

Midterm 1 October 14, 2015 Duration: 50 minutes

This test has 4 questions on 5 pages, each worth 10 points, for a total of 40 points.

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, except question #3 where the answer alone is sufficient.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices (including calculators, phones, etc.)

First Name: Solutions Last Name: _____

Student No.: _____ Section: _____

Signature: _____

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

2 marks

1. (a) Two lines L_1 and L_2 in the x, y, z coordinate system are given in symmetric form as follows:

$$L_1 : \frac{x}{5} = \frac{y-2}{-1} = \frac{z}{-1}, \quad L_2 : \frac{x-1}{1} = \frac{y-3}{1} = \frac{z+1}{-1},$$

These lines intersect at a single point. Find that point.

Answer: $(0, 2, 0)$

Solution: Insert $x = -5y + 10$ in $x - 1 = y - 3$ to get $6y = 12$, so $y = 2$. Then $x = (-5)(2) + 10 = 0$, and $z = -x/5 = 0$, so the intersection point is $(0, 2, 0)$.

3 marks

- (b) Find a normal vector to the plane containing the above lines L_1 and L_2 .

Answer: $\langle 1, 2, 3 \rangle$

Solution: The lines have directions given by the vectors $\langle 5, -1, -1 \rangle$ and $\langle 1, 1, -1 \rangle$. Their cross product, which is normal to the plane, is $\langle 5, -1, -1 \rangle \times \langle 1, 1, -1 \rangle = \langle 2, 4, 6 \rangle$, and we simplify this to the parallel vector $\langle 1, 2, 3 \rangle$.

2 marks

- (c) Find the equation of the plane containing the above lines L_1 and L_2 (write as an equation in terms of x, y, z).

Answer: $x + 2y + 3z = 4$

Solution: The equation of the plane must have the form $x + 2y + 3z + d = 0$. The point $(0, 2, 0)$ is on the plane, so $d = -4$.

3 marks

- (d) (This part is not related to parts (a), (b), (c).) Find the angle between the plane $x + y = 0$ and the vector $\vec{a} = \langle 1, 0, -1 \rangle$.

Answer: $\pi/6$ or 30°

Solution: A normal vector to the plane is $\vec{n} = \langle 1, 1, 0 \rangle$. The angle θ between \vec{a} and \vec{n} obeys

$$\cos \theta = \frac{\vec{a} \cdot \vec{n}}{|\vec{a}| |\vec{n}|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2},$$

so $\theta = \frac{\pi}{3}$. The angle between \vec{a} and the plane is the complementary angle $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$.

2 marks

2. (a) Let $u(x, y) = x^{4m} + \frac{x}{y} + e^m$, where m is a fixed integer. Compute the partial derivatives u_x and u_y .

$$\text{Answer: } u_x = 4mx^{4m-1} + \frac{1}{y}, \quad u_y = -\frac{x}{y^2}.$$

4 marks

- (b) Consider the wave equation $u_{tt} = c^2 u_{xx}$. Let m be an integer and λ a real number. Is $u(x, t) = \cos(mx) \sin(\lambda t)$ a solution for the wave equation? The options are:

Yes, for any values of λ, m (give a proof).

Yes, but only for special values of λ, m (describe those values).

No, not for any λ, m (explain why not).

$$\text{Answer: Yes, but only if } \lambda = \pm cm.$$

Solution: We compute the partial derivatives:

$$\begin{aligned} u_x &= -m \sin mx \sin \lambda t, & u_{xx} &= -m^2 \cos mx \sin \lambda t, \\ u_t &= \lambda \cos mx \cos \lambda t, & u_{tt} &= -\lambda^2 \cos mx \sin \lambda t. \end{aligned}$$

Substituting into the wave equation,

$$-\lambda^2 \cos mx \sin \lambda t = c^2 (-m^2 \cos mx \sin \lambda t).$$

Since $\cos mx \sin \lambda t$ is not generally zero, we must have $-\lambda^2 = -c^2 m^2$, or $\lambda = \pm cm$.

2 marks

- (c) Let $v(x, t) = e^{(x+t)/2}$. Compute the partial derivatives v_{xx} and v_t .

$$\text{Answer: } v_{xx} = \frac{1}{4} e^{(x+t)/2}, \quad v_t = \frac{1}{2} e^{(x+t)/2}$$

Solution:

$$v_x = \frac{1}{2} e^{(x+t)/2}, \quad v_{xx} = \frac{1}{4} e^{(x+t)/2}, \quad v_t = \frac{1}{2} e^{(x+t)/2}.$$

2 marks

- (d) Find a solution to the forced heat equation $u_t - u_{xx} = 4e^{(x+t)/2}$.

$$\text{Answer: } u(x, t) = 16e^{(x+t)/2}$$

Solution: From part (c),

$$v_t - v_{xx} = \frac{1}{2} e^{(x+t)/2} - \frac{1}{4} e^{(x+t)/2} = \frac{1}{4} e^{(x+t)/2},$$

so $u(x, t) = 16v(x, t)$ obeys

$$u_t - u_{xx} = 16(v_t - v_{xx}) = (16)(1/4)e^{(x+t)/2}.$$

10 marks

3. Match each function with its contour plot. In the contour plots, the *values* of the contours are evenly spaced.

$$f_1(x, y) = x^2 - y \quad \text{_____}$$

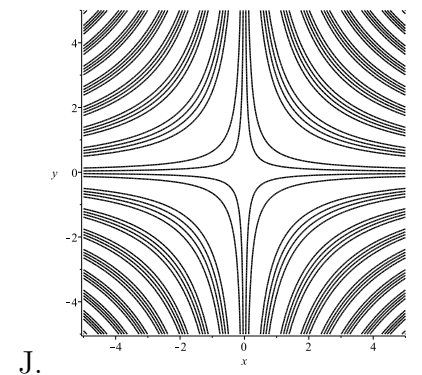
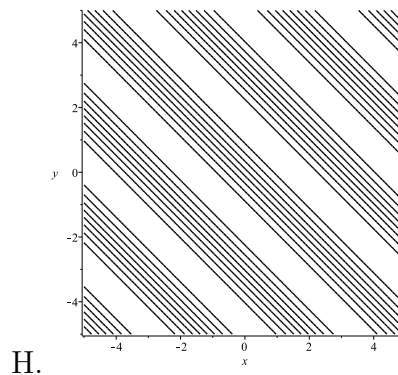
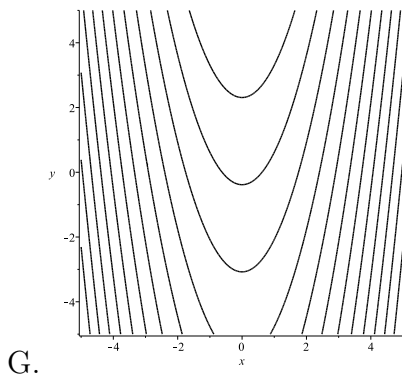
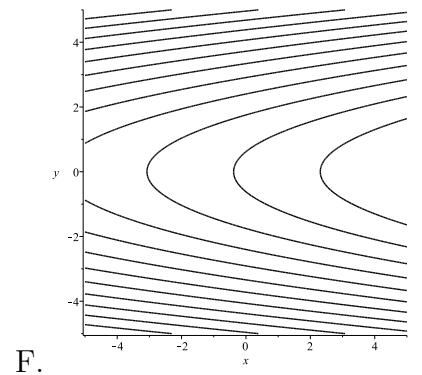
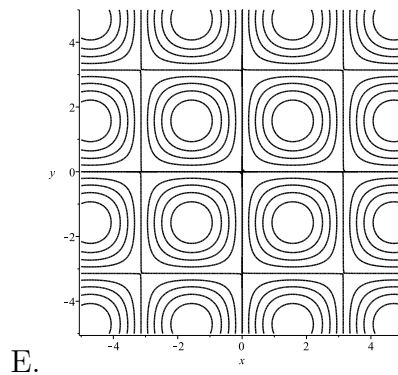
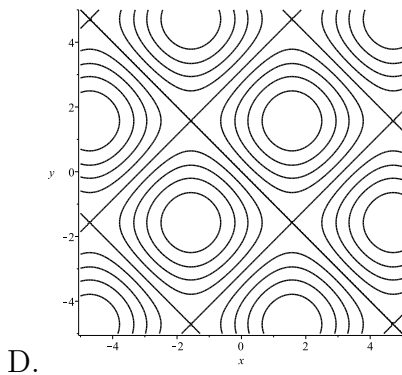
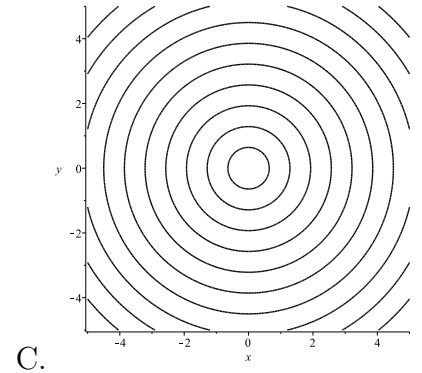
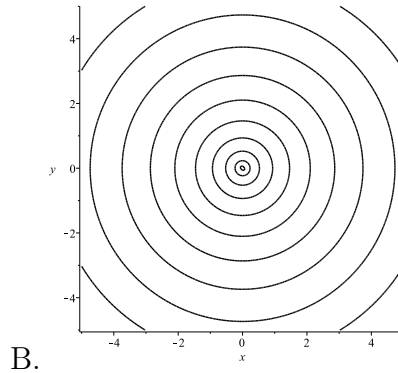
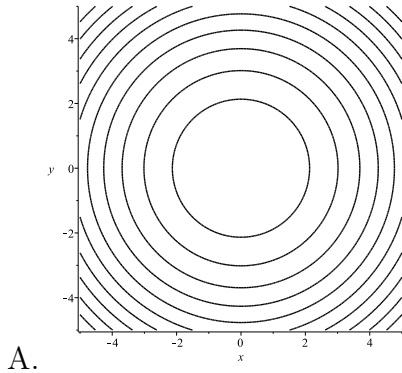
$$f_2(x, y) = \sin(x) \sin(y) \quad \text{_____}$$

$$f_3(x, y) = \sin(xy) \quad \text{_____}$$

$$f_4(x, y) = x^2 + y^2 \quad \text{_____}$$

$$f_5(x, y) = \sqrt{x^2 + y^2} \quad \text{_____}$$

Answer: f_1 G, f_2 E, f_3 J, f_4 A, f_5 C



5 marks 4. (a) The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has area $A = \pi ab$. I compute the area of the ellipse using measurements $a = 10$ cm and $b = 5$ cm for its axes. However, I am only confident of each of my measurements up to ± 1 mm error. Use differentials to estimate the maximal possible error in the computed area.

Answer: $3\pi/2$ cm²

Solution: Computing differentials, we find

$$dA = \pi b da + \pi a db.$$

With $a = 10$ and $b = 5$,

$$dA = 5\pi da + 10\pi db.$$

We know that both $|\Delta a|, |\Delta b| \leq 0.1$ cm. To maximize the potential error, we input $da = db = 0.1$ and find the maximum error has magnitude

$$dA = 1.5\pi \text{ cm}^2.$$

5 marks (b) Consider the function

$$f(x, y) = (y - 1)e^{\cos(xy-y)}.$$

Using a linear approximation, estimate the value of $f(1.1 + \frac{\pi}{2}, 0.9)$.

Answer: -0.1

Solution: The linear approximation near $(1 + \frac{\pi}{2}, 1)$ is

$$f(1.1 + \frac{\pi}{2}, 0.9) \approx f(1 + \frac{\pi}{2}, 1) + f_x(1 + \frac{\pi}{2}, 1)(0.1) + f_y(1 + \frac{\pi}{2}, 1)(-0.1).$$

To evaluate the right-hand side, we use $f(1 + \frac{\pi}{2}, 1) = 0$, and

$$f_x(x, y) = (y - 1)e^{\cos(xy-y)} \cdot (-\sin(xy - y) \cdot y)$$

$$f_y(x, y) = e^{\cos(xy-y)} + (y - 1) \cdot e^{\cos(xy-y)} \cdot (-\sin(xy - y) \cdot (x - 1)),$$

and hence

$$f_x(1 + \frac{\pi}{2}, 1) = 0, \quad f_y(1 + \frac{\pi}{2}, 1) = 1.$$

Thus our approximation is

$$f(1.1 + \frac{\pi}{2}, 0.9) \approx 0 + (0)(0.1) + (1)(-0.1) = -0.1.$$