vanishes at $n+2$ points x and x_k, $k = 0, 1, \ldots, n$. $\implies$ ϕ' vanishes at $n+1$ points $\xi_0, \ldots, \xi_n$ between these points $\implies$ ϕ'' vanishes at n points between these new points, and so on until $\phi^{(n+1)}$ vanishes at an (unknown) point ξ in (x_0, x_n). But

$$\phi^{(n+1)}(t) = e^{(n+1)}(t) - \frac{e(x)}{\pi(x)}\pi^{(n+1)}(t) = f^{(n+1)}(t) - \frac{e(x)}{\pi(x)}(n+1)!$$

since $p_n^{(n+1)}(t) \equiv 0$ and because $\pi(t)$ is a monic polynomial of degree $n+1$. The result then follows immediately from this identity since $\phi^{(n+1)}(\xi) = 0$. □

Example: $f(x) = \log(1+x)$ on $[0, 1]$. Here, $|f^{(n+1)}(\xi)| = n!/(1+\xi)^{n+1} < n!$ on $(0, 1)$. So $|e(x)| < |\pi(x)| n!/(n+1)! \leq 1/(n+1)$ since $|x - x_k| \leq 1$ for each x, x_k, $k = 0, 1, \ldots, n$, in $[0,1] \implies |\pi(x)| \leq 1$. This is probably pessimistic for many x, e.g. for $x = \frac{1}{2}$, $\pi(\frac{1}{2}) \leq 2^{-(n+1)}$ as $|\frac{1}{2} - x_k| \leq \frac{1}{2}$.

This shows the important fact that the error can be large at the end points, an effect known as the "Runge phenomena" (Carl Runge, 1901). There is a famous example due to Runge, where the error from the interpolating polynomial approximation to $f(x) = (1+x^2)^{-1}$ for $n+1$ equally-spaced points on $[-5,5]$ diverges near ± 5 as n tends to infinity: try `demo_03_runge.m` from the website.

Building Lagrange interpolating polynomials from lower degree ones.

Notation: Let $Q_{i,j}$ be the interpolating polynomial at x_k, $k = i, \ldots, j$.

Theorem.

$$Q_{i,j}(x) = \frac{(x - x_i)Q_{i+1,j}(x) - (x - x_j)Q_{i,j-1}(x)}{x_j - x_i} \tag{3}$$

Proof. Let $s(x)$ denote the right-hand side of (3). Because of uniqueness, we wish to show that $s(x_k) = f_k$ and that the $s(x)$ is of the correct degree; left as exercises. □

Comment: this can be used as the basis for constructing interpolating polynomials. In books: may find topics such as the Newton form and divided differences.

Generalisation: given data f_i and g_i at distinct x_i, $i = 0, 1, \ldots, n$, with $x_0 < x_1 < \cdots < x_n$, can we find a polynomial p such that $p(x_i) = f_i$ and $p'(x_i) = g_i$?

Theorem. There is a unique polynomial $p_{2n+1} \in \Pi_{2n+1}$ such that $p_{2n+1}(x_i) = f_i$ and $p'_{2n+1}(x_i) = g_i$ for $i = 0, 1, \ldots, n$.

Construction: given $L_{n,k}(x)$ in (1), let

$$\begin{aligned} H_{n,k}(x) = {} & [L_{n,k}(x)]^2(1 - 2(x - x_k)L'_{n,k}(x_k)) \\ \text{and } K_{n,k}(x) = {} & [L_{n,k}(x)]^2(x - x_k). \end{aligned}$$

Then

$$p_{2n+1}(x) = \sum_{k=0}^{n}[f_k H_{n,k}(x) + g_k K_{n,k}(x)] \tag{4}$$

interpolates the data as required. The polynomial (4) is called the **Hermite interpolating polynomial**.

Theorem. Let p_{2n+1} be the Hermite interpolating polynomial in the case where $f_i = f(x_i)$ and $g_i = f'(x_i)$ and f has at least $2n+2$ smooth derivatives. Then, for every $x \in [x_0, x_n]$,

$$f(x) - p_{2n+1}(x) = [(x - x_0)(x - x_1)\cdots(x - x_n)]^2 \frac{f^{(2n+2)}(\xi)}{(2n+2)!},$$

where $\xi \in (x_0, x_n)$ and $f^{(2n+2)}$ is the $(2n+2)$nd derivative of f.