Powers from five terms in arithmetic progression

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1 Introduction

A celebrated theorem of Erdős and Selfridge [5] asserts that a product of consecutive nonzero integers can never be a perfect power. More generally, the techniques of [5] have been extended and refined by Győry [6] and Saradha [9] to prove that the Diophantine equation

$$n(n+1)(n+2)\cdots(n+k-1) = by^l \tag{1}$$

has only the solution $(n,k,b,y,l) = (48,3,6,140,2)$ in positive integers n,k,b,y and l, where $k,l \geq 2$, $P(b) \leq k$ and $P(y) > k$. Here, $P(m)$ denotes the greatest prime factor of the integer m (where, for completeness, we write $P(\pm 1) = 1$ and $P(0) = \infty$). Rather surprisingly, no similar conclusion is available for the frequently studied generalization of this equation to products of consecutive terms in arithmetic progression

$$n(n+d)(n+2d)\cdots(n+(k-1)d) = by^l, \tag{2}$$

under the assumption $P(b) \leq k$, or the weaker $P(b) < k$, for even a *single* value of k. Here, to avoid trivialities, we must suppose that $\gcd(n,d) = 1$. Further, if $k = 3$ or 4, we should note that equation (2) has, in fact, infinitely many solutions with $l = 2$.

In this short note, we will deduce a partial analogue of the theorem of Erdős-Selfridge, Győry and Saradha, for the smallest possible value of k, namely $k = 5$. To be precise, we will prove the following :

Theorem 1.1 *If n and d are coprime nonzero integers, then the Diophantine equation*

$$n(n+d)(n+2d)(n+3d)(n+4d) = by^l \tag{3}$$

has no solutions in nonzero integers b, y and l with $l \geq 2$ and $P(b) \leq 3$.

This is a slight sharpening of a special case of Theorem 1.2 of [2] (see also [7]), where a result is obtained for equation (2) for $4 \leq k \leq 11$, under the more restrictive assumption that $P(b) < k/2$. The techniques of [2] or [7] are inadequate, however, to derive Theorem 1.1. The key to our current improvement is a new idea for obtaining, from solutions to (2), corresponding solutions to ternary equations of signature $(l,l,3)$.

There is an abundant literature on equations (1) and (2) and their generalizations. An excellent survey of the current state of play in this area is that of Shorey [13]; much of the recent progress is, in fact, due to the work of Shorey and his collaborators (see e.g. [6], [7], [8], [9], [10], [11], [12], [14], [15], [16]). It does not appear, however, that any techniques available at the present time, including those introduced here, allow one to conclude that equation (2) (with $P(b) < k$) has even finitely many solutions in, for example, the case $k = 6$.

2 The proof of Theorem 1.1

Our proof will, for the most part, follow along similar lines to those in [2] (in particular, as in [2], our arguments will rely on the fact that a solution to (3) is closely connected to solutions to related ternary Diophantine equations). Indeed, Theorem 1.2 of [2] implies the current Theorem 1.1 unless we have, assuming, as we may, that l is prime,

$$l \geq 7 \text{ and } P(b) = 3. \tag{4}$$

Let us suppose that we have a solution to equation (3) in nonzero integers n, d, y, b and l satisfying the conditions of Theorem 1.1 (where we additionally assume, without loss of generality, that d is positive). From the fact that $\gcd(n, d) = 1$, we may write

$$n + id = b_i y_i^l \text{ for } 0 \leq i \leq 4, \tag{5}$$

where b_i and y_i are integers with $P(b_i) \leq 3$. To guarantee that such a representation is unique, we will further assume that each b_i is lth power free and positive. From (4), we may, in fact, suppose that

$$\max\{P(b_i)\} = 3.$$

Let us begin by considering the case when $P(b_2) = 3$. Then the identity

$$(n + d)(n + 3d) - n(n + 4d) = 3d^2$$

implies that

$$b_1 b_3 (y_1 y_3)^l - b_0 b_4 (y_0 y_4)^l = 3d^2,$$

where

$$\gcd(b_1 b_3 y_1 y_3, b_0 b_4 y_0 y_4) = 1 \text{ and } P(b_0 b_1 b_3 b_4) \leq 2.$$

We may now appeal to a result of ternary Diophantine equations of signature $(l, l, 2)$; this is a special case of Theorems 1.1 and 1.2 of [1].

Proposition 2.1 *Let $l \geq 7$ be prime and $\alpha \neq 1$ be a nonnegative integer. Then the Diophantine equation*

$$a^l + 2^\alpha b^l = 3c^2$$

has no solutions in nonzero coprime integers (a, b, c) with $ab \neq \pm 1$.

From this, we conclude that necessarily

$$b_0 = b_4 = 2 \text{ and } y_0 = -1, \ y_4 = 1.$$

This implies that $n = -2$ and $d = 1$, contradicting the fact that $y \neq 0$ in (3).

To complete the proof of Theorem 1.1, it remains to treat the situation when

$$\max\{P(b_0), P(b_1), P(b_3), P(b_4)\} = 3$$

(so that either 3 divides both n and $n+3d$, or 3 divides both $n+d$ and $n+4d$). In this case, we will apply the "smallest" of a family of identities that one may use to transfer information about putative solutions to equations of the shape (2), to corresponding ternary equations of signature $(l, l, 3)$ (as opposed to the equations of signature $(l, l, 2)$ or (l, l, l) used in [2]). The latter may, hopefully, be treated by the methods developed in [3] and [4]. In our case, we will appeal to the identity

$$(n+d)(n+d)(n+4d) - n(n+3d)(n+3d) = 4d^3.$$

This implies the equation

$$b_1^2 b_4 \left(y_1^2 y_4\right)^l - b_0 b_3^2 \left(y_0 y_3^2\right)^l = 4d^3,$$

whereby, dividing out a suitable power of 2, we find a solution in nonzero integers (a, b, c) to one of

$$a^l + 3^\beta b^l = 2^\alpha c^3 \;\; \alpha \geq 1, \; \beta \geq 3, \; \gcd(a, 3b) = 1, \tag{6}$$

or

$$Aa^l + Bb^l = c^3, \; AB = 2^\alpha 3^\beta, \; \alpha \geq 1, \; \beta \geq 3, \; \gcd(Aa, Bb) = 1. \tag{7}$$

The first of these equations may be handled by an immediate application of results from [3]. Indeed, Theorem 1.5 of that paper implies that (6) has no solutions in nonzero integers whatsoever. For equation (7), we argue in a similar fashion. Supposing, renaming if necessary, that $3 \mid B$, we consider the "Frey" elliptic curve

$$E = E(a, b, c) \;:\; y^2 + 3cxy + Bb^l y = x^3$$

with (see Lemma 2.1 of [3]) discriminant $27AB^3(ab^3)^l$ and conductor $2 \cdot 3^\delta \cdot \prod_{p \mid ab} p$, for $\delta \in \{0, 1\}$ (where here we suppose that $\gcd(ab, 6) = 1$). From Lemma 3.4 of [3], since $l \geq 7$ is prime, the corresponding mod l Galois representation

$$\rho_l^E : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_l),$$

on the l-torsion $E[l]$ of E, is unramified outside of $2, 3$ and l and necessarily arises from a cuspidal newform f of weight $2 \cdot 3^\delta$ and trivial Nebentypus character. Noting that the modular curves $X_0(N)$ have genus 0 for all N dividing 6, we conclude that such a newform cannot exist. This completes the proof of Theorem 1.1.

□

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