Simultaneous Approximation to Pairs of Algebraic Numbers

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Abstract

The author uses an elementary lemma on primes dividing binomial coefficients and estimates for primes in arithmetic progressions to sharpen a theorem of J. Rickert on simultaneous approximation to pairs of algebraic numbers. In particular, it is proven that

$$\max\left\{\left|\sqrt{2}-\frac{p_1}{q}\right|,\left|\sqrt{3}-\frac{p_2}{q}\right|\right\} > 10^{-10}q^{-1.8161}$$

for p_1, p_2 and q integral. Applications of these estimates are briefly discussed.

1 Introduction

Effective lower bounds for rational approximation to algebraic numbers and their applications to diophantine equations are widely known in the literature (see e.g. [1], [2], [3], [4], [7], [8], [9], [10], [11], [12], [13] and [15]). Via Padé approximation, Baker [1, 2] was able to show, for example, that

$$\left|\sqrt[3]{2} - \frac{p}{q}\right| > 10^{-6} q^{-2.955}$$

for all positive integers p and q and relate this to solutions of the equation

$$x^3 - 2y^3 = u.$$

Subsequently, Baker [3] derived bounds of the form

$$\max_{1 \leq i \leq m} \left\{ \left| \theta_i - \frac{p_i}{q} \right| \right\} > q^{-\lambda} \tag{1.1}$$

for certain algebraic $\theta_1, \theta_2, \ldots \theta_m, \lambda = \lambda(\theta_1, \ldots \theta_m)$ and $p_1, \ldots p_m, q$ positive integers with $q > q_0(\lambda, \theta_1, \ldots \theta_m)$. Simultaneous approximation results have also been considered by Chudnovsky [8], Osgood [13], Fel'dman [10, 11] and Rickert [15], the last three of whom dealt with algebraic numbers of the form

$$(\theta_1, \theta_2, \ldots \theta_m) = (r_1^\nu, r_2^\nu, \ldots r_m^\nu) \tag{1.2}$$

for $r_1, r_2, \ldots r_m$ and ν rational. In particular, Rickert showed that

$$\max \left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{3} - \frac{p_2}{q} \right| \right\} > 10^{-7} q^{-1.913} \tag{1.3}$$

for p_1, p_2 and q positive integers.

Recently, the author was able to sharpen the work of Osgood, Fel'dman and Rickert in the situation described in (1.2). In [5], we stated our results in full generality, leaving all constants "effectively computable" rather than explicit. Here, we will present a completely explicit version of our theorem in the special case considered by Rickert. Our sharpening depends upon bounds on the Chebyshev function

$$\theta(x) = \sum_{p \leq x} \log(p)$$

from Schoenfeld [16]. Specifically, we show that

$$\max \left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{3} - \frac{p_2}{q} \right| \right\} > 10^{-10} q^{-1.8161} \tag{1.4}$$

holds for any positive integers p_1, p_2 and q (compare to (1.3)).

We also give bounds for simultaneous approximation to pairs of numbers of the form $(1 - 1/N)^{1/4}, (1 + 1/N)^{1/4}$. These are analogous to the results of Rickert, but are strengthened by application of a combination of the aforementioned work of Schoenfeld with bounds on primes in arithmetic progressions due to Ramaré and Rumely [14]. In a forthcoming paper [6], the author applies these results to the problem of solving certain related norm form equations.

2 Our Approximating Forms

The work that follows depends upon the specific nature of the (equal-weighted) Padé approximants to the system of functions

$$1, (1+a_1x)^\nu, \ldots (1+a_mx)^\nu.$$

These were investigated by Rickert in [15], through consideration of the integral

$$I_i(x) = \frac{1}{2\pi i} \int_\gamma \frac{(1+zx)^k(1+zx)^\nu}{(z-a_i)(A(z))^k}\, dz \qquad (0 \le i \le m)$$

where $0 = a_0, a_1, \ldots a_m$ are distinct integers, k a positive integer, ν a positive rational, $A(z) = \prod_{i=0}^m (z-a_i)$ and γ a closed, counter-clockwise contour containing $a_0, a_1, \ldots a_m$. In fact, he showed that one may write

$$I_i(x) = \sum_{j=0}^m p_{ij}(x)(1+a_jx)^\nu \qquad (0 \le i \le m) \tag{2.1}$$

where the $p_{ij}(x)$'s are polynomials in x with rational coefficients and degree at most k. To be precise,

$$p_{ij}(x) = \sum \binom{k+\nu}{h_j} (1+a_jx)^{k-h_j} x^{h_j} \prod_{\substack{l=0 \\ l \ne j}}^m \binom{-k_{il}}{h_l} (a_j - a_l)^{-k_{il}-h_l}$$

where $\sum$ refers to the sum over all nonnegative $h_0, \ldots h_m$ with $h_0 + \ldots + h_m = k + \delta_{ij} - 1$ for δ_{ij} the Kronecker delta. Taking $x = 1/N$ in (2.1), Rickert deduced measures for simultaneous rational approximation to

$$(1+a_1/N)^\nu, \ldots (1+a_m/N)^\nu$$

by appealing to

Lemma 2.1 *Let $\theta_1, \dots \theta_m$ be arbitrary real numbers. Suppose there exist positive real numbers l, p, L and P $(L > 1)$ such that for each positive integer k, we can find integers p_{ijk} $(0 \le i, j \le m)$ with nonzero determinant,*

$$|p_{ijk}| \le pP^k \quad (0 \le i, j \le m)$$

and

$$\left|\sum_{j=0}^{m} p_{ijk}\theta_j\right| \le lL^{-k} \quad (0 \le i \le m).$$

Then we may conclude that

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \dots \left|\theta_m - \frac{p_m}{q}\right|\right\} > cq^{-\lambda}$$

it for all integers $p_1, \dots p_m$ and q, where

$$\lambda = 1 + \frac{\log(P)}{\log(L)}$$

and

$$c^{-1} = 2\ (m+1)\ p\ P\ (\max(1, 2l))^{\lambda-1}.$$

For simplicity's sake, we will follow Rickert's exposition closely in determining upper bounds for $|p_{ij}(1/N)|$ and $|I_i(1/N)|$. Using more precise asymptotics (via, for instance, the saddle point method) fails to yield marked improvements.

We restrict ourselves to the case when $m = 2, a_0 = 0, a_1 = 1, a_2 = -1, x = 1/N$ and $\nu = 1/n$, where $N \ge 2$ and $n \ge 2$. Then

Lemma 2.2

$$|p_{ij}(1/N)| \le 1.55 \left(\frac{N\sqrt{3}+2}{N\sqrt{3}-\sqrt{3}} \right)^{1/n} \left(\frac{3\sqrt{3}}{2}(1+\frac{2}{N\sqrt{3}}) \right)^{k}$$

Proof: This bound is a consequence of the proof of Rickert's Lemma 4.1 in [15]. ∎

We also have

Lemma 2.3

$$|I_i(1/N)| \le c(n) \left(\frac{27}{4}(N^3 - N) \right)^{-k}$$

where $c(n)$ can be taken as $27/32$ if $n \ge 2$ and as $135/256$ if $n \ge 4$.

Proof: The result follows from Lemma 4.2 in [15] upon noting that

$$\left| \binom{k+1/n}{3k} \right| \le \frac{27}{64} \left(\frac{4}{27} \right)^{k}$$

for $n \ge 2$, and

$$\left| \binom{k+1/n}{3k} \right| \le \frac{135}{512} \left(\frac{4}{27} \right)^{k}$$

for $n \ge 4$. ∎

3 Coefficients of Our Approximants

To sharpen Rickert's bounds, we study the polynomials $p_{ij}(x)$ more closely. We have

Lemma 3.1 *If k is a positive integer, then*

(a) *If $\nu = 1/2$, then $2^{3k-1}\ p_{ij}(x) \in \mathbb{Z}[x]$*

(b) *If $\nu = 1/4$, then $2^{4k-1}\ p_{ij}(x) \in \mathbb{Z}[x]$*

Proof: The first part follows directly from Rickert's Lemma 4.3. The second is similar; from Lemma 4.1 in [8], we have that if $h_0 > 0$, then

$$2^{3h_0-1}\binom{k+1/4}{h_0}$$

is an integer. Since $a_0 = 0, a_1 = 1$ and $a_2 = -1$, at most one term in the product

$$\prod_{l \neq j}(a_l - a_j)^{-k_{il}-h_l}$$

is not equal to one in modulus, whence, taking

$$M = \max\ \ \{2k, 3h_0 - 1 + k + \max\{h_1, h_2\}\}$$

we have that

$$2^M\ \binom{k+1/4}{h_0}\ (a_1 - a_2)^{-k_{il}-h_l}$$

is an integer for $l = 1$ or 2. Since $h_0 + h_1 + h_2 \leq k$, it follows that $M \leq 4k-1$, concluding the proof. ■

It turns out that these resulting polynomials have integer coefficients possessing large common factors. To exploit this fact, we utilize the following special case of a result of the author (Lemma 4.1 in [5]) :

Lemma 3.2 *Define for $1 \leq r < n, (r, n) = 1$ and $\{x\} = x - [x]$, $S(r)$ to be the set of primes p with $p > \sqrt{nk+1}$, $(p, nk) = 1$, $pr \equiv 1 \bmod n$ and*

$\left\{\frac{k-1}{p}\right\} > \max\left(\frac{2n-r}{2n}, \frac{r}{n}\right)$. *Then if* $p \in S(r)$,

$$ord_p\left(\binom{k+1/n}{h_0}\binom{k+h_1-1}{h_1}\binom{k+h_2-1}{h_2}\right) \geq 1$$

for all nonnegative integers h_0, h_1 *and* h_2 *with* $h_0+h_1+h_2 = k$ *or* $k-1$.

Define $P_2(k)$ to be the product over all primes p with $p > \sqrt{2k+1}, (p, 2k) = 1$ and $\{(k-1)/p\} > 3/4$. Fixing $\nu = 1/2$, it follows from (2.2) and Lemma 3.2 that $P_2(k)$ divides the greatest common divisor, say $\Pi_2(k)$, of all the coefficients of the $2^{3k-1}p_{ij}(x)$ $(0 \leq i, j \leq 2)$. Similarly, define $P_4(k)$ to be the product over all primes p with either $p \equiv 1 \bmod 4$, $p > \sqrt{4k+1}, (p, 4k) = 1$ and $\{(k-1)/p\} > 7/8$, or $p \equiv 3 \bmod 4$, $p > \sqrt{4k+1}, (p, 4k) = 1$ and $\{(k-1)/p\} > 3/4$. If $\nu = 1/4$, then $P_4(k)$ divides the greatest common divisor, say $\Pi_4(k)$, of the coefficients of the $2^{4k-1}p_{ij}(x)(0 \leq i, j \leq 2)$. We have

Lemma 3.3 *If* k *is a positive integer, then*

$$(a) \qquad \Pi_2(k) > \frac{1}{168}(3/2)^k$$

and

$$(b) \qquad \Pi_4(k) > \frac{1}{679}(4/3)^k.$$

Proof: (a) From our prior remarks, we may write

$$\Pi_2(k) \geq P_2(k).$$

Define $J_l(k)$ to be the open interval $\left(\frac{k-1}{l}, \frac{4(k-1)}{4l-1}\right)$ for l a positive integer. Then, by definition,

$$P_2(k) \geq \prod_{l=1}^{\left[\frac{k-1}{\sqrt{2k+1}}\right]} \prod_{\substack{p\in J_l(k) \\ (p,2k)=1}} p.$$

Firstly, suppose that $k \geq 15656$. Then, applying two results of Schoenfeld [16] (namely, Corollary 2 to Theorem 7 and the Note added in proof), we have

$$\sum_{p\in J_1(k)} \log(p) > 0.988828\left(\frac{4}{3}(k-1.1)\right) - 1.000081(k-1)$$

$$\sum_{p\in J_2(k)} \log(p) > 0.981682\left(\frac{4}{7}(k-1.1)\right) - 1.000081\left(\frac{k-1}{2}\right)$$

$$\sum_{p\in J_3(k)} \log(p) > 0.976870\left(\frac{4}{11}(k-1.1)\right) - 1.000081\left(\frac{k-1}{3}\right)$$

and

$$\sum_{p\in J_4(k)} \log(p) > 0.973344\left(\frac{4}{15}(k-1.1)\right) - 1.000081\left(\frac{k-1}{4}\right).$$

Since $k \geq 15656$, these estimates imply that

$$\sum_{1\leq l\leq 4} \sum_{\substack{p\in J_l(k) \\ (p,2k)=1}} \log(p) > 0.41k > \log(3/2)\ k$$

whence

$$\Pi_2(k) > (3/2)^k.$$

If, however, $1 \leq k \leq 15655$, we first use a double precision Maple V program to calculate

$$\sum_{l=1}^{\left[\frac{k-1}{\sqrt{2k+1}}\right]} \sum_{\substack{p\in J_l(k) \\ (p,2k)=1}} \log(p)$$

for each such k. In the instances when this quantity fails to exceed $\log(3/2)\, k$ (the largest occurence of which corresponds to the value $k = 270$), we explicitly calculate $\Pi_2(k)$, finding that in all cases

$$\Pi_2(k) > \frac{1}{168}(3/2)^k$$

where the extreme is obtained when $k = 30$.

(b) As before, we have

$$\Pi_4(k) \geq P_4(k)$$

and defining, for each positive integer l, the intervals $M_l(k)$ and $N_l(k)$ by

$$M_l(k) = \left(\frac{k-1}{l}, \frac{8(k-1)}{8l-1} \right)$$

and

$$N_l(k) = \left[\frac{8(k-1)}{8l-1}, \frac{4(k-1)}{4l-1} \right),$$

it follows that

$$P_4(k) \geq \prod_{l=1}^{\left[\frac{k-1}{\sqrt{4k+1}}\right]} \left(\prod_{\substack{p \in M_l(k) \\ (p,2k)=1}} p \right) \left(\prod_{\substack{p \in N_l(k) \\ p \equiv 3 \bmod 4 \\ (p,k)=1}} p \right). \tag{3.1}$$

Suppose that $k \geq 85000$. Then we may estimate

$$\sum_{\substack{p \in M_l(k) \\ (p,2k)=1}} \log(p)$$

as in (a), finding that

$$\sum_{l=1}^{7} \sum_{\substack{p \in M_l(k) \\ (p,2k)=1}} \log(p) > 0.1857k.$$

To deal with the final product in (3.1), we utilize recent work of Ramaré and Rumely [14] on bounding the function

$$\theta(x,k,l) = \sum_{\substack{p \le x \\ p \equiv l \bmod k}} \log(p).$$

For our purposes, we require only

Lemma 3.4 *(a) If $x \le 10^{10}$, then*

$$|\theta(x,4,3) - x/2| \le 1.034832\sqrt{x}.$$

(b) If $x > 10^{10}$, then

$$|\theta(x,4,3) - x/2| \le 0.001119x.$$

We therefore have, for $k \ge 85000$,

$$\sum_{\substack{p \in N_1(k) \\ p \equiv 3 \bmod 4}} \log(p) > 0.993852\left(\frac{2(k-1.1)}{3}\right) - 1.006641\left(\frac{4(k-1)}{7}\right)$$

$$\sum_{\substack{p \in N_2(k) \\ p \equiv 3 \bmod 4}} \log(p) > 0.990608\left(\frac{2(k-1.1)}{7}\right) - 1.009721\left(\frac{4(k-1)}{15}\right)$$

$$\sum_{\substack{p \in N_3(k) \\ p \equiv 3 \bmod 4}} \log(p) > 0.988227\left(\frac{2(k-1.1)}{11}\right) - 1.012037\left(\frac{4(k-1)}{23}\right)$$

and

$$\sum_{\substack{p \in N_4(k) \\ p \equiv 3 \bmod 4}} \log(p) > 0.986252\left(\frac{2(k-1.1)}{15}\right) - 1.013975\left(\frac{4(k-1)}{31}\right)$$

whence

$$\sum_{l=1}^{4} \sum_{\substack{p \in N_l(k) \\ p \equiv 3 \bmod 4 \\ (p,k)=1}} \log(p) > 0.1053k.$$

It follows that

$$\Pi_2(k) \geq P_2(k) \geq e^{(0.1857+0.1053)k} > (4/3)^k.$$

If $1 \leq k < 85000$, we calculate, via Maple V, the series

$$\sum_{l=1}^{\left[\frac{k-1}{\sqrt{4k+1}}\right]} \left(\sum_{\substack{p \in M_l(k) \\ (p,2k)=1}} \log(p) \; + \sum_{\substack{p \in N_l(k) \\ p \equiv 3 \bmod 4 \\ (p,k)=1}} \log(p) \right).$$

For $k \geq 474$, this quantity is smaller than $\log(4/3)\, k$. If $k \leq 473$, we explicitly compute the value $\Pi_4(k)$ and find that

$$\Pi_4(k) > \frac{1}{679}(4/3)^k$$

where $\Pi_4(k)\ (3/4)^k$ is minimal for $k = 31$. ∎

4 Simultaneous Approximation Results

We are now ready to prove

Theorem 4.1 *If $N \geq 13$, then*

$$\max\left\{ \left| \sqrt{1-\frac{1}{N}} - \frac{p_1}{q} \right|, \left| \sqrt{1+\frac{1}{N}} - \frac{p_2}{q} \right| \right\} > \left(1.7 \times 10^6\ N\right)^{-1} q^{-\lambda}$$

for all positive integers p_1, p_2 *and* q, *where*

$$\lambda = 1 + \frac{\log(8\sqrt{3}N + 16)}{\log\left(\frac{81}{64}(N^2-1)\right)}$$

and

Corollary 4.2 *If* p_1, p_2 *and* q *are integers, then*

$$\max\left\{\left|\sqrt{2} - \frac{p_1}{q}\right|, \left|\sqrt{3} - \frac{p_2}{q}\right|\right\} > 10^{-10} q^{-1.8161}.$$

We also have

Theorem 4.3 *If* $N \geq 4$ *then*

$$\max\left\{\left|\sqrt[4]{1 - \frac{1}{N}} - \frac{p_1}{q}\right|, \left|\sqrt[4]{1 + \frac{1}{N}} - \frac{p_2}{q}\right|\right\} > \left(3.4 \times 10^{10}\ N\right)^{-1} q^{-\lambda}$$

for all positive integers p_1, p_2 *and* q, *where*

$$\lambda = 1 + \frac{\log(18\sqrt{3}N + 36)}{\log\left(\frac{9}{16}(N^2-1)\right)}.$$

To prove Theorem 4.1, we apply Lemma 2.1 to the real numbers (setting $\nu = 1/2$)

$$\theta_1 = \sqrt{1 - \frac{1}{N}}, \quad \theta_2 = \sqrt{1 + \frac{1}{N}}$$

and the integers

$$p_{ijk} = 2^{3k-1}\ N^k\ \Pi_2(k)^{-1}\ p_{ij}(1/N).$$

By Lemma 3.4 of [15], $\det(p_{ijk})$ is nonzero,while Lemmas 2.2 and 3.3 ensure that

$$|p_{ijk}| \leq \frac{651}{5}\left(\frac{\sqrt{3}N + 2}{\sqrt{3}N - \sqrt{3}}\right)^{1/2} (8\sqrt{3}N + 16)^k \quad (0 \leq i, j \leq 2).$$

Since Lemmas 2.3 and 3.3 together yield the inequality

$$\left| p_{i0k} + p_{i1k}\sqrt{1-\frac{1}{N}} + p_{i2k}\sqrt{1+\frac{1}{N}} \right| \leq \frac{567}{8}\left(\frac{81}{64}(N^2-1)\right)^{-k}$$

for $0 \leq i \leq 2$, we may conclude, from Lemma 2.1, that

$$\max\left\{ \left| \sqrt{1-\frac{1}{N}} - \frac{p_1}{q} \right|, \left| \sqrt{1+\frac{1}{N}} - \frac{p_2}{q} \right| \right\} > cq^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(8\sqrt{3}N+16)}{\log\left(\frac{81}{64}(N^2-1)\right)}$$

and

$$c^{-1} = \frac{1984}{45}\left(\frac{567}{4}\right)^{\lambda}(\sqrt{3}N+2)\left(\frac{\sqrt{3}N+2}{\sqrt{3}N-\sqrt{3}}\right)^{1/2}.$$

The desired result follows from the inequality

$$c^{-1}/N < 1.7 \times 10^6$$

which, for $N \geq 13$, is readily obtained by calculus.

Corollary 4.2 is almost immediate. We take $N = 49$ in Theorem 4.1 and replace p_1, p_2 and q by $4p_2, 5p_1$ and $7q$. We therefore have

$$\max\left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{3} - \frac{p_2}{q} \right| \right\} > \frac{7}{10}\,(8.33 \times 10^7)^{-1}(7q)^{-\lambda} \qquad (4.1)$$

where

$$\lambda = 1 + \frac{\log(392\sqrt{3}+16)}{\log(6075/2)} \sim 1.816066.$$

Since the right hand side of (4.1) exceeds $10^{-10}q^{-1.8161}$, we conclude as stated.

The proof of Theorem 4.3 is similar. We take $\nu = 1/4$,

$$\theta_1 = \sqrt[4]{1-\frac{1}{N}}, \qquad \theta_2 = \sqrt[4]{1+\frac{1}{N}}$$

and

$$p_{ijk} = 2^{4k-1}\ N^k\ \Pi_4(k)^{-1}\ p_{ij}(1/N).$$

Then, as before, $\det(p_{ijk}) \neq 0$,

$$|p_{ijk}| \leq \frac{21049}{40} \left(\frac{\sqrt{3}N+2}{\sqrt{3}N-\sqrt{3}} \right)^{1/4} (18\sqrt{3}N+36)^k \quad (0 \leq i, j \leq 2)$$

and

$$\left| p_{i0k} + p_{i1k}\sqrt[4]{1-\frac{1}{N}} + p_{i2k}\sqrt[4]{1+\frac{1}{N}} \right| \leq \frac{91665}{512} \left(\frac{9}{16}(N^2-1) \right)^{-k}$$

for $0 \leq i \leq 2$. We conclude that

$$\max \left\{ \left| \sqrt[4]{1-\frac{1}{N}} - \frac{p_1}{q} \right|, \left| \sqrt[4]{1+\frac{1}{N}} - \frac{p_2}{q} \right| \right\} > cq^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(18\sqrt{3}N+36)}{\log\left(\frac{9}{16}(N^2-1)\right)}$$

and

$$c^{-1} = \frac{7936}{25} \left(\frac{91665}{512} \right)^{\lambda} (\sqrt{3}N+2) \left(\frac{\sqrt{3}N+2}{\sqrt{3}N-\sqrt{3}} \right)^{1/4}.$$

Theorem 4.3 obtains from the inequality

$$c^{-1}/N < 3.4 \times 10^{10}$$

which holds for all $N \geq 4$. We note that Theorems 4.1 and 4.3 give improvements upon the trivial Liouville bounds for all values of N satisfying the stated hypotheses (i.e. for $N \geq 13$ and $N \geq 4$, respectively).

5 Concluding Remarks

The exponent for q in (1.4) can be further improved to ~ 1.79155 by using more precise estimates for $|p_{ij}(1/N)|$ and $|I_i(1/N)|$ and noting that we can replace the quantity $\frac{1}{168}(3/2)^k$ in Lemma 3.3 by

$$c(\delta)\ e^{(-\gamma-\psi(3/4)-\delta)k}$$

for any $\delta > 0$, where $c(\delta)$ is positive and effectively computable, γ is Euler's constant and $\psi(x)$ is the derivative of $\log(\Gamma(x))$. Numerically, one has

$$e^{-\gamma-\psi(3/4)} \sim 1.663.$$

For details, the reader is directed to [5].

Regarding the relation between these results and diophantine equations, one may use Corollary 4.2, arguing as in [15], to show that all integer solutions of the simultaneous Pell-type equations

$$x^2 - 2z^2 = u,\ y^2 - 3z^2 = v$$

satisfy

$$\max\{|x|, |y|, |z|\} \leq \left(10^{10}\ \max\{|u|, |v|\}\right)^{5.5}.$$

This strengthens the work of Rickert [15], who proved that, in the same situation,

$$\max\{|x|, |y|, |z|\} \leq \left(10^{7}\ \max\{|u|, |v|\}\right)^{12}.$$

The connection between Theorem 4.3 and solving certain norm form equations is discussed at greater length in [5] and [6].

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