Midterm 1 Duration: 50 minutes

This test has 4 ***questions*** *on* ***6 pages****, each worth 10 points, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, except question #3 where the answer alone is sufficient.
- Continue on the closest blank page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: ____________________

Student-No: ____________________ Section: ____________________

Signature: ____________________

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

6 marks 1. (a) Find the equation of the plane containing the point $(2, 0, -1)$ and containing the line $x = 1 - t$, $y = 2t$, $z = 2 + 2t$.

Answer: $6x + y + 2z = 10$

Solution: Let P be the point $(2, 0, -1)$. We set $t = 0$ and $t = 1$ to see that two points on the line are $Q = (1, 0, 2)$ and $R = (0, 2, 4)$. Two vectors in the plane are $\overrightarrow{PQ} = \langle -1, 0, 3 \rangle$ and $\overrightarrow{PR} = \langle -2, 2, 5 \rangle$. Their cross product is $\langle -1, 0, 3 \rangle \times \langle -2, 2, 5 \rangle = \langle -6, -1, -2 \rangle$, and this vector is normal to the plane. The vector $\langle 6, 1, 2 \rangle$ is also normal. The equation of the plane therefore has the form $6x + y + 2z = d$. We substitute $(x, y, x) = (2, 0, -1)$ to find that $d = 10$, so the plane is $6x + y + 2z = 10$.

2 marks (b) Find the parametric equations of the line which is normal to the plane $x + 2z = 1$ and which contains the point $(0, -1, 0)$.

Answer: $x = t$, $y = -1$, $z = 2t$

Solution: The plane has normal $\langle 1, 0, 2 \rangle$ and this defines the direction of the line, whose parametric equations are therefore $x = t$, $y = -1$, $z = 2t$.

2 marks (c) Find the acute angle between the planes $x + y = -5$ and $y - z = 12$.

Answer: $\frac{\pi}{3}$, or $60°$

Solution: Normal vectors to the planes are $\vec{n}_1 = \langle 1, 1, 0 \rangle$ and $\vec{n}_2 = \langle 0, 1, -1 \rangle$. The angle between the planes is the angle θ between the normals, which obeys

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

Therefore $\theta = \frac{\pi}{3}$, or $60°$.

2 marks 2. (a) Let $f(x,y) = e^{xy} + x^2y + \cos(x)$. Compute the partial derivatives f_x and f_y.

Answer: $f_x = ye^{xy} + 2xy - \sin x$,
$f_y = xe^{xy} + x^2$.

Solution: Partial differentiation simply gives $f_x = ye^{xy} + 2xy - \sin x$, $f_y = xe^{xy} + x^2$.

4 marks (b) Find all values of the constant s such that $f(x,y) = e^{sx}\cos(3y)$ satisfies the Laplace equation $f_{xx} + f_{yy} = 0$.

Answer: $s = \pm 3$

Solution: We compute the partial derivatives:

$$f_x = se^{sx}\cos(3y), \quad f_{xx} = s^2e^{sx}\cos(3y),$$
$$f_y = -3e^{sx}\sin(3y), \quad f_{yy} = -9e^{sx}\cos(3y).$$

Therefore,

$$f_{xx} + f_{yy} = (s^2 - 9)e^{sx}\cos(3y),$$

and this equals zero when $s^2 = 9$, i.e., $s = \pm 3$.

4 marks (c) The equation $z^3 - z + 2xy - y^2 = 0$ determines a function $z = f(x,y)$ implicitly near $(x,y) = (2,4)$ with $f(2,4) = 1$. Find the derivative $\frac{\partial z}{\partial y}$ at the point $(x,y) = (2,4)$.

Answer: 2

Solution: We differentiate implicitly to obtain

$$3z^2\frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} + 2x - 2y = 0,$$

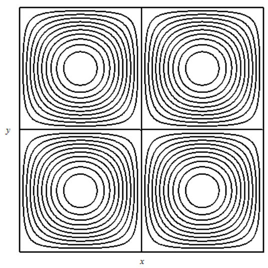
which gives

$$\frac{\partial z}{\partial y} = -\frac{2x - 2y}{3z^2 - 1}.$$

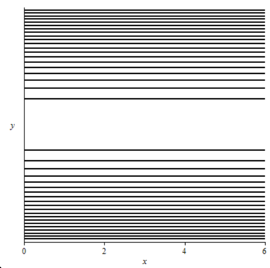
At $(x,y) = (2,4)$, where $z = 1$, the right-hand side is 2.

10 marks

3. Consider the following 10 contour plots, 5 graphs, and 5 functions. Each contour plot is the contour plot of one of the 5 graphs, or of one of the 5 functions. Match each contour plot with the corresponding graph or function. In the 10 contour plots, the x axis is horizontal and the y axis is vertical. In the 5 graphs, the z axis is vertical, and the y and x axes are the other two with the x axis rightmost.



A.



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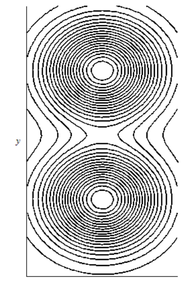

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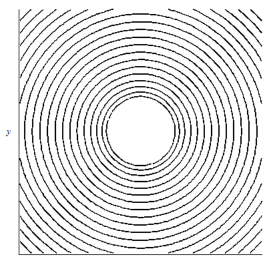

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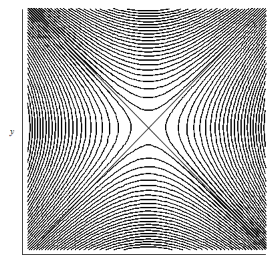

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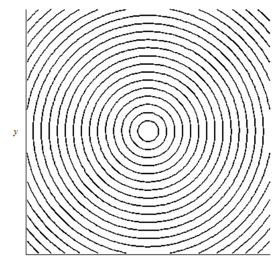

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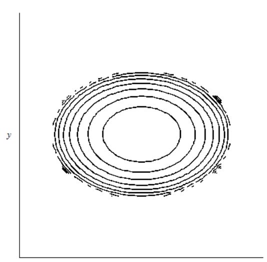

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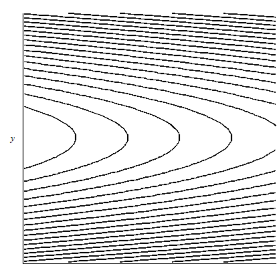

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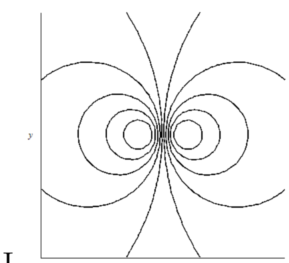
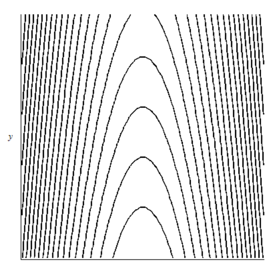
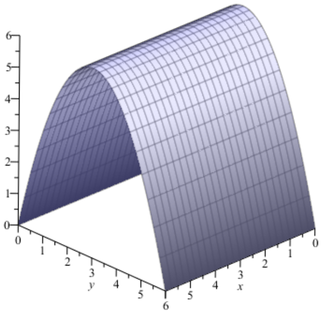


I.



J.



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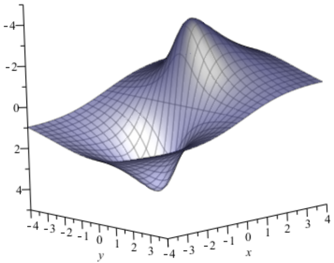

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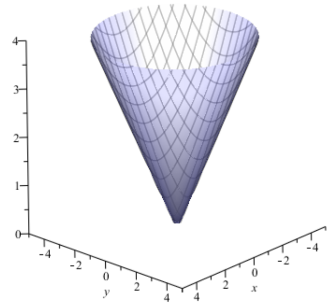

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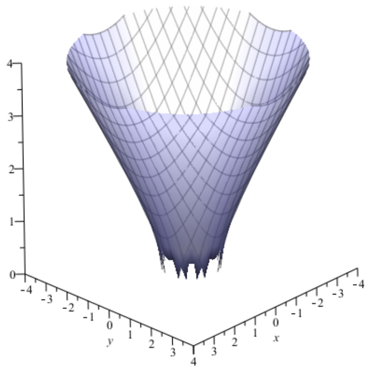

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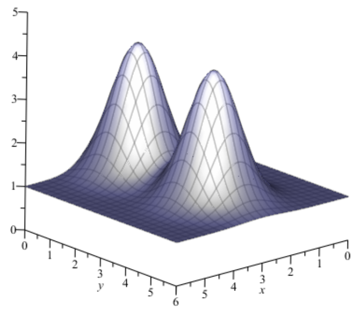

5.

6. $f(x, y) = \sqrt{1 - 3x^2 - 6y^2}$.
7. $f(x, y) = 3y + 2x^2$.
8. $f(x, y) = 1 - x^2 + y^2$.
9. $f(x, y) = 1 + \sin(x)\sin(y)$.
10. $f(x, y) = 3x + 2y^2$.

Write your answer in order as Aa, Bb, Cc, ..., Jj where a is the number of the function corresponding to contour plot A, b to B, and so on.

Answer: A9, B1, C5, D4, E8, F3, G6, H10, I2, J7

5 marks 4. (a) Find the equation of the plane tangent to $z = \sqrt{36 - 4x^2 - 9y^2}$ at the point $(x, y, z) = (2, 1, \sqrt{11})$.

Answer: $z = -\frac{1}{\sqrt{11}}(8x + 9y - 36)$

Solution: The tangent plane has equation $z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. Computation gives $f_x = -4x/z$, $f_y = -9y/z$, and substitution then gives

$$z = \sqrt{11} - \frac{8}{\sqrt{11}}(x - 2) - \frac{9}{\sqrt{11}}(y - 1) = -\frac{1}{\sqrt{11}}(8x + 9y - 36).$$

5 marks (b) Three positive numbers, each less than 10, are multiplied together. The values of the three numbers are known with respective errors at most 0.1, 0.05 and 0.04. *Using differentials*, estimate the maximum possible error in the computed product.

Answer: 19

Solution: Let x, y, z be the three numbers, and let $P = xyz$ be their product. Then

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz = yzdx + xzdy + xydz.$$

This is maximal when $x = y = z = 10$, with $dx = 0.1$, $dy = 0.05$, $dz = 0.04$, in which case it equals $10^2(0.1 + 0.05 + 0.04) = 19$.