The University of British Columbia

Final Examination - December 17, 2012

Mathematics 253

All Sections

Closed book examination | Time: 3.0 hours

Last Name Solutions **First Name** ____________ **Signature** ________________

Student Number ____________________ **Section #:** ________________

Special Instructions:

No books, notes, or calculators are allowed.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		10
2		10
3		10
4		10
5		12
6		12
7		15
8		21
Total		100

Problem 1 [10 points]. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

1. **3 points.** Find the value of the constant k such that $u(x,t) = e^{kt}\cos(2x)$ is a solution of the heat equation $u_t = u_{xx}$.

$u_t = ke^{kt}\cos(2x)$

$u_x = -2e^{kt}\sin(2x)$

$u_{xx} = -4e^{kt}\cos(2x)$

$u_t = u_{xx} \Rightarrow -4e^{kt}\cos(2x) = ke^{kt}\cos(2x)$

$\Rightarrow k = -4$

Answer: $k = -4$

2. **3 points.** Let P be the plane tangent to the ellipsoid $2x^2 + 3y^2 + z^2 = 9$ at the point $(-1, 1, 2)$. Find the point where P intersects the y-axis.

$f(x,y,z) = 2x^2+3y^2+z^2$ $\quad \vec{\nabla} f = \langle 4x, 6y, 2z \rangle$

$\vec{N} = \vec{\nabla} f(-1,1,2) = \langle -4, 6, 4 \rangle$

plane: $-4(x+1) + 6(y-1) + 4(z-2) = 0$

y axis: $-4(0+1) + 6(y-1) + 4(0-2) = 0$

$\Rightarrow 6(y-1) = 12 \Rightarrow y = 3$

Answer: $y = 3$

3. **4 points.** Suppose that $e^{xyz} = x + y + z$. Find $\frac{\partial z}{\partial x}$ at the point $(-1, 2, 0)$.

$\frac{\partial}{\partial x}\left(e^{xyz}\right) = \frac{\partial}{\partial x}(x+y+z)$

$yz\, e^{xyz} + xy e^{xyz} \frac{\partial z}{\partial x} = 1 + \frac{\partial z}{\partial x}$ at $(-1,2,0)$

$0 \cdot e^0 + (-2)e^0 \frac{\partial z}{\partial x} = 1 + \frac{\partial z}{\partial x}$

$-2\frac{\partial z}{\partial x} = 1 + \frac{\partial z}{\partial x}$

$\Rightarrow 1 = -3\frac{\partial z}{\partial x}$

Answer: $\frac{\partial z}{\partial x} = -\frac{1}{3}$

Problem 2 [10 points]. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

1. **5 points.** Let α be the acute angle between the plane $z = 4x + 2y$ and the plane tangent to the surface $z = x^2 + y^2$ at the point $(1, 2, 5)$. Find $\cos(\alpha)$.

Normal vectors : $0 = 4x + 2y - z \Rightarrow N_1 = \langle 4, 2, -1 \rangle$

$0 = x^2 + y^2 - z = f(x,y,z) \qquad \vec{\nabla} f = \langle 2x, 2y, -1 \rangle$

$\vec{N}_2 = \vec{\nabla} f(1,2,5) = \langle 2, 4, -1 \rangle$

$$\cos(\alpha) = \frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1||\vec{N}_2|} = \frac{\langle 4,2,-1\rangle \cdot \langle 2,4,-1\rangle}{\sqrt{4^2+2^2+1^2}\sqrt{4^2+2^2+1^2}} = \frac{17}{21}$$

Answer: $\frac{17}{21}$

2. **5 points.** Find all points (x, y, z) where the hyperboloid $x^2 + y^2 - z^2 = 1$ and the plane $x + y - z = 1$ meet at right angles.

Normal vectors: $\vec{N}_1 = \langle 2x, 2y, -2z \rangle \quad \vec{N}_2 = \langle 1, 1, -1 \rangle$

$0 = \vec{N}_1 \cdot \vec{N}_2 = 2x + 2y + 2z$

solve
$$\boxed{\begin{aligned} 0 &= 2x + 2y + 2z \\ 1 &= x + y - z \\ 1 &= x^2 + y^2 - z^2 \end{aligned}} \rightarrow \begin{aligned} 0 &= x + y + z \\ 1 &= x + y - z \\ \hline -1 &= 2z \end{aligned}$$

Answer: $(1, -\frac{1}{2}, -\frac{1}{2})$ $(-\frac{1}{2}, 1, -\frac{1}{2})$

1st 2 eqns $\Rightarrow \quad z = -\frac{1}{2} \quad x + y = \frac{1}{2}$

$\Rightarrow \quad 1 = x^2 + (\frac{1}{2} - x)^2 - (-\frac{1}{2})^2$

$\Rightarrow \quad 1 = x^2 + \frac{1}{4} - x + x^2 - \frac{1}{4}$

$\Rightarrow \quad 2x^2 - x - 1 = 0$

$\Rightarrow \quad (2x + 1)(x - 1) = 0 \qquad x = 1 \text{ or } x = -\frac{1}{2}$

$\Downarrow \qquad \Downarrow$

$y = -\frac{1}{2} \qquad y = 1$

Problem 3 [10 points]. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

1. **5 points.** Let $f(x,y) = x^4 + xy - 5y^2$. Consider the directional derivative of f at the point (a, a^3) in the direction $\langle \frac{3}{5}, \frac{4}{5} \rangle$. For what values of a is this directional derivative equal to zero?

$$\nabla f = \langle 4x^3 + y,\ x - 10y \rangle$$

$$0 = (D_{\langle \frac{3}{5}, \frac{4}{5} \rangle} f)(a, a^3) = \langle 4a^3 + a^3,\ a - 10a^3 \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle$$

$$= \frac{3}{5}(5a^3) + \frac{4}{5}(a - 10a^3)$$

$$= 3a^3 + \frac{4}{5}a - 8a^3$$

$$0 = -5a^3 + \frac{4}{5}a$$

$$5a^3 = \frac{4}{5}a$$

$$\Rightarrow a = 0 \text{ or}$$

$$a^2 = \frac{4}{25}$$

$$\Rightarrow a = \pm \frac{2}{5}$$

Answer: $a = 0, \frac{2}{5}$, or $-\frac{2}{5}$

2. **5 points.** Use linear approximation to estimate the distance from the origin to the point $(6.02, 7.99)$. Express your answer as a single number in decimal form.

$$f(x,y) = \sqrt{x^2 + y^2} \qquad f_x = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$f_x(6,8) = \frac{6}{10} \qquad f_y(6,8) = \frac{8}{10} \qquad f(6,8) = 10$$

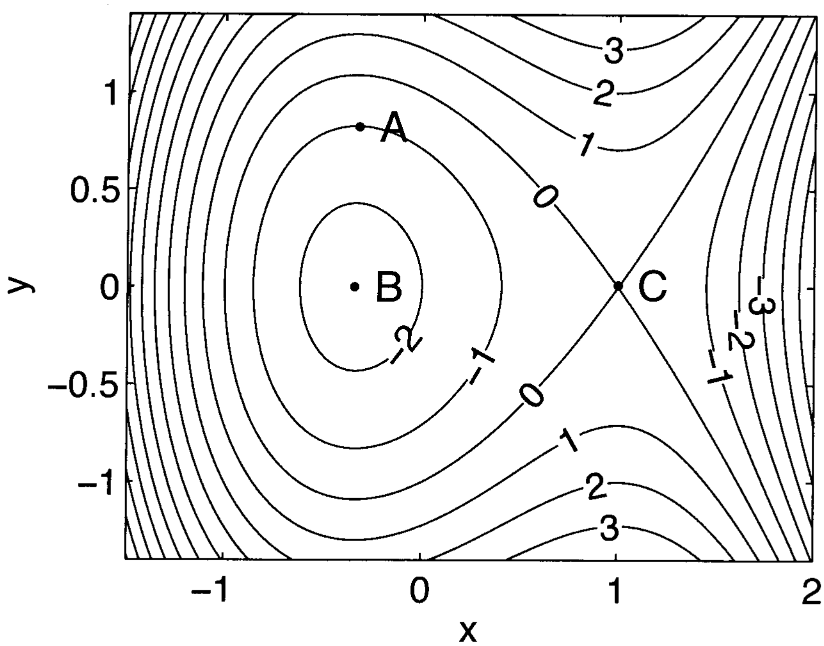
$$f(x,y) \approx 10 + \frac{6}{10}(x-6) + \frac{8}{10}(y-8)$$

$$= 10 + \frac{6}{10}(0.02) + \frac{8}{10}(-0.01)$$

$$= 10 + 0.012 - 0.008$$

$$= 10.004$$

Answer: 10.004



Problem 4 [10 points]. A contour plot of a function $f(x, y)$ is shown above. Answer the following questions concerning the points A, B, and C marked on the plot. For each question **CIRCLE THE CORRECT ANSWER:**

1. Point A.
   (a) Point A is a (i) local minimum, (ii) local maximum, (iii) saddle point, (iv) not a critical point.
   (b) At point A, f_x is (i) positive, (ii) negative, (iii) zero
   (c) At point A, f_y is (i) positive, (ii) negative, (iii) zero
   (d) At point A, f_{yy} is (i) positive, (ii) negative, (iii) zero

2. Point B.
   (a) Point B is a (i) local minimum, (ii) local maximum, (iii) saddle point, (iv) not a critical point.
   (b) At point B, f_x is (i) positive, (ii) negative, (iii) zero
   (c) At point B, f_{xx} is (i) positive, (ii) negative, (iii) zero

3. Point C.
   (a) Point C is a (i) local minimum, (ii) local maximum, (iii) saddle point, (iv) not a critical point.
   (b) At point C, f_{xx} is (i) positive, (ii) negative, (iii) zero
   (c) At point C, f_{yy} is (i) positive, (ii) negative, (iii) zero

Problem 5 [12 points]. Let D be the region which lies inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.

1. **5 points.** Sketch the circles and shade the region D. Find the coordinates of the points of intersection.

intersection:

$y^2 = 1 - x^2$

$(x-1)^2 + 1 - x^2 = 1$

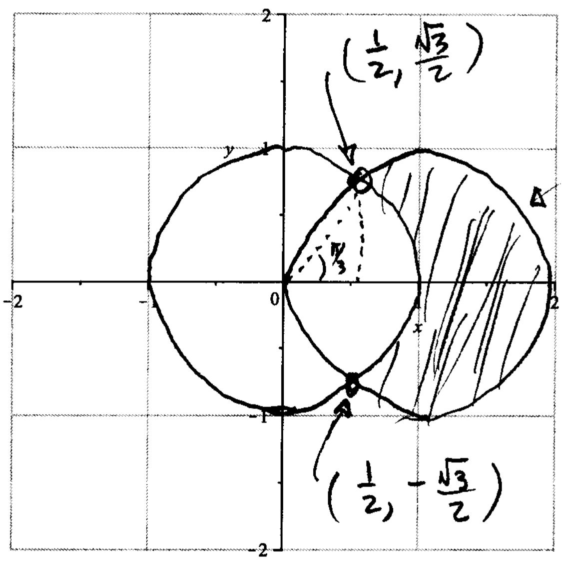
$x^2 - 2x + 1 - x^2 = 0$

$-2x + 1 = 0$

$x = \frac{1}{2}$ $\quad \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

$y^2 = \frac{3}{4}$

$y = \pm \frac{\sqrt{3}}{2}$



$x^2 - 2x + 1 + y^2 = 1$

$x^2 + y^2 = 2x$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

2. **7 points.** Evaluate the double integral

$$\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dA$$

Use polar coords

$$\int_{\theta=-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{r=1}^{2\cos\theta} \frac{1}{r}\, r\,dr\,d\theta$$

$$= \int_{\theta=-\pi/3}^{\pi/3} (2\cos\theta - 1)\,d\theta = \Big[2\sin\theta - \theta\Big]_{-\pi/3}^{\pi/3}$$

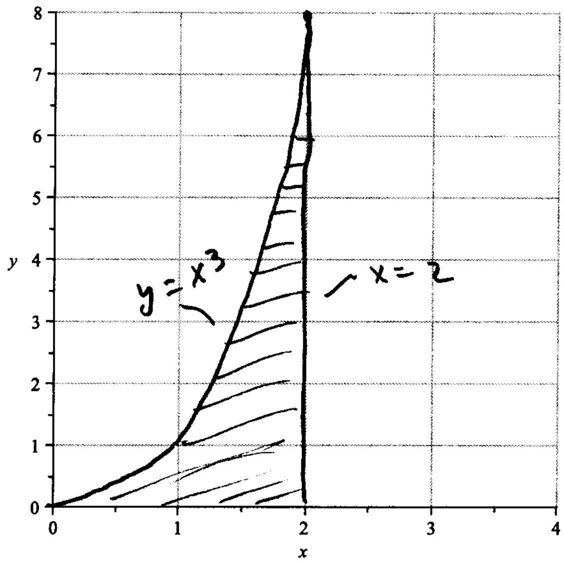
$$= 2\left(\frac{\sqrt{3}}{2}\right) - \frac{\pi}{3} - \left(2\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{3}\right)$$

$$= \boxed{2\sqrt{3} - \frac{2\pi}{3}}$$

Problem 6 [12 points].
Consider the iterated integral

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \cos(x^4)\, dx\, dy$$

1. **5 points.** Sketch the region of integration on the graph provided:



$x = \sqrt[3]{y}$

$\Longleftrightarrow$

$y = x^3$

2. **7 points**. Evaluate the integral.

$$\int_{x=0}^{2} \int_{y=0}^{x^3} \cos(x^4)\, dy\, dx$$

$$= \int_{x=0}^{2} \Big[y\Big]_0^{x^3} \cos(x^4)\, dx = \int_0^2 x^3 \cos(x^4)\, dx$$

$$= \left[\frac{1}{4}\sin(x^4)\right]_0^2 = \boxed{\frac{1}{4}\sin(16)}$$

Problem 7 [15 points].
Consider the function $f(x,y) = 2x^2 + 3y^2 - 4x - 5$ on the domain $x^2 + y^2 \leq 16$.

1. **4 points**. Find and classify the critical points of f on the interior of its domain.

$f_x = 4x - 4 = 0 \quad f_y = 6y = 0 \quad \Rightarrow$ crit pt. $(1,0)$

$f_{xx} = 4 \quad f_{yy} = 6 \quad f_{xy} = 0$

$D = f_{xx}f_{yy} - f_{xy}^2 = 24 > 0 \quad f_{xx} > 0$

$\boxed{(1,0) \text{ is a local minimum.}}$

2. **8 points.** Use Lagrange multipliers to identify the coordinates of all possible extrema of f on the boundary of the domain.

Boundary is $g(x,y) = x^2+y^2 = 16$

$f_x = \lambda g_x \qquad 4x - 4 = \lambda 2x$

$f_y = \lambda g_y \qquad 6y = \lambda 2y \Rightarrow y = 0$ or $\lambda = 3$

$x^2 + y^2 = 16$

if $y = 0$ then $x^2 = 16 \Rightarrow x = \pm 4$ so $\boxed{(4,0) \quad (-4,0)}$ are possible

if $\lambda = 3$ then $4x - 4 = 6x \Rightarrow -4 = 2x \Rightarrow x = -2$

$x = -2 \Rightarrow 4 + y^2 = 16 \Rightarrow y^2 = 12 \Rightarrow y = \pm\sqrt{12} = \pm 2\sqrt{3}$

so $\boxed{(-2, 2\sqrt{3}) \quad (-2, -2\sqrt{3})}$ 4 possibilities

Problem 7 continued.

3. **3 points.** Determine the absolute maximum and absolute minimum values of f on its domain.

abs. max/min must occur at one of the points

$(1,0)$, $(4,0)$, $(-4,0)$, $(-2, 2\sqrt{3})$, or $(-2,-2\sqrt{3})$

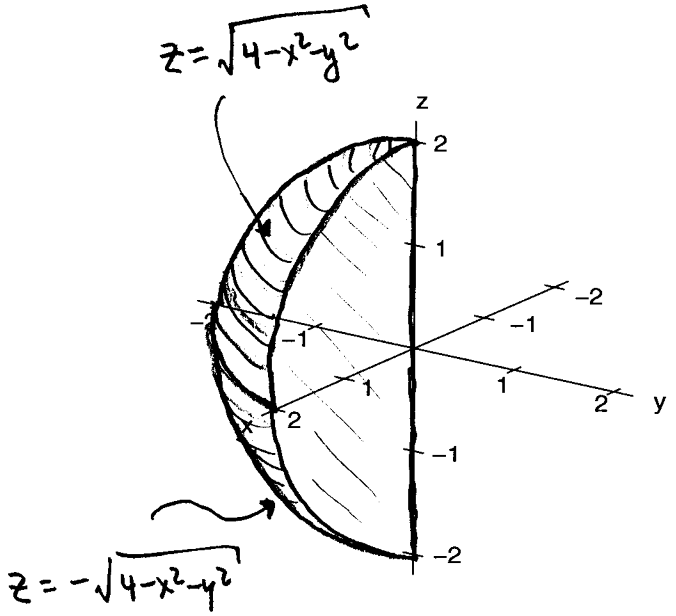
(x,y)	$f(x,y)$
$(1,0)$	$2-4-5 = -7$
$(4,0)$	$32-16-5 = 11$
$(-4,0)$	$32+16-5 = 43$
$(-2,2\sqrt{3})$	$8+36+8-5 = 47$
$(-2,-2\sqrt{3})$	$8+36+8-5 = 47$

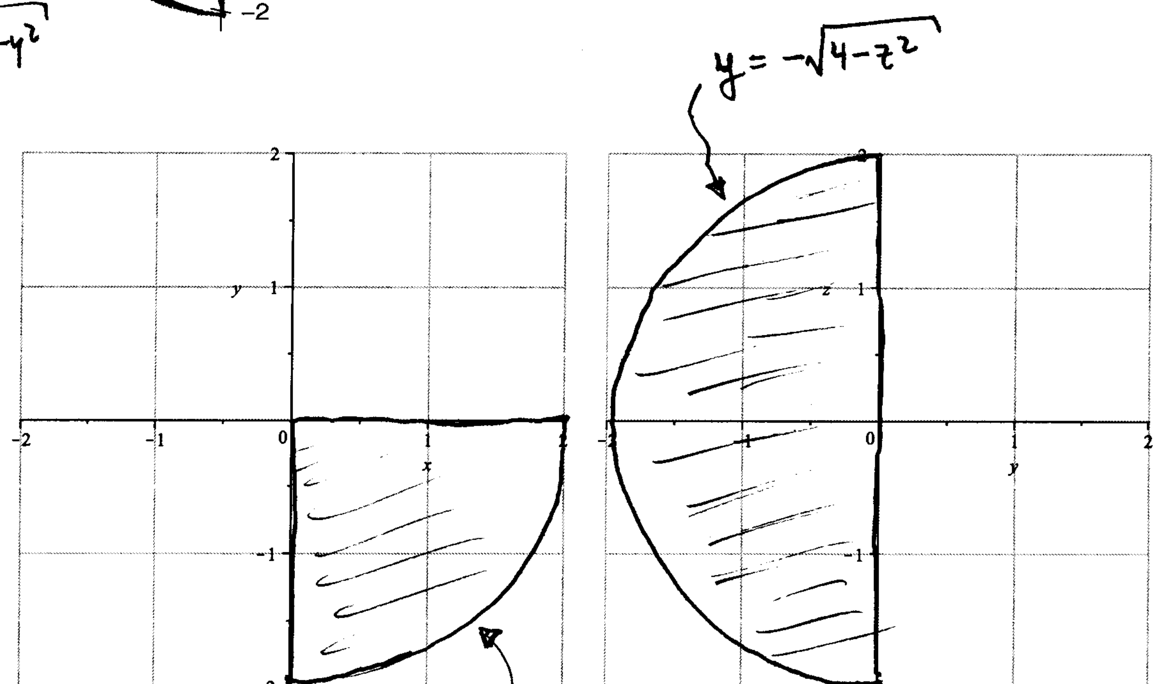
max value $= 47$

min value $= -7$

Problem 8 [21 points]. Let E be the solid region inside of the sphere $x^2 + y^2 + z^2 = 4$ where the x-coordinate is positive and the y-coordinate is negative.

1. **3 points.** Sketch the region E, you may use the given set of axes or draw your own. If you draw your own, include labels and coordinate values. Sketch the projection of E onto the xy-plane, and its projection onto the yz-plane using the graphs provided.





Projection of E onto the xy-plane.

Projection of E onto the yz-plane.

$y = -\sqrt{4-x^2}$

Problem 8 continued:

2. Find the limits of integration for the triple integral $\iiint_E f(x,y,z)\,dV$ in the coordinates given below. **3 points each..**

$$\int_{x=0}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=0} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

$$\int_{z=-2}^{z=2} \int_{y=-\sqrt{4-z^2}}^{y=0} \int_{x=0}^{x=\sqrt{4-y^2-z^2}} f(x,y,z)\,dx\,dy\,dz$$

$$\int_{\theta=\frac{3\pi}{2}}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

$$\int_{\phi=0}^{\phi=\pi} \int_{\theta=\frac{3\pi}{2}}^{\theta=2\pi} \int_{\rho=0}^{\rho=2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\,\sin\phi\,d\rho\,d\theta\,d\phi$$

Problem 8 continued:

3. **4 points.** Compute the integral $\iiint_E x\,dV$.

$$\int_{\phi=0}^{\pi}\int_{\theta=\frac{3\pi}{2}}^{2\pi}\int_{\rho=0}^{2} \rho\cos\theta\sin\phi \; \rho^2\sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi}\int_{\theta=\frac{3\pi}{2}}^{2\pi} \left[\frac{1}{4}\rho^4\right]_0^2 \cos\theta\sin^2\phi\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi} 4\,[\sin\theta]_{\frac{3\pi}{2}}^{2\pi} \;\frac{1}{2}(1-\cos 2\phi)\, d\phi$$

$$= 4(0-(-1))\left[\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi\right]_0^{\pi} = 4\cdot\left(\frac{\pi}{2}-0\right) = \boxed{2\pi}$$

4. **2 points.** Use the result of part 3 to determine the coordinates of the center of mass of the region E (assuming it has uniform density). You may use symmetry arguments, and use the fact that a sphere of radius R has volume $\frac{4\pi}{3}R^3$.

E is $\frac{1}{4}$ of a sphere so $\mathrm{Vol}(E) = \frac{1}{4}\left(\frac{4\pi}{3}(2)^3\right) = \frac{8\pi}{3}$

$$(\bar{x},\bar{y},\bar{z}) = \frac{1}{\mathrm{vol}(E)}\left(\iiint_E x\,dV, \iiint_E y\,dV, \iiint_E z\,dV\right)$$

$$= \frac{3}{8\pi}\left(2\pi, \; -2\pi, \; 0\right)$$

from part 3

since object is symmetric under $x \leftrightarrow -y$ we have $\bar{x} = -\bar{y}$

since object is symmetric under $z \leftrightarrow -z$, $\bar{z}=0$

$\therefore$ $\boxed{(\bar{x},\bar{y},\bar{z}) = \left(\frac{3}{4}, \frac{-3}{4}, 0\right)}$