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Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$

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Abstract. Following an approach originally due to Mahler and sharpened by Chudnovsky, we develop an explicit version of the multi-dimensional "hypergeometric method" for rational and algebraic approximation to algebraic numbers. Consequently, if a, b and n are given positive integers with $n \geqq 3$, we show that the equation of the title possesses at most one solution in positive integers x, y. Further results on Diophantine equations are also presented. The proofs are based upon explicit Padé approximations to systems of binomial functions, together with new Chebyshev-like estimates for primes in arithmetic progressions and a variety of computational techniques.

1. Introduction

A classical problem in Diophantine approximation is to determine, for a given irrational number θ and real $\varepsilon > 0$, positive constants $c = c(\theta, \varepsilon)$ and $\lambda = \lambda(\theta)$ such that the inequality

$$\left|\theta - \frac{p}{q}\right| > cq^{-\lambda-\varepsilon} \tag{1.1}$$

is satisfied by all integers p and q with $q > 0$. If θ is an algebraic number of degree n, then a result of Liouville implies that we may take $\lambda = n$ in (1.1), a fact which established the existence of transcendental numbers. In 1909, Thue [Th1] showed that the exponent in Liouville's theorem could be replaced by $\lambda = n/2 + 1$. Consequently, if $F(x, y)$ is an irreducible binary form (i.e. homogeneous polynomial in $\mathbb{Z}[x, y]$) of degree $n \geqq 3$, the Thue equation

$$F(x, y) = m$$

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has, for a fixed non-zero integer m, at most finitely many solutions in integers x and y. Subsequently, Thue's theorem has been sharpened by Siegel [Si1]

$$\left(\lambda = \min\left\{\frac{n}{s+1} + s\colon 0 \leqq s \leqq n-1\right\}\right),$$

Dyson [Dy] and Gelfond [G] ($\lambda = \sqrt{2n}$) and, finally, Roth [Ro] ($\lambda = 2$), extended to p-adic valuations by Mahler [Ma2] and Ridout [Rid], and generalized to simultaneous approximation by Schmidt [Schm]. Other than Liouville's original theorem, however, each of these results is ineffective in the sense that it is not possible to explicitly compute the constant $c(\theta, \varepsilon)$ from the given proof.

In the intervening years, three distinct methods have arisen to derive effective improvements upon Liouville's theorem, based upon Padé approximation to hypergeometric functions (see e.g. [Ba1], [Ba2], [Ba3], [Be2], [Be3], [Ch], [Ea], [Ma1]), the method of Thue-Siegel (see e.g. [Bo1], [Bo2], [BM]) or lower bounds for linear forms in logarithms of algebraic numbers (see e.g. [Ba4], [F]). These correspond to producing certain "auxilliary polynomials" (depending on the algebraic number θ) in $1, 2$ or many variables, respectively. While the last two of these methods are strong enough to imply improvements upon Liouville's exponent $\lambda = n$ in (1.1) for all algebraic numbers θ of degree at least 3, the first is characterized by sharper bounds with smaller implied constants, though necessarily in a restricted setting.

In this paper, we will develop a general version of the "hypergeometric method". Our approach will follow closely that of Mahler [Ma1], together with a number of refinements stemming from work of Baker [Ba3] and Chudnovsky [Ch]. In most situations, our results, in contrast to [Ch], will be completely explicit, with an eye towards applications to Diophantine equations. In fact, we will postpone a discussion of the general applicability of our methods and of allied results for nonarchimedean valuations to a future paper [Be4], concentrating instead upon bounds for algebraic numbers of relatively small degree.

As mentioned previously, improvements upon Liouville's theorem have profound implications for Diophantine equations. In particular, lower bounds for rational approximation to numbers of the form $\sqrt[n]{a/b}$ are equivalent to bounds upon integer solutions (x, y) to diagonal Thue equations of the shape

$$ax^n - by^n = c. \tag{1.2}$$

If $n = 3$ and $c = \pm 1$, then Delone [De] and Nagell [N] (see also [DF]), independently, applied what is now termed Skolem's p-adic method together with information on fundamental units in cubic number fields to show, for a and b nonzero integers, that (1.2) possesses at most one solution in positive integers (x, y). Similarly, Ljunggren [Lj2] (see also [Ta]) deduced a like result in the case $n = 4$ and $c = \pm 1$. For larger values of n, by sharpening Padé approximation-based techniques of Thue [Th2] and Siegel [Si2], Domar [Do] was able to show, with a few exceptions, that (1.2) with $c = \pm 1$ in fact possesses at most two positive solutions. One should note that while the results of [De], [N] and [Lj2] provide an explicit description of a purported solution (x, y) to (1.2) if $n = 3$ or 4 and $c = \pm 1$, the same is not true of [Do]. For further results on equation (1.2), the reader is directed to [Ev], [Hy], [Mi], [Mo] and [Mu].

Recently, B. M. M. de Weger and the author [BdW] (see also [Mi]) strengthened Domar's theorem by showing that, if $ab \neq 0, n \geqq 3$ and $c = \pm 1$, then (1.2) has at most one solution in positive integers (x, y), except possibly for those (a, b, n) (where, without loss of generality, we assume that $b > a \geqq 1$) with

$$b = a + 1, \quad 2 \leqq a \leqq \min\{0.3n, 83\} \quad \text{and} \quad 17 \leqq n \leqq 347. \tag{1.3}$$

As an application of the main results of the paper at hand, we are able to sharpen these conclusions, proving

Theorem 1.1. *If a, b and n are integers with $ab \neq 0$ and $n \geqq 3$, then the equation*

$$|ax^n - by^n| = 1 \tag{1.4}$$

has at most one solution in positive integers (x, y).

This implies that the equation

$$(a+1)x^n - ay^n = 1$$

has, for $a \geqq 1$ and $n \geqq 3$, precisely the solution $(x, y) = (1, 1)$ in positive integers (if $a = 1$, this is a consequence of a result of Darmon and Merel [DM]). As far as we know, this is the first instance of a complete solution of a parametrized family of Thue equations of arbitrary degree. We note that the methods of [BdW], which include lower bounds for linear forms in logarithms of algebraic numbers together with results from [Be3] and various computational techniques, differ from those of the current paper in that they appeal to what is essentially the "one-dimensional" version of the results derived here.

Another classical Diophantine problem, arising in a variety of contexts, is to solve the equation

$$\frac{x^n - 1}{x - 1} = y^m \tag{1.5}$$

where x, y, n and m are integers with $x > 1, y > 1, n > 2$ and $m > 1$. An as yet unproven conjecture regarding this equation is that the only such solutions (x, y, n, m) are those given by $(3, 11, 5, 2)$, $(7, 20, 4, 2)$ and $(18, 7, 3, 3)$. Indeed, it is not even known if (1.5) has finitely many solutions of this form. For a detailed history of early algebraic approaches to (1.5), the reader is directed to the book of Ribenboim [Rib]. More recent studies of this equation have utilized the hypergeometric method (e.g. [SS]) or lower bounds for linear forms in logarithms (e.g. [ST] or [Bu]; the first of these uses archimedean estimates, the second p-adic). By application of Theorem 1.1, we may show

Corollary 1.2. *If $x > 1$, $y > 1$, $n > 2$ and $m > 1$ are integers, then:*

(a) *If $n \equiv 1 \pmod{m}$, it follows that the only solution to equation (1.5) satisfies $(x, y, n, m) = (3, 11, 5, 2)$.*

(b) *There are no solutions to (1.5) with $x = z^m$ for $z \in \mathbb{Z}$.*

(c) *There are no solutions to* (1.5) *with* $x = z^2$ *for* $z \in \mathbb{Z}$.

(d) *If* $\omega(n)$ *denotes the number of distinct prime factors of* n, *then there are no solutions to* (1.5) *with* $\omega(n) > m - 2$.

The first of these results was stated as a corollary in Le [Le2], but the proof is erroneous since Lemma 3 of [Le2] is false (this also invalidates the claims of Yu and Le in [LY]; see the comment in [Yu]). Similarly, the results of Le [Le3] on equation (1.5) must be regarded as unproven since Lemma 1 and Lemma 2 of that paper are both incorrect; indeed the fact that the equation $X^2 - 3Y^2 = 11^2$ possesses integral solutions while $X^2 - 3Y^2 = 11$ is insoluble serves to contradict Lemma 1 while, in the notation of [Le3], taking $(a, b, n, k, X, Y) = (1, 2, 3, 47, 63, 50)$ contradicts Lemma 2. Our proof uses Theorem 1.1 in conjunction with work of Ljunggren [Lj1]. Part (b) is Theorem 1 of [Le1], but follows very easily from Theorem 1.1. Part (c) is a slight sharpening of work of Saradha and Shorey [SS], who deduced a like result under the hypothesis that

$$z \geqq 32 \quad \text{or} \quad z \in \{2, 3, 4, 8, 9, 16, 27\}.$$

Finally, part (d) obtains from arguments of Shorey [Sh1] and [Sh2], upon application of parts (a) and (b).

The outline of this paper is the following. In the next section, we begin by proving a pair of technical lemmata which imply effective bounds for rational and algebraic approximation to real θ, under certain specific conditions. To obtain these bounds, it is necessary to construct families of "approximating" polynomials. In Section 3, we carry out this construction by appealing to the theory of Padé approximation to binomial functions. We also derive a number of explicit, essentially sharp bounds for these polynomials. In Section 4, we study the p-adic valuations of products of the gamma function evaluated at rational points. This enables us to explicitly describe a factor $\Delta_{m,n,r}$ that arises in considering the coefficients of our "approximating" polynomials at nonarchimedean places. In Section 5, we derive upper and lower bounds for sums of logarithms of primes in arithmetic progression, in fixed intervals. This combines information about zeros of certain Dirichlet L-functions with results obtained by sieving. These bounds are applied in Section 6, together with a variety of computations, to majorize the term $\Delta_{m,n,r}$ considered in Section 4. In Section 7, we state and prove our Main Theorem on rational approximation to algebraic numbers. Section 8 contains data regarding computation of continued fraction expansions to numbers of the form $\sqrt[n]{1 + 1/a}$, completing the proof of Theorem 1.1. In Section 9, we turn our attention to approximation of algebraic numbers by algebraic numbers of fixed degree. Section 10 contains explicit formulae for certain "characteristic numbers" introduced by Chudnovsky [Ch] and related to $\Delta_{m,n,r}$ of Section 4, at least for small values of the parameter m. Finally, in Section 11, we prove Corollary 1.2.

2. Folklore lemmata

To derive lower bounds for rational approximation to a given real number θ or to a linear form in powers of θ, it suffices to produce sequences of good simultaneous rational approximations to $1, \theta, \theta^2, \ldots, \theta^m$ for a positive integer m, which satisfy certain independence conditions. To be precise, we can use the following "folklore lemma". The first part is an explicit version of Lemma 3.2 of [Ch] (see also Lemma 1 of [NS]) while the second is essentially a combination of Lemma 6 of [Ba3] and the proof of the main theorem of that paper.

Lemma 2.1. *Let θ, c, d, C and D be positive real numbers and r_0 and k positive integers, with θ irrational and $C, D > 1$. Further, suppose, for each positive integer r with $r \geqq r_0$, we can find a sequence of polynomials*

$$P_{i,r}(x) = \sum_{j=0}^{k} a_{ij} x^j$$

for $0 \leqq i \leqq k$, with $a_{ij} \in \mathbb{Z}$, the matrix (a_{ij}) nonsingular,

$$|a_{ij}| \leqq cC^r$$

and

$$|P_{i,r}(\theta)| \leqq dD^{-r}.$$

1. *If t is any real number satisfying $t > 1$ and $tdD^{r_0-1} \geqq 1$, p and q are nonzero integers with $q \geqq D^{\frac{r_0-1}{k}}(td)^{-\frac{1}{k}}$ and $\delta = \max\{|\theta|, |p/q|, 1\}$, then*

$$\left|\theta - \frac{p}{q}\right| > \left(\frac{t}{t-1}\,\frac{k(k+1)}{2}\delta^{k-1} cC(td)^{\frac{\log(C)}{\log(D)}}\right)^{-1} q^{-\lambda_1}$$

where

$$\lambda_1 = k\left(1 + \frac{\log(C)}{\log(D)}\right).$$

2. *If $r_0 = 1$, $C^{k-1} < D$ and $x_0, x_1, \ldots, x_k$ are integers, not all zero, with absolute value at most X, then*

$$\left|\sum_{i=0}^{k} x_i \theta^i\right| > \left(2k^{k/2} c^k C^k (2k^{\frac{1}{2}k+1} c^{k-1} d)^{\lambda}\right)^{-1} X^{-\lambda_2}$$

where

$$\lambda_2 = \frac{k \log C}{\log(D/C^{k-1})}.$$

Proof. We will first prove part 1. Choose r to be the smallest positive integer such that $D^r > tdq^k$. To see that $r \geqq r_0$, note that $q \geqq D^{\frac{r_0-1}{k}}(td)^{-\frac{1}{k}}$ and so $tdq^k \geqq D^{r_0-1}$. We therefore have $D^r \leqq tdDq^k$ and so

$$C^r = D^{r\left(\frac{\log(C)}{\log(D)}\right)} \leqq C(td)^{\frac{\log(C)}{\log(D)}} q^{k\left(\frac{\log(C)}{\log(D)}\right)}. \tag{2.1}$$

Since (a_{ij}) is nonsingular, we can find some i, $0 \leqq i \leqq k$, for which $P_{i,r}(p/q) \neq 0$. Thus

$$\frac{1}{q^k} \leqq |P_{i,r}(p/q)| \leqq |P_{i,r}(p/q) - P_{i,r}(\theta)| + |P_{i,r}(\theta)| < |P_{i,r}(p/q) - P_{i,r}(\theta)| + \frac{1}{tq^k}$$

whereby

$$|P_{i,r}(p/q) - P_{i,r}(\theta)| > \left(1 - \frac{1}{t}\right) q^{-k}.$$

On the other hand, we have

$$|P_{i,r}(p/q) - P_{i,r}(\theta)| = \left| \int_{p/q}^{\theta} P'_{i,r}(x)\, dx \right| \leqq \frac{k(k+1)}{2} \delta^{k-1}\, cC^r \left| \theta - \frac{p}{q} \right|.$$

It follows, therefore, that

$$\left| \theta - \frac{p}{q} \right| > \left(\frac{t}{t-1} \frac{k(k+1)}{2} \delta^{k-1}\, cC^r \right)^{-1} q^{-k}$$

and so the desired result obtains from (2.1).

Let us next turn our attention to part 2. From Lemma 6 of [Ba3], under the hypotheses of our lemma, we have

$$\left| \sum_{i=0}^{k} x_i \theta^i \right| > \left(k^{k/2} (cC^r)^k \right)^{-1} - kdD^{-r} X (cC^r)^{-1}.$$

Let us choose r to be the smallest positive integer such that

$$C^{(k-1)r} D^{-r} < (2k^{\frac{1}{2}k+1} c^{k-1}\, dX)^{-1}$$

as is possible from the fact that $C^{k-1} < D$. It follows that

$$\left| \sum_{i=0}^{k} x_i \theta^i \right| > \left(2k^{k/2} (cC^r)^k \right)^{-1}$$

and, from our choice of r, we have

$$(C^{k-1}/D)^r \geqq (C^{k-1}/D)(2k^{\frac{1}{2}k+1} c^{k-1}\, dX)^{-1}.$$

We thus have

$$C^{kr} \leqq C^k (2k^{\frac{1}{2}k+1} c^{k-1}\, dX)^{\lambda_2}$$

and the result follows as claimed. $\square$

As noted by Baker [Ba3], a result of the nature of part 2 of Lemma 2.1 leads almost immediately to a bound for approximation to θ by algebraic numbers of fixed degree. In Section 9, we will derive such a bound. In order to apply the previous lemmata, for various algebraic θ, we need to construct sequences of polynomials $P_{i,r}(x)$ with the requisite properties. To accomplish this, we turn to the theory of Padé approximation.

3. Padé approximants to binomial functions

Let us suppose that $f_1(z), f_2(z), \ldots, f_m(z)$ are defined by formal power series at $z = 0$. If $n_1, \ldots, n_m$ are nonnegative integers and

$$A_1(z|n_1, \ldots, n_m), \ldots, A_m(z|n_1, \ldots, n_m)$$

polynomials in z of degree at most $n_1, \ldots, n_m$, respectively, then we will call the $A_i(z|n_1, \ldots, n_m)$ *simultaneous Padé approximants* for the system $f_i(z)$ if

$$\sum_{i=1}^{m} A_i(z|n_1, \ldots, n_m) f_i(z) = R(z|n_1, \ldots, n_m)$$

possesses a zero of order

$$\sum_{i=1}^{m} (n_i + 1) - 1$$

at $z = 0$ (see Mahler [Ma3]). In this section, we will explicitly construct simultaneous Padé approximants to systems of binomial functions, following Mahler [Ma1] (see also Baker [Ba3] and Chudnovsky [Ch]). These approximants will yield the polynomials needed for application of Lemma 2.1. It should be noted that, in essence, this construction dates back to work of Hermite [He].

Let us suppose that $m \geqq 2$ is an integer and set

$$\phi(\zeta) = \prod_{k=1}^{m} \prod_{h=0}^{\rho_k - 1} (\zeta - \omega_k - h)$$

and

$$R\left(z \,\middle|\, \begin{matrix} \omega_1 & \cdots & \omega_m \\ \rho_1 & \cdots & \rho_m \end{matrix}\right) = \frac{(-1)^{\sigma-1}\Gamma(\rho_1)\cdots\Gamma(\rho_m)}{2\pi\sqrt{-1}} \int_\gamma \frac{(1-z)^\zeta \, d\zeta}{\phi(\zeta)}$$

where γ is a closed, positively oriented contour enclosing all the poles of the integrand, $\rho_1, \ldots, \rho_m$ are positive integers, $\omega_1, \ldots, \omega_m$ are complex numbers such that $\omega_i - \omega_j$ are nonintegral for all $i \neq j$ and $\sigma = \rho_1 + \cdots + \rho_m$.

Cauchy's residue theorem implies, then, that

$$R\left(z \,\middle|\, \begin{matrix} \omega_1 & \cdots & \omega_m \\ \rho_1 & \cdots & \rho_m \end{matrix}\right) = \sum_{j=1}^{m} A_j\left(z \,\middle|\, \begin{matrix} \omega_1 & \cdots & \omega_m \\ \rho_1 & \cdots & \rho_m \end{matrix}\right)(1-z)^{\omega_j}$$

where

$$A_j\left(z\left|\begin{matrix}\omega_1 & \cdots & \omega_m\\ \rho_1 & \cdots & \rho_m\end{matrix}\right.\right) = (-1)^{\sigma-1}\Gamma(\rho_1)\cdots\Gamma(\rho_m)\sum_{h=0}^{\rho_j-1}\frac{(1-z)^h}{\phi'(\omega_j+h)}.$$

We may readily observe that $R\left(z\left|\begin{matrix}\omega_1 & \cdots & \omega_m\\ \rho_1 & \cdots & \rho_m\end{matrix}\right.\right)$ has a zero at $z=0$ of order $\sigma-1$ (see e.g. [Ma1]). Further, from the preceding equation, the $A_j\left(z\left|\begin{matrix}\omega_1 & \cdots & \omega_m\\ \rho_1 & \cdots & \rho_m\end{matrix}\right.\right)$ (for $1 \leqq j \leqq m$) are polynomials in z of degree ρ_j-1 and thus it follows that we have constructed a system of simultaneous Padé approximants to the system of functions $(1-z)^{\omega_j}$ $(1 \leqq j \leqq m)$.

Let us now define, for $1 \leqq i, j \leqq m$,

$$R_i\left(z\left|\begin{matrix}\omega_1 & \cdots & \omega_m\\ \rho_1 & \cdots & \rho_m\end{matrix}\right.\right) = R\left(z\left|\begin{matrix}\omega_1 & \cdots & \omega_m\\ \rho_1+\delta_{i1} & \cdots & \rho_m+\delta_{im}\end{matrix}\right.\right)$$

and

$$A_{ij}\left(z\left|\begin{matrix}\omega_1 & \cdots & \omega_m\\ \rho_1 & \cdots & \rho_m\end{matrix}\right.\right) = A_j\left(z\left|\begin{matrix}\omega_1 & \cdots & \omega_m\\ \rho_1+\delta_{i1} & \cdots & \rho_m+\delta_{im}\end{matrix}\right.\right)$$

where δ_{ij} is the Kronecker delta function.

We show that the system of polynomials $A_{ij}\left(z\left|\begin{matrix}\omega_1 & \cdots & \omega_m\\ \rho_1 & \cdots & \rho_m\end{matrix}\right.\right)$ $(1 \leqq i, j \leqq m)$ is, in a certain sense, independent, a fact that will be useful in establishing that the corresponding matrix (a_{ij}) in Lemma 2.1 is nonsingular. To be precise, from Mahler [Ma1], we have

$$\det_{1\leqq i,j\leqq m}\left(A_{ij}\left(z\left|\begin{matrix}\omega_1 & \cdots & \omega_m\\ \rho_1 & \cdots & \rho_m\end{matrix}\right.\right)\right) = \pm\prod_{\substack{h,k=1\\ h\neq k}}^{m}\frac{\Gamma(\omega_h-\omega_k)\Gamma(\rho_k)}{\Gamma(\rho_k+\omega_h-\omega_k)}z^{\sigma}$$

and so, if $z \neq 0$, we may conclude that

$$\det_{1\leqq i,j\leqq m}\left(A_{ij}\left(z\left|\begin{matrix}\omega_1 & \cdots & \omega_m\\ \rho_1 & \cdots & \rho_m\end{matrix}\right.\right)\right) \neq 0. \tag{3.1}$$

Next, let us examine the coefficients of the polynomials $A_{ij}\left(z\left|\begin{matrix}\omega_1 & \cdots & \omega_m\\ \rho_1 & \cdots & \rho_m\end{matrix}\right.\right)$ more closely. If we write

$$F\left(\zeta\left|\begin{matrix}\omega\\ \rho\end{matrix}\right.\right) = \prod_{h=0}^{\rho-1}(\zeta-\omega-h) = \frac{\Gamma(\zeta-\omega+1)}{\Gamma(\zeta-\omega-\rho+1)}$$

then it follows that

$$\phi(\zeta) = \prod_{k=1}^{m} F\left(\zeta \,\middle|\, \begin{matrix} \omega_k \\ \rho_k \end{matrix}\right)$$

and

$$\phi'(\omega_j + h) = F'\left(\omega_j + h \,\middle|\, \begin{matrix} \omega_j \\ \rho_j \end{matrix}\right) \prod_{\substack{k=1 \\ k \neq j}}^{m} F\left(\omega_j + h \,\middle|\, \begin{matrix} \omega_k \\ \rho_k \end{matrix}\right) \tag{3.2}$$

for $1 \leqq j \leqq m$ and $0 \leqq h \leqq \rho_j - 1$. Further, we may readily observe that

$$\frac{\Gamma(\rho_j)}{F'\left(\omega_j + h \,\middle|\, \begin{matrix} \omega_j \\ \rho_j \end{matrix}\right)} = (-1)^{\rho_j - h - 1} \binom{\rho_j - 1}{h} \tag{3.3}$$

and, if $j \neq k$,

$$\frac{\Gamma(\rho_k)}{F\left(\omega_j + h \,\middle|\, \begin{matrix} \omega_k \\ \rho_k \end{matrix}\right)} = \frac{\Gamma(\rho_k)\Gamma(\omega_j - \omega_k + h - \rho_k + 1)}{\Gamma(\omega_j - \omega_k + h + 1)}. \tag{3.4}$$

Let us now suppose that r is a positive integer. Suppressing dependence on the complex numbers $\omega_1, \ldots, \omega_m$, we write

$$R_i(z, r) = R_i\left(z \,\middle|\, \begin{matrix} \omega_1 & \cdots & \omega_m \\ r+1 & \cdots & r+1 \end{matrix}\right)$$

and

$$A_{ij}(z, r) = A_{ij}\left(z \,\middle|\, \begin{matrix} \omega_1 & \cdots & \omega_m \\ r+1 & \cdots & r+1 \end{matrix}\right).$$

It follows from (3.2), (3.3) and (3.4), then, that

$$A_{ij}(z, r) = (-1)^{mr+m+r+\delta_{ij}} \sum_{h=0}^{r+\delta_{ij}} a_{i,j,h,r}(1-z)^h \tag{3.5}$$

where

$$a_{i,j,h,r} = (-1)^h \binom{r+\delta_{ij}}{h} \prod_{\substack{k=1 \\ k \neq j}}^{m} b_{j,k,h,r+\delta_{ik}} \tag{3.6}$$

for

$$b_{j,k,h,r} = \frac{\Gamma(r+1)\Gamma(\omega_j - \omega_k + h - r)}{\Gamma(\omega_j - \omega_k + h + 1)}. \tag{3.7}$$

We note at this juncture that there is a slight discrepancy between these and the corresponding formulae in [Ch] (equations (6.1), (6.2) and the first displayed equation on page 359 of that paper).

Here and henceforth, we will suppose that n is a positive integer with $n > m$, $z < 0$ is real and that

$$\omega_k = \frac{k-1}{n} \quad \text{for } 1 \leqq k \leqq m. \tag{3.8}$$

To apply Lemma 2.1, we need to deduce upper bounds for $|R_i(z,r)|$, $|A_{ij}(z,r)|$ and for rational numbers $\Delta_{m,n,r}$ such that the polynomials

$$\Delta_{m,n,r} A_{ij}(z,r)$$

have integral, rather than rational, coefficients. For the first two of these, we derive asymptotically sharp estimates, following arguments of Mahler [Ma1] and Chudnovsky [Ch]. For the third, we again follow [Ch], but require a much more explicit version of the deliberations undertaken there. We treat this problem in Section 4.

Let us first obtain a bound for $|R_i(z,r)|$. Following Mahler [Ma1], we may write

$$R_i(z,r) = \int_0^z dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-2}} dt_{m-1} R_i(z|t_1,\ldots,t_{m-1})$$

where

$$R_i(z|t_1,\ldots,t_{m-1}) = \prod_{h=1}^{m} \left(\frac{t_{h-1}-t_h}{1-t_{h-1}}\right)^{r+\delta_{ih}} \prod_{h=2}^{m} (1-t_{h-1})^{1/n-1}.$$

Here, we have

$$z \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_{m-1} \leqq 0 \tag{3.9}$$

and we define $t_0 = z$ and $t_m = 0$. It follows that

$$|R_i(z,r)| \leqq \frac{|z|^{m-1}}{(m-1)!} \max\left\{\frac{t_i - t_{i-1}}{1-t_i}\right\} \max\left\{\prod_{h=1}^{m} \frac{t_h - t_{h-1}}{1-t_{h-1}}\right\}^r$$

where the maxima are taken over variables t_l $(0 \leqq l \leqq m)$ satisfying (3.9). Now

$$\max_{z \leqq t_{i-1} \leqq t_i \leqq 0} \left\{\frac{t_i - t_{i-1}}{1-t_i}\right\} = |z|$$

and so, arguing as in the proof of Corollary 2.3 of [Ch], we arrive at

Lemma 3.1. *Suppose that z is a negative real number and that r, i and m are positive integers. Then it follows that*

$$|R_i(z,r)| \leqq \frac{|z|^m}{(m-1)!} |1-(1-z)^{1/m}|^{mr}.$$

It appears to be rather less simple to obtain an asymptotically sharp bound for $|A_{ij}(z,r)|$. We begin by deriving upper bounds for the $|b_{j,k,h,r}|$ defined in (3.7). In fact, if $0 \leqq h \leqq r$, we claim that $|b_{j,k,h,r}|$ has roughly the same order of magnitude as the binomial coefficient $\binom{r}{h}$. To see this, consider

$$c_{j,k,h,r} = |b_{j,k,h,r}| \binom{r}{h}^{-1} = \left| \frac{\Gamma(\omega_j - \omega_k + h - r)\Gamma(h+1)\Gamma(r-h+1)}{\Gamma(\omega_j - \omega_k + h + 1)} \right|.$$

We wish to maximize $c_{j,k,h,r}$ for fixed r, $0 \leqq h \leqq r$, $1 \leqq j,k \leqq m$, $k \neq j$. Since

$$\Gamma(z)\Gamma(-z) = \frac{-\pi}{z\sin(\pi z)},$$

we have

$$\left| \frac{\Gamma(\omega_j - \omega_k + h - r)}{\Gamma(\omega_j - \omega_k + h + 1)} \right| = \left| \frac{\Gamma(\omega_k - \omega_j - h)}{\Gamma(\omega_k - \omega_j + r - h + 1)} \right|$$

and so $c_{j,k,h,r} = c_{k,j,r-h,r}$. It suffices, then, to assume that $\omega_j - \omega_k > 0$, say, via (3.8), $\omega_j - \omega_k = a/n$ with $1 \leqq a \leqq m-1$.

Now, if $0 \leqq h \leqq r-1$, then

$$\frac{c_{j,k,h,r}}{c_{j,k,h+1,r}} = \frac{r-h}{r-h-a/n} \frac{h+1+a/n}{h+1} > 1$$

and so

$$\max_{0 \leqq h \leqq r} c_{j,k,h,r} = c_{j,k,0,r} := c(a/n, r).$$

Since

$$c(a/n, r) = \left| \frac{\Gamma(a/n - r)\Gamma(r+1)}{\Gamma(a/n + 1)} \right|$$

and the function

$$f(x) = \left| \frac{\Gamma(x-r)}{\Gamma(x+1)} \right|$$

is concave up on the interval $(0,1)$, it follows that

$$\max_{1 \leqq a \leqq m-1} c(a/n, r) = \max\{c(1/n, r), c((m-1)/n, r)\}.$$

Further, $c(a/n, 1) = \dfrac{n^2}{a(n-a)}$ and

$$\frac{c(a/n, r+1)}{c(a/n, r)} = \frac{r+1}{r+1-a/n} < \left(1 + \frac{1}{r}\right)^{a/n},$$

where the last inequality follows via calculus (since $0 < a/n < 1$). We may therefore conclude, using induction, that

$$c(a/n, r) \leqq \frac{n^2}{a(n-a)} r^{a/n}$$

and so

$$c_{j,k,h,r} \leqq \Phi_{m,n,r} = \max\left\{\frac{n^2}{n-1} r^{1/n}, \frac{n^2}{(m-1)(n-m+1)} r^{(m-1)/n}\right\}$$

for $0 \leqq h \leqq r$. Thus,

$$|b_{j,k,h,r}| \leqq \Phi_{m,n,r} \binom{r}{h}$$

for $0 \leqq h \leqq r$. Now, assuming $i \neq j$, the factor for $k = i$ in the product (3.6) can be estimated by

$$|b_{j,k,h,r+1}| \leqq \Phi_{m,n,r+1} \binom{r+1}{h} < 2(r+1)\Phi_{m,n,r} \binom{r}{h}.$$

It follows that

$$|A_{ij}(z,r)| \leqq \sum_{h=0}^{r} |a_{i,j,h,r}|(1-z)^h \leqq 2(r+1)\Phi_{m,n,r}^{m-1} \sum_{h=0}^{r} \binom{r}{h}^m (1-z)^h$$

and so, if $i \neq j$,

$$|A_{ij}(z,r)| \leqq 2(r+1)\Phi_{m,n,r}^{m-1}\left(1 + (1-z)^{1/m}\right)^{mr}.$$

Next suppose that $i = j$. Then, for $0 \leqq h \leqq r$,

$$|a_{i,j,h,r}| = \binom{r+1}{h} \prod_{\substack{k=1 \\ k \neq j}}^{m} |b_{j,k,h,r}| \leqq (r+1)\Phi_{m,n,r}^{m-1} \binom{r}{h}^m$$

while

$$|a_{i,j,r+1,r}| = \left|\prod_{\substack{k=1 \\ k \neq j}}^{m} \frac{\Gamma(r+1)\Gamma(\omega_j - \omega_k + 1)}{\Gamma(\omega_j - \omega_k + r + 2)}\right|.$$

Since $\Gamma(x) < 1/x$ for $x \in (0,1)$ and $\Gamma(x)$ is increasing for $x \in (2, \infty)$, we have

$$|a_{i,j,r+1,r}| < \frac{1}{\left|\omega_j - \omega_k + 1\right|^{m-1}} \leqq \left(\frac{n}{n-m+1}\right)^{m-1}.$$

Thus

$$|a_{i,j,r+1,r}|(1-z)^{r+1} < \left(\frac{n}{n-m+1}\right)^{m-1}(1-z)\left(1+(1-z)^{1/m}\right)^{mr}$$

and so, since $r \geqq 1$ and $\left(\frac{n}{m-1}\right)^{m-1} \geqq 3$, we obtain

Lemma 3.2. *Suppose that $n > m \geqq 2$ are integers and that $z < 0$ is real. Then if*

$$\Phi_{m,n,r} = \max\left\{\frac{n^2}{n-1}r^{1/n}, \frac{n^2}{(m-1)(n-m+1)}r^{(m-1)/n}\right\},$$

we may conclude that

$$|A_{ij}(z,r)| \leqq \left(1 + \max\left\{1, \frac{1-z}{6}\right\}\right)(r+1)\Phi_{m,n,r}^{m-1}\left(1+(1-z)^{1/m}\right)^{mr}.$$

While for the principal application of this paper (i.e. Theorem 1.1), we will always take the parameter m to be even, it should be noted that, for odd values of m, Lemma 3.2 is not quite asymptotically sharp. For the sake of completeness, we mention the following effective (but nonexplicit) result of Chudnovsky (Theorem 2.1 of [Ch]):

Lemma 3.3. *Suppose that z is a negative real number and that r, i, j and m are positive integers with $1 \leqq i, j \leqq m$. Further, for $z_1, z_2 \in \mathbb{C}$, define*

$$|z_1 \ominus z_2| = \max\{|z_1 - \varepsilon z_2| : \varepsilon^m = 1\}.$$

Then we may conclude that

$$\lim_{r\to\infty} \frac{1}{r} \log|A_{ij}(z,r)| = m \log|1 \ominus (1-z)^{1/m}|.$$

Let $\Delta_{m,n,r}$ be the smallest positive rational number for which

$$\Delta_{m,n,r} A_{ij}(z,r) \in \mathbb{Z}[z]$$

for $1 \leqq i, j \leqq m$ and define (as in [Ch])

$$Chr_n^m = \limsup_{r\to\infty} \frac{1}{r} \log \Delta_{m,n,r}.$$

Then, taking $\theta = (b/a)^{1/n}$, $k = m-1$ and

$$P_{i-1,r}(x) = \sum_{j=1}^{m} a^{r+1} \Delta_{m,n,r} A_{ij}\left(\frac{a-b}{a}, r\right) x^{j-1}$$

for $1 \leqq i \leqq m$, we may apply Lemmata 2.1, 3.1 and 3.3 and inequality (3.1) (which implies that the corresponding matrix (a_{ij}) is nonsingular), to conclude

Theorem 3.4 (Chudnovsky; Proposition 6.2 of [Ch]). *Let $b > a$ be positive, relatively prime integers and suppose that n and m are integers with $n > m \geqq 2$ and that $\varepsilon > 0$. Then assuming*

$$(\sqrt[m]{b} - \sqrt[m]{a})^m e^{Chr_n^m} < 1,$$

one may conclude that

$$|(b/a)^{1/n} - p/q| > |q|^{-\lambda-\varepsilon}$$

for $|q| \geqq q_0(a,b,n,\varepsilon)$, where

$$\lambda = (m-1)\left\{1 - \frac{\log\big((\sqrt[m]{b} \ominus \sqrt[m]{a})^m e^{Chr_n^m}\big)}{\log\big((\sqrt[m]{b} - \sqrt[m]{a})^m e^{Chr_n^m}\big)}\right\}$$

and $q_0(a,b,n,\varepsilon)$ is effectively computable.

We note that this result may be readily extended to provide irrationality measures of this form for $(a/b)^{s/n}$ where s is a positive integer, relatively prime to n (as is done in [Ch]).

Careful choice of the parameter m (depending on a, b and n) allows us to derive strong effective irrationality measures in a quite general setting. For full flexiblity of application to algebraic numbers in radical extensions of the form $\alpha^{1/n}$, we need to have m grow with n. As a particular example, in the simplest case defined by (1.3), we have, for given values of m, the following irrationality measures λ_m for $\theta = (3/2)^{1/17}$:

m	λ_m	m	λ_m	m	λ_m
2–3	17	6	14.39	9	16.34
4	15.06	7	15.21	10	16.63
5	14.32	8	15.50	11–16	17

We note that, in this situation, Theorem 3.4 fails to improve upon Liouville's Theorem for $2 \leqq m \leqq 3$ and $11 \leqq m \leqq 16$.

The remainder of this paper is devoted to bounding the quantity Chr_n^m and studying the Diophantine consequences of such bounds. In [Ch], very little is said regarding this problem in the cases where m exceeds some fixed constant, particularly in case m grows as a function of n. In fact, as mentioned above, for applications it is almost always necessary to have $m \gg \log n$ or even $m \gg n$. While Chudnovsky (Proposition 6.6 of [Ch]) demonstrates that

$$Chr_n^m \leqq (m-1)\left(\frac{2n}{\phi(n)} \sum_{\substack{a=1 \\ (a,n)=1}}^{n} \frac{1}{a} - 1 - \log n - \sum_{p|n} \frac{\log p}{p-1}\right),$$

whereby

$$Chr_n^m \ll (m-1)\log n$$

for n prime, the "truth" for such n is more likely to be

$$Chr_n^m \ll (m-1)\log(en/m).$$

In a future paper [Be4], we will prove such an inequality under the further assumption that m/n exceeds a small absolute constant, which enables us to substantially sharpen the results of Evertse [Ev] on the equation $ax^n - by^n = c$. In the current paper, we merely provide evidence for this assertion by deriving an explicit version of Theorem 3.4 for prime $n \leqq 347$.

4. Arithmetic properties of the coefficients

In this section, we turn our attention to estimating the p-adic valuations of the (rational) coefficients of the polynomials $A_{ij}(z,r)$, in particular deriving an upper bound upon the quantity $\Delta_{m,n,r}$ defined in the previous section. From now on, we will assume for simplicity (and, for our applications, without loss of generality) that n is prime. Throughout, we will denote by $[x]$ the greatest integer not exceeding a real number x and set $\{x\} = x - [x]$ (so that $0 \leqq \{x\} < 1$). If a is an integer, we define $\mathrm{ord}_p(a)$ to be the highest power of a prime p which divides a and, if $r = a/b$ is rational, we take

$$\mathrm{ord}_p(a/b) = \mathrm{ord}_p(a) - \mathrm{ord}_p(b).$$

We will have use of the following lemma of Chudnovsky (Lemma 4.5 of [Ch]):

Lemma 4.1. *Suppose that u, v, s and n are integers with $u < v$ and $1 \leqq s < n$, and that p is a prime, not dividing n and satisfying $p^2 > \max\{|nu-s|, |nv-s|\}$. Choose $k \in \mathbb{N}$ such that $kn \equiv s \pmod p$ and $k \leqq p$. Then*

$$\mathrm{ord}_p\big((nu-s)\big(n(u+1)-s\big)\cdots(nv-s)\big) = \left[\frac{v-k}{p}\right] - \left[\frac{u-1-k}{p}\right].$$

Here and henceforth, for positive integers m, n and r, we will define

$$\Omega_{m,n,r} = \max\{\sqrt{nr+n+m}, 2n\}$$

and suppose that p is prime with $p > \Omega_{m,n,r}$. Let us define, for fixed $j \in \{1, 2, \ldots, m\}$,

$$S_j = \{j - l \colon 1 \leqq l \leqq m, l \neq j\}.$$

Choose integers $s_1, s_2, \ldots, s_{m-1} \in S_j$, ordered such that if integers u_l and t_l are chosen to satisfy

$$u_l = \frac{t_l p + s_l}{n} \tag{4.1}$$

with $1 \leqq t_l \leqq n-1$, then

$$1 \leqq t_1 < t_2 < \cdots < t_{m-1} \leqq n-1. \tag{4.2}$$

With s_l defined in this manner, for $1 \leqq l \leqq m-1$, we now let

$$g_{j,k,h,r} = r! \prod_{v=-h}^{r-h} (nv - j + k)^{-1}. \tag{4.3}$$

Applying the preceding lemma under the assumption that $0 \leqq h \leqq r+1$, we have

$$\mathrm{ord}_p\, g_{j,k,h,r} = -\left[\frac{r-h-u_{\psi(j,k)}}{p}\right] + \left[\frac{-h-1-u_{\psi(j,k)}}{p}\right] + \left[\frac{r}{p}\right],$$

where $1 \leqq \psi(j,k) \leqq m-1$ is chosen such that $s_{\psi(j,k)} = j-k$. If p divides $h+1+u_{\psi(j,k)}$, then

$$\mathrm{ord}_p\, g_{j,k,h,r} = \left\{\frac{r-h-u_{\psi(j,k)}}{p}\right\} - \left\{\frac{r}{p}\right\} - \frac{1}{p}$$

while, if p fails to divide $h+1+u_{\psi(j,k)}$,

$$\mathrm{ord}_p\, g_{j,k,h,r} = \left\{\frac{r-h-u_{\psi(j,k)}}{p}\right\} + \left\{\frac{h+u_{\psi(j,k)}}{p}\right\} - \left\{\frac{r}{p}\right\} - 1.$$

In both cases, we find that

$$\mathrm{ord}_p\, g_{j,k,h,r} = \begin{cases} -1 & \text{if } \left\{\frac{r}{p}\right\} \geqq \left\{\frac{h+u_{\psi(j,k)}}{p}\right\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Also,

$$\mathrm{ord}_p \binom{r}{h} = \begin{cases} 0 & \text{if } \left\{\frac{r}{p}\right\} \geqq \left\{\frac{h}{p}\right\}, \\ 1 & \text{otherwise.} \end{cases} \tag{4.5}$$

Let us define $s_0 = t_0 = u_0 = \psi(j,j) = 0$, for $1 \leqq j \leqq m$, set $s_{l+m} = s_l$ and $t_{l+m} = t_l + n$ for $0 \leqq l \leqq m-1$, and write

$$u_{l+m} = \frac{(t_l+n)p + s_l}{n} = u_l + p.$$

Further, define intervals

$$I_k = \left[1 - \frac{u_k}{p}, 1 - \frac{u_{k-1}}{p}\right)$$

for $1 \leqq k \leqq m$. We claim that these intervals are disjoint and cover $[0,1)$. To see this, note that

$$u_{k+1} - u_k = \frac{(t_{k+1} - t_k)p + s_{k+1} - s_k}{n}$$

and so (4.2) and $|s_{k+1} - s_k| \leqq m - 1$ imply that

$$u_{k+1} - u_k > \frac{p-m}{n} > \frac{2n-m}{n} > 1, \tag{4.6}$$

whence

$$0 = u_0 < u_1 < \cdots < u_{m-1} < u_m = p. \tag{4.7}$$

The main result of this section is

Proposition 4.2. *Let α, n, m and r be positive integers with $n > m > \alpha \geqq 1$ and p be a prime with $p > \Omega_{m,n,r}$. If, for some h, i and j with $0 \leqq h \leqq r$ and $1 \leqq i, j \leqq m$, we have $\operatorname{ord}_p a_{i,j,h,r} = -\alpha$, where $a_{i,j,h,r}$ is as defined in (3.6), then*

$$\left\{\frac{r}{p}\right\} \geqq \min_{1 \leqq k \leqq m} \left\{\frac{u_{k+\alpha} - u_k - 1}{p}\right\}.$$

Conversely, if

$$\left\{\frac{r}{p}\right\} \geqq \min_{1 \leqq k \leqq m} \left\{\frac{u_{k+\alpha} - u_k}{p}\right\},$$

then there exist h, i and j with $0 \leqq h \leqq r$ and $1 \leqq i, j \leqq m$ for which $\operatorname{ord}_p a_{i,j,h,r} \leqq -\alpha$.

Proof. Suppose $p > \Omega_{m,n,r}$ and $\operatorname{ord}_p a_{i,j,h,r} = -\alpha$ for some $1 \leqq \alpha \leqq m-1$. For fixed j, we have

$$\{s_l \colon 1 \leqq l \leqq m-1\} = \{j - l \colon 1 \leqq l \leqq m, l \neq j\}$$

where (4.1) and (4.2) are satisfied. Upon noting that

$$\Gamma(z+1) = z\Gamma(z)$$

for $z \neq 0, -1, -2, \ldots$, it follows from (3.6) and (3.7) that

$$a_{i,j,h,r} = \pm \binom{r + \delta_{ij}}{h} n^{r(m-1)+m-\delta_{ij}} \prod_{\substack{k=1 \\ k \neq j}}^{m} g_{j,k,h,r+\delta_{ik}} \tag{4.8}$$

where $g_{j,k,h,r+\delta_{ik}}$ is as in (4.3). Since $p > \Omega_{m,n,r} \geqq 2n$, p fails to divide n and so (4.4) and (4.5) imply that

$$\operatorname{ord}_p a_{i,j,h,r} = 1 - N_{i,j,h,p} \tag{4.9}$$

where $N_{i,j,h,p}$ denotes the number of elements in the set

$$\left\{k \in \mathbb{N}\colon 1 \leqq k \leqq m, \left\{\frac{r+\delta_{ik}}{p}\right\} \geqq \left\{\frac{h+u_{\psi(j,k)}}{p}\right\}\right\}.$$

Suppose now that h is an integer with $0 \leqq h \leqq r+\delta_{ij}$ and $\left\{\frac{h}{p}\right\} \in I_\kappa$, where $1 \leqq \kappa \leqq m$. Then we claim that

$$(4.10) \qquad \left\{\frac{h+u_\kappa}{p}\right\} < \left\{\frac{h+u_{\kappa+1}}{p}\right\} < \cdots < \left\{\frac{h+u_{m-1}}{p}\right\}$$
$$< \left\{\frac{h+u_0}{p}\right\} < \left\{\frac{h+u_1}{p}\right\} < \cdots < \left\{\frac{h+u_{\kappa-1}}{p}\right\}$$

where u_i is as defined in (4.1). To see this, note that, since $\left\{\frac{h}{p}\right\} < 1 - \frac{u_{\kappa-1}}{p}$, we have, from (4.7), that

$$\left\{\frac{h}{p}\right\} + \left\{\frac{u_l}{p}\right\} < 1$$

for $0 \leqq l \leqq \kappa - 1$. Therefore

$$\left\{\frac{h+u_0}{p}\right\} < \left\{\frac{h+u_1}{p}\right\} < \cdots < \left\{\frac{h+u_{\kappa-1}}{p}\right\}$$

which implies (4.10) for $\kappa = m$. On the other hand, if $\kappa < m$, since $\left\{\frac{h}{p}\right\} \geqq 1 - \frac{u_\kappa}{p}$, (4.7) yields

$$\left\{\frac{h}{p}\right\} + \left\{\frac{u_l}{p}\right\} \geqq 1 - \frac{u_\kappa}{p} + \frac{u_l}{p} \geqq 1$$

for $\kappa \leqq l < m$. It follows, for these values of l, that

$$\left\{\frac{h+u_l}{p}\right\} = \left\{\frac{h}{p}\right\} + \left\{\frac{u_l}{p}\right\} - 1$$

whence

$$\left\{\frac{h+u_\kappa}{p}\right\} < \left\{\frac{h+u_{\kappa+1}}{p}\right\} < \cdots < \left\{\frac{h+u_{m-1}}{p}\right\}.$$

Since

$$\left\{\frac{h+u_{m-1}}{p}\right\} = \left\{\frac{h}{p}\right\} + \frac{u_{m-1}}{p} - 1 < \left\{\frac{h}{p}\right\},$$

we conclude as stated.

Combining (4.9) and (4.10), we therefore have that $\mathrm{ord}_p\, a_{i,j,h,r} = -\alpha$ precisely when

$$\left\{\frac{h + u_{\kappa+\alpha}}{p}\right\} \leqq \left\{\frac{r + \Xi_{i,j,h,\alpha}}{p}\right\} < \max\left(1, \left\{\frac{h + u_{\kappa+\alpha+1}}{p}\right\}\right),$$

where

$$\Xi_{i,j,h,\alpha} = \begin{cases} 1 & \text{if } \psi(j,i) \equiv \kappa + \alpha \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\left\{\frac{h}{p}\right\} \in I_\kappa$ implies that

$$\left\{\frac{h}{p}\right\} \geqq 1 - \frac{u_\kappa}{p},$$

we therefore have, if $\mathrm{ord}_p\, a_{i,j,h,r} = -\alpha$, that

$$\left\{\frac{r + \Xi_{i,j,h,\alpha}}{p}\right\} \geqq \left\{\frac{h + u_{\kappa+\alpha}}{p}\right\} = \left\{\frac{h}{p}\right\} + \frac{u_{\kappa+\alpha}}{p} - 1 \geqq \frac{u_{\kappa+\alpha} - u_\kappa}{p}.$$

To prove the converse, suppose that k_0 is such that

$$\frac{u_{k_0+\alpha} - u_{k_0}}{p} = \min_{1 \leqq k \leqq m} \left\{\frac{u_{k+\alpha} - u_k}{p}\right\}$$

and that

$$\left\{\frac{r}{p}\right\} \geqq \frac{u_{k_0+\alpha} - u_{k_0}}{p}.$$

Let us choose $0 \leqq h_0 \leqq r$ such that $h_0 \equiv -u_{k_0} \pmod{p}$. We claim that $\mathrm{ord}_p\, a_{i,j,h_0,r} \leqq -\alpha$. Indeed,

$$\left\{\frac{h_0}{p}\right\} = 1 - \frac{u_{k_0}}{p}$$

and so

$$\left\{\frac{r}{p}\right\} \geqq \left\{\frac{h_0}{p}\right\} + \frac{u_{k_0+\alpha}}{p} - 1 = \left\{\frac{h_0 + u_{k_0+\alpha}}{p}\right\}.$$

Now, from (4.10), since $\{h_0/p\} \in I_{k_0}$, we have

$$\left\{\frac{h_0 + u_{k_0}}{p}\right\} < \cdots < \left\{\frac{h_0 + u_{k_0-1}}{p}\right\}$$

and thus, applying (4.9) and (4.10), we conclude as desired. $\square$

Noting that the t_k's remain fixed when p runs through a given residue class modulo n, we may define

$$d_{a,\alpha} = \min_{1 \leqq k \leqq m} (t_{\kappa+\alpha} - t_k)$$

for $p \equiv a \pmod{n}$, $1 \leqq a \leqq n-1$. From the proof of the preceding proposition, if $\operatorname{ord}_p a_{i,j,h,r} = -\alpha$, we have

$$\left\{\frac{r + \Xi_{i,j,h,\alpha}}{p}\right\} \geqq \frac{u_{\kappa+\alpha} - u_\kappa}{p} = \frac{t_{\kappa+\alpha} - t_\kappa}{n} + \frac{s_{\kappa+\alpha} - s_\kappa}{pn}$$

and since $|s_{\kappa+\alpha} - s_\kappa| < m$ and $m < n$, it follows that

$$\left\{\frac{r + \Xi_{i,j,h,\alpha}}{p}\right\} > \frac{d_{a,\alpha}}{n} - \frac{1}{p}.$$

We therefore obtain

Corollary 4.3. *Let α, n, m and r be positive integers with $n > m > \alpha \geqq 1$. Further, let p be a prime satisfying $p > \Omega_{m,n,r}$ and $p \equiv a \pmod{n}$ with $1 \leqq a \leqq n-1$. If $\operatorname{ord}_p a_{i,j,h,r} = -\alpha$, for some $0 \leqq h \leqq r$, where $a_{i,j,h,r}$ is as defined in* (3.6), *then it follows that either p divides $(r+1)(r+2)$ or*

$$\left\{\frac{r+2}{p}\right\} > \frac{d_{a,\alpha}}{n}.$$

Some simplifying observations are in order. Firstly, we note, for fixed α, m, n, r and prime $p \equiv a \pmod{n}$, that the integer $d_{a,\alpha}$ is in fact independent of the parameter j. To see this, first extend the definition of the sets S_j defined earlier to

$$\tilde{S}_j = \{j - l \colon 1 \leqq l \leqq m\} = S_j \cup \{0\}$$

and define

$$T_{j,p} = \{t \in \mathbb{Z} \colon tp \equiv -s \pmod{n} \text{ for some } s \in \tilde{S}_j\}.$$

Then we may readily show that $T_{j+1,p}$ is obtained from $T_{j,p}$ by subtracting $\bar{p}$ from each of its elements, where $p\bar{p} \equiv 1 \pmod{n}$ and $1 \leqq \bar{p} \leqq n-1$. Since the set of differences $\{t_{\kappa+\alpha} - t_k\}$ for $1 \leqq k \leqq m$, corresponding to a fixed j is just the analogous set of differences obtained from $T_{j,p}$ upon ordering this set, we conclude as desired.

We may also note that if $a_1 \equiv -a_2 \pmod{n}$, then, for fixed m and n, we have $d_{a_1,\alpha} = d_{a_2,\alpha}$. This follows immediately upon noticing that $T_{j,p_1} = -T_{j,p_2}$, for primes $p_i \equiv a_i \pmod{n}$. Together, these observations enable us to compute $d_{a,\alpha}$ under the assumption that $1 \leqq a \leqq \frac{n-1}{2}$ and with t_k chosen such that $t_k \equiv a^{-1}k \pmod{n}$ for $1 \leqq t_k \leqq n-1$ and $1 \leqq k \leqq m-1$ (with, again, $t_0 = 0$ and $t_{l+m} = t_l + n$ for $0 \leqq l \leqq m-1$). By way of example, for $n = 17$ and $m = 6$, the relevant values of $d_{a,\alpha}$ are as given in the following table:

	$d_{1,\alpha}$	$d_{2,\alpha}$	$d_{3,\alpha}$	$d_{4,\alpha}$	$d_{5,\alpha}$	$d_{6,\alpha}$	$d_{7,\alpha}$	$d_{8,\alpha}$
$\alpha = 1$	1	1	1	1	1	2	2	2
$\alpha = 2$	2	2	5	4	4	5	5	4
$\alpha = 3$	3	8	6	5	7	8	7	6
$\alpha = 4$	4	9	11	9	10	11	10	8
$\alpha = 5$	5	10	12	13	13	14	12	10

while for $n = 23$ and $m = 8$, we have

	$d_{1,\alpha}$	$d_{2,\alpha}$	$d_{3,\alpha}$	$d_{4,\alpha}$	$d_{5,\alpha}$	$d_{6,\alpha}$	$d_{7,\alpha}$	$d_{8,\alpha}$	$d_{9,\alpha}$	$d_{10,\alpha}$	$d_{11,\alpha}$
$\alpha = 1$	1	1	1	1	1	1	1	2	2	2	2
$\alpha = 2$	2	2	2	5	5	4	4	5	5	4	4
$\alpha = 3$	3	3	8	6	6	5	7	8	7	7	6
$\alpha = 4$	4	11	9	11	10	8	10	11	10	9	8
$\alpha = 5$	5	12	10	12	14	11	13	14	12	11	10
$\alpha = 6$	6	13	16	17	15	15	16	17	15	16	12
$\alpha = 7$	7	14	17	18	19	19	19	20	18	18	14

Let us next observe that $\{r/p\} > a/b$ is equivalent, for $0 < a/b < 1$, to

$$\frac{r}{N+1} < p < \frac{r}{N + a/b}$$

for the nonnegative integer $N = [r/p]$. It follows, then, that the product of primes p satisfying $p > \Omega_{m,n,r}$ which divide the denominator of a given $a_{i,j,h,r}$ and fail to divide $(r+1)(r+2)$, up to requisite multiplicities, divides

$$\exp\left(\sum_{a=1}^{n-1}\sum_{\alpha=1}^{m-1}\sum_{N=0}^{N_0}\left(\theta\left(\frac{r+2}{N + d_{a,\alpha}/n}, n, a\right) - \theta\left(\frac{r+2}{N+1}, n, a\right)\right)\right). \tag{4.11}$$

Here

$$\theta(x, n, a) = \sum_{\substack{p \leqq x \\ p \equiv a \pmod{n}}} \log p$$

and N_0 denotes the smallest positive integer such that $\dfrac{r+2}{N_0+1} < \sqrt{nr+n+m}$.

Let us now begin the process of explicitly bounding $\Delta_{m,n,r}$. From (4.8), it is clear that

$$\Delta_{m,n,r} \leqq (n^{r+1} s_{n,r})^{1-m} \Delta_0 \Delta_1 \tag{4.12}$$

where $s_{n,r}$ denotes the n-part of $r!$ (i.e. for n prime, $s_{n,r} = n^{\mathrm{ord}_n r!}$) and, suppressing dependence upon n, m and r, $\Delta_0\Delta_1$ is the least common multiple of the denominators of the coefficients $a_{i,j,h,r}$ given by (4.8) for $0 \leqq h \leqq r + \delta_{ij}$ and $1 \leqq i, j \leqq m$, where Δ_0 consists of those primes p with $p \leqq \Omega_{m,n,r}$ while Δ_1 is comprised of those with $p > \Omega_{m,n,r}$. Here, we assume that the rational coefficients $a_{i,j,h,r}$ are in reduced form.

Let us first deduce a lower bound upon $s_{n,r}$. Note that

$$s_{n,r} = n^{\mathrm{ord}_n r!} = n^{\left[\frac{r}{n}\right] + \left[\frac{r}{n^2}\right] + \cdots}$$

and so, if $\rho = \left[\dfrac{\log r}{\log n}\right]$, we have

$$\frac{\log s_{n,r}}{\log n} = \sum_{t=1}^{\rho} \left(\frac{r}{n^t} - \left\{ \frac{r}{n^t} \right\} \right) \geqq \sum_{t=1}^{\rho} \frac{r+1}{n^t} - \rho = \frac{(r+1)(1 - n^{-\rho})}{n-1} - \rho.$$

Writing $\rho = \dfrac{\log r}{\log n} - \gamma$, where $0 \leqq \gamma < 1$, and noting that

$$\frac{2 - \dfrac{r+1}{r} n^{\gamma} + \gamma(n-1)}{n-1} > \frac{1 - \dfrac{n}{r}}{n-1},$$

this implies that

$$s_{n,r} \geqq \begin{cases} r^{-1} n^{\frac{r-1}{n-1}} & \text{if } n \leqq r, \\ 1 & \text{if } n > r. \end{cases} \tag{4.13}$$

Let us next deal with Δ_0. Suppose that p divides the denominator of some $g_{j,k,h,r}$ (or, possibly, $g_{j,k,h,r+1}$). It follows that we have $(p, n) = 1$. Since there are at most $[r/p^t] + 1$ terms of the form $nl - s_k$ (for $-h \leqq l \leqq r - h$) with $\mathrm{ord}_p(nl - s_k) \geqq t$ (for $t = 1, 2, \ldots$), we conclude that the order to which p divides the denominator of $g_{j,k,h,r+\delta_{ik}}$ is bounded above by

$$\left[\frac{\log(nr+n+m)}{\log p}\right] + \left[\frac{r+\delta_{ik}}{p}\right] + \left[\frac{r+\delta_{ik}}{p^2}\right] + \cdots.$$

On the other hand,

$$\mathrm{ord}_p\big((r+\delta_{ik})!\big) = \left[\frac{r+\delta_{ik}}{p}\right] + \left[\frac{r+\delta_{ik}}{p^2}\right] + \cdots$$

and so it follows that

$$\log \Delta_0 \leqq (m-1) \sum_{p \leqq \Omega_{m,n,r}} \log(nr+n+m). \tag{4.14}$$

Now from Corollary 1 of Rosser and Schoenfeld [RS], we have

$$\pi(x) < \frac{1.25506x}{\log x} \quad \text{for } x > 1$$

where $\pi(x)$ denotes the number of primes p with $p \leqq x$, and so

$$\sum_{p \leqq \Omega_{m,n,r}} \log(nr + n + m) < 2.52\Omega_{m,n,r}. \tag{4.15}$$

For larger primes (i.e. with $p > \Omega_{m,n,r}$), we apply Corollary 4.3. Since, for such primes, we have

$$\mathrm{ord}_p\, \Delta_1 \leqq m - 1,$$

it follows, from (4.11), (4.12), (4.14) and (4.15), that

$$\Delta_{m,n,r} \leqq \left(\frac{(r+2)(r+1)e^{2.52\Omega_{m,n,r}}}{n^{r+1}s_{n,r}}\right)^{m-1} \Delta_2 \tag{4.16}$$

where $\Delta_2 = \Delta_2(m,n,r)$ satisfies

$$\log \Delta_2 = \sum_{a=1}^{n-1} \sum_{\alpha=1}^{m-1} \sum_{N=0}^{N_0} \left(\theta\left(\frac{r+2}{N + d_{a,\alpha}/n}, n, a\right) - \theta\left(\frac{r+2}{N+1}, n, a\right)\right). \tag{4.17}$$

To derive an upper bound upon Δ_2, we note that

$$\sum_{N=0}^{N_0} \left(\theta\left(\frac{r+2}{N + d_{a,\alpha}/n}, n, a\right) - \theta\left(\frac{r+2}{N+1}, n, a\right)\right)$$

is bounded above by

$$\sum_{N=0}^{N_1} \theta\left(\frac{r+2}{N + d_{a,\alpha}/n}, n, a\right) - \sum_{N=0}^{N_1 - 1} \theta\left(\frac{r+2}{N+1}, n, a\right)$$

for each positive integer N_1 (see e.g. [Ea]). If $\delta_n(x)$ is a positive real-valued function for which

$$\max_{1 \leqq a \leqq n-1} \frac{n-1}{x} \left|\theta(x,n,a) - \frac{x}{n-1}\right| < \delta_n(x), \tag{4.18}$$

then we have

$$\log \Delta_2 \leqq \frac{r+2}{n-1} \sum_{a=1}^{n-1} \sum_{\alpha=1}^{m-1} \left(\sum_{N=0}^{N_1} \frac{1 + \delta_n\left(\frac{r+2}{N + d_{a,\alpha}/n}\right)}{N + d_{a,\alpha}/n} - \sum_{N=0}^{N_1 - 1} \frac{1 - \delta_n\left(\frac{r+2}{N+1}\right)}{N+1} \right) \tag{4.19}$$

It remains to derive suitable $\delta_n(x)$ to satisfy (4.18).

5. Chebyshev-like estimates for primes in arithmetic progressions

The deliberations of Section 4 motivate the need to establish reasonably precise upper and lower bounds for the function $\theta(x, n, a)$. From the theory of the distribution of primes in arithmetic progressions, we have

$$\theta(x, n, a) \sim \frac{x}{\phi(n)}$$

where $\phi(n)$ denotes Euler's totient function. For our purposes, however, we require more than asymptotics. Following the arguments of Ramaré and Rumely [RR], we combine a result on zero-free regions for certain Dirichlet L-functions (Theorem 3.6.2 of [RR]) with explicit computations designed to verify that the zeros of these functions in the critical strip, with imaginary part bounded by 1000, lie on the half-line (that is, satisfy the Generalized Riemann Hypothesis to height 1000, or, more succinctly, GRH(1000)).

To carry out these calculations, we employed a number of Turbo Pascal routines written by Professor Rumely, which he was kind enough to provide to us. Very roughly, these work as follows. Program "L", after computing character values for the L-functions with a given conductor and root numbers for the associated functional equations, computes the zeros for these L-functions. This is done by producing Taylor series expansions of the related partial zeta functions, via Euler-MacLaurin summation. Brent's linear/quadratic search algorithm is then used to find zeros on the half-line for each L-functions while Brent's max/min algorithm for finding intervening maxima/minima is used in conjunction with Laguerre's method to identify (possible) roots off the half-line. Programs "V" and "ZCHECK" are then used to validate the list of zeros produced. For a more detailed exposition of these algorithms, the reader is directed to the paper of Rumely [Ru]. Our computations were carried out over the course of a number of weeks on a variety of Sun Sparc 4 and Sparc 20 machines, running algorithms "L", "V" and "ZCHECK" for all L-functions associated with Dirichlet characters with prime conductor n between 73 and 347. Explicit data on the zeros encountered, their pair correlation, etc. is available from the author on request. For the purposes of this paper, however, we need only note the following

Theorem 5.1 (Rumely and Sun). *Every L-function associated to a Dirichlet character with prime conductor n for $73 \leqq n \leqq 347$ satisfies* GRH(1000). *That is, the nontrivial zeros of such an L-function with imaginary part bounded in modulus by* 1000 *have real part* $1/2$.

We apply this result to give us Chebyshev-type estimates for $\theta(x, n, a)$ with relatively large values of x. Let us define

$$\varepsilon_n = \max_{x \geqq 10^{11}} \max_{1 \leqq a \leqq n-1} \max_{1 \leqq y \leqq x} \frac{n-1}{x} \left| \theta(y, n, a) - \frac{y}{n-1} \right|. \tag{5.1}$$

To deduce upper bounds upon ε_n for $3 \leqq n \leqq 347$, we apply Theorems 4.3.1 and 4.3.2 of [RR] (to be precise, we modify Theorem 4.3.2 by replacing

$$\left(1 + (1 + \delta)^{m+1}\right)^m \delta^{-m}$$

with the term $A(m, \delta)$ from Theorem 4.3.1; the argument leading to Theorem 4.3.2 indicates that this is a valid substitution). In these theorems, we take $H = 10000$ (for $5 \leqq n \leqq 13$), $H = 2500$ (for $19 \leqq n \leqq 71$) and $H = 1000$ (for $71 \leqq n \leqq 347$) (where these choices are permissible by Theorem 5.1 above and Theorem 2.1.1 of [RR]). Further, in the notation of [RR], we take $x_0 = 10^{11}$, $m = 9$ (for $n = 5, 7$, $19 \leqq n \leqq 37$ or $73 \leqq n \leqq 101$), $m = 10$ (for $n = 11, 13$, $41 \leqq n \leqq 71$ or $103 \leqq n \leqq 277$), $m = 11$ (for $281 \leqq n \leqq 347$) and $0.0048 < \delta < 0.0054$ (for $73 \leqq n \leqq 347$), $0.0018 < \delta < 0.0021$ (for $19 \leqq n \leqq 71$) and $0.00046 < \delta < 0.00049$ (for $5 \leqq n \leqq 13$), chosen to optimize our bound. If $n = 3$ or $n = 17$, we simply apply Theorem 1 of [RR] with $x_0 = 10^{10}$. We conclude

Theorem 5.2. *If $3 \leqq n \leqq 347$ is prime and ε_n is as defined in* (5.1), *then $\varepsilon_n < \tilde{\varepsilon}_n$ where $\tilde{\varepsilon}_n$ is given in the following table*:

n	$\tilde{\varepsilon}_n$	n	$\tilde{\varepsilon}_n$	n	$\tilde{\varepsilon}_n$	n	$\tilde{\varepsilon}_n$
3	0.002238	67	0.018873	151	0.046304	241	0.060570
5	0.002686	71	0.019435	157	0.047254	251	0.062161
7	0.003007	73	0.033698	163	0.048204	257	0.063116
11	0.003606	79	0.034706	167	0.048837	263	0.064073
13	0.003893	83	0.035374	173	0.049787	269	0.065030
17	0.010746	89	0.036369	179	0.050737	271	0.065349
19	0.011296	97	0.037686	181	0.051054	277	0.066307
23	0.011980	101	0.038341	191	0.052637	281	0.066943
29	0.012968	103	0.038664	193	0.052954	283	0.067261
31	0.013290	107	0.039305	197	0.053587	293	0.068851
37	0.014244	109	0.039625	199	0.053904	307	0.071081
41	0.014869	113	0.040265	211	0.055806	311	0.071719
43	0.015176	127	0.042496	223	0.057710	313	0.072038
47	0.015788	131	0.043132	227	0.058345	317	0.072677
53	0.016702	137	0.044084	229	0.058663	331	0.074915
59	0.017613	139	0.044402	233	0.059298	337	0.075876
61	0.017917	149	0.045987	239	0.060252	347	0.077478

We now turn our attention to the problem of bounding $\theta(x, n, a)$ for smaller values of x. Let us define

$$\theta_n = \max_{1 \leqq a \leqq n-1} \max_{0 < x \leqq 10^{11}} \frac{1}{\sqrt{x}} \left| \theta(x, n, a) - \frac{x}{n-1} \right|.$$

We obtain the values for θ_n with $2 \leqq n \leqq 397$ through sieving. As noted in [RR], the function $\left(\theta(x, n, a) - x/(n-1)\right)/\sqrt{x}$ is monotone decreasing between jumps at primes. From this observation, it is a straightforward matter to compute θ_n, as described in [RR]. To accomplish this, we used code generously provided by Enrico Bombieri, written in C and implemented on a Sparc Ultra. This code is available from the author on request. The total amount of computation required to produce the values for θ_n with n prime between 2 and 400 was approximately 400 hours. In the table that follows, we list values for n and θ_n (the latter rounded up in the sixth decimal place), together with the progression for which this maximum is attained and the corresponding value $\tilde{x}$. This last integer satisfies $\tilde{x} = p_j$ (i.e. the jth prime) where either

$$\theta_n = \frac{1}{\sqrt{p_{j-1}}} \left| \theta(p_{j-1}, n, a) - \frac{p_{j-1}}{n-1} \right|$$

or

$$\theta_n = \lim_{x \to p_j^-} \frac{1}{\sqrt{x}} \left| \theta(x, n, a) - \frac{x}{n-1} \right|$$

as appropriate.

n	θ_n	a	$\tilde{x}$	n	θ_n	a	$\tilde{x}$
2	2.071993	1	1423	173	0.720103	7	353
3	1.798158	1	69991	179	0.720622	7	1439
5	1.412480	4	349	181	0.720787	7	1093
7	1.116838	4	24470870029	191	0.721560	7	389
11	0.976421	5	726270803	193	0.721705	7	4253
13	1.017317	10	65095932067	197	0.721987	7	401
17	1.001057	8	6395663	199	0.722123	7	1201
19	1.001556	6	461687	211	0.722887	7	2539
23	0.973114	12	793489	223	0.723568	7	2237
29	0.793283	9	2039	227	0.723779	7	461
31	0.853475	8	16773763751	229	0.723881	7	1381
37	0.867916	26	4058619751	233	0.724081	7	1871
41	0.818620	29	30239497	239	0.724369	7	4787
43	0.832936	25	6547405001	241	0.724461	7	971
47	0.744386	34	2000700217	251	0.724902	7	509
53	0.829958	36	5813	257	0.725150	7	521
59	0.710444	58	25841	263	0.725387	7	2111
61	0.719386	1	9212953	269	0.725613	7	1621
67	0.728237	24	48679198759	271	0.725686	7	1091
71	0.750488	11	41333	277	0.725899	7	1669
73	0.759154	72	4817	281	0.726036	7	569
79	0.730952	26	38932253	283	0.726103	7	2837
83	0.703220	7	173	293	0.726425	7	593
89	0.705420	7	541	307	0.726839	7	3691
97	0.713661	79	2206247	311	0.726951	7	1873
101	0.709028	7	613	313	0.727005	7	1259
103	0.709547	7	419	317	0.727113	7	641
107	0.710525	7	863	331	0.727468	7	1993
109	0.710988	7	443	337	0.727611	7	2029
113	0.711863	7	233	347	0.727839	7	701
127	0.714487	7	769	349	0.727883	7	5591
131	0.715133	7	269	353	0.727969	7	4243
137	0.716031	7	281	359	0.728095	7	2161
139	0.716619	121	90124089259	367	0.728257	7	3677
149	0.717609	7	2689	373	0.728373	7	1499
151	0.717847	7	2423	379	0.728486	7	1523
157	0.718525	7	2833	383	0.728559	7	773
163	0.719154	7	659	389	0.728666	7	2341
167	0.719547	7	1009	397	0.728804	7	2389

6. Endgame computations

We are now in position to obtain explicit upper bounds upon $\Delta_{m,n,r}$, where we restrict attention to primes $17 \leqq n \leqq 347$ and take $m = \left[\frac{n+1}{3}\right]$, so that, in all cases, m is even. We have

Proposition 6.1. *If $\Delta_{m,n,r}$ is as defined previously, $17 \leqq n \leqq 347$ is prime and $m = \left[\frac{n+1}{3}\right]$, then*

$$\log \Delta_{m,n,r} < c_1(n)r + d(n)$$

for all $r \geqq 1$, while

$$\log \Delta_{m,n,r} < c_2(n)r$$

for all $r \geqq r_0(n)$, where the last constant is effectively computable. The constants $c_1(n), c_2(n)$ and $d(n)$ are given in the following table:

n	$c_1(n)$	$c_2(n)$	$d(n)$	n	$c_1(n)$	$c_2(n)$	$d(n)$
17	8.93	8.21	13.06	167	139.95	104.28	82.87
19	9.40	8.64	15.46	173	146.07	108.37	87.71
23	13.03	11.89	17.66	179	151.40	111.65	83.92
29	17.39	15.78	29.95	181	152.20	112.19	91.69
31	17.92	16.25	30.55	191	163.78	119.73	84.40
37	21.92	19.73	32.51	193	164.81	120.46	91.51
41	25.83	23.13	36.08	197	170.17	123.89	104.53
43	26.62	23.85	33.95	199	170.80	124.24	110.41
47	30.46	27.13	40.16	211	183.12	132.00	124.02
53	34.78	30.78	35.37	223	195.74	139.85	112.93
59	39.18	34.46	48.34	227	201.15	143.15	116.91
61	39.96	35.14	55.93	229	202.11	143.77	100.61
67	44.76	39.16	43.56	233	207.50	147.00	102.49
71	48.36	42.04	54.80	239	213.74	150.70	105.66
73	52.83	42.68	48.11	241	214.95	151.54	95.14
79	58.27	46.87	54.65	251	226.83	158.55	115.64
83	62.70	50.15	49.64	257	233.75	162.75	113.23
89	67.56	53.69	60.29	263	240.15	166.42	119.49
97	73.71	58.30	62.14	269	246.54	170.03	124.75
101	78.29	61.59	50.36	271	247.72	170.79	134.21
103	79.16	62.30	60.85	277	254.62	174.83	119.17
107	83.55	65.38	50.84	281	260.46	178.17	116.79
109	84.18	65.84	58.97	283	261.67	178.95	118.21
113	89.22	69.51	77.93	293	274.23	186.00	129.73
127	100.47	77.36	72.61	307	289.00	194.23	124.89
131	105.34	80.75	71.51	311	294.70	197.29	130.14

n	$c_1(n)$	$c_2(n)$	$d(n)$	n	$c_1(n)$	$c_2(n)$	$d(n)$
137	111.44	85.11	79.94	313	296.38	198.48	130.18
139	112.15	85.61	77.27	317	302.73	202.11	134.63
149	122.53	92.62	85.82	331	317.41	209.93	147.69
151	123.41	93.27	89.04	337	324.63	213.85	139.95
157	129.07	97.06	81.61	347	338.02	221.00	133.98
163	134.80	100.86	93.64				

Proof. Note that the values for $c_2(n)$ follow from direct application of the asymptotics for $\theta(x,n,a)$ to equations (4.16) and (4.17). To prove the stated results for $c_1(n)$, we consider various ranges for the parameter r. Firstly, for large r, we apply inequalities (4.16) and (4.19), together with the Chebyshev-type estimates for primes in arithmetic progression derived in Section 5. In fact, if we suppose that $r \geqq 3 \times 10^6$, then defining

$$\delta_n(x) = \begin{cases} \tilde{\varepsilon}_n & \text{if } x \geqq 10^{11}, \\ \dfrac{\theta_n(n-1)}{\sqrt{x}} & \text{if } x < 10^{11}, \end{cases}$$

where $\tilde{\varepsilon}_n$ and θ_n are as in Section 5, we readily obtain the desired bounds from (4.16) and (4.19), upon choosing $2 \leqq N_1 \leqq 16$ to optimize (4.19). For smaller values of r, we must work rather harder. To deal with these cases, we begin by precomputing a table of primes up to 2×10^8, via a sieve, and sorting them into arithmetic progressions. Firstly, if $50000 \leqq r < 3 \times 10^6$, we apply inequality (4.16) and compute Δ_2 from equation (4.17). To do this, for each value of N between 0 and N_0, we consider the primes in our table between $\dfrac{r+2}{N+1}$ and $\dfrac{r+2}{N+1/n}$ and sum their logarithms up to multiplicities determined modulo n (noting that there are at most $n-1$ relevant subintervals of

$$\left(\frac{r+2}{N+1}, \frac{r+2}{N+1/n}\right)$$

to deal with). An important observation is that as we increment r, the term

$$\theta\left(\frac{r+2}{N+d_{a,\alpha}/n}, n, a\right) - \theta\left(\frac{r+2}{N+1}, n, a\right)$$

changes by at most the addition of the logarithm of a single prime from our table if there is a prime $p \equiv a \pmod{n}$ in the interval

$$\left(\frac{r+2}{N+d_{a,\alpha}/n}, \frac{r+3}{N+d_{a,\alpha}/n}\right)$$

and the subtraction of such a prime, if there is one in the interval

$$\left(\frac{r+2}{N+1}, \frac{r+3}{N+1}\right).$$

We are thus able to obtain the value of Δ_2 corresponding to $r+1$ from that corresponding to r, without recomputation. Similarly, we may readily compute $\Delta_2(m,n,r+\delta)$ from $\Delta_2(m,n,r)$ for small integral δ. Unfortunately, this approach is still too slow for our purposes as it potentially requires

$$2N_0(m-1)(n-1) \sim 2m\sqrt{rn}$$

additions and subtractions to obtain $\Delta_2(m,n,r+1)$ from $\Delta_2(m,n,r)$. To speed this up, we observe that if $\delta \in \mathbb{N}$ and $s \in \{r, r+1, \dots, r+\delta\}$, then

$$\log \Delta_2(m,n,s) \leqq \log \Delta_2(m,n,r) + \sum_{a=1}^{n-1} \sum_{\alpha=1}^{m-1} \sum_{N=0}^{N_0} \chi(a,\alpha,N,n,r,\delta)$$

where

$$\chi(a,\alpha,N,n,r,\delta) = \theta\left(\frac{r+2+\delta}{N+d_{a,\alpha}/n}, n, a\right) - \theta\left(\frac{r+2}{N+d_{a,\alpha}/n}, n, a\right).$$

It follows, if we find that

$$\log \Delta_{m,n,r} < \big(c_1(n) - \varepsilon(n)\big) r$$

for some $\varepsilon(n) > 0$, and also, for some positive integer δ, that

$$S = \sum_{a=1}^{n-1} \sum_{\alpha=1}^{m-1} \sum_{N=0}^{N_0} \chi(a,\alpha,N,n,r,\delta) < \varepsilon(n) r + c_1(n)\delta, \tag{6.1}$$

then we verify Proposition 6.1 for all $s \in \{r, r+1, \dots, r+\delta\}$.

To implement this observation, let us assume that $\delta > n$ and write

$$S = S_1 + S_2 + S_3$$

where

$$S_1 = \sum_{a=1}^{n-1} \sum_{\alpha=1}^{m-1} \sum_{\delta/2 \leqq N \leqq N_0} \chi(a,\alpha,N,n,r,\delta),$$

$$S_2 = \sum_{a=1}^{n-1} \sum_{\alpha=1}^{m-1} \sum_{1 \leqq N < \delta/2} \chi(a,\alpha,N,n,r,\delta)$$

and

$$S_3 = \sum_{a=1}^{n-1} \sum_{\alpha=1}^{m-1} \chi(a,\alpha,0,n,r,\delta).$$

To bound the S_i's, we use the trivial estimate

$$\theta(B,n,a) - \theta(A,n,a) \leqq \log B\big(\pi(B,n,a) - \pi(A,n,a)\big)$$

where $\pi(x,n,a)$ denotes the number of primes $p \equiv a \pmod{n}$ with $p \leqq x$. If $N \geqq \delta/2$, then, for fixed $d \in \mathbb{N}$,

$$\frac{\delta}{N+\frac{d}{n}} < 2$$

and so

$$\sum_{a=1}^{n-1}\left(\pi\left(\frac{r+2+\delta}{N+d_{a,\alpha}/n},n,a\right)-\pi\left(\frac{r+2}{N+d_{a,\alpha}/n},n,a\right)\right) \leqq 1.$$

Since there are fewer than n possible values for $d_{a,\alpha}$ and since the $d_{a,\alpha}$ are strictly increasing in α, it follows that

$$S_1 \leqq (n-1)\sum_{\delta/2 \leqq N \leqq N_0} \log\left(\frac{r+2+\delta}{N+d_{a,m-1}/n}\right).$$

From the fact that $r \geqq 50000$, we have

$$N \leqq N_0 \leqq \frac{r+2}{\sqrt{nr+n+m}} \leqq \frac{r+2}{\sqrt{n(r+1)}} < \frac{r}{2n}$$

and so

$$\frac{r+2+\delta}{N+d_{a,m-1}/n} \leqq \frac{r+2+\delta}{N+1/n} < \frac{r+\delta}{N}$$

whence we may conclude that

$$S_1 < (n-1)\sum_{\delta/2 \leqq N \leqq N_0} \log\left(\frac{r+\delta}{N}\right). \tag{6.2}$$

Similarly, we obtain

$$S_2 \leqq (n-1)(m-1)\left(\sum_{1 \leqq N < \delta/2} \log\left(\frac{r+\delta}{N}\right) + \sum_{1 \leqq N < \delta/n}\left[\frac{\delta}{nN}\right]\log\left(\frac{r+\delta}{N}\right)\right). \tag{6.3}$$

To bound S_3, we argue somewhat less naively. Recall the following theorem of Brun-Titschmarsh type, due to Montgomery and Vaughan [MV]:

Theorem 6.2. *If x and y are positive real numbers and a and n are relatively prime integers with $1 \leqq n < y \leqq x$, then*

$$\pi(x+y,n,a) - \pi(x,n,a) < \frac{2y}{\phi(n)\log(y/n)}.$$

It follows, since we assume $\delta > n$, that setting

$$\xi(a, \alpha, n, \delta) = \min\left\{ \left[\frac{2n\delta}{(n-1)\, d_{a,\alpha} \log(\delta/d_{a,\alpha})} \right], \left[\frac{\delta}{d_{a,\alpha}} \right] + 1 \right\},$$

we have

$$\chi(a, \alpha, 0, n, r, \delta) \leqq \xi(a, \alpha, n, \delta) \log\left(\frac{(r+2+\delta)n}{d_{a,\alpha}} \right)$$

whereby

$$S_3 \leqq \sum_{a=1}^{n-1} \sum_{\alpha=1}^{m-1} \frac{2n\delta \log\big((r+2+\delta)n/d_{a,\alpha}\big)}{(n-1)\, d_{a,\alpha} \log\big(\delta/d_{a,\alpha}\big)}.$$

By calculus, since $\delta > n > d_{a,\alpha} \geqq 1$, we may show that

$$\frac{\log\big((r+2+\delta)n/d_{a,\alpha}\big)}{d_{a,\alpha} \log\big(\delta/d_{a,\alpha}\big)} \leqq \frac{\log\big((r+2+\delta)n\big)}{\log(\delta)}$$

and so, since $n \geqq 17$,

$$S_3 \leqq (n-1)(m-1) \frac{17\delta}{8 \log \delta} \log\big((r+2+\delta)n\big). \tag{6.4}$$

Given r, δ, n and m, then, we may thus readily bound S by using (6.2), (6.3) and (6.4) and appealing to the inequality

(6.5)

$$\sum_{A \leqq N \leqq B} \log\left(\frac{r+\delta}{N}\right) < (B-A+1)\log(r+\delta) - B(\log B - 1) + (A-1)\big(\log(A-1) - 1\big).$$

To illustrate this, let us consider the situation with $n = 17, m = 6$ and $r = 100000$ (so that $N_0 = 76$). We compute $\Delta_2(6, 17, 10^5)$ explicitly and find that

$$\log \Delta_2(6, 17, 10^5) = 2321042.99325\ldots$$

and so, from (4.13) and (4.16),

$$\log \Delta_{6,17,10^5} < 832485.44.$$

This implies that we may take $\varepsilon = 0.6$ and so, choosing $\delta = 50$, we find that (6.2), (6.3), (6.4) and (6.5) yield the inequality

$$S = S_1 + S_2 + S_3 < 60000$$

whence (6.1) is satisfied. This proves Proposition 6.1 for $10^5 \leqq r \leqq 10^5 + 50$. Applying like arguments for the remaining cases with $r \geqq 50000$, we conclude, after lengthy computations, that

$$\frac{1}{r} \log \Delta_{m,n,r} < c_1(n)$$

for these values.

For $1000 \leqq r < 50000$, we compute Δ_2 from (4.17) as before, but treat the small primes p satisfying

$$p \leqq \Omega_{m,n,r} = \max\{\sqrt{nr + n + m}, 2n\}$$

more carefully. In fact, we calculate Δ_0 explicitly from the definition for these cases. We find, as before, that

$$\frac{1}{r} \log \Delta_{m,n,r} < c_1(n)$$

for all m, n under consideration and $1000 \leqq r < 50000$.

Finally, for each pair m and n and each positive integer r with $1 \leqq r \leqq 1000$, we explicitly compute

$$\frac{1}{r} \log \Delta_{m,n,r}$$

and verify that it fails to exceed $c_1(n)$, except for a number of small values of r, the largest of which is $r = 41$, corresponding to $n = 31$. In fact, we have

$$\frac{1}{r} \log \Delta_{m,n,r} \leqq c_1(n)$$

for all $r > 3$ (if $223 \leqq n \leqq 347$), $r > 6$ (if $109 \leqq n \leqq 211$), $r > 8$ (if $79 \leqq n \leqq 107$), etc. For the remaining exceptional cases, we verify that

$$\frac{1}{r} \log \Delta_{m,n,r} - c_1(n) < \frac{1}{r} d(n).$$

The maximal values for

$$\frac{1}{r} \log \Delta_{m,n,r} - c_1(n)$$

correspond to $r = 1$ or $r = 2$ except for those n with $17 \leqq n \leqq 41$, $n = 47$ and $59 \leqq n \leqq 73$. In all cases, the maxima occur for $r \leqq 23$. The actual tabulated values are available from the author upon request. This completes the proof of Proposition 6.1.

These calculations required a total of roughly four thousand hours of computing time on a collection of Sun Sparc 4, Sparc 20 and Sparc Ultra machines using code written in C. They have since been checked using Pari GP and, for some of the computations, Maple V. In no cases did we encounter discrepancies. □

7. The main theorem

We are now in position to prove the main theorem of our paper, namely:

Theorem 7.1. *Let $b > a$ be positive, relatively prime integers and suppose that $n, c_1(n)$ and $d(n)$ are as in Proposition* 6.1 *and $m = \left[\frac{n+1}{3}\right]$. If we have*

$$(\sqrt[m]{b} - \sqrt[m]{a})^m e^{c_1(n)} < 1,$$

then, if p and $q > 0$ are integers, we may conclude that

$$\left|\left(\frac{b}{a}\right)^{1/n} - \frac{p}{q}\right| > \left(3.15 \times 10^{24}(m-1)^2 n^{m-1} e^{c_1(n)+d(n)} (\sqrt[m]{b} + \sqrt[m]{a})^m\right)^{-1} q^{-\lambda}$$

where

$$\lambda = (m-1)\left\{1 - \frac{\log\left((\sqrt[m]{b} + \sqrt[m]{a})^m e^{c_1(n)+1/20}\right)}{\log\left((\sqrt[m]{b} - \sqrt[m]{a})^m e^{c_1(n)}\right)}\right\}.$$

Proof. We take $\theta = (b/a)^{1/n}$, $k = m - 1$ and

$$P_{i-1,r}(x) = \sum_{j=1}^{m} a^r \Delta_{m,n,r} A_{ij}\left(\frac{a-b}{a}, r\right) x^{j-1}$$

for $1 \leqq i \leqq m$ in Lemma 2.1, so that inequality (3.1) implies the nonsingularity of the matrix (a_{ij}) where

$$a_{ij} = a^r \Delta_{m,n,r} A_{ij}\left(\frac{a-b}{a}, r\right).$$

Let us note first that, if $n = 17$, we may assume, without loss of generality, that $1 < b/a \leqq 3/2$. To see this suppose that $b/a = \tau > 1$, so that

$$\lambda > 5\left(1 - \frac{\log\left((\tau^{1/6} + 1)^6 a e^{8.98}\right)}{\log\left((\tau^{1/6} - 1)^6 a e^{8.93}\right)}\right).$$

If $\tau > 3/2$ and $a \geqq 3$, it follows that $\lambda > 17$ and so Theorem 7.1 is a consequence of Liouville's theorem. If $\tau \geqq 2$ and $a \geqq 1$, we reach a like conclusion. Similarly, if $n = 19$ (so that, again, $m = 6$), we may also assume that $1 < b/a \leqq 3/2$, since

$$\lambda > 5\left(1 - \frac{\log\left((\tau^{1/6}+1)^6 a e^{9.45}\right)}{\log\left((\tau^{1/6}-1)^6 a e^{9.40}\right)}\right)$$

and the pairs of inequalities $\tau > 3/2$ and $a \geqq 4$; or $\tau > 5/3$ and $a \geqq 2$; or $\tau > 2$ and $a \geqq 1$ each imply that $\lambda > 19$. Carrying out this argument for larger values of n, we may assume that b/a is bounded above by

$$\begin{cases} 3/2 & \text{if } 17 \leqq n \leqq 23, \\ 2 & \text{if } 29 \leqq n \leqq 109, \\ 3 & \text{if } 113 \leqq n \leqq 347. \end{cases}$$

Combining Lemma 3.1 and Proposition 6.1, we have

$$|P_{i,r}(\theta)| \leqq \frac{\left(\frac{b-a}{a}\right)^m}{(m-1)!} e^{c_1(n)r+d(n)} (b^{1/m} - a^{1/m})^{mr}$$

for $0 \leqq i \leqq m-1$, and so, after routine calculations, we find that we may always take $d = e^{17.75}$ and

$$D^{-1} = e^{c_1(n)} (b^{1/m} - a^{1/m})^m$$

in Lemma 2.1. From Lemma 3.2 and Proposition 6.1,

$$\left| a^r \Delta_{m,n,r} A_{ij}\left(\frac{a-b}{a}, r\right)\right| \leqq 2(r+1)\phi_{m,n,r}^{m-1} e^{c_1(n)r+d(n)} (b^{1/m} + a^{1/m})^{mr}$$

for all values of i and j and thus we may readily show that

$$c = 60.38 n^{m-1} e^{d(n)}$$

and

$$C = e^{c_1(n)+1/20} (b^{1/m} + a^{1/m})^m$$

are valid choices in Lemma 2.1. Since we assume $b/a \leqq 3$, we have either

$$\max\{|\theta|, |p/q|\}^{m-2} < 3^{1/3}$$

or that the conclusion of Theorem 7.1 follows directly from Liouville's theorem. Taking

$$t = \frac{n}{n-m+1}$$

and $r_0 = 1$ in Lemma 2.1 and noting that the inequality in Theorem 7.1 is trivial if

$$\frac{\log C}{\log D} \geqq \frac{n-m+1}{m-1},$$

verifying that

$$60.38 \times 3^{1/3} \times e^{1/20} \frac{n}{m-1} \left(\frac{n}{n-m+1} e^{17.75} \right)^{\frac{n-m+1}{m-1}}$$

is majorized by 3.15×10^{24} for the values of m and n under consideration, we conclude as desired. $\square$

To complement this result, we mention the following theorem of [Be3], which corresponds to the case $m = 2$ in Theorem 3.4:

Theorem 7.2. *For integer n, define the constant $c_3(n)$ by*

n	$c_3(n)$	n	$c_3(n)$	n	$c_3(n)$	n	$c_3(n)$	n	$c_3(n)$
3	2.03	11	1.67	23	1.53	41	1.45	59	1.40
4	1.62	13	1.65	29	1.51	43	1.43	61	1.39
5	1.84	17	1.58	31	1.51	47	1.44	67	1.38
7	1.76	19	1.56	37	1.46	53	1.40	71	1.36

Suppose that a, b, s and n are positive integers with $b > a$, $1 \leqq s < n/2$, $(s,n) = 1$ and n occuring in the above table. Define, for $c \in \mathbb{Z}$,

$$\kappa(c,n) = \prod_{p|n} p^{\max\{\mathrm{ord}_p(n/c)+1/(p-1),0\}}.$$

If, further, we have that

$$(\sqrt{b}+\sqrt{a})^{2(n-2)} > (b-a)^{2(n-1)} \left(\frac{\kappa(b-a,n)}{c_3(n)} \right)^n,$$

then we can conclude that

$$\left| \left(\frac{b}{a} \right)^{s/n} - \frac{p}{q} \right| > a^{-1} (10^{10} q)^{-\lambda}$$

with

$$\lambda = 1 + \frac{\log\left(\frac{\kappa(b-a,n)}{c_3(n)} (\sqrt{b}+\sqrt{a})^2 \right)}{\log\left(\frac{c_3(n)}{(b-a)^2 \kappa(b-a,n)} (\sqrt{b}+\sqrt{a})^2 \right)}.$$

If one is content with effective rather than explicit bounds, we deduce a result as above, valid for suitably large p/q, only with $c_3(n)$ replaced by e^{G_n}, where

$$G_n = -\gamma - \frac{2}{\phi(n)} \sum_{\substack{1 \leqq r < n/2 \\ (r,n)=1}} \psi\Big(\frac{n-r}{n}\Big)$$

for γ the Euler-Mascheroni constant, $\phi(n)$ Euler's totient function and $\psi(z)$ the derivative of the logarithm of $\Gamma(z)$. This is equivalent to Theorem 5.3 of [Ch], upon noting that

$$e^{G_n + Chr_n^2} = \kappa(b-a,n) \prod_{p|n} p^{\min\{\mathrm{ord}_p(b-a),\, \mathrm{ord}_p n + 1/(p-1)\}}$$

(see [Be1] for a proof).

8. Continued fraction expansions

From (1.3), to complete the proof of Theorem 1.1, we are led to consider the equation

$$|ax^n - (a+1)y^n| = 1 \tag{8.1}$$

with $2 \leqq a \leqq \min\{0.3n, 83\}$ and n prime, $17 \leqq n \leqq 347$. If (x, y) is a positive solution to (8.1), then we have

$$\left|\sqrt[n]{1+\frac{1}{a}} - \frac{x}{y}\right| < \frac{1}{any^n} \tag{8.2}$$

so that x/y is an exceptionally good rational approximation to $\sqrt[n]{1+1/a}$. We wish to show that all such solutions have $x/y = 1$. We first establish this for relatively small values of x and y, proving

Proposition 8.1. *If a and n are positive integers with $n \geqq 3$, then equation* (8.1) *has no solution in integers x and y with*

$$1 < \max\{|x|, |y|\} \leqq 10^{5000}.$$

We note that the upper bound here is quite a bit stronger than what we actually require to complete the proof of Theorem 1.1; computationally, however, it is not significantly more difficult to derive than an upper bound of, say, 10^{100}.

To prove Proposition 8.1, we consider the initial 10500 partial quotients in the continued fraction expansions to each of the 2954 $\sqrt[n]{1+1/a}$, where a and n satisfy (1.3) with n prime. Note that it is possible to carry out these computations entirely in integer arithmetic, using the structure of the minimal polynomial $f_{a,n}(x) = (a+1)x^n - a$ (as in, e.g., [LT]). For all but very small values of n, however, it appears to be much more economical to compute the decimal expansion of $\sqrt[n]{1+1/a}$ to high precision and then simply apply the Euclidean algorithm. We employ Pari GP for this purpose on a Sun Sparc 4 machine.

From a theorem of Kuzmin (see e.g. [Kh] and [LT]), one has that, for almost all real numbers θ, the probability that the nth partial quotient of θ is a positive integer k is given by

$$P(k) = \log\left(\frac{(k+1)^2}{k(k+2)}\right)/\log 2$$

or, equivalently, the probability that the nth partial quotient of θ is at most k is

$$Q(k) = 1 - \frac{\log\left(\frac{k+2}{k+1}\right)}{\log 2}.$$

If one adopts the philosophy that such a heuristic should be valid for any "reasonably defined" real number, unless one has specific knowledge (e.g. about its continued fraction expansion, boundedness of partial quotients, etc.) to the contrary, we might expect that the probability of a given irrational of the form $\sqrt[n]{1+1/a}$ (with $n \geqq 3$) possessing a partial quotient exceeding 10^6 among its first 10500 partial quotients should be about 0.015. Defining $m(a,n)$ to be the largest partial quotient among the first 10500 of $\sqrt[n]{1+1/a}$, upon considering the 2954 cases of a and n defined by (1.3) with prime n, this leads to an expectation of 44.4108... cases for which $m(a,n)$ exceeds 10^6. In fact, we find precisely 44 such situations, detailed in the following table:

a	n	$m(a,n)$	a	n	$m(a,n)$	a	n	$m(a,n)$
2	37	1777871	20	137	1648510	42	227	3682553
2	347	1836676	21	107	2880067	42	239	2589120
4	257	1815210	21	193	10564244	45	163	2336300
6	73	1151630	22	179	1014469	46	191	18935003
6	113	17850737	22	223	1599154	52	277	7948681
7	71	2942356	23	227	1844526	57	227	2130943
8	227	1747852	23	271	1367976	57	293	1233501
8	281	2521773	28	281	1202167	60	317	2108163
11	127	3877985	30	331	1989372	61	251	1009130
11	131	8905394	36	131	2518701	62	317	2695313
12	97	2398474	38	151	2330107	62	337	2598313
12	337	6345557	39	211	1642509	64	241	1225326
15	293	4181758	39	293	1863042	67	233	1085533
16	293	1251851	40	283	1789726	76	263	1125489
18	271	1330647	41	269	1114732			

Further analysis of our data for the 2954 cases in question does nothing to undermine the belief that continued fraction expansions of these numbers behave as expected.

Now, we observe from (8.2) that a positive solution to (8.1) corresponds to a convergent in the continued fraction expansion to $\sqrt[n]{1+1/a}$. For such a convergent p_i/q_i, we have, as an immediate corollary of Theorem 13 of [Kh],

$$\left|\sqrt[n]{1+\frac{1}{a}}-\frac{p_i}{q_i}\right| > \frac{1}{(a_{i+1}+2)q_i^2}$$

where a_{i+1} is the $(i+1)$st partial quotient in the aforementioned continued fraction expansion. It therefore follows from (8.2) that a solution (x, y) to (8.1) (with $x/y = p_i/q_i$) induces a partial quotient a_{i+1} satisfying

$$a_{i+1} \geqq anq_i^{n-2} - 1. \tag{8.3}$$

This standard folklore argument allows us to reduce the problem of searching for small solutions to (8.1) to a routine examination of a list of the initial partial quotients in the continued fraction expansion to $\sqrt[n]{1+1/a}$.

For each of the 2954 pairs (a, n) under consideration, we note that none of the first five convergents p_i/q_i in the continued fraction expansion to $\sqrt[n]{1+1/a}$ yields a solution to (8.1) other than with $p_i/q_i = 1$. Since we always find that $q_5 \geqq 254$ (where equality is obtained for $(a, n) = (4, 19)$), inequality (8.3) implies that we require a partial quotient exceeding 10^{37} in order to contradict Theorem 8.1. Since, for each of our 2954 examples, we find that q_{10500} exceeds 10^{5000}, upon examination of the above table, we conclude as desired, finishing the proof of Proposition 8.1.

We now apply Theorem 7.1 with (a, b, n) as in (1.3). For example, if

$$(a, b, n) = (2, 3, 17),$$

we find from Theorem 7.1 that

$$\left|\left(\frac{3}{2}\right)^{1/17} - \frac{p}{q}\right| > (6.3 \times 10^{43})^{-1} q^{-16.08}$$

for all $p, q > 0$. Combining this with (8.2) implies that every solution to

$$|3x^{17} - 2y^{17}| = 1$$

satisfies

$$\max\{|x|, |y|\} < 8.78 \times 10^{45}.$$

Proposition 8.1 thus yields the desired result. We argue similarly for the other values of (a, b, n) under consideration, in each case deducing an inequality which permits application of Proposition 8.1. This completes the proof of Theorem 1.1.

9. Approximation by algebraic numbers

If β is an algebraic number, we will denote by $H(\beta)$ the maximum modulus of the coefficients of the minimal polynomial of β over $\mathbb{Q}$. Further, let $q(\beta)$ be the smallest positive integer such that $q(\beta)\beta$ is an algebraic integer. We have, as a direct generalization of Liouville's theorem,

Lemma 9.1. *If θ is an algebraic number of degree $n \geqq 3$, $k \in \mathbb{N}$ and $\alpha \neq \theta$ is algebraic of degree at most k, then*

$$|\theta - \alpha| \geqq k^{-2}(k+1)^{1-n} q(\theta)^{-kn}(1+|\theta|)^{1-k}\bigl(1+H(\theta)\bigr)^{k(1-n)} H(\alpha)^{-n}.$$

Proof. This follows from Satz 3 on page 7 and Hilfssatz 15 on page 74 of Schneider [Schn]. $\square$

We compare this to the strong, though ineffective, result of Schmidt [Schm]:

Theorem 9.2. *If θ is algebraic, k is a positive integer and $\varepsilon > 0$, then there exist at most finitely many algebraic numbers α of degree at most k such that*

$$|\theta - \alpha| < H(\alpha)^{-k-1-\varepsilon}.$$

One may show that this theorem is essentially best possible (i.e. the exponent $-k-1$ cannot in general be replaced by a larger constant). If $k = 1$, Theorem 9.2 reduces to Roth's Theorem.

To derive an effective improvement upon Lemma 9.1 that approaches Theorem 9.2 in strength, in the special case where $\theta = (b/a)^{1/n}$, we apply part 2 of Lemma 2.1, together with the estimates leading to Theorem 3.4. We have

Theorem 9.3. *Suppose that a, b, n and m are positive integers with $n > m \geqq 2$ and $b > a$. Further define $|z_1 \ominus z_2|$ for $z_1, z_2 \in \mathbb{C}$ and Chr_n^m as in Section 3, and suppose that*

$$(\sqrt[m]{b} \ominus \sqrt[m]{a})^{m(m-2)}(\sqrt[m]{b} - \sqrt[m]{a})^m e^{(m-1)Chr_n^m} < 1.$$

It follows, if α is algebraic of degree strictly less than m, that there exists an effectively computable constant $c = c(a, b, n, m)$ such that

$$|(b/a)^{1/n} - \alpha| > cH(\alpha)^{-\chi-1}$$

where

$$\chi = \frac{(1-m)\log\bigl((\sqrt[m]{b} \ominus \sqrt[m]{a})^m e^{Chr_n^m}\bigr)}{\log\bigl((\sqrt[m]{b} \ominus \sqrt[m]{a})^{m(m-2)}(\sqrt[m]{b} - \sqrt[m]{a})^m e^{(m-1)Chr_n^m}\bigr)}.$$

Proof. Let us note that the lower bound for $|(b/a)^{1/n} - \alpha|$ is an easy consequence of Lemma 9.1 if α has degree $\geqq n-1$, so that we may in fact extend the above conclusion to include approximation of $(b/a)^{1/n}$ by algebraic numbers of arbitrary degree. If we suppose that the minimal polynomial of α over $\mathbb{Z}$ is given by

$$P(z) = \sum_{i=0}^{m-1} x_i z^i$$

where $x_i \in \mathbb{Z}$, then the combination of part 2 of Lemma 2.1 with the estimates of Section 3 implies that

$$\left| \sum_{i=0}^{m-1} x_i (b/a)^{i/n} \right| \gg H(\alpha)^{-\chi}$$

where the implicit effective constant depends at most upon a, b, n and m and χ is as above. Since Hilfssatz 15 of [Schn] yields the inequality

$$\left| \sum_{i=0}^{m-1} x_i (b/a)^{i/n} \right| \leqq (m-1)^2 \left(1 + (b/a)^{\frac{m-2}{n}}\right)^{m-2} H(\alpha) |(b/a)^{1/n} - \alpha|,$$

we conclude as desired. □

10. Effective results

Before proceeding with the proofs of our corollaries, we will say a few words about effective rather than explicit results (though, as previously mentioned, we will, for the most part, postpone such discussions to a future paper [Be4]). From Proposition 4.2 and the arguments leading to (4.16) and (4.17), we have

$$Chr_n^m = \frac{2}{n-1} \sum_{a=1}^{\frac{n-1}{2}} \sum_{\alpha=1}^{m-1} \sum_{N=0}^{\infty} \left(\frac{1}{N + d_{a,\alpha}/n} - \frac{1}{N+1} \right) - \frac{(m-1)n}{n-1} \log n.$$

To simplify this expression, define $\psi(z)$ to be the derivative of the logarithm of the gamma function, whereby

$$\sum_{N=0}^{N_0} \left(\frac{1}{N+a} - \frac{1}{N+b} \right) = \psi(N_0 + a + 1) - \psi(N_0 + b + 1) + \psi(b) - \psi(a).$$

It follows that

$$Chr_n^m = \frac{2}{n-1} \sum_{a=1}^{\frac{n-1}{2}} \sum_{\alpha=1}^{m-1} \left(\psi(1) - \psi(d_{a,\alpha}/n) \right) - \frac{(m-1)n}{n-1} \log n$$

and so, from the fact that

$$\psi(1) = -\gamma$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant, we have

$$Chr_n^m = -\frac{2}{n-1} U_{m,n} - (m-1)\left(\gamma + \frac{n \log n}{n-1} \right)$$

with

$$U_{m,n} = \sum_{a=1}^{\frac{n-1}{2}} \sum_{\alpha=1}^{m-1} \psi(d_{a,\alpha}/n).$$

For small values of m, we may achieve quite explicit characterizations of Chr_n^m. In particular, we have (assuming $n > m$ is prime)

$$U_{2,n} = \sum_{j=1}^{[n/2]} \psi\left(\frac{j}{n}\right)$$

whence the functional equation

$$\psi(z) - \psi(1-z) = -\pi \cot(\pi z)$$

in combination with

$$\psi(nz) = \log(n) + \frac{1}{n}\sum_{j=0}^{n-1} \psi(z + j/n)$$

(where we take $z = 1/n$) implies (see e.g. Chudnovsky [Ch]) that

$$Chr_n^2 = \frac{\pi}{n-1}\sum_{j=1}^{[n/2]} \cot\frac{\pi j}{n}.$$

Similarly, we have, for prime $n > 3$,

$$U_{3,n} = \sum_{j=1}^{[n/3]}\left\{\psi\left(\frac{j}{n}\right) + \psi\left(\frac{2j}{n}\right)\right\} + \sum_{j=[n/3]+1}^{[n/2]}\left\{\psi\left(\frac{n-j}{n}\right) + \psi\left(\frac{n-2j}{n}\right)\right\}.$$

To see this, note that, arguing as in Section 4, we may assume that $\frac{n-1}{2}$ distinct values of a with $1 \leqq a \leqq n-1$ are such that there exist $\tilde{a}$ with $\tilde{a} \equiv a^{-1} \pmod{n}$ and $1 \leqq \tilde{a} \leqq \frac{n-1}{2}$. It follows, in the notation of that section, that we have

$$t_1 = \tilde{a}, \quad t_2 = 2\tilde{a}, \quad t_3 = n, \quad t_4 = \tilde{a} + n, \quad t_5 = 2\tilde{a} + n.$$

Therefore

$$d_{a,1} = \min(\tilde{a}, n - 2\tilde{a})$$

and

$$d_{a,2} = \min(n - \tilde{a}, 2\tilde{a})$$

whence

$$d_{a,1} = \begin{cases} \tilde{a} & \text{if } \tilde{a} \leqq [n/3], \\ n - 2\tilde{a} & \text{if } \tilde{a} > [n/3] \end{cases}$$

and

$$d_{a,2} = \begin{cases} 2\tilde{a} & \text{if } \tilde{a} \leqq [n/3], \\ n - \tilde{a} & \text{if } \tilde{a} > [n/3]. \end{cases}$$

The result thus follows from the arguments leading to Corollary 4.3. We also have

$$\begin{aligned} U_{4,n} = & \sum_{j=1}^{[n/4]} \left\{ \psi\left(\frac{j}{n}\right) + \psi\left(\frac{2j}{n}\right) + \psi\left(\frac{3j}{n}\right) \right\} \\ & + \sum_{j=[n/4]+1}^{[n/3]} \left\{ \psi\left(\frac{n-j}{n}\right) + \psi\left(\frac{n-2j}{n}\right) + \psi\left(\frac{n-3j}{n}\right) \right\} \\ & + \sum_{j=[n/3]+1}^{[2n/5]} \left\{ \psi\left(\frac{n-j}{n}\right) + \psi\left(\frac{j}{n}\right) + \psi\left(\frac{3j-n}{n}\right) \right\} \\ & + \sum_{j=[2n/5]+1}^{[n/2]} \left\{ \psi\left(\frac{n-j}{n}\right) + \psi\left(\frac{j}{n}\right) + \psi\left(\frac{n-2j}{n}\right) \right\}, \end{aligned}$$

$$\begin{aligned} U_{5,n} = & \sum_{j=1}^{[n/5]} \left\{ \psi\left(\frac{j}{n}\right) + \psi\left(\frac{2j}{n}\right) + \psi\left(\frac{3j}{n}\right) + \psi\left(\frac{4j}{n}\right) \right\} \\ & + \sum_{j=[n/5]+1}^{[n/4]} \left\{ \psi\left(\frac{n-j}{n}\right) + \psi\left(\frac{n-2j}{n}\right) + \psi\left(\frac{n-3j}{n}\right) + \psi\left(\frac{n-4j}{n}\right) \right\} \\ & + \sum_{j=[n/4]+1}^{[2n/7]} \left\{ \psi\left(\frac{n-j}{n}\right) + \psi\left(\frac{n-2j}{n}\right) + \psi\left(\frac{j}{n}\right) + \psi\left(\frac{4j-n}{n}\right) \right\} \\ & + \sum_{j=[2n/7]+1}^{[n/3]} \left\{ \psi\left(\frac{n-j}{n}\right) + \psi\left(\frac{n-2j}{n}\right) + \psi\left(\frac{j}{n}\right) + \psi\left(\frac{n-3j}{n}\right) \right\} \\ & + \sum_{j=[n/3]+1}^{[2n/5]} \left\{ \psi\left(\frac{2j}{n}\right) + \psi\left(\frac{4j-n}{n}\right) + \psi\left(\frac{j}{n}\right) + \psi\left(\frac{3j-n}{n}\right) \right\} \\ & + \sum_{j=[2n/5]+1}^{[n/2]} \left\{ \psi\left(\frac{2n-3j}{n}\right) + \psi\left(\frac{n-j}{n}\right) + \psi\left(\frac{2n-4j}{n}\right) + \psi\left(\frac{n-2j}{n}\right) \right\} \end{aligned}$$

and

$$\begin{aligned} U_{6,n} = & \sum_{j=1}^{[n/6]} \left\{ \psi\left(\frac{j}{n}\right) + \psi\left(\frac{2j}{n}\right) + \psi\left(\frac{3j}{n}\right) + \psi\left(\frac{4j}{n}\right) + \psi\left(\frac{5j}{n}\right) \right\} \\ & + \sum_{j=[n/6]+1}^{[n/5]} \left\{ \psi\left(\frac{n-j}{n}\right) + \psi\left(\frac{n-2j}{n}\right) + \psi\left(\frac{n-3j}{n}\right) + \psi\left(\frac{n-4j}{n}\right) + \psi\left(\frac{n-5j}{n}\right) \right\} \\ & + \sum_{j=[n/5]+1}^{[2n/9]} \left\{ \psi\left(\frac{n-j}{n}\right) + \psi\left(\frac{n-2j}{n}\right) + \psi\left(\frac{n-3j}{n}\right) + \psi\left(\frac{j}{n}\right) + \psi\left(\frac{5j-n}{n}\right) \right\} \end{aligned}$$

$$+\sum_{j=[2n/9]+1}^{[n/4]}\left\{\psi\left(\frac{n-j}{n}\right)+\psi\left(\frac{n-2j}{n}\right)+\psi\left(\frac{n-3j}{n}\right)+\psi\left(\frac{j}{n}\right)+\psi\left(\frac{n-4j}{n}\right)\right\}$$

$$+\sum_{j=[n/4]+1}^{[2n/7]}\left\{\psi\left(\frac{n-j}{n}\right)+\psi\left(\frac{2j}{n}\right)+\psi\left(\frac{5j-n}{n}\right)+\psi\left(\frac{j}{n}\right)+\psi\left(\frac{4j-n}{n}\right)\right\}$$

$$+\sum_{j=[2n/7]+1}^{[n/3]}\left\{\psi\left(\frac{n-j}{n}\right)+\psi\left(\frac{2j}{n}\right)+\psi\left(\frac{n-2j}{n}\right)+\psi\left(\frac{j}{n}\right)+\psi\left(\frac{n-3j}{n}\right)\right\}$$

$$+\sum_{j=[n/3]+1}^{[3n/8]}\left\{\psi\left(\frac{2j}{n}\right)+\psi\left(\frac{n-j}{n}\right)+\psi\left(\frac{j}{n}\right)+\psi\left(\frac{n-2j}{n}\right)+\psi\left(\frac{3j-n}{n}\right)\right\}$$

$$+\sum_{j=[3n/8]+1}^{[2n/5]}\left\{\psi\left(\frac{2j}{n}\right)+\psi\left(\frac{n-j}{n}\right)+\psi\left(\frac{j}{n}\right)+\psi\left(\frac{n-2j}{n}\right)+\psi\left(\frac{2n-5j}{n}\right)\right\}$$

$$+\sum_{j=[2n/5]+1}^{[3n/7]}\left\{\psi\left(\frac{2n-3j}{n}\right)+\psi\left(\frac{n-j}{n}\right)+\psi\left(\frac{j}{n}\right)+\psi\left(\frac{3j-n}{n}\right)+\psi\left(\frac{5j-2n}{n}\right)\right\}$$

$$+\sum_{j=[3n/7]+1}^{[n/2]}\left\{\psi\left(\frac{2n-3j}{n}\right)+\psi\left(\frac{n-j}{n}\right)+\psi\left(\frac{j}{n}\right)+\psi\left(\frac{2n-4j}{n}\right)+\psi\left(\frac{n-2j}{n}\right)\right\}.$$

From this last expression, we observe that there are two typographical errors in Example 6.5 of [Ch] (i.e. the value Chr_{11}^{6}).

11. Proof of Corollary 1.2

The proof of Corollary 1.2(a) for $m \geqq 3$ is immediate from Theorem 1.1 upon noting that (1.5) yields, if $n = km + 1$, the equation

$$x(x^k)^m - (x-1)y^m = 1.$$

For $m = 2$, the result is a consequence of work of Ljunggren [Lj1] who proved that equation (1.5) with $m = 2$ possesses only the solutions $(3, 11, 5)$ and $(7, 20, 4)$ in positive integers (x, y, n) with $x > 1$, $y > 1$ and $n > 2$. Similarly, Corollary 1.2(b) follows from noting that if $x = z^m$, we have

$$(z^n)^m - (z^m - 1)y^m = 1$$

and so the equation

$$X^m - (z^m - 1)Y^m = 1$$

has the two positive solutions $(X, Y) = (z, 1)$ and $(X, Y) = (z^n, y)$, contradicting Theorem 1.1. Combining these two results with the arguments leading to Theorem 3 of [Sh1] and Theorem 2 of [Sh2] immediately implies Corollary 1.2(d).

Let us now consider Corollary 1.2(c). We note that it is possible to produce a direct proof of this result from Theorem 7.1 in conjunction with lower bounds for linear forms in logarithms of two algebraic numbers, say those due to Laurent, Mignotte and Nesterenko [LMN]. Instead, we will appeal directly to Theorem 1.1 and the arguments of Saradha and Shorey [SS] (which also utilize the results of [LMN]). As noted in [SS], there is no loss of generality in assuming that n and m are odd primes with $n \geqq 5$. If there is a solution to (1.5) with $x = z^2$, then we have

$$\frac{z^n - 1}{z - 1} = Y^m, \quad \frac{z^n + 1}{z + 1} = X^m \tag{11.1}$$

where X and Y are positive, relatively prime integers with $X, Y > 1$. It follows immediately that

$$(z+1)x^m - (z-1)y^m = 2 \tag{11.2}$$

has the two distinct solutions in positive integers given by $(x, y) = (1, 1)$ and

$$(x, y) = (X, Y).$$

Theorem 1.1 thus implies that z is even. By Theorem 1 of [SS], we may therefore restrict attention to equation (11.2) with

$$z \in \{6, 10, 12, 14, 18, 20, 22, 24, 26, 28, 30\}.$$

We wish to show, for these values of z, that (11.2) has no solutions in positive integers x and y with $xy > 1$. To do this, we require a number of lemmata.

Lemma 11.1. *If (x, y) is a solution in positive integers to (11.2) with $xy > 1$, then $y \geqq \big((z-1)m + 3\big)/2$.*

Proof. Since we assume that $xy > 1$, we have that $y > x > 0$ and so

$$(z+1)x^m - (z-1)(x+1)^m \geqq 2$$

whence

$$2x^m - (z-1)\sum_{h=0}^{m-1} \binom{m}{h} x^h \geqq 2.$$

The desired conclusion follows upon noting that

$$y - 1 \geqq x \geqq \big((z-1)m + 1\big)/2. \quad \square$$

Lemma 11.2. *If equation (11.2) with $z \in \{6, 10, 12, 14, 18, 20, 22, 24, 26, 28, 30\}$ possesses a solution (x, y) in positive integers with $xy > 1$, then $m \leqq 587$.*

Proof. We apply a pair of theorems from a paper of Mignotte [Mi] on the equation $ax^n - by^n = c$. These results, in turn, are obtained from the bounds for linear forms in two logarithms in [LMN].

Firstly, we use Theorem 2 of [Mi] with (in the notation of that paper)

$$a = z+1, \quad b = z-1, \quad c = 2, \quad A = z+1$$

and

$$\lambda = \log\left(1 + \frac{\log(z+1)}{\log\left(\dfrac{z+1}{z-1}\right)}\right).$$

Then we may conclude that if (11.2) has a solution (x, y) in positive integers with $xy > 1$, then $m \leqq 7521$. Next, we apply Theorem 1 of [Mi] where the inequality $m \leqq 7521$ permits us to take (again, in the notation of that paper) $h = 5\lambda$ and $U = 24.2$. If we let $z = 6$, we conclude, then, that

$$(m - 432)\log y - 380\log^{1/2} y - 127 < 0.$$

If $m \geqq 593$, it follows that $y \leqq 1164$, contradicting Lemma 11.1. Since we may restrict attention to prime values of m, we therefore have $m \leqq 587$. For the larger values of z, we argue similarly, in fact obtaining stronger bounds upon m in all cases. □

Lemma 11.3. *Let a be a positive integer with* $\gcd(a, m) = 1$. *If $z = am \pm 1$, then the equations* (11.1) *have no solution in integers (x, y, z, m, n) with $x, y > 1$ and $m, n > 2$.*

Proof. This is immediate from the proof of Lemma 8 of [SS] upon application of Corollary 1.2(a). □

To conclude the proof of Corollary 1.2(c), we need to deal with the values of m not excluded by Lemma 11.2 (i.e. $m \leqq 587$). For a fixed z and m, let us suppose that equations (11.1) possess a positive solution. Suppose, further, that we can find a prime p satisfying

(i) $p \equiv 1 \pmod{m}$, say $p = am + 1$,

(ii) if k is the smallest positive integer for which $z^{2k} \equiv 1 \pmod{p}$, then

$$\gcd(k, 6) > 1,$$

and

(iii) if S_1 is the set of residues n modulo $p - 1$ for which

$$\left(\frac{z^n - 1}{z - 1}\right)^a \equiv 1 \pmod{p}$$

is solvable and S_2 is the corresponding set for the congruence

$$\left(\frac{z^n + 1}{z + 1}\right)^a \equiv 1 \pmod{p},$$

then if $n \in S_1 \cap S_2$, we have that $n \equiv 1 \pmod{m}$.

We claim that this implies a contradiction. To see this, note first that if (X, Y, n) is a positive solution to (11.1), then p fails to divide XY. If this were not the case, we would have

$$z^{2n} \equiv 1 \pmod{p}$$

and so $k|n$ where k is as in (ii). Since $\gcd(k, 6) > 1$, this contradicts the fact that $\gcd(n, 6) = 1$. It follows that

$$X^{p-1} = X^{am} \equiv 1 \pmod{p} \quad \text{and} \quad Y^{p-1} = Y^{am} \equiv 1 \pmod{p}$$

whence

$$\left(\frac{z^n - 1}{z - 1}\right)^a \equiv 1 \pmod{p} \quad \text{and} \quad \left(\frac{z^n + 1}{z + 1}\right)^a \equiv 1 \pmod{p}.$$

Since, by assumption (iii), we have $n \equiv 1 \pmod{m}$, applying Corollary 1.2(a) leads to the desired contradiction.

While it is by no means clear that, given z and m, we can in fact find a prime p with properties (i), (ii) and (iii), for all the cases under consideration, this turns out to be a relatively easy matter. For example, if $z = 6$ and $m = 43$, if we take $p = 1033$, we find that $p = 24 \times 43 + 1$ and (in the notation of (ii)) $k = 86$. The 24-th roots of unity modulo 1033 are

$$\pm 1, \pm 14, \pm 135, \pm 176, \pm 195, \pm 196, \pm 231, \pm 355, \pm 369, \pm 398, \pm 407, \pm 500$$

and so

$$S_1 = \{1, 16, 35, 38 \pmod{43}\}$$

and

$$S_2 = \{1, 8, 10, 22, 25 \pmod{43}\}$$

whereby $n \equiv 1 \pmod{43}$, contradicting Corollary 1.2(a). We provide a complete list of primes p in the case that $z = 6$, noting that Lemma 11.2 and Lemma 11.3 allow us to suppose that $11 \leqq m \leqq 587$, m prime:

m	p	m	p	m	p	m	p	m	p	m	p
11	67	83	997	179	1433	277	1663	389	9337	499	19961
13	313	89	1069	181	2897	281	3373	397	6353	503	3019
17	103	97	3881	191	2293	283	1699	401	3209	509	4073
19	229	101	809	193	3089	293	1759	409	4909	521	16673
23	277	103	619	197	3547	307	5527	419	17599	523	6277
29	233	107	857	199	3583	311	1867	421	6737	541	9739
31	373	109	2617	211	8863	313	1879	431	3449	547	8753

m	p	m	p	m	p	m	p	m	p	m	p
37	223	113	2713	223	2677	317	32969	433	5197	557	4457
41	739	127	3049	227	5449	331	1987	439	13171	563	13513
43	1033	131	787	229	2749	337	5393	443	2659	569	6829
47	283	137	823	233	2797	347	2083	449	14369	571	9137
53	2969	139	1669	239	1913	349	30713	457	16453	577	3463
59	709	149	1193	241	4339	353	14827	461	2767	587	10567
61	367	151	907	251	4519	359	10771	463	5557		
67	1609	157	7537	257	1543	367	6607	467	2803		
71	569	163	2609	263	1579	373	15667	479	3833		
73	877	167	5011	269	2153	379	4549	487	7793		
79	1423	173	1039	271	1627	383	4597	491	3929		

The situation for $z \in \{10, 12, 14, 18, 20, 22, 24, 26, 28, 30\}$ is similar. Refining the argument leading to Lemma 11.2 together with a result analogous to Lemma 19 of [SS] allows us to improve the upper and lower bounds upon m for these values of z. In all cases, we can again find prime p satisfying (i), (ii) and (iii). This completes the proof of Corollary 1.2(c).

12. Concluding remarks

While this manuscript was in preparation, the author learned that Corollaries 1.2(a) and 1.2(c) have been obtained independently by Mignotte (private communication) and by Bugeaud, Mignotte, Roy and Shorey [BMRS], respectively, using quite different techniques. Also, Professor Mignotte informed the author that a more careful application of the results of [LMN] allows one to substantially reduce the number of cases from those defined by (1.3). In fact, this approach allows one to sharpen the inequalities in (1.3) to $a \leqq 8$ and $n \leqq 61$, obviating the need for many of the extensive computations described in Sections 5 and 6, at least in order to prove Theorem 1.1.

It is also worth noting that while the proof of Theorem 1.1 implicitly relies upon the theory of linear forms in logarithms of algebraic numbers (through its appeal to Theorem 1.1 of [BdW]), more uniform versions of the estimates obtained in this paper would permit the removal of this dependence.

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References

[Ba1] *A. Baker*, Rational approximations to certain algebraic numbers, Proc. London Math. Soc. (3) **14** (1964), 385–398.

[Ba2] *A. Baker*, Rational approximations to $\sqrt[3]{2}$ and other algebraic numbers, Quart. J. Math. Oxford Ser. (2) **15** (1964), 375–383.

[Ba3] *A. Baker*, Simultaneous rational approximations to certain algebraic numbers, Proc. Cambridge Phil. Soc **63** (1967), 693–702.

[Ba4] *A. Baker*, Transcendental Number Theory, Cambridge Univ. Press, Cambridge 1975.

[Be1] *M. Bennett*, Simultaneous rational approximation to binomial functions, Trans. Amer. Math. Soc. **348** (1996), 1717–1738.

[Be2] *M. Bennett*, Effective measures of irrationality for certain algebraic numbers, J. Austral. Math. Soc. **62** (1997), 329–344.

[Be3] *M. Bennett*, Explicit lower bounds for rational approximation to algebraic numbers, Proc. London Math. Soc. **75** (1997), 63–78.

[Be4] *M. Bennett*, Rational approximation to algebraic numbers of small height II: generalities, in preparation.

[BdW] *M. Bennett* and *B. M. M. de Weger*, On the Diophantine equation $|ax^n - by^n| = 1$, Math. Comp. **67** (1998), 413–438.

[Bo1] *E. Bombieri*, On the Thue-Siegel theorem, Acta Math. **148** (1982), 255–296.

[Bo2] *E. Bombieri*, Effective Diophantine approximation on $\mathbb{G}_m$, Ann. Scu. Norm. Sup. Pisa **20** (1993), 61–89.

[BM] *E. Bombieri* and *J. Mueller*, On effective measures of irrationality for $\sqrt[r]{a/b}$ and related numbers, J. reine angew. Math. **342** (1983), 173–196.

[Bu] *Y. Bugeaud*, Linear forms in p-adic logarithms and the diophantine equation $\frac{x^n-1}{x-1} = y^q$, Math. Proc. Cambridge Philos. Soc. **127** (1999), 373–381.

[BMRS] *Y. Bugeaud*, *M. Mignotte*, *Y. Roy* and *T. N. Shorey*, The equation $\frac{x^n-1}{x-1} = y^q$, Math. Proc. Cambridge Philos. Soc. **127** (1999), 353–372.

[Ch] *G. V. Chudnovsky*, On the method of Thue-Siegel, Ann. Math. (II) **117** (1983), 325–382.

[DM] *H. Darmon* and *L. Merel*, Winding quotients and some variants of Fermat's Last Theorem, J. reine angew. Math. **490** (1997), 81–100.

[De] *B. N. Delone*, Solution of the indeterminate equation $X^3q + Y^3 = 1$, Izv. Akad. Nauk SSR (6) **16** (1922), 253–272.

[DF] *B. N. Delone* and *D. K. Faddeev*, The Theory of Irrationalities of the Third Degree, Transl. Math. Monogr. **10**, Am. Math. Soc., 1964.

[Do] *Y. Domar*, On the Diophantine equation $|Ax^n - By^n| = 1, n \geqq 5$, Math. Scand. **2** (1954), 29–32.

[Dy] *F. Dyson*, The approximation to algebraic numbers by rationals, Acta. Math. **79** (1947), 225–240.

[Ea] *D. Easton*, Effective irrationality measures for certain algebraic numbers, Math. Comp. **46** (1986), 613–622.

[Ev] *J.-H. Evertse*, On the equation $ax^n - by^n = c$, Comp. Math. **47** (1982), 289–315.

[F] *N. I. Fel'dman*, An effective refinement in the exponent in Liouville's theorem (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 985–1002.

[G] *A. O. Gelfond*, Transcendental and Algebraic Numbers, English transl. by L. F. Boron, Dover Publications Inc., New York 1960.

[He] *Ch. Hermite*, Sur la généralisation des fractions continues algébriques, Ann. Mat. Pura Appl. **21** (1893), 289–308.

[Hy] *S. Hyyrö*, Über die Gleichung $ax^n - by^n = z$ und das Catalansche Problem, Ann. Acad. Sci. Fenn. Ser. A I **355** (1964), 50 pp.

[Kh] *A. Khintchine*, Continued Fractions, P. Noordhoff Ltd., Groningen 1963, 3rd edition.

[LT] *S. Lang* and *H. Trotter*, Continued fractions for some algebraic numbers, J. reine angew. Math. **255** (1972), 112–134.

[LMN] *M. Laurent*, *M. Mignotte* and *Y. Nesterenko*, Formes linéaires en deux logarithmes et déterminants d'interpolation, J. Number Th. **55** (1995), 285–321.

[Le1] *Maohua Le*, A note on the diophantine equation $(x^m-1)/(x-1) = y^n + 1$, Math. Proc. Cambridge Philos. Soc. **116** (1994), 385–389.

[Le2] *Maohua Le*, A note on perfect powers of the form $x^{m-1} + \cdots + x + 1$, Acta Arith. **69** (1995), 91–98.

[Le3] *Maohua Le*, A diophantine equation concerning finite groups, Pacific J. Math. **169** (1995), 335–341.

[LY] *Li Yu* and *Maohua Le*, On the Diophantine equation $(x^m-1)/(x-1) = y^n$, Acta Arith. **73** (1995), 363–366.

[Lj1] *W. Ljunggren*, New propositions about the indeterminate equation $\frac{x^n-1}{x-1} = y^q$ (Norwegian), Norske Mat. Tidsskrift **25** (1943), 17–20.

[Lj2] *W. Ljunggren*, On an improvement of a theorem of T. Nagell concerning the Diophantine equation $Ax^3 + By^3 = C$, Math. Scand. **1** (1953), 297–309.

[Ma1] *K. Mahler*, Ein Beweis des Thue-Siegelschen Satzes über die Approximation algebraischer Zahlen für binomische Gleichungen, Math. Ann. **105** (1931), 267–276.

[Ma2] *K. Mahler*, Zur Approximation algebraischer Zahlen, I: Über den grössten Primteiler binärer Formen, Math. Ann. **107** (1933), 691–730.

[Ma3] *K. Mahler*, Perfect systems, Compos. Math. **19** (1968), 95–166.

[Mi] *M. Mignotte*, A note on the equation $ax^n - by^n = c$, Acta Arith. **75** (1996), 287–295.

[MV] *H. L. Montgomery* and *R. C. Vaughan*, The large sieve, Mathematika **20** (1973), 119–134.

[Mo] *L. J. Mordell*, Diophantine Equations, Academic Press, London 1969.

[Mu] *J. Mueller*, Counting solutions of $|ax^r - by^r| \leqq h$, Quart. J. Math. Oxford (2) **38** (1987), 503–513.

[N] *T. Nagell*, Solution complète de quelques équations cubiques à deux indéterminées, J. Math. (9) **4** (1925), 209–270.

[NS] *Y. Nesterenko* and *T. N. Shorey*, Perfect powers in products of integers from a block of consecutive integers (II), Acta Arith. **76** (1996), 191–198.

[RR] *O. Ramaré* and *R. Rumely*, Primes in arithmetic progressions, Math. Comp. **65** (1996), 397–425.

[Rib] *P. Ribenboim*, Catalan's Conjecture, Academic Press, London 1994.

[Rid] *D. Ridout*, The p-adic generalization of the Thue-Siegel-Roth theorem, Mathematika **5** (1958), 40–48.

[RS] *J. B. Rosser* and *L. Schoenfeld*, Approximate formulas for some functions of prime numbers, Ill. J. Math. **6** (1962), 64–94.

[Ro] *K. F. Roth*, Rational approximations to algebraic numbers, Mathematika **2** (1955), 1–20, corr., ibid. 168.

[Ru] *R. Rumely*, Numerical computations concerning the ERH, Math. Comp. **61** (1993), 415–440.

[SS] *N. Saradha* and *T. N. Shorey*, The equation $(x^n - 1)/(x - 1) = y^q$ with x square, Math. Proc. Cambridge Philos. Soc. **125** (1999), 1–19.

[Schm] *W. M. Schmidt*, Simultaneous approximation to algebraic numbers by rationals, Acta Math. **125** (1970), 189–201.

[Schn] *Th. Schneider*, Einführung in die transzendenten Zahlen, Springer Verlag, Berlin-Göttingen-Heidelberg 1957.

[Sh1] *T. N. Shorey*, Perfect powers in values of certain polynomials at integer points, Math. Proc. Cambridge Philos. Soc. **99** (1986), 195–207.

[Sh2] *T. N. Shorey*, On the equation $z^q = (x^n - 1)/(x - 1)$, Indag. Math. **89** (1986), 345–351.

[ST] *T. N. Shorey* and *R. Tijdeman*, New applications of Diophantine approximations to Diophantine equations, Math. Scand. **39** (1976), 5–18.

[Si1] *C. L. Siegel*, Approximation algebraischer Zahlen, Mat. Z. **10** (1921), 173–213.

[Si2] *C. L. Siegel*, Die Gleichung $ax^n - by^n = c$, Math. Ann. **114** (1937), 57–68.

[Ta] *V. Tartakovskiĭ*, Auflösung der Gleichung $x^4 - \rho y^4 = 1$, Izv. Akad. Nauk SSR (6) **20** (1926), 301–324.

[Th1] *A. Thue*, Über Annäherungswerte algebraischer Zahlen, J. reine angew. Math. **135** (1909), 284–305.

[Th2] *A. Thue*, Berechnung aller Lösungen gewisser Gleichungen von der Form, Vid. Skrifter I Mat.-Naturv. Klasse (1918), 1–9.

[Yu] *Ping-Zi Yuan*, Comment: A note on perfect powers of the form $x^{m-1} + \cdots + x + 1$, Acta Arith. **69** (1995), 91–98, by Maohua Le and On the Diophantine equation $(x^m - 1)/(x - 1) = y^n$, Acta Arith. **69** (1995), 363–366, by Li Yu and Maohua Le, Acta Arith. **83** (1998), 199.

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