Introduction to admissible representations of p-adic groups

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Chapter II. Smooth representations

The theory of representations of $\mathfrak{p}$-adic groups started off by looking at unitary representations on Hilbert spaces. After [Jacquet-Langlands:1970] introduced the category of admissible representations, the subject largely lost its analytical flavour and became quite algebraic—in many aspects not very different from the theory of representations of finite groups.

In this chapter I'll define smooth and admissible representations of arbitrary locally profinite groups and prove their basic properties. I shall generally take the coefficient ring to be a commutative Noetherian ring $\mathcal{R}$, assumed to contain $\mathbb{Q}$. The point of allowing representations with coefficients in a ring like $\mathcal{R}$ is to allow dealing with families of representations in a reasonable way. For example, suppose $\mathfrak{k}$ to be a $\mathfrak{p}$-adic field and G to be the multiplicative group $\mathfrak{k}^\times$. The unramified characters of G are those trivial on $\mathfrak{o}^\times$, and may be identified with characters of $\mathbb{Z}$. In this case it is useful to take $\mathcal{R}$ to be the ring $\mathbb{C}[z^{\pm 1}]$, the group algebra of $\mathbb{Z}$, whose maximal ideals parametrize its complex characters. The group acts on this through the left or right regular representation.

Occasionally $\mathcal{R}$ will be taken to be a field. Often, it will be a field $\mathbb{D}$, which I assume throughout to be an algebraically closed field of characteristic 0. In places I shall take $\mathbb{D}$ to be $\mathbb{C}$. There are good reasons, as we shall see, for not making this choice always. One might be called philosophical—the theory of representations of a locally profinite group is essentially a matter of algebra, and the choice of $\mathbb{D}$ emphasizes this.

Throughout this chapter, G will be a locally profinite group. At the beginning it will be more or less arbitrary. However, I'll assume always that it possesses a countable basis of neighbourhoods of the identity—i.e. that it is what I call in [Casselman:2018] a König group. Also, it will be countable at infinity in the sense that the discrete set G/K is countable for one, hence all, compact open subgroups K.

Thus this chapter will generally present results valid without significant assumptions about the structure of G. This necessarily excludes almost everything interesting, but includes many basic items. The current version incorporates parts of [Bernstein:1992].

Contents

1. Admissible representations and the Hecke algebra [hecke.tex]

A **smooth** G module over $\mathcal{R}$ is a representation (π, V) of G on an $\mathcal{R}$-module V such that each v in V is fixed by an open subgroup of G. A smooth representation (π, V) is said to be **admissible** if for each open subgroup K in G the subspace V^K of vectors fixed by elements of K is finitely generated over $\mathcal{R}$. If $\mathcal{R}$ is a field, this just means that V^K has finite dimension.

The subspace of smooth vectors in any representation of G is stable under G, since if v lies in V^K then $\pi(g)v$ lies in $V^{gKg^{-1}}$.

The group G acts on functions on G by the left and right regular representations:

$$R_g f(x) = f(xg), \quad L_g f(x) = f(g^{-1}x) .$$

I'll call a locally constant function f on G **uniformly** locally constant if there exists a compact open subgroup K of G such that $R_k f = L_k f$ for all k in K. The left and right actions of G commute, hence give rise to a smooth representation $\mathcal{L}$ of $G \times G$ on the space of all locally constant functions that preserves the uniformly locally constant ones.

The rational **Hecke algebra** $\mathcal{H}_{\mathbb{Q}}(G)$ is the ring of smooth rational measures of compact support on G, in which multiplication is convolution. Let

$$\mathcal{H}_{\mathcal{R}} = \mathcal{R} \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}} .$$

♡ [smooth-distribution] Normally, the ring $\mathcal{R}$ will be implicit, and $\mathcal{H}_{\mathcal{R}}$ will be written as just $\mathcal{H}$. It is a consequence of Proposition I.6.6 that given a choice of rational Haar (right-invariant) measure dx, smooth $\mathcal{R}$-valued functions may be identified with smooth $\mathcal{R}$-valued distributions:

$$\varphi \longmapsto D_\varphi = \varphi(x)\, dx ,$$

so that

$$\langle D_\varphi, f \rangle = \mathrm{meas}(K) \sum_{G/K} \varphi(x) f(x)$$

if both f and φ are fixed on the right by elements of K.

Suppose that D_1 and D_2 are two smooth measures of compact support, corresponding to smooth functions φ_1 and φ_2. Then

$$\begin{aligned}
\pi(D_1)\pi(D_2)v &= \int_G \varphi_1(x)\pi(x) \left(\int_G \varphi_2(y)\pi(y)v\, dy \right) dx \\
&= \int_{G\times G} \varphi_1(x)\varphi(y)\pi(xy)v\, dx\, dy \\
&= \int_G \varphi_2(y) \left(\int_G \varphi_1(x)\pi(xy)v\, dx \right) dy \\
&= \int_G \varphi_2(y) \left(\int_G \varphi_1(zy^{-1})\pi(z)v\, dz \right) dy \\
&= \int_G \left(\int_G \varphi_1(zy^{-1})\varphi_2(y)\, dz \right) \pi(z)v\, dz \\
&= \pi(D_\varphi)v
\end{aligned}$$

where

$$\varphi(z) = \int_G \varphi_1(zy^{-1})\varphi_2(y)\, dy .$$

The measure D_φ is also smooth and of compact support. It is the **convolution** $D_1 * D_2$. This product makes $\mathcal{H}$ into a ring.

The Hecke algebra $\mathcal{H}$ does not have a multiplicative unit unless G is compact. This is the source of some technical problems.

There exists for every smooth representation (π, V) of G a canonical homomorphism from the ring $\mathcal{H}$ to $\mathrm{End}_{\mathcal{R}}(V)$. Fix for the moment a right-invariant Haar measure dx on G. Suppose D to be D_φ. For v in V define

$$\pi(D)v = \int_G \varphi(x)\pi(x)v \, dx \, .$$

If v is fixed by the elements of the compact open group K and φ is fixed by elements of K with respect to the right regular representation, this is also

$$\mathrm{meas}(K) \sum_{G/K} \varphi(x)\pi(x)v \, .$$

This definition depends only on D_φ. We can in fact characterize, if not define, $\pi(D)$ solely in terms of the distribution D. If F is a linear function on V, then $\Phi(x) = F\big(\pi(x)v\big)$ is a locally constant function on G, and we may apply D to it. Then

$$F\big(\pi(D)v\big) = \int_G \varphi(x) F\big(\pi(x)v\big) \, dx = \langle D, \Phi \rangle \, .$$

If K is a compact open subgroup of G, the Hecke algebra $\mathcal{H}$ contains the subalgebra $\mathcal{H}(G/\!/K)$ of distributions invariant under left and right multiplication by elements of K. This algebra has as basis the measures $\mu_{KgK/K}$ of $\mathcal{H}$:

$$\langle \mu_{KgK/K}, f \rangle = \frac{1}{\mathrm{meas}(K)} \int_{KgK} f(x) \, dx$$

as g varies over $K\backslash G/K$. In particular

$$\pi(\mu_{K/K}) = \frac{1}{\mathrm{meas}(K)} \int_K \pi(k) \, dk$$

amounts to projection onto K-fixed vectors. If π is any smooth representation of G then every element of $\mathcal{H}(G/\!/K)$ takes V^K into itself, and

(II.1.1) [kgk/k]

$$\begin{aligned} \pi(\mu_{KgK/K}) &= \sum_{KgK/K} \pi(x)\pi(\mu_{K/K}) \\ &= \sum_{K/K\cap gKg^{-1}} \pi(kg)\pi(\mu_{K/K}) \\ &= |KgK/K| \cdot \pi(\mu_{K/K})\pi(g)\pi(\mu_{K/K}) \, . \end{aligned}$$

since the map from $K/K \cap gKg^{-1}$ to KgK/K, taking k to kg, is a bijection.

The projection $\mu_{K/K}$ is the unit of $\mathcal{H}(G/\!/K)$.

For every closed subgroup H of G, define $V(H)$ to be the subspace of V generated by the $\pi(h)v - v$ for h in H. This subspace of V is characterized by the property that every H-equivariant linear map from V to a vector space on which H acts trivially factors through $V/V(H)$. If H is compact, something better happens:

[projection] **II.1.2. Proposition.** *For any compact open subgroup K and smooth representation V, we have an equality*

$$V(K) = \big\{ v \in V \mid \pi(\mu_{K/K})v = 0 \big\}$$

and a direct sum decomposition

$$V = V(K) \oplus V^K \, .$$

Proof. If v is fixed by K_* then

$$\pi(\mu_{K/K})v = \frac{1}{[K\colon K_*]} \sum_{K/K_*} \pi(k)v$$

and

$$v = \frac{1}{[K\colon K_*]} \sum_{K/K_*} v \; .$$

If we subtract the first from the second, we get

$$v - \pi(\mu_{K/K})v = \frac{-1}{[K\colon K_*]} \sum_{K/K_*} \big(\pi(k)v - v\big)$$

[abelian-smooth] **II.1.3. Proposition.** *Suppose*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

to be an exact sequence of G-representations. If the representation on V is smooth so is that on U and W, and for every compact open subgroup K the sequence

$$0 \longrightarrow U^K \longrightarrow V^K \longrightarrow W^K \longrightarrow 0$$

is also exact.

[exactness-cor] **II.1.4. Corollary.** *If V is smooth, it is admissible if and only if both U and W are.*

Thus the categories of smooth and admissible representations are abelian.

[restriction-to-K] **II.1.5. Proposition.** *Suppose K to be a fixed compact open subgroup of G, and suppose $\mathcal{R}$ to be $\mathbb{D}$. A smooth representation defined over $\mathcal{R}$ is admissible if and only if the restriction of π to K is a direct sum of irreducible representations, each occurring with finite multiplicity.*

One has to be a bit careful, because an irreducible representation over a field might become reducible over a field extension. This is not a problem if the field is algebraically closed.

Proof. Choose a sequence of compact open subgroups K_n normal in K and with $\{1\}$ as limit. Then $V = V(K_n) \oplus V^{K_n}$. But every smooth finite-dimensional representation of K factors through some K_n.

As I remarked earlier, in many ways things do not behave here very differently from how they do for finite groups.

The following is trivial, but best to state formally:

[irr-hom] **II.1.6. Proposition.** *If (π_i, V_i) are two smooth representations of G over $\mathbb{D}$, then*

$$\mathrm{Hom}_G(V_1, V_2) = \begin{cases} \mathbb{D} & \text{if } \pi_1 \text{ is isomorphic to } \pi_2 \\ 0 & \text{otherwise.} \end{cases}$$

[fg-field] **II.1.7. Lemma.** *Suppose that G possesses a countable basis $\{K_n\}$ of compact open subgroups such that $\mathcal{H}_{\mathbb{Q}}(G/\!/K_n)$ is a finitely generated ring. Then every admissible representation (π, V) may be defined over a field F generated over $\mathbb{Q}$ by a countable set of generators.*

We shall see eventually that this hypothesis holds when G is the group of rational points on a Zariski-connected reductive group defined over a $\mathfrak{p}$-adic field.

2. The centre of G [hecke.tex]

In this section, assume the coefficient ring to be $\mathbb{D}$.

If (π, V) is an admissible representation of G then each space V^K is stable under the centre Z_G of G. The subgroup $Z_G \cap K$ acts trivially on it.

[schur-scalars] **II.2.1. Proposition.** *If (π, V) is irreducible the centre acts as scalar multiplication by a single character.*

Proof. Suppose $V^K \neq \{0\}$. Since V^K has finite dimension and $\mathbb{D}$ is algebraically closed, there exists a character ω of Z_G with values in $\mathbb{D}^\times$ such that the subspace

$$V^K(\omega) = \big\{ v \in V^K \bigm| \pi(z)v = \omega(z) \cdot v \text{ for all } z \in Z_G \big\} \neq \{0\} .$$

If U is the subspace of all v in V satisfying this equation, it is both non-trivial and stable under G, hence equal to all of V.

In general, I call an admissible representation **centrally simple** if this occurs. If Z_G acts through the character ω then π is called an ω-representation. For any central character ω with values in $\mathbb{D}^\times$ the Hecke algebra $\mathcal{H}_{\mathbb{D},\omega}$ is that of uniformly smooth functions on G compactly supported modulo Z_G such that

$$f(zg) = \omega(z) f(g) .$$

If π is centrally simple with central character ω it becomes a module over the Hecke algebra $\mathcal{H}_{\omega^{-1}}$:

$$\pi(f)v = \int_{G/Z_G} f(x)\pi(x)v \, dx ,$$

which is well defined since $f(zx)\pi(zx) = f(x)\pi(x)$.

But now I generalize this somewhat.

If C is a commuting set of linear operators acting on a vector space U of dimension n and γ a map from C to $\mathbb{D}$, let

$$U(\!(\gamma)\!) = \big\{ v \in U \bigm| \big(c - \gamma(c)\big)^n v = 0 \text{ for some } n \text{ and all } c \in C \big\} .$$

[commuting] **II.2.2. Lemma.** *Suppose C to be any commuting set of linear operators acting on the finite dimensional vector space U over $\mathbb{D}$. There exists a direct sum decomposition of U into non-zero spaces $U(\!(\gamma)\!)$.*

It is called the **primary decomposition** of U.

Proof. The slight technical problem is that no assumption is made on the size of C.

Suppose at first that C is made up of a single element c. Let $P(x) = \prod (x - \gamma_i)^{m_i}$ be the characteristic polynomial of c. For m larger than all m_i

$$\prod\nolimits_i (c - \gamma_i)^m = 0 .$$

Since there exist polynomials $a(x)$ and $b(x)$ such that

$$1 = a(x)(x - \gamma_1)^m + b(x) \prod\nolimits_{i \neq 1} (x - \gamma_i)^m ,$$

a simple inductive argument implies that U is the direct sum of primary eigenspaces with respect to c.

If C is finite a similar inductive argument will imply the Lemma. For each finite subset of C the primary decomposition of U assigns to that subset a partition of $n = \dim(U)$, that determined by the dimensions of its subspaces $U(\!(\gamma)\!)$. There are only a finite number of these partitions; choose one of greatest length. Suppose it belongs to the subset S in C. Any larger subset must determine a refinement of the decomposition for S, and hence must actually be the same. The decomposition for S is therefore one for all of C.

From this follows immediately:

[centre] **II.2.3. Proposition.** *If (π, V) is a finitely generated admissible representation of G, the restriction of π to Z_G is a direct sum of primary components $V(\!(\omega)\!)$, where the ω vary over a finite set of homomorphisms from Z_G to $\mathbb{D}^\times$.*

The characters ω occurring in this decomposition are called the **central characters** of π.

3. The contragredient [hecke.tex]

Suppose (π, V) to be a smooth representation of G. Let

$$\widehat{V} = \mathrm{Hom}_{\mathcal{R}}(V, \mathcal{R}) .$$

The group G acts on $\widehat{V}$ according to the recipe

$$\langle \widehat{\pi}(g)\widehat{v}, v\rangle = \langle \widehat{v}, \pi(g^{-1})v\rangle .$$

The point is that the canonical pairing is G-invariant. The **contragredient** representation $(\widetilde{\pi}, \widetilde{V})$ is that on the smooth vectors in $\widehat{V}$.

[K-fixed-dual] **II.3.1. Proposition.** *Suppose (π, V) to be a smooth representation of G and K a compact open subgroup of G. Restriction of f to V^K is an isomorphism of $\widetilde{V}^K$ with $\mathrm{Hom}_{\mathcal{R}}(V^K, \mathcal{R})$.*

Proof. Because $V = V^K \oplus V(K)$ and the functions in $\widetilde{V}^K$ are precisely those annihilating $V(K)$.

From the exact sequence of R-modules

$$\mathcal{R}^n \longrightarrow V^K \longrightarrow 0$$

we deduce

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}}(V^K, \mathcal{R}) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^n, \mathcal{R}) \cong \mathcal{R}^n .$$

Therefore $\widetilde{V}^K$ is finitely generated over $\mathcal{R}$, and $\widetilde{\pi}$ is again admissible.

In general, $\widetilde{V}$ may be very small. However, in special circumstances it will be sufficiently large. I'll call (π, V) **free** over $\mathcal{R}$ if each V^K is free over $\mathcal{R}$. There are many examples of such representations.

[free-c] **II.3.2. Lemma.** *If V is admissible and free over $\mathcal{R}$, then the contragredient is also free over $\mathcal{R}$, and the canonical map from V into the contragredient of its contragredient is an isomorphism.*

Furthermore:

[contraexact] **II.3.3. Corollary.** *Suppose U, V, W all to be free over $\mathcal{R}$. If*

$$0 \to U \to V \to W \to 0$$

is a short exact sequence of admissible representations, then so is

$$0 \to \widetilde{W} \to \widetilde{V} \to \widetilde{U} \to 0 .$$

Suppose U and V both to be smooth representations. Given a G-equivariant map from U to V, we get by duality one from $\widehat{V}$ to $\widehat{U}$.

[dualdual] **II.3.4. Proposition.** *Suppose U smooth, V admissible, both free over $\mathcal{R}$. The canonical map defined above:*

$$\mathrm{Hom}_G(U, V) \longrightarrow \mathrm{Hom}_G(\widetilde{V}, \widetilde{U})$$

is an isomorphism.

Proof. We can define an inverse. Dualizing once again we get an inverse map

$$\mathrm{Hom}_G\big(\widetilde{\widetilde{U}}, \widetilde{\widetilde{V}}\big)\,.$$

But U is canonically embedded in $\widetilde{\widetilde{U}}$ and since V is admissible we know that V may be identified with $\widetilde{\widetilde{V}}$, so we get an inverse map

$$\mathrm{Hom}_G\big(\widetilde{V}, \widetilde{U}\big) \longrightarrow \mathrm{Hom}_G(U, V)\,.$$

Remark. I am not completely happy with my treatment of the contragredient when the coefficient ring is not a field, and I am not at all sure what the right approach should be. There are certainly some cases in which the current treatment is entirely satisfactory.

4. Representations of a group and of its Hecke algebra [hecke.tex]

What is the relationship between a smooth representation of G and the associated representation of its Hecke algebra $\mathcal{H}$?

[hecke-same] **II.4.1. Proposition.** *Suppose (π_i, V_i) are two smooth representations of G. Then*

$$\mathrm{Hom}_G(V_1, V_2) = \mathrm{Hom}_{\mathcal{H}}(V_1, V_2)\,.$$

A mild technical problem here is that $\mathcal{H}$ has no multiplicative identity.

Proof. Any G-homomorphism is clearly a homomorphism of modules over the Hecke algebra as well. So suppose now that one is given a map F of modules over the Hecke algebra. Suppose v in V_1, g in G, and choose a compact open subgroup K fixing v, $\pi_1(g)v$, $F(v)$, and $\pi_2(g)F(v)$. Then

$$\begin{aligned} F\big(\pi_1(g)v\big) &= \frac{F\big(\pi_1(\mu_{KgK})v\big)}{|KgK/K|} \\ &= \frac{\pi_2(\mu_{KgK})F(v)}{|KgK/K|} \\ &= \pi_2(g)F(v)\,. \end{aligned}$$

A smooth representation is said to be **co-generated** by a subspace U if every non-zero G-stable subspace of V intersects U non-trivially. This is dual to the condition of generation, in the following sense:

[co-generation] **II.4.2. Lemma.** *Suppose $\mathcal{R}$ to be a field and K to be a compact open subgroup of G. An admissible representation (π, V) is generated by V^K if and only if its smooth contragredient is co-generated by $\widetilde{V}^K$.*

Proof. Suppose that V is generated by V^K, and suppose U to be a G-stable subspace of $\widetilde{V}$ with $U \cap \widetilde{V}^K = U^K = 0$. If $U^\perp$ is the annihilator of U in $\widetilde{\widetilde{V}} = V$, then $\big(V/U^\perp\big)^K = \widetilde{U}^K = 0$. Thus $V^K = (U^\perp)^K$, and since V^K generates V, $V = U^\perp$ and $U = 0$. The converse argument is similar.

[two-hecke] **II.4.3. Proposition.** *Suppose that (π_i, V_i) are two smooth representations of G and that K is a compact open subgroup of G. If*

(a) the space V_1 is generated as a G-space by V_1^K;
(b) the space V_2 is co-generated as a G-space by V_2^K.

Then the canonical map

$$\mathrm{Hom}_G(V_1, V_2) \longrightarrow \mathrm{Hom}_{\mathcal{H}(G//K)}(V_1^K, V_2^K)$$

is an isomorphism.

These conditions are satisfied if $V_1 = V_2$ is irreducible and K is small enough, for example.

Proof. If F lies in $\mathrm{Hom}_G(V_1, V_2)$ then for any f in $\mathcal{H}$ we have

$$F\big(\pi_1(f)v\big) = \pi_2(f)F(v)$$

for every f in $\mathcal{H}$ and v in V_1^K. Conversely, if we are given F in $\mathrm{Hom}_{\mathcal{H}(G//K)}(V_1^K, V_2^K)$ then since V_1^K generates V_1 this formula will serve to define a G-map from V_1 to V_2 once we know that

if v lies in V_1^K, f in $\mathcal{H}$, and $\pi_1(f)v = 0$ then $\pi_2(f)F(v) = 0$.

But if $\pi_1(f)v = 0$ then for every h in $\mathcal{H}$

$$\pi_1(\mu_{K/K} * h)\pi_1(f)v = \pi_1(\mu_{K/K} * h * f * \mu_{K/K})v = 0\,.$$

Since F is assumed to be $\mathcal{H}(G//K)$-equivariant,

$$\pi_2(\mu_{K/K} * h * f * \mu_{K/K})F(v) = \pi_2(\mu_{K/K} * h)\pi_2(f)F(v) = 0$$

for every h in $\mathcal{H}$. This means that the G-space U generated by $\pi_2(f)F(v)$ satisfies $U^K = \{0\}$. This means by assumption that it $U = 0$.

[hk-irr] **II.4.4. Proposition.** *Suppose $\mathcal{R}$ to be $\mathbb{D}$ and (π, V) to be a smooth representation of G. Then*

(a) if π is irreducible then V^K is an irreducible module over $\mathcal{H}(G//K)$ for all K;
♡ [two-hecke] *(b) if V satisfies conditions (a) and (b) of Proposition II.4.3 and V^K is an irreducible module over $\mathcal{H}(G//K)$ then π is irreducible.*

Proof. Suppose (π, V) to be irreducible, and let U be any non-trivial $\mathcal{H}(G//K)$-stable subspace of V^K. Since V is irreducible, U must generate V as a G-space, so every v in V is of the form $\sum c_i\pi)g_iu_i$ with u_i in U. But then for v in V^K

$$v = \pi(\mu_{K/K})v = \sum c_i\pi(\mu_{K/K})\pi(g_i)u_i = \frac{c_i}{|KgK/K|}\pi(Kg_iK)u_i$$

which lies in U since U is assumed to be stable under $\mathcal{H}(G//K)$. So $V^K \subseteq U$.

♡ [two-hecke] Conversely, assume conditions (a) and (b) of Proposition II.4.3 to hold for V, and assume V^K irreducible. If U is any non-zero G-stable subspace of V then by (b) $U^K \neq 0$ must be a submodule of V^K, but will equal it because of irreducibility. But (a) implies that then $U = V$.

[hk-irr-jacobson] **II.4.5. Proposition.** *Suppose $\mathcal{R}$ to be $\mathbb{D}$ and (π, V) to be an irreducible admissible representation of G, K compact and open in G. The homomorphism from $\mathcal{H}(G//K)$ to $\mathrm{End}_{\mathbb{D}}(V^K)$ is surjective.*

♡ [density] *Proof.* This follows from Theorem II.12.1 in the Appendix to this Chapter.

Does every finite-dimensional module over $\mathcal{H}(G//K)$ arise as the space V^K for some admissible V? And
♡ [two-hecke] more particularly one satisfying the conditions (a) and (b) of Proposition II.4.3? We can obtain a partial answer to these questions. It is motivated by a simple observation. Let V be an admissible representation of G, $U = V^K$. To each v in V we can assign the function

$$F_v\colon\ G \longrightarrow U, \quad g \longmapsto \pi(\mu_{K/K})\pi(g)v$$

Then $f * F_v = \pi(f)F_v$ for every f in $\mathcal{H}(G//K)$, and the map from V to $C^\infty(G,U)$ is equivariant with respect to the right regular action of G.

Conversely, if U is a finite-dimensional representation of $\mathcal{H}(G//K)$, define I_U to be the space of all functions $F: G \to U$ such that $f * F = \pi(f)F$ for all f in the Hecke algebra. There is a canonical embedding of U itself into this, and let V be the subspace of I_U generated by this copy. It is not hard to verify that $V^K = U$, and that V is also co-generated by U. But whether this representation of G is admissible presumably depends on G.

5. Characters [characters.tex]

In this section, suppose $\mathcal{R} = \mathbb{D}$.

If (π, V) is admissible then for every f in $\mathcal{H}(G)$ the trace of $\pi(f)$ is well defined since it may be identified with an operator on some V^K, which is finite-dimensional. This defines the character of π as a linear functional on the Hecke algebra.

[character] **II.5.1. Proposition.** *If the (π_i, V_i) make up a finite set of inequivalent irreducible admissible representations of G then their characters are linearly independent.*

♡ [hk-irr] *Proof.* Choose K so small that $V_i^K \neq 0$ for all i. They then form, according to Proposition II.4.4 and
♡ [two-hecke] Proposition II.4.3, inequivalent modules over $\mathcal{H}(G//K)$. Because the π_i are all distinct as well as irreducible,
♡ [density] Theorem II.12.1 implies that the map from the Hecke algebra into $\prod \mathrm{End}(U_i)$ is surjective. Suppose now that

$$\sum c_i \cdot \mathrm{TR}_i = 0\,,$$

which means that

$$\sum c_i \cdot \mathrm{TR}\big(\pi_i(f)\big) = 0$$

for all f in the Hecke algebra. But then we can choose f in the Hecke algebra such that $\pi_i(f) = I$ but all the other $\pi_j(f) = 0$, which implies that $c_i = 0$.

♡ [abelian-smooth] The following is a consequence of Proposition II.1.3:

[char-exact] **II.5.2. Proposition.** *If*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is an exact sequence of admissible G-spaces, then the character of V is the sum of the characters of U and V.

It implies easily one half of this refinement:

[jh] **II.5.3. Corollary.** *Two admissible representations of finite G-length have the same Jordan-Hölder factors if and only if they have the same characters.*

Proof. It remains to be seen that if U and V have the same characters then they have the same Jordan-Hölder factors. But for this, by the previous result, it suffices to see that the semi-simplifications of U and V are
♡ [character] isomorphic. But this follows from Proposition II.5.1 and an induction argument.

6. Tensor products [characters.tex]

Suppose the (π, V_i) $(i = 1, 2)$ to be smooth representations of G_i, with $\mathbb{D} = \mathcal{R}$. The group $G_1 \times G_2$ then acts on the tensor product:

$$(\pi_1 \otimes_{\mathcal{R}} \pi_2)(g_1, g_2)\colon\ v_1 \otimes v_2 \longmapsto \pi(g_1)v_1 \otimes \pi(g_2)v_2\,.$$

[tensor-prod] **II.6.1. Proposition.** *If the π_i are irreducible, so is $\pi_1 \otimes_{\mathcal{R}} \pi_2$. Every irreducible representation of $G_1 \times G_2$ is of this form.*

Proof. Let $G = G_1 \times G_2$. Suppose $V \ne 0$ to be any G-stable subspace of $V_1 \otimes V_2$, and $U \ne 0$ an irreducible G_1-stable subspace of V. As a representation of G_1, V is a direct sum of copies of V_1, so that by ♡ [irr-hom] Proposition II.1.6 the representation on U is isomorphic to that on V_1, and we may as well assume that $U = V_1$. and the canonical map from $U \otimes \mathrm{Hom}_{\mathcal{R}}(U, V)$ to V is an isomorphism. For the same reasons, the canonical map from $U \otimes \mathrm{Hom}_G(U, W)$ to W is an isomorphism. For the same reasons again V is isomorphic to $U \otimes \mathrm{Hom}_G(U, V)$. But the embedding of W into V induces an embedding of $\mathrm{Hom}_G(U, W)$ into $\mathrm{Hom}_G(U, V)$. The second is isomorphic to V_2 as a representation of G_2, and since V_2 is irreducible this embedding is an isomorphism. Hence $W = V$.

Similarly, if V is any irreducible representation of $G_1 \times G_2$ and $U \ne 0$ is any irreducible G_1-stable subspace, then V is isomorphic to $U \otimes \mathrm{Hom}_{\mathcal{R}}(U, V)$, and $\mathrm{Hom}(U, V)$ is an irreducible representation of G_2.

The group $G_1 \times G_2$ also acts on the space $\mathrm{Hom}_{\mathbb{D}}(V_1, V_2)$. The pair (g_1, g_2) takes f to

$$\pi_2(g_1) \cdot f \cdot \pi_1(g_2^{-1})\,.$$

(Confusing right and left is an inherent problem in this business.) The subspace of homomorphisms invariant with respect to the diagonal copy of G is just $\mathrm{Hom}_G(V_1, V_2)$. There is a canonical map from $V_2 \otimes \widehat{V}_1$ to the space $\mathrm{Hom}_{\mathbb{D}}(V_1, V_2)$ taking $v_2 \otimes \widehat{v}_1$ to the linear map

$$u \longmapsto \langle \widehat{v}, u \rangle v\,.$$

The image consists of the maps of finite rank.

7. Matrix coefficients [matrix-c.tex]

If (π, V) is an admissible representation the **matrix coefficient** associated to the pair v in V, $\widetilde{v}$ in $\widetilde{V}$ is the $\mathcal{R}$-valued function

$$F_{\widetilde{v},v} = \langle \widetilde{v}, \pi(g)v \rangle\,,$$

which is uniformly smooth. Let $\mathcal{A}(\pi)$ be the space of smooth functions spanned by the matrix coefficients of π. It is a smooth representation of $G \times G$ (one factor acting on the left, one on the right), and the map from $\widetilde{V} \otimes V$ to $C(G)$ is $G \times G$-equivariant. In particular, for any fixed $\widetilde{v}$ the map

$$F_{\widetilde{v}}\colon\ v \longmapsto F_{\widetilde{v},v}$$

is equivariant from V to $C(G)$, with respect to the right regular action.

Now suppose that $\mathcal{R} = \mathbb{D}$. Let $\mathcal{A}(G) = \mathcal{A}_{\mathbb{D}}(G)$ be the space of uniformly smooth functions on G contained in a $G \times G$-stable admissible subrepresentation of $C^\infty(G)$. This is what Harish-Chandra called the space of **automorphic forms** on G.

[mc] **II.7.1. Proposition.** *Suppose F to be a smooth function on G fixed on left and right by elements of some compact open subgroup. The following are equivalent:*

(a) the function F is contained in some $\mathcal{A}(\pi)$ with π admissible;
(b) the space $A_L(F)$ spanned by all L_gF is an admissible L_G-representation;
(c) the space $A_R(F)$ spanned by all R_gF is an admissible R_G-representation;
(d) the function F lies in $\mathcal{A}(G)$.

Proof. Suppose that $V = A_R(F)$ is an admissible representation of G. We want to find to see $\widetilde{v}$ in $\widetilde{V}$ and v in V such that

$$F(g) = \langle \widetilde{v}, R_g v \rangle \,.$$

It is natural to take $v = F$, and which case this equation can be written as

$$[R_gF](1) = \langle \widetilde{v}, R_gF \rangle \,.$$

This suggests defining

$$\langle \widetilde{v}, f \rangle = f(1)$$

for all f in V. Then

$$\langle R_k\widetilde{v}, f \rangle = \langle \widetilde{v}, R_{k^{-1}} f \rangle = f(k^{-1}) = [L_k f](1)$$

for all f in V. But the left action of G commutes with the right action, so if $L_kF = F$ for all k in the compact open subgroup K then $L_kv = v$ for all v in A_RF, and $R_k\widetilde{v} = v$, so that $\widetilde{v}$ lies in $\widetilde{V}$.

8. Compact representations [matrix-c.tex]

In this section, suppose $\mathcal{R} = \mathbb{D}$.

Suppose (π, V) to be a smooth representation of G possessing a central character ω. All of its matrix coefficients are then eigenfunctions for Z_G. Using terminology introduced by [Bernstein:1992], I'll call π **compact** if all of its matrix coefficients $\langle \widetilde{v}, \pi(g)v \rangle$ ($\widetilde{v}$ in $\widetilde{V}$, v in V) have compact support modulo Z_G. We'll see later that, according to a theorem of [Jacquet:1971], that these can be characterized as **super-cuspidal** representations (in the terminology of Harish-Chandra) when G is reductive. The following result is implicit in the proof of Theorem I.6 in [Bernstein:1992].

[compact-fd] **II.8.1. Proposition.** *Suppose (π, V) to be compact, K a compact open subgroup of G, v in V. Then the space*

$$\big\{\pi(\mu_{K/K})\pi(g)v \,\big|\, g \in G\big\}$$

has finite dimension.

Proof. Suppose that it did not have finite dimension. Suppose v fixed by the compact open subgroup $K_\circ$. One could then choose an infinite number of linearly dependent vectors $v_i = \pi(\mu_{K/K})\pi(g_i)v$. The union $\bigcup g_iK_\circ$ must then be infinite modulo Z_G. The v_i are all in V^K, so we may find $\widetilde{v}$ in $\widetilde{V}^K$ such that $\langle \widetilde{v}, v_i \rangle = 1$ for all i. This contradicts the assumption that the support of matrix coefficients is compact modulo Z_G.

[B-eq] **II.8.2. Corollary.** *A smooth ω-central representation is compact if and only if the function $\pi(\mu_{K/K})\pi(g)v$ has compact support on G.*

[compact-admissible] **II.8.3. Corollary.** *A finitely generated compact representation is admissible.*

[cuspidals-projective] **II.8.4. Proposition.** *Suppose $\mathcal{R} = \mathbb{D}$. An irreducible compact representation is projective and injective in the category of smooth representation of G with the same central character.*

Proof. I'll begin the proof here but postpone part of it to a later section. It is motivated by the analogous case of modules over a commutative ring with unit, in proving that a module is projective if and only if it is a summand of a free module. But in our case there are some minor difficulties because the Hecke algebra is not commutative and does not possess a unit.

Let ω be the central character of π. By definition, the representation (π, V) embeds into the right regular representation of G on $\mathcal{H}_\omega$. Identify V with its image. I'll prove later that *there exists an equivariant projection from $\mathcal{H}_\omega$ onto V, which is to say an equivariant surjection $\mathcal{P}$ such that $\mathcal{P}(v) = v$ for v in V.*

It will be of the form

$$\mathcal{P}(f) = R_{f^\vee} v_0$$

for some particular $v_0 \neq 0$ in V. Here $f^\vee(g) = f(g^{-1})$. Given this, we now know V to be a summand of $\mathcal{H}_\omega$.

Suppose now that (σ, U) is an arbitrary smooth representation of G and that we are given a surjection $U \to V$ of smooth representations. Choose u_0 in U mapping to v_0. Let Π be the map from $\mathcal{H}_\omega$ to U taking f to $\sigma(f^\vee)u_0$. This is G-equivariant. The diagram

images/h.eps

$$\begin{tikzcd} & \mathcal{H}_\omega \arrow[dl, "\Pi"'] \arrow[dr, "\mathcal{P}"] & \\ U \arrow[rr, "F"] & & V \end{tikzcd}$$

is commutative, and the restriction of Π to the image of V in $\mathcal{H}_\omega$ amounts to a splitting of the surjection. This shows that V is projective.

♡ [compact-admissible] Injectivity follows from the projectivity of its contragredient, since according to Corollary II.8.3 π is admissible and hence the smooth dual of $\widetilde{V}$ is the same as V.

9. Unitary representations [unitary.tex]

In this section I take $\mathcal{R}$ to be $\mathbb{C}$.

If V is a vector space over $\mathbb{C}$, its **conjugate** $\overline{V}$ is the same additive group, but with the new complex multiplication

$$c \circ v = \overline{c}\, v\,.$$

By definition, an element of $\widehat{\overline{V}}$ is a function $f\colon V \to \mathbb{C}$ such that $f(cv) = \overline{c}\, f(v)$, whereas the space $\overline{\widehat{V}}$ is that of all $\mathbb{C}$-linear maps from V to $\mathbb{C}$, but with the multiplication $[c \circ f](v) = \overline{c} f(v)$.

[conj-dual] **II.9.1. Lemma.** *The map taking f to $\overline{f}$ is an isomorphism of $\widehat{\overline{V}}$ with $\overline{\widehat{V}}$.*

A **Hermitian pairing** of two spaces U and V is a $\mathbb{C}$-linear function $H\colon U \otimes \overline{V} \to \mathbb{C}$. Equivalently, it is an $\mathbb{R}$-linear function taking $u \otimes v$ to $u \bullet v$ such that

$$(cu) \bullet v = u \otimes (\overline{c}\, v) = c(u \bullet v)\,.$$

It is also equivalent to a linear map $\iota = \iota_H\colon U \to \widehat{\overline{V}}$:

$$\iota\colon\ u \longmapsto [v \longmapsto u \bullet v]$$

characterized by the equation

$$u \bullet v = \langle \iota(v), u \rangle\,.$$

A **Hermitian form** on a complex vector space U is a Hermitian pairing of U with itself. If H takes $u \otimes v$ to $u \bullet v$ then the map

$${}^t\overline{H}\colon u \otimes v \longmapsto \overline{v \bullet u}$$

is also a Hermitian form, and this defines an involution on the space of Hermitian forms on U. The form is called **symmetric** if ${}^t\overline{H} = H$, or in other words

$$v \bullet u = \overline{u \bullet v}$$

for all u, v. Thsi involution is a linear transformation over $\mathbb{R}$. The space of all Hermitian forms is the direct sum of ± 1 eigenspaces. These are isomorphic, since $H \mapsto i\,H$ swaps them. A form is **symmetric** when it lies in the $+1$-eigenspace.

[symmetric-cond] **II.9.2. Lemma.** *A form is symmetric if and only if $u \bullet u$ is real for all u.*

Proof. The well known polarization identity asserts that

$$y \bullet x = (1/4)\Big(\big((x+y)\bullet(x+y) - (x-y)\bullet(x-y)\big) - i\big((x+iy)\bullet(x+iy) - (x-iy)\bullet(x-iy)\big)\Big) \,.$$

Verifying this is straightforward. Swapping x and y, it implies that

$$y \bullet x = \overline{x \bullet y}$$

if every $u \bullet u$ is real.

I'll call the form **non-degenerate** if ι is an injection. Of course if V is finite-dimensional this happens if and only if ι is an isomorphism.

Suppose that the form is non-degenerate, that V is finite-dimensional, and that the form is symmetric. The map ι satisfies the equation

[iota-symm] **(II.9.3)**
$$\langle \iota(v), \iota^{-1}(\widetilde{v})\rangle = \overline{\langle \widetilde{v}, v\rangle} \,.$$

Also in these circumstances the form on V gives rise to one on $\widehat{V}$:

[vdotv-conj] **(II.9.4)**
$$\widehat{u} \bullet \widehat{v} = \iota^{-1}(\widehat{v}) \bullet \iota^{-1}(\widehat{u}) = \langle \widehat{u}, \iota^{-1}(v)\rangle = \overline{\langle \widehat{v}, \iota^{-1}(u)\rangle}, \quad .$$

If $c \neq 0$ is real, and ι is replaced by $c \cdot \iota$, then $u \bullet v$ is replaced by $c(u \bullet v)$, and ι^{-1} is replaced by $(1/c)\iota^{-1}$.

[vxv-conj] **(II.9.5)**
$$(\widehat{u} \otimes u) \bullet (\widehat{v} \otimes v) = \langle \widehat{u}, \iota^{-1}(\widehat{v})\rangle \cdot \langle \iota(v), u\rangle$$

is therefore canonical.

A **unitary** representation of G is one with a positive definite G-invariant Hermitian inner product. According ♡ [symmetric-cond] to Lemma II.9.2, it is necessarily symmetric. This implies also that ι is G-equivariant. Unitary representations are important in our subject because they are the ones that appear in orthogonal decompositions of arithmetic quotients, and this has arithmetic consequences. In one classic example, unitarity is related to Ramanujan's conjecture.

If (π, V) is admissible, then any G-invariant form on V induces Hermitian forms on each V^K and the image of ι in $\widehat{V^K}$. Therefore we may apply the previous discussion to V itself.

[schurs-unitary] **II.9.6. Lemma.** *If (π, U) and (ρ, V) are both irreducible unitary admissible representations, then*

$$\mathrm{Hom}(U, \widehat{\widehat{V}}) = \begin{cases} \mathbb{C} & \text{if } \pi \text{ is isomorphic to } \rho \\ 0 & \text{otherwise.} \end{cases}$$

[unitary-sum] **II.9.7. Proposition.** *Every admissible unitary representation is a countable direct sum of irreducible unitary representations, each occurring with finite multiplicity.*

This requires the assumption that G possesses a countable basis of neighbourhoods of the identity.

It is easy to see that the matrix coefficients of a unitary representation are bounded. A much stronger condition on matrix coefficients is fundamental. Suppose π to be an irreducible representation with central character ω. It is said to be **square-integrable modulo the centre Z_G of G** if $|\omega| = 1$ (i.e. its central character is unitary) and every matrix coefficient is square-integrable on G/Z_G. A compact representation with unitary central character is square-integrable modulo the centre, for example.

[irr-sqint] **II.9.8. Proposition.** *If π is an irreducible admissible representation of G, then it is square-integrable if and only if a single non-zero matrix coefficient is square-integrable.*

A square-integrable representation may be embedded into $\mathrm{L}^2(G)$, and it is unitary. More precisely:

[sqint-unitary] **II.9.9. Proposition.** *Suppose (π, V) to be an irreducible, square-integrable, admissible representation. For $\widetilde{v}_0 \neq 0$ in $\widetilde{V}$ the pairing*

$$u \bullet v = \int_{G/Z_G} \langle \pi(g)u, \widetilde{v}_0 \rangle \overline{\langle \pi(g)v, \widetilde{v}_0 \rangle}\, dg$$

defines a G-invariant positive definite inner product on V.

10. Schur orthogonality [schur.tex]

Throughout this section, $\mathcal{R}$ will be a field. At the beginning it will be $\mathbb{C}$. Assume also:

Restriction to Z_G induces an isomorphism of $\mathrm{Hom}(G, \mathbb{R}^\times_{>0})$ with $\mathrm{Hom}(Z_G, \mathbb{R}^\times_{>0})$.

This is true, for example, if G is the group of rational points on a Zariski-connected reductive group defined over a local field. As a consequence of this assumption:

[assumption-cor] **II.10.1. Lemma.** *(1) If χ is any character of Z_G, there exists a character ρ of G such that $\chi \cdot \rho$ is unitary on Z_G. (2) The group G is unimodular.*

The following is the basic version of Schur orthogonality:

[schur-unitary-2] **II.10.2. Proposition.** (Unitary Schur orthogonality) *If π is square integrable modulo Z_G then for some $\gamma_\pi > 0$*

$$\int_{G/Z_G} \langle \widetilde{u}, \pi(g)u \rangle \overline{\langle \widetilde{v}, \pi(g)v \rangle}\, dg = \gamma_\pi (\widetilde{u} \otimes u) \bullet (\widetilde{v} \otimes v)\,.$$

The term on the right is the canonical inner product defined earlier.

♡ [schurs-unitary] *Proof.* An immediate consequence of Lemma II.9.6.

This has more general consequences. Suppose now that $\mathcal{R} = \mathbb{D}$. Suppose (π, V) to be any irreducible admissible representation of G, with central character ω. Matrix coefficients define an equivariant embedding of $\widetilde{\pi} \otimes \pi$ into $C(G)$:

$$\Phi_{\hat{v} \otimes v}(g) = \langle \widetilde{v}, \pi(g)v \rangle\,.$$

Dually, if f is any function in $\mathcal{H}_{\omega^{-1}}$ Then the integral

[mc-int] **(II.10.3)**
$$\int_{G/Z_G} f(g) \Phi_{\hat{v} \otimes v}(g)\, dg$$

defines a map from $\mathcal{H}_{\omega^{-1}}$ to the dual of $\widetilde{\pi} \otimes \pi$, which is (canonically) $\pi \otimes \widetilde{\pi}$. This last space may be identified with the linear operators in $\mathrm{End}(V)$ of finite rank, and it is easy to see:

[endo-int] **II.10.4. Lemma.** *The image of f with respect to the map from $\mathcal{H}_{\omega^{-1}}$ to $\mathrm{End}(V)$ is the same as $\pi(f)$.*

This image is in some sense the Fourier transform of f evaluated at the representation π. Of course this depends on the particular realization of π. More canonically, one might choose the Fourier transform to be in the image of $\pi(f)$, considered as an element of $V \otimes \widetilde{V}$, with respect to the matrix coefficient map. Directly in terms of an element of $\mathrm{End}(V)$ this is the function

$$\mathrm{trace}\,(\pi(g^{-1})f)\,.$$

From this discussion follows the first part of:

[schurs-cuspidal] **II.10.5. Proposition.** *If (π, U) and (ρ, V) are irreducible, compact representations of G with the same central character, then*

$$\int_G \langle \widetilde{u}, \pi(g)u\rangle\langle\widetilde{\pi}(g)\widetilde{v}, v\rangle dg$$

is equal to 0 if π and ρ are not isomorphic, and equal to

$$c_\pi\langle\widetilde{v}, u\rangle\langle\widetilde{u}, v\rangle$$

for some constant $c_\pi \neq 0$ if they are isomorphic.

Proof. It remains to prove that $c_\pi \neq 0$.

♡ [fg-field] This can be reduced to the unitary case. Because of Lemma II.1.7, the representation π may be defined over a field with a countable number of elements. As is well known, it may then be embedded into $\mathbb{C}$. The proof that $c_\pi \neq 0$ depends on this, and in the rest of this section I'll prove the theorem when $\mathcal{R} = \mathbb{C}$.

♡ [assumption-cor] The formula remains the same if π and ρ are replaced by $\pi\cdot\chi$ and $\rho\cdot\chi$, where χ is a character of G. According to Lemma II.10.1, we may therefore assume the comon central character to be unitary, and both become square-integrable modulo Z_G.

♡ [schur-unitary-2] We may now apply Proposition II.10.2, and we may assume that $\pi = \rho$. This gives us

$$\int_G \langle\widetilde{u}, \pi(g)u\rangle\overline{\langle\widetilde{v}, \pi(g)v\rangle}\, dg = \gamma_\pi(\widetilde{u}\otimes u)\bullet(\widetilde{v}\otimes v)\,.$$

for all $\widetilde{u}$ etc. What we want now is to transform this to

$$\int_G \langle\widetilde{u}, \pi(g)u\rangle\langle\widetilde{\pi}(g)\widetilde{w}, w\rangle\, dg = \gamma_\pi\langle\widetilde{v}, w\rangle\langle\widetilde{u}, w\rangle\,.$$

For this, in order to get the right hand sides to look similar we must set $\widetilde{v} = \iota(w)$, $v = \iota^{-1}(\widetilde{w})$. But then one of the terms in the first integral becomes (since π is unitary)

$$\langle\widetilde{\pi}(g)\iota(w), \iota^{-1}(\widetilde{w}\rangle = \langle\iota(\pi(g)w), \iota^{-1}(\widetilde{w}\rangle\,.$$

♡ [vdotv-conj] But by (II.9.4) this is the same as

$$\overline{\langle\widetilde{w}, \pi(g)w\rangle}\,.$$

It is good to keep in mind:

[degree-is-dim] **II.10.6. Proposition.** *Suppose G compact with total measure 1, and let (π, V) be an irreducible admissible representation. Then $c_\pi = 1/\dim(\pi)$.*

The representation is necessarily of finite dimension. For this reason, the positive constant $1/c_\pi$ is called the **formal degree** of π.

♡ [schurs-cuspidal] *Proof.* Let (e_i) be a basis of V, $\widehat{e}_j$ the dual. Apply Proposition II.10.5:

$$\int_G \langle\widehat{e}_i, \pi(g)e_j\rangle\,\langle\widehat{e}_j, \pi(g^{-1})e_k\rangle\, dg = c_\pi\langle\widehat{e}_i, e_k\rangle\langle\widehat{e}_j, e_j\rangle = c_\pi\langle\widehat{e}_i, e_k\rangle\,.$$

for all i, j, k. Sum over j, set $i = k$. We are looking at a diagonal entry of the matrix product $\pi(k)\pi(k^{-1})$. We deduce that $c_\pi\,\dim(\pi) = 1$.

♡ [cuspidals-projective] Now to conclude the proof of Proposition II.8.4.

We are given an irreducible compact representation (π, V), and a vector $\widetilde{v}_0$ in $\widetilde{V}$. Fix v_0 in V such that $\langle \widetilde{v}_0, v_0 \rangle = d_\pi$. Map f in $\mathcal{H}_\omega$ to

$$\mathcal{P}f = R_{f^\vee} v_0 \,.$$

This is a G-equivariant map from $\mathcal{H}_\omega$ to V. We compute

$$\begin{aligned} \mathcal{P}\gamma_v &= R_{\gamma_v^\vee} v_0 \\ &= \int_{G/Z} \gamma_v^\vee(x)\pi(x)v_0\,dx \\ &= \int_{G/Z} \langle \pi(x^{-1})v, \widetilde{v}_0 \rangle \pi(x) v_0\,dx \,. \end{aligned}$$

This last is an element of V. But according to Schur orthogonality

$$\Big\langle \int_{G/Z} \langle \pi(x^{-1})v, \widetilde{v}_0\rangle \pi(x)v_0\,dx, \widetilde{v} \Big\rangle = \int_{G/Z} \langle \pi(x^{-1})v, \widetilde{v}_0\rangle \langle \pi(x)v_0, \widetilde{v}\rangle\,dx = \frac{1}{d_\pi}\langle v, \widetilde{v}\rangle \langle v_0, \widetilde{v}_0\rangle \,.$$

11. Induced representations [induced.tex]

If H is a closed subgroup of G and (σ, U) is a smooth representation of H, the **unnormalized** smooth representation *Ind*$(\sigma \,|\, H, G)$ **induced** by σ is the right regular representation of G on the space of all uniformly smooth functions $f\colon G \to U$ such that

$$f(hg) = \sigma(h)f(g)$$

for all h in H, g in G. Let

$$\delta_{H\backslash G} = \delta_H \backslash \delta_G \,.$$

The **normalized** induced representation is

$$\mathrm{Ind}(\sigma \,|\, H, G) = \mathit{Ind}\big(\sigma\delta_{H\backslash G}^{-1/2} \,\big|\, H, G\big) \,.$$

Why the δ-factor? Well, $\mathrm{Ind}(\delta_{H\backslash G}^{1/2})$ is the space of smooth functions on $H\backslash G$. The normalization is motivated by ♡ [one-densities] Corollary I.7.7, which asserts that $\mathrm{Ind}(\delta_{H\backslash G}^{-1/2})$ is isomorphic to that of smooth one-densities. The symmmetry between $\delta^{\pm 1/2}$ suggests a useful duality.

The **compactly supported induced representation** Ind_c is on the analogous space of functions of compact support on G modulo H.

[induced-admissible] **II.11.1. Proposition.** *If $H\backslash G$ is compact and (σ, U) admissible then $\mathrm{Ind}(\sigma \,|\, H, G)$ is an admissible representation of G.*

The hypothesis holds when G is a reductive $\mathfrak{p}$-adic group and H a parabolic subgroup.

Proof. If $H\backslash G/K$ is the disjoint union of cosets HxK (for x in a finite set X), then the map

$$f \longmapsto \big(f(x)\big)$$

is a linear isomorphism

[ind-restr-K] **(II.11.2)** .
$$\mathrm{Ind}_c(\sigma \,|\, H, G)^K \cong \oplus_{x\in X} U^{H\cap xKx^{-1}}$$

Suppose (π, V) to be a smooth representation of G, (σ, U) one of H. The map

$$\Lambda\colon \mathrm{Ind}(\sigma \mid H, G) \to U$$

taking f to $f(1)$ is an H-morphism from $\mathrm{Ind}(\sigma)$ to $\sigma\delta_H^{1/2}\delta_G^{-1/2}$. If we are given a G-morphism from V to $\mathrm{Ind}(\sigma \mid H, G)$ then composition with Λ induces an H-morphism from V to $\sigma\delta_H^{1/2}\delta_G^{-1/2}$.

[frobenius] II.11.3. Proposition. (Frobenius reciprocity) *If π is a smooth representation of G and σ one of H then evaluation at 1 induces a canonical isomorphism*

$$\mathrm{Hom}_G\big(\pi, \mathrm{Ind}(\sigma \mid H, G)\big) \to \mathrm{Hom}_H\big(\pi, \sigma\delta_H^{1/2}\delta_G^{-1/2}\big)\ .$$

For F in $\mathrm{Ind}(\widetilde{\sigma} \mid H, G)$ and f in $\mathrm{Ind}_c(\sigma \mid H, G)$ the product $\langle F(g), f(g)\rangle$ lies in $\mathrm{Ind}_c(\delta_{H\backslash G}^{-1/2})$. On this space, fix a positive G-invariant integral

$$\int_{H\backslash G} \varphi(x)\, dx\ .$$

[ind-duality] II.11.4. Proposition. *The pairing*

$$\langle F, f\rangle = \int_{H\backslash G} \langle F(x), f(x)\rangle\, dx$$

defines an isomorphism of $\mathrm{Ind}(\widetilde{\sigma} \mid H, G)$ with the smooth dual of $\mathrm{Ind}_c(\sigma \mid H, G)$.

In particular, if $\mathcal{R} = \mathbb{C}$ and σ is unitary so is $\mathrm{Ind}(\sigma \mid H, G)$.

12. *Appendix.* Semi-simple algebra `[induced.tex]`

In this section I summarize relevant results from [Bourbaki:1958]. Suppose R to be any algebra over $\mathbb{D}$. The main item is this:

[density] II.12.1. Theorem. *Suppose given a finite collection of irreducible, finite-dimensional R-modules V_i that are pair-wise non-isomorphic. The canonical map from R to $\prod_i \mathrm{End}_{\mathbb{D}}(V_i)$ is surjective.*

This is Corollaire 1 to Proposition 4 of §4.3 in [Bourbaki:2011]. It is (as we shall see) elementary, but it is not easy to extract a succinct account from the literature. The proof I give here proceeds in several steps.

A **semi-simple** module over R is any direct sum of irreducible modules.

[semi-simple] II.12.2. Lemma. *A finite-dimensional R-module V is semi-simple if and only if every submodule is a summand.*

Proof. The only non-trivial thing to prove is that any submodule U is a summand. This will be shown by induction on the codimension of U. If it is 0, there is nothing to prove.

Suppose V to be the sum $\bigoplus V_i$ of irreducibles. If all the V_i are contained in U then $V = U$. Otherwise, say V_1 is not contained in U. Since it is irreducible, we must have $V_1 \cap U = \{0\}$. The projection

$$U \longrightarrow V/V_1 = \bigoplus_{i\neq 1} V_i$$

is then injective. The codimension of the image of U has smaller codimension than that of U in V, so induction U possesses an R-stable complement W in V/V_1. But W may be identified with a submodule in V, and $W \oplus V_1$ is a complement in V. ▯

The **commutant** R' of R in $\mathrm{End}_{\mathbb{D}}(V)$ is the ring of operators commuting with it. The **bicommutant** R'' is the commutant of the commutant. It contains R.

[bicommutant] **II.12.3. Lemma.** *Given a finite collection of irreducible R-modules V_i of finite dimension that are pair-wise non-isomorphic. If $V = \bigoplus V_i$, the canonical map from the bicommutant of R to $\prod_i \mathrm{End}_{\mathbb{D}}(V_i)$ is surjective.*

Proof. This is because the commutant is the direct product of copies of $\mathbb{D}$, one for each representation.

♡ [density] Theorem II.12.1 will now follow from:

[bicommutant-density] **II.12.4. Lemma.** *Given any semi-simple R-module V of finite dimension, the image of the bicommutant in $\mathrm{End}_{\mathbb{D}}(V)$ is the same as that of R.*

Proof. Suppose (e_i) (for $1 \le i \le m$) to be a basis of V. It must be shown that if ρ lies in the bicommutant, there exists r in R such that $r(e_i) = \rho(e_i)$ for all i.

♡ [semi-simple] The representation of R on $W = V^m$ is also semisimple. Let $e = (e_i)$ be diagonally embedded. By Lemma II.12.2 the submodule $U = R{\cdot}e$ is a direct summand of W. The projection from V onto U lies in the commutant R' of W, so the bicommutant R'' takes W into itself. But this means that $R'' \cdot e = R \cdot e$, which implies that there for every ρ in R'' there exists r in R such that $r(e) = \rho(e)$.

13. References [induced.tex]

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