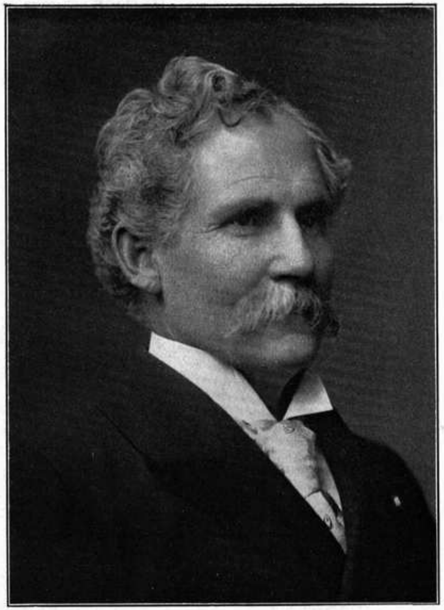

ELISHA SCOTT LOOMIS

The Pythagorean Proposition

Its Proofs Analyzed and Classified

And

Bibliography of Sources

For DATA of

The Four Kinds of Proofs

By

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Printed in the United States of America.

DEDICATED

To

The Masters and Wardens Association of the 22nd Masonic District of the Most Worshipful Grand Lodge of Free and Accepted Masons of Ohio.

Most Faithfully, Fraternally and with profound respect,

ELISHA S. LOOMIS, 32°.

1321 W. 111th Street
Cleveland, Ohio.
September 18, 1926.

This work was given by the Author, Brother Elisha S. Loomis, 32°, to the Masters and Wardens Association of the 22nd Masonic District of the Most Worshipful Grand Lodge of Free and Accepted Masons of Ohio and Published by said Association.

Foreword

THE object of this work is to present to the future investigator, under one cover, simply and concisely, what is known relative to the Pythagorean proposition, and to set forth certain established facts concerning the proofs and the geometric figures pertaining thereto.

It establishes that:

First, that there are but *four* kinds of proofs for the Pythagorean proposition, viz:

I.—Those based upon Linear Relations (implying the Time Concept)—the Algebraic Proofs.

II.—Those based upon Comparison of Areas (implying the Space Concept)—the Geometric Proofs.

III.—Those based upon Vector Operation (implying the Direction Concept)—the Quarternionic Proofs.

IV.—Those based upon Mass and Velocity (implying the Force Concept)—the Dynamic Proofs.

Second, that the number of Algebraic proofs is limitless.

Third, that there are only ten types of geometric figures from which a Geometric Proof can be deduced.

This third fact is not mentioned nor implied by any work consulted by the author of this treatise, but which, once established, becomes the basis for the classification of all possible geometric proofs.

Fourth, that the number of geometric proofs is limitless.

Fifth, that no trigonometric proof is possible.

By consulting the Table of Contents any investigator can determine in what field his proof falls, and then, by reference to the text, he can find out wherein it differs from what has already been established.

With the hope that this simple exposition of this historically renowned and mathematically fundamental proposition, without which the science of Trigonometry and all that it implies would be impossible, may interest many minds and prove helpful and suggestive to the student, the teacher and the future original investigator, to each and to all who are seeking more light, the author, through the medium of The Masters and Wardens Association of the 22nd Masonic District of the Most Worshipful Grand Lodge of Free and Accepted Masons of Ohio, sends it forth.

ELISHA S. LOOMIS.

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and Worshipful Master J. Burt Clough, Heights Lodge No. 633.

Acknowledgments

"Every man builds upon his predecessors."

My predecessors made this work possible, and may those who make further investigations relative to this renowned proposition do better than their predecessors have done.

The author herewith expresses his obligations:

To the many who have preceded him in this field, and whose text and proof he has acknowledged herein on the page where such proof is found;

To those who, upon request, courteously granted him permission to make use of such proof, or refer to the same;

To the following Journals and Magazines whose owners so kindly extended to him permission to use proofs found therein, viz:

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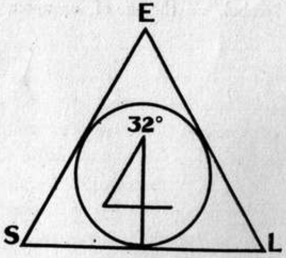
Abbreviations and Contractions

a-square = square upon the shorter leg.
A. R. Colburn = Arthur R. Colburn LL. M., Dist. of Columbia Bar.
b-square = square upon the longer leg.
const'd = constructed.
const. = construct.
cos = cosine.
Edward's Geom. = Edward's Elements of Geometry.
eq. = equation.
eq's = equations.
Fig. or fig. = figure.
h-square = square upon the hypotenuse.
Jury Whipper = Jury Whipper's "46 Beweise der Pythagoreaischen Lehrsatzes," 1880.
Jour. Ed'n = Journal of Education.
Math. = Mathematical.
Math. Mo. = The American Mathematical Monthly.
Mo. = Monthly.
No. or no. = number.
Olney's Geom. = Olney's Elements of Geometry, University edition.
outw'ly = outwardly.
par. = parallel.
paral. = parallelogram.
perp. = perpendicular.
prop'l = proportional.
p. = page.
pt. = point.
quad. = quadrilateral.
rt. = right.
rt. tri. = right triangle.
rect. = rectangle.

Sci. Am. Sup. = Scientific American Supplement.
sec = secant.
sin = sine.
sq. = square.
sq's = squares.
tang = tangent.
∴ = therefore.
tri. = triangle.
tri's = triangles.
trap. = trapezoid.
v. = volume.
HE^2, or any like symbol, = the sq. of, or upon, the line HE, or like symbol.

ὁμιλεῖν τῷ Θεῷ



GOD GEOMETRIZES CONTINUALLY~*PLATO.*

Freemasonry

FREEMASONRY is a very ancient and respectable institution and embraces as members "men of every rank and condition of life, of every nation and clime, and of every religion which acknowledges a Supreme Being and has faith in the immortality of the soul; it stands preeminent among the institutions established for the improvement of mankind—as far above other secret associations in usefulness as it is beyond them in age."

Authentic history holds that the commencement of secret moral associations had an origin as ancient as that of the 38 noted Pyramids of Egypt (2100 B. C.).

The triumph of mind over matter came through the constructural monuments, and there is no speculation in the declaration that these builders formed themselves into an association for mutual protection and improvement at this early date, and tradition informs us that "this union of scientific men (architects) differed from the Freemasons of today in little more than in name."

The triad, architecture, astronomy and geometry, took the first rank as sciences in early Egyptian times, and here we may look for the origin of the Masonic society.

As the Masonic author, George S. Blackie, says: Doubtless at first it was a mutual improvement association simply, and those only would be admitted whose occupation was subsidiary to the great art." In time the Egyptian priest-hood 'ambitious and anxious to acquire all knowledge which could give them a further hold on a superstitious people sought to participate in the learning and wisdom of the architects.' And, "once admitted to the fraternity, they connected the mythology of their country and their metaphysical speculations concerning the nature of God with the exclusively scientific teachings of the builders, thereby producing that combination of science and theology which forms such a conspicuous part of the principles of Freemasonry. . . . The fraternity and priest craft soon became one, imparting their knowledge in symbolic and hieroglyphic instruction, accompanied by particular rites and ceremonies," and under an oath of secrecy.

Profane history records that this esoteric knowledge of Egypt, through its secret societies, found a permanent footing in all the commercial cities throughout the countries on and about the Mediterranean sea, and especially in Tyre whose king, in 1019 B. C., aided King Solomon in the building of the temple of Jerusalem.

And "the Holy Scriptures inform us that Hiram, King of Tyre, assisted King Solomon in his work with materials and operatives, and that he sent to superintend the latter a cunning artificer in brass and iron, Hiram, the son of a widow of Tyre."

Therefore the 'opinion that Freemasons—that is members of a secret order of scientific architecture—flourished at the building of King Solomon's temple, is not so absurd as is often supposed, and truly these recorded facts relieve Freemasons from the charge of credulity.'

A careful investigation of the great historic orders, viz: of the Dionysia, 15th century B. C., of the Eleusinia, 14th century B. C., and of the Essenes, 1st century B. C., shows that each bore a striking similarity with modern Masonry as to tenets and rites; that each claimed esoteric wisdom and learning—the foundation on which Freemasonry of today rests.

These facts and the additional fact that all learning and knowledge during these early centuries were in the keeping of the priests of the accepted religion of their times, and as these priests now controlled these scientific and secret orders, their only school of learning, making them the foundation of their religious institutions, it follows that Pythagoras, a leading metaphysician, mathematician and teacher of his day (569-470 B. C.) being highly educated in all Egyptian, Zoroastrian and Babylonian lore, must have been initiated into these secrets and mysteries, and that therefore he was an acknowledged and accepted Freemason as Freemasons were then known.

Adorned with Virtue, Mercy and Justice, the three great attributes of Freemasonry, he became the founder of the Pythagorean school of philosophy, mathematics and religion, wherein arose the

famous oppositions of philosophy, known as the 10 antitheses of Pythagorean teaching, namely: 1, limited and unlimited; 2, even and odd; 3, one and many; 4, right and left; 5, male and female; 6, rest and motion; 7, straight and crooked; 8, light and darkness; 9, good and evil; 10, square and rectangle. These antitheses gave rise to deep metaphysical speculations in the schools of philosophy of Greece and elsewhere, (See Plato, e. g., The Republic), and even today their implications are the riddles of thinkers.

The Pythagoreans taught that "Deity is the one, the Original Unity, the Infinite, out of which all finite things have come."

As a teacher, after having become proficient in all fields of wisdom of his epoch, overtopping all like a Galileo, a Newton, Pythagoras "laid great stress on the discipline of the will into obedience, temperance, silence, self-examination, simplicity in personal attire, and self restraint in all its forms,"—and these are the cardinal virtues of present day Freemasonry.

In the field of astronomy he anticipated Copernicus by making the sun the center of the cosmos. In geometry he enunciated and demonstrated the renowned theorem known to us as the 47th proposition of the first book of Euclid's Elements wherein we learn that: *The square described upon the hypotenuse of a right-angled triangle is equal to the sum of the squares described upon the other two sides.*

Without this proposition, it being the only geometric theorem referred to in the ritual of all Free and Accepted Masons, the initiate would never have heard: "Geometry, the first and noblest of the sciences, is the basis on which the superstructure of Freemasonry is erected. By geometry we discover how the planets move in their respective orbits, and demonstrate their various revolutions. By it we account for the return of the seasons. By it we discover the power, wisdom and goodness of the Grand Artificer of the Universe."

Finally, as stated by that great mason, Thomas Holland: "Principalities and powers, monarchies, thrones, diplomacy, and even religious fanaticism never has been, and never will be, able to obliterate Freemasonry, and I believe it is destined to come to the front and perform works of magnitude hitherto unknown."

FRATERNITY, EQUALITY AND HUMAN LIBERTY

The Pythagorean Proposition

THIS celebrated proposition is one of the most important theorems in the whole realm of geometry and is known in history as the 47th proposition, that being its number in the first book of Euclid's Elements.

It is also (erroneously) sometimes called the Pons Asinorum. Although the practical application of this theorem was known long before the time of Pythagoras he, doubtless, generalized it from an Egyptian rule of thumb ($3^2 + 4^2 = 5^2$) and first demonstrated it about 540 B. C., from which fact it is generally known as the Pythagorean Proposition. This famous theorem has always been a favorite with geometricians.

Many purely geometric demonstrations of this famous theorem are accessible to the teacher, as well as an unlimited number of proofs based upon the algebraic method of geometric investigation. Also quarternions and dynamics furnish a few proofs.

No doubt many other proofs than these now known will be resolved by future investigators, for the possibilities of the algebraic and geometric relations implied in the theorem are limitless.

This theorem with its many proofs is a striking illustration of the fact that there is more than *one* way of establishing the same truth.

But before proceeding to the methods of proof, the following historical account translated from a monograph by Jury Whipper, published in 1880, and entitled "46 Beweise des Pythagoraischen Lehrsatzes," may prove both interesting and profitable.

Whipper acknowledges his indebtedness to F. Graap who translated it out of the Russian. It is as follows: "One of the weightiest propositions in geometry if not the weightiest with reference to its deductions and applications is doubtless the so-called Pythagoreon proposition."

The Greek text is as follows:

'Εν τοίς ὀρθογωνίοις τὸ ἀπὸ τῆς τήν ὀρθήν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστί τοῖς ἀπὸ τῶν τήν ὀρθήν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις.

The Latin reads: In rectangulis triangulis quadratum, quod a latere rectum angulum subtendente describitur, aequale est eis, quae a lateribus rectum angulum continentibus describuntur.

German: In den rechtwinkeligen Dreiecken ist das Quadrat, welches von der dem rechten Winkel gegenuber liegenden Seite beschrieben Wird, den Quadraten, welche von den ihn umschliessenden Seiten beschrieben werden, gleich.

According to the testimony of Proklos the demonstration of this proposition is due to Euclid who adopted it in his elements (I, 47). The method of the Pythagorean demonstration remains unknown to us. It is undecided whether Pythagoras himself discovered this characteristic of the right triangle, or learned it from Egyptian priests, or took it from Babylon: regarding this opinions vary.

According to that one most widely disseminated Pythagoras learned from the Egyptian priests the characteristics of a triangle in which one leg = 3 (designating Osiris), the second = 4 (designating Iris), and the hypotenuse = 5 (designating Horus): for which reason the triangle itself is also named the Egyptian or Pythagorean.

The characteristics of such a triangle, however, were known not to the Egyptian priests alone, the Chinese scholars also knew them. "In Chinese history," says Mr. Skatschkow, "great honors are awarded to the brother of the ruler Uwan, Tschou-Gun, who lived 1100 B. C.: he knew the characteristics of the right triangle (perfected) made a map of the stars, discovered the compass and determined the length of the meridian and the equator.

Another scholar (Cantor) says: this emperor wrote or shared in the composition of a mathematical treatise in which were discovered the fundamental features, ground lines, base lines, of mathematics,

in the form of a dialogue between Tchou-Gun and Schan-Gao. The title of the book is: Tschaou pi, i. e., the high of Tschao. Here too are the sides of a triangle already named legs as in the Greek, Latin German and Russian languages.

Here are some paragraphs of the 1st chapter of the work. Tschou-Gun once said to Schan-Gao: "I learned, sir, that you know numbers and their applications, for which reason I would like to ask how old Fo-chi determined the degrees of the celestial sphere. There are no steps on which one can climb up to the sky, the chain and the bulk of the earth are also inapplicable; I would like for this reason, to know how he determined the numbers."

Schan-Gao replied: "The art of counting goes back to the circle and square."

If one divides a right triangle into its parts the line which unites the ends of the sides when the base = 3, the altitude = 4 is 5.

Tschou-Gun cried out: "That is indeed excellent."

It is to be observed that the relations between China and Babylon more than probably led to the assumption that this characteristic was already known to the Chaldeans. As to the geometrical demonstration it comes doubtless from Pythagoras himself. In busying with the addition of the series he could very naturally go from the triangle with sides 3, 4 and 5, as a single instance to the general characteristics of the right triangle.

After he observed that addition of the series of odd numbers ($1 + 3 = 4$, $1 + 3 + 5 = 9$, etc.) gave a series of squares, Pythagoras formulated the rule for finding, logically, the sides of a right triangle: Take an odd number (say 7) which forms the shorter side, square it ($7^2 = 49$), subtract one ($49 - 1 = 48$), halve the remainder ($48 \div 2 = 24$); this half is the longer side, and this increased by one ($24 + 1 = 25$), is the hypotenuse.

The ancients recognized already the significance of the Pythagorean proposition for which fact may serve among others as proof the account of Diogenes Laertius and Plutarch concerning Pythagoras. The latter is said to have offered (sacrificed) the Gods an ox in gratitude after he learned the notable characteristics of the right triangle. This story is without doubt a fiction, as sacrifice of animals, i. e., blood-shedding, antagonizes the Pythagorean teaching.

During the middle ages this proposition which was also named *inventum hecatombe dignum* (in-as-much as it was even believed that a sacrifice of a hecatomb—100 oxen—was offered) won the honor-designation *Magister matheseos*, and the knowledge thereof was some decades ago still the proof of a solid mathematical training (or education). In examinations to obtain the master's degree this proposition was often given; there was indeed a time, as is maintained, when from every one who submitted himself to the test as master of mathematics a new (original) demonstration was required.

This latter circumstance, or rather the great significance of the proposition under consideration was the reason why numerous demonstrations of it were thought out.

The collection of demonstrations which we bring in what follows,* must, in our opinion, not merely satisfy the simple thirst for knowledge, but also serve as important aids in the teaching of geometry. The variety of demonstrations, even when some of them are finical, must demand in the learners the development of rigidly logical thinking, must show them how many sidedly an object can be considered, and spur them on to test their abilities in the discovery of like demonstrations for the one or the other proposition."

Brief biographical information concerning Pythagoras.

"The birthplace of Pythagoras was the island of Samos; there the father of Pythagoras, Mnessarch, obtained citizenship for services which he had rendered the inhabitants of Samos during a time of famine. Accompanied by his wife Pithay, Mnessarch frequently

*Note.—There were but 46 different demonstrations in the monograph by Jury Whipper, which 46 are among the classified collection found in this work.

traveled in business interests; during the year 569 A. C. he came to Tyre; here Pythagoras was born. At eighteen Pythagoras, secretly, by night, went from (left) Samos, which was in the power of the tyrant Polycrates, to the island Lesbos to his uncle who welcomed him very hospitably. There for two years he received instruction from Ferekid who with Anaksimander and Thales had the reputation of a philosopher.

After Pythagoras had made the religious ideas of his teacher his own, he went to Anaksimander and Thales in Miletus (549 A. C.). The latter was then already 90 years old. With these men Pythagoras studied chiefly cosmography, i. e., Physics and Mathematics.

Of Thales it is known that he borrowed the solar year from Egypt; he knew how to calculate sun and moon eclipses, and determine the elevation of a pyramid from its shadow; to him also are attributed the discovery of geometrical projections of great import; e. g., the characteristic of the angle which is inscribed and rests with its sides on the diameter, as well as the characteristics of the angle at the base of an (equilateral) isosceles triangle.

Of Anaksimander it is known that he knew the use of the dial in the determination of the sun's elevation; he was the first who taught geography and drew geographical maps on copper. It must be observed too, that Anaksimander was the first prose writer, as down to his day all learned works were written in verse, a procedure which continued longest among the East Indians.

Thales directed the eager youth to Egypt as the land where he could satisfy his thirst for knowledge. The Phoenician priest college in Sidon must in some degree serve as preparation for this journey. Pythagoras spent an entire year there and arrived in Egypt 547.

Although Polikrates who had forgiven Pythagoras's nocturnal flight addresses to Amasis a letter in which he commended the young scholar, it cost Pythagoras as a foreigner, as one unclean, the most incredible toil to gain admission to the priest caste which only unwillingly initiated even their own people into their mysteries or knowledge.

The priests in the temple Heliopolis to whom the king in person brought Pythagoras declared it impossible to receive him into their midst, and directed him to the oldest priest college at Memphis, this commended him to Thebes. Here somewhat severe conditions were laid upon Pythagoras for his reception into the priest caste; but nothing could deter him. Pythagoras performed all the rites, and all tests, and his study began under the guidance of the chief priest Sonchis.

During his 21 years stay in Egypt Pythagoras succeeded not only in fathoming and absorbing all the Egyptian but also became sharer in the highest honors of the priest caste.

In 527 Amasis died; in the following (526) year in the reign of Psammenit, son of Amasis, the Persian king Kambis invaded Egypt and loosed all his fury against the priest caste.

Nearly all members thereof fell into captivity, among them Pythagoras, to whom as abode Babylon was assigned. Here in the center of the world commerce where Bactrians, Indians, Chinese, Jews and other folk came together, Pythagoras had during 12 years stay opportunity to acquire those learnings in which the Chaldeans were so rich.

A singular accident secured Pythagoras liberty in consequence of which he returned to his native land in his 56th year. After a brief stay on the island Delos where he found his teacher Ferekid still alive, he spent a half year in a visit to Greece for the purpose of making himself familiar with the religious, scientific and social condition thereof.

The opening of the teaching activity of Pythagoras, on the island of Samos, was extraordinarily sad; in order not to remain wholly without pupils he was forced even to pay his sole pupil, who was also

named Pythagoras, a son of Eratokles. This led him to abandon his thankless land and seek a new home in the highly cultivated cities of Magna Graecia (Italy).

In 510 Pythagoras came to Kroton. As is known it was a turbulent year. Tarquin was forced to flee from Rome, Hippias from Athens; in the neighborhood of Kroton, in Sibaris, insurrection broke out.

The first appearance of Pythagoras before the people of Kroton began with an oration to the youth wherein he rigorously but at the same time so convincingly set forth the duties of young men that the elders of the city entreated him not to leave them without guidance (counsel). In his second oration he called attention to law abiding and purity of morals as the butresses of the family. In the two following orations he turned to the matrons and children. The result of the last oration in which he specially condemned luxury was that thousands of costly garments were brought to the temple of Hera, because no matron could make up her mind to appear in them on the street.

Pythagoras spoke captivatingly, and it is for this reason not to be wondered at that his orations brought about a change in the morals of Kroton's inhabitants; crowds of listeners streamed to him. Besides the youth who listened all day long to his teaching some 600 of the worthiest men of the city, matrons and maidens, came together at his evening entertainments; among them was the young, gifted and beautiful Theana, who thought it happiness to become the wife of the 60 year old teacher.

The listeners divided accordingly into disciples, who formed a school in the narrower sense of the word, and into auditors, a school in the broader sense. The former, the so-called mathematicians were given the rigorous teaching of Pythagoras as a scientific whole in logical succession from the prime concepts of mathematics up to the highest abstraction of philosophy; at the same time they learned to regard everything fragmentary in knowledge as more harmful than ignorance even.

From the mathematicians must be distinguished the auditors (university extensioners) out of whom subsequently were formed the Pythagoreans. These took part in the evening lectures only in which nothing rigorously scientific was taught. The chief themes of these lectures were: ethics, immortality of the soul, and transmigration—metempsychology.

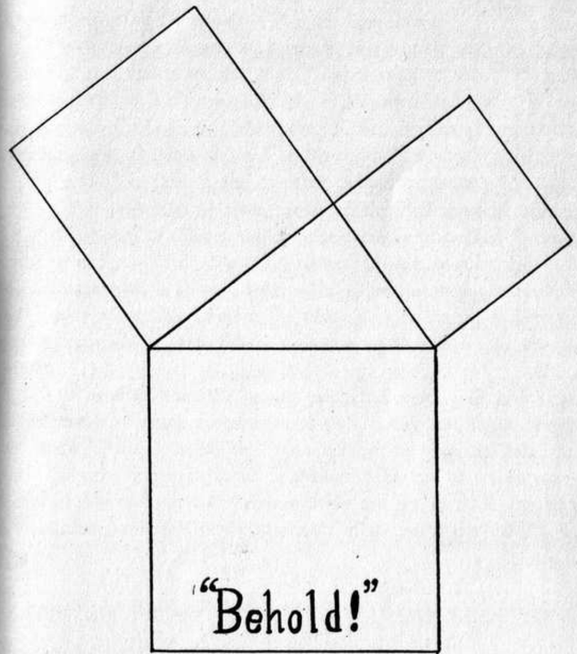
About the year 490 when the Pythagorean school reached its highest splendor—brilliancy—a certain Hypasos who had been expelled from the school as unworthy put himself at the head of the democratic party in Kroton and appeared as accuser of his former colleagues. The school was broken up, the property of Pythagoras was confiscated and he himself exiled.

The subsequent 16 years Pythagoras lived in Tarentum, but even here the democratic party gained the upper hand in 474 and Pythagoras a 95 year old man must flee again to Metapontus where he dragged out his poverty-stricken existence 4 years more. Finally democracy triumphed there also; the house in which was the school was burned, many disciples died a death of torture and Pythagoras himself with difficulty having escaped the flames died soon after in his 99th year."*

*Note.—The above translation is that of Dr. Theodore H. Johnston, Principal (1907) of the West High School, Cleveland, O.

Come and take choice of all my Library.

—*Titus Andronicus.*



Viam Inveniam aut Faciam.

Methods of Proof

The type and form of a figure necessarily determine the possible argument of a derived proof; hence, as an aid for reference, an order of arrangement of the proofs is of great importance.

The order of arrangement herein is, only in part, my own, being a modification and extension of the classification of the 100 proofs given by Prof. B. F. Yanney, A. M., of Wooster University, Wooster, O., and Prof. J. A. Calderhead, B. Sc., of Curry University, Pittsburg, Pa., as published in The American Mathematical Monthly, Vol's III-VI, 1896-9, and of other published proofs.

In this exposition of some proofs of the Pythagorean theorem the aim has been to classify and arrange them as to method of proof and type of figure used; to give the name, in case it has one, by which the demonstration is known; to give the name and page of the journal, magazine or text wherein the proof may be found, if known; and occasionally to give other interesting data relative to certain proofs.

It is assumed that the person using this work will know the fundamentals of plane geometry, and that, having the figure before him, he will readily supply the "reasons why" for the steps taken as, often from the figure, the proof is obvious; therefore only such statements of construction and demonstration are set forth in the text as is necessary to establish the argument of the particular proof.

I.

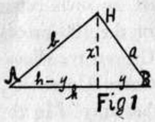
ALGEBRAIC PROOFS THROUGH LINEAR RELATIONS

A.—Similar Right Triangles.

From linear relations of similar right triangles it may be proven that, *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*

And since the algebraic square is the measure of the geometric square, the truth of the proposition as just stated involves the truth of the proposition as stated under Geometric Proofs through comparison of areas. A few algebraic proofs are the following:

One



In the right triangle ABH, Fig. 1, draw a perp. from H to AB, and call its foot C. The triangles ABH, HAC and BHC are similar. And since, from three similar and related triangles, there are possible nine simple proportions, these proportions and their resulting equations are:

(1) $a : x = b : h - y \therefore ah - ay = bx$.
(2) $a : y = b : x \therefore ax = by$.
(3) $x : y = h - y : x \therefore x^2 = hy - y^2$.
(4) $a : x = h : b \therefore ab = hx$.
(5) $a : y = h : a \therefore a^2 = hy$.
(6) $x : y = b : a \therefore ax = by$.
(7) $b : h - y = h : b \therefore b^2 = h^2 - hy$.
(8) $b : x = h : a \therefore ab = hx$.
(9) $h - y : x = b : a \therefore ah - ay = bx$.

Since equations (1) and (9) are identical, also (2) and (6), and (4) and (8), there remain but six different equations, and the problem becomes, how may these six equations be combined so as to give the desired relation $h^2 = a^2 + b^2$, which geometrically interpreted is $AB^2 = BH^2 + HA^2$.

In this proof *One*, and in every case hereafter, as in proof *Seventeen*, etc., the symbol AB^2, or a like symbol, signifies $\overline{AB}^2$.

1st.—Legendre's Solution.

a. From no single equation of the above nine can the desired relation be determined, and there is but one combination of two equations which will give it; viz., (5) and (7). Adding these gives $h^2 = a^2 + b^2$.

b. See Davies Legendre, p. 112,
Journal of Education, 1888, V. XXV, p. 404, fig. V.
Heath's Math. Monograph, No. 1, p. 19, proof III.
or any late text on geometry.

c. Since equations (5) and (7) are implied in the principle that homologous sides of similar triangles are proportional it follows that the truth of this important proposition is but a corollary to the more general law of similarity.

2nd.—Other Solutions.

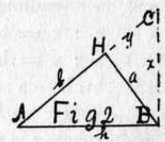
a. By the law of combinations there are possible 20 sets of three equations out of the six different equations. Rejecting all sets containing (5) and (7), and all sets containing dependent equations, there are remaining 13 sets from which the elimination of x and y may be accomplished in 44 different ways, each giving a distinct proof for the relation $h^2 = a^2 + b^2$.

b. See the American Math. Monthly, V. III, p. 66.
or Edward's Geometry, p. 157, fig. 15.

c. W. W. Rouse Ball, of Trinity College, Cambridge, Eng., seems to think Pythagoras knew of this proof.

Two

Produce AH to C so that CB will be perpendicular to AB.



The triangles ABH, CAB and BCH are similar. See Fig. 2.

From the continued proportion, $b : h : a = a : x : y = h : b + y : x$, nine different simple proportions are possible, viz:

(1) $b : h = a : x$.
(2) $b : a = a : y$.
(3) $h : a = x : y$.
(4) $b : h = h : b + y$.
(5) $b : a = h : x$.
(6) $h : a = b + y : x$.
(7) $a : x = h : b + y$.
(8) $a : y = h : x$.

(9) $x : b + y = y : x$, from which six different equations are possible as in *One* above.

1st—Solutions From Sets of Two Equations.

a. As in *One,* there is but one set of two equations, which will give the relation $h^2 = a^2 + b^2$.

b. See Math. Mo., V. III, p. 66.

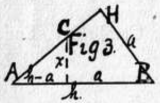
2nd—Solution From Sets of Three Equations.

a. As in 2nd under proof *One,* fig. 1, there are 13 sets of three eq's, giving 44 distinct proofs that give $h^2 = a^2 + b^2$.

b. See Math. Mo. V. III, p. 66.

c. Therefore from three similar rt. tri's so related that any two have one side in common there are 90 ways of proving that $h^2 = a^2 + b^2$.

Three



In Fig. 3 take on BA, BD = BH, and CD perp. to AB, forming the two similar tri's ABH and CAD.

a. From the continued proportion $a : x = b : h - a = h : b - x$ the simple proportions and their resulting eq's are:

(1) $a : x = b : h - a \therefore ah - a^2 = bx$,

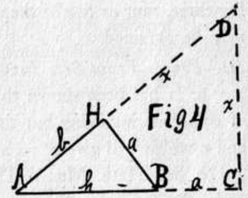
(2) $a : x = h : b - x \therefore ab - ax = hx$,

(3) $b : h - a = h : b - x \therefore b^2 - bx = h^2 - ah$.

As there are but three equations and as each equation contains the unknown x in the 1st degree, there are possible but three solutions giving $h^2 = a^2 + b^2$.

b. See Math. Mo., V. III, p. 66.

Four



In Fig. 4 extend AB to C making BC = BH, and draw CD perp. to AC. Produce AH to D, forming the two similar tri's ABH and ADC.

a. From the continued proportion $b : h + a = a : x = h : b + x$ three equations are possible giving, as in fig. 3, three proofs.

b. See Math. Mo., V. III, p. 67.

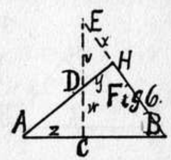
Five

Draw AC the bisector of the angle HAB, and CD perp. to AB, forming the similar tri's ABH and BCD. Then $CB = a - x$ and $DB = h - b$.

a. From the continued proportion $h : a - x = a : h - b : x$ three equations are possible giving, as in fig. 3, three proofs for $h^2 = a^2 + b^2$.

b. Original with the author, Feb. 23, 1926.

Six



Through D, any pt. in either leg of the rt. triangle ABH, draw DC perp. to AB and extend it to E a pt. in the other leg produced, thus forming the four similar rt. tri's ABH, BEC, ACD and EHD. From the continued proportion $(AB = h) : (BE = a + x) : (ED = v) : (DA = b - y) = (BH = a) : (BC = h - z) : (DH = y) : (DC = w) = (HA = b) : (CE = v + w) : (HE = x) : (CA = z)$, eighteen simple proportions and eighteen different equations are possible.

From no single equation nor from any set of two eq's can the relation $h^2 = a^2 + b^2$ be found but from combination of eq's involving three, four or five of the unknown elements u, w, x, y, z, solutions may be obtained.

1st—Proofs From Sets Involving Three Unknown Elements.

a. It has been shown that there is possible but one combination of equations involving but three of the unknown elements, viz., x, y and z which will give $h^2 = a^2 + b^2$.

b. See Math. Mo. V. III, p. III.

2nd.—Proofs From Sets Involving Four Unknown Elements.

a. There are possible 114 combinations involving but four of the unknown elements each of which will give $h^2 = a^2 + b^2$.

See Math. Mo., V. III, p. III.

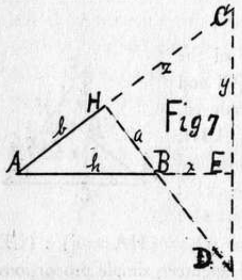
3rd.—Proofs From Sets Involving All Five Unknown Elements.

a. Similarly, there are 4749 combinations involving all five of the unknowns, from each of which $h^2 = a^2 + b^2$ can be obtained.

b. See Math. Mo., V. III, p. 112.

c. Therefore the total no. of proofs from the relations involved in fig. 6 is 4864.

Seven



Produce AB to E, fig. 7, and through E draw, perp. to AE, the line CED meeting AH produced in C and HB produced in D, forming the four similar rt. tri's ABH, DBE, CAE and CDH.

a. As in fig. 6, eighteen different equations are possible from which there are also 4864 proofs.

b. Therefore the total no. of ways of proving that $h^2 = a^2 + b^2$ from 4 similar rt. tri's related as in fig's 6 and 7 is 9728.

c. As the pt. E approaches the pt. B, fig. 7, approached fig. 2, above, and becomes fig. 2, when E falls on B.

d. Suppose E falls on AB so that CE cuts HB between H and B; then we will have 4 similar rt. tri's involving 6 unknowns. How many proofs will result?

Eight

In fig. 8 produce BH to D, making BD = BA, and E, the middle pt. of AD, draw EC parallel to AH, and join BE, forming the 7 similar rt. triangles AHD, ECD, BED, BEA, BCE, BHF and AEF, but six of which need consideration, since tri's BED and BEA are congruent and, in symbolization, identical.

From these 6 different rt. triangles, sets of 2 tri's may be selected in 15 different ways, sets of 3 tri's may be selected in 20 different ways, sets of 4 tri's may be selected in 15 different ways, sets of 5 tri's may be selected in 6 different ways, and sets of 6 tri's may be selected in 1 way, giving, in all, 57 different ways in which the 6 triangles may be combined.

But as all the proofs derivable from the sets of 2, 3, 4, or 5 tri's are also found among the proofs from the set of 6 triangles, an investigation of this set will suffice for all.

In the 6 similar rt. tri's, let AB = h, BH = a, HA = b, DE = EA = x, BE = y, FH = z and BF = v, whence $EC = \frac{b}{2}$, $DH = h - a$, $DC = \frac{h - a}{2}$, $EF = y - v$, $BE = \frac{h + a}{2}$, $AD = 2x$ and $AF = b - z$, and from these data the continued proportion is

$$b : b/2 : y : (h + a)/2 : a : x$$
$$= h - a : (h - a)/2 : x : b/2 : z : y - v$$
$$= 2x : x : h : y : v : b - z.$$

From this continued proportion there result 45 simple proportions which give 28 different equations, and, as ground-work for determining the number of proofs possible, they are here tabulated.

(1) $b : b/2 = h - a : (h - a)/2$, whence $1 = 1$. Eq. 1.
(2) $b : b/2 = 2x : x$, whence $1 = 1$. Eq. 1.
(3) $h - a : (h - a)/2 = 2x : x$, whence $1 = 1$. Eq. 1^3.
(4) $b : y = h - a : x$, whence $bx = (h - a)y$. Eq. 2.
(5) $b : y = 2x : h$, whence $2xy = bh$. Eq. 3.
(6) $h - a : x = 2x : h$, whence $2x^2 = h^2 - ah$. Eq. 4.
(7) $b : (a + h)/2 = h - a : b/2$, whence $b^2 = h^2 - a^2$. Eq. 5.
(8) $b : (h + a)/2 = 2x : y$, whence $(h + a)x = by$. Eq. 6.
(9) $h - a : b/2 = 2x : y$, whence $bx = (h - a)y$. Eq. 2.
(10) $b : a = h - a : z$, whence $bz = (h - a)a$. Eq. 7.
(11) $b : a = 2x : v$, whence $2ax = bv$. Eq. 8.
(12) $h - a : z = 2x : v$, whence $2xz = (h - a)v$. Eq. 9.
(13) $b : x = h - a : y - v$, whence $(h - a)x = b(y - v)$. Eq. 10.
(14) $b : x = 2x : b - z$, whence $2x^2 = b^2 - bz$. Eq. 11.
(15) $h - a : y - v = 2x : b - z$, whence $2(y - v)z = (h - a)(b - z)$. Eq. 12.
(16) $b/2 : y = (h - a)/2 : x$, whence $bx = (h - a)y$. Eq. 2.
(17) $b/2 : y = x : h$, whence $2xy = bh$. Eq. 3.
(18) $(h - a)/2 : x = x : h$, whence $2x^2 = h^2 - ah$. Eq. 4^2.
(19) $b/2 : (h + a)/2 = (h - a)/2 : b/2$, whence $b^2 = h^2 - a^2$. Eq. 5^2.
(20) $b/2 : (h + a)/2 = x : y$, whence $(h + a)x = by$. Eq. 6.
(21) $(h - a)/2 : b/2 = x : y$, whence $bx = (h - a)y$. Eq. 2^4.
(22) $b/2 : a = (h - a)/2 : z$, whence $bz = (h - a)a$. Eq. 7^2.
(23) $b/2 : a = x : v$, whence $2ax = bv$. Eq. 8^2.
(24) $(h - a)/2 : z = x : v$, whence $2xz = (h - a)v$. Eq. 9^2.
(25) $b/2 : x = (h - a)/2 : y - v$, whence $(h - a)x = b(y - v)$. Eq. 10^2.

(26) $b/2 : x = x : b - z$, whence $2x^2 = b^2 - bz$. Eq. 11^2.
(27) $(h - a)/2 : y - v = x : b - z$, whence $2(y - v)x = (h - a)(b - z)$. Eq. 12^2.
(28) $y : (h + a)/2 = x : b/2$, whence $(h + a)x = by$. Eq. 6^3.
(29) $y : (h + a)2 = h : y$, whence $2y^2 = h^2 + ah$. Eq. 13.
(30) $x : b/2 = h : y$, whence $2xy = bh$. Eq. 3^3.
(31) $y : a = x : z$, whence $wx = yz$. Eq. 14.
(32) $y : a = h : v$, whence $vy = ah$. Eq. 15.
(33) $x : z = h : v$, whence $vx = hz$. Eq. 16.
(34) $y : x = x : y - v$, whence $x^2 = y(y - v)$. Eq. 17.
(35) $y : x = h : b - z$, whence $hx = y(b - z)$. Eq. 18.
(36) $x : y - v = h : b - z$, whence $(b - z)x = h(y - v)$. Eq. 19.
(37) $(h + a)/2 : a = b/2 : z$, whence $(h + a)z = ab$. Eq. 20.
(38) $(h + a)/2 : a = y : v$, whence $2by = (h + a)v$. Eq. 21.
(39) $b/2 : z = y : v$, whence $2yz = bv$. Eq. 22.
(40) $(h + a)/2 : x = b/2 : y - v$, whence $bx = (h + a)(y - v)$. Eq. 23.
(41) $(h + a)/2 : x = y : b - z$, whence $2xy = (h + a)(b - z)$. Eq. 24.
(42) $b/2 : y - v = y : b - z$, whence $2y(y - v) = b^2 - bz$. Eq. 25.
(43) $a : x = z : y - v$, whence $xz = a(y - v)$. Eq. 26.
(44) $a : x = v : b - z$, whence $vx = a(b - z)$. Eq. 27.
(45) $z : y - v = v : b - z$, whence $v(y - v) = (b - z)z$. Eq. 28.

The symbol 2^4, see (21), means that equation 2 may be derived from 4 different proportions. Similarly for 6^3, etc.

Since a definite no. of sets of dependent equations, three equations in each set, is derivable from a given continued proportion and since these sets must be known and dealt with in establishing the no. of possible proofs for $h^2 = a^2 + b^2$, it becomes necessary to determine the no. of such sets. In any continued proportion the symbolization for the no. of such sets, three equations in each set, is $\frac{n^2(n+1)}{2}$, in which *n* signifies the no. of simple ratios in a member of the continued prop'n. Hence for the above continued proportion there are derivable 75 such sets of dependent equations. They are:

(1), (2), (3); (4), (5), (6); (7), (8), (9); (10), (11), (12); (13), (14), (15); (16), (17), (18); (19), (20), (21); (22), (23), (24); (25), (26), (27); (28), (29), (30); (31), (32), (33); (34), (35), (36); (37), (38), (39); (40), (41), (42); (43), (44), (45); (1), (4), (16); (1), (7), (19); (1) (10), (22); (1), (13), (25); (4), (7), (28); (4), (10), (31); (4), (13), (34); (7), (10), (37); (7), (13), (40); (10), (13), (43); (16), (19), (20); (16), (22), (31); (16), (25), (34); (19), (22), (37); (19), (25), (40); (22), (25), (43); (28), (31), (37); (28), (34), (40); (31), (34), (43); (37), (40), (43); (2), (5), (17); (2), (8), (20); (2), (11), (23); (2), (14), (26); (5), (8), (29); (5), (11), (32); (5), (14), (35); (8), (11), (38); (8), (14), (41); (11), (14), (44); (17), (20), (29); (17), (23), (32); (17), (26), (35); (20), (23), (38); (20), (26), (41); (23), (26), (44); (29), (32), (38); (29), (35), (41); (32), (35), (44); (38), (41), (44); (3), (6), (18); (3), (9), (21); (3), (12), (24); (3), (15), (27); (6), (9), (30); (6), (12), (33); (6), (15), (36); (9), (12), (36); (9), (15), (42); (12), (15), (45); (18), (21), (30); (18), (24), (33); (18), (27), (36); (21), (24), (39); (21), (27), (42); (24), (27), (45); (30), (33), (39); (30), (36), (42); (33), 36), (45); (39), (42), (45).

These 75 sets expressed in the symbolization of the 28 equations give but 49 sets as follows:

1, 1, 1; 2, 3, 4; 2, 5, 6; 7, 8, 9; 10, 11, 12; 6, 13, 3; 14, 15, 16; 17, 18, 19; 20, 21, 22; 23, 24, 25; 26, 27, 28; 1, 2, 2; 1, 5, 5; 1,

7, 7; 1, 10, 10; 1, 6, 6; 2, 7, 14; 2, 10, 17; 5, 7, 20; 5, 10, 23; 7, 10, 26; 6, 14, 20; 6, 17, 23; 14, 17, 26; 20, 23, 26; 1, 3, 3; 1, 8, 8; 1, 11, 11; 3, 8, 15; 3, 11, 18; 6, 8, 21; 6, 11, 24; 8, 11, 27; 13, 15, 21; 13, 18, 24; 15, 18, 27; 21, 24, 27; 1, 4, 4; 1, 9, 9; 1, 12, 12; 4, 9, 16; 4, 12, 19; 2, 9, 22; 2, 12, 25; 9, 12, 28; 3, 16, 22; 3, 19, 25; 16, 19, 28; 22, 25, 28.

Since eq. 1 is an identity and eq. 5 gives, at once, $h^2 = a^2 + b^2$, there are remaining 26 equations involving the 4 unknowns x, y, z and v, and proofs may be possible from sets of equations involving x and y, x and z, x and v, y and z, y and v, z and v, x, y and z, x, y and v, x, z and v, y, z and v, and x, y, z and v.

1st.—Proofs From Sets Involving Two Unknowns.

a. The two unknowns, x and y, occur in the following five equations, viz., 2, 3, 4, 6 and 13, from which but one set of two, viz., 2 and 6, will give $h^2 + a^2 = b^2$, and as eq. 2 may be derived from 4 different proportions and equation 6 from 3 different proportions, the no. of proofs from this set are 12.

Arranged in sets of three we get,

2^4, 3^3, 13 giving 12 other proofs;
(2, 3, 4) a dependent set—no proof;
2^4, 4^2, 13 giving 8 other proofs;
(3, 6, 13) a dependent set—no proof;
3^3, 4^2, 6^3 giving 18 other proofs;
4^2, 6^3, 13 giving 6 other proofs;
3^3, 4^2, 13 giving 6 other proofs.

Therefore there are 62 proofs from sets involving x and y.

b. Similarly, from sets involving x and z there are 8 proofs, the equations for which are 4, 7, 11, and 20.

c. Sets involving x and v give no additional proofs.

d. Sets involving y and z give 2 proofs, but the equations were used in a and b, hence cannot be counted again, they are 7, 13 and 20.

e. Sets involving y and v give no proofs.

f. Sets involving z and v give same results as *d*.

Therefore the no. of proofs from sets involving two unknowns is 70, making, in all 72 proofs so far, since $h^2 = a^2 + b^2$ is obtained directly from two different prop's.

2nd.—Proofs From Sets Involving Three Unknowns.

a. The three unknowns x, y and z occur in the following 11 equations, viz., 2, 3, 4, 6, 7, 11, 13, 14, 18, 20 and 24, and from these 11 equations sets of four can be selected in $\frac{11 \cdot 10 \cdot 9 \cdot 8}{\lfloor 4}$ = 330 ways, each of which will give one or more proofs for $h^2 = a^2 + b^2$. But as the 330 sets, of four equations each, include certain sub-sets heretofore used, certain dependent sets of three equations each found among those in the above 75 sets, and certain sets of four dependent equations, all these must be determined and rejected; the proofs from the remaining sets will be proofs additional to the 72 already determined.

Now, of 11 consecutive things arranged in sets of 4 each, any *one* will occur in $\frac{10 \cdot 9 \cdot 8}{\lfloor 3}$ or 120 of the 330 sets, any *two* in $\frac{9 \cdot 8}{\lfloor 2}$ or 36 of the 330, and any *three* in $\frac{8}{1}$, or 8 of the 330 sets. Therefore any sub-set of two equations will be found in 36, and any of three equations in 8, of the 330 sets.

But some one or more of the 8 may be some one or more of the 36 sets; hence a sub-set of two and a sub-set of three will not necessarily cause a rejection of 36 + 8 = 44 of the 330 sets.

The sub-sets which gave the 70 proofs are:
2, 6, for which 36 sets must be rejected;
7, 20, for which 35 sets must be rejected, since
7, 20, is found in *one* of the 36 sets above;
2, 3, 13, for which 7 other sets must be rejected, since
2, 3, 13 is found in *one* of the 36 sets above;

2, 4, 13, for which 6 other sets must be rejected;
3, 4, 6, for which 7 other sets must be rejected;
4, 6, 13 for which 6 other sets must be rejected;
3, 4, 13, for which 6 other sets must be rejected;
4, 7, 11, for which 7 other sets must be rejected; and
4, 11, 20, for which 7 other sets must be rejected;
for all of which 117 sets must be rejected.

Similarly the dependent sets of three, which are 2, 3, 4; 3, 6, 13; 2, 7, 14; 6, 14, 20; 3, 11, 18; 6, 11, 24; and 13, 18, 24; cause a rejection of 6 + 6 + 6 + 6 + 8 + 7 + 8, or 47 more sets.

Also the dependent sets of four, and not already rejected, which are, 2, 4, 11, 18; 3, 4, 7, 14; 3, 6, 18, 24; 3, 13, 14, 20; 3, 11, 13, 24; 6, 11, 13, 18; and 11, 14, 20, 24, cause a rejection of 7 more sets. The dependent sets of *fours* are discovered as follows: take any two dependent sets of threes having a common term as 2, 3, 4, and 3, 11, 18; drop the common term 3, and write the set 2, 4, 11, 18; a little study will disclose the 7 sets named, as well as other sets already rejected; e. g., 2, 4, 6, 13. Rejecting the 117 + 49 + 7 = 171 sets there remain 159 sets, each of which will give one or more proofs, determined as follows. Write down the 330 sets, a thing easily done, strike out the 171 sets which must be rejected, and, taking the remaining sets one by one, determine how many proofs each will give; e. g., take the set 2, 3, 7, 11; write it thus 2^4, 3^3, 7^2, 11^2, the exponents denoting the different proportions from which the respective equations may be derived; the product of the exponents, $4 \times 3 \times 2 \times 2 = 48$, is the number of proofs possible for that set. The set 6^3, 11^2, 18^1, 20^1 gives 6 proofs, the set 14^1, 18^1, 20^1, 24^1 gives but 1 proof; etc.

The 159 sets, by investigation, give 1231 proofs.

b. The three unknowns x, y and v occur in the following twelve equations, —2, 3, 4, 6, 8, 10, 11, 13, 15, 17, 21 and 23, which give 495 different sets of 4 equations each, many of which must be rejected for same reasons as in *a*. Having established a method in *a*, we leave details to the one interested.

c. Similarly for proofs from the eight equations containing x, z and v, and the seven eq's containing y, z and v.

3rd.—Proofs From Sets Involving The Four Unknowns x, y, z and v.

a. The four unknowns occur in 26 equations; hence there are $\frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}{\lfloor 5} = 65780$ different sets of 5 equations each. Rejecting all sets containing sets heretofore used and also all remaining sets of five dependent equations of which 2, 3, 9, 19, 28, is a type, the remaining sets will give us many additional proofs, the determination of which involves a vast amount of time and labor if the method given in the preceeding pages is followed. If there be a shorter method, I am unable, as yet, to discover it; neither am I able to find anything by any other investigator.

4th.—Special Solutions.

a. By an inspection of the 45 simple proportions given above, it is found that certain proportions are worthy of special consideration as they give equations from which very simple solutions follow.

From proportions (7) and (19) $h^2 = a^2 + b^2$ follows immediately. Also from the pairs (4) and (18), and (10) and (37), solutions are readily obtained.

b. Hoffman's solution.

Joh. Jos. Ign. Hoffman made a collection of over 30 proofs, publishing the same in "Der Pythagoraische Lehrsatz," 2nd edition Mainz, 1821, of which the solution from (7) is one. He selects the two triangles, (see fig. 8), AHD and BCE, from which $b : (h + a)/2 = h - a : b/2$ follows, giving at once $h^2 = a^2 + b^2$.

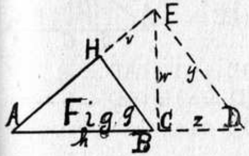
See Jury Whipper's 46 proofs, 1880, p. 40, fig. 41.

c. Similarly from the two triangles BCE and ECD $b/2 : (h + a)/2 = (h - a)/2 : b/2$, $h^2 = a^2 + b^2$. Also from the

three triangles AHD, BEA, and BCE proportions (4) and (8) follow, and from the three triangles AHD, BHE and BCE proportions (10) and (37) give at once $h^2 = a^2 + b^2$.

See Math. Mo., V. III, p. 169-70.

Nine



In fig. 9, produce AB to any pt. D, from D draw a perp. DE to AH produced, and from E drop the perp. EC, thus forming the 4 similar rt. tri's ABH, AED, ECD and ACE.

From the homologous sides of these similar triangles the following continued proportion results:

$(AH = b) : (AE = b + v) : (EC = w) : (AC = h + x)$
$= (BH = a) : (DE = y) : (CD = z) : (EC = w)$
$= (AB = h) : (AD = h + x + z) : (DE = y) : (AE = b + v)$.

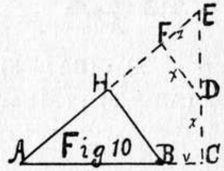
a. From this continued prop'n 18 simple proportions are possible, giving, as in fig. 6, several thousand proofs.

b. See Math. Mo., V. III, p. 171.

Ten

a. In fig. 10 are three similar rt. tri's, ABH, EAC and DEF, from which the continued proportion

$(HA = b) : (AC = h + v) : (DF = DC = x)$
$= (HB = a) : (CE = y) : (FE = z)$
$= (AB = h) : (AE = h + v + z) : (DE = y - x)$ follows giv-

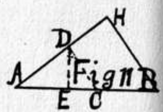


ing 9 simple proportions from which many more proofs for $h^2 = a^2 + b^2$ may be obtained.

b. See Math. Mo. V. III, p. 171.

Eleven

a. In fig. 11, four similar rt. triangles are obvious; they are: ABH, ACD, CDE and DAE, from which the continued proportion



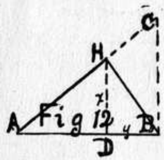
$(BH = a) : (CD = DH = v) : (EC = y) : (DE = x)$
$= (HA = b) : (DA = b - v) : (DE = x) : (AE = z)$
$= (AB = h) : (AC = z + y) : (CD = v) : (AD = b - v)$
follows; 18 simple proportions are possible from which many more proofs for $h^2 = a^2 + b^2$ result.

By an inspection of the 18 proportions it is evident that they give no simple equations from which easy solutions follow, as was found in the investigation of fig. 8, as in *a* under proof *Eight*.

See Math. Mo., V. III, p. 171.

Twelve

a. The construction of fig. 12 gives five similar rt. triangles, which are: ABH, AHD, HBD, ACB and BCH, from which the continued prop'n



$(BH = a) : (HD = x) : (BD = y) : (CB = \frac{a^2}{x}) : (CH = \frac{ay}{x})$
$= (HA = b) : (DA = h - y) : (DH = x) : (BA = h) : (HB = a)$
$= (AB = h) : (AH = b) : (HB = a) : (AC = b + \frac{ay}{x}) : (BC = \frac{a^2}{x})$
follows, giving 30 simple proportions from which only 12 different

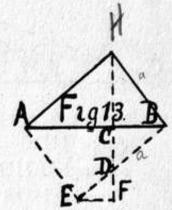
equations result. From these 12 equations several proofs for $h^2 = a^2 + b^2$ are obtainable.

b. In fig. 9, when C falls on B it is obvious that the graph become that of fig. 12. Therefore the solution of fig. 12, is only a particular case of fig. 9; also note that several of the proofs of case 12 are identical with those of case 1, proof *One*.

c. The above is an original method of proof by the author of this work.

Thirteen

In fig. 13, calling the vertex of the rt. angle H, complete the parallelogram HE and draw HF perp. to, and EF par. with AB, respectively, forming the six similar rt. tri's, BHA, HCA, BCH, AEB, DCB and DFE, from which 45 simple proportions are obtainable, resulting in several thousand more possible proof for $h^2 = a^2 + b^2$, only one of which we mention.



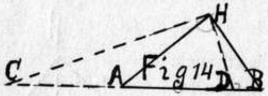
a. (1) From tri's DBH and BHA,
$DB : (BH = a) = (BH = a) : (HA = b); \therefore DB = \frac{a^2}{b}$
and (2) $HD : (AB = h) = (BH = a) : (HA = b); \therefore HD = \frac{ah}{b}$.
(3) From tri's DFE and BHA,
$DF : (EB - DB) = (BH = a) : (AB = h)$.
or $DF : b^2 - \frac{a^2}{b} = a : h; \therefore DF = a(\frac{b^2 - a^2}{bh})$
(4) Tri. $ABH = \frac{1}{2}$ par. $HE = \frac{1}{2} AB \times HC = \frac{1}{2} ab$
$= \frac{1}{2} [AB (\frac{AC + CF}{2})] = \frac{1}{2} [AB (\frac{HD + DF}{2})] = \frac{1}{4} [h (\frac{ah}{b} + a(\frac{b^2 - a^2}{bh}))]$

$= \frac{ah^2}{4b} + \frac{ab}{4} - \frac{a^3}{4b}$ ∴ (5) $\frac{1}{2}ab = \frac{ah^2 + ab^2 - a^3}{4b}$, whence (6) $h^2 = a^2 + b^2$.

b. This particular proof was produced by Prof. D. A. Lehman, Prof. of Math. at Baldwin University, Berea, O., Dec. 1899.

c. Also see Math. Mo., V. VII, No. 10, p. 228.

Fourteen



Construction. In the rt. tri. ABC, take AD = AH = AC, and draw HD and HC.

Proof. Tri's CAH and HAD are isosceles. Angle CHD is a rt. angle, since A is equidistant from C, D and H.

Angle HDB = angle CHD + angle DCH.

= angle AHD + 2 angle CHA = angle CHB.

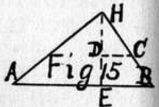
∴ tri's HDB and CHB are similar, having angle DBH in common and angle DHB = angle ACH.

∴ CB : BH = BH : DB. or h + b : a = a : h — b.

Whence $h^2 = a^2 + b^2$.

a. See Math. Teacher, Dec., 1925. Credited to Alvin Knoer, a Milwaukee High School pupil.

Fifteen



In fig. 15 the const's is obvious giving four similar right triangles ABH, AHE, HBE and HCD, from which the continued proportion (BH = a) : (HE = x) : (BE = y) : (CD = y/2)
= (HA = b) : (EA = h — y) : (EH = x) : (DH = x/2)
= (AB = h) : (AH = b) : (HB = a) : (HC = a/2) follows, giving 18 simple proportions.

a. From the two simple proportions

(1) a : y = h : a and

(2) b : h — y = h : b we get easily $h^2 = a^2 + b^2$.

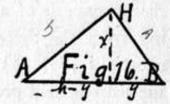
b. This solution is original with the author, but, like cases 11 and 12, it is subordinate to case 1.

c. As the number of ways in which three or more similar right triangles may be constructed so as to contain related linear relations with but few unknowns involved is unlimited, so the number of possible proofs therefrom must be unlimited.

Sixteen

The two following proofs, differing so much, in method, from those preceding, are certainly worthy of a place among selected proofs.

1st.—This proof rest on the axiom, "The whole is equal to the sum of its parts."



a. Let AB = h, BH = a and HA = b, in the rt. tri. ABH, and let HC, C being the pt. where the perp. from H intersects the line AB, be perp to AB. Suppose $h^2 = a^2 + b^2$. If $h^2 = a^2 + b^2$, then $a^2 = x^2 + y^2$ and $b^2 = x^2 + (h - y)^2$, or $h^2 = x^2 + y^2 + x^2 + (h - y^2)$
$= y^2 + 2x^2 + (h - y)^2 = y^2 + 2y(h - y) + (h - y)^2 = [y + (h - y)]^2$
∴ h = y + (h — y), i. e., AB = BC + CA, which is true
∴ the supposition is true, or $h^2 = a^2 + b^2$.

b. This proof is one of Joh. Hoffmann's 30 proofs. See Jury Whipper, 1880, p. 38, fig. 37.

2nd.—This proof is the "Reductio ad Absurdum" proof.

a. $h^2 <, =,$ or $> (a^2 + b^2)$. Suppose it is less. Then, since $h^2 = [(h - y) + y]^2 = [(h - y) + x^2 \div (h - y)]^2$ and $b^2 = [ax \div (h - y)]^2$, then $[(h - y) + x^2 \div (h - y]^2$

$< [ax \div (h - y)]^2 + a^2.$

$\therefore [x^2 + (h - y)^2]^2 < a^2 [x^2 + (h - y)^2].$

$\therefore a^2 > x^2 + (h - y)^2$, which is absurd. For, if the supposition be true, we must have $a^2 < x^2 + (h - y)^2$, as is easily shown.

Similarly, the supposition that $h^2 > a^2 + b^2$, will be proven false.

Therefore it follows that $h^2 = a^2 + b^2$.

b. See Math. Mo., V. III, p. 170.

B.—THE MEAN PROPORTIONAL PRINCIPLE

The mean proportional principle, leading to equivalency of areas of triangles and parallelograms, is very prolific in proofs.

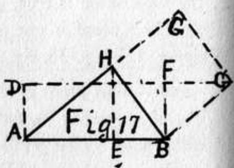
By rejecting all similar right triangles other than those obtained by dropping a perpendicular from the vertex of the right angle to the hypotenuse of a right triangle and omitting all equations resulting from the three similar right triangles thus formed, save only equations (3), (5) and (7), as given in proof *One,* we will have limited our field greatly. But in this limited field the proofs possible are many, of which a few very interesting ones will now be given.

In every figure under B we will let h= the hypotenuse, a = the shorter leg, and b = the longer leg of the given right triangle ABC.

Seventeen

Join AC forming the tri. ABC.

One square constructed outwardly on leg a. Through C, fig. 17, draw CD par. to AB, meeting the perp. AD at D. Join AC forming the tri. ABC.



Tri. ABC = ½ sq. BG = ½ rect. BD.

$\therefore$ BG = a^2 = rect. BD = sq. EF + rect. ED

= sq. EF + (EA × EB = HE^2 = sq. EF + HE^2.

But tri's HBE and ABH are similar.

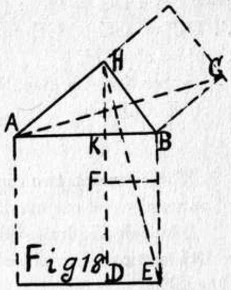
$\therefore$, if in tri. HBE, $HB^2 = BE^2 + HE^2$, then in its similar, the tri. ABH, $AB^2 = BH^2 + HA^2$.

$\therefore h^2 = a^2 + b^2$.

a. See Scientific American Supplement, V. 70, p. 382, Dec. 10, 1910, fig. 7,—one of the proofs of Arthur E. Colburn, LL. M., of Dist. of Columbia Bar.

Eighteen

Two squares constructed outwardly. Draw HD perp. to AB, join A and C, H and E, and complete the sq. BF.

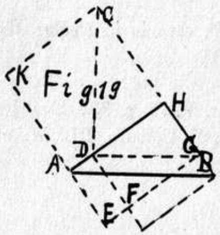


Rect. BD = 2 tri. EBH = 2 tri. ABC = sq. HC = sq. BF + (rect. FE = GE × GB = AK × KB = HK^2) = sq. BF + HK^2.

But tri's HBK and ABH are similar. Then as in fig. 17, we have, at once, $h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., V. 70, p. 359, Dec. 3, 1910. Credited to A. E. Colburn.

Nineteen



Two squares, one on AH const'd outwardly, the other on HB overlapping the given triangle.

Take HD = HB and cons't rt. tri. CDG. Then tri's CDH and ABH are equal. Draw GE par. to AB meeting GKA produced at E.

Rect. GK = rect. GA + sq. HK = (HA = HC) HG + sq. HK = HD^2 + sq. HK

Now GC : DC = DC : (HC = GE)

$\therefore$ DC^2 = GC $\times$ GE = rect. GK = sq. HK + sq. DB = AB^2

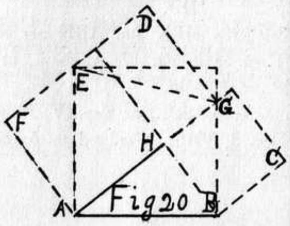
$\therefore$ $h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., V. 70, p. 382, Dec. 10, 1910. Credited to A. E. Colburn.

Twenty

Three squares, two constructed outwardly and one overlapping.

Through G draw GD par. to BH meeting FE produced at D. Draw EG.



Tri. AGE is common to sq. BE and rect. AD.

$\therefore$ tri. AGE = ½ sq. BE = ½ rect. AD

$\therefore$ sq. BE = rect. AD.

Rect. AD = sq. HF + (rect. HD = sq. HC, see fig. 19).

$\therefore$ sq. BE = sq. HF + sq. HC or

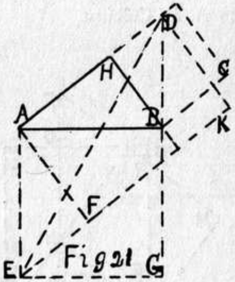
$h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., V. 70, p. 382, Dec. 10, 1910. Credited to A. E. Colburn.

Twenty-One

Three squares, two constructed outwardly, and the one AH overlapping.

Extend GB to D, draw DK par. to HB and draw DE.



Tri. AEF = tri. ABH, having AF = AH, and sides respectively perp. to each other.

Tri. DAE = ½ rect. AK = ½ sq. AG.

$\therefore$ sq. AG = rect. AK = sq. HF + (rect. HK = DK $\times$ DH = HA $\times$ HD = HB^2 = sq. HC) = sq. HF = sq. HC.

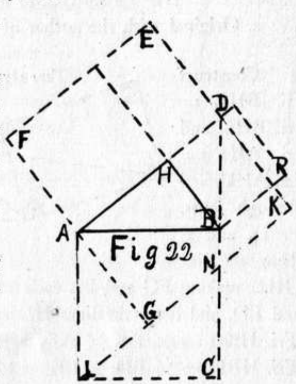
$\therefore$ $h^2 = a^2 + b^2$.

a. See Am. Sci. Sup., V. 70, p. 383, Dec. 10, 1910. Credited to A. E. Colburn.

Twenty-Two

Three squares all constructed outwardly.

Extend CB to D, meeting AH produced at D, and through D draw ED par. to HB, making DK = AH. Draw KL and produce FA to G.



Tri. ALG = tri. DNK, and tri. LCN = tri. ABD

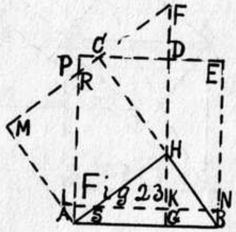
$\therefore$ rect. GD = sq. LB, having polygon AGNB in common.

$\therefore$ sq. AC = rect. AK = rect. AE = sq. HF + rect. HE = sq. HF + sq.

HP, see fig. 20. $\therefore h^2 = a^2 + b^2$.

a. See Am. Sci. Sup., V. 70, p. 383, Dec. 10, 1910. Credited to A. E. Colburn.

Twenty-Three

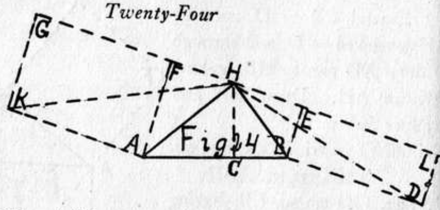


Two square, one on AH outwardly, and one, LE, with sides = AB, transposed.

Draw through H, perp. to AB, GH and produce it to meet MC produced at F. Take HK = GB, and through K draw LN par. and equal to AB. Complete the transposed sq. LE. Sq. LE = rect. DN + rect. DL = (DK × KN = LN × KN = AB × AG = HB^2) + (rect. LD = paral. AF = sq. AC) for tri. FCH = tri. RMA and tri. CPR = tri. SLA. $\therefore$ sq. LE = HB^2 + sq. AC, or $h^2 = a^2 + b^2$.

a. Original with the author of this work, Feb. 2, 1926.

Twenty-Four



Construct tri. BHE = tri. BHC and tri. AHF = tri. AHC, and through pts. F, H, and E draw the line GHL, making FG and EL each = AB, and complete the rect's FK and ED, and draw the lines HD and HK.

Tri. HKA = ½ AK × AF = ½ AB × AC — ½ AH^2.

Tri. HBD = ½ BD × BE = ½ AB × BC = ½ HB^2.

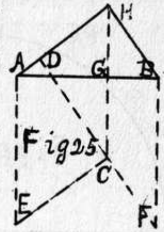
Whence AB × AC = AH^2 and AB × BC = HB^2.

Adding, we get AB × AC + AB × BC = AB (AC + BC) = AB^2, or $AB^2 = BH^2 + HA^2$.

$\therefore h^2 = a^2 + b^2$.

a. Original with the author, discovered Jan. 31, 1926.

Twenty-Five

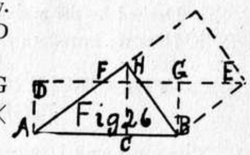


Construction. Draw HC, AE and BF each perp. to AB, making each equal to AB. Draw EC and FCD. Tri's ABH and HCD are equal and similar.

Figure FCEBHA = paral. CB + paral. CA = CH × GB + CH × GA = AB × GB + AB × AG = $HB^2 + HA^2$ = AB (GB + AG) = AB × AB = AB^2.

a. See Math. Teacher, V. XVI, 1915. Credited to Geo. G. Evans, Charleston High School, Boston, Mass.

Twenty-Six



One square constructed outwardly.

Draw perp's HC and AD, and take AD = CB.

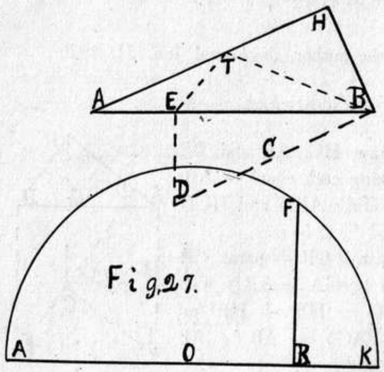
Paral BEFA = rect. AG = AB × BG = AB × BC = BH^2. And AB × AC = AH^2

Adding, we have $BH^2 + AH^2$ = AB × BC + AB × AC = AB (BC + AC) = AB^2.

$\therefore h^2 = a^2 + b^2$.

a. See Jury Whipper's Pythagoraische Lehrsatz, 1880, p. 39, fig. 38. Credited to Oscar Werner, as recorded in "Archiv. d. Math. und Phys.," Grunert, 1855.

Twenty-Seven



Construction. Through B draw BD = 2BH, and par. to HA. From D draw perp to AB, as DE. Find mean prop'l between AB and AE which is BF. From A, on AH, lay off AT = AH, lay off AT = BF. Draw TE and TB, forming the two similar tri's AET and ATB, from which AT : AB = AE : AT, or $(b - a)^2 = h(h - EB)$, whence $EB = \frac{h - (b - a)^2}{h}$ (1). Also EB : AH = BD : AB. $\therefore EB = \frac{2ab}{h}$ (2). Equaling we have $\frac{h - (b - a)^2}{h} = \frac{2ab}{h}$, whence $h^2 = a^2 + b^2$.

a. Devised by the author Feb. 28, 1926.

b. Here we introduce the circle in finding mean prop'l.

C.—THE CIRCLE IN CONNECTION WITH THE RIGHT TRIANGLE

I.—Through the Use of One Circle.

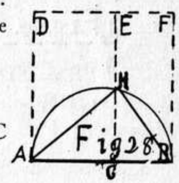
From certain Linear Relations of the Chord, Secant and Tangent in conjunction with a right triangle, or with similar related right triangles, it may also be proven that: *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*

And since the algebraic is the measure or transliteration of the geometric square the truth by any proof through the algebraic method involves the truth of the geometric method.

Furthermore these proofs through the use of circle elements are true, not because of straight-line properties of the circle, but because of the law of similarity, as each proof may be reduced to the proportionality of the homologous sides of similar triangles, the circle being a factor only in this, that the homologous angles are measured by equal arcs.

(1) The Method by Chords.

Twenty-Eight



H is any pt. on the semicircle BHA. ∴ the tri. ABH is a rt. triangle. Complete the sq. AF and draw the pert. EHC.

$BH^2 = AB \times BC$ (mean proportional)

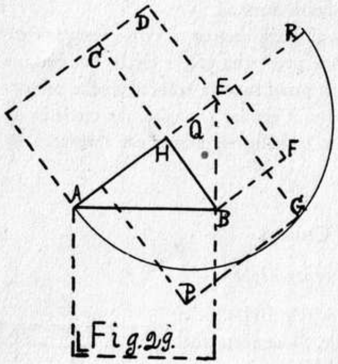
$AH^2 = AB \times AC$ (mean proportional)

Sq. AF = rect. BE + rect. AE = $AB \times BC + AB \times AC = BH^2 + AH^2$.

$\therefore h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., V. 70, p. 383, Dec. 10, 1910. Credited to A. E. Colburn.

Twenty-Nine



Take ER = ED and Bisect HE. With Q as center describe semi-circle AGR. Complete sq. EP. Rect. HD = HC × HE = HA × HE = HB² = sq. HF. EG is a mean proportional between EA and (ER = ED).

∴ sq. EP = rect. AD = sq. AC + sq. HF.

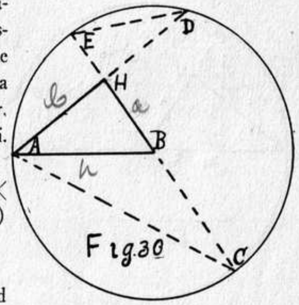
But AB is a mean prop'l between EA and (ER = ED. ∴ EG = AB. ∴ sq. BL = sq. AC + sq. HF.

$\therefore h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., V. 70, p. 359, Dec. 3, 1910. Credited to A. E. Colburn.

Thirty

In any circle upon any diameter, EC in fig. 30, take any distance from the center less than the radius, as BH. At H draw a choral AD perp. to the diameter, and join AB forming the rt. tri. ABH.



a. Now HA × HD = HC × HE, or $b^2 = (h + a)(h - a)$

$\therefore h^2 = a^2 + b^2$.

b. By joining A and C, and E and D, two similar rt. tri's are formed, giving HC : HA = HD : HE, or, again, $b^2 = (h + a)(h - a)$.

$\therefore h^2 = a^2 + b^2$.

But by joining C and D, the tri. DHC = tri. AHC, and since the tri. DEC is a particular case of *One,* fig. 1, as is obvious, the above proof is subordinate to, being but a particular case of the proof of *One.*

c. See Edwards' Geometry, p. 156, fig. 9, and Journal of Education, 1887, V. XXV, p. 404, fig. VII.

Thirty-One

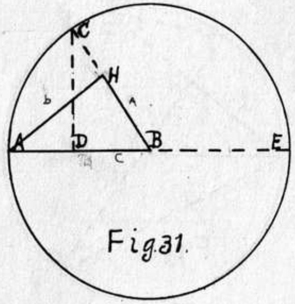


Fig. 31.

The construction is easily perceived.

a. Since CD is a mean prop'l between AD and DE, we have directly, as CD = AH, $b^2 = (h - a)(h + a) = h^2 - a^2$. $\therefore h^2 = a^2 + b^2$.

b. By analysis and comparison it is obvious, by substituting for ABH its equal tri. CBD, that this solution is subordinate to that under 30.

c. See Journal of Education, 1888, V. XXVII, p. 327, 21st proof, or Heath's Math. Monograph, No. 2, p. 30, 17th of the 26 proofs there given.

Thirty-Two

In any circle draw any chord as AC perp. to any diameter as BD, and join A and B, B and C, and C and D, forming the three similar rt. tri's ABH, CBH and DBC.

a. Whence AB : DB = BH : BC, giving AB $\times$ BC = DB $\times$ BH = (DH + HB) BH = DH $\times$ BH + BH^2 = AH $\times$ HC + BH^2; or $h^2 = a^2 + b^2$.

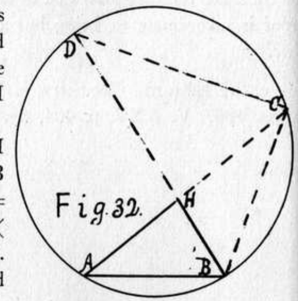


Fig. 32.

b. Fig. 32 is closely related to fig. 30.

c. For solutions see Edwards' Geom. p. 156, fig. 10, Journal of Education, 1887, V. XXVI, p. 21, fig. 14, Heath's Math. Monographs, No. 1, p. 26, and Math. Mo., V. III, p. 300, solution XXI.

Thirty-Three

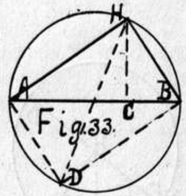


Fig. 33.

The construction is obvious. From the similar triangles HDA and HBC, we have HD : HB = AD : CB, or HD $\times$ CB = HB $\times$ AD (1).

In like manner, from the similar triangles DHB and AHC, HD $\times$ AC = AH $\times$ DB (2). Adding (1) and (2), HD $\times$ AB = HB $\times$ AD + AH $\times$ DB (3).
$\therefore h^2 = a^2 + b^2$.

a. See Halsted's Elementary Geom. for Eq. (3), p. 202. See Edwards' Geom., p. 158, fig. 17, and Math. Mo., V. IV, p. 11.

Thirty-Four

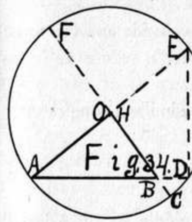


Fig. 34.

In fig. 34, on any diameter construct any rt. tri. as ABH, and produce the sides to chords. Draw ED. Then in the similar rt. tri. ABH and AED.

AB : AE = AH : AD, or h : b + HE = b : h + BD,

$\therefore h(h + BD) = b(b + HE) = b^2 + bHE = b^2 + HF \times HC = b^2 + HC^2$ (1).

Now conceive AD to revolve on A as a center until D coincides with C, when AB = AD = AC = h, BD = 0, and HB = HC = a. Substituting in (1) we have $h^2 = a^2 + b^2$.

a. This is the solution of G. I. Hopkins of Manchester, N. H. See his Plane Geom., p. 92, art. 427, and Jour. of Ed., 1888, V. XXVII, p. 327, 16th prob. Also Heath's Math. Monographs, No. 2, p. 28, proof XV.

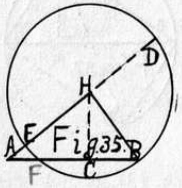
b. Special case. When FC is a diameter we get (1) BC = (b + a) (b − a)/h, and (2) BC = $2b^2$/h − h.

Equating, $\therefore h^2 = a^2 + b^2$.

c. See Math. Mo., V. III, p. 300.

(2) The Method by Secants.

Thirty-Five



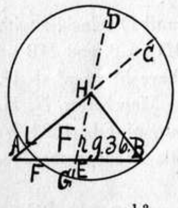
The construction of Fig. 35, has the vertex of the rt. angle of the given rt. tri. at the center of the circle whose radius is the shorter leg.

The secants and their external segments bring reciprocally proportional, (see any plane geometry), letting F be the pt. where the circle intersects the line AB, we have, AD : AB = AF : AE, or $b + a : h = (h - 2CB = h - \frac{2a^2}{h}) : b - a$, whence $h^2 = a^2 + b^2$.

a. In case $b = a$, the points A, E and F coincide and the proof still holds; for substituting b for a the above prop'n reduces to $h^2 - 2a^2 = 0$; $\therefore h^2 = 2a^2$ as it should.

b. By joining E and B, and F and D, the similar triangles upon which the above rests are formed.

Thirty-Six



In fig. 36 E the middle pt. of AB will fall between A and F, at F, or between F and B, as HB is less than, equal to, or greater than HE. Hence there are three cases; but investigation of one case—when E falls between F and B—is sufficient. Join L an B; and F and C, making the two similar triangles AFC and ALB, whence $h : b + a = b - a : AF$; $\therefore AF = \frac{b^2 - a^2}{h}$ (1).

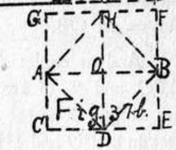
Join F and G, and B and D making the two similar tri's FGE and BDE, whence $\frac{1}{2}h : a - \frac{1}{2}h = a + \frac{1}{2}h : FE$, whence FE =

$\frac{a^2 - \frac{1}{4}h^2}{\frac{1}{2}h}$, (2). Adding (1) and (2) gives $\frac{1}{2}h = \frac{a^2 + b^2 - \frac{1}{2}h^2}{h}$ whence $h^2 = a^2 + b^2$.

a. The above solution is given by Krueger, in "Aumerkungen uber Hrn. geh. R. Wolf's Auszug aus der Geometrie," 1746. Also see Jury Whipper, p. 41, fig. 42, and Math. Mo., V. IV, p. 11.

b. When G falls midway between F and B, then fig. 36 becomes fig. 35. Therefore cases 35 and 36 closely related.

Thirty-Seven



In fig. 37a, take HF = HB. With B as center, and BF as radius describe semicircle DEG, G being the pt. where the circle intersects AB. Produce AB to D, and draw FG, FB, BE to AH produced, and DE, forming the similar tri's AGF and AED, from which $(AG = x) : (AF = y) = (AE = y + 2FH) : (AD = x + 2BG) = y + 2z : x + 2r$ whence $x^2 + 2rx = y^2 + 2yz$ (1).

But if, see fig. 37b, HA = HB, (sq. $GE = h^2$) = (sq. $HB = a^2$) + (4 tri. AHG = sq. $HA = b^2$), whence $h^2 = a^2 + b^2$; then, (see fig. 37a) when BF = BG, we will have $BG^2 = HB^2 + HF^2$, or $r^2 = z^2 + z^2$, (since z = FH). (2).

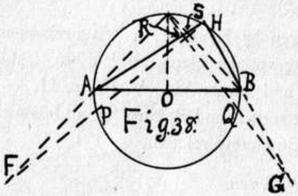
(1) + (2) = (3) $x^2 + 2rx + r^2 = y^2 + 2yz + z^2 + z^2$

or (4) $(x + r)^2 = (y + z)^2 + z^2$

$\therefore$ (5) $h^2 = a^2 + b^2$, since $x + r = AB = h$, $y + z = AH = b$, and $z = HB = a$.

a. See Jury Whipper, p. 36, where Whipper also credits it to Joh. Hoffmann. See also Whipper, p. 37, fig. 34, for another statement of same proof.

Thirty-Eight



In fig. 38, in the circle whose center is O, and whose diameter is AB, erect the perp. DO, D being the mid-point of the arc BHA, join D to A and B, produce DA to F, making AF = AH, and produce HB to G making BG = BD, thus forming the two isosceles tri's FHA and DGB; also the two isosceles tri's ARD and BHS. As angle DAH = 2 angle at F, and angle HBD = 2 angle at G, and as angle DAH and angle HBD are measured by same arc HD, then angle at F = angle at G. ∴ arc AP = arc QB.

And as angles ADR and BHS have same measure, ½ of arc APQ, and ½ of arc BQP, respectively, then tri's ARD and BHS are similar, R is the intersection of AH and DG, and S the intersection of BD and HF. Now since tri's FSD and GHR are similar, being equiangular, we have, DS : DF = HR : HG. ∴ DS : (DA + AF) = HR : (HB + BG),

∴ DS : (DA + AH) = HR : (HB + BD),

∴ DS : (2BR + RH) = HR : (2BS + SD)

∴ (1) $DS^2 + 2DS \times BS = HR^2 + 2HR \times BR$.

And (2) $HA^2 = (HR + RA)^2 = HR^2 + 2HR \times RA + RA^2$
$= HR^2 + 2HR \times RA + AD^2$

(3) $HB^2 = BS^2 = (BD - DS)^2 = BD^2 - 2BD \times DS + DS^2$
$= AD^2 - (2BD \times DS - DS^2)$
$= AD^2 - 2\ (BS + SD)\ DS + DS^2$
$= AD^2 - 2BS \times SD - 2DS^2 + DS^2$
$= AD^2 - 2BS \times DS - DS^2 = AD^2 - (2BS \times DS - DS^2)$

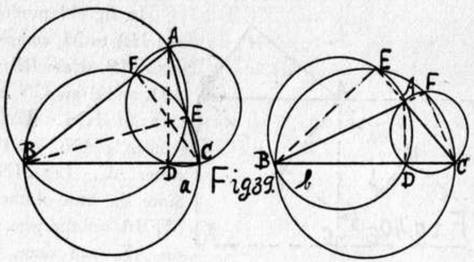
(2) + (3) = (4) $HB^2 + HA^2 = 2AD^2$. But as in proof, fig. 37b, we found, (eq. 2), $r^2 = z^2 + z^2 = 2z^2$.

∴ $2AD^2$, (in fig. 38) $= AB^2$.

∴ $h^2 = a^2 + b^2$.

a. See Jury Whipper, p. 44, fig. 43, and there credited to Joh. Hoffmann, one of his 30 solutions.

Thirty-Nine



In fig. 39, let BCA be any triangle, and let AD, BE and CF be the three perpendiculars from the three verticles, A, B and C, to the three sides, BC, CA and AB, respectively. Upon AB, BC and CA as diameters describe circumferences, and since the angles ADC, BEC and CFA are rt. angles, the circumferences pass through the points D and E, F and E, and F and D, respectively.

Since BC × BD = BA × BF, CB × CD = CA × CE, and AB × AF = AC × AE, therefore

$[BC \times BD + CB \times CD = BC\ (BD + CD) = BC^2]$
$= [BA \times BF + CA \times CE = BA^2 \pm AB \times AF + CA^2 \pm AC \times AE = AB^2 + AC^2 + 2AB \times AF\ (\text{or } 2AC \times AE)]$.

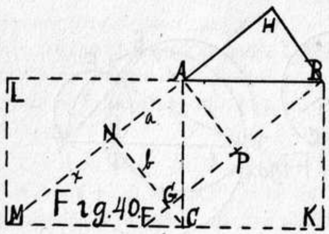
When the angle A is acute (see fig. a) or obtuse (see fig. b) the sign is — or + respectively.

And as angle A approaches 90°, AF and AE approach 0, and at 90° they become 0, and we have $BC^2 = AB^2 + AC^2$.

$\therefore$ when A = a rt. angle $h^2 = a^2 + b^2$.

a. See Olney's Elements of Geometry, University Edition, Part III, p. 252, art. 671, and Heath's Math. Monographs, No. 2, p. 35, proof XXIV.

Forty



In fig. 40 produce KC and HA to M, complete the rect. MB, draw BF par. to AM, and draw CN and AP perp to HM. Draw the semicircle ANC on the diameter AC. Let MN = x. Since the area of the paral. MFBA = the area of the sq AK, and since, by the Theorem for the measurement of a parallelogram, (see fig. 204a), we have (1) sq. AC = (BF × AP = AM × AP) = a(a + x). But, in tri. MCA, CN is a mean proportional between AN and NM.

$\therefore$ (2) $b^2 = ax$.

(1) — (2) = (3) $h^2 - b^2 = a^2 + ax - ax = a^2$.

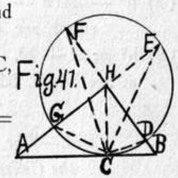
$\therefore h^2 = a^2 + b^2$.

a. This proof is No. 99 of A. R. Colburn's 108 solutions, being devised Nov. 1, 1922.

(3) THE METHOD BY TANGENTS.

1st.—The Hypotenuse As A Tangent.

Forty-One



The construction of fig. 41 is evident, and side b may be >, =, or < a.

From the similar triangles ACG and AEC, we have,

AC : AE = AG : AC, or AC : b + r = b — r : AC;

$\therefore$ (1) $AC^2 = b^2 - r^2$.

From the similar tri's CBD and BFC, we get

(2) $CB^2 = a^2 - r^2$.

From the similar rt. tri's BCH and HCA, we get

(3) $BC \times AC = r^2$

$\therefore$ (4) $2BC \times AC = 2r^2$. (1) + (2) + (4) gives

(5) $AC^2 + 2AC \times BC + BC^2 = a^2 + b^2 = (AC + BC)^2 = AB^2$

$\therefore h^2 = a^2 + b^2$.

a. See Math. Mo., V. III, p. 300.

Forty-Two

Having constructed fig. 42, from the similar tri's ACD and AHC, we get, calling OC = r, (AC = h — a) : (AH = b) = (AD = b — 2r) : (AC = h — a)

$\therefore$ (1) $(h - a)^2 = b^2 - 2br$. But

(2) $a^2 = a^2$.

(1) + (2) = (3) $(h - a)^2 + a^2 = a^2 + b^2 - 2br$, or $(h - a)^2 + 2br + a^2 = a^2 + b^2$

Also (AC = h — a) : (AH = b) = (OC = OH = r) : (HB = a), whence

(4) (h — a) a = br.

$\therefore$ (5) $(h - a)^2 + 2(h - a)r + a^2 = a^2 + b^2$

$\therefore$ (6) $h^2 = a^2 + b^2$.

or, in (3) above, expand and factor gives

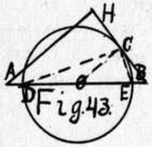
(7) $h^2 - 2a(h - a) = a^2 + b^2 - 2br$. Sub. for $a(h - a)$ its equal, see (4) above, and collect, we have

(8) $h^2 = a^2 + b^2$.

a. See Math. Mo. V. IV, p. 81.

2nd.—The Hypotenuse A Secant Which Passes Through the Center of the Circle and One or Both Legs Tangents.

Forty-Three



The construction is evident, having BH, the shorter leg a tangent. From the similar tri's BCE and BDC, we get, BC : BD = BE : BC, whence $BC^2 = BD \times BE = (BO + OD)BE = (BO + OC)BE$. (1)

From similar tri's OBC and ABH, we get

OB : AB = OC : AH, whence $OB/h = r/b$; $\therefore BO = \frac{hr}{b}$ (2)

BC : BH = OC : AH, whence $BC = \frac{ar}{b}$ (3)

Substituting (2) and (3) in (1), gives,

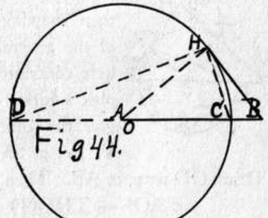
$$\frac{a^2 r^2}{b^2} = \left(\frac{hr}{b} + r\right)BE = \left(\frac{hr + br}{b}\right)(BO - OC) = \left(\frac{hr + br}{b}\right)\left(\frac{hr + br}{b}\right) \quad (4)$$

whence $h^2 = a^2 + b^2$.

Special cases of no. 43 often met with are:

(a) When, in fig. 43, O coincides with A.

Forty-Four

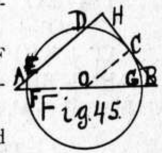


From the similar triangles BHC and BDH, we get, h — b : a = a : h + b, whence directly $h^2 = a^2 + b^2$.

a. This case is found in: Heath's Math. Monographs, No. p. 22, proof VII, Hopkins' Plane Geom. p. 92, fig. IX, Journal of Education, 1887, V. XXVI, p. 21, fig. VIII, Math. Mo., V. III, p. 229, Jury Whipper, 1880, p. 39, fig. 39, where he says it is found in Hubert's Elements of Algebra, Wurceb, 1792, also in Whipper, p. 40, fig. 40, as one of Joh. Hoffmann's 30 proofs. Also by Richardson in Runkle's Mathematical (Journal) Monthly, No. 11, 1859. Many persons, independent of above sources, have found this proof.

(b) When O, fig. 44, is the middle point of AB.

Forty-Five



In this case let HB < HA, and employ tangent HC and secant HE, whence $HC^2 = HE \times HD = AD \times AE = AG \times AF = BF \times BG = BC^2$. Now employing like argument as in case 43 above, we get $h^2 = a^2 + b^2$.

(c) When O is the middle point of AB, and HB = HA, HB and HA are tangents, and AG = BF, secants. Argument same as (b), by applying theory of limits.

(d) When O is any pt. in AB, and the two legs are tangents. This is only another form of No. 43 above, the general case. But as the general case gives, see proof, case above, $h^2 = a^2 + b^2$, therefore the special *must* be true, whence in this case (c) $h^2 = a^2 + b^2$. Or if a *proof* by explicit argument is desired, proceed as in case 43.

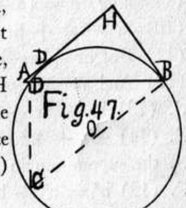
Forty-Six

By proving the general case, as in fig. 43, and then showing that some case is only a particular of the general, and therefore true immediately, is here contrasted with the following long and complex solution of this (d) particular case. I now give the solution found in The Am. Math. Mo., V. IV, p. 80.

"Draw OD perp. to AB. Then, $AT^2 = AE \times AF = AO^2 - EO^2$
$= AO^2 - TH^2$ (1)
$BP^2 = BF \times BE = BO^2 - FO^2 = BO^2 - HP^2$ (2)
Now, AO : OT = AD : OD;
$\therefore AO \times OD = OT \times AD$.
And, since OD = OB, OT = TH = HP, and AD = AT + TD
= AT + BP.
$\therefore AT \times TH + HP \times BP = AO \times OB$ (3)
Adding (1), (2), and $2 \times$ (3),
$AT^2 + BP^2 + 2AT \times TH + 2HP \times BP = AO^2 - TH^2 +$
$BO^2 - HP^2 + 2AO \times OB$;
$\therefore AT^2 + 2AT \times TH + FH^2 + BP^2 + 2BP \times HP + HP^2 =$
$AO^2 + 2AO \times OB + BO^2$.
$\therefore (AT + TH)^2 + (BP + CP)^2 = (AO + OB)^2$.
$\therefore AH^2 + BH^2 = AB^2$." Q. E. D.
$\therefore h^2 = a^2 + b^2$.

3rd.—The Hypotenuse A Secant Not Passing Through the Center of the Circle, and Both Legs Tangents.

Forty-Seven



Through B draw BC parallel to HA, making BC = 2BH; with O, the middle point of BC, as center, describe a circumference, tangent at B and E, (In the fig., the D on AH should be E.) and draw CD, forming the two similar rt. tri's ABH and BDC, whence BD : (AH = b) = (BC = 2a) : (AB = h)

from which, $DB = \frac{2ab}{h}$. (1)

Now, by the principal of tang and sec relations,
$(AE^2 = [b - a]^2) = (AB = h)\ (AD = h - DB)$, whence

$$DB = h - \frac{(b - a)^2}{h} \quad (2)$$

Equating (1) and (2) gives $h^2 = a^2 + b^2$.

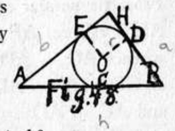
a. If the legs HB + HA are equal, by theory of limits same result obtains.

b. Math. Mo., V. IV, p. 81, No. XXXII.

c. See twenty-seventh proof above, and observe that this forty-seventh proof is superior to it.

4th.—Hypotenuse and Both Legs Tangents.

Forty-Eight



The construction, fig. 48, has the three sides of the rt. tri. ABH all tangents. Denote AB by h, BH by a and HA by b; also OD by r.

Now, (1) $h + 2r = a + b$.
(2) $h^2 + 4hr + 4r^2 = a^2 + 2ab + b^2$.
(3) Now if $4hr + 4r^2 = 2ab$, then $h^2 = a^2 + b^2$.
(4) Suppose $4hr + 4r^2 = 2ab$.
(5) $4r(h + r) = 2ab$; $\therefore 2r(h + r) = ab$.
(1) = (6) $2r = a + b - h$. (6) in (5) gives
(7) $(a + b - h)\ (h + r) = ab$.

(8) $h(a + b - h - r) + ar + br = ab$.
(1) = (9) $r = (a + b - h - r)$. (9) in (8) gives
(10) $hr + ar + br = ab$.
(11) But $hr + ar + br = 2$ area tri. ABC
(12) And $ab = 2$ area tri. ABC
$\therefore$ (13) $hr + ar + br = ab$
$\therefore$ (14) $4hr + 4r^2 = 2bc$
$\therefore$ the supposition in (4) is true.
$\therefore$ (15) $h^2 = a^2 + b^2$.

a. This solution was devised by the author Dec. 13, 1901, *before* I received Vol. VIII, 1901, p. 258, Am. Math. Mo., where a like solution is given.

b. By drawing a line OC, in fig. 48, we have the geom. fig. from which, May, 1891, Dr. L. A. Bauer, of Carnegie Institute, Wash., D. C., deduced a proof through the equations

(1) Area of tri. $ABH = \frac{1}{2}r(h + a + b)$, and

(2) $HD + HE = a + b - h$. See pamphlet: On Rational Right-Angled Triangles, Aug., 1912, by Artemus Martin for the Bauer proof. In same pamphlet is still another proof attributed to Lucius Brown of Hudson, Mass.

b. For this same fig. 48 there is another proof known as the "Harmonic Proportion Proof."

From the similar tri's AHF and ADH,
$AH : AD = AF : AH$, or $AC : AD = AF : AE$
whence $AC + AD : AF + AE = AD : AE$
or $CD : CF = AD : AE$,
and $AC - AD = AF - AE = AD : AE$,
or $DE : EF = AD : AE$.
$\therefore CD : CF = DC : EF$.
or $(h + b - a) : (h + b + a) = (a - h + b) : (a + h + b)$
$\therefore$ by expanding and collecting, we get
$h^2 = a^2 + b^2$.

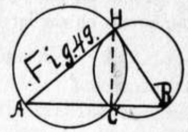
c. See Olney's Elements of Geometry, University Edition, p. 312, art. 971, or Schuyler's Elements of Geometry, p. 353, exercise 4. Also Math. Mo., V. IV, p. 12, proof XXVI.

Remark.—By ingenious devices, some if not all, of these in which the circle has been employed can be proved without the use of the circle—not nearly so easily perhaps, but proved. The figure, without the circle, would suggest the device to be employed. By so doing new proofs may be discovered.

II.—Through the Use of Two Circles.

Forty-Nine

Construction. Upon the legs of the rt. tri. ABH, as diameters construct circles and draw HC, forming three similar rt. tri. ABH, HBC and HAC.

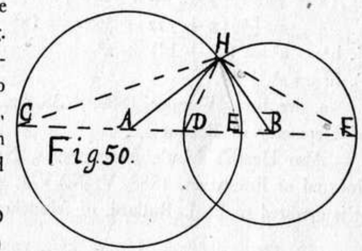


Then we get $h : b = b : AC \therefore hAC = b^2$ (1)
Also we get $h : a = a : BC$. $\therefore hBC = a^2$ (2)
(1) + (2) = (3) $h^2 = a^2 + b^2$.

a. See Edwards' Elements of Geom., p. 161, fig. (34) and Math. Mo., V. IV, p. 11.

Fifty

With the legs of the rt. tri. ABH as radii (fig. 50) describe circumferences, and extend AB to C and F. Draw HC, HD, HE and HF. From the similar tri's AHF and HDH,



$AF : AH = AH : AD$
$\therefore b^2 = AF \times AD$ (1)

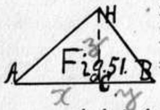
From the similar tri's CHB and HEB,
$CB : HB = HB : BE \therefore a^2 = CB \times BE$ (2)
(1) + (2) = (3) $a^2 + b^2 = CB \times BE + AF \times AD$
$= (h + b)(h - b) + (h + a)(h - a)$
$= h^2 - b^2 + h^2 - a^2$;
or (4) $2h^2 = 2a^2 + 2b^2 \therefore h^2 = a^2 + b^2$.

a. Math. Mo., V. IV, p. 12, and also on p. 12 is a proof by Richardson. But it is much more difficult than the above method.

D.—RATIO OF AREAS

As in the three preceeding divisions, so here in D we must rest our proofs on similar rt. triangles.

Fifty-One



In the figure, draw HC perp. to AB, forming the three similar triangles, ABH, AHC and HBC, and let AB = h, BH = a, HB = b, CA = x, CB = y, and HC = z.

Since similar surfaces are proportional to the squares of their homologous dimensions, therefore,

$[\frac{1}{2}(x + y)z \div \frac{1}{2}yz = h^2 \div a^2] = [\frac{1}{2}yz \div \frac{1}{2}xz = a^2 \div b^2]$
$= [\frac{1}{2}(x + y)z \div \frac{1}{2}yz = (a^2 + b^2)\ a^2]$
$\therefore h^2 \div a^2 = (a^2 + b^2) \div a^2$
$\therefore h^2 = a^2 + b^2$.

a. See Jury Whipper, 1880, p. 38, fig. 36 as found in Elements of Geometry of Bezout;

Also Heath's Math. Monographs, No. 2, p. 29, proof XVI; Journal of Education, 1888, V. XXVII, p. 327, 19th proof, where it is credited to L. J. Bullard, of Manchester, N. H.

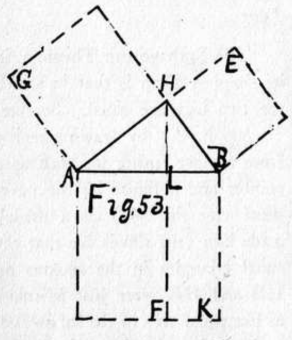
Fifty-Two

In fig. 52 draw AG and BG par. respectively to BH and AH, and through H draw FE par. to AB, and complete the rect. on AB, thus forming the similar tri's BHE, HAF and BAG. Denote the areas of these tri's by x, y and z respectively.

Then $z : y : x = h^2 : a^2 : b^2$
But it is obvious that $z = x + y$
$\therefore h^2 = a^2 + b^2$.

a. Original with the author, March 26, 1926, 10 p. m.

Fifty-Three



The construction of fig. 53 is evident. Since the triangles ABH, AHL, and HBL are similar, so also the squares AK, BE and HG, and since similar polygons are to each other as the squares of their homologous dimensions, we have
tri. ABH : tri. HBL : tri. AHL
$= h^2 : a^2 : b^2$
= sq. AK : sq. BE : sq. HG.
But tri. ABH = tri. HBL + tri. AHL.
$\therefore$ sq. AK = sq. BE + sq. HG $\therefore h^2 = a^2 + b^2$.

a. Devised by the author, July 1, 1901.

b. Another solution by the author is:

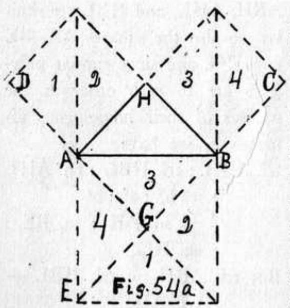
[...]se, by equation (5), see fig. 1, proof *One*, $BH^2 = BA \times BL =$ [...]. LK, and in like manner, $AH^2 = AB \times AL =$ rect. AF,

$\therefore$ sq. BK = rect. LK + rect. AF = sq. BE + sq. HG,
$\therefore h^2 = a^2 + b^2$.

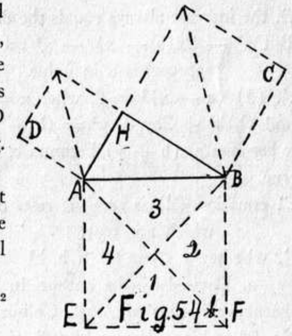
c. This principle of "mean proportional" can be made use of in many of the here-in-after figures among the Geometric Proofs, thus giving variations as to the proof of said figures. Also many other figures may be constructed based upon the use of the "mean proportional" relation; hence all such proofs, since they result from an algebraic relationship of corresponding lines of similar triangles, must be classed as algebraic proofs.

E.—ALGEBRAIC PROOF, THROUGH THEORY OF LIMITS.

Fifty-Four



The Pythagorean Theorem, in its simplest form is that in which the two legs are equal. Socrates (b. 500 B. C.), by drawing replies from a slave, using his staff as a pointer and a figure on the pavement (see fig. 54a) as a model, made him (the slave) see that the equal triangles in the squares on HB and HA were just as many as like equal tri's in the sq. on AB, as is evident by inspection. (See Plato's Dialogues, Meno. Vol. I, pp. 256-260, Edition of 1883, Jowett's translation, Chas. Scribner and Sons).

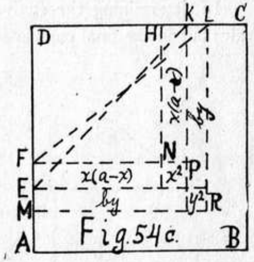


Starting with fig. 54a, and decreasing the length of AH, which necessarily increases the length of HB, since AB remains constant, we decrease the sq. HD and increase the sq. HC (see fig. 54b).

Now we are to prove that the sum of the two variable squares, sq. HD and sq. HC will equal the constant sq. AF.

We have, fig. 54a, $h^2 = a^2 + b^2$ (1)

But let side AH, fig. 54a, be diminished as by x, thus giving AH, fig. 54b, or better, FD, fig 54c, and let DK be increased by y, as determined by the hypotenuse h remaining constant,



Now, fig. 54c, when a = b,
$a^2 + b^2 = 2$ area of sq. DP,
And when $a < b$, we have
$(a - x)^2$ = area of sq. DN, and
$(b + y)^2$ = area of sq. DR.
Also $c^2 - (b + y)^2 = (a - x)^2$
= area of MABCLR, or
$(a - x)^2 + (b + y)^2 = c^2$ (2).
Is this true? Suppose it is; then, after reducing (2) — (1) = (3)
$-2ax + x^2 + 2by + y^2 = 0$
or (4) $2ax - x^2 = 2by + y^2$, which shows that the area by which (a^2 = sq. DP) is diminished = the area by which b^2 is increased. See graph 54c.

DA = AB = c
DE = DK = a = b
DF = a — x
DL = b + y
FE = HK = x
KL = EM = y
EK = FL = h

∴ the increase always equals the decrease.

But $a^2 - 2x(a - x) - x^2 = (a - x)^2$ approaches 0 when x approaches a in value.

∴ (5) $(a - x)^2 = 0$, when $x = a$, which is true and (b) $b^2 + 2by + y^2 = (b + y)^2 = c^2$, when $x = a$, for when x becomes a, $(b + y)$ becomes c, and so, we have $c^2 = c^2$ which is true.

∴ equation (2) is true; it rests on the eq's (5) and (6), both of which are true.

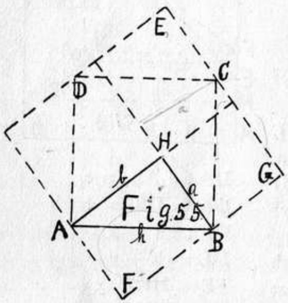
∴ whether $a <= \text{or} > b$, $h^2 = a^2 + b^2$.

a. Devised by the author, in Dec. 1925. Also a like proof to the above is that of A. R. Colburn, devised Oct. 18, 1922, and is No. 96 in his collection of 108 proofs.

F.—ALGEBRAIC-GEOMETRIC PROOFS.

In determining the equivalency of areas these proofs are algebraic; but in the final comparison of areas they are geometric.

Fifty-Five



The construction, see fig. 55, being made, we have sq. FE $= (a + b)^2$.

But sq. FE = sq. AC + 4 tri. ABH

$= h^2 + 4\frac{ab}{2} = h^2 + 2ab$.

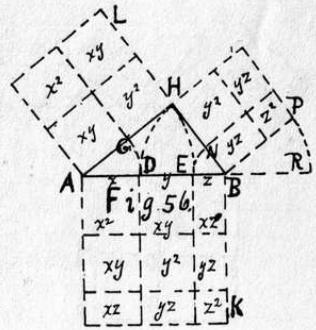
Equating, we have

$h^2 + 2ab = (a + b)^2 = a^2 + 2ab + b^2$

∴ $h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., V. 70, p. 382, Dec. 10, 1910, credited to A. R. Colburn, Washington, D. C.

Fifty-Six



With A as center and AH as radius describe arc HE: with B as center and BH as center describe arc HD; with B as center describe arc EN; with A as center describe arc GD. Draw the parallel lines as indicated. By inspecting the figure it becomes evident that if $y^2 = 2xz$, then the theorem holds. Now, since AH is a tangent and AR is a chord of same circle,

$AH^2 = AR \times AD$, or $(x + y)^2 = x(2y + 2z) = x^2 + 2xy + 2xz$.

Whence $y^2 = 2xz$.

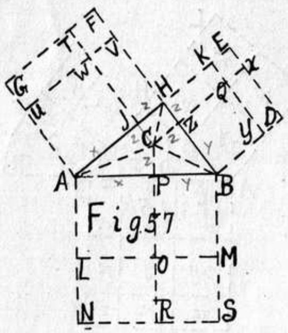
∴ sq. AK $= [(x^2 + y^2 + 2xy) = \text{sq. AL}] + [(z^2 + 2yz + (2xz = y^2)] = \text{sq. HP}$.

∴ $h^2 = a^2 + b^2$.

a. See Sci. Am. Supt., V. 84, p. 362, Dec. 8, 1917, and credited to A. R. Colburn. It is No. 79 in his (then) 91 proofs.

b. This proof is a fine illustration of the flexibility of geometry. Its value lies, not in a repeated proof of the many times established fact, but in the effective marshaling and use of the elements of a proof, and even more also in the better insight which it gives us to the interdependence of the various theorems of geometry.

Fifty-Seven



Draw the bisectors of angles A, B and H, and from their common point C draw the perp's CR, CX and CT; take AL = AU = AP, and BZ = BP, and draw lines UV par. to AH, LM par. to AB and KY par. to BH. Let AJ = AP = x, BZ = BP = y, and HZ = HJ = z = CJ = CP = CZ.

Now 2 tri. ABH = HB × HA
= (x + z) (y + z)
= xy + xz + yz + z²
= rect. PM + rect. HW + rect. HQ + sq. KX.

But 2 tri. ABH = 2AP × CP + 2BP × CP + (2 sq. HC = 2PC²)
= 2xz + 2yz + 2z²
= 2 rect. HW + 2 rect. HQ + 2 sq. KX.

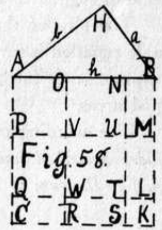
∴ rect. PM = rect. HW + rect. HQ + sq. KX.

Now sq. AS = (sq. AO = sq. AW) + (sq. OS = sq. BQ) + (2 rect. PM = rect. HW + 2 rect. HQ + 2 sq. HK)
= sq. HG + sq. HD. ∴ $h^2 = a^2 + b^2$.

a. This proof is due to Mr. F. S. Smedley, a photographer, of Berea, O., June 10, 1901.

Also see Jury Whipper, 1880, p. 34, fig. 31, credited to E. Mollmann, as given in "Archives d. Mathematik, u. Ph. Grunert," 1851, for fundamentally the same proof.

Fifty-Eight



In fig. 58 take AN and AQ = AH, and KM and KR = BH, and through N and R, and Q and M, draw NS and RO par. to KB, and QL par. to KB, and QL and MP par. to AB. Then it follows that

CK = h, CS = b, RK = a, CR = h — a,
SK = h — b, and RS = a + b — h.

Now sq. AK = CK² = CS² + RK² — RS² + 2CR × SK, or

$h^2 = b^2 + a^2 - (a + b - h)^2 + 2(h - a)(h - b)$
$= b^2 + a^2 - a^2 - b^2 - h^2 - 2ab + 2ah + 2bh + 2h^2 - 2ah - 2bh + 2ab = h^2.$

∴ 2CR × SK = RS², or
$2(h - a)(h - b) = (a + b - h)^2$, or
$2h^2 + 2ab - 2ah - 2bh = a^2 + b^2 + h^2 + 2ab - 2ah - 2bh.$
∴ $h^2 = a^2 + b^2$.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with author Apr. 23, 1926, 2 p. m.

II.—GEOMETRIC PROOFS.

All geometric proofs must result from the comparison of areas—the foundation of which is superposition.

As the possible number of algebraic proofs has been shown to be limitless, so it will be conclusively shown that the possible number of geometric proofs through dissection and comparison of congruent or equivalent areas is also "absolutely unlimited."

The geometric proofs are classified under ten type-forms, as determined by the figure, and only a limited number under each type will be given, among them being the more important, the better known and the recently devised or new ones.

The references to the authors in which the proof, or figure, is found or suggested, are arranged chronologically so far as possible.

The idea of throwing the suggested proof into the form of a single equation is my own; by means of it every essential element of the proof is set forth, as well as the comparison of the equivalent or equal areas.

The wording of the theorem for the geometric proof is: *The square described upon the hypotenuse of a right-angled triangle is equal to the sum of the squares described upon the other two sides.*

TYPES

It is obvious that the three squares constructed upon the three sides of a right-angled triangle can have eight different positions, as per selections. Let us designate the square upon the hypotenuse by *h,* the square upon the shorter side by *a,* and the square upon the other side by *b,* and set forth the eight arrangements; they are:

1. All squares h, a and b exterior.
2. a and b exterior and h interior.
3. h and a exterior and b interior.
4. h and b exterior and a interior.
5. a exterior and h and b interior.
6. b exterior and h and a interior.
7. h exterior and a and b interior.
8. All squares h, a and b interior.

By exterior is meant constructed outwardly.

By interior is meant constructed overlapping the given right triangle.

The arrangement designated above constitutes the first eight of the following ten geometric types.

Also for some selected figures for proving Euclid I, 47, the reader is referred to H. d'Andre, N. H. Math. (1846) Vol. 5, p. 324.

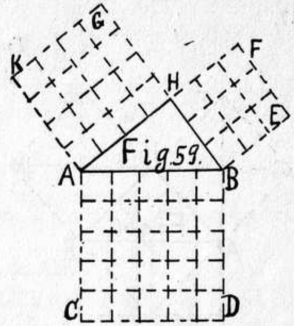
A.

This type includes all proofs derived from the figure determined by constructing squares upon each side of a right-angled triangle, each square being constructed outwardly from the given triangle.

The proofs under this type are classified as follows:

(a) Those proofs in which pairs of the dissected parts are congruent.

One



Particular case — illustrative rather than demonstrative.

The sides are to each other as, 3, 4, 5 units. Then sq. AD contains 25 sq. units, HE 9 sq. units and HK 16 sq. units. Now it is evident that the no. of unit squares in the sq. AD = the sum of the unit squares in the squares HE and HK.

$\therefore$ square AD = sq. HE + sq. HK.

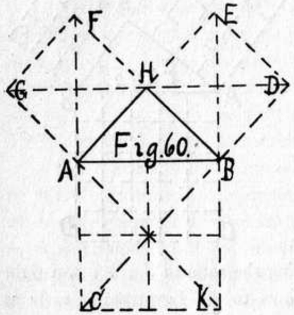
a. That by the use of the lengths 3, 4, and 5, or length having the ratio of 3 : 4 : 5, a right angled triangle is formed was known to the Egyptians as early as 2000 B. C., for at that time there existed professional "rope-fasteners"; they were employed to construct right angles which they did by placing three pegs so that a rope measuring off 3, 4 and 5 units would just reach around them. This method is in use today by carpenters and masons; sticks 6 and 8 feet long from the two sides and a "ten-foot" stick forms the hypotenuse, thus completing a right-angled triangle, hence establishing the right angle.

But granting that the early Egyptians formed right angles in the "rule of thumb" manner described above, it does not follow, in fact it is not believed, that they knew the area of the square upon the hypotenuse to be equal to the sum of the areas of the squares upon the other two sides.

The discovery of this fact is credited to Pythagoras, a renowned philosopher and teacher, born at Samos about 570 B. C., after whom the theorem is called "The Pythagorean Theorem." (See p. 25).

b. See Hill's Geometry for Beginners, p. 153; Ball's History of Mathematics, pp. 7-10; Heath's Math. Monographs, No. 1, pp. 15-17.

Two



Another particular case is illustrated by fig. 60, in which BH = HA, showing 16 equal triangles.

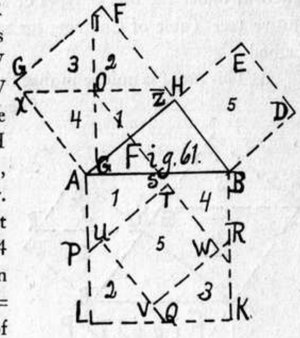
Since the sq. AK contains 8 of these triangles,

∴ sq. AK = sq. HD + sq. HG.

a. See Beeman and Smith's New Plane and Solid Geometry, p. 103, fig. 1.

b. For this and many other demonstrations by dissection, see H. Perigal, in Messenger of Mathematics, 1873, V. 2, p. 103.

Three



In fig. 61, through P, Q, R and S, the centers of the sides of the sq. AK draw PT and RV par. to AH, and QU and SW par. to BH, and through O, the center of the sq. HG, draw XH par. to AB and IY (In fig. 61, change G on line AH to Y.) par. to AL, forming 8 congruent quadrilaterals; viz., 1, 2, 3 and 4 in sq. AK, and 1, 2, 3 and 4 in sq. HG, and sq. 5 in sq. AK = sq. (5 = HD). The proof of their congruency is evident, since, in the paral. ⊙B, (SB = SA) = (OH = OG = AP since AP = AS).

(Sq. AK = 4 quad. APTS + sq. TV) = (sq. HG = 4 quad. OYHZ) + sq. HD.

∴ sq. on AB = sq. on BH + sq. on AH.

a. See Mess. Math., Vol. 2, 1873, p. 104, by Henry Perigal, F. R. A. S., etc., MacMillan and Co., London and Cambridge. Here H. Perigal shows the great value of proof by dissection, and suggests its application to other theorms also. Also see Jury Whipper, 1880, p. 50, fig. 46; Ebene Geometrie, Von G. Mahler, Leipzig, 1897, p. 58, fig. 71, and School Visitor, V. III, 1882, p. 208, fig. 1, for a particular application of the above demonstration.

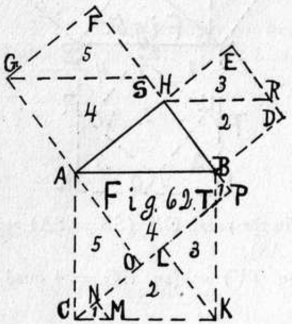
b. See Todhunter's Euclid for a simple proof extracted from a paper by De Morgan, in Vol. I of the Quarterly Journal of Math., and reference is also made there to the work "Der Pythagoraische Lehrsatz," Mainz, 1821, by J. J. I. Hoffmann.

c. By the above dissection any two squares may be transformed into one square, a fine puzzle for pupils in plane geometry.

d. Hence any case in which the three squares are exhibited, as set forth under the first 9 types of II, Geometric Proofs, A to J inclusive (see Table of Contents for said types) may be proved by this method.

d. This proof is unique in that the smaller sq. HD is not dissected.

Four



In fig. 62, on CK construct tri. CKL = tri. ABH; produce CL to P making LP = BH and take LN = BH; draw NM, AO and BP each perp. to CP; at any angle of the sq. GH, as F, construct a tri. GSF = tri. ABH, and from any angle of the sq. HD, as H, with a radius = KM, determine the pt. R and draw HR, thus dissecting the sq's, as per figure.

It is readily shown that sq. AK = (tri. CMN = tri. BTP) + (trap. NMKL = trap. DRHB) + (tri. KTL = tri. HRE) + (quad. AOTB + tri. BTP = trap. GAHS) + tri. ACO = tri. GSF) = (trap. DRHB + tri. HRE = sq. BE) + (trap. GAHS + tri. GSF = sq. AF) = sq. BE + sq. AF.

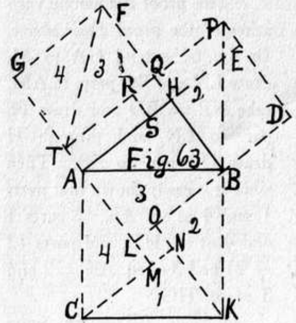
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This dissection and proof were devised by the author to establish the Law of Dissection, by which, no matter how the three squares are arranged, or placed, their resolution into the respective parts as numbered in fig. 62, can be readily obtained.

b. In many of the geometric proofs herein the reader will observe that the above dissection, wholly or partially, has been employed. Hence these proofs are but variation of this general proof.

March 18, 1926. E. S. LOOMIS

Five



In fig. 63 conceive rect. TS cut off from sq. AF and placed in position of rect. QE, AS coinciding with HE; then DEP is a st. line since these rect. were equal by construction. The rest of the construction and dissection is evident.

sq. AK = (tri. CKN = tri. PBD) + (tri. KBO = tri. BPQ) + (tri. BAL = tri. TFQ) + (tri. ACM = tri. FTG) + (sq. LN = sq. RH) = sq. BE + rect. QE + rect. GQ + sq. RH = sq. BE + sq. GH.

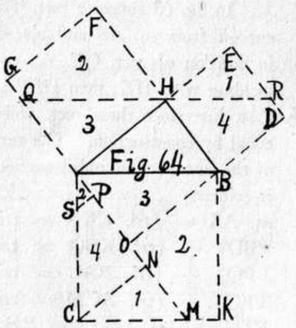
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author after having carefully analyzed the esoteric implications of Bhaskara's "Behold!" proof—see proof *One Hundred Fifty-Four*, fig. 212.

b. The reader will notice that this dissection contains some of the elements of the preceding dissection, that it is applicable to all three-square figures like the preceding, but that it is not so simple or fundamental, as it requires a transposition of one part of the sq. GH, — the rect. TS —, to the sq. HD, — the rect. in position QE— so as to form the two congruent rect's GQ and QD.

c. The student will note that all geometric proofs hereafter, which make use of dissection and congruency, are fundamentally only variations of the proofs established by proofs *Three, Four* and *Five,* and that all other geometric proofs are based, either partially or wholly on the equivalency of the corresponding pairs of parts of the figures under consideration.

Six

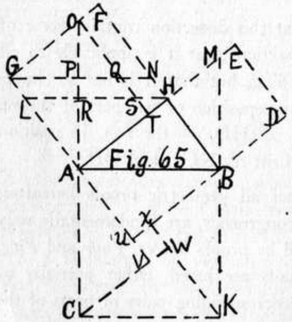


This proof is a simple variation of the proof *Four* above. In fig. 64 extend GA to M, draw LN and BO perp. to AM, take NP = BD and draw PS par. to LN, and through H draw QR par. to AB. Then since it is easily shown that parts 1 and 4 of sq. AK = parts 1 and 4 of sq. HD, and parts (2 + 2) and 3 of sq. AK = 2 and 3 of sq. HG,

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author March 28, 1926 to obtain a figure more readily constructed than fig. 62.

Seven



In fig. 65, produce CA to O, KB to M, GA to V, making AV = AG, DB to U, and draw KX and CW par. resp. to BH and AH, GN and HL par. to AB, and OT par. to FB.

Sq. AK = [tri. CKW = (tri. HLA = trap. BDEM + tri. NST)] + [tri. KBX = tri. GNF = (trap. OQNF + tri. BMH)] + (tri. BAU = tri. OAT) + (tri. ACV = tri. AOG) + (sq. VX = paral.

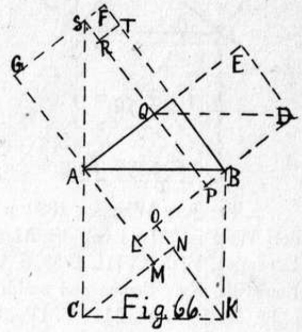
SN) = sq. BE + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with author March 28, 1926, 9:30 p. m.

b. A variation of the proof *Five* above.

Eight



In fig. 66 produce CA to S, draw SP par. to FB, take HT = HB, draw TR par. to HA, produce GA to M, making AM = AG, produce DB to L, draw KO and CN par. resp. to BH and AH, and draw QD.

Rect. RH = rect. QB.

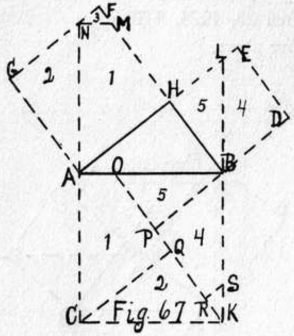
Sq. AK = (tri. CKN = tri. ASG) + (tri. KBO = tri. SAQ) + (tri. BAL = tri. DQP) + (tri. ACM = tri. QDE) + (sq. LN = sq. ST) = rect. PE + rect. GQ + sq. ST = sq. BE + rect. QB + rect. GQ + sq. ST = sq. BE + sq. GH.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with author March 28, 1926, 10 a. m.

b. This is another variation of fig. 63.

Nine



In fig. 67, the dissection is evident and shows that parts 1, 2 and 3 in sq. AK are congruent to parts 1, 2 and 3 in sq. HG; also so that parts 4 and 5 in sq. AK are congruent to parts 4 and 5 in sq. HD.

∴ (sq. AK = parts 1 + 2 + 3 + 4 + 5) = (sq. HG = parts 1 + 2 + 3) + (sq. HD = parts 4 + 5).

∴ sq. on AB = sq. on BH + sq. on AH.

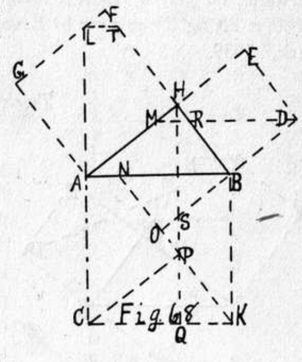
a. See Jury Whipper, 1880, p. 27, fig. 24, as given by Dr. Rudolf Wolf in "Handbook der Mathematik, etc.," 1869; Journal of Education, V. XXVIII, 1888, p. 17, 27th proof, by C. W. Tyron, Louisville, Ky.; Beman and Smith's Plane and Solid Geom., 1895, p. 88, fig. 5; Math. Mo., V. IV, 1897, p. 169, proof XXXIX; and Heath's Math. Monographs, No. 2, p. 33, proof. XXII. Also The School Visitor, V. III, 1882, p. 209, for an application of it to a particular case.

Ten

In fig. 68 the construction is readily seen, as also the congruency of the corresponding dissected parts, from which square AK = (quad. CPNA = quad. LAHT) + (tri. CKP = tri. ALG) + (tri. BOK = quad. DEHR + tri. TFL) + (tri. NOB = tri. RBD).

∴ square upon AB = sq. upon BH + sq. upon AH.

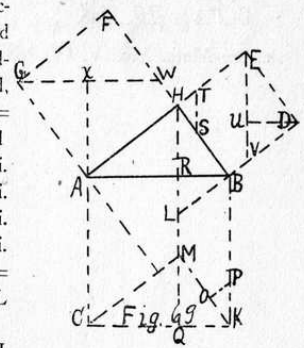
a. See Math. Mo., V. IV, 1897, p. 169, proof. XXXVIII.



Eleven

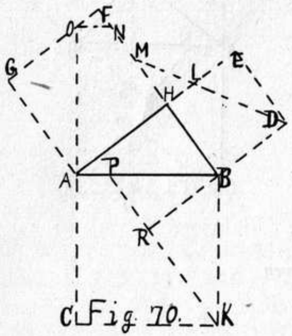
The construction and dissection of fig. 69 is perceived, and the congruency of the corresponding parts can be easily established, and we find that sq. AK = (quad. ANMR = quad AHWX) + (tri. CNA = tri. WFG) + (tri. CQM = tri. AXG) + (tri. MQK = tri. EDU) + (tri. POK = tri. THS) + pentagon BLMOP = pentagon ETSBV) + (tri. BRL = tri. DUV).

∴ sq. upon AB = sq. upon BH + sq. upon AH.



a. Original with the author of this work, August 9, 1900. Afterwards, on July 4, 1901, I found same proof in Jury Whipper, 1880, p. 28, fig. 25, as given by E. von Littrow in "Popularen Geometrie," 1839.

Twelve



In fig. 70, extend CA to O, and draw ON and KP par. to AB and BH respectively, and extend DB to R. Take BM = AB and draw DM. Then we have sq. AB = (trap. ACKP = trap. OABN = pentagon OGAHN) + (tri. BRK = trap. BDLH + tri. MHL = tri. OFN) + (tri. PRB = tri. LED).

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 170, proof. XLIV.

Thirteen

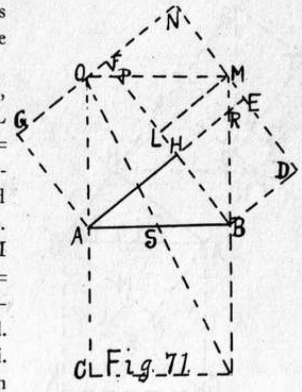
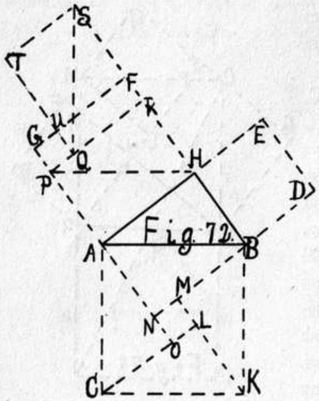


Fig 71 objectifies the lines to be drawn and how they are drawn is readily seen. Since tri. OMN = tri. ABC, tri. MPL = tri. BRH, tri. BML = tri. AOG, and tri. OSA = tri. KSB, (K is the pt. of intersection of the lines MB and OS) then sq. AK = trap. ACKS + tri. KSB = tri. KOM = trap. BMOS + tri. OSA = quad. AHPO + tri. ABH + tri. BML + tri. MPL = quad. AHPO + tri. OMN + tri. AOG + tri. BRH = (pentagon AHPOG + tri. OPF) + (trap. PMNF = trap. RBDE) + tri. BRH = sq. HG + sq. HD.

∴ sq. upon AB = sq. upon HD + sq. upon AH.

a. See Sci. Am. Sup., V. 70, p. 383, Dec. 10, 1910. It is No. 14 of A. R. Colburn's 108 proofs.

Fourteen



In fig. 66, extend GA and DB to O and N, and KM and CL par. to AG and AH, and extend BF to S, making FS = BH; complete the sq. SU and draw NP and PR par. respectively to AB and AH; join SQ. Then it follows that sq. AK = 4 tri. BAN + sq. NL = rect. AR + rect. TR + sq. GQ = rect. AR + rect. QF + sq. GQ + (sq. TF = sq. ND) = sq. HG + sq. HD.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

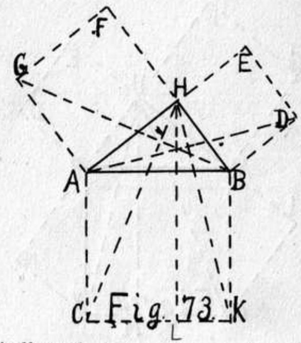
a. This proof is credited to Miss E. A. Coolidge, a blind girl. See Journal of Education, V. XXVIII, 1888, p. 17, 26th proof.

b. The reader will note that this proof employs exactly the same dissection and arrangement as found in the solution by the Hindu mathematician, Bhaskara. See fig. 212, proof *One Hundred Fifty-Four.*

(b) Those proofs in which pairs of the dissected parts are shown to be equivalent.

As the triangle is fundamental in the determination of the equality.. of two areas, Euclid's proof will be given first place.

Fifteen



In fig. 73 draw HL (The perp. from H to CK intersects CK at L.) perp. to CK, and draw HC, HK, AD and BG. Sq. AK = rect. AL + rect. BL = 2 tri. HAC + 2 tri. HBK = 2 tri. GAB + 2 tri. DBA = sq. GH + sq. HD.

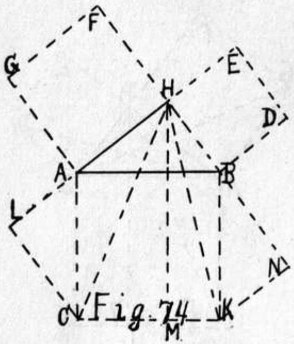
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Euclid, about 300 B. C. discovered the above proof, and it has found a place in every standard text on geometry. Logically no better proof can be devised than Euclid's.

For the old descriptive form of this proof see Elements of Euclid by Todhunter, 1887, Prop. 47, Book I. For a modern model proof, second to none, see Beman and Smith's New Plane and Solid Geometry, 1899, p. 102, Prop. VIII, Book II. Also see Heath's Math. Monographs, No. 1, 1900, p. 18, proof I.

b. I have noticed lately two or three American texts on geometry in which the above proof does not appear. I suppose the author wishes to show his originality or independence—possibly up-to-dateness. He shows something else. The leaving out of Euclid's proof is like the play of Hamlet with Hamlet left out.

Sixteen



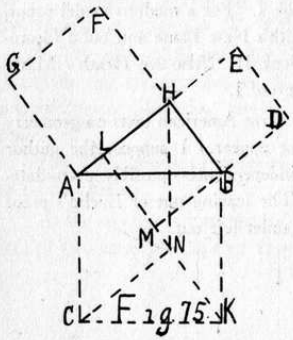
In fig. 74 extend HA and HB and draw the perp's CL and KN; also draw the perp. HM and the lines HC and HK. Then it is evident that tri's ABH, CAL and BKN are equal.

Now sq. BK = rect. AM + rect. BM = 2 tri. HAC + 2 tri. HBK = HA × CL + HB × KN = sq. HG + sq. HD.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Elements of Geom., p. 155, fig. (4).

Seventeen



In fig. 75 draw KL par. to BH, CN par. to AH, extend DB to M, and draw HN. Then it is evident that sq. AK = hexagon ACNKBH = par. ACNH + par. HNKB = AH × LN + BH × HL = sq. HG + sq. HD.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom. 1895, p. 161, fig. (32).

b. In each of the 39 figures given by Edwards the author hereof devised the proof as found herein.

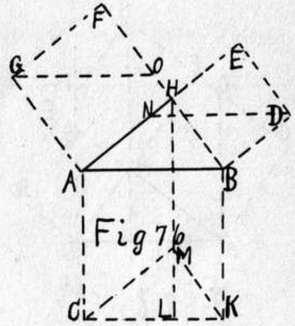
Eighteen

In fig. 76 the construction is evident.

Sq. AK = rect. BL + rect. AL = paral. BM + paral. AM = paral. BN + paral. AO = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 160, fig. (28); Ebene Geometrie von G. Mahler, Leipzig, 1897, p. 80, fig. 60; and Math. Mo., V. IV, 1897, p. 168, proof. XXXIV.



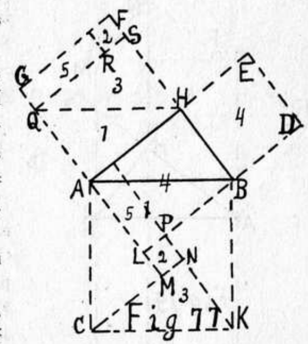
Nineteen

In fig. 77 the construction is evident, as well as the parts containing like numerals.

Sq. AK = tri. BAL + tri. CKN + sq. LN + (tri. ACM + tri. KBP = tri. HQA + tri. QHS + sq. RF + (rect. HL = sq. HP + rect. AP = sq. HD + rect. GR) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Heath's Math. Monographs, No. 2, p. 33, proof XXI.



Twenty

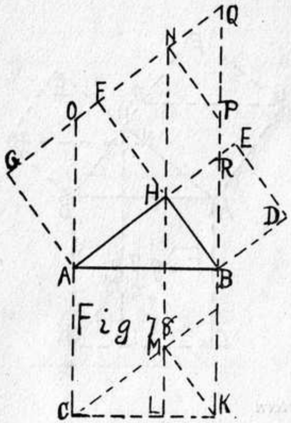
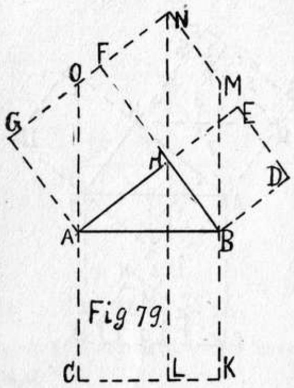


Fig. 78 suggests its construction, as all lines drawn are either perp. or par. to a side of the given tri. ABH. Then we have sq. AK = rect. BL + rect. AL = paral. BHMK + paral. AHMC = paral. BHNP + paral. AHNO = sq. HD + sq. HG. ∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This is known as Haynes's proof; see Math. Magazine, Vol. I, 1882, p. 25, and School Visitor, V. IX, 1888, p. 5, proof IV.

Twenty-One



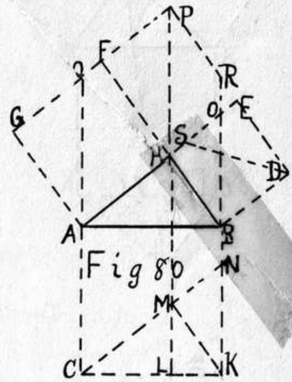
The construction of fig. 79 is easily seen.

Sq. AK = rect. BL + rect. AL = paral. BHNM + paral. AHNO = sq. HD + sq. HG. ∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This is Lecchio's proof, 1753. See jury Whipper, 1880, p. 26, fig. 22, (Historical Note); Olney's Geom., 1872, part III, p. 251, 5th method; Jour. of Education, V. XXV, 1887, p. 404, fig. III; Hopkins's Plane Geom., 1891, p. 91, fig. II; Edwards's Geom., 1895, p. 159, fig. (25); Math. Mo., V. IV, 1897, p. 169, XL; Heath, Math. Monographs, No. 1, 1900, p. 22, proof VI.

b. The above proof is but a particular case of Pappus' Theorem.

Twenty-Two



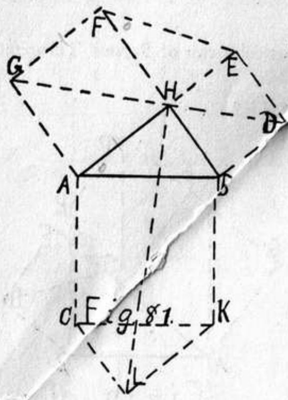
In fig. 80 extend DE and GF to P, CA and KB to Q and R respectively, draw CN par. to AH, and draw PL and KM perp. to AB and CN respectively. Take HO = ES and draw DS.

Sq. AK = tri. KNM + hexagon HCKMNB = tri. BOH + pentagon ACNBH = tri. DSE + pentagon QBORP = tri. DES + paral. AHPQ + quad. PHOR = sq. HG + tri. DES + paral. BP — tri. BOH = sq. HG + tri. DES + trap. HBDS = sq. HG + sq. HD.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 170, proof XLV.

Twenty-Three

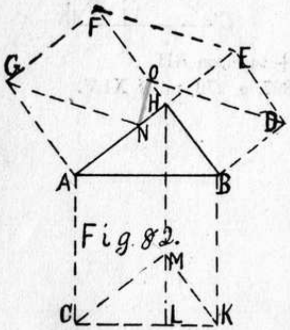


The construction of fig. 81 needs no explanation; from it we get sq. AK + 2 tri. ABH = hexagon ACLKBH = 2 quad. ACLH = 2FEDG = hexagon ABDEFG = sq. AF + sq. BE + 2 tri. ABH.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 32, fig. 29, as found in "Aufangsgrunden der Geometrie" von Tempelhoff, 1769.

Twenty-Four

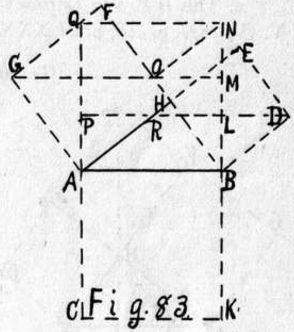


In fig. 82 take BO = AH and AN = BH, and complete the figure; we will have sq. AK = rect. BL + rect. AL = paral. BKMH + paral. ACMH = paral. DEFO + paral. FGNE = sq. DH + sq. GH.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 158, fig. (21), and Math. Mo., V. IV, 1897, p. 169, proof XLI.

Twenty-Five

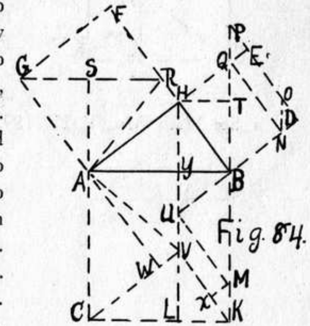


In fig. 83 extend CA to Q and complete sq. QB. Draw GM and DP each par. to AB, and draw NO perp. to BF. This construction gives sq. AB = sq. AN = rect. AL + rect. PN = paral. BDRA + (rect. AM = paral. GABO) = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 158, fig. (29), and Math. Mo., V. IV, 1897, p. 168, proof XXXV.

Twenty-Six

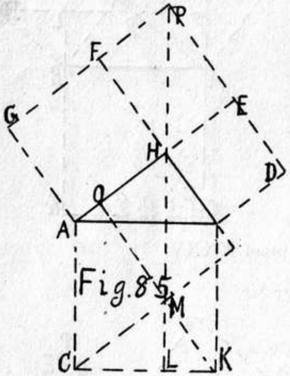


In fig. 84 extend KB to meet DE produced at P, draw QN par. to DE, NO par. to BP, GR and HT par. to AB, extend CA to S, draw HL par. to AC, CV par. to AH, KV and MU par. to BH, MX par. to AH, extend GA to W, DB to U, and draw AR and AV. Then we will have sq. AK = tri. ACW + tri. CVL + quad. AWVY + tri. VKL + tri. KMX + trap. UVXM + tri. MBU + tri. BUY = (tri. GRF + tri. AGS + quad. AHRS) + (tri. BHT + tri. OND + trap. NOEQ + tri. QBN + tri. HQT) = sq. BE + sq. AT.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This is E. von Littrow's proof, 1839; see also Math. Mo., V. IV, 1897, p. 169, proof XXXVII.

Twenty-Seven



In fig. 85 extend GH and DE to P, HB (B is the vertex opp. the side AH of tri. ABC, and N is the intersection of the lines HB and CM) to N, draw PL perp. to AB, CN par. to AH and KO par. to BH; whence sq. AK = [(trap. HACN — tri. MNH = paral. ACMH = rect. AL) = (trap. AHPG — tri. HPF = sq. AG)] + [(trap. HOKB — tri. MHO = paral. HMKB = rect. BL) = (trap. HBDP — tri. PHE = sq. HD)].

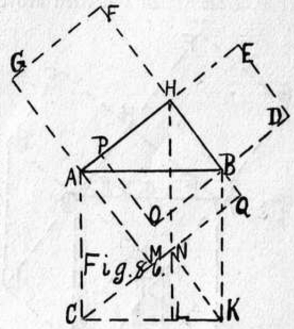
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 169, proof XLII.

Twenty-Eight

In fig. 86, construct sq. HM = sq. HG, sq. HO = sq. HD, draw HL perp. to AB, and draw CM and KN. From this construction, we get sq. AK = rect. BL + rect. AL = paral. HNKB + paral. HACN = sq. BP + sq. HM = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

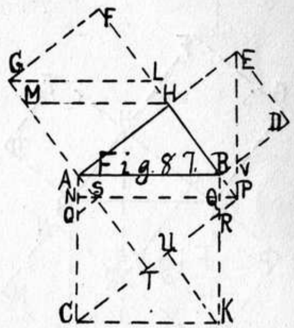


a. Vieth's proof—see Jury Whipper, 1880, p. 24, fig. 19, as given by Vieth, in "Aufangsgrunden der Mathematik," 1805; also Math. Mo., V. IV, 1897, p. 169, proof XXXVI.

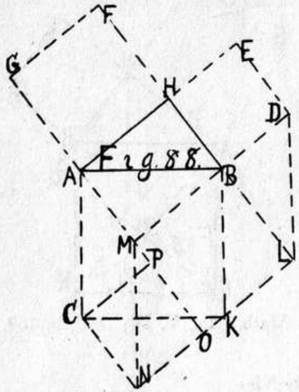
Twenty-Nine

In fig. 87, construct the sq. HT, draw GL, HM, and PN par. to AB; also KU par to BH, OS par. to AB, and join EP. By analysis we find that sq. AK = (trap. CTSO + tri. KRU) + [tri. CKU + quad. STRQ + (tri. SON = tri. PRQ) + rect. BQ] = (trap. EHBV + tri. EVD) + [tri. GLF + tri. HMA + (paral. SB = paral. ML)] = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.



a. After three days of analyzing and classifying solutions based on the A type of figure, the above dissection occurred to me, July 16, 1890, from which I devised above proof.



Thirty

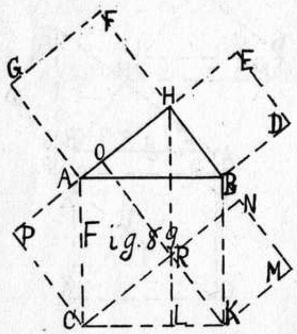
In fig. 88, through K draw NL par. to AH, extend HB to L, GA to O, DB to M, draw DL and MN par. to BK, and CN par. to AO.

Sq. AK = hexagon ACNKBM = paral. CM + paral. KM = sq. CO + sq. ML = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 157, fig. (16).

Thirty-One

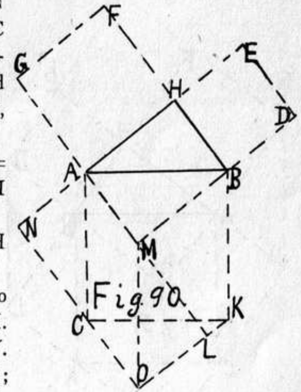


In fig. 89 extend HB to M making BM = AH, HA to P making AP = BH, draw CN and KM each par. to AH, CP and KO each perp to AH, and draw HL perp. to AB. Sq. AK = rect. BL + rect. AL = paral. RKBH + paral. CRHA = sq. RM + sq. CO = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 169, proof XLIII.

Thirty-Two



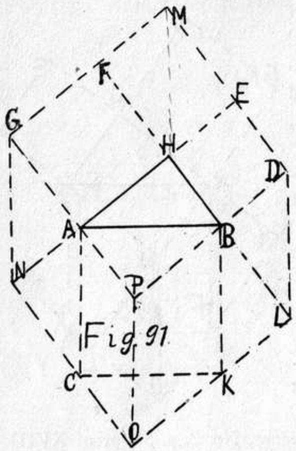
In fig. 90 extend HA to N making AN = BH, through C draw NO par. to BH and making CO = BH, draw AL = and par. to NO, extend DB to M, draw MO and OK.

Sq. AK = hexagon ACOKBM = paral. COMA + paral. OKBM = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This proof is credited to C. French, Winchester, N. H. See Journal of Education, V. XXVIII, 1888, p. 17, 23d proof; Edwards's Geom., 1895, p. 159, fig. (26); Heath's Math. Monographs, No. 2, p. 31, proof XVIII.

Thirty-Three

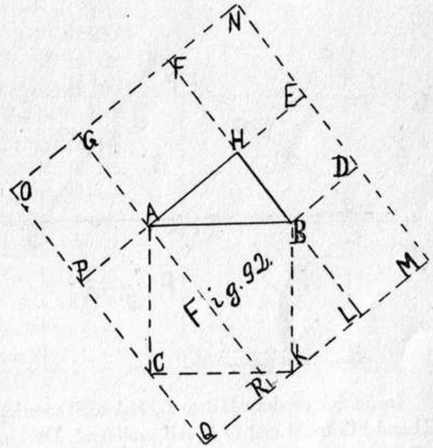


In fig. 91 complete the squares AK, HD and HG, also the paral's FE, GC, AO, PK and BL. From these we find that sq. AK = hexagon ACOKBP = paral. OPGN — paral. CAGN + paral. POLD — paral. BKLD = paral. LDMH — (tri. MAE + tri. LDB) + paral. GNHM — (tri. GNA + tri. HMF) = sq. HD + sq. HG. ∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Olney's Geom., University Edition, 1872, p. 251, 8th method; Edwards's Geom., 1895, p. 160, fig. (30).

Thirty-Four

In fig. 92 extend GF and DE to N, complete the square NQ, and extend HA to P, GA to R and HB to L.



From these dissected parts of the sq. NQ we see that sq. AK + (4 tri. ABH + rect. HM + rect. GE + rect. OA) = sq. NQ = (rect. PR = sq. HD + 2 tri. ABH) + (rect. AL = sq. HG + 2 tri. ABH) + rect. HM + rect. GE + rect. AO, = sq. AK + (4 tri. ABH + rect. HM + rect. GE + rect. OA — 2 tri. ABH — 2 tri. ABH — rect. HM — rect. GE — rect. OA = sq. HD + sq. HG.

∴ sq. AK = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Credited by Hoffmann, in "Der Pythagoraische Lehrsatz," 1821, to Henry Boad, of London, Eng. See Jury Whipper, 1880, p. 18, fig. 12.

Thirty-Five

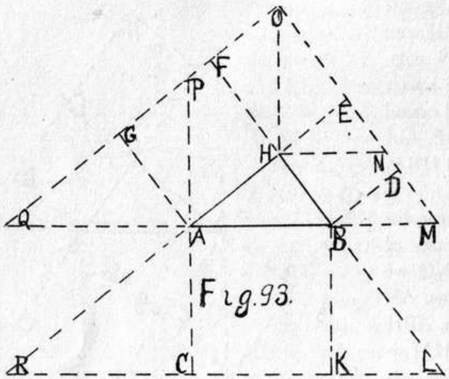


Fig. 93.

In fig. 93, produce HB to L, HA to R meeting CK prolonged, ED and FG to M and Q in AB prolonged, DE and GF to O, CA to P, and draw HN par. to AB and OH par. to BK. Then sq. AK + tri. RAC + tri. BLK + tri. ABH = tri. RLH = (tri. ONH = tri. BLK) + (paral. HM = sq. HD) + (tri. QPA = tri. RAC) + (paral. HP = sq. HG) + tri. ABH.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 30, fig. 28a.

Thirty-Six

In fig. 94 extend HB and CK to L, AB and ED to M, DE and GF to O, CA and KB to P and N respectively and draw PA. Now observe that (quad. CLBA = sq. HK + tri. BLK) = [quad. BMNF = hexagon AHBMOP = (tri. EMB = tri. BLK) + paral. BO = sq. HD) + (paral. AO = sq. AF)].
∴ sq. upon AB = sq. upon BH + sq. upon AH.

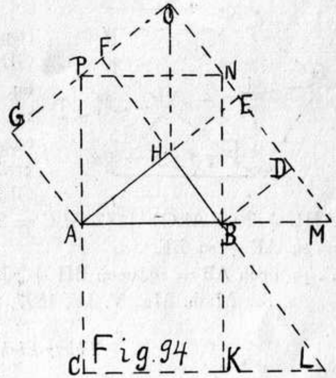


Fig. 94

a. Devised by the author, July 7, 1901, but suggested by fig. 28b, in Jury Whipper, 1880, p. 31.

b. By omitting, from the fig., the sq. AK, and the tri's BLK and BMD, an algebraic proof through the mean proportional is easily obtained.

B.

This type includes all proofs derived from the figure in which the square constructed upon the hypotenuse overlaps the given triangle and the squares constructed upon the legs as in type A.

Thirty-Seven

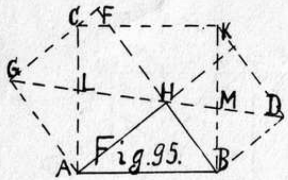


Fig. 95

Assuming the three squares constructed, as in fig. 95, draw GD—it must pass through H. Sq. AK = trap. ABML = 2 tri. AHL + 2 tri. ABH + 2 tri. HBM = 2 tri. AHL + 2 (tri. ACG = tri. ALG + tri. GLC) + 2 tri. HBM = (2 tri. AHL + 2 tri. ALG) + (2 GLC = 2 tri. DMB) + 2 tri. HBM = sq. AF + sq. BE.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 250, proof XLIX.

Thirty-Eight

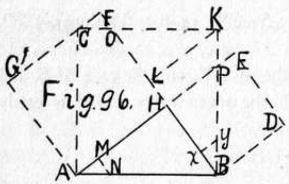


Fig. 96.

In fig. 96 draw KL par. to AH, take HM = BH, and draw MN par. to BH.

Sq. AK = tri. ANM + trap. MNBH + tri. BKL + tri. KOL + quad. MHOC = (tri. COF + tri. ACG + quad. MHOC) + (trap. PEDB + tri. BPH) = sq. AF + sq. HD.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 250, proof L.

b. If XY is drawn in place of MN, (LX = HB), the proof is prettier, although same in principle.

Thirty-Nine

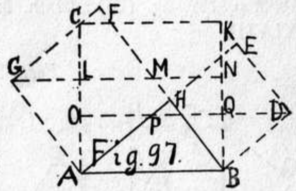


Fig. 97.

In fig. 97 draw GN and OD par. to AB.

Sq. AK = rect. AQ + rect. OK = paral. AD + rect. AN = sq. BE + paral. AM = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 250, XLVI.

Forty

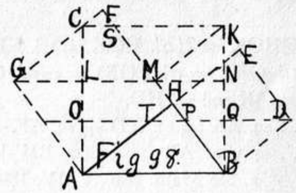


Fig 98.

In fig. 98, draw GN and DR par. to AB and KM par. to AH. R is the pt. of intersection of AG and DO.

Sq. AK = rect. AQ + rect. RN + rect. LK = (paral. DA = sq. BE) + (paral. RM = pentagon RTHMG + tri. CSF) + (paral. GMKC = trap. GMSC + tri. TRA) = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 250, proof XLVII.

Forty-One

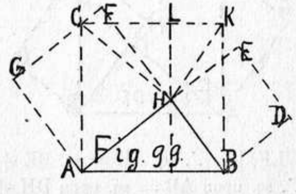


Fig. 99.

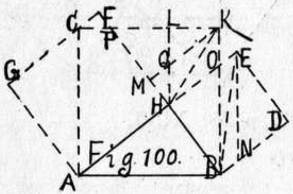
In fig. 99 draw LM through H perp. to AB, and draw HK and HC.

Sq. AK = rect. LB + rect. LA = 2 tri. KHB + 2 tri. CAH = sq. AD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 12, proof V; Edwards's Geom., 1895, p. 159, fig. (23); Math. Mo., V. IV, 1897, p. 250, proof XLVIII.

Forty-Two



In fig. 100 draw HL par. to BK, KM par. to AH, and draw EB and KH.

Sq. AK = (tri. ABH = tri. ACG) + quad. AHPC common to sq. AK and sq. AF + (tri. HQM = tri. CPF) + (tri. KPM = tri. END) + [paral. QHOK = 2(tri. HOK = tri. KHB — tri. OHB = tri. EHB — tri. OHB = tri. EOB) = paral. OBNE] + tri. OHB common to sq. AK and sq. HD.

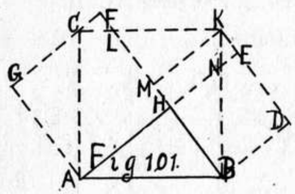
∴ sq. AK = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 250, proof LI.

b. See Sci. Am. Sup., V. 70, 1910, p. 382, for a geometric proof, unlike the above proof, but based upon a similar figure of the B type.

Forty-Three

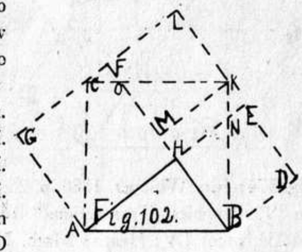


In fig. 101, extend DE to K, and draw KM perp to FB. Sq. AK = (tri. ABH = tri. ACG) + quad. AHLC common to sq. AK and sq. AF + (tri. KLM = tri. BNH) + [tri. BKM = tri. KBD = trap. BDEN + (tri. KNE = tri. CLF)]. ∴ sq. AK = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 161, fig. (36); Math. Mo., V. IV, 1897, p. 251, proof LII.

Forty-Four



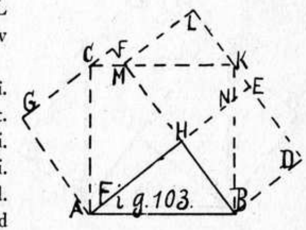
In fig. 102 extend GF to L making FL = HB and draw KL and KM respectively par. to BH and AH.

Sq. AK = (tri. ABH = tri. CKL = trap. BDEN + tri. COF) + (tri. BKM = tri. ACG) + (tri. KOM = tri. BNH) + quad. AHOC common to sq. AK and sq. AF = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 251, proof LVII.

Forty-Five



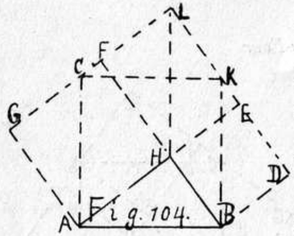
In fig. 103 extend DE to L making KL = HN, and draw ML.

Sq. AK = (tri. ABH = tri. ACG) + (tri. BMK = ½ rect. BL = [trap. BDEN + (tri. KNE = tri. CMF) + (tri. MKL = tri. BNH)] + quad. AHMC common to sq. AK and sq. AF = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 158, fig. (18).

Forty-Six

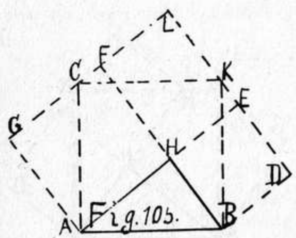


In fig. 104 extend GF and DE to L and draw LH.
Sq. AK = hexagon AHBKLC = paral. HK + paral. HC = sq. HD + sq. HC.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author, July 7, 1901. See Olney's Geom. university edition, 1872, p. 250, fig. 374; Jury Whipper, 1880, p. 25, fig. 20b, as given by M. v. Ash, in "Philosophical Transactions," 1683; Math. Mo., V. IV, 1897, p. 251, proof LV; Heath's Math. Monographs, No. 1, 1900, p. 24, proof IX.

b. By extending LH to AB, an algebraic proof can be readily devised, thus increasing the no. of figures for simple proofs.

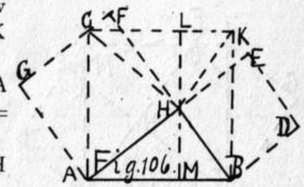
Forty-Seven



In fig. 105 extend GF and DE to L.
Sq. AK = pentagon ABDLG — (3 tri. ABH = tri. ABH + rect. LH) = sq. HD + sq. AF.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Journal of Education, 1887, V. XXVI, p. 21, fig. X.

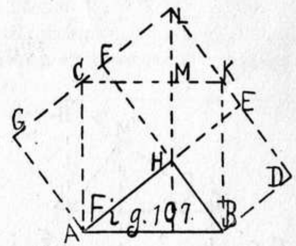
Forty-Eight



In fig. 106, through H draw LM perp. to AB, and draw HK and HC.
Sq. AK = rect. LB + rect. LA = 2 tri. HBK + 2 tri. AHC = sq. HD + sq. HG.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Sci. Am. Sup., V. 70, p. 383, Dec. 10, 1910, being No. 16 in A. R. Colburn's 108 proofs.

Forty-Nine

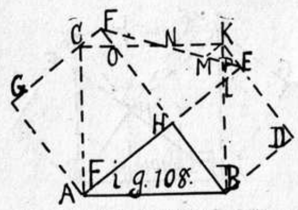


In fig. 107 extend GF and DE to L, and through H draw LN, N being the pt. of intersection of NH and AB.
Sq. AK = rect. MB + rect. MA = paral. HK + paral. HC = sq. HD + sq. HG.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 13, fig. 5b, and p. 25, fig. 21, as given by Klagel in "Encyclopaedie," 1808; Edwards's Geom., 1895, p. 156, fig. (7); Ebene Geometrie, von G. Mahler, 1897, p. 87, art. 11; Math. Mo., V. IV, 1897, p. 251, LIII.

b. This figure will give an algebraic proof.

Fifty



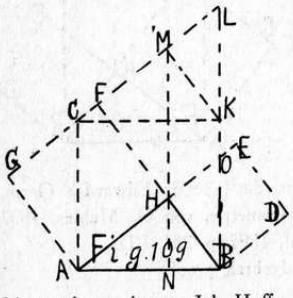
In fig. 108 extend DE to K, draw FE, and draw KM par. to AH.

Sq. AK = (tri. ABH = tri. ACG) + quad. AHOC common to sq. AK and sq. AF + tri. BLH common to sq. AK and sq. HD + [quad. OHLK = OHLPN + (tri. PKM = tri. PLE) + (tri. MKN = tri. OFN) = tri. FEH = tri. KBD = (trap. BDEL + tri. COF)] = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Math. Mo., V. IV, 1897, p. 251, proof LVI.

Fifty-One



In fig. 109 extend GF and BK to L, through draw MN par. to BK, and draw KM.

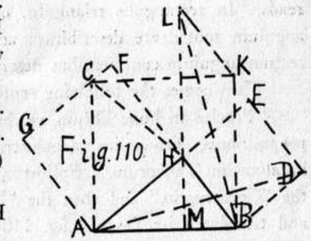
Sq. AK = paral. AOLC = paral. HL + paral. HC = (paral. HK = sq. AD) + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 27, fig. 23, where it says that this proof was given to Joh. Hoffmann, 1800, by a friend; also Math. Mo., 1897, V. IV, p. 251, proof LIV.

b. From this figure an algebraic proof is easily devised.

Fifty-Two



In fig. 110 draw KL par. and equal to BH, through H draw LM par. to BK, and draw AD, LB and CH.

Sq. AK = rect. MK + rect. MC = (paral. HK = 2 tri. BKL = 2 tri. ABD = sq. BE) + (2 tri. AHC = sq. AF).

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This figure and proof is taken from the following work, now in my library, the title page of which is:

"Euclides Elementorum Geometricorum
Libros Tredecim
Isidorum et Hypsiclem
& Recentiores de Corporibus Regularibus, &
Procli
Propositiones Geometricas

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Claudius Richards
e Societate Jesu Sacerdos, patria Ornacensis in libero Comitatu
Burgundae, & Regius Mathematicarum
Professor: dicantique
Philippo IIII. Hispaniarum et Indicarum Regi Cathilico.
Antwerpiae,
ex Officina Hiesonymi Verdussii. M. DC. XLV.
Cum Gratia & Privilegio"

The figures of this book are all grouped together at the end of the volume. The above figure is numbered 62, and is constructed for "Propositio XLVII," in "Librum Primum," which proposition reads, "In rectangulis triangulis, quadratum quod a latere rectum angulum subtedente describitur; aequuale est eis, quae a lateribus rectum angulum continentibus describuntur quadratis."

Then comes the following sentence:

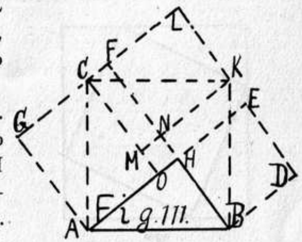
"Proclus in hunc librum, celebrat Pythagoram Authorem huius propositionis, pro cuius demonstratione dicitur Diis Sacrificasse hecatombam Taurorum." Following this comes the "Supposito," then the "Constructio," and then the "Demonstratio," which condensed and translated is: (as per fig. 110) triangle BKL equals triangle ABD; square BE equals twice triangle ABM and rectangle MK equals twice triangle BKL; therefore rectangle MK equals square BE. Also square AG equals twice triangle AHC; rectangle HM equals twice triangle CAH; therefore square AG equals rectangle HM. But square BK equals rectangle KM plus rectangle CM. Therefore square BK equals square AG plus square BD.

The work from which the above is taken is a book of 620 pages, 8 inches by 12 inches, bound in vellum, and, though printed in 1645 A. D., is well preserved. It once had a place in the Sunderland Library, Blenheim Palace, England, as the book plate shows—on the book plate is printed—"From the Sunderland Library, Blenheim Palace, Purchased, April, 1882."

I found the book in a second-hand book store in Toronto, Canada, and on July 15, 1891, I purchased it. E. S. Loomis.

The work has 408 diagrams, or geometric figures, is entirely in Latin, and highly embellished.

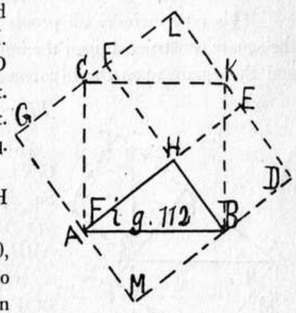
Fifty-Three



In fig. 111 extend GF to L making FL = BH, draw KL, and draw CO and KM par. to BH and AH respectively.
Sq. AK = (tri. ABH = tri. ACG) + tri. CAO common to sq's AK and AF + sq. MH common to sq's AK and AF + [pentagon MNBKC = rect. ML = rect. MF + (sq. NL = sq. HD)] = sq. HD + sq. HG.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

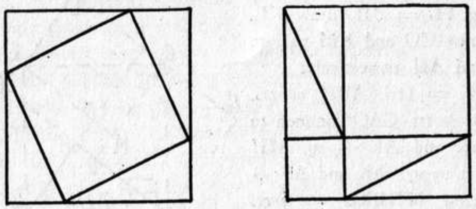
a. Devised by the author July 30, 1900.

Fifty-Four



In fig. 112 produce GF and DE to L, and GA and DB to M.
Sq. AK + 4 tri. ABH = sq. GD = sq. HD + sq. HG + (rect. HM = 2 tri. ABH) + (rect. LH = 2 tri. ABH) whence sq. AK = sq. HD + sq. HG.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 17, fig. 10, and is credited to Henry Boad, as given by Johann Hoffmann, in "Der Pythagoraische Lehrsatz," 1821; also see Edwards's Geom., 1895, p. 157, fig. (12). Heath's Math. Monographs, No. 1, 1900, p. 18, fig. 11; also attributed to Pythagoras, by W. W. Rouse Ball. Also see Pythagoras and his Philosophy in Sect. II, Vol. 10, p. 239, 1904, in proceedings of Royal Society of Canada, wherein the figure appears as follows:

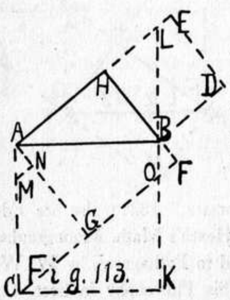


LOOK

C.

This type includes all proofs derived from the figure in which the square constructed upon the longer leg overlaps the given triangle and the square upon the hypotenuse.

Fifty-Five



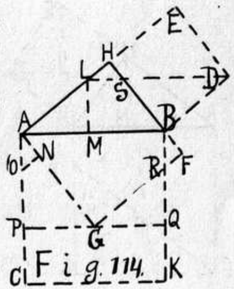
In fig. 113 extend KB to L, take GN = BH and draw MN par. to AH.
Sq. AK = quad. AGOB common to sq's AK and AF + (tri. COK = tri. ABH + tri. BLH) + (trap. CGNM = trap. BDEL) + (tri. AMN = tri. BOF) = sq. HD + sq. HG.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 268, proof LIX.

Fifty-Six

In fig. 114, draw DL par. to AB, through G draw PQ par. to CK, take GN = BH, draw ON par. AH and LM perp. to AB.
Sq. AK = quad. AGRB common to sq's AK and AF + (tri. ANO = tri. BRF) + (quad. OPGN = quad. LMBS) + (rect. PK = paral. ABDL = sq. BE) + (tri. GRQ = tri. AML) = sq. BE + sq. AF.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

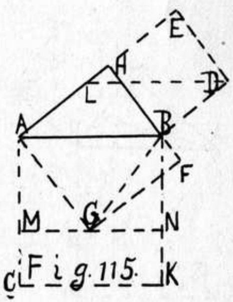
a. Devised by the author July 20, 1900.



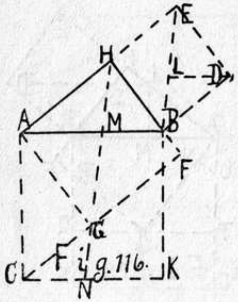
Fifty-Seven

In fig. 115 through G and D draw MN and DL each par. to AB, and draw GB.
Sq. AK = rect. MK + rect. MB = paral. AD + 2 tri. BAG = sq. BE + sq. AF.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 268, proof LXII.



Fifty-Eight



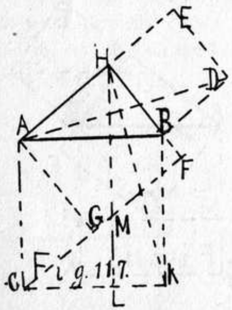
In fig. 116 extend FG to C, draw EB, and through G draw HN, and draw DL par. to AB.

Sq. AK = 2[quad. ACNM = (tri. CGN = tri. DBL) + tri. GAM common to sq. AK and AF + (tri. ACG = tri. ABH = tri. AMH + tri. ELD)] = 2 tri. AGH + 2 tri. BDE = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 268, proof LXIII.

Fifty-Nine

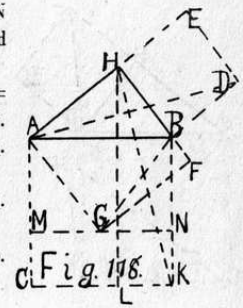


In fig. 117 extend FG to C, draw HL par. to AC, and draw AD and HK. Sq. AK = rect. BL + rect. AL = (2 tri. KBH = 2 tri. ABD + paral. ACMH = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 11, II; Math. Mo., V. IV, 1897, p. 267, proof LVIII.

Sixty



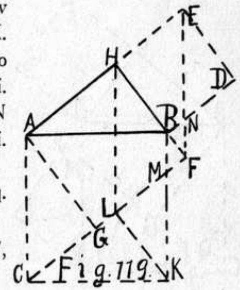
In fig. 118 through G draw MN par. to AB, draw HL perp. to CK, and draw AD, HK and BG.

Sq. AK = rect. MK + rect. AN = (rect. BL = 2 tri. KBH = 2 tri. ABD) + 2 tri. AGB = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 268, proof LXI.

Sixty-One



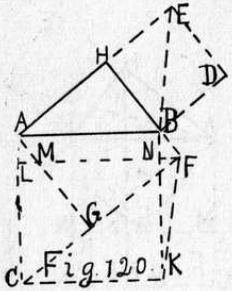
In fig. 119, extend FG to C, draw HL par. to BK, and draw EF and LK.

Sq. AK = quad. AGMB common to sq's AK and AF + (tri. ACG = tri. ABH) + (tri. CKL = trap. EHBN + tri. BMF) + (tri. KML = tri. END) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 268, proof LXIV.

Sixty-Two

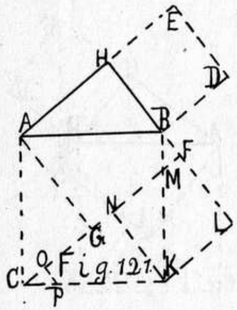


In fig. 120 draw FL par. to AB, extend FG to C, and draw EB and FK. Sq. AK = (rect. LK = 2 tri. CKF = 2 tri. ABE = 2 tri. ABH + 2 tri. HBE = tri. ABH + tri. FMG + sq. HD) + (rect. AN = paral. MB).

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 269, LXVII.

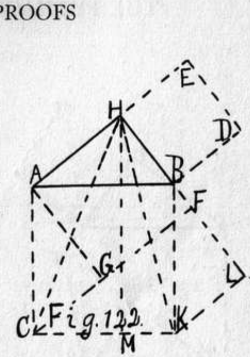
Sixty-Three



In fig. 121 extend FG to C, HB to L, draw KL par. to AH, and take NO = BH and draw OP and NK par. to BH.
Sq. AK = quad. AGMB common to sq's AK and AF + (tri. ACG = tri. ABH) + (tri. CPO = tri. BMF) + (trap. PKNO + tri. KMN = sq. NL = sq. HD) = sq. HD + sq. AF.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 157, fig. (14).

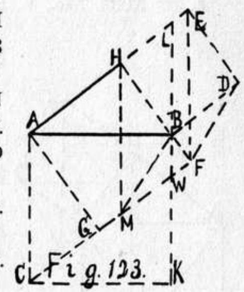
Sixty-Four



In fig. 122 extend HB to L making FL = BH, draw HM perp. to CK and draw HC and HK.
Sq. AK = rect. BM + rect. AM = 2 tri. KBH + 2 tri. HAC = sq. HD + sq. HG.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 161, fig. (37).

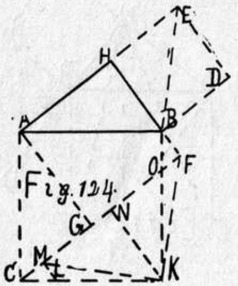
Sixty-Five



In fig. 123 extend KB and FG respectively to L and C, draw LF and HM respectively par. to BK and draw MB and FD.
Sq. AK = paral. ACNL = paral. HN + paral. HC = (2 tri. BHM = 2 tri. DEF = sq. HD) + sq. HG = sq. HD + sq. HG.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 269, proof IXIX.

Sixty-Six



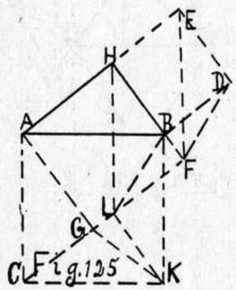
In fig. 124 extend FG to C, draw KN par. to BH, take NM = BH, draw ML par. to HB, and draw MK KF and BE.

Sq. AK = quad. AGOB common to sq's AK and AF + (tri. ACG = tri. ABH) + (tri. CLM = tri. BOF) + [(tri. LKM = tri. OKF) + tri. KON = tri. BEH] + (tri. MKN = tri. EBD) = (tri. BEH + tri. EBD) + (quad. AGOB + tri. BOF + tri. ABC) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 269, proof LXVIII.

Sixty-Seven



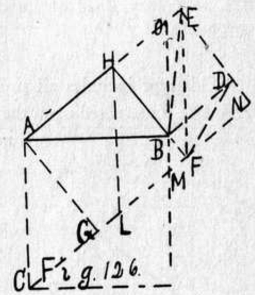
In fig. 125 extend FG to H, draw HL par. to AC, KL par. to HB, and draw KG, LB, FD and EF.

Sq. AK = quad. AGLB common to sq's AK and AF + (tri. ACG = tri. ABH) + (tri. CKG = tri. EFD = ½ sq. HD) + (tri. GKL = tri. BLF) + (tri. BLK = ½ paral. HK = ½ sq. HD) = (½ sq. HD + ½ sq. HD) + (quad. AGLB + tri. ABH + tri. BLF) = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 268, proof LXV.

Sixty-Eight



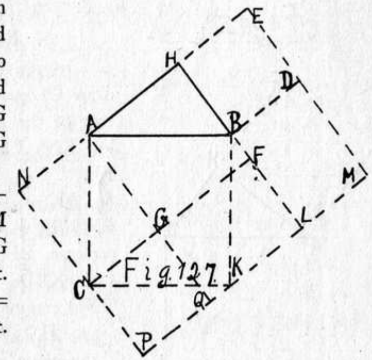
In fig. 126 extend FG to C and N, making FN = BD, KB to O, (K being the vertex opp. A in the sq. CB) draw FD, FE and EB, and draw HL par. to AC.

Sq. AK = paral. ACMO = paral. HM + paral. HC = [(paral. EHLF = rect. EF) — (paral. EOMF = 2 tri. EBF = 2 tri. DBF = rect. DF) = sq. HD] = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. IV, 1897, p. 268, proof LXVI.

Sixty-Nine



In fig. 127 through C and K draw NP and PM par. respectively to BH and AH, and extend ED to M, HF to L, AG to Q, HA to N and FG to C.

Sq. AK + rect. HM + 4 tri. ABH = rect. NM = sq. HD + sq. HG + (rect. NQ = rect. HM) + (rect. GL = 2 tri. ABH) + (rect. BM = 2 tri. ABH).

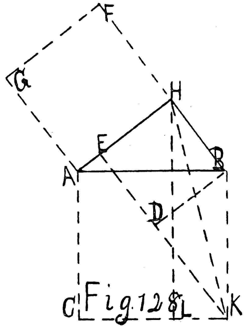
∴ sq. AK = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Credited by Joh. Hoffman, in "Der Pythagoraische Lehrsatz," 1821, to Henry Boad of London; see Jury Whipper, 1880, p. 19, fig. 13.

D.

This type includes all proofs derived from the figure in which the square constructed upon the shorter leg overlaps the given triangle and the square upon the hypotenuse.

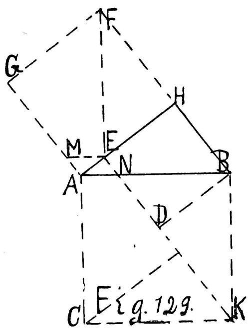


Seventy

In fig. 128 extend ED to K, draw HL perp. to CK and draw HK.
Sq. AK = rect. BL + rect. AL = (2 tri. BHK = sq. HD + (sq. HE by Euclid's proof).
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 11, fig. 3.

Seventy-One



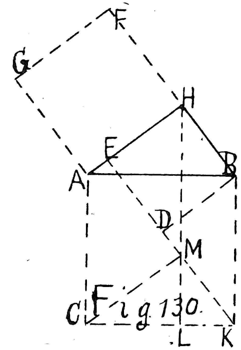
In fig. 129 extend ED to K, draw CL par. to AH, (L being the foot of the perp. from C upon the line KN) EM par. to AB and draw FE.
Sq. AK = (quad. ACLN = quad. EFGM) + (tri. CKL = tri. ABH = trap. BHEN + tri. EMA) + (tri. KBD = tri. FEH) + tri. BND common to sq's AK and HD, = sq. HD + sq. AF.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 155, fig. (2).

Seventy-Two

In fig. 130 extend ED to K, draw HL par. to AC, and draw CM.
Sq. AK = rect. BL + rect. AL = paral. HK + paral. HC = sq. HD + sq. HG.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

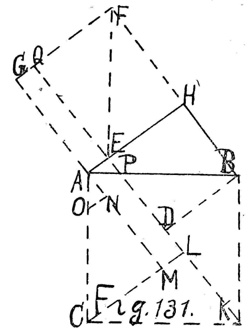
a. Devised by the author Aug. 1, 1900.



Seventy-Three

In fig. 131 extend ED to K and Q, draw CL perp to EK, extend GA to M, take MN = BH, draw NO par. to AH, and draw FE.
Sq. AK = (tri. CKL = tri. FEH) + (tri. KBD = tri. EFQ) + (trap. AMLP + tri. AON = rect. GE) + tri. BPD common to sq's AK and BE + (trap. CMNO = trap. BHEP) = sq. HD + sq. HG.
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a Original with the author, August 1, 1900.



Seventy-Four

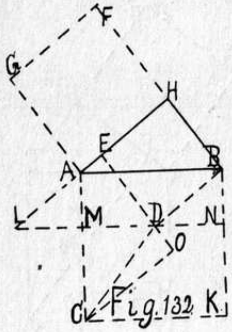


Fig. 132

In fig. 132, through D draw LN par. to AB, extend ED to K, and draw HL and CD.

Sq. AH = (rect. AN = paral. AD = sq. DH) + (rect. MK 2 tri. DCK = sq. GH).

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Contrived by the author, August 1, 1900.

b. As in types A, B and C, many other proofs may be derived from the D type of figure.

Seventy-Five

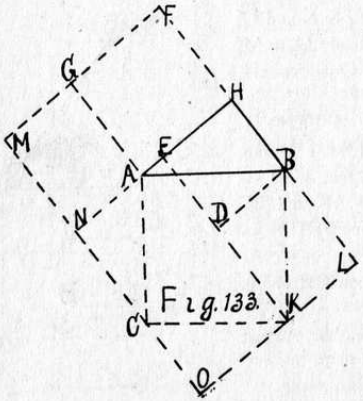


Fig. 133.

In fig. 133 extend FB and FG to L and M making BL = AH and GM = BH, complete the rectangle FO and extend HA to N, and ED to K.

Sq. AK + rect. MH + 4 tri. ABH = rect. FO = sq. HD + sq. HG + (rect. NK = rect. MH) + (rect. MA = 2 tri. ABH) + (rect. DL = 2 tri. ABH); collecting we have sq. AK = ss. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Credited to Henry Boad by Joh. Hoffmann, 1821; see Jury Whipper, 1880, p. 20, fig. 14.

E.

This type includes all proofs derived from the figure in which the squares constructed upon the hypotenuse and the longer leg overlap the given triangle.

Seventy-Six

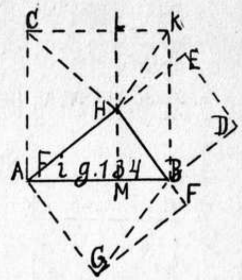


Fig. 134

In fig. 134, through A draw LM par. to KB, and draw GB, HK and HC.

Sq. AK = rect. LB + rect. LA = (2 tri. HBK = sq. HD) + (2 tri. CAH = 2 tri. BAG = sq. AF).

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 14, VI; Edwards's Geom., 1895, p. 162, fig. (38); Math. Mo., V. V, 1898, p. 74, proof LXXV.

Seventy-Seven

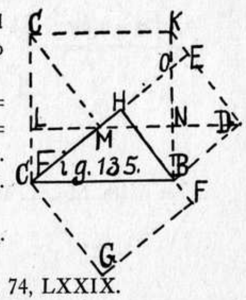


Fig. 135.

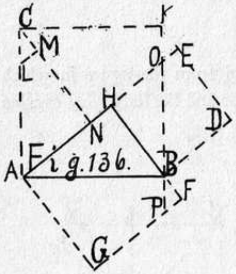
In fig. 135 extend DE to K and draw DL and CM par. respectively to AB and BH.

Sq. AK = (rect. LB = paral. AD = sq. BE) + (rect. LK = paral. CD = trap. CMEK = trap. AGFB) + (tri. KDN = tri. ABC) = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. V, 1898, p. 74, LXXIX.

Seventy-Eight



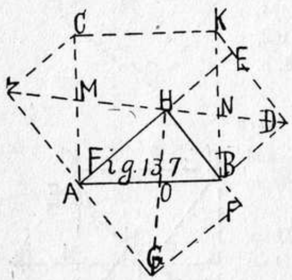
In fig. 136 extend KB to P, draw CN par. to HB, take NM = HB, and draw ML par. to AH.

Sq. AK = (quad. NOKC = quad. GPBA) + (tri. CLM = tri. BPF) + (trap. ANML = trap. BDEO) + tri. ABH common to sq's AK and AF + tri. BOH common to sq's AK and HD = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Math. Mo., V. V, 1898, p. 74, proof LXXVII.

Seventy-Nine



In fig. 137, extend DE to K, GA to L, draw CL par. to AH, and draw LD and HG.

Sq. AK = 2[trap. ABNM = tri. AOH common to sq's AK and AF + (tri. AHM = tri. AGO) + tri. HBN common to sq's AK and HD + (tri. BHO = tri. BDN) = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

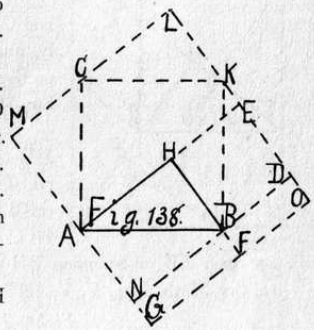
a. See Math. Mo., V. V, 1898, p. 74, proof LXXVI.

Eighty

In fig. 138 GF and ED to O and complete the rect. MQ. Extend DB to N.

Sq. AK + rect. NO + 4 tri. ABH = rect. MO = sq. HD + sq. AF + rect. BO + [rect. AL = (rect. HN = 2 tri. ABH) + (sq. HG = 2 tri. ABH + rect. NF)], which coll'd gives sq. AK = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.



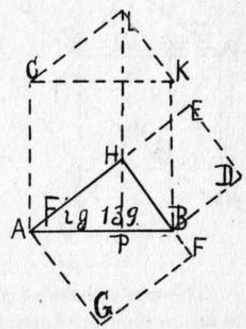
a. Credited to Henry Boad by Joh. Hoffman, in "Der Pythagoraische Lehrsatz," 1821; see Jury Whipper, 1880, p. 21, fig. 15.

Eighty-One

In fig. 139 draw CL and KL par. respectively to AH and BH, and draw, through H, LP.

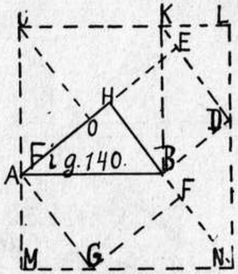
Sq. AK = hexagon AHBKLC = paral. LB + paral. LA = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.



a. Devised by the author, March 12, 1926.

Eighty-Two

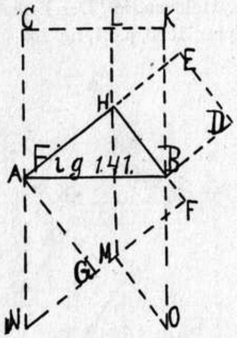


In fig. 140 extend CA, HB, DE and CK to M, N, K and L respectively, and draw MN, LN and CO respectively par. to AB, KB and HB. Sq. AK + 2 tri. AGM + 3 tri. GNF + trap. AGFB = rect. CN = sq. HD + sq. HG + 2 tri. AGM + 3 tri. GNF + trap. COEK. which coll'd gives sq. AK = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH +sq. upon AH.

a. See Math. Mo., V. V, 1898, p. 74, proof LXXVIII.

Eighty-Three



In fig. 141 extend KB and CA respectively to O and N, through H draw LM par. to KB, and draw GN and MO respectively par. to AH and BH. Sq. AK = rect. LB + rect. LA = paral. BHMO + paral. HANM = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author, August 1, 1900.

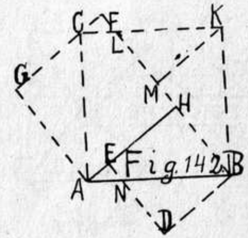
b. Many other proofs are derivable from this type of figure.

c. An algebraic proof is easily obtained from fig. 141.

F.

This type includes all proofs derived from the figure in which the squares constructed upon the hypotenuse and the shorter leg overlap the given triangle.

Eighty-Four



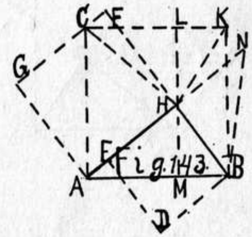
In the fig. 142 draw KM par. to AH.

Sq. AK = (tri. BKM = tri. ACG) + (tri. KLM = tri. BND) + quad. AHLC common to sq's AK and AF + (tri. ANE = tri. CLF) + trap. NBHE common to sq's AK and EB = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. The Journal of Education, V. XXVIII, 1888, p. 17, 24th proof, credits this proof to J. M. McCready, of Black Hawk, Wis.; see Edwards's Geom., 1895, p. 89, art. 73; Heath's Math. Monographs, No. 2, 1900, p. 32, proof XIX.

Eighty-Five



In fig. 143 extend AH to N making HN = HE, through H draw LM par. to BK, and draw BN, HK and HC.

Sq. AK = rect. LB + rect. LA = (2 tri. HBK = 2 tri. HBN = sq. HD) + (2 tri. CAH = 2 tri. AHC = sq. HG) = sq. HD + sq. HG.

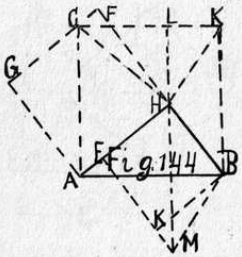
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author, August 1, 1900.

b. An algebraic proof may be resolved from this figure.

c. Other geometric proofs are easily derived from this form of figure.

Eighty-Six



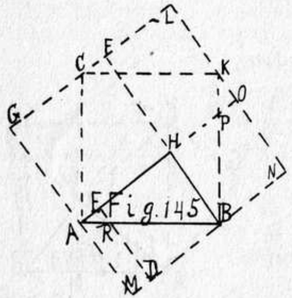
In fig. 144 draw LH perp. to AB and extend it to meet ED produced and draw MB, HK and HC.

Sq. AK = rect. LB + rect. LA = (paral. HMBK = 2 tri. MBH = sq. BE) + (2 tri. CAH = 2 tri. AHC = sq. AF) = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. see Jury Whipper, 1880, p. 14, fig. 7.

Eighty-Seven



In fig. 145 extend GA and BD to M, complete the square ML, and extend AH to O.

Sq. AK + 4 tri. ABH = sq. LM = sq. HD + sq. HG + 3 tri. ABH = (trap. BNOP + tri. ARE = tri. ABH), which collected gives sq. AK = sq. HD + sq. HG.

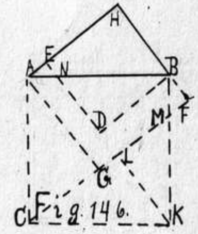
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 17, fig. 11, where it is credited, by Johann Hoffmann, in "Der Pythagoraische Lehrsatz," 1821, to Henry Boad.

G.

This type includes all proofs derived from figures in which the squares constructed upon the two legs overlap the given triangle.

Eighty-Eight



In fig. 146 extend FG to C and draw KL par. to AG.

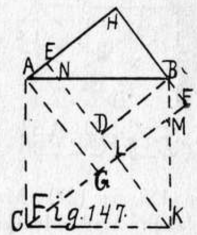
Sq. AK = quad. AGMB common to sq's AK and AF + (tri. ACG = tri. ABH) + (tri. HKL = trap. NBHE + tri. BMF) + (tri. KML = tri. CND) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 161, fig. (33); Math. Mo., V. V, 1898, p. 73, proof LXX.

b. In Sci. Am. Sup., V. 70, p. 359, Dec. 3, 1910, is a proof by A. R. Colburn, by use of above figure, but the argument is not that given above.

Eighty-Nine



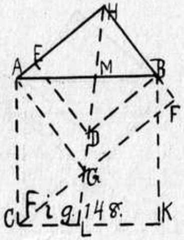
In fig. 147 extend FG to C and ED to K.

Sq. AK = (tri. ACG = tri. ABH of sq. HG) + (tri. CKL = trap. NBHE + tri. BMF) + (tri. KBD = tri. BDN of sq. HD + trap. LMBD common to sq's AK and HG) + pentagon AGLDB common to sq's AK and HG) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 159, fig. (24); Sci. Am. Sup., V. 70, p. 382, Dec. 10, 1910, for a proof by A. R. Colburn on same form of figure.

Ninety



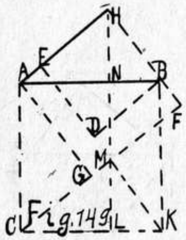
In fig. 148 extend FG to C and draw HG extended to L.

Sq. AK = 2[trap. ACLM = tri. GMA common to sq's AK and AF + (tri. ACG = tri. AMH of sq. AF + tri. HMB of sq. HD) + (tri. CLG = tri. BMD of sq. HD)] = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. V, 1898, p. 73, proof LXXII.

Ninety-One



In fig. 149 extend FG to C, ED to K and draw HL par. to BK.

Sq. AK = rect. BL + rect. AL = (paral. MKBH = sq. HD) + (paral. CMHA = sq. HG) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Journal of Education, V. XXVII, 1888, p. 327, fifteenth proof by M. Dickinson, Winchester, N. H.; Edwards's Geom., 1895, p. 158, fig. (22); Math. Mo., V. V, 1898, p. 73, proof LXXI; Heath's Math. Monographs, No. 2, p. 28, proof XIV.

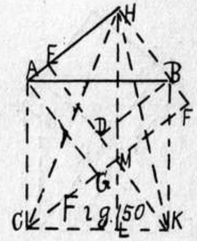
b. An algebraic proof is easily devised from this figure.

Ninety-Two

In fig. 150 extend ED and FG to K and C respectively, draw HL perp. to CK, and draw HC and HK.

Sq. AK = rect. BL + rect. AL = (paral. MKBH = 2 tri. KBH = sq. HD) + (paral. CMHA = 2 tri. CHA = sq. HG) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.



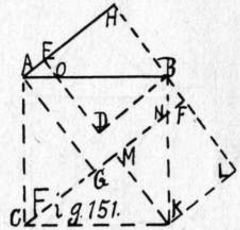
a. See Jury Whipper, 1880, p. 12, fig. 4.

b. This proof is only a variation of the one preceeding.

c. From this figure an algebraic proof is obtainable.

Ninety-Three

In fig. 151 extend FG to C, HF to L making FL = HB, and draw KL and KM respectively par. to AH and BH.

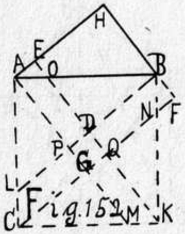


Sq. AK = { [(tri. CKM = tri. BKL) — tri. BNF = trap. OBHE] + (tri. KMN = tri. BOD) = sq. HD } + [(tri. ACG = tri. ABH) + (tri. BOD + hexagon AGNBDO) = sq. HG] = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. As taken from "Philosphia et Mathesis Universa, etc.", Ratisbonae, 1774, by Reichenberger; see Jury Whipper, 1880, p. 29, fig. 27.

Ninety-Four

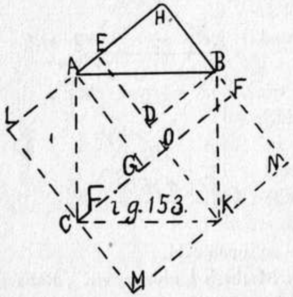


In fig. 152 extend AG, ED, BD and FG to M, K, L and C respectively. Sq. AK = 4 tri. ALP + 4 quad. LCGP + sq. PQ + tri. AOE — (tri. BNE = tri. AOE) = (2 tri. ALP + 3 quad. LCGP + sq. PQ + tri. AOE = sq. HG) + (2 tri. ALP + quad. LCGP — tri. AOE = sq. HD) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 29, fig. 26, as given by Reichenberger, in Philosophic et Mathesis Universa, etc.", Ratisbonae, 1774.

Ninety-Five



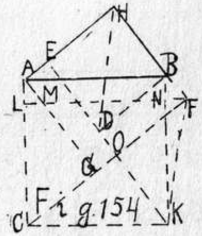
In fig. 153 extend HF and HA respectively to N and L, and complete the sq. HM, and extend ED to K and BG to C.

Sq. AK + 4 tri. ABH = sq. HM = (sq. FK = sq. HD) + sq. HG + (rect. LG = 2 tri. ABH) + (rect. OM = 2 tri. ABH) whence sq. AK = sq. HD + sq. HG.

∴ sq. upon AK = sq. upon BH + sq. upon AH.

a. Similar to Henry Boad's proof, London, 1733; see Jury Whipper, 1880, p. 16, fig. 9; Math. Mo., V. V, 1898, p. 74, proof LXXIV.

Ninety-Six



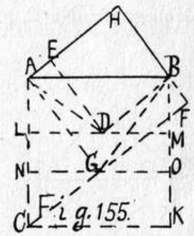
In fig. 154 extend FG and ED to C and K respectively, draw FL par. to AB, and draw HD and FK.

Sq. AK = (rect. AN = paral. MB) + (rect. LK = 2 tri. CKF = 2 tri. CKO + 2 tri. FOK = tri. FMG + tri. ABH + 2 tri. DBH) = sq. HD + sq. HG.

∴ sq upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. V, 1898, p. 74, proof LXXIII.

Ninety-Seven



In fig. 155 produce FG to C, through D and G draw LM and NO par. to AB, and draw AD and BG.

Sq. AK = rect. NK + rect. AO = (rect. AM = 2 tri. ADB = sq. HD) + (2 tri. GBA = sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This is No. 15 of A. R. Colburn's 108 proofs; see his proof in Sci. Am. Sup., V. 70, p. 383, Dec. 10, 1910.

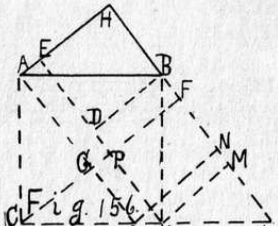
b. An algebraic proof from this figure is easily obtained.

2 tri. BAD = $hx = a^2$ (1)

2 tri. BAG = $h(h - x) = b^2$ (2)

(1) + (2) = (3) $h^2 = a^2 + b^2$. E. S. L.

Ninety-Eight



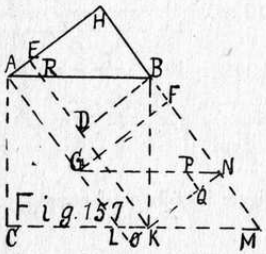
In fig. 156 produce HF and CK to L, ED to K, and AG to O, and draw KM and ON par. to AH.

Sq. AK = paral. AOLB = [trap. AGFB + (tri. OLM = tri. ABH) = sq. HG] + { rect. GN = [tri. CLF — (tri. COG = tri. KLM) — (tri. OLN = tri. CKP)] = sq. FK = sq. HD } = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This proof is due to Prin. Geo. M. Phillips, Ph. D., of the West Chester State Normal School, Pa., 1875; see Heath's Math. Monographs, No. 2, p. 36, proof XXV.

Ninety-Nine



In fig. 157 extend CK and HF to M, ED to K, and AG to O making GO = HB, draw ON par. to AH, and draw GN.

Sq. AK = paral. ALMB = paral. GM + paral. AN = [(tri. NGO — tri. NPQ = trap. RBHE) + (tri. KMN = tri. BRD)] = sq. HD + sq. HG.

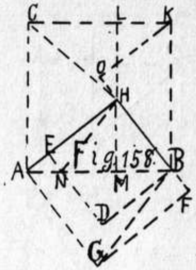
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised by the author, March 14, 1926.

H.

This type includes all proofs devised from the figure in which the squares constructed upon the hypotenuse and the two legs overlap the given triangle.

One Hundred



In figure 158 draw LM par. to KB, draw HK, HC, HN and GB, and produce BH to O making BO = AH, and draw KO.

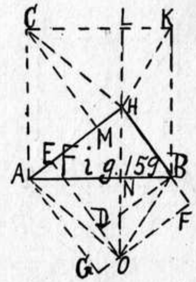
Sq. AK = rect. LB + rect. LA = (2 tri. KHB = 2 tri. BHA = sq. HD) + (2 tri. CAH = 2 tri. AGB = sq. AF) = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author. Afterwards the first part of it was discovered to be the same as the solution in Math. Mo., V. V, 1898, p. 78, proof LXXXI.

b. This figure gives readily an algebraic proof.

One Hundred One



In figure 159 extend ED to O, draw AO, OB, HK and HC, and draw, through H, LO perp. to AB, and draw CM perp. to AH.

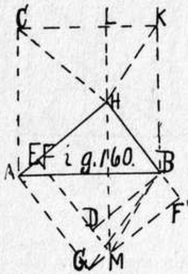
Sq. AK = rect. LB + rect. LA = (paral. HOBK = 2 tri. OBH = sq. HD) + (paral. CAOH = 2 tri. OHA = sq. HG) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Olney's Geom., 1872, Part III, p. 251, 6th method; Journal of Education, V. XXVI, 1887, p. 21, fig. XIII; Hopkins's Geom., 1891, p. 91, fig. VI; Edwards's Geom., 1895, p. 160, fig. (31); Math. Mo., V. V, 1898, p. 74, proof LXXX; Heath's Math. Monographs, No. 1, 1900, p. 26, proof XI.

b. From this figure one can deduce an algebraic proof.

One Hundred Two



In fig. 160 draw LM perp. to AB through H, extend ED to M, and draw BG, BM, HK and HC.

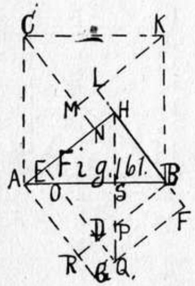
Sq. AK = rect. LB + rect. LA = (paral. KHMB = 2 tri. MBH = sq. HD) + (2 tri. ABH = 2 tri. BAG = sq. HG) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 15, fig. 8.

b. An algebraic proof follows from the "mean prop'l" principle.

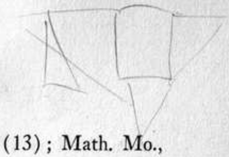
One Hundred Three



In fig. 161 extend ED to Q, BD to R, draw HQ perp. to AB, CN perp. to AH, KM perp. to CN, and extend BH to L.

Sq. AK = tri. ABH common to sq's AK and HG + (tri. BKL = trap. HEDP of sq. HD + tri. QPD of sq. HG) + (tri. KCM = tri. BAR of sq. HG) + (tri. CAN = trap. QFBP of sq. HG + tri. HPB of sq. HD) + (sq. MH = sq. RQ) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.



a. See Edwards's Geom., 1895, p. 157, fig. (13); Math. Mo., V. V, 1898, p. 74, proof LXXXII.

One Hundred Four

In fig. 162 extend ED to P, draw HP, draw CM perp. to AH, and KL perp. to CM.

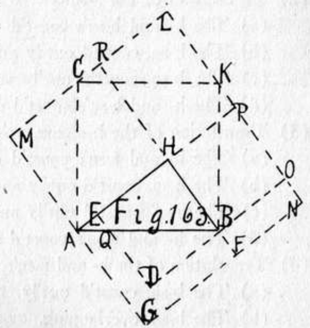
Sq. AK = tri. ANE common to sq's AK and NG + trap. ENBH common to sq's AK and HD + (tri. BOH = tri. BND of sq. HD) + (trap. KLMO = trap. AGPN) + (tri. KCL = tri. PHE of sq. HG) + (tri. CAM = tri. HPF of sq. HG) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author August 3, 1890.

b. Many other proofs may be devised from this type of figure.

One Hundred Five



In fig. 163 extend GA to M, GF to N, and complete rect. MN, and extend DB and AH respectively to O and P.

Sq. AK + rect. BN + 3 tri. ABH + trap. AGFB = rect. MN = (sq. HO = sq. HD) + sq. HG + rect. BN + [rect. AL = (rect. HL = 2 tri. ABH) + (sq. AR = tri. ABH + trap. AGFB)] = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 22, fig. 16, credited by Joh. Hoffmann in "Der Pythagoraische Lehrsatz," 1821, to Henry Boad, of London, Eng.

I

This type includes all proofs derived from a figure in which there has been a translation from its normal position of one or more of the constructed squares.

Symbolizing the hypotenuse-square by h, the shorter-leg-square by a, and the longer-leg-square by b, we find, by inspection, that there are *seven* distinct cases possible in this I-type of figure, and that each of the first three cases have *four* possible arrangements, each of the second three cases have *two* possible arrangements, and the seventh case has but one arrangement, thus giving 19 sub-types, as follows:

(1) Translation of the h-square, with
  (a) The a- and b-squares constructed outwardly.
  (b) The a-sq. const'd out'ly and the b-sq. overlapping.
  (c) The b-sq. const'd out'ly and the a-sq. overlapping.
  (d) The a- and b-sq's const'd overlapping.
(2) Translation of the a-square, with
  (a) The h- and b-sq's const'd out'ly.
  (b) The h-sq. const'd out'ly and the b-sq. overlapping.
  (c) The b-sq. const'd out'ly and the h-sq. overlapping.
  (d) The h- and b-sq's const'd overlapping.
(3) Translation of the b-square, with
  (a) The h- and a-sq's const'd out'ly.
  (b) The h-sq. const'd out'ly and the a-sq. overlapping.
  (c) The a-sq. const'd out'ly and the h-sq. overlapping.
  (d) The h- and a-sq's const'd overlapping.
(4) Translation of the h- and a-sq's, with
  (a) The b-sq. const'd out'ly.
  (b) The b-sq. overlapping.
(5) Translation of the h- and b-sq's with

  (a) The a-sq. const'd out'ly.
  (b) The a-sq. const'd overlapping.
(6) Translation of the a- and b-sq's, with
  (a) The h-sq. const'd out'ly.
  (b) The h-sq. const'd overlapping.
(7) Translation of all three, h-, a- and b-squares.

From the sources of proofs consulted, I discovered that only 8 out of the possible 19 cases had received consideration. To complete the gap of the 11 missing ones I have devised a proof for each missing case, as by the Law of Dissection (see fig. 62, proof *Four*) a proof is readily produced for any position of the squares. Like Agassiz's student, after proper observation he found the law, and then the arrangement of parts (scales) produced desired results.

CASE (1), (a).

One Hundred Six

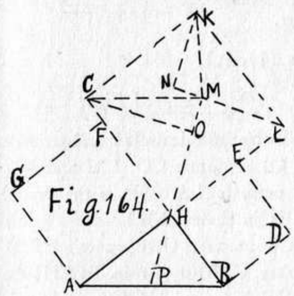


Fig. 164.

In fig. 164 the sq. upon the hypotenuse, hereafter called the h— sq., has been translated to the position HK. From P the middle pt. of AB draw PM making HM = AH; draw LM, KM, and CM; draw KN = LM, perp. to LM produced, and CO = AB, perp. to HM.

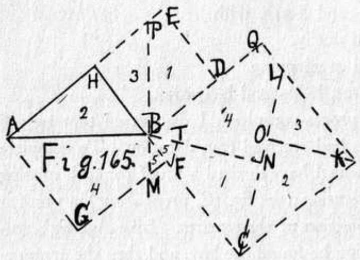
Sq. HK = (2 tri. HMC = HM × CO = sq. AH) + (2 tri. MLK = ML × KN = sq. BH) = sq. BH + sq. AH.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author, August 4, 1900. Several other proofs from this figure is possible.

Case (1), (b).

One Hundred Seven



In fig. 165, the position of the sq's are evident, as the b— sq. overlaps and the h— sq. is translated to right of normal position. Draw EM perp. to AB through B, take KL = EB, draw LC, and BN and KO perp. to LC, and FB perp. to BN.

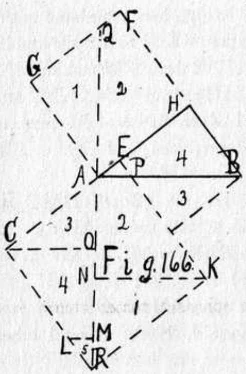
Sq. BK = (trap. FCNB = trap. PBDE) + (tri. CKO = tri. ABH) + (tri. KLO = tri. BPH) + (quad. BOLQ + tri. BTF = trap. GFBA) = sq. BH + sq. AH.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. One of my dissection devices.

Case (1), (c).

One Hundred Eight



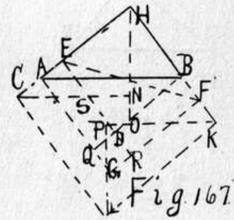
In fig. 166 draw LA and produce it to Q, and draw CO, LM and KN each perp. to LA. (D is the vertex opp. H in the sq. EB.)

Sq. CK = (tri. CAB = tri. BPD) + (trap. CLMO + trap. BPEH) + (tri. KRN = tri. AQG) + (quad. NKEA + tri. RML = trap. AHTQ) = sq. HD + sq. CK.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

Case (1), (d).

One Hundred Nine



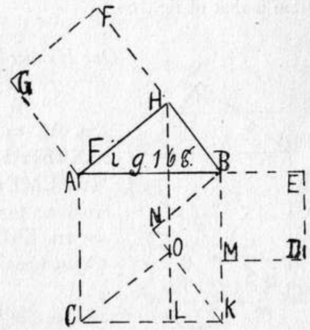
In fig. 167 draw HO and LP each equal to and perp. to AB, draw CN, and KP each par. to AB, draw EF, and produce ED to R and BD to Q.

Sq. CK = (tri. KLP = trap. BHES of sq. HD + tri. AES of sq. AG) + (tri. HKO = trap. AQDS of sq. HG + tri. BSD of sq. HD) + (tri. CHA = tri. FEH of sq. HG) + (tri. LCT = tri. EFR of sq. HD) + (tri. CHN = tri. FEH of sq. HG) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

Case (2), (a).

One Hundred Ten



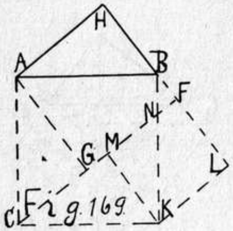
In fig. 168 with sq's placed as in the figure, draw HL perp. to CK, CO and BN par. to AH, making BN = BH, and draw KN.

Sq. AK = rect. BL + rect. AL = (paral. OKBH = sq. BD) + (paral. COHA = sq. AF) = sq. BD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

Case (2), (b).—For which are more proofs extant than for any other of these 19 cases—Why? Because of the ready dissection of the resulting figures.

One Hundred Eleven



In fig. 169 extend FG to C.

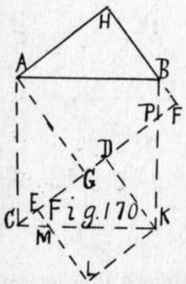
Sq. AK = (pentagon AGMKB = quad. AGNB common to sq's AK and AF + tri. KNM common to sq's AK and FK) + (tri. ACG = tri. BNF + trap. NKLF) + (tri. CKM = tri. ABH) = sq. FK + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Hill's Geom. for Beginners, 1886, p. 154, proof I; Beeman and Smith's New Plane and Solid Geom., 1899, p. 104, fig. 4.

b. This figure is of special interest as the sq. ML may occupy 15 other positions having common vertex with sq. AK and its sides coincident with side or sides produced of sq. HG. One such solution is that of fig. 164.

One Hundred Twelve



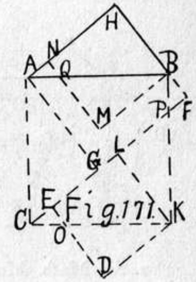
In fig. 170 extend FG to C.

Sq. AK = quad. AGNB common to sq's AK and AF + (tri. ACG = tri. ABH) + (tri. CME = tri. BNF) + (trap. EMKD common to sq's AK and EK) + (tri. KND = tri. KML) = sq. DL + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 161, fig. (35).

One Hundred Thirteen



In fig. 171 extend FG to C and const. sq. HM = sq. LD, the sq. translated.

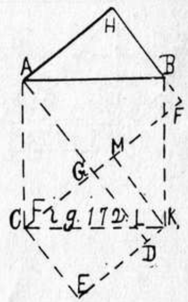
Sq. AK = (tri. ACG = tri. ABH) + (tri. COE = tri. BPF) + (trap. EOKL common to both sq's AK and LD, or = trap. NQBH) + (tri. KPL = tri. KOD = tri. BQM) + [(tri. BQM + polygon AGPBMQ) = quad. AGPB common to sq's AK and AF] = sq. LD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Sci. Am. Sup., V. 70, p. 359, Dec. 3, 1910, by A. R. Colburn.

b. I think it better to omit sq. HM (not necessary), and thus reduce it to proof above.

One Hundred Fourteen



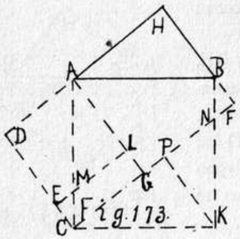
In fig. 172 extend ED to K and draw KM par. to BH.

Sq. AK = quad. AGNB common to sq's AK and AF + (tri. ACG = tri. ABH) + (tri. CKM = trap. CEDL + tri. BNF) + (tri. KNM = tri. CLG) = sq. GE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 156, fig. (8).

One Hundred Fifteen



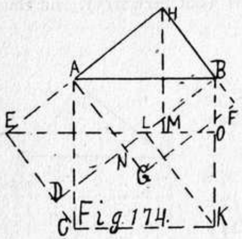
In fig. 173 extend DE to C and draw KN par. to HB.

Sq. AK = quad. AGNB common to sq's AK and HG + (tri. ACG = tri. CAD = trap. EMAD + tri. BNF) + (tri. CKN = tri. ABH) + (tri. KNP = tri. AML) = sq. DL + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. VI, 1899, p. 33, proof LXXXVI.

One Hundred Sixteen



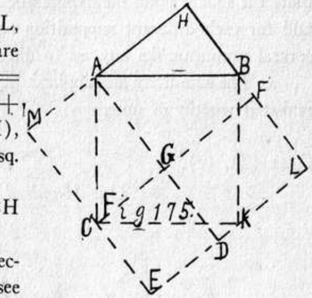
In fig. 174 extend ED to C, DN to B, and draw EO par. to AB, KL perp. to DB and HM perp. to EO.

Sq. AK = rect. AO + rect. CO = paral. AELB + paral. ECKL = sq. AD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. VI, 1899, p. 33, LXXXVIII.

One Hundred Seventeen

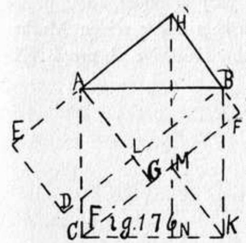


In fig. 175 extend HF to L, HA to M and complete the square HE. Sq. AK + 4 tri. ABH = sq. HE = sq. CD + sq. AF + (2 rect. GL = 4 tri. ABH), whence sq. AK = sq. CD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This is one of the conjectured proofs of Pythagoras; see Ball's Short Hist. of Math., 1888, p. 24; Hopkins's Plane Geom., 1891, p. 91, fig. IV; Edwards's Geom., 1895, p. 162, fig. (39); Beman and Smith's New Plane Geom, 1899, p. 103, fig. 2; Heath's Math. Monographs, N. 1, 1900, p. 18, proof II.

One Hundred Eighteen



In fig. 176 extend FG to C, draw HN perp. to CK and KM par. to HB.

Sq. AK = rect. BN + rect. AN = paral. BHMK + paral. HACM = sq. AD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. VI, 1899, p. 33, proof LXXXVII.

b. In this figure the given triangle may be either ACG, CKM, HMF or BAL; taking either of these four triangles several proofs for each is possible. Again, by inspection, we observe that the given triangle may have any one of seven other positions within the square AGFH, right angles coinciding. Furthermore the square upon the hypotenuse may be constructed overlapping, and for each different supposition as to the figure there will result several proofs unlike any, as to dissection, given heretofore.

c. The simplicity and applicability of figures under Case (2), (b) makes it worthy of note.

CASE (2), (c).

One Hundred Nineteen

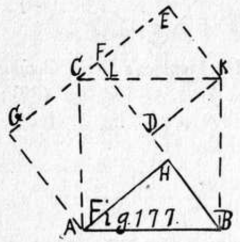


Fig. 177

In fig. 177 ED being the sq. translated, the construction is evident.

Sq. AK = quad. AHLC common to sq's AK and AF + (tri. ABC = tri. ACG) + (tri. BKD = trap. LKEF + tri. CLF) + tri. KLD common to sq's AK and ED = sq. ED + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 22, fig. 17, as given by von Hauff, in "Lehrbegriff der reinen Mathematik," 1803; Heath's Math. Monographs, 1900, No. 2, proof XX.

CASE (2), (d).

One Hundred Twenty

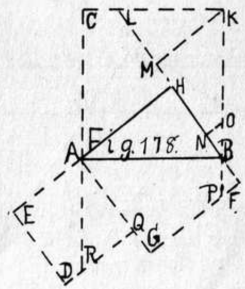


Fig. 178

In fig. 178 extend KB to P, CA to R, BH to L, draw KM perp. to BL, take MN = HB, and draw NO par. to AH.

Sq. AK = tri. ABH common to sq's AK and AF + (tri. BON = tri. BPF) + (trap. NOKM = trap. DRAE) + (tri. KLM = tri. ARQ) + (quad. AHLC = quad. AGPB) = sq. AD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. VI, 1899, p. 34, proof XC.

One Hundred Twenty-One

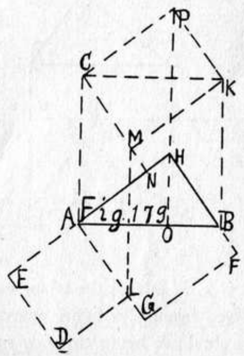


Fig. 179

In fig. 179 upon CK const. tri. CKP = tri. ABH, draw CN par. to BH, KM par. to AH, draw ML and through H draw PO.

Sq. AK = rect. KO + rect. CO = (paral. PB = paral. CL = sq. AD) + (paral. PA = sq. AF) = sq. AD + sq. AF.

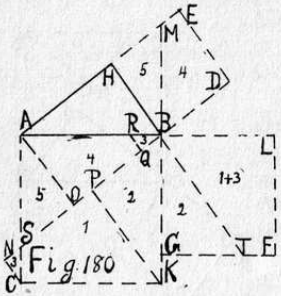
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author, July 28, 1900.

b. An algebraic proof comes readily from this figure.

CASE (3), (a).

One Hundred Twenty-Two



In fig. 180 produce DB to N, HB to F, KB to M, and draw CN, AO, KP and RQ perp. to NB.

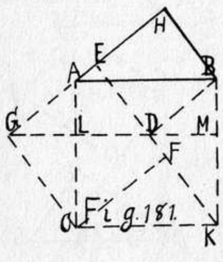
Sq. AK = (quad. CKPS + tri. BRQ = trap. TBLF) + (tri. KBP = tri. TBG) + (trap. OQRA = trap. BMED) + (tri. ASO = tri. BMH) = sq. HD + sq. GL.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised for missing Case (3), (a), March 17, 1926.

CASE (3), (b).

One Hundred Twenty-Three



In fig. 181 extend ED to K and through D draw GM par. to AB.

Sq. AK = rect. AM + rect. CM = (paral. GB = sq. HD) + (paral. CD = sq. GF) = sq. EB + sq. EC.

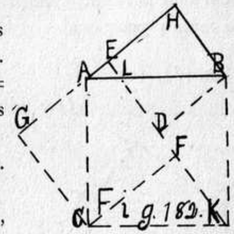
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. VI, 1899, p. 33, proof LXXXV.

b. This figure furnishes an algebraic proof.

c. If any of the triangles congruent to tri. ABH is taken as the given triangle, a figure expressing a different relation of the squares is obtained, hence covering some other case of the 19 possible cases.

One Hundred Twenty-Four



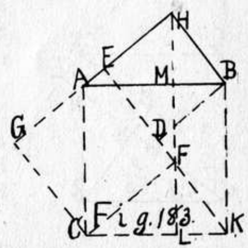
In fig. 182 extend EF to K.

Sq. AK = quad. ACFL common to sq's AK and GF + (tri. CKF = trap. LBHE + tri. ALE) + (tri. KBD = tri. CAG) + tri. BDL common to sq's AK and HD = sq. HD + sq. AK.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Olney's Geom., Part III, 1872, p. 250, 2nd method; Jury Whipper, 1880, p. 23, fig. 18; proof by E. Forbes, Winchester, N. H., as given in Jour. of Ed'n, V. XXVIII, 1888, p. 17, 25th proof; Jour. of Ed'n, V. XXV, 1887, p. 404, fig. II; Hopkins's Plane Geom., 1891, p. 91, fig. III; Edwards's Geom., 1895, p. 155, fig. (5); Math. Mo., V. VI, 1899, p. 33, proof LXXXIII; Heath's Math. Monographs, No. 1, 1900, p. 21, proof V; Geometric Exercises in Paper Folding, by T. Sundara Row, fig. 13, p. 14 of 2nd Edition of The Open Court Pub. Co., 1905. Every teacher of geometry should use this paper folding proof.

One Hundred Twenty-Five



In fig. 183 extend EF to K, and HL perp. to CK.

Sq. AK = rect. BL + rect. AL = paral. BF + paral. AF = sq. HD + sq. GF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. VI, 1899, p. 33, proof LXXXIV.

b. Fig. 183 is fig. 182 with extra

line HL, fig. 182 gives a proof by congruency, while fig. 183 gives a proof by equivalency, and it also gives an algebraic proof by use of the mean proportional.

CASE (3), (c).

One Hundred Twenty-Six

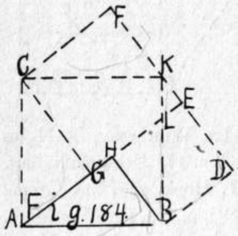


Fig. 184

In fig. 184 the construction is evident, FG being the translated b-square. Sq. AK = quad. GLKC common to sq's AK and CE + (tri. CAG = trap. BDEL + tri. KLE) + (tri. ABH = tri. CKF) + tri. BLH common to sq's AK and HD = sq. HD + sq. CE.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Halsted's Elements of Geom., 1895, p. 78, theorem XXXVII; Edwards's Geom., 1895, p. 156, fig. (6); Heath's Math. Monographs, No. 1, 1900, p. 27, proof XIII.

One Hundred Twenty-Seven

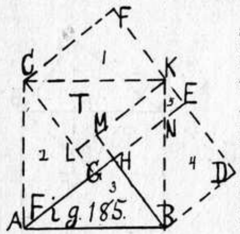


Fig. 185

In fig. 185 draw KL perp. to CG and extend BH to M.

Sq. AK = (tri. ABH = tri. CKF) + tri. BNH common to sq's AK and HD + (quad. CGNK = sq. LH + trap. MHNK + tri. KCL) common to sq's AK and FG + (tri. CAG = trap. BDEN + tri. KNE) = sq. HD + sq. FG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Sci. Am. Sup., V. 70, p. 383, Dec. 10, 1910, in which proof A. R. Colburn makes T the given triangle, and then substitutes part 2 for part 1, part 3 for parts 4 and 5, thus showing sq. AK = sq. HD + sq. FG.

CASE (3), (d).

One Hundred Twenty-Eight

In fig. 186 produce AH to O, draw CN par. to HB, and extend CA to G. Sq. AK = trap. EMBH common to sq's AK and HD + (tri. BOH = tri. BMD) + (quad. NOKC = quad. FMAG) + (tri. CAN = tri. GAL) + tri. AME common to sq's AK and EG = sq. HD + sq. LF.

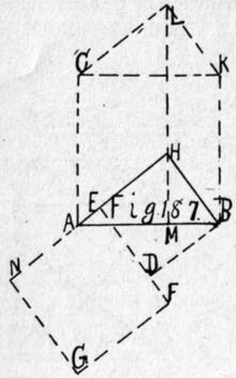
∴ sq. upon AB = sq. upon BH + sq. upon AH.

Fig. 186

a. See Math. Mo., V. VI, 1899, p. 34, proof LXXXIX.

b. As the relative position of the given triangle and the translated square may be indefinitely varied, so the number of proofs must be indefinitely great, of which the following is one.

One Hundred Twenty-Nine



In fig. 187 draw LM through H. Sq. AK = rect. KM + rect. CM = paral. KH + paral. CH = sq. HD + (sq. on AH = sq. NF).

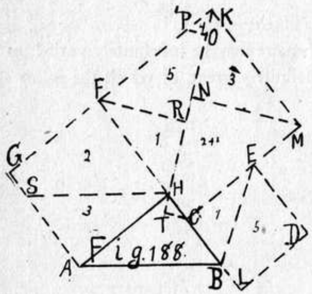
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author, July 28, 1900.

b. An algebraic solution may be devised from this figure.

CASE (4), (a).

One Hundred Thirty



In fig. 188 extend KH to T making NT = AH, draw TC, draw FR, MN and PO perp. to KH, and draw HS par. to AB.

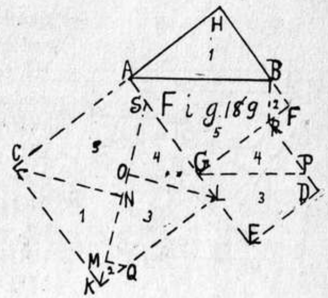
Sq. CK = (quad. CMNH + tri. KPO = quad. SHFG) + (tri. MKN = tri. HSA) + (trap. FROP = trap. EDLB) + (tri. FHR = tri. ECB) = sq. CD + sq. GH.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised by author for case (4), (a) March 18, 1926.

CASE (4), (b).

One Hundred Thirty-One



In fig. 189 draw GP par. to AB, take LS = AH, draw KS, draw LO, CN and QM perp. to KS, and draw BR.

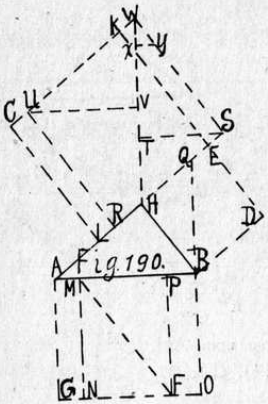
Sq. AK = (tri. CKN = tri. ABH) + (tri. QKM = tri. BRF) + (trap. QLOM = trap. PGED) + (tri. LSO = tri. GPR) + (quad. COSA = quad. AGRB) = sq. GD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised by author for Case (4), (b).

Case (5), (a).

One Hundred Thirty-Two



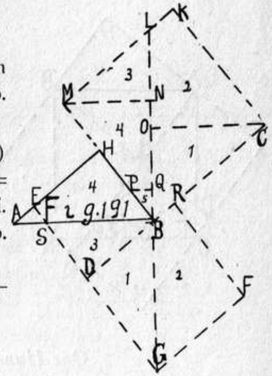
In fig. 190 CE and AF are the translated sq's; produce GF to O and complete the sq. MO; produce HE to S and complete the sq. US; produce OB to Q, draw MF, draw WH, draw ST and UV perp. to WH, and take TX = HB and draw XY perp. to WH. Since sq. MO = sq. AF, and sq. US = sq. CE, and since sq. RW = (quad. URHV + tri. WYX = trap. MFOB + (tri. HST = tri. BQH) + (trap. TSYX = trap. BDEQ) + tri. UVW = tri. MFN) = sq. HD + (sq. NB = sq. AF).

∴ sq. RW = sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised March 18, 1926, for Case (5), (a).

Case (5), (b).

One Hundred Thirty-Three



In fig. 191 draw GL through B, and draw PQ, CO and MN perp. to BL.

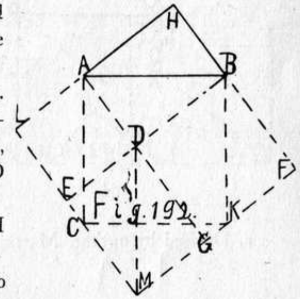
Sq. BK = (tri. CBO = tri. BGD) + (quad. OCKL + tri. BPO = trap. GFRB) + (tri. MLN = tri. BSD) + (trap. PQNM = trap. SEHB) = sq. HD + sq. DF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised for Case (5), (b).

Case (6), (a).

One Hundred Thirty-Four



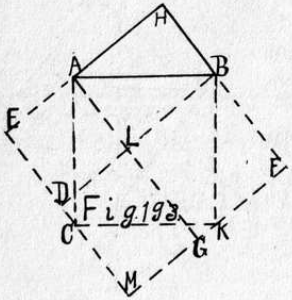
In fig. 192 extend ED and FG to M thus completing the sq. HM, and draw DM.

Sq. AK + 4 tri. ABC = sq. HM = sq. LD + sq. DF + (2 rect. HD = 4 tri. ABC), from which sq. AK = sq. LD + sq. DF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This proof is credited to M. McIntosh of Whitwater, Wis. See Jour. of Ed'n, 1888, V. XXVII, p. 327, seventeenth proof.

One Hundred Thirty-Five



In fig. 193 complete the sq. on EH.

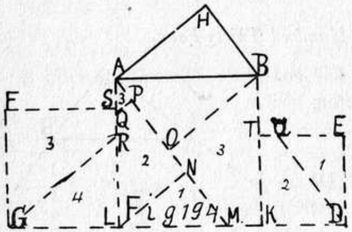
Sq. HM — (4 tri. ABC = 2 rect. HL) = sq. AK = sq. EL + sq. LF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jour. of Ed'n, 1887, V. XXVI, p. 21, fig. XII.

b. Another proof is $h^2 = (a + b)^2 - 2ab = a^2 + b^2$.

One Hundred Thirty-Six

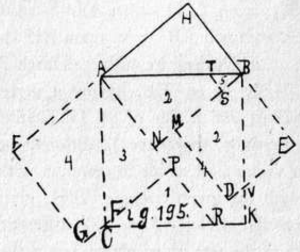


In fig. 194, the translation is evident. And since, by proof *Six,* fig. 64, the parts 1, 2, (3 + 3) and 4 of sq. AK are congruent with corresponding parts of sq's KE and LF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised by author, March 28, 1926, 3 p. m.

One Hundred Thirty-Seven

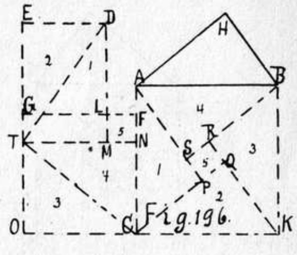


In fig. 195, the translation and construction is evident. Sq. AK = (tri. CRP = tri. BVE) + (trap. ANST = trap. BMDV) + (quad. NRKB + tri. BST = trap. AFGC) + tri. ACP common to sq. AK and AG = sq. ME + sq. FP.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised by author, March 26, 1926, 10 p. m.

One Hundred Thirty-Eight



In fig. 196 the sq. on AH is translated to position of GC, and the sq. on HB to position of GD. Draw AP and KR par. to BH making each = AH, draw CQ and BS par. to AH, take CN = BH and draw NT par. to AB, continue DL to M, and draw DT and TC. Consider the two sq's EL and GC as the two rect's EM and TC, and the sq. LN.

Sq. AK = (tri. ACP = tri. DTM) + (tri. CKQ = tri. TDE) + (tri. KBR = tri. CTO) + (tri. BAS = tri. TCN) + (sq. SQ = sq. LN) = sq. EL + sq. GC.

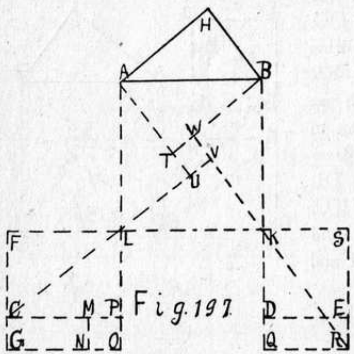
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised by author, March 22, 1926.

b. As sq. EL, having a vertex and a side in common with a vertex and a side of sq. GC, either externally (as in fig. 194), or internally, may have 12 different positions, and as sq. GC may have a vertex and a side in common with the fixed sq. AK, or in common with the given triangle ABH, giving 15 different positions, there is possible 180 — 3 = 177 different figures, hence 176 proofs other than the one given above, using the dissection as used here, and 178 more proofs by using the dissection as given in proof *Four*, fig. 62.

c. This proof is a variation of that given in proof *Five*, fig. 63.

One Hundred Thirty-Nine



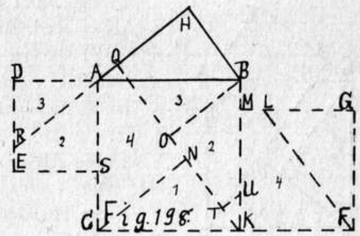
In fig. 197 the construction is evident, as FO is the translation of the sq. on AH, and KE is the translation of the sq. on BH.

Since rect. CN = rect. QE, we have sq. AK = (tri. LKV = tri. CLP) + (tri. KBW = tri. LCF) + (tri. BAT = tri. KRQ) + (tri. ALU = tri. RKS) + (sq. TV = sq. MO) = rect. KR + rect. FP + sq. MO = sq. KE + sq. FO.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised by author, March 30, 1926, 8 p. m.

One Hundred Forty



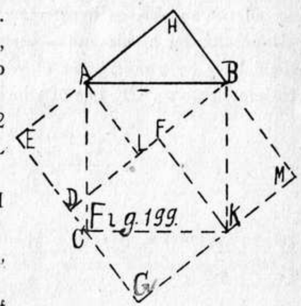
In figure 198 the translation and construction is evident.

Parts 2 and 3 of sq. AK = parts 2 and 3 of sq. DS, parts 1 and 4 + 4 of sq. AK = parts 1 and 4 of sq. KG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised by author, March 27, 1926, 10:40 p. m.

One Hundred Forty-One



In fig. 199 complete the sq. on EH, draw BD par. to AH, and draw AL and KF perp. to DB.

Sq. HG — (4 tri. ABH = 2 rect. HL) = sq. AK = sq. EL + sq. DK.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

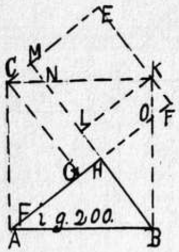
a. See Edwards's Geom., 1895, p. 158, fig. (19).

b. By changing position of sq. FG, many other proofs might be obtained.

c. This is a variation of proof, fig. 153.

CASE (6), (b).

One Hundred Forty-Two



In fig. 200 the construction is evident. Sq. AK = (tri. ABH = trap. KEMN + tri. KOF) + (tri. BOH = tri. KLN) + quad. GOKC common to sq's AK and CF + (tri. CAG = tri. CKE) = sq. MK + sq. CF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Hopkins's Plane Geom., 1891, p. 92, fig. VIII.

b. By drawing a line EH, a proof through parallelogram, may be obtained. Also an algebraic proof.

c. Also any one of the other three triangles, as CAG may be called the given triangle, from which other proofs would follow. Furthermore since the tri. ABH may have seven other positions leaving side of sq. AK as hypotenuse, and the sq. MK may have 12 positions having a side and a vertex in common with sq. CF, we would have 84 proofs, some of which have been or will be given; etc., etc., as to sq. CF, one of which is the next proof.

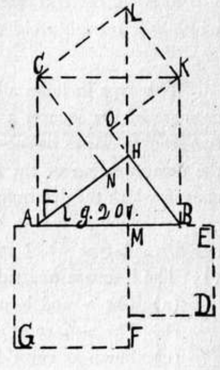
One Hundred Forty-Three

In fig. 201, through H draw LM, and draw CN par. to BH and KO par. to AH.

Sq. AK = rect. KM + rect. CM = paral. KH + paral. CH = HB × KO + AH × CN = sq. on BH + sq. on AH = sq. MD + sq. MG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

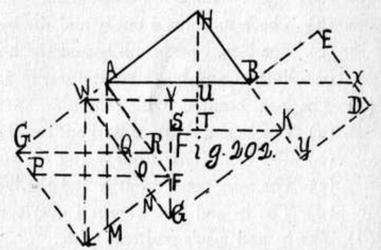
a. Original with the author January 31, 1926, 3 p. m.



CASE (7), (a).

One Hundred Forty-Four

In fig. 202 extend AB to X, draw WU and KS each = to AH and par. to AB, CV and HT perp. to AB, GR and FP par. to AB, and LW and AM perp. to AB.



Sq. AK = (tri. KCS + tri. FPL = trap. BYDX + tri. FON) + (tri. HKT = tri. GRA = tri. BEX + trap. RAWQ) + (tri. WUH = tri. LWG) + (tri. CWV = tri. WLN) + (sq. VT = paral. QOFR) = sq. BD + sq. GF.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author, August 8, 1900.

b. As in fig. 200 many other arrangements are possible each of which will furnish proof or proofs.

J.

This type includes all proofs derived from figures in which one or more of the squares are not graphically represented. There are two leading classes or sub-types in this type—first, the class in which the determination of the proof is based upon a square; second, the class in which the determination of the proof is based upon a triangle.

As in the I-type, so here, by inspection we find 6 sub-classes in our first sub-type which may be symbolized thus:

(1) The h-square omitted, with
  (a) The a- and b-squares const'd outwardly—3 cases.
  (b) The a-sq. const'd out'ly and the b-sq. overlapping—3 cases.
  (c) The b-sq. const'd out'ly and the a-sq. overlapping—3 cases.
  (d) The a- and b-squares overlapping.—3 cases.
(2) The a-sq. omitted, with
  (a) The h- and b-sq's const'd out'ly—3 cases.
  (b) The h-sq. const'd out'ly and the b-sq. overlapping—3 cases.
  (c) The b-sq. const'd out'ly and the h-sq. overlapping—3 cases.
  (d) The h- and b-sq's const'd out'ly and overlapping—3 cases.
(3) The b-sq. omitted, with
  (a) The h- and a-sq's const'd out'ly—3 cases.
  (b) The h-sq. const'd out'ly and the a-sq. overlapping—3 cases.
  (c) The a-sq. const'd out'ly and the h-sq. overlapping—3 cases.
  (d) The h- and a-sq's const'd overlapping—3 cases.
(4) The h- and a-sq's omitted, with
  (a) The b-sq. const'd out'ly,
  (b) The b-sq. const'd overlapping,
  (c) The b-sq. translated—in all 3 cases.
(5) The h- and b-sq'd omitted, with
  (a) The a-sq. const'd out'ly,
  (b) The a-sq. const'd overlapping,

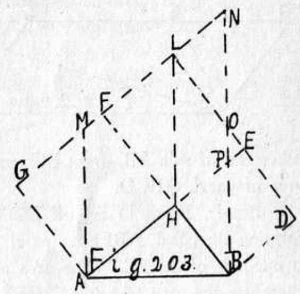
  (c) The a-sq. translated—in all 3 cases.
(6) The a- and b-sq's omitted, with
  (a) The h-sq. const'd out'ly,
  (b) The h-sq. const'd overlapping,
  (c) The h-sq. translated—in all 3 cases.

The total of these enumerated cases is 45. We shall give but a few of these 45, leaving the remainder to the ingenuity of the interested student.

(A)—Proofs determined by arguments based upon a square.

Case (1), (a).

One Hundred Forty-Five



In fig. 203 produce GF and DE to N and L respectively and draw AM, HL and BN perp. to AB. The tri. AMG = tri. ABH.

Sq. HD + sq. GH = (paral. HO = paral. LP) + paral. MH = paral. MP = AM $\times$ AB = AB $\times$ AB = $(AB)^2$.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised by author for case (1), (a), March 20, 1926.

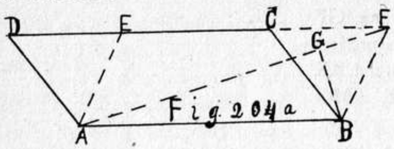
b. See proof *Fifty-One,* fig. 109. By omitting lines CK and HN in said figure we have fig. 203. Therefore proof No. 145 is only a variation of proof No. 51, fig. 109.

Analysis of proofs given will show that many supposedly new proofs are only modifications of some more fundamental proof.

Case (1), (b).

While this case may be proved in some other way, we have selected the following as being quite unique. It is due to the ingenuity of Mr. Arthur R. Colburn of Washington, D. C., and is No. 97 of his 108 proofs.

It rests upon the following Theorem on Parallelogram, which is: "If from one end of the side of a parallelogram a straight line be drawn to any point in the opposite side, or the opposite side extended, and a line from the other end of said first side be drawn perpendicular to the first line, or its extension, the product of these two drawn lines will measure the area of the parallelogram." Mr. Colburn formulated this theorem and its use is discussed in Vol. 4, p. 45, of the "Mathematics Teacher," Dec., 1911. I have not seen his proof, but have demonstrated it as follows:



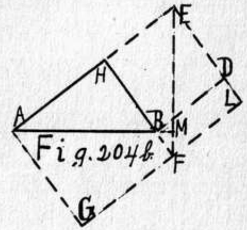
In the paral. ABCD, from the end A of the side AB, draw AF to side DC produced, and from B, the other end of side AB, draw BG perp. to AF. Then AF × BG = area of paral. ABCD.

Proof: From D lay off DE = CF, and draw AE and BF forming the paral. ABFE = paral. ABCD. ABF is a triangle and is one-half of ABFE. The area of tri. FAB = ½ FA × BG; therefore the area of paral. ABFE = 2 times the area of the tri. FAB, or FA × BG. But the area of paral. ABFE = area of paral. ABCD.

∴ AF × BG measures the area of paral. ABCD. Q. E. D.

By means of this Parallelogram Theorem the Pythagorean Theorem can be proved in many cases, of which here is one.

One Hundred Forty-Six



In fig. 204b extend GF and ED to L completing the paral. AL, draw FE and extend AB to M. Then by the paral. theorem:

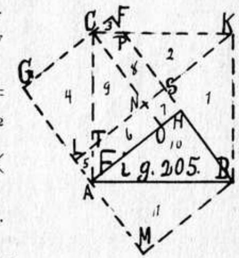
(1) EF × AM = AE × AG.
(2) EF × BM = FL × BF
(1) — (2) = (3) EF(AM — BM) = AE × AG — FL × BF
(3) = (4) (EF = AB) × AB = AGFH + BDEH
or sq. AB = sq. HG + sq. HD
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This is No. 97 of A. R. Colburn's 108 proofs.

b. By inspecting this figure we discover in it the five dissected parts as set forth by my Law of Dissection. See proof *Four,* fig. 62.

Case (2), (c).

One Hundred Forty-Seven



In fig. 205 produce GA to M making AM = HB, draw BM, and draw KL par. to AH and CO par. to BH.

Sq. AK = 4 tri. ABH + sq. NH = $4 \times \frac{AH \times BH}{2} + (AH - BH)^2$

$= 2AH \times BH + AH^2 - 2AH \times BH + BH^2 = BH^2 + AH^2$.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

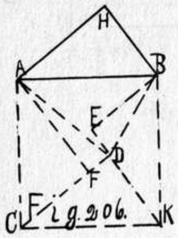
a. Original with author, March, 1926.

b. See Sci. Am. Sup., V. 70, p. 383, Dec. 10, 1910, Fig. 17, in which Mr. Colburn makes use of the tri. BAM.

c. Another proof by author is obtained by comparison and substitutions of dissected parts as numbered.

CASE (6), (a). This is a popular figure with authors.

One Hundred Forty-Eight



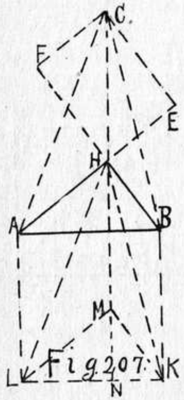
In fig. 206 draw CD and KD par. respectively to AH and BH, draw AD and BD, and draw AF perp. to CD and BE perp. to KD extended.

Sq. AK = 2 tri. CDA + 2 tri. BDK = CD × AF + KD × EB = $CD^2 + KD^2$.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author, August 4, 1900.

One Hundred Forty-Nine



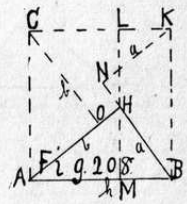
In fig. 207 extend AH to E making HE = HB, extend BH to F making HF = HA, through H draw CN perp. to AB, draw LM and KM par. respectively to AH and BH, and, having completed the rect. FE, draw CA, CB, HL and HK.

Sq. AK = rect. BN + rect. AN = paral. BM + paral. AM = (2 tri. HMK = 2 tri. CHB = sq. BH) + (2 tri. HAL = 2 tri. CAH = sq. AH).

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with author March 26, 1926, 9 p. m.

One Hundred Fifty



In fig. 208 complete the sq. AK overlapping the tri. ABH, draw through H the line LM perp. to AB, extend BH to N making BN = AH, and draw KN perp. to BN, and CO perp. to AH. Then, by the parallelogram theorem, Case (1), (b),

Sq. AK = paral. KM + paral. CM = (BH × KN = a^2) + (AH × CO = b^2) = $a^2 + b^2$.

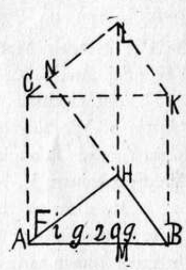
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Teacher, Vol. 4, p. 45, 1911, where the proof is credited to Arthur H. Colburn.

b. See fig. 212—which is more fundamental, proof No. 150 or proof No. 154?

c. See fig. 65 and fig. 213.

One Hundred Fifty-One



In fig. 209 draw CL and KL par. to AH and BH respectively, and through H draw LM.

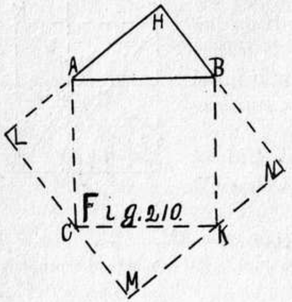
Sq. AK = rect. KM + rect. LM = paral. KH + paral. CH = BH × NL + AH × NH = $BH^2 + AH^2$.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. This is known as Haynes's Solution. See the Math. Magazine, V. I, p. 60, 1882. Also said to have been discovered in 1877 by Geo. M. Phillips, Ph. D., Prin. of the West Chester State Normal School, Pa. See Heath's Math. Monographs, No. 2, p. 38, proof XXVI.

b. An algebraic proof is easily obtained.

One Hundred Fifty-Two



In fig. 210 extend HB to N and complete the sq. HM. Sq. AK = sq. HM $-4\dfrac{HB \times HA}{2} = (LA + AH)^2 - 2HB \times HA = LA^2 + 2LA \times AH + AH^2 - 2HB \times HA = BH^2 + AH^2$.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. Credited to T. P. Stowell, of Rochester, N. Y. See The Math. Magazine, V. I, 1882, p. 38; Olney's Geom., Part III, 1872, p. 251, 7th method; Jour. of Ed'n, V. XXVI, 1877, p. 21, fig. IX; also V. XXVII, 1888, p. 327, 18th proof, by R. E. Binford, Independence, Texas; The School Visitor, V. IX, 1888, p. 5, proof II; Edwards's Geom., 1895, p. 159, fig. (27); Math. Mo., V. VI, 1899, p. 70, proof XCIV; Heath's Math. Monographs, No. 1, 1900, p. 23, proof VIII; Sci. Am. Sup., V. 70, p. 359, fig. 4, 1910.

b. For algebraic solutions, see p. 2, in a pamphlet by Artemus Martin of Washington, D. C., Aug. 1912, entitled "On Rational Right-Angled Triangles"; and a solution by A. R. Colburn, in Sci. Am. Supplement, V. 70, p. 359, Dec. 3, 1910.

c. By drawing the line HM, and considering the part of the figure to the right of said line HM, we have the figure from which the proof known as Garfield's Solution follows—see proof *One Hundred Fifty-Seven,* fig. 215.

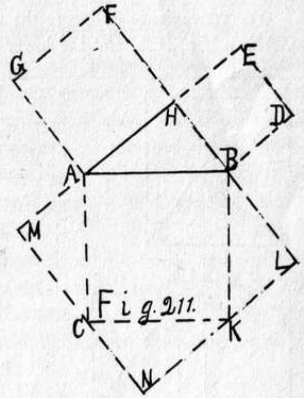
One Hundred Fifty-Three

In fig. 211, extend HA to M and complete the sq. ML. Sq. AK = sq. ML $-4\dfrac{HB \times HA}{2} = (HB + HA)^2 - 2HB \times HA = HB^2 + 2HB \times HA + HA^2 - 2HB \times HA$ = sq. HB + sq. HA.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 35, fig. 32, as given in "Hubert's Rudimenta Algebrae," Wurceb, 1762.

b. This is but a variation of proof *Ninety-Five,* fig. 153.



CASE (6), (b).

One Hundred Fifty-Four

For convenience designate the upper part of fig. 212, i. e., the sq. AK, as fig. 212a, and the lower part as 212b.

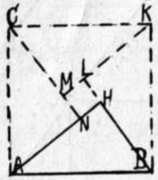
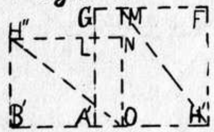


Fig. 212.



In fig. 212a the construction is evident, for 212b is made from the dissected parts of 212a. GH′ is a sq. each side of which = AH, LB′ is a sq. each side of which = BH.

Sq. AK = 2 tri. ABH + 2 tri. ABH + sq. MH = rect. B′N + rect. OF + sq. LM = sq. B′L + sq. A′F.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Hopkins's Plane Geom., 1891, p. 91, fig. V; Am. Math. Mo., V. VI, 1899, p. 69, XCI; Beman and Smith's New Plane Geom., 1899, p. 104, fig. 3; Heath's Math. Monographs, No. 1, 1900, p. 20, proof IV. Also Mr. Bodo. M. DeBeck, of Cincinnati, O., about 1905 without knowledge of any previous solution discovered above form of figure and devised a proof from it.

b. History relates that the Hindu Mathematician Bhaskara, born 1114 A. D., discovered the above proof and followed the figure with the single word "Behold," not condescending to give other than the figure and this one word for proof. And history furthermore declares that the Geometers of Hindustan knew the truth and proof of this theorem centuries before the time of Pythagoras—may he not have learned about it while studying Indian lore at Babylon?

Whether he gave fig. 212b as well as fig. 212a, as I am of the opinion he did, many late authors think not; with the two figures, 212a and 212b, side by side, the word "Behold!" may be justified,

especially when we recall that the tendency of that age was to keep secret the discovery of truth for certain purposes and from certain classes; but with the figure 212b omitted, the act is hardly defensible —not any more so than "See?" would be after figure 210.

Again, authors who give 212a and "Behold!", fail to tell their readers whether Bhaskara's proof was geometric or algebraic. Why this silence on so essential a point? For, if algebraic, the figure 212a is enough as the next two proofs show. I now quote from Beman and Smith: "The inside square is evidently $(b - a)^2$, and each of the four triangles is $\frac{1}{2}$ ab; ∴ $h^2 - 4 \times \frac{1}{2}ab = (b - a)^2$, whence $h^2 = a^2 + b^2$."

It is conjectured that Pythagoras had discovered the geometric proof above. Be it so, Bhaskara discovered it independently, as also did Wallis, an English Mathematician, in the 17th century, and so reported, Miss Coolidge, the blind girl, a few years ago: see proof *Fourteen,* fig. 72.

One Hundred Fifty-Five

In fig. 213 draw CN par. to BH, KM par. to AH, and extend BH to L.

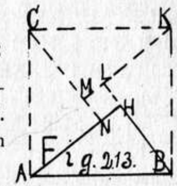


Fig. 213.

$$\text{Sq. } AK = 4\,\frac{HB \times HA}{2} + \text{sq. } MH = 2HB \times HA + (AH - BH)^2 = 2HB \times HA + HA^2 - 2HB \times HA + HB^2 = HB^2 + HA^2.$$

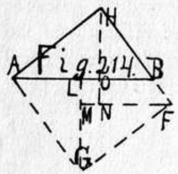
∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Olney's Geom., Part III, 1872, p. 250, 1st method; Jour. of Ed'n, V. XXV, 1887, p. 404, fig. IV, and also fig. VI; Jour. of Ed'n, V. XXVII, 1888, p. 327, 20th proof, by R. E. Binford, of Independence, Texas; Edwards's Geom., 1895, p. 155, fig. (3); Math. Mo., V. VI, 1899, p. 69, proof XCII; Sci. Am. Sup., V. 70, p. 359, Dec. 3, 1910, fig. 1.

b. A study of the many proofs by Arthur R. Colburn, LL. M., of Dist. of Columbia Bar, establishes the thesis, so often reiterated in this work, that figures may take any form and position so long as they include triangles whose sides bear a rational algebraic relation to the sides of the given triangle, or whose dissected areas are so related, through equivalency that $h^2 = a^2 + b^2$ results.

Case (6), (c).

One Hundred Fifty-Six



In fig. 214 produce HB to F and complete the sq. AF. Draw GL perp. to AB, FM par. to AB, and HN perp. to AB.

Sq. $AF = AH^2 = 4\frac{AO \times HO}{2} + [LO^2 = (AO - HO)^2] = 2AO \times HO + AO^2 - 2AO \times HO + HO^2 = AO^2 + HO^2 = (AO = AH^2 \div AB)^2 + (HO = AH \times HB \div AB)^2 = AH^4 \div AB^2 + AH^2 \times HB^2 \div AB^2 = AH^2(AH^2 + HB^2) \div AB^2$.

$AH^2(AH^2 + HB^2) \div AB^2$.

$\therefore 1 = (AH^2 + BH^2) \div AB^2$. $\therefore AB^2 = BH^2 + AH^2$.

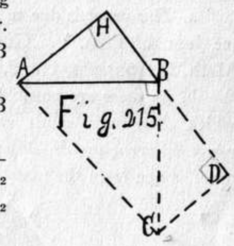
$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. VI, 1899, p. 69, proof CIII.

b. The reader will observe that this proof proves too much, as it first proves that $AH^2 = AO^2 + HO^2$, which is the truth sought. Triangles ABH and AHO are similar, and what is true as to the relations of the sides of tri. AHO must be true, by the law of similarity, as to the relations of the sides of tri. ABH.

(B) Proofs based upon a triangle through the calculations and comparison of equivalent areas.

One Hundred Fifty-Seven



In fig. 215 extend HB to D making BD = AH, through D draw DC par. to AH and equal to BH, and draw CB and CA.

Area of trap. CDHA = area of ACB + 2 area of ABH.

$\therefore \frac{1}{2}(AH + CD)HD = \frac{1}{2} AB^2 + 2 \times \frac{1}{2} AH \times HB$ or $(AH + HB)^2 = AB^2 + 2AH \times HB$, whence $AB^2 = BH^2 + AH^2$.

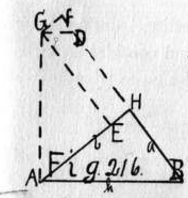
$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. This is the "Garfield Demonstration,"—hit upon by the General in a mathematical discussion with other M. C.'s about 1876. See Jour. of Ed'n, V. III, 1876, p. 161; The Math. Magazine, Vol. I, 1882, p. 7; The School Visitor, V. IX, 1888, p. 5, proof III; Hopkins's Plane Geom., 1891, p. 91, fig. VII; Edwards's Geom., 1895, p. 156, fig. (11); Heath's Math. Monographs, No. 1, 1900, p. 25, proof X.

b. For extension of any triangle, see V. Jelinek, Casopis, 28 (1899) 79—; Fschr. Math., (1899) 456.

c. See No. 153, fig. 211.

One Hundred Fifty-Eight



In fig. 216 extend BH to F making HF = AH, erect AG perp. to AB making AG = AB, draw GE par. to HB and GD par. to AB. Since tri's ABH and GDF are similar, $GD = h(1 - a/b)$, and $FD = a(1 - a/b)$.

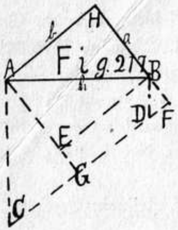
Area of fig. ABFG = area ABH + area AHFG = area ABDG + area GDF. $\therefore \frac{1}{2}ab + \frac{1}{2}b[b + (b - a)] = \frac{1}{2}h[h + h(1 -$

$a/b)] + \frac{1}{2}a(b - a)\ (1 - a/b)$, (1). Whence $h^2 = a^2 + b^2$.
$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. This proof is due to J. G. Thompson, of Winchester, N. H.; see Jour. of Ed'n, V. XXVIII, 1888, p. 17, 28th proof; Heath's Math. Monographs, No. 2, p. 34, proof XXIII.

b. As there are possible several figures of above type, in each of which there will result two similar triangles, there are possible many different proofs, differing only in shape of figure. The next proof is one from the many.

One Hundred Fifty-Nine



In fig. 217 produce HB to F making HF = HA, through A draw AC perp. to AB making AC = AB, draw CF, AG par. to HB, BE par. to AH, and BD perp. to AB. Since tri's ABH and BDF are similar, we find that $DF = a(1 - a/b)$ and $BD = h(1 - a/b)$.

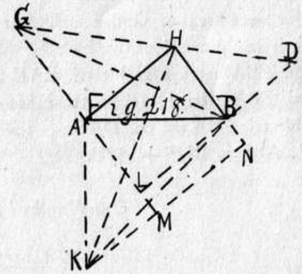
Area of trap. CFHA = 2 area ABH + area trap. AGFB = area ABH + area trap. ACDB + area BDF.

Whence area ACG + area AGFB = area ACDB + area BDF or $\frac{1}{2}ab + \frac{1}{2}b[b + (b - a)] = \frac{1}{2}h[h + h(1 - a/b)] + \frac{1}{2}a(b - a)(1 - a/b)$.

This equation is equation (1) in the preceding solution, as it ought to be, since, if we draw BE par. to AH and consider only the figure below the line AB, calling the tri. ACG the given triangle, we have identically fig. 217 above.
$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. Original with the author, August, 1900. See also Jour. of Ed'n, V. XXVIII, 1888, p. 17, 28th proof.

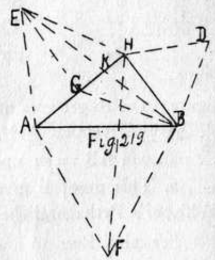
One Hundred Sixty



In fig. 218 extend HB to N making HN = AB, draw KN, KH and BG, extend GA to M and draw BL par. to AH. Tri. KBA + tri. ABH = quad. BHAK = (tri. HAK = tri. GAB) + (tri. DGB = tri. BHG) = quad. ABDG = tri. HBD + tri. GAH + tri. ABH, whence tri. BAK = tri. HBD + tri. GAH.
$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Jury Whipper, 1880, p. 33, fig. 30, as found in the works of Joh. Hoffmann.

One Hundred Sixty-One

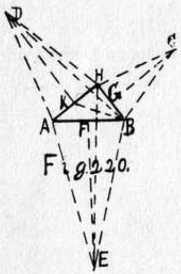


In fig. 219 construct the three equilateral triangles upon the three sides of the given triangle ABH, and draw EB and FH, draw EG perp. to AH, and draw GB.

Since EG and HB are parallel, tri. EBH = tri. BEG = ½ tri. ABH.
$\therefore$ tri. GBH = tri. HEG.
(1) Tri. HAF = tri. EAB = tri. EAK + (tri. BGA = ½ tri. ABH) + (tri. BKG = tri. EKH) = tri. EAH + ½ tri. ABH.
(2) In like manner, tri. BHF = tri. DHB + ½ tri. ABH.
(1) + (2) = (3) (tri. HAF + tri. BHF = tri. BAF + tri.

ABH) = tri. EAH + tri. DHB + tri. ABH, whence tri. FBA = tri. EAH + tri. DHB.
But since areas of similar surfaces are to each other as the squares of their like dimensions, we have
tri. FBA : tri. DHB : tri. EAH = $AB^2 : BH^2 : AH^2$, whence
tri. FBA : tri. DHB + tri. EAH = $AB^2 : BH^2 + AH^2$.
But tri. FBA = tri. DAH + tri. EAH.
$\therefore AB^2 = BH^2 + AH^2$.

One Hundred Sixty-Two



In fig. 220 from the middle points of AB, BH and HA draw the three perp's FE, GC and KD, making FE = 2AB, GC = 2BH and KD = 2HA, complete the three isosceles tri's EBA, CHB and DAH, and draw EH, BK and DB.

Since these tri's are respectively equal to the three sq's upon AB, BH and HA, it remains to prove tri. EBA = tri. CHB + tri. DAH. The proof is same as that in fig. 219, hence proof for 220 is a variation of proof for 219.

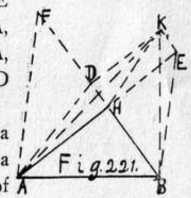
a. Devised by the author, because of the figure, so as to get area of tri. EBA = AB^2, etc.

$\therefore AB^2 = BH^2 + AH^2$.
$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. This proof is given by Joh. Hoffmann; see his solution in Whipper's Pythagoraische Lehrsatz, 1880, pp. 45-48.

See, also, Beeman and Smith's New Plane and Solid Geometry, 1899, p. 105, ex. 207.

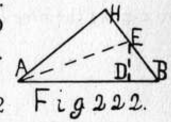
One Hundred Sixty-Three



In fig. 221 produce AH to E making HE = HB, produce CH to F making HF = HA, draw BK perp. to AB making BK = BA, KD par. to AH, and draw EB, KH, KA, AD and AF. BD = AB and KD = HB.
Area of tri. ABK = (area of tri. KHB = area of tri. EHB) + (area of tri. AHK = area of tri. AHD) + (area of ABH = area of ADF).
$\therefore$ area of ABK = area of tri. EHB + area of tri. AHF.
$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards's Geom., 1895, p. 158, fig. (20).

One Hundred Sixty-Four



In fig. 222 take AD = AH, draw ED perp. to AB, and draw AE. Tri's ABH and BED are similar, whence DE = AH × BD ÷ HB. But DB = AB — AH.

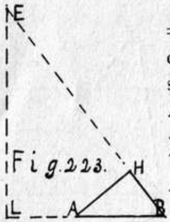
Area of tri. ABH = ½ AH × BH = $2\dfrac{AD \times ED}{2}$ + ½ ED × DB = AD × ED + ½ ED × DB = $\dfrac{AH^2(AB - AH)}{BH} + \frac{1}{2}\dfrac{AH(NB - AH)^2}{BH}$

$\therefore BH^2 = 2AH \times AB - 2AH^2 + AB^2 + AH^2 - 2AH \times AB$.
$\therefore AB^2 = BH^2 + AH^2$.
$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. VI, 1899, p. 70, proof XCV.

b. See proof *Five,* fig. 5, under I, Algebraic Proofs, for an algebraic proof.

One Hundred Sixty-Five



In fig. 223 produce BA to L making AL = AH, at L draw EL perp. to AB, and produce BH to E. The tri's ABH and EBL are similar.

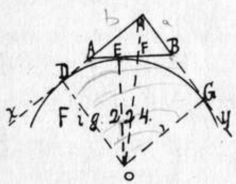
Area of tri. ABH $= \frac{1}{2}$ AH $\times$ BH $= \frac{1}{2}$ LE $\times$ LB $-$ LE $\times$ LA $= \frac{1}{2}\frac{AH(AH + AB)^2}{BH} - \frac{AH^2(AH + AB)}{BH}$, whence $AB^2 = BH^2 + AH^2$.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. VI, 1899, p. 70, proof XCVI.

b. This and the preceeding proof are the converse of each other. The two proofs teach that if two triangles are similar and so related that the area of either triangle may be expressed principally in terms of the sides of the other, then either triangle may be taken as the principal triangle, giving, of course, as many solutions as it is possible to express the area of either in terms of the sides of the other.

One Hundred Sixty-Six



In fig. 224 produce HA and HB and describe the arc of a circle tang. to HX, AB and HY. From O, its center, draw to points of tangency, OG, OE and OD, and draw OH.

Area of sq. DG $= r^2 = \frac{1}{2}ab + (2\frac{BEXV}{2} + 2\frac{AEXV}{2} = \frac{1}{2}ab + hr$.

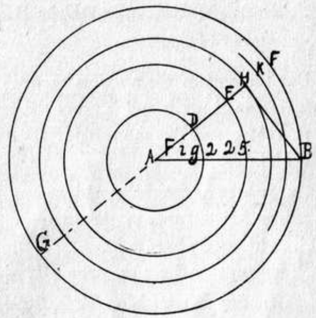
But since $2r = h + a + b$, $\therefore r = \frac{1}{2}(h + a + b)$.

$\therefore \frac{1}{4}(h + a + b)^2 = \frac{1}{2}ab + h(h + a + b)$, whence $h^2 = a^2 + b^2$.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. This proof is original with Prof. B. F. Yanney, Wooster University, O. See Math. Mo., V. VI, 1899, p. 70, XCVII.

One Hundred Sixty-Seven



In fig. 225 let AE = BH. Since the area of a circle is πr^2, if it can be proven that the circle whose radius is AB = the circle whose radius AH + the circle whose radius is AE, the truth sought is established.

It is evident, if the triangle ABH revolves in the plane of the paper about A as a center, that the area of the circle generated by AB will equal the area of the circle generated by AH plus the area of the annulus generated by HF.

Hence it must be shown, if possible, that the area of the annulus is equal to the area of the circle whose radius is AE.

Let AB = h = AF, AH = b, BH = a, AD = ½ BH = r, HK = KF, and AK = mr, whence GH = h + b, AK $= \frac{h + b}{2}$ = mr, HF = h − b, HK = KF $= \frac{h - b}{2}$

Now (GH = h + b) : (BH = 2r) = (BH = 2r) : (HF = h − b) (1)

whence $h = \sqrt{b^2 + 4r^2}$ and $b = \sqrt{h^2 - 4r^2}$.

$\therefore \frac{h + b}{2} = \frac{\sqrt{b^2 + 4r^2} + b}{2}$ = mr, whence $b = r(m - \frac{1}{m})$, and $\frac{h+b}{2} = \frac{h+\sqrt{h^2-4r^2}}{2} = mr$, whence $h = r(m+\frac{1}{m})$.

$\therefore \frac{a-b}{2} = \frac{r(m+\frac{1}{m}) - r(m-\frac{1}{m})}{2} = \frac{r}{m} = HK$. Now

since $(AD = r) : (AK = mr) = (HK = \frac{r}{m}) : (AD = r)$, (2)

$\therefore AD : AK = HF : AE$, or $2\pi\ AD : 2\pi\ AK = HF : AE$,

$\therefore 2\pi\ AK \times HF = 2\pi\ AD \times AE$, or $2\pi\ (\frac{h+b}{2})\ HF = \pi\ AE \times AE$.

But the area of the annulus equals ½ the sum of the circumferences where radii are h and b times the width of the annulus or HF.

$\therefore$ the area of the annulus HF = the area of the circle where radius is HB.

$\therefore$ the area of the circle with radius AB = the area of the circle with radius AH + area of the annulus.

$\therefore \pi h^2 = \pi a^2 + \pi b^2$.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math. Mo., V. I, 1894, p. 223, the proof by Andrew Ingraham, President of the Swain Free School, New Bedford, Mass.

b. This proof, like that of proof *One Hundred Fifty-Six,* fig. 214, proves too much, since both equations (1) and (2) imply the truth sought. The author, Professor Ingraham, does not show his readers how he determined that $HK = \frac{r}{m}$, hence the implication is hidden; in (1) we have directly $h^2 - b^2 = (4r^2 = a^2)$.

Having begged the question in both equations, (1) and (2), Professor Ingraham has, no doubt, unconsciously, fallen under the formal fallacy of *petitio principii.*

From the preceeding array of proofs it is evident that the algebraic and geometric proofs of this most important truth are as unlimited in number as are the ingenious resources and ideas of the mathematical investigator.

NO TRIGONOMETRIC PROOFS

Facing forward the thoughtful reader may raise the question: Are there any proofs based upon the science of trigonometry or analytical geometry?

There are no trigonometric proofs, because all the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean Theorem; because of this theorem we say $\sin^2 a + \cos^2 A = 1$, etc. Trigonometry *is* because the Pythagorean Theorem *is.*

As Descartes made the Pythagorean theorem the basis of his method of analytical geometry, no independent proof can here appear. Analytical Geometry is Euclidian Geometry treated algebraically and hence involves all principles already established.

Therefore in analytical geometry all relations concerning the sides of a right angled triangle imply or rest directly upon the Pythagorean theorem as is shown in the equation, viz., $x^2 + y^2 = r^2$.

And The Calculus being but an algebraic investigation of geometric variables by the method of limits it accepts the truths of geometry as established, and therefore furnishes no new proof, other than that, if squares be constructed upon the three sides of a variable oblique triangle, as any angle of the three approaches a right angle the square on the side opposite approaches in area the sum of the squares upon the other two sides.

But not so with quaternions, or vector analysis. It is a mathematical science which introduces a new concept not employed in any of the mathematical sciences mentioned heretofore,—the concept of direction.

And by means of this new concept the complex demonstrations of old truths are wonderfully simplified, or new ways of reaching the same truth are developed.

We here give four quarternionic proofs of the Pythagorean Proposition. Other proofs are possible.

III.—QUARTERNIONIC PROOFS

One

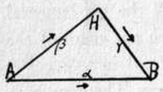


Fig. 226

In fig. 226 designate the sides as to distance and direction by A, B and G. Now, by the principle of direction $A = B + G$; also since the angle at H is a right angle, $2SBG = 0$. (S signifies Scalar, Hardy, p. 6.)

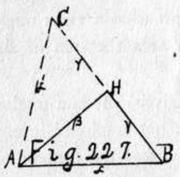
(1) $A + B = G$. $(1)^2 = (2)$ $A^2 = B^2 + 2SBG + G^2$;

(2) reduced $= (3)$ $\therefore A^2 = B^2 + G^2$, considered as lengths.

$\therefore$ sq. upon AB = sq. upon AH + sq. upon HB.

a. See Hardy's Elements of Quarternions, 1881, p. 82, art. 54, 1; also Jour. of Ed'n, V. XXVII, 1888, p. 327, Twenty-second proof.

Two



In fig. 227, extend BH to C making HC = HB and draw AC. As vectors $AB = AH + HB$, or $A = B + G$ (1). Also $AC = AH + HC$, or $A = B - G$ (2).

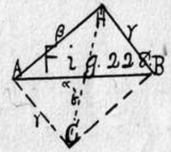
Squaring (1) and (2) and adding, we have $A^2 + A^2 = 2B^2 + 2G^2$. Or as lengths, $AB^2 + AC^2 = 2AH^2 + 2AB^2$. But AB = AC.

$\therefore AB^2 = AH^2 + HB^2$.

$\therefore$ sq. upon AB = sq. upon AH + sq. upon HB.

a. This is James A. Calderhead's solution. See Math. Mo., V. VI, 1899, p. 71, proof XCIX.

Three



In fig. 228 complete the rect. HC and draw HC. As vectors $AB = AH + HB$, or $A = B + G$ (1) $HC = HA + AC$, or $A' = -B + G$ (2).

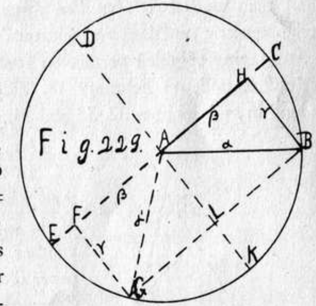
Squaring (1) and (2) and adding, gives $A^2 + A'^2 = 2B^2 + 2G^2$. Or considered as lines, $AB^2 + HC^2 = 2AH^2 + 2HB^2$. But HC = AB.

$\therefore AB^2 = AH^2 + HB^2$.

$\therefore$ sq. upon AB = sq. upon AH^2 + sq. upon HB^2.

a. Another of James A. Calderhead's solutions. See Math. Mo., V. VI, 1899, p. 71, proof C.

Four



In fig. 229 the construction is evident, as angle GAK = —angle BAK. The radius being unity, LG and LB are sines of GAK and BAK.

As vectors, $AB = AH + HB$, or $A = B + G$ (1). Also $AG = AF + FG$ or $A' = -B + G$ (2). Squaring (1) and (2) and adding, gives $A^2 + A'^2 = 2B^2 + 2G^2$. Or considering the vectors as distances, $AB^2 + AG^2 = 2AH^2 + 2HB^2$, or $AB^2 = AH^2 + HB^2$.

$\therefore$ sq. upon AB = sq. upon AH + sq. upon BH.

a. Original with the author, August, 1900.

b. Other solutions from the trigonometric right line function figure (see Schuyler's Trigonometry, 1873, p. 78, art. 85) are easily devised through vector analysis.

IV.—DYNAMIC PROOFS.

The Science of Dynamics, since 1910, is a claimant for a place as to a few proofs of the Pythagorean Theorem.

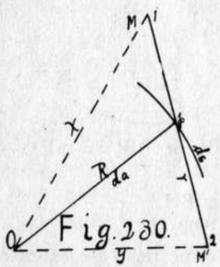
In Science, New Series, Oct. 7, 1910, V. 32, pp. 863-4, Professor Edwin F. Northrup, Palmer Physical Laboratory, Princeton, N. J., through equilibrium of forces, establishes the formula $h^2 = a^2 + b^2$.

In V. 33, p. 457, Mr. Mayo D. Hersey, of the U. S. Bureau of Standards, Washington, D. C., says that, if we admit Professor Northrup's proof, then the same result may be established by a much simpler course of reasoning based on certain simple dynamic laws.

Then in V. 34, pp. 181-2, Mr. Alexander MacFarlane, of Chatham, Ontario, Canada, comes to the support of Professor Northrup, and then gives two very fine dynamic proofs through the use of trigonometric functions and quarternionic laws.

Having obtained permission from the editor of Science, Mr. J. McK. Cattell, on February 18, 1926, to make use of these proofs found in said volumes 32, 33 and 34, of Science, they now follow.

One



In fig. 230, O-p is a rod without mass which can be revolved in the plane of the paper about O as a center. 1-2 is another such rod in the plane of the paper of which p is its middle point. Concentrated at each end of the rod 1-2 are equal masses m and m' each distant r from p.

Let R equal the distance O-p, X = O-1, y = O-2. When the system revolves about O as a center, the point

p will have a linear velocity, r = ds/dt = da/dt = RW, where ds is the element of the arc described in time dt, da is the differential angle through which O-p turns, and W is the angular velocity.

1. Assume the rod 1-2 free to turn on p as a center. Since m at 1 and m' at 2 are equal and equally distant from p, p is the center of mass. Under these conditions $E' = \frac{1}{2}(2m)V^2 = m R^2 W^2$ (1).

2. Conceive rod, 1-2, to become rigorously attached at p. Then, as O-p revolves about O with angular velocity W, 1-2 also revolves about p with like angular velocity. By making attachment at p rigid the system is forced to take on an additional kinetic energy, which can be only that, which is a result of the additional motion now possessed by m at 1 and by m' at 2, in virtue of their rotation about p as a center. This added kinetic energy is $E'' = \frac{1}{2}(2m)r^2W^2 = mr^2 W^2$. (2). Hence total kinetic energy is $E = E' + E'' = mW^2(R^2 + r^2)$. (3).

3. With the attachment still rigid at p, the kinetic energy of m at 1 is, plainly, $E_0' = \frac{1}{2} mx^2 W^2$. (4). Likewise $E_0'' = \frac{1}{2} m y^2 W^2$. (5).

∴ the total kinetic energy must be $E = E_0' + E_0'' = \frac{1}{2} m W^2 (x^2 + y^2)$. (6).

∴ (3) = (6), or $\frac{1}{2} (x^2 + y^2) = R^2 + r^2$. (7).

In (7) we have a geometric relation of some interest, but in a particular case when x = y, that is, when line 1-2 is perpendicular to line O-p, we have as a result $x^2 = R^2 + r^2$. (8).

∴ sq. upon hypotenuse = sum of squares upon the two legs of a right triangle.

Then in Vol. 33, p. 457, on March 24, 1911, Mr. Mayo D. Hersey says: "while Mr. R. F. Deimal holds that equation (7) above expresses a geometric fact—I am tempted to say 'accident'—which text books raise to the dignity of a theorem." He further says, "Why not let it be a simple one? For instance, if the force F whose rectangular components are x and y, acts upon a particle of mass m until it has imparted the velocity q whose components in the same plane are u and v, then the work done upon the particle $x = \frac{1}{2} m u^2$ while the work done by $y = \frac{1}{2} m v^2$. But the work done by the components is identical with the work $\frac{1}{2} m q^2$ done by their resultants. Equating, we have $\frac{1}{2} m q^2 = \frac{1}{2} m u^2 + \frac{1}{2} m v^2$, or $q^2 = u^2 + v^2$. But the velocity components u and v are the two legs of a right triangle of which q is the hypotenuse, so that here again is our Pythagorean relation.

Answering Professor Northrup—see Science, Vol. 34, p. 181, Mr. Alexander FacFarlane, Chathem, Ontario, says: "In reply to Dr. Northrup's question,—is a dynamical proof possible—that looking at the question from the point of view of vector-analysis, or rather of the algebra of space, I would answer, Yes."

But says Mr. MacFarlane, he could with ease deduce the more general proposition (Euclid II, 12 and 13).

His proof is merely the reverse of the following reasoning. I look upon the x, y, R, r and —v of his diagram as vectors.

The kinetic energy of the first mass is $\frac{1}{2} m (xW)^2 = \frac{1}{2} m W^2 x^2$; and similarly that of the second is $\frac{1}{2} m W^2 y^2$. But $x^2 = R^2 + r^2 + 2 \cos R\, r$ and $y^2 = R^2 + (-r)^2 - 2 \cos R\, r$, where cos R r denotes the rectangle formed by R and the projection of r along R. Hence $\frac{1}{2} m W^2 (x^2 + y^2) = \frac{1}{2} (2m) (R^2 + r^2) W^2 = \frac{1}{2} x\, 2m R^2 W^2 + \frac{1}{2} x\, 2m r^2 W^2$.

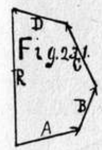
Here we pass from the one to the other expression for the kinetic energy of the system by means of the extended Pythagorean Theorem; on the other hand Dr. Northrup can deduce from the expression for the kinetic energy of the system the truth of this geometric theorem.

This same principle, $E = \frac{1}{2} m v^2$ has an important bearing on the fundamental principles of vector analysis; it places the orthodox quarternionist in a corner from which there is no escape.

Because E is assumed in mathematical analysis to be positive and $\frac{1}{2}$m is positive, it follows from the established principles of analysis

that v^2 must be positive; consequently, to hold that the square of a simple vector is negative is to contradict the established conventions of mathematical analysis.

The quarternionist tries to get out by saying that after all v is not a velocity having direction, but merely a speed. To this I reply that $E = \cos \int m\, v\, d\, v = \frac{1}{2} m v^2$, and that these expressions v and dv are both vectors having directions which are different.



Recently (in the Bulletin of the Quarternion Association) I have been considering what may be called the generalization of the Pythagorean Theorem.

Let A, B, C, D, etc., fig. 231, denote vectors having any direction in space, and let R denote the vector from the origin of A to the terminal of the last vector; then the generalization of the P. T. is $R^2 = A^2 + B^2 + C^2 + D^2 + 2(\cos AB + \cos AC + \cos AD) + 2(\cos BC + \cos BD) + 2(\cos CD) +$ etc., where cos AB denotes the rectangle formed by A and the projection of B parallel to A. The theorem of P. is limited to two vectors A and B which are at right angles to one another, giving $R^2 = A^2 + B^2$. The extension given in Euclid removes the condition of perpendicularity, giving $R^2 = A^2 + B^2 + \cos AB$.

Space geometry gives $R^2 = A^2 + B^2 + C^2$ when A, B, C are othogonal, and $R^2 = A^2 + B^2 + C^2 + 2 \cos AB + 2 \cos AC + 2 \cos BC$ when that condition is removed.

Further, space-algebra gives a complementary theorem, never dreamed of by either Pythagoras or Euclid.

Let V denote in magnitude and direction the resultant of the directed areas enclosed between the broken lines A + B + C + D and the resultant line R, and let sin AB denote in direction and magnitude the area enclosed between A and the projection of B which is perpendicular to A; then the complementary theorem is

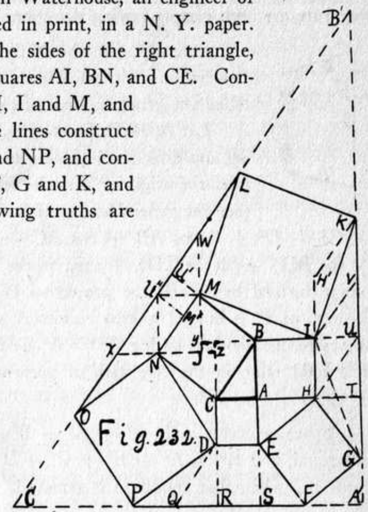
4V = 2(sin AB + sin AC + sin AD +   ) + 2(sin BC + sin BD +   ) + 2(sin CD +   ) + etc.

THE PYTHAGOREAN CURIOSITY

The following is reported to have been taken from a note book of Mr. John Waterhouse, an engineer of N. Y. City. It appeared in print, in a N. Y. paper. in July, 1899. Upon the sides of the right triangle, fig. 232, construct the squares AI, BN, and CE. Connect the points E and H, I and M, and N and D. Upon these lines construct the squares EG, MK and NP, and connect the points P and F, G and K, and L and O. The following truths are demonstrable.

1. Square BN = square CE + square AJ. (Euclid).

2. Triangle HAE = triangle IBM = triangle DCN = triangle CAB, since HA = BI and EA = MY, EA = DC and HA = NZ, and HA = BA and EA = CA.



3. Lines HI and GK are parallel, for, since angle GHI = angle IBM, ∴ triangle HGI = triangle BMI, whence IG = IM = IK. Again extend HI to H′ making IH′ = IH, and draw H′K, whence triangle IHG = triangle IH′K, each having two sides and the included angle respectively equal. ∴ the distances from G and K to the line HH′ are equal. ∴ the lines HI and GK are parallel. In

like manner it may be shown that DE and PF, also MN and LO, are parallel.

4. GK = 4HI, for HI = TU = GT = UV = VK (since VK is homologous to BI in the equal triangles VKI and BIM). In like manner it can be shown that PF = 4DE. That LO = 4MN is proven as follows: triangles LWM and IVK are equal; therefore the homologous sides WM and VK are equal. Likewise OX and QD are equal each being equal to MN. Now in tri. WJX, MJ and XN = NJ; therefore M and N are the middle points of WJ and XJ; therefore WX = 2MN; therefore LO = 4MN.

5. The three trapezoids HIGK, DEPF and MNLO are each equal to 5 times the triangle CAB. The 5 triangles composing the trapezoid HIGK are each equal to the triangle CAB, each having the same base and altitude as triangle CAB. In like manner it may be shown that the trapezoid DEPF, so also the trapezoid MNLO, equals 5 times the triangle CAB.

6. The square MK + the square NP = 5 times the square EG or BN. For the square on MI = the square on MY + the square on YI + $(2AB)^2 + AC^2 = 4AB^2 + AC^2$; and the square on ND + the square on NZ + the square ZD = $AB^2 + (2AC)^2 = AB^2 + 4AC^2$. Therefore the square MK + the square NP = $5AB^2 + 5AC^2 = 5(AB^2 + AC^2) = 5BC^2$ = 5 times the square BN.

7. The bisector of the angle A′ passes through the vertex A; for AS = AT. But the bisector of the angle B′ or C′, does not pass through the vertex B, or C. Otherwise BU would equal BU′, whence NU″ + U″M would equal NM + U″M′; that is, the sum of the two legs of a right triangle would equal the hypotenuse + the perpendicular upon the hypotenuse from the right angle. But this is impossible. Therefore the bisector of the angle B′ does not pass through the vertex B.

8. The square on LO = the sum of the squares on PF and GK;

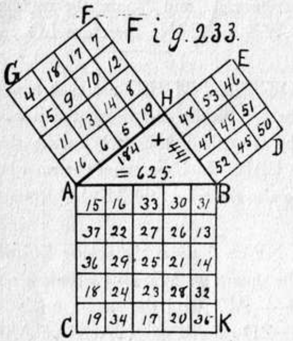
for LO : PF : GK = BC : CA : AB.

9. Etc, etc.

See Casey's Sequel to Euclid, 1900, Part I, p. 16.

PYTHAGOREAN MAGIC SQUARES

One



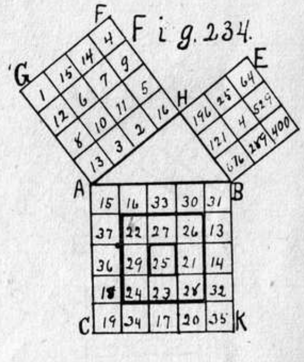
The sum of any row, column or diagonal of the square AK is 125; hence the sum of all the numbers in the square is 625. The sum of any row, column or diagonal of square GH is 46, and of HD is 147; hence the sum of all the numbers in the square GH is 184, and in the square HD is 441. Therefore the magic square AK (625) = the magic square HD (441) + the magic square HG (184).

Formulated by the author, July, 1900.

Two

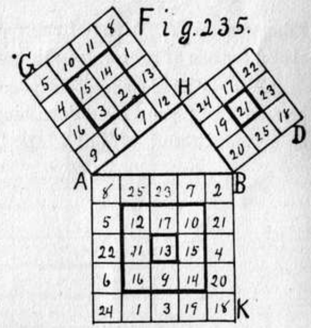
The square AK is composed of 3 magic squares, 5^2, 15^2 and 25^2. The square HD is a magic square each number of which is a square. The square HG is a magic square formed from the first 16 numbers. Furthermore, observe that the sum of the nine square numbers in the square HD equals 48^2 or 2304, a square number.

Formulated by the author, July, 1900.



Three

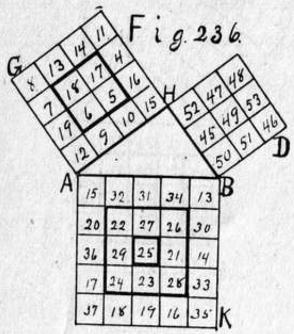
The sum of all the numbers in square (AK = 325) + the sum of all the numbers in square (HD = 189) + the sum of all the numbers in square (HG + 136).
Square AK is made up of 13, 3 × (3 × 13), and 5 × (5 × 13); square HD is made up of 21, 3 × (3 × 21), and square HG is made up of 4 × 34 — each row, column and diagonal, and the sum of the four inner numbers.



Many other magic squares of this type giving 325, 189 and 136 for the sums of AK, HD and HG respectively may be formed.

This one was formed by Prof. Paul A. Towne, of West Edmeston, N. Y.

Four



The sum of numbers in sq. (AK = 625) = the sum of numbers in sq. (HD = 441) + the sum of numbers in sq. (HG = 184).

Sq. AK gives $1 \times (1 \times 25)$; $3 \times (3 \times 25)$; and $5 \times (5 \times 25)$, as elements; sq. HD gives $1 \times (1 \times 49)$; $3 \times (3 \times 49)$ as elements; and sq. HG gives 1×46 and 3×46, as elements.

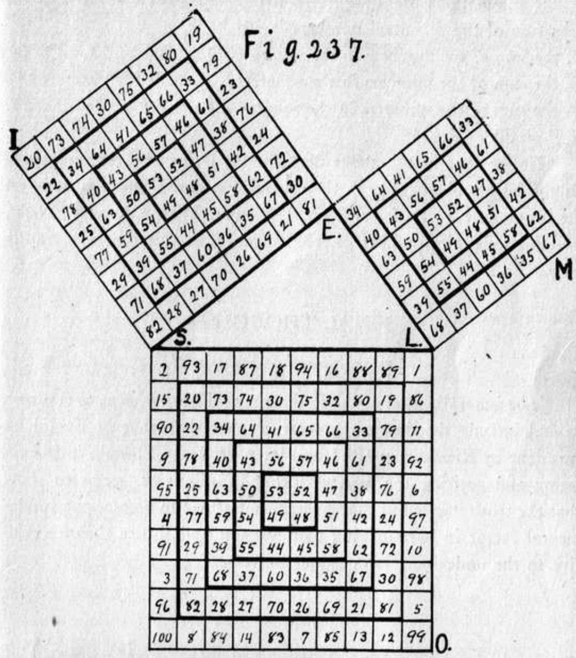
This one also was formed by Professor Towne, of West Edmeston, N. Y. Many of this type may be formed. See fig. 233, above, for one of my own of this type.

Also see Mathematical essays and Recreations, by Herman Schubert, in The Open Court Publishing Co., Chicago, 1898, p. 39, for an extended theory of The Magic Square.

Five

Observe the following series:

The sum of the inner 4 numbers is $1^2 \times 202$; of the 16-square, $2^2 \times 202$; of the 36-square, $3^2 \times 202$; of the 64-square, $4^2 \times 202$; and of the 100-square, $5^2 \times 202$.



"On the hypotenuse and legs of the right angled triangle, ESL, are constructed the concentric magic squares of 100, 64, 36 and 16.

The sum of the two numbers at the extremities of the diagonals, and of all lines, horizontal and diagonal, and of the two numbers equally distant from the extremities, is 101. The sum of the numbers in the diagonals and lines of each of the four concentric magic squares is 101 multiplied by half the number of cells in boundary lines; that is, the summations are 101×2; 101×3; 101×4; 101×5. The sum of the 4 central numbers is 101×2.
$\therefore$ the sum of the numbers in the square (SO $= 505 \times 10 = 5050$)
$=$ the sum of the numbers in the square (EM $= 303 \times 6 = 1818$)
$+$ the sum of the numbers in the square (EI $= 404 \times 8 = 3232$).
$505^2 = 303^2 + 404^2$.

Notice that in the above diagram the concentric magic squares on the legs is identical with the central concentric magic squares on the hypotenuse." Professor Paul A. Towne, West Edmeston, N. Y.

An indefinite number of magic squares of this type are readily formed.

FINAL THOUGHT

Is it an all-embracing truth?

The generalization of the Pythagorean Theorem so as to conform to and include the data of geometries other than that of Euclid, as was done by Riemann in 1854, and later, 1915, by Einstein in formulating and positing the general theory of relativity, seems to show that the truth implied in this theorem is destined to become the fundamental factor in harmonizing past, present and future theories relative to the underlying laws of our universe.

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3. Names of persons for whom a proof has been named or to whom a proof has been credited are:

4. Names of other persons mentioned:

INDEX OF NAMES

"Exegi monumentum, aere perennius
Regali que situ pyramidum altius,
Quod non imber edax, non aquilo impotens
Possit diruere aut innumerabilis
Annorum series et fuga temporum.
Non omnis morior." —*Horace.*

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