PRIME FACTORS OF CONSECUTIVE INTEGERS

MARK BAUER AND MICHAEL A. BENNETT

ABSTRACT. This note contains a new algorithm for computing a function $f(k)$ introduced by Erdős to measure the minimal gap size in the sequence of integers all of whose prime factors exceed k. This algorithm enables us to show that $f(k)$ is not monotone, verifying a conjecture of Ecklund and Eggleton.

1. INTRODUCTION

A classic theorem of Sylvester [15] (see also [13]) states that if n and k are positive integers, with $n > k$, then the product of k consecutive integers

$$\Pi_{n,k} = n(n+1)\cdots(n+k-1)$$

is necessarily divisible by a prime $p > k$ (i.e. $P(\Pi_{n,k}) > k$, where $P(m)$ denotes greatest prime divisor of a positive integer m). If we therefore, following Erdős [6], define $f(k)$ to be the least integer with the property that

$$P\left(\Pi_{n,f(k)}\right) > k$$

whenever $n > k$, it follows that $f(k) \leq k$. Asymptotically, one may in fact show that

$$\frac{\log k \, (\log\log k)(\log\log\log\log k)}{(\log\log\log k)^2} \ll f(k) \ll \frac{k \log\log\log k}{\log k \, (\log\log k)},$$

via work of Rankin [12] and Shorey [14], respectively. Standard heuristics for the size of gaps between consecutive primes lead one to expect that, in fact, the actual order of magnitude of $f(k)$ is $(\log k)^2$.

The explicit computation of $f(k)$ for $k \leq 10$ was carried out by Utz [16] (resolving a conjecture of Erdős [6]) and subsequently extended to $k \leq 42$ by Lehmer [11], to $k \leq 46$ by Ecklund and Eggleton [2], and to $k \leq 73$ by Ecklund, Eggleton and Selfridge [3], [4]. We tabulate these results as follows :

k	$f(k)$	k	$f(k)$	k	$f(k)$
1	1	$5-12$	4	$47-58$	8
2	2	$13-40$	6	$59-60$	9
$3-4$	3	$41-46$	7	$61-73$	14

In [2], Ecklund and Eggleton posed the question of whether or not $f(k)$ is monotone and conjectured that it is not – in fact they expressed a strong belief that $f(113) = 14$ and $f(114) = 13$ (and that $f(127) = 12$). In this short note, we will present a new algorithm for computing $f(k)$ which will enable us to confirm Ecklund and Eggleton's suspicions. In particular, we prove the following :

Theorem 1.1. *If $74 \leq k \leq 178$, then the values of $f(k)$ are as follows :*

k	$f(k)$	k	$f(k)$
$74 - 113$	14	$115 - 150$	12
114	13	$151 - 178$	14

The fact that $f(114) < f(113)$ contradicts an assertion of Utz [16] to the effect that "if k is composite, the value of $f(k)$ is that of $f(p)$, where p is the largest prime smaller than k". That this statement might potentially be false was already noted by Ecklund and Eggleton [2].

2. THE ALGORITHM

For fixed m and t, we will in fact consider the problem of classifying those positive integers n for which

$$(2.1) \qquad\qquad P\left(\Pi_{n,m}\right) \leq p_t$$

where p_t is the t-th prime.

Instead of proceeding as in Lehmer [11], where first inequality (2.1) is treated in case $m = 2$ and then extended to larger values of m, we will argue as follows. For any given $m \geq 3$, we will split $4\left[m/4\right]$ (or $m - 1$, if $m \equiv -1 \pmod 4$) of the indices $0, \cdots, m - 1$ into disjoint pairs of the form $(i, i+2)$, where $i \equiv 0, 1 \pmod 4$. There are

$$\left[\frac{m}{4}\right] + \left[\frac{m+1}{4}\right]$$

such pairs. Choose t_0 such that

$$\pi(m - 1) \leq t_0 \leq t.$$

If $m \geq 3$, we can thus find an $i \in [0, m-3]$ such that the product $(n+i)(n+i+2)$ is divisible by at most

$$(2.2) \qquad\qquad N = \left[\frac{t - t_0}{\left[\frac{m}{4}\right] + \left[\frac{m+1}{4}\right]}\right]$$

of the primes in

$$(2.3) \qquad\qquad \{p_{t_0+1}, \cdots, p_t\}.$$

Writing $D > 1$ for the largest square-free integer dividing $(n + i)(n + i + 2)$ and setting

$$DY^2 = (n + i)(n + i + 2) \text{ and } X = n + i + 1,$$

we find that $X^2 - DY^2 = 1$, whereby, if (T_1, U_1) are the smallest positive integers for which $x^2 - Dy^2 = 1$, there necessarily exists a positive integer d for which $(X, Y) = (T_d, U_d)$, where

$$T_d + \sqrt{D}\, U_d = \left(T_1 + \sqrt{D}\, U_1\right)^d.$$

Classic results on primitive divisors (see e.g. Carmichael [1]; in this setting, this probably dates back rather earlier to Lucas) ensure that, at least for $d > 12$, there is a prime dividing U_d that does not divide DU_j for $j < d$. Since work of Lehmer

[9] guarantees that if an odd prime p divides some U_d, then it necessarily divides U_j where

$$(2.4) \qquad j \mid \frac{p - \left(\frac{D}{p}\right)}{2},$$

we conclude, from the assumption $P(U_d) \leq p_t$, that

$$(2.5) \qquad d \leq \max\left\{\frac{p_t + 1}{2}, 12\right\}.$$

Our algorithm thus proceeds as follows. For each of the

$$(2.6) \qquad M = -1 + 2^{t_0} \sum_{j=0}^{N} \binom{t - t_0}{j}$$

values of D under consideration (trivially omitting $D = 1$), we first compute the corresponding fundamental solution (T_1, U_1). Since we assume that $P(T_d U_d D) \leq p_t$ and since $T_1 U_1 \mid T_d U_d$, we need only consider those D for which both $P(T_1 U_1) \leq p_t$ and with DU_1 divisible by at most N of the primes in (2.3). For each such D, we can compute (T_d, U_d) for all d satisfying (2.5), via recursion say, and check whether $P(T_d U_d) \leq p_t$ and whether DU_d is divisible by at most N primes in (2.3). Actually, we can usually get away with restricting our attention to prime values of d (since $T_p U_p$ divides $T_d U_d$ whenever $p \mid d$), and moreover employ (2.4) to further reduce our computations.

Essentially, what this algorithm accomplishes is to find all positive integers r such that $P(r^3 - r) \leq p_t$, under the additional constraint that $r^2 - 1$ is divisible by at most N primes in (2.3). Having carried this out, in order to find all solutions to (2.1), it is left to see whether $P(r \pm j) \leq p_t$ holds for at least m consecutive integers of the form $r \pm j$ satisfying $j \leq m - 2$. Each r satisfying these criteria yields an integer n such that $P(\Pi_{n,m}) \leq p_t$, and possibly several such n with $n \in [r - (m-2), r + (m-2)]$.

The calculation of $f(k)$ proceeds in two stages from here. Using the list of integers r that are output by the algorithm, we can find all integers n satisfying $P(\Pi_{n,m}) \leq k$. If all such n are at most k in size, then this establishes an upper bound for $f(k)$, namely, $f(k) \leq m$. If we already know (say via naive search) an integer $n > k$ such that $P(\Pi_{n,m-1}) \leq k$, then $f(k) \geq m$ and we may conclude that $f(k) = m$. Otherwise, we rerun the algorithm to classify those integers with $P(\Pi_{n,m-1}) \leq k$, and, if necessary, iterate the process, at each stage replacing $m - i$ by $m - i - 1$.

If, on the other hand, our list of n for which $P(\Pi_{n,m}) \leq k$ contains an $n > k$ such that $P(\Pi_{n,m}) \leq k$, then $f(k) > m$. To determine $f(k)$ explicitly, we merely examine all n with this property, and find, for each such n, the largest integer m_n with the property that $P(\Pi_{n,m_n}) \leq k$. Then $f(k) = \max_n\{m_n\} + 1$.

3. Computations

By way of example, to find the values of n for which $P(\Pi_{n,12}) \leq 173$, we take $t = 40$ and $m = 12$. Our goal is to choose t_0 in such a way that the number of discriminants we need to consider, M as defined in (2.6), is minimized. In general, it

is easy to show that the optimal choice for t_0 comes from taking either $t_0 = \pi(m-1)$ or from choosing t_0 to be the minimal integer with $t_0 \geq \pi(m-1)$ satisfying

$$t_0 \equiv t + 1 \left(\mod \left[\frac{m}{4} \right] + \left[\frac{m+1}{4} \right] \right).$$

In the case $(t, m) = (40, 12)$, this leads to $t_0 = 5$, with corresponding number of discriminants $M = 12293376$.

Proceeding as described above, we find that 2692 of these discriminants yield a suitable T_1 and U_1 satisfying both $P(T_1 U_1) \leq 173$ and $D\,U_1$ divisible by at most 5 of the primes between 13 and 173. Corresponding to these, we find that 183 pairs (T_d, U_d) (10 of which have $d > 1$) lead to n with $P(\Pi_{n,12}) \leq 173$. The values for r found using the algorithm satisfy $r \leq 175$, $295 \leq r \leq 303$ and $319 \leq r \leq 327$. Calculating the corresponding n enables us to conclude as follows :

Proposition 3.1. *The positive integer n for which*

$$P\left(\Pi_{n,12} \right) \leq 173$$

are precisely those with either $n \leq 167$ or $n \in \{294, 295, 318, 319\}$.

From our remarks at the close of the preceding section and the fact that

$$P\left(\Pi_{294,13} \right) = 151 \quad \text{and} \quad P\left(\Pi_{318,13} \right) = 163,$$

we conclude that $f(k) = 14$ for $151 \leq k \leq 178$. Similarly, from $P\left(\Pi_{200,11} \right) = 103$ and $P\left(\Pi_{114,13} \right) = 61$, it follows that, for $61 \leq k \leq 113$, we have $f(k) = 14$, while $f(114) = 13$ and, for $115 \leq k \leq 150$, $f(k) = 12$, as suspected by Ecklund and Eggleton. This completes the proof of Theorem 1.1.

4. Concluding remarks

The computations leading to Proposition 3.1 were split and run in parallel (the algorithm is clearly well suited to parallelization) on a Mac PowerPC dual core 2.5 GHz processor, taking a total of just over 58 hours of processing time. Computationally, as one might expect, the costly step is the computation of fundamental units in real quadratic fields; the complexity of this part of the algorithm is well understood (see e.g [17]). The number of fields under consideration, M, is typically, for a good choice of t_0, substantially smaller than $2^t - 1$, the analogous quantity in Lehmer's approach (at least provided $k > 3$). Indeed, if we suppose that t is a fixed constant multiple of m, say $t = \kappa m$ with $\kappa > 1$, and take $t_0 = \pi(m-1)$, then we have

$$M = 2^{(1+o(1))\frac{m}{\log m}},$$

which compares rather favourably with $2^{\kappa m}$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALGARY, CALGARY AB
E-mail address: `mbauer@math.ucalgary.ca`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER BC
E-mail address: `bennett@math.ubc.ca`