Midterm 1 October 14, 2015 Duration: 50 minutes

This test has 4 ***questions*** *on* ***5 pages****, each worth 10 points, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, except question #3 where the answer alone is sufficient.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices (including calculators, phones, etc.)

First Name: Solutions Last Name: __________

Student No.: __________ Section: __________

Signature: __________

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

2 marks

1. (a) Two lines L_1 and L_2 in the x, y, z coordinate system are given in symmetric form as follows:

$$L_1 : \frac{x}{5} = \frac{y-2}{-1} = \frac{z}{-1}, \qquad L_2 : \frac{x-1}{1} = \frac{y-3}{1} = \frac{z+1}{-1},$$

These lines intersect at a single point. Find that point.

Answer: $(0, 2, 0)$

Solution: Insert $x = -5y + 10$ in $x - 1 = y - 3$ to get $6y = 12$, so $y = 2$. Then $x = (-5)(2) + 10 = 0$, and $z = -x/5 = 0$, so the intersection point is $(0, 2, 0)$.

3 marks

(b) Find a normal vector to the plane containing the above lines L_1 and L_2.

Answer: $\langle 1, 2, 3 \rangle$

Solution: The lines have directions given by the vectors $\langle 5, -1, -1 \rangle$ and $\langle 1, 1, -1 \rangle$. Their cross product, which is normal to the plane, is $\langle 5, -1, -1 \rangle \times \langle 1, 1, -1 \rangle = \langle 2, 4, 6 \rangle$, and we simplify this to the parallel vector $\langle 1, 2, 3 \rangle$.

2 marks

(c) Find the equation of the plane containing the above lines L_1 and L_2 (write as an equation in terms of x, y, z).

Answer: $x + 2y + 3z = 4$

Solution: The equation of the plane must have the form $x + 2y + 3z + d = 0$. The point $(0, 2, 0)$ is on the plane, so $d = -4$.

3 marks

(d) (This part is not related to parts (a), (b), (c).) Find the angle between the plane $x + y = 0$ and the vector $\vec{a} = \langle 1, 0, -1 \rangle$.

Answer: $\pi/6$ or $30°$

Solution: A normal vector to the plane is $\vec{n} = \langle 1, 1, 0 \rangle$. The angle θ between $\vec{a}$ and $\vec{n}$ obeys

$$\cos\theta = \frac{\vec{a} \cdot \vec{n}}{|\vec{a}|\,|\vec{n}|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2},$$

so $\theta = \frac{\pi}{3}$. The angle between $\vec{a}$ and the plane is the complementary angle $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$.

2 marks

2. (a) Let $u(x,y) = x^{4m} + \frac{x}{y} + e^m$, where m is a fixed integer. Compute the partial derivatives u_x and u_y.

Answer: $u_x = 4mx^{4m-1} + \frac{1}{y}$, $\quad u_y = -\frac{x}{y^2}$.

4 marks

(b) Consider the wave equation $u_{tt} = c^2 u_{xx}$. Let m be an integer and λ a real number. Is $u(x,t) = \cos(mx)\sin(\lambda t)$ a solution for the wave equation? The options are:

Yes, for any values of λ, m (give a proof).

Yes, but only for special values of λ, m (describe those values).

No, not for any λ, m (explain why not).

Answer: Yes, but only if $\lambda = \pm cm$.

Solution: We compute the partial derivatives:

$$u_x = -m \sin mx \sin \lambda t, \quad u_{xx} = -m^2 \cos mx \sin \lambda t,$$
$$u_t = \lambda \cos mx \cos \lambda t, \quad u_{tt} = -\lambda^2 \cos mx \sin \lambda t.$$

Substituting into the wave equation,

$$-\lambda^2 \cos mx \sin \lambda t = c^2 \left(-m^2 \cos mx \sin \lambda t\right).$$

Since $\cos mx \sin \lambda t$ is not generally zero, we must have $-\lambda^2 = -c^2 m^2$, or $\lambda = \pm cm$.

2 marks

(c) Let $v(x,t) = e^{(x+t)/2}$. Compute the partial derivatives v_{xx} and v_t.

Answer: $v_{xx} = \frac{1}{4}e^{(x+t)/2}$, $v_t = \frac{1}{2}e^{(x+t)/2}$

Solution:

$$v_x = \frac{1}{2}e^{(x+t)/2}, \quad v_{xx} = \frac{1}{4}e^{(x+t)/2}, \quad v_t = \frac{1}{2}e^{(x+t)/2}.$$

2 marks

(d) Find a solution to the forced heat equation $u_t - u_{xx} = 4e^{(x+t)/2}$.

Answer: $u(x,t) = 16e^{(x+t)/2}$

Solution: From part (c),

$$v_t - v_{xx} = \frac{1}{2}e^{-(x+t)/2} - \frac{1}{4}e^{(x+t)/2} = \frac{1}{4}e^{(x+t)/2},$$

so $u(x,t) = 16v(x,t)$ obeys

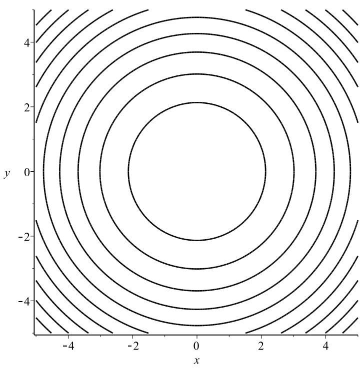
$$u_t - u_{xx} = 16(v_t - v_{xx}) = (16)(1/4)e^{(x+t)/2}.$$

10 marks 3. Match each function with its contour plot. In the contour plots, the *values* of the contours are evenly spaced.

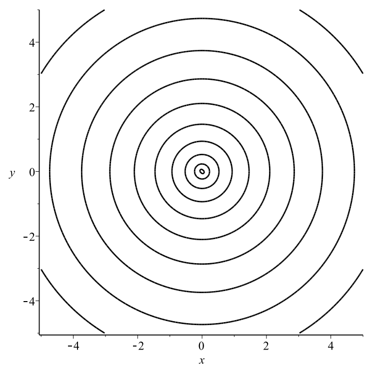
$f_1(x,y) = x^2 - y$ ______ $f_2(x,y) = \sin(x)\sin(y)$ ______ $f_3(x,y) = \sin(xy)$ ______

$f_4(x,y) = x^2 + y^2$ ______ $f_5(x,y) = \sqrt{x^2+y^2}$ _____

Answer: f_1G, f_2E, f_3J, f_4A, f_5C



A.



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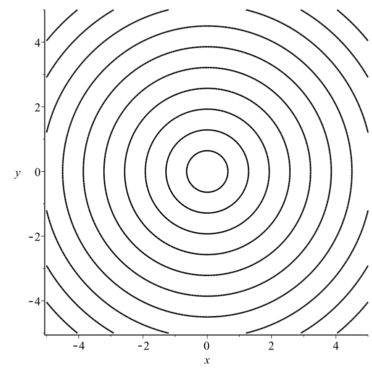

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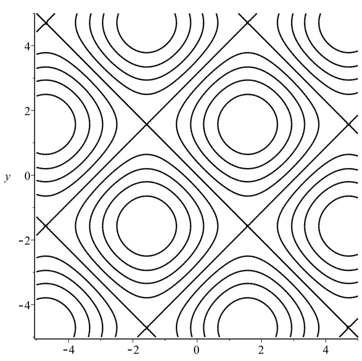

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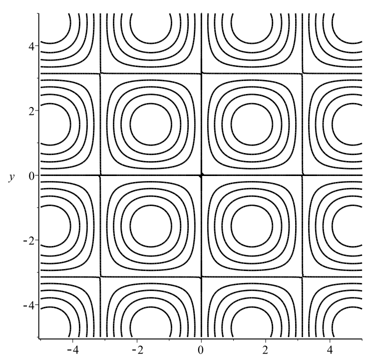

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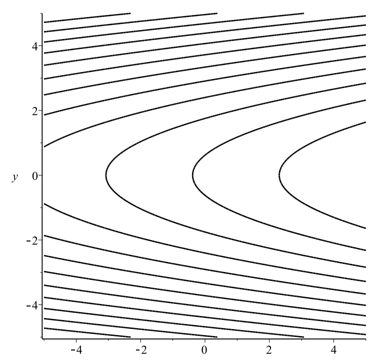

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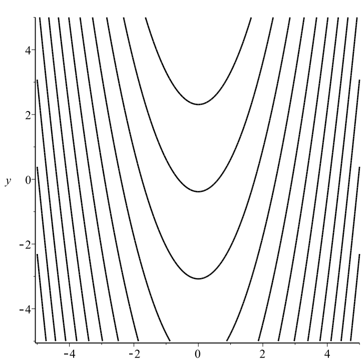

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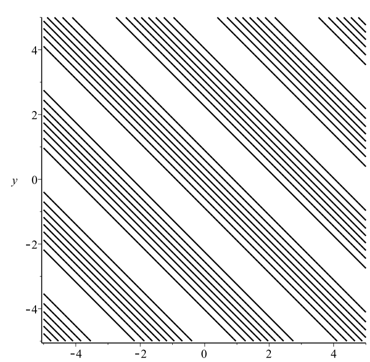

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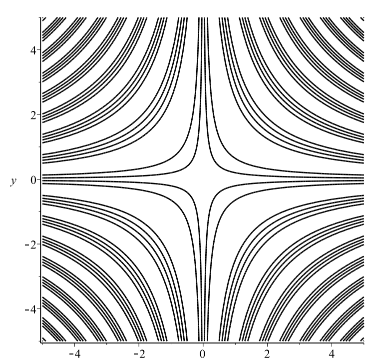

J.

5 marks 4. (a) The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has area $A = \pi ab$. I compute the area of the ellipse using measurements $a = 10$ cm and $b = 5$ cm for its axes. However, I am only confident of each of my measurements up to ± 1 mm error. Use differentials to estimate the maximal possible error in the computed area.

Answer: $3\pi/2$ cm^2

Solution: Computing differentials, we find

$$dA = \pi b\, da + \pi a\, db.$$

With $a = 10$ and $b = 5$,

$$dA = 5\pi da + 10\pi db.$$

We know that both $|\Delta a|, |\Delta b| \leq 0.1$ cm. To maximize the potential error, we input $da = db = 0.1$ and find the maximum error has magnitude

$$dA = 1.5\pi \text{ cm}^2.$$

5 marks (b) Consider the function

$$f(x, y) = (y - 1)e^{\cos(xy-y)}.$$

Using a linear approximation, estimate the value of $f(1.1 + \frac{\pi}{2}, 0.9)$.

Answer: -0.1

Solution: The linear approximation near $(1 + \frac{\pi}{2}, 1)$ is

$$f(1.1 + \frac{\pi}{2}, 0.9) \approx f(1 + \frac{\pi}{2}, 1) + f_x(1 + \frac{\pi}{2}, 1)(0.1) + f_y(1 + \frac{\pi}{2}, 1)(-0.1).$$

To evaluate the right-hand side, we use $f(1 + \frac{\pi}{2}, 1) = 0$, and

$$f_x(x, y) = (y - 1)e^{\cos(xy-y)} \cdot (-\sin(xy - y) \cdot y)$$
$$f_y(x, y) = e^{\cos(xy-y)} + (y - 1) \cdot e^{\cos(xy-y)} \cdot (-\sin(xy - y) \cdot (x - 1)),$$

and hence

$$f_x(1 + \frac{\pi}{2}, 1) = 0, \quad f_y(1 + \frac{\pi}{2}, 1) = 1.$$

Thus our approximation is

$$f(1.1 + \frac{\pi}{2}, 0.9) \approx 0 + (0)(0.1) + (1)(-0.1) = -0.1.$$