The Lefschetz Trace Formula for the Moduli Stack of Principal Bundles

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Contents

1 Introduction

1.1 General Concepts

First we will give a general introduction to the ideas of this thesis.

The Lefschetz Trace Formula

The classical Lefschetz trace formula reads as follows:

$$\operatorname{tr} F \mid H^*(X, \mathbb{Q}) = \sum_{F(x)=x} \iota_F(x).$$

Here X is a compact oriented C^∞ manifold and

$$H^*(X, \mathbb{Q}) = \bigoplus_{p=0}^{\dim X} H^p(X, \mathbb{Q})$$

is the singular cohomology of X with values in the rational numbers $\mathbb{Q}$. Furthermore, $F : X \to X$ is a C^∞ map having only non-degenerate fixed points. The trace of the $\mathbb{Q}$-linear map $F : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ induced by $F : X \to X$ on cohomology is by definition

$$\operatorname{tr} F \mid H^*(X, \mathbb{Q}) = \sum_{p=0}^{\dim X} (-1)^p \operatorname{tr} F \mid H^p(X, \mathbb{Q}).$$

For a fixed point x (i.e. $F(x) = x$) we denote by $\iota_F(x) = \pm 1$ its index.[1]

If X is a complex manifold and F is holomorphic, then the index is always positive, so that we have

$$\operatorname{tr} F \mid H^*(X, \mathbb{Q}) = \#\{x \in X \mid F(x) = x\}.$$

[1] More precisely, $\iota_F(x) = \operatorname{sign} \det(DF(x) - 1)$, where $DF(x) : T_x X \to T_x X$ is the Jacobian of F at x. A fixed point x is non-degenerate if and only if $\det(DF(x) - 1) \neq 0$.

There is also an algebraic version of this formula. In the algebraic case, X is a complete smooth variety over an algebraically closed field k, and $F : X \to X$ is a morphism of varieties having non-degenerate fixed points. The formula reads:

$$\operatorname{tr} F \mid H^*(X_{\text{ét}}, \mathbb{Q}_\ell) = \#\{x \in X \mid F(x) = x\}.$$

The only difference from the analytic version is that we use the étale cohomology of X with values in the ℓ-adic rational numbers $\mathbb{Q}_\ell$, instead of singular cohomology.

A very interesting special case arises when the variety X is defined over a finite field: Let $\mathbb{F}_q$ be the field with q elements and let k be an algebraic closure of $\mathbb{F}_q$. That the variety X over k is *defined over* $\mathbb{F}_q$ means simply that equations defining X can be found that have coefficients in $\mathbb{F}_q$. For example, X could be defined by equations with integral coefficients that we reduce modulo q. So in the case that X is defined over $\mathbb{F}_q$, the *geometric Frobenius* $F_q : X \to X$ acts on X and its ℓ-adic cohomology. The geometric Frobenius F_q acts on X simply by raising coordinates to the q^{th} power: $F_q(x_q, \ldots, x_n) = (x_1^q, \ldots, x_n^q)$. Note that the fixed points of F_q are precisely those points of X with coordinates in $\mathbb{F}_q$. The geometric Frobenius automatically has only non-degenerate fixed points. Hence in this case the trace formula reads:

$$\operatorname{tr} F_q \mid H^*(X_{\text{ét}}, \mathbb{Q}_\ell) = \#X(\mathbb{F}_q).$$

Here $X(\mathbb{F}_q)$ is the set of points of X with coordinates in $\mathbb{F}_q$.

For technical reasons we reformulate this formula in terms of the *arithmetic Frobenius* Φ_q. On $H^*(X_{\text{ét}}, \mathbb{Q}_\ell)$, Φ_q acts simply as the inverse of F_q. For Φ_q the trace formula reads:

$$q^{\dim X} \operatorname{tr} \Phi_q \mid H^*(X_{\text{ét}}, \mathbb{Q}_\ell) = \#X(\mathbb{F}_q). \tag{1}$$

This follows from Poincaré duality. One advantage of (1) is that it remains true if X is assumed to be a smooth variety that is not necessarily complete[2], (without having to pass to cohomology with compact supports). A rigorous definition of the arithmetic Frobenius is given in Definition 2.4.1. A proof of (1) can be found as Proposition 2.4.3.

Algebraic Stacks

Algebraic stacks relate to algebraic varieties in the same way groupoids relate to sets. A groupoid is a category all of whose morphisms are isomorphisms. A set X is considered as a groupoid (also denoted X) by taking for objects the elements of X and for morphisms only the identity morphisms. A group G is considered as a groupoid (denoted BG) with one object whose automorphism group is G. A G-set X is considered as a groupoid (denoted X_G or $[X/G]$) by taking as objects the elements of X and for the set of morphisms from x to y ($x, y \in X$) the transporter:

$$\operatorname{Hom}_{X_G}(x, y) = \operatorname{Trans}_G(x, y) = \{g \in G \mid gx = y\}.$$

For a groupoid $\mathfrak{X}$ we define

$$\#\mathfrak{X} = \sum_{\xi \in [\mathfrak{X}]} \frac{1}{\# \operatorname{Aut} \xi}, \tag{2}$$

where the sum is taken over the set of isomorphism classes of $\mathfrak{X}$, and for an isomorphism class ξ, $\operatorname{Aut} \xi$ is the automorphism group of any representative of ξ. If $\# \operatorname{Aut} \xi$ happens to be infinite, we set $\frac{1}{\# \operatorname{Aut} \xi}$ equal to zero. In case $\mathfrak{X} = X$ is a set, $\#X$ is just the number of elements of X. If $\mathfrak{X} = BG$, for a group G, we have $\#BG = \frac{1}{\#G}$. If $\mathfrak{X} = X_G$, for a G-set X, we have $\#\mathfrak{X} = \frac{\#X}{\#G}$ by the orbit formula, at least if one of X or G is finite.

[2] and not even necessarily separated

For an algebraic variety X, the set of morphisms from another variety S into X, denoted $X(S) = \mathrm{Hom}(S, X)$, is simply a set. For an algebraic stack $\mathfrak{X}$, $\mathfrak{X}(S)$ is always a groupoid. Recall that a variety X over k can in fact be completely described by the collection of sets $X(S)$, where S ranges over all k-varieties (or more generally k-schemes). Similarly, an algebraic stack $\mathfrak{X}$ is defined by the collection of groupoids $\mathfrak{X}(S)$, S ranging over all k-schemes. This is the way in which algebraic stacks arise as a generalization of the concept of algebraic varieties. The most basic consequence of this definition is that algebraic stacks form a 2-category instead of an ordinary 1-category, as do algebraic varieties.[3]

Clearly, every algebraic variety X over k is an algebraic k-stack. If G is an algebraic group we get a corresponding algebraic stack, denoted BG, called the *classifying stack of* G. For a k-variety S, $BG(S)$ is the groupoid of principal G-bundles[4] with base S. On the other hand, we have a morphism $\pi : \{pt\} \to BG$, where $\{pt\} = \operatorname{Spec} k$ is the one-point variety. π is actually a principal G-bundle, in fact the *universal* principal G-bundle. Both of these properties are analogous to the classifying space of a Lie group in homotopy theory; hence the notation BG. If X is a variety on which the algebraic group G acts via $\sigma : G \times X \to X$, then there exists an algebraic stack X_G, characterized by the fact that the following diagram is both 2-cartesian and 2-cocartesian:

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & X_G \end{array}$$

So X_G can be considered as a quotient of X by G. Here $\pi : X \to X_G$ is a principal G-bundle. The object X_G always exists as algebraic stack, even if the quotient of X by G as variety does not exist at all. Moreover, X_G is always smooth if X and G are (see [19]). Note that as a special case, BG is simply the stack associated to the trivial action of G on $\{pt\} = \operatorname{Spec} k$.

Algebraic stacks are the proper framework for considering moduli problems. Whenever the objects that one wants to classify by a moduli space have automorphisms, one encounters serious problems if one restricts oneself to algebraic varieties (or schemes or even algebraic spaces). For example, let X be an algebraic curve over the algebraically closed field k. If $k = \mathbb{C}$, the complex numbers, this just means that X is a Riemann surface. Consider the moduli problem for vector bundles of rank n and degree d on X. Then a moduli variety exists only for *stable* vector bundles. Even if one restricts attention to stable vector bundles it is difficult to construct a universal vector bundle over the moduli space. In fact, a universal bundle only exists if n and d are coprime. This comes from the fact that even the stable vector bundles have automorphisms, the trivial automorphisms coming from multiplication by scalars (elements of k^*). Using algebraic stacks, all these problems disappear: There exists an algebraic stack $\mathfrak{X}$ such that for any k-scheme S, $\mathfrak{X}(S)$ is the groupoid of families of vector bundles of rank n and degree d on X, parametrized by S. The algebraic stack $\mathfrak{X}$ is smooth and a universal vector bundle exists over $\mathfrak{X}$.

For the precise definition of algebraic stacks, we refer to [19]. In accordance with [19] an algebraic stack is an algebraic stack as defined by M. Artin in [2]. The stacks defined by P. Deligne and D. Mumford in [6] will be called Deligne-Mumford stacks. The significance of a stack being a Deligne-Mumford stack is that the objects it classifies have no infinitesimal automorphisms.

The Trace Formula for Algebraic Stacks

Let us return to the situation where k is an algebraic closure of the finite field $\mathbb{F}_q$. Assume that we are given a smooth algebraic k-stack $\mathfrak{X}$ defined over the finite field $\mathbb{F}_q$. I propose the following

[3]For an Introduction to 2-categories see [9].

[4]Note that in general, the categories $BG(S)$ and $B(G(S))$ are not equivalent. Only if $H^1(S, G) = 0$, do we have equivalence. In fact, BG is obtained by 'stackafying' (the analogue of sheafafying) the prestack $S \mapsto B(G(S))$.

trace formula:

Conjecture 1.1.1 *The arithmetic Frobenius Φ_q satisfies the following trace formula:*

$$\boxed{q^{\dim \mathfrak{X}} \operatorname{tr} \Phi_q \mid H^*(\mathfrak{X}_{\mathrm{sm}}, \mathbb{Q}_\ell) = \#\mathfrak{X}(\mathbb{F}_q)}$$

Recall that $\mathfrak{X}(\mathbb{F}_q)$ is a groupoid so that $\#\mathfrak{X}(\mathbb{F}_q)$ is defined as in (2) above. For algebraic stacks we have replaced the étale cohomology by the smooth cohomology. For algebraic varieties the smooth cohomology is equal to the étale cohomology, but for algebraic stacks, the étale topology is not fine enough. Note that Conjecture 1.1.1 is a direct generalization of formula (1) so that Conjecture 1.1.1 is true if $\mathfrak{X}$ happens to be a smooth algebraic variety. The general problem is to find reasonable assumptions on $\mathfrak{X}$ that imply the truth of Conjecture 1.1.1. A very remarkable feature of this formula is that neither side is necessarily a finite sum. Instead, the formula asserts the convergence of two series and the equality of their limits. This is the main difference of Conjecture 1.1.1 from formula (1). In this thesis the truth of Conjecture 1.1.1 is established in some interesting special cases. For example, the main result of Section 2, Theorem 2.4.5, establishes the trace formula for the case of smooth Deligne-Mumford stacks of finite type. In this case the sums on both sides of the equation are finite; no limiting process is needed.

The Case $\mathfrak{X} = X_G$

Let G be a smooth algebraic group defined over $\mathbb{F}_q$ and let X be a smooth algebraic variety defined over $\mathbb{F}_q$, endowed with a G-action (also defined over $\mathbb{F}_q$). Then the algebraic stack $\mathfrak{X} = X_G$ is defined over $\mathbb{F}_q$. In the case that G is connected, by Lemma 3.5.6 Conjecture 1.1.1 reduces to

$$q^{\dim X - \dim G} \operatorname{tr} \Phi_q \mid H^*(X_{G\,\mathrm{sm}}, \mathbb{Q}_\ell) = \frac{\#X(\mathbb{F}_q)}{\#G(\mathbb{F}_q)}. \tag{3}$$

$H^*(X_{G\,\mathrm{sm}}, \mathbb{Q}_\ell)$ should be considered as the algebraic analogue of the G-equivariant cohomology of X. So (3) can be thought of as a Lefschetz trace formula for equivariant cohomology. In this thesis, the truth of (3) is established if G is a linear algebraic group. (See Theorem 3.5.7.)

The main ingredient in the proof of Theorem 3.5.7 is the Leray spectral sequence for a fibering of algebraic stacks (see Theorem 3.3.12). Consider, for example, the case $G = \mathbb{G}_{\mathrm{m}}$, the multiplicative group and $X = \{pt\} = \operatorname{Spec} k$, so that $\mathfrak{X} = B\mathbb{G}_{\mathrm{m}}$. We define $\mathbb{G}_{\mathrm{m}}$ by the property that $\mathbb{G}_{\mathrm{m}}(S) = \Gamma(S, \mathcal{O}_S)^*$ for any k-scheme S. In particular, $\mathbb{G}_{\mathrm{m}}(\mathbb{C}) = \mathbb{C}^*$. We can think of $\mathbb{G}_{\mathrm{m}}$ as the algebraic analogue of the circle group S^1. We have:

$$H^j(\mathbb{G}_{\mathrm{m}}, \mathbb{Q}_\ell) = \left\{ \begin{array}{ll} \mathbb{Q}_\ell \text{ with } \Phi_q \text{ acting trivially} & \text{for } j = 0 \\ \mathbb{Q}_\ell \text{ with } \Phi_q \text{ acting via multiplication with } \frac{1}{q} & \text{for } j = 1 \\ 0 & \text{otherwise} \end{array} \right\} \tag{4}$$

The Leray spectral sequence for the fibering $\{pt\} \to B\mathbb{G}_{\mathrm{m}}$ reads as follows:

$$H^i(B\mathbb{G}_{\mathrm{m}}, \mathbb{Q}_\ell) \otimes H^j(\mathbb{G}_{\mathrm{m}}, \mathbb{Q}_\ell) \Longrightarrow H^{i+j}(\{pt\}_{\text{ét}}, \mathbb{Q}_\ell) = \left\{ \begin{array}{ll} \mathbb{Q}_\ell & \text{if } i+j=0 \\ 0 & \text{otherwise} \end{array} \right.$$

So from this spectral sequence together with (4) we get

$$\begin{aligned} H^0(B\mathbb{G}_{\mathrm{m}}, \mathbb{Q}_\ell) &= \mathbb{Q}_\ell \text{ with } \Phi_q \text{ acting trivially} \\ H^1(B\mathbb{G}_{\mathrm{m}}, \mathbb{Q}_\ell) &= 0 \end{aligned}$$

and

$$H^i(B\mathbb{G}_{\mathrm{m}}, \mathbb{Q}_\ell) \otimes H^1(\mathbb{G}_{\mathrm{m}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^{i+2}(B\mathbb{G}_{\mathrm{m}}, \mathbb{Q}_\ell) \otimes H^0(\mathbb{G}_{\mathrm{m}}, \mathbb{Q}_\ell)$$

for all $i \geq 2$. So in general:

$$H^i(B\mathbb{G}_m, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & \text{for } i \text{ even} \\ 0 & \text{for } i \text{ odd} \end{cases} \tag{5}$$

where Φ_q acts on $H^i(B\mathbb{G}_m, \mathbb{Q}_\ell)$ via multiplication with $\frac{1}{q^{i/2}}$. This is analogous to the fact that BS^1, the classifying space of S^1, is the infinite dimensional complex projective space and has the same Betti numbers as those given by (5). Even though $B\mathbb{G}_m$ is of finite type and has finite dimension ($\dim B\mathbb{G}_m = -1$) it has the cohomology of an infinite dimensional space.

This exhibits a fundamental difference between algebraic stacks and algebraic varieties: Whereas algebraic varieties (just like C^∞ manifolds) always have finite dimensional cohomology, this is not necessarily the case for algebraic stacks. This is the main difficulty in proving Conjecture 1.1.1. One has to make sense of the trace of Φ_q on the infinite dimensional graded algebra $H^*(\mathfrak{X}_{sm}, \mathbb{Q}_\ell)$.

In the case of $B\mathbb{G}_m$ we get from (5):

$$\operatorname{tr} \Phi_q \mid H^*(B\mathbb{G}_m, \mathbb{Q}_\ell) = \sum_{i=0}^{\infty} \frac{1}{q^i}.$$

So in this case $\operatorname{tr} \Phi_q \mid H^*(\mathfrak{X}_{sm}, \mathbb{Q}_\ell)$ *converges*. According to (3) the limit should be

$$\frac{q^{\dim \mathbb{G}_m}}{\#\mathbb{G}_m(\mathbb{F}_q)} = \frac{q^1}{\#\mathbb{F}_q^*} = \frac{q}{q-1},$$

which it is. This proves the trace formula for the case of $B\mathbb{G}_m$.

In general, we choose an embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ and consider $H^*(\mathfrak{X}_{sm}, \mathbb{C}) = H^*(\mathfrak{X}_{sm}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}$. Thus the eigenvalues of the arithmetic Frobenius Φ_q become complex numbers and we are able to prove the convergence of the series $\operatorname{tr} \Phi_q | H^*(\mathfrak{X}_{sm}, \mathbb{C})$, at least in the case that G may be embedded into GL_n for some n. (See Section 3.5.)

Here it also becomes apparent why we use the arithmetic Frobenius instead of the geometric Frobenius. The trace of the geometric Frobenius does not converge.

1.2 G-bundles over a curve

The Canonical Parabolic Subgroup

Let X be an algebraic curve over the field k (for example a Riemann surface over $\mathbb{C}$). Let G be a reductive algebraic group over k, for example one of the classical matrix groups. The second part of this thesis is devoted to the study of the algebraic stack of principal G-bundles on X. We will denote this stack by $\mathfrak{H}^1(X, G)$. So for any k-scheme S we have

$$\mathfrak{H}^1(X, G)(S) = \text{groupoid of principal } G\text{-bundles on } X \times S.$$

If, for example, $G = GL_n$, the general linear group, then principal G-bundles can be thought of as vector bundles of rank n.

Before we go any further we have to define the *degree* of a principal G-bundle E on X. For a principal G-bundle E, the *degree* $d(E)$ is the homomorphism

$$\begin{aligned} d(E) : \mathsf{X}(G) &\longrightarrow \mathbb{Z} \\ \chi &\longmapsto \deg(E \times_{G,\chi} \mathcal{O}_X) \end{aligned}$$

where $\mathsf{X}(G)$ is the character group of G and $\deg(E \times_{G,\chi} \mathcal{O}_X)$ is the degree of the line bundle on X associated to E via the character χ. Let $\mathfrak{H}^1_d(X, G)$ be the stack of G-bundles of degree d.

The final result of this thesis is that Conjecture 1.1.1 holds for $\mathfrak{X} = \mathfrak{H}^1_d(X, G)$, modulo a conjecture which we are able to verify in many cases (see Theorem 8.4.22). Towards that goal we prove a number of results that are interesting in their own right. The new difficulty in dealing with the stack $\mathfrak{H}^1_d(X, G)$ is that it is not of finite type. This means that not only the left hand side of the formula of Conjecture 1.1.1 is an infinite series, but the right hand side is also. There are infinitely many isomorphism classes of G-bundles of degree d, but fortunately, their automorphism groups grow very large so that the sum

$$\#\mathfrak{H}^1_d(X, G)(\mathbb{F}_q) = \sum_{E \in H^1_d(X,G)(\mathbb{F}_q)} \frac{1}{\operatorname{Aut} E(\mathbb{F}_q)}$$

converges. Here $H^1_d(X, G)(\mathbb{F}_q)$ is the set of isomorphism classes of G-bundles of degree d that are defined over $\mathbb{F}_q$.

The main tool for studying the stack $\mathfrak{H}^1_d(X, G)$ is a natural stratification on it. Every principal G-bundle E on X has (besides the degree) two important invariants attached to it: The *degree of instability* $\deg_i(E)$ and the *type of instability* $t_i(E)$. The degree of instability is a non-negative integer. The type of instability is more complicated. It is also a finer invariant than the degree of instability, and so the type if instability induces a finer stratification on $\mathfrak{H}^1_d(X, G)$ than the degree of instability does.

Let E be a principal G-bundle on X. Then $\operatorname{Aut}_G(E) = E \times_{G,Ad} G$, the sheaf of automorphisms of E, is a reductive group scheme over X. The type of instability of E only depends on this reductive group scheme $\operatorname{Aut}_G(E)$. For example, consider the case $G = GL(n)$. The type of instability of a vector bundle E is then the *Harder-Narasimhan Polygon* which is defined in terms of the canonical flag $0 < E_1 < \ldots < E_r = E$ on E. As noted above, $\operatorname{Aut}(E) = GL(E)$ is a relative group variety over X. All fibers of $\operatorname{Aut}(E) \to X$ are isomorphic to $GL(n)$ as group varieties. The canonical flag defines a subgroup scheme $P \subset \operatorname{Aut}(E)$, by taking those automorphisms of E respecting the canonical flag. The completeness of the flag variety of E over X implies that $\operatorname{Aut}(E)/P$ is proper over X, i.e. that P is a *parabolic subgroup of* $\operatorname{Aut}(E)$. So the canonical flag can be thought of as being induced by a canonical parabolic subgroup P of $\operatorname{Aut}(E)$. This motivates the study of reductive group schemes over X and their parabolic subgroups.

Reductive Group Schemes Over A Curve

Let G be a reductive group scheme[5] over the curve X over the field k. Then the Lie algebra $\mathfrak{g}$ of G is a vector bundle of degree zero on X. For a parabolic subgroup $P \subset G$ (i.e. a smooth subgroup scheme such that G/P is proper over X) we define the *degree of* P to be the degree of the Lie algebra $\mathfrak{p}$ of P: $\deg P = \deg \mathfrak{p}$, where $\mathfrak{p}$ is considered simply as a vector bundle over X. We define the *degree of instability* $\deg_i(G)$ *of* G to be the maximal degree of its parabolic subgroups.

$$\deg_i(G) = \max_{\substack{P \subset G \\ \text{parabolic}}} \deg P.$$

This is easily seen to be finite. If $\deg_i(G) = 0$ we call G *semi-stable.* For the case of a vector bundle, the degree of instability turns out to be twice the area of the Harder-Narasimhan polygon.

To G we have an associated scheme $\operatorname{Dyn}(G)$, the scheme of Dynkin diagrams of G, which is finite étale over X. To fix notation, let $\{\mathfrak{v}_1, \ldots, \mathfrak{v}_r\}$ be the connected components of $\operatorname{Dyn} G$. Every parabolic subgroup $P \subset G$ defines an open and closed subscheme of $\operatorname{Dyn} G$, the *type*[6] $t(P)$, so $t(P)$ can be considered as a subset $t(P) \subset \{\mathfrak{v}_1, \ldots, \mathfrak{v}_r\}$. To $\mathfrak{v}_i \in t(P)$ we can define in a

[5] G has now taken the place of $\operatorname{Aut}_G(E)$.
[6] See Remark 6.2.1

canonical way a vector bundle $W(P, \mathfrak{v}_i)$. It can be obtained as a factor of a suitable filtration of the unipotent radical of P. Its degree $n(P, \mathfrak{v}_i) = \deg W(P, \mathfrak{v}_i)$ is called the *numerical invariant of the parabolic subgroup P with respect to the component of its type $\mathfrak{v}_i$.*

In this thesis we prove that there exists a unique parabolic subgroup $P \subset G$ satisfying:

i. The numerical invariants of P are positive

ii. $P/R_u(P)$ is semi-stable.

(See Theorem 6.4.4.) Here $R_u(P)$ is the unipotent radical of P, the largest unipotent normal subgroup of P. The quotient $P/R_u(P)$ is a reductive group scheme over X. In the case that $G = \mathrm{Aut}(E)$ for a vector bundle E and P is the parabolic subgroup corresponding to a flag $0 < E_1 < \ldots < E_r = E$ of E, then $P/R_u(P)$ is the automorphism group of the associated graded object $E_1 \oplus E_2/E_1 \oplus \ldots \oplus E/E_{r-1}$. The parabolic subgroup P of Theorem 6.4.4 is called the *canonical parabolic subgroup* of G. If P is the canonical parabolic subgroup of G, then the formal sum

$$\sum_{\mathfrak{v} \in t(P)} n(P, \mathfrak{v})\mathfrak{v} \quad \in \quad \text{free abelian group on } \{\mathfrak{v}_1, \ldots, \mathfrak{v}_r\}$$

is called the *type of instability of G.*

We also prove that the canonical parabolic subgroup P of G is the largest element in the set of parabolics of maximal degree in G. In particular, the degree of the canonical parabolic subgroup is the degree of instability of G.

Root Systems and Convex Solids

We prove Theorem 6.4.4 by reducing the study of parabolic subgroups of G to the study of *parabolic subsets* (or in another terminology *facets*) of the root system of G. First we reduce the question of unique existence of a canonical parabolic subgroup to the case that G is rationally trivial. This means that one (and hence any) generic maximal torus of G is split: Let K be the function field (the field of meromorphic functions) of X. A generic maximal torus is a maximal torus of the generic fiber G_K of G. For a generic maximal torus $T \subset G_K$ to be split, means that $T \cong \mathbb{G}_{\mathrm{m}}{}_K^n = K^{*n}$ for some n.

So assume that G is rationally trivial and let $T \subset G_K$ be a generic maximal torus. Then we get an associated root system $\Phi = \Phi(G_K, T)$. The set Φ is a subset of the character group $\mathsf{X}(T)$ of T and can be characterized as follows:

$$\mathfrak{g}_K = \mathfrak{g}_{K,0} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{K,\alpha}$$

where $\mathfrak{g}_{K,\alpha} = \{A \in \mathfrak{g}_K \mid Ad(t)A = \alpha(t)A \text{ for all } t \in T\}$ and $Ad : G \to GL(\mathfrak{g})$ is the adjoint representation of G. Let V be the $\mathbb{R}$-vector subspace of $\mathsf{X}(T) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by Φ. Then Φ is a *root system* in V.

Now since G_K is the generic fiber of the reductive group scheme G over X, $\Phi = \Phi(G_K, T)$ has an additional structure on it: I have called this structure a *complementary convex solid.* It comes about as follows: The Borel subgroups[7] of G containing T are in bijection with the Weyl chambers[8] of Φ. For any Borel subgroup B of G containing T, let $d(B)$ be the element of V^* (the dual space of V) such that

$$\langle \alpha, d(B) \rangle = \deg L(B, \alpha)$$

[7]Borel subgroups are parabolic subgroups that are solvable. They are minimal parabolic subgroups.

[8]The Weyl chambers of a root system Φ are the facets of maximal dimension. The facets of Φ are the equivalence classes of the equivalence relation on V defined by the set of hyperplanes $(H_\alpha)_{\alpha \in \Phi}$, where H_α is the hyperplane orthogonal to α. The facets of Φ are in bijection with the parabolic subsets of Φ and the parabolic subgroups of G containing T.

for all $\alpha \in \Phi$. Here $L(B, \alpha)$ is a line bundle on X associated to B and α in a canonical way. The components of the type of B, $\mathfrak{v}_1, \ldots, \mathfrak{v}_r$ are in bijection with the elements of the basis $\alpha_1, \ldots, \alpha_r$ of Φ defined by B and in fact $L(B, \alpha_i) = W(B, \mathfrak{v}_i)$ for $i = 1, \ldots, r$. For positive roots $\alpha \in \Phi^+$, $L(B, \alpha)$ is constructed as a factor in a suitable filtration of the unipotent radical $R_u(B)$ of B. To stay within the context of root systems, we index the vectors $d(B)$ by the Weyl chambers of Φ. So let $\mathfrak{C}$ be the set of Weyl chambers of Φ. A *complementary convex solid* for Φ is a collection of vectors $(d(\mathfrak{c}))_{\mathfrak{c} \in \mathfrak{C}}$ of V^* satisfying the following two axioms (see Definition 5.2.1):

i. If λ is a fundamental dominant weight with respect to both the Weyl chambers $\mathfrak{c}$ and $\mathfrak{d}$, then
$$\langle \lambda, d(\mathfrak{c}) \rangle = \langle \lambda, d(\mathfrak{d}) \rangle.$$

ii. If $\mathfrak{c}$ is a Weyl chamber and α a simple root with respect to $\mathfrak{c}$ and $\mathfrak{d} = \sigma_\alpha(\mathfrak{c})$ is the Weyl chamber obtained from $\mathfrak{c}$ by reflection about the hyperplane orthogonal to α, then
$$\langle \alpha, d(\mathfrak{c}) \rangle \leq \langle \alpha, d(\mathfrak{d}) \rangle.$$

Most of Section 6 is devoted to proving that in the above situation these two axioms are indeed satisfied.

The convex solid giving rise to the name of this structure is actually the convex hull F of the vectors $d(\mathfrak{c})$, $\mathfrak{c} \in \mathfrak{C}$. The family $(d(\mathfrak{c}))_{\mathfrak{c} \in \mathfrak{C}}$ can be reconstructed from F. We call the root system with complementary convex solid (Φ, d) *semi-stable* if $0 \in F$. We also define the *degree* and the *numerical invariants* for any facet of (Φ, d). Our main result on a root system with complementary convex solid (Φ, d) is that (Φ, d) has a unique *special facet* P. It is the largest element in the set of facets of maximal degree. (See Corollaries 5.3.15 and 5.3.17.)

For the facet P to be *special* means that

i. all numerical invariants of P are positive

ii. the reduced root system with complementary convex solid (Φ_P, d_P) obtained from (Φ, d) by *reduction to* P is semi-stable.

The convexity of F plays a crucial role in the proof of this result, since the special facet turns out to be the facet containing the unique point of F closest to the origin of V^*.

If (Φ, d) is the root system with complementary convex solid associated to the generic maximal torus T of G, Corollary 5.3.15 gives (via the correspondence between facets of Φ and parabolic subgroups of G containing T) a parabolic subgroup P of G, special (or 'canonical') with respect to T. The existence of the canonical parabolic subgroup follows easily from this. One of the main observations that makes this proof work is that any two parabolic subgroups of G contain in their intersection *generically* a maximal torus of G.

I would like to point out Conjecture 6.4.7 to the effect that $H^0(X, \mathfrak{g}/\mathfrak{p}) = 0$, where $\mathfrak{p}$ is the Lie algebra of the canonical parabolic subgroup P of G. The significance of this conjecture is essentially that the canonical parabolic subgroup is *rigid*. The truth of this conjecture would improve many of the results of Sections 7 and 8.

Families of Reductive Group Schemes

The following results lay the foundation for the stratification of $\mathfrak{H}^1_d(X, G)$ mentioned above:

i. Let G be a reductive group scheme over the curve X over the scheme S. (For example, G could be the automorphism group of a family of vector bundles parametrized by S.) Then the degree of instability $\deg_i(s) = \deg_i(G_s)$ is *upper semicontinuous* on S.

ii. Let $G/X/S$ be as before and assume that the degree of instability of G is constant on S. Then, if we pass to a cover of S that is universally homeomorphic to S, there exists a parabolic subgroup $P \subset G$ such that for every $s \in S$, P_s is the canonical parabolic subgroup of G_s.

Now let G be a reductive group scheme over the curve X over the field k. From now on we will assume that G and all its twists ${}^E G$, for G-torsors E, satisfy Conjecture 6.4.7. In other words, we assume that G satisfies Condition (§) (see Definition 8.4.7).

Let $\mathfrak{H}^1_d(X,G)_{\leq m}$ be the algebraic stack of principal G-bundles of degree d and degree of instability less than or equal to m. By (i) $\mathfrak{H}^1_d(X,G)_{\leq m}$ is an open substack of $\mathfrak{H}^1_d(X,G)$. We also prove that $\mathfrak{H}^1_d(X,G)_{\leq m}$ is of finite type (Theorem 8.2.6) and even of the form Y_H for some Deligne-Mumford stack Y and a linear algebraic group H acting on Y. Hence the trace formula Conjecture 1.1.1 holds for $\mathfrak{H}^1_d(X,G)_{\leq m}$. To prove the trace formula for all of $\mathfrak{H}^1_d(X,G)$, we use the exhaustion by the various $\mathfrak{H}^1_d(X,G)_{\leq m}$. To make this process work, we need information on $\mathfrak{H}^1_d(X,G)_m$, the stack of principal G-bundles on X of degree of instability equal to m. By (ii) $\mathfrak{H}^1_d(X,G)_m$ decomposes as follows:

$$\mathfrak{H}^1_d(X,G)_m = \coprod_{\mathfrak{v}} \mathfrak{H}^1_d(X,G)_{\mathfrak{v}}.$$

Here $\mathfrak{H}^1_d(X,G)_{\mathfrak{v}}$ is the stack of principal G-bundles of type of instability $\mathfrak{v}$, the disjoint sum being taken over all types of intability that give rise to the degree of instability m. We also prove that $\mathfrak{H}^1_d(X,G)_{\mathfrak{v}}$ has the same cohomology as $\mathfrak{H}^1_{d'}(X,G')$, for a suitable reductive group scheme G' on X and degree d'. This group G' can be obtained by taking $G' = P/R_u(P)$, where P is the canonical parabolic of $\mathrm{Aut}_G(E)$ for any G-bundle E of type of instability $\mathfrak{v}$. The groups G' that occur in this manner are finite in number, so that we get good control over $\mathfrak{H}^1_d(X,G)_m$, making it possible to prove the absolute convergence of $\operatorname{tr}\Phi_q|H^*(\mathfrak{H}^1_d(X,G)_{\mathrm{sm}},\mathbb{C})$. We obtain a proof of the trace formula, which reads in this case:

$$\boxed{q^{(g-1)\dim_X G}\operatorname{tr}\Phi_q \mid H^*(\mathfrak{H}^1_d(X,G)_{\mathrm{sm}},\mathbb{C}) = \sum_{E\in H^1_d(X,G)(\mathbb{F}_q)} \frac{1}{\#\mathrm{Aut}(E)(\mathbb{F}_q)}}$$

where g is the genus of the curve X. Note that the cases in which G satisfies Condition (§) include the following:

i. $G = GL_n$, so that $\mathfrak{H}^1_d(X,G)$ is the stack of vector bundles of rank n and degree d.

ii. G is rationally trivial and the genus of X is zero or one.

iii. $\mathrm{Dyn}\, G$ is connected.

2 Deligne-Mumford Stacks

Introduction

In this section we prove the Lefschetz trace formula for a smooth Deligne-Mumford stack $\mathfrak{X}$ of finite type over a finite field $\mathbb{F}_q$ (Theorem 2.4.5). The proof contains several steps. First we consider the case of a certain subclass of algebraic $\mathbb{F}_q$-stacks. These are the stacks we call 'tractable' (see Definition 2.2.3). This terminology is not standard.

The reason for introducing these 'tractable' stacks is the following. If $\mathfrak{X}$ is a tractable stack, then the coarse moduli space X of $\mathfrak{X}$ is very nice (i.e. a smooth variety) and the projection $\mathfrak{X} \to X$

is also very nice (i.e. étale). So it is possible to deduce the trace formula for 'tractable' stacks from the trace formula for smooth varieties. The trace formula for smooth varieties is reviewed in Proposition 2.4.3. The trace formula for tractable stacks is Lemma 2.4.4. The remarkable fact is that for a tractable stack $\mathfrak{X}$, the structure morphism $\mathfrak{X} \to X$ (X the coarse moduli space) induces an isomorphism on the level of ℓ-adic cohomology (see Proposition 2.2.8). On the other hand, we have $\#X(\mathbb{F}_q) = \sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut} \xi}$, a result which is due to Serre. This is the contents of Section 2.3. So both sides of the trace formula yield the same value for $\mathfrak{X}$ and X.

To deduce the trace formula for general smooth Deligne-Mumford stacks from that for tractable stacks, we need two results. First, every non-empty reduced Deligne-Mumford stack contains a non-empty tractable open substack. Secondly, we need the Gysin sequence (Corollary 2.1.3) for smooth pairs of algebraic k-stacks. The existence of the Gysin sequence is the contents of Section 2.1.

Note that in the case of a Deligne-Mumford stack $\mathfrak{X}$ the ℓ-adic cohomology algebra $H^*(\overline{\mathfrak{X}}_{\mathrm{sm}}, \mathbb{Q}_\ell)$ is finite dimensional over $\mathbb{Q}_\ell$. We have $H^p(\overline{\mathfrak{X}}_{\mathrm{sm}}, \mathbb{Q}_\ell) = 0$ for $p > 2\dim \mathfrak{X}$. So in this case, unlike for the general case, there is no limiting process needed to make sense of the trace formua.

Let S be a scheme. An *algebraic S-stack* is an algebraic S-stack in the sense of [19]. Only concepts not explained in [19] will be reviewed here. Note in particular, that all algebraic stacks (and hence all schemes and all algebraic spaces) are assumed to be quasi-separated.

2.1 Purity

Let k be a separably closed field.

Definition 2.1.1 Let $\mathfrak{Z}$ and $\mathfrak{X}$ be smooth k-stacks and $i : \mathfrak{Z} \to \mathfrak{X}$ a closed immersion. We call the pair $i : \mathfrak{Z} \to \mathfrak{X}$ a *smooth pair of algebraic k-stacks of codimension c* if for every connected component $\mathfrak{Z}'$ of $\mathfrak{Z}$, letting $\mathfrak{X}'$ be the connected component of $\mathfrak{X}$ such that $i(\mathfrak{Z}') \subset \mathfrak{X}'$, we have

$$\dim \mathfrak{Z}' + c = \dim \mathfrak{X}'.$$

Let $i : \mathfrak{Z} \to \mathfrak{X}$ be a smooth pair of k-stacks and let $\mathfrak{U} = \mathfrak{X} - \mathfrak{Z}$. Let F be an abelian sheaf on $\mathfrak{X}_{\mathrm{sm}}$. We denote the p^{th} *cohomology of $\mathfrak{X}_{\mathrm{sm}}$ with values in F and support in $\mathfrak{Z}$* by $H^p_{\mathfrak{Z}}(\mathfrak{X}_{\mathrm{sm}}, F)$. The group $H^p_{\mathfrak{Z}}(\mathfrak{X}_{\mathrm{sm}}, F)$ is the p^{th} derived functor of

$$F \longmapsto \ker(\Gamma(\mathfrak{X}_{\mathrm{sm}}, F) \to \Gamma(\mathfrak{U}_{\mathrm{sm}}, F))$$

evaluated at F. We have a long exact sequence

$$\ldots \longrightarrow H^p_{\mathfrak{Z}}(\mathfrak{X}_{\mathrm{sm}}, F) \longrightarrow H^p(\mathfrak{X}_{\mathrm{sm}}, F) \longrightarrow H^p(\mathfrak{U}_{\mathrm{sm}}, F) \longrightarrow \ldots \tag{6}$$

Let $j : \mathfrak{U}_{\mathrm{sm}} \to \mathfrak{X}_{\mathrm{sm}}$ be the inclusion morphism. Then we denote the right derived functors of

$$F \longmapsto \ker(F \to j_* j^* F)$$

by $\mathcal{H}^p_{\mathfrak{Z}}(\mathfrak{X}_{\mathrm{sm}}, F)$. The sheaf $\mathcal{H}^p_{\mathfrak{Z}}(\mathfrak{X}_{\mathrm{sm}}, F)$ is the sheaf on $\mathfrak{X}_{\mathrm{sm}}$ associated to the presheaf

$$U \longmapsto H^p_{\mathfrak{Z}_U}(U_{\mathrm{sm}}, F|U),$$

where U ranges over the smooth $\mathfrak{X}$-schemes. There is an E_2 spectral sequence

$$H^p(\mathfrak{X}_{\mathrm{sm}}, \mathcal{H}^q_{\mathfrak{Z}}(\mathfrak{X}_{\mathrm{sm}}, F)) \Longrightarrow H^{p+q}_{\mathfrak{Z}}(\mathfrak{X}_{\mathrm{sm}}, F). \tag{7}$$

Let $\Lambda = \mathbb{Z}/(n)$ where n is prime to $\operatorname{char}(k)$.

Proposition 2.1.2 (Cohomological Purity) *Let $i : \mathfrak{Z} \to \mathfrak{X}$ be a smooth pair of k-stacks of codimension c. Let F be a locally constant sheaf of Λ-modules on $\mathfrak{X}_{\mathrm{sm}}$. Then*

$$\mathcal{H}^p_{\mathfrak{Z}}(\mathfrak{X}_{\mathrm{sm}}, F) = \begin{cases} 0 & \text{for } p \neq 2c \\ i_* i^* F(-c) & \text{for } i = 2c \end{cases} .$$

Proof. Since the isomorphisms whose existence we claim are also claimed to be canonical, the question of their existence is local on $\mathfrak{X}_{\mathrm{sm}}$. So we may pass to a smooth presentation $X \to \mathfrak{X}$ of $\mathfrak{X}$ and thus assume that $i : \mathfrak{Z} \to \mathfrak{X}$ is a smooth pair of *k-schemes* of codimension c. Then our proposition follows from [20, VI, Theorem 5.1, Remark 5.2 and Corollary 6.4]. □

Corollary 2.1.3 (Gysin Sequence) *Let $i : \mathfrak{Z} \to \mathfrak{X}$ be a smooth pair of k-stacks of codimension c. Let F be a locally constant sheaf of Λ-modules on $\mathfrak{X}_{\mathrm{sm}}$. Then we have a long exact sequence*

$$\ldots \longrightarrow H^{p-2c}(\mathfrak{Z}_{\mathrm{sm}}, i^* F(-c)) \longrightarrow H^p(\mathfrak{X}_{\mathrm{sm}}, F) \longrightarrow H^p(\mathfrak{U}_{\mathrm{sm}}, F) \longrightarrow \ldots$$

Proof. From the spectral sequence (7) and purity we get

$$H^{p+2c}_{\mathfrak{Z}}(\mathfrak{X}_{\mathrm{sm}}, F) = H^p(\mathfrak{Z}_{\mathrm{sm}}, i^* F(-c)).$$

So the sequence (6) gives us the Gysin sequence. □

2.2 'Tractable Stacks'

For an S-stack $\mathfrak{X}$ we define $\mathfrak{Aut}(\mathfrak{X})$ to be the S-stack given by the following 2-cartesian diagram

$$\begin{array}{ccc} \mathfrak{Aut}(\mathfrak{X}) & \longrightarrow & \mathfrak{X} \\ \downarrow & \boxed{2} & \downarrow \Delta \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array}$$

Note that the two 1-morphisms $\mathfrak{Aut}(\mathfrak{X}) \rightrightarrows \mathfrak{X}$ are 2-isomorphic but not canonically so. Therefore it is unambiguous to discuss the properties of $\mathfrak{Aut}(\mathfrak{X}) \to \mathfrak{X}$ without specifying which of the two morphisms we mean.

Remark 2.2.1 Let k be a field and let $\mathfrak{X}$ be an algebraic k-stack, locally of finite type. If $\mathfrak{Aut}\,\mathfrak{X} \to \mathfrak{X}$ is flat and $\mathfrak{X} \to X$ is the coarse moduli space of $\mathfrak{X}$, then X is an algebraic space locally of finite type and $\mathfrak{X} \to X$ is faithfully flat and locally of finite type (see [19, Corollaire 4.9]).

Let k be a field and let $\mathfrak{X}$ be an algebraic k-stack, locally of finite type. Assuming that $\mathfrak{Aut}\,\mathfrak{X} \to \mathfrak{X}$ is flat, we define $B\,\mathfrak{Aut}\,\mathfrak{X}$ to be the image of the morphism

$$\mathfrak{X} \xrightarrow{\Delta} \mathfrak{X} \times \mathfrak{X}.$$

So we have a faithfully flat, locally of finite type presentation

$$\mathfrak{X} \longrightarrow B\,\mathfrak{Aut}\,\mathfrak{X}$$

of $B\,\mathfrak{Aut}\,\mathfrak{X}$ making $B\,\mathfrak{Aut}\,\mathfrak{X}$ into an algebraic k-stack, locally of finite type. We clearly have a diagram

$$\begin{array}{ccc} \mathfrak{Aut}(\mathfrak{X}) & \longrightarrow & \mathfrak{X} \\ \downarrow & \boxed{2} & \downarrow \\ \mathfrak{X} & \longrightarrow & B\,\mathfrak{Aut}\,\mathfrak{X} \end{array}$$

which is both 2-cartesian and 2-cocartesian, justifying the notation $B\,\mathfrak{Aut}\,\mathfrak{X}$. Letting X be the coarse moduli space of $\mathfrak{X}$ we have a diagram

$$\begin{array}{ccc} B\,\mathfrak{Aut}(\mathfrak{X}) & \longrightarrow & \mathfrak{X} \\ \downarrow & \boxed{2} & \downarrow \\ \mathfrak{X} & \longrightarrow & X \end{array}$$

which is also both 2-cartesian and 2-cocartesian.

Definition 2.2.2 An algebraic S-stack $\mathfrak{X}$ is called a *Deligne-Mumford stack* if the diagonal

$$\mathfrak{X} \xrightarrow{\Delta} \mathfrak{X} \times \mathfrak{X}$$

is unramified.

Definition 2.2.3 Let k be any field. Then we call $\mathfrak{X}$ *tractable* if it satisfies the following conditions:

i. $\mathfrak{X}$ is an algebraic Deligne-Mumford stack of finite type over k.

ii. $\mathfrak{Aut}(\mathfrak{X}) \to \mathfrak{X}$ is finite étale.

iii. The coarse moduli space of $\mathfrak{X}$ is a smooth k-variety.

Note 2.2.4 If $\mathfrak{X}$ is a tractable k-stack then $\mathfrak{X}$ is smooth integral of finite type over k. If X is the coarse moduli space of $\mathfrak{X}$ then the natural morphism $\mathfrak{X} \to X$ is étale.

Proof. Since $\mathfrak{Aut}\,\mathfrak{X} \to \mathfrak{X}$ is finite étale, the same holds for $\mathfrak{X} \to B\,\mathfrak{Aut}\,\mathfrak{X}$. That in turn implies that $B\,\mathfrak{Aut}\,\mathfrak{X} \to \mathfrak{X}$ is at least étale. Again by descent, $\mathfrak{X} \to X$ is étale. Since X is smooth, $\mathfrak{X}$ is so too. It remains to prove that $\mathfrak{X}$ is connected. But this is clear because X is a variety. □

Lemma 2.2.5 *Let X be an integral normal k-scheme of finite type. Let $f : Y \to X$ be a separated étale morphism of finite type. Then there exists a non-empty open subset $U \subset X$ such that the induced morphism $f : f^{-1}(U) \to U$ is finite étale.*

Proof. Without loss of generality we may assume that Y is connected. Then Y is integral. Let L be the function field of Y and let Y' be the normalization of X in L. Then Y is naturally an open subscheme of Y' (see [20, I, Theorem 3.21]).

$$\begin{array}{ccc} Y & \xrightarrow{i} & Y' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f'} \\ & & X \end{array}$$

Let $Z = Y' - i(Y)$. Then f' being finite implies that $f'(Z)$ is closed in X. Let $U = X - f'(Z)$. Then U is non-empty, because it contains the generic point of X. Since $f'^{-1}(U) \subset i(Y)$ we have that $f : f^{-1}(U) \to U$ is a base change of f' and hence finite. □

Proposition 2.2.6 *Let $\mathfrak{X}$ be a non-empty reduced Deligne-Mumford stack, locally of finite type over the field k. Then there exists a non-empty open substack $\mathfrak{X}' \subset \mathfrak{X}$ which is tractable.*

Proof. Let $\rho : Y \to \mathfrak{X}$ be a smooth presentation, where Y is a k-scheme, locally of finite type. Since $\mathfrak{X}$ is reduced, Y is also. So Y contains a non-empty open subscheme Y' that is a smooth k-variety. Let $\mathfrak{X}' \subset \mathfrak{X}$ be the image of Y' under ρ. The stack $\mathfrak{X}'$ is then an open substack of $\mathfrak{X}$ that is smooth and of finite type over k. So upon replacing $\mathfrak{X}$ by $\mathfrak{X}'$ we may assume that $\mathfrak{X}$ is of finite type and has a presentation $\rho : Y \to \mathfrak{X}$ where Y is a smooth k-variety. Consider the morphism $\mathrm{Aut}(\rho) \to Y$ induced via the following base change:

$$\begin{array}{ccc} \mathrm{Aut}(\rho) & \longrightarrow & Y \\ \downarrow & \boxed{2} & \downarrow \rho \\ \mathfrak{Aut}\,\mathfrak{X} & \longrightarrow & \mathfrak{X} \end{array}$$

The morphism $\mathrm{Aut}(\rho) \to Y$ is representable, of finite type and separated. Hence $\mathrm{Aut}(\rho)$ is a separated algebraic k-space of finite type. By generic flatness ([12, Théorème 6.9.1]) there exists an open subset $Y' \subset Y$ such that the induced morphism $\mathrm{Aut}(\rho') \to Y'$ is flat. Here ρ' denotes the restriction of ρ to Y'. So upon replacing Y by Y' and $\mathfrak{X}$ by the image $\mathfrak{X}'$ of ρ' we may assume that $\mathrm{Aut}(\rho) \to Y$ is flat. Since $\mathfrak{X}$ is a Deligne-Mumford stack, $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ and hence $\mathrm{Aut}(\rho) \to Y$ is unramified. Therefore, $\mathrm{Aut}(\rho) \to Y$ is étale and separated. By lemma 2.2.5 there exists an open subset $Y' \subset Y$ such that $\mathrm{Aut}(\rho') \to Y'$ is finite étale. So without loss of generality we may assume that $\mathrm{Aut}(\rho) \to Y$ is finite étale. The same holds then for $\mathfrak{Aut}\,\mathfrak{X} \to \mathfrak{X}$.

We have now satisfied conditions i and ii. To satisfy condition iii consider the coarse moduli space X of $\mathfrak{X}$. According to Remark 2.2.1 X is an algebraic k-space of finite type. Since $\mathfrak{Aut}\,\mathfrak{X} \to \mathfrak{X}$ is finite étale, $\mathfrak{X} \to X$ is étale. In particular, X is smooth over k. By [18, II, Proposition 6.7] there is a non-empty open subspace $X' \subset X$ that is a scheme. Restricting X' even further, we may assume that X' is a smooth k-variety. Let $\mathfrak{X}'$ be the base change

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow & \boxed{2} & \downarrow \\ X' & \longrightarrow & X \end{array}$$

Clearly, $\mathfrak{X}'$ is an open substack of $\mathfrak{X}$ that satisfies even condition iii. □

Comparison of the Cohomology with the Coarse Moduli Space

Let k be a separably closed field.

Lemma 2.2.7 *Let $\mathfrak{X}$ be a tractable k-stack with coarse moduli variety X. Then there exists a finite group G and an étale epimorphism $Y \to X$ of k-schemes yielding a 2-cartesian diagram*

$$\begin{array}{ccc} (BG)_Y & \longrightarrow & Y \\ \downarrow & \boxed{2} & \downarrow \\ \mathfrak{X} & \longrightarrow & X. \end{array}$$

Proof. Let $Y \to \mathfrak{X}$ be an étale presentation of $\mathfrak{X}$. Let G' be the pullback of $\mathfrak{Aut}\,\mathfrak{X}$ to Y:

$$\begin{array}{ccc} G' & \longrightarrow & Y \\ \downarrow & \boxed{2} & \downarrow \\ \mathfrak{Aut}\,\mathfrak{X} & \longrightarrow & \mathfrak{X}. \end{array}$$

Then G is a finite étale group scheme over Y. So after passing to an étale cover of Y, we may assume that G' is constant, say equal to G_Y, for a finite group G (considered as a finite étale group scheme over k). Then the following diagram is also 2-cartesian:

$$\begin{array}{ccc} (BG)_Y & \longrightarrow & Y \\ \downarrow & \boxed{2} & \downarrow \\ B\,\mathfrak{Aut}\,\mathfrak{X} & \longrightarrow & \mathfrak{X}. \end{array}$$

This implies the lemma. □

Proposition 2.2.8 *Let $\mathfrak{X}$ be a tractable k-stack with coarse moduli variety X. Then we have*

$$H^*(X, \mathbb{Q}_\ell) = H^*(\mathfrak{X}, \mathbb{Q}_\ell).$$

Proof. Let $\pi : \mathfrak{X} \to X$ denote the structure morphism. It suffices to prove that

$$R^i \pi_* \mathbb{Q}_\ell = \begin{cases} \mathbb{Q}_\ell & \text{for } i = 0 \\ 0 & \text{for } i > 0, \end{cases}$$

where we work with $\mathbb{Q}_\ell$-sheaves. This question is local in X, so by Lemma 2.2.7 we may assume that $\mathfrak{X} = (BG)_X$, for a finite group G. But then $R^i \pi_* \mathbb{Q}_\ell$ is clearly constant, equal to $H^i(BG_{\mathrm{sm}}, \mathbb{Q}_\ell)$. But $H^i(BG_{\mathrm{sm}}, \mathbb{Q}_\ell)$ is nothing but the cohomology of the finite group G with values in the field $\mathbb{Q}_\ell$. This proves the proposition. $\square$

2.3 Gerbes Over a Finite Field

Proposition 2.3.1 *Let $\mathfrak{G}$ be an algebraic $\mathbb{F}_q$-gerbe, étale and of finite type over $\mathbb{F}_q$. Then $\mathfrak{G}$ is neuter.*

Proof. We will use the following cartesian diagram

$$\begin{array}{ccc} \coprod_{i=1}^n S_i & \xrightarrow{\sigma} & \operatorname{Spec} \mathbb{F}_{q^n} \\ {\scriptstyle \pi} \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{F}_{q^n} & \longrightarrow & \operatorname{Spec} \mathbb{F}_q \end{array}$$

where for every $i = 1, \ldots, n$ the scheme S_i is just a copy of $\operatorname{Spec} \mathbb{F}_{q^n}$. The morphism π is defined by $\pi | S_i = \mathrm{id}_{\operatorname{Spec} \mathbb{F}_{q^n}}$ for all $i = 1, \ldots, n$. The morphism σ is defined by $\sigma | S_i = \operatorname{Spec}(\phi^i)$ where $\phi : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ is the Frobenius automorphism defined by $\phi(\alpha) = \alpha^q$ for $\alpha \in \mathbb{F}_{q^n}$.

Since $\mathfrak{G}$ is étale over $\mathbb{F}_q$, there exists an étale presentation $\rho : X_0 \to \mathfrak{G}$, such that X_0 is a finite étale $\mathbb{F}_q$-scheme. Let X_1 be defined by the following 2-cartesian diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{p_2} & X_0 \\ {\scriptstyle p_1} \downarrow & \boxed{2} & \downarrow {\scriptstyle \rho} \\ X_0 & \xrightarrow{\rho} & \mathfrak{G} \end{array}$$

The algebraic space X_1 is then also a finite étale $\mathbb{F}_q$-scheme. Hence there exists an $n > 0$ such that for every point ξ of the underlying topological space of X_1 there exists an $x \in X_1(\mathbb{F}_{q^n})$ factoring through ξ. Now choose an arbitrary $\mathbb{F}_{q^n}$-valued point x_0 of X_0. We will construct a morphism $z : \coprod_{i=1}^n S_i \to X_1$ yielding two commutative diagrams (here written as one diagram).

$$\begin{array}{ccc} \coprod_{i=1}^n S_i & \underset{\pi}{\overset{\sigma}{\rightrightarrows}} & \operatorname{Spec} \mathbb{F}_{q^n} \\ {\scriptstyle z} \downarrow & & \downarrow {\scriptstyle x_0} \\ X_1 & \underset{p_1}{\overset{p_2}{\rightrightarrows}} & X_0 \end{array}$$

Consider the morphism $\delta = (p_1, p_2)$ of $\mathbb{F}_q$-schemes

$$X_1 \xrightarrow{\delta} X_0 \times X_0$$

which comes via base change from

$$\mathfrak{G} \xrightarrow{\Delta} \mathfrak{G} \times \mathfrak{G}.$$

Since $\mathfrak{G}$ is a gerbe, Δ and hence δ is an epimorphism. By our choice of n this implies that

$$X_1(\mathbb{F}_{q^n}) \xrightarrow{\delta} X_0(\mathbb{F}_{q^n}) \times X_0(\mathbb{F}_{q^n})$$

is surjective. So for every $i = 1, \ldots, n$ we may choose $z_i \in X_1(\mathbb{F}_{q^n})$ such that $\delta(z_i) = (x_0, x_0 \circ \operatorname{Spec} \phi^i)$. Defining the restriction of z to S_i to be equal to z_i, we have constructed z as required. Using z it is easy to verify that $\sigma^*(\rho(x_0)) = \pi^*(\rho(x_0))$. Thus $\rho(x_0)$ desceds to the required section of $\mathfrak{G} \to \mathbb{F}_q$. $\square$

Proposition 2.3.2 *Let G be a finite étale group scheme over $\mathbb{F}_q$. Then*

$$\sum_{\xi \in [BG(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut} \xi} = 1.$$

Proof. The groupoid $BG(\mathbb{F}_q)$ is the category of finite étale $\mathbb{F}_q$-schemes X endowed with an action $\sigma : X \times G \to X$ such that the diagram

$$\begin{array}{ccc} X \times G & \xrightarrow{\sigma} & X \\ {\scriptstyle p_1} \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{F}_q \end{array}$$

is cartesian. Let $\mathcal{C}$ be the category defined as follows. Objects of $\mathcal{C}$ are pairs (X, ϕ) where X is a finite set and $\phi : X \to X$ is a permutation. Morphisms in $\mathcal{C}$ are defined by

$$\operatorname{Hom}_{\mathcal{C}}((X, \phi), (Y, \psi)) = \{\alpha : X \to Y \mid \psi\alpha = \alpha\phi\}.$$

Then we have an equivalence of categories

$$\begin{array}{rcl} (\text{finite étale } \mathbb{F}_q\text{-schemes}) & \longrightarrow & \mathcal{C} \\ X & \longmapsto & (\overline{X}, \phi) \\ (X \to Y) & \longmapsto & (\overline{X} \to \overline{Y}). \end{array}$$

Here $\overline{X} = X(\overline{\mathbb{F}}_q)$, where $\overline{\mathbb{F}}_q$ is some fixed algebraic closure of $\mathbb{F}_q$, and $\phi : \overline{X} \to \overline{X}$ is the Frobenius. Using this equivalence of categories we can reinterpret $BG(\mathbb{F}_q)$ as the groupoid $B(\overline{G}, \phi)$ of objects (X, ψ) of $\mathcal{C}$ endowed with an action $\sigma : (X, \psi) \times (\overline{G}, \phi) \to (X, \psi)$ such that the underlying action of $\overline{G}$ on X is simply transitive. Now the finite group $\overline{G}$ acts on its underlying set via

$$\begin{array}{rcl} \overline{G} \times \overline{G} & \longrightarrow & \overline{G} \\ (g, \alpha) & \longmapsto & g\alpha\phi(g)^{-1}. \end{array} \tag{8}$$

Denote the groupoid defined by this action by $\widetilde{G}$. For an object of $\widetilde{G}$, i.e. an element α of $\overline{G}$, we define an object (X_α, ψ_α) of $\mathcal{C}$ as follows. The set X_α is just the set $\overline{G}$ and $\psi_\alpha : X_\alpha \to X_\alpha$ is defined by $\psi_\alpha(g) = \alpha\phi(g)$, for $g \in \overline{G}$. Now we define a right action of $(\overline{G}, \phi)$ on (X_α, ψ_α) via right translations in $\overline{G}$. Let $\alpha, \beta \in \overline{G}$ be two objects of $\widetilde{G}$. For a morphism $\theta : \alpha \to \beta$, i.e. an element $\theta \in \operatorname{Trans}(\alpha, \beta)$ which is just an element of $\overline{G}$ satisfying $\theta\alpha = \beta\phi(\theta)$ we define a morphism $\theta : (X_\alpha, \psi_\alpha) \to (X_\beta, \psi_\beta)$ by left multiplication with θ inside $\overline{G}$. This morphism respects the $(\overline{G}, \phi)$-actions on (X_α, ψ_α) and (X_β, ψ_β). So we have defined a functor

$$\widetilde{G} \longrightarrow B(\overline{G}, \phi).$$

This functor is easily seen to be fully faithful. To prove essential surjectivity, let (X, ψ) be an object of $\mathcal{C}$ with a right $(\overline{G}, \phi)$-action making X a simply transitive $\overline{G}$-set. Choose an element

$x_0 \in X$ and $\alpha \in \overline{G}$ such that $x_0\alpha = \psi(x_0)$. Then $g \mapsto x_0 g$ defines an isomorphism of (X_α, ψ_α) with (X, ψ). Thus our functor is an equivalence of groupoids and we are reduced to proving that

$$\sum_{\xi \in [\widetilde{G}]} \frac{1}{\# \operatorname{Aut} \xi} = 1.$$

Or in other words, that

$$\sum_{\overline{G}\alpha} \frac{1}{\# \operatorname{Stab}_{\overline{G}}(\alpha)} = 1,$$

where the sum is taken over all orbits under the action (8) of $\overline{G}$ on its underlying set. But this is simply the orbit equation. □

Corollary 2.3.3 *Let $\mathfrak{G}$ be an algebraic $\mathbb{F}_q$-gerbe, étale and of finite type over $\mathbb{F}_q$. Then*

$$\sum_{\xi \in [\mathfrak{G}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut} \xi} = 1.$$

Proof. According to Proposition 2.3.1 $\mathfrak{G}$ is neuter. So let E be an object of the category $\mathfrak{G}(\mathbb{F}_q)$. Let G be its sheaf of automorphisms, which can be defined by the following 2-cartesian diagram

$$\begin{array}{ccc} G & \longrightarrow & \mathbb{F}_q \\ \downarrow & \boxed{2} & \downarrow {\scriptstyle (E,E)} \\ \mathfrak{G} & \xrightarrow{\Delta} & \mathfrak{G} \times \mathfrak{G} \end{array}$$

Since Δ is representable and étale of finite type, we see that G is a finite étale group scheme over $\mathbb{F}_q$. Now $\mathfrak{G}$ is isomorphic to BG and the claim follows from Proposition 2.3.2. □

Corollary 2.3.4 (Serre) *Let $\mathfrak{X}$ be an algebraic Deligne-Mumford stack of finite type such that $\mathfrak{Aut}\,\mathfrak{X} \to \mathfrak{X}$ is étale with coarse moduli space X. Then we have*

$$\sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut}(\xi)} = \# X(\mathbb{F}_q).$$

Proof. Let $\pi : \mathfrak{X} \to X$ be the structure morphism. Then π is étale of finite type. Let $x \in X(\mathbb{F}_q)$ be a point and let $\mathfrak{G}_x$ be the $\mathbb{F}_q$-gerbe defined by the pullback

$$\begin{array}{ccc} \mathfrak{G}_x & \longrightarrow & \mathbb{F}_q \\ \downarrow & \boxed{2} & \downarrow {\scriptstyle x} \\ \mathfrak{X} & \xrightarrow{\pi} & X \end{array}$$

Clearly, $\mathfrak{G}_x$ is an étale $\mathbb{F}_q$-gerbe of finite type. We have a 2-cartesian diagram of set theoretic groupoids

$$\begin{array}{ccc} \mathfrak{G}_x(\mathbb{F}_q) & \longrightarrow & \{\varnothing\} \\ \downarrow & \boxed{2} & \downarrow {\scriptstyle x} \\ \mathfrak{X}(\mathbb{F}_q) & \longrightarrow & X(\mathbb{F}_q) \end{array}$$

Hence we have

$$\begin{aligned} \sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut} \xi} &= \sum_{x \in X(\mathbb{F}_q)} \sum_{\xi \in [\mathfrak{G}_x(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut} \xi} \\ &= \# X(\mathbb{F}_q) \end{aligned}$$

which finishes the proof. □

2.4 The Trace Formula for Deligne-Mumford Stacks

Let $\mathbb{F}_q$ be the field with q elements. Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. Let $\mathfrak{X}$ be an algebraic $\mathbb{F}_q$-stack of finite type, $\overline{\mathfrak{X}} = \mathfrak{X} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. We want to examine the action of the arithmetic Frobenius on the cohomology $H^i(\overline{\mathfrak{X}}_{\text{sm}}, \mathbb{Q}_\ell)$. So let $\varphi_q : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q; a \mapsto a^q$ be the Frobenius. We get induced morphisms $\operatorname{Spec} \varphi_q : \operatorname{Spec} \overline{\mathbb{F}}_q \to \operatorname{Spec} \overline{\mathbb{F}}_q$ and $\mathrm{id}_{\mathfrak{X}} \times \operatorname{Spec} \varphi_q : \overline{\mathfrak{X}} \to \overline{\mathfrak{X}}$. Let us abbreviate $\mathrm{id}_{\mathfrak{X}} \times \operatorname{Spec} \varphi_q$ by $\widehat{\varphi}_q : \overline{\mathfrak{X}} \to \overline{\mathfrak{X}}$. The morphism $\widehat{\varphi}_q$ induces a morphism of topoi, which we again denote by $\widehat{\varphi}_q : \overline{\mathfrak{X}}_{\text{sm}} \to \overline{\mathfrak{X}}_{\text{sm}}$. Let F be a sheaf on $\mathfrak{X}_{\text{sm}}$. By functoriality of cohomology $\widehat{\varphi}_q$ induces a homomorphism $\widehat{\varphi}_q^* : H^i(\overline{\mathfrak{X}}_{\text{sm}}, F) \to H^i(\overline{\mathfrak{X}}_{\text{sm}}, \widehat{\varphi}_q^* F)$. Since F is defined over $\mathfrak{X}_{\text{sm}}$ we have $\widehat{\varphi}_q^* F = F$ and an induced homomorphism $\widehat{\varphi}_q^* : H^i(\overline{\mathfrak{X}}_{\text{sm}}, F) \to H^i(\overline{\mathfrak{X}}_{\text{sm}}, F)$.

Definition 2.4.1 The homomorphism thus constructed is called the *arithmetic Frobenius* acting on $H^i(\overline{\mathfrak{X}}_{\text{sm}}, F)$ and is denoted by

$$\Phi_q : H^i(\overline{\mathfrak{X}}_{\text{sm}}, F) \to H^i(\overline{\mathfrak{X}}_{\text{sm}}, F).$$

Taking $F = \mathbb{Z}/\ell^{n+1}\mathbb{Z}$, passing to the limit and tensoring with $\mathbb{Q}_\ell$ we obtain $\Phi_q : H^i(\overline{\mathfrak{X}}_{\text{sm}}, \mathbb{Q}_\ell) \to H^i(\overline{\mathfrak{X}}_{\text{sm}}, \mathbb{Q}_\ell)$. Taking $F = \mu_{\ell^{n+1}}$ and $\mathfrak{X} = \mathbb{F}_q$, we get $\Phi_q : \mu_{\ell^{n+1}} \to \mu_{\ell^{n+1}}$ which is given by $\Phi_q(\zeta) = \zeta^q$. Passing to the limit and tensoring with $\mathbb{Q}_\ell$, we see that Φ_q acts on $\mathbb{Q}_\ell(1)$ by multiplication with q. On $\mathbb{Q}_\ell(c)$ the endomorphism Φ_q acts by multiplication with q^c.

Lemma 2.4.2 *Let $\mathfrak{X}$ be an algebraic stack over the finite field $\mathbb{F}_q$. Then for any integer c we have*

$$H^p(\overline{\mathfrak{X}}_{\text{sm}}, \mathbb{Q}_\ell(c)) = H^p(\overline{\mathfrak{X}}_{\text{sm}}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(c)$$

as $\mathbb{Q}_\ell$-vector spaces with Φ_q-action. In particular

$$\operatorname{tr} \Phi_q | H^p(\overline{\mathfrak{X}}_{\text{sm}}, \mathbb{Q}_\ell(c)) = q^c \operatorname{tr} \Phi_q | H^p(\overline{\mathfrak{X}}_{\text{sm}}, \mathbb{Q}_\ell).$$

Proof. Clear. □

Proposition 2.4.3 *Let X be a smooth variety over the finite field $\mathbb{F}_q$. Then we have*

$$q^{\dim X} \operatorname{tr} \Phi_q | H^*(\overline{X}_{\text{sm}}, \mathbb{Q}_\ell) = \# X(\mathbb{F}_q).$$

Proof. Let n be the dimension of X. The usual trace formula for the geometric Frobenius F_q reads as follows.

$$\sum_{i=0}^{2n} (-1)^i \operatorname{tr} F_q | H^i_c(\overline{X}_{\text{ét}}, \mathbb{Q}_\ell) = \# X(\mathbb{F}_q). \tag{9}$$

Consider the cup product pairing

$$\begin{array}{rcl} H^i_c(\overline{X}_{\text{ét}}, \mathbb{Q}_\ell) \times H^{2n-i}(\overline{X}_{\text{ét}}, \mathbb{Q}_\ell(n)) & \longrightarrow & \mathbb{Q}_\ell \\ (\xi, \eta) & \longmapsto & (\xi, \eta) = \operatorname{tr}(\xi \cup \eta) \end{array} \tag{10}$$

which is perfect by Poincaré duality. The cup product satisfies $F_q(\xi) \cup F_q(\eta) = F_q(\xi \cup \eta)$. Hence we have $(F_q \xi, F_q \eta) = (\xi, \eta)$. The arithmetic Frobenius Φ_q acts as the inverse of F_q on $H^j(\overline{X}_{\text{ét}}, \mathbb{Q}_\ell(m))$ for any j, m. Hence Φ_q is the transpose of F_q with respect to the pairing (10). Since the trace of the transpose of a matrix is the same as the trace of the original matrix, we get from (9)

$$\sum_{i=0}^{2n} (-1)^i \operatorname{tr} \Phi_q | H^{2n-i}(\overline{X}_{\text{ét}}, \mathbb{Q}_\ell(n)) = \# X(\mathbb{F}_q)$$

which is equivalent to

$$q^n \sum_{i=0}^{2n} (-1)^i \operatorname{tr} \Phi_q | H^i(\overline{X}_{\text{ét}}, \mathbb{Q}_\ell) = \# X(\mathbb{F}_q).$$

Since smooth and étale cohomology coincide for X, the proposition follows. $\square$

Lemma 2.4.4 *Let $\mathfrak{X}$ be a tractable $\mathbb{F}_q$-stack. Then we have*

$$q^{\dim \mathfrak{X}} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{X}}_{\text{sm}}, \mathbb{Q}_\ell) = \sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut}(\xi)}.$$

Proof. By Proposition 2.2.8 we have for any $p \geq 0$

$$H^p(\overline{X}_{\text{sm}}, \mathbb{Q}_\ell) = H^p(\overline{\mathfrak{X}}_{\text{sm}}, \mathbb{Q}_\ell),$$

where X is the coarse moduli space of $\mathfrak{X}$. Since X is a smooth $\mathbb{F}_q$-variety we have by Proposition 2.4.3 the following:

$$q^{\dim X} \operatorname{tr} \Phi_q | H^*(\overline{X}_{\text{sm}}, \mathbb{Q}_\ell) = \# X(\mathbb{F}_q).$$

By Corollary 2.3.4 we have

$$\sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut}(\xi)} = \# X(\mathbb{F}_q).$$

So together this implies the trace formula for $\mathfrak{X}$. $\square$

Theorem 2.4.5 *Let $\mathfrak{X}$ be a smooth Deligne-Mumford stack of finite type and constant dimension over the finite field $\mathbb{F}_q$. Then we have*

$$q^{\dim \mathfrak{X}} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{X}}_{\text{sm}}, \mathbb{Q}_\ell) = \sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut}(\xi)}.$$

Proof. Let $|\mathfrak{X}|$ be the set of points of $\mathfrak{X}$. The Zariski topology on $|\mathfrak{X}|$ is defined by calling $U \subset |\mathfrak{X}|$ open if there exists an open substack $\mathfrak{U} \subset \mathfrak{X}$ such that $U = |\mathfrak{U}|$. We then have a bijection between open subsets of $|\mathfrak{X}|$ and open substacks of $\mathfrak{X}$. The set $|\mathfrak{X}|$ with the Zariski topology is a noetherian topological space, because $\mathfrak{X}$ is of finite type over a field. Let

$$\begin{aligned} \mathfrak{M} &= \{U \subset |\mathfrak{X}| \mid \text{The theorem holds} \\ &\qquad \text{for the open substack } \mathfrak{U} \text{ corresponding to } U\}. \end{aligned}$$

Then $\mathfrak{M}$ has a maximal element, say U. Let $\mathfrak{U}$ be the corresponding open substack of $\mathfrak{X}$. We will prove that $\mathfrak{U} = \mathfrak{X}$.

So assume that $\mathfrak{U} \neq \mathfrak{X}$, and let $\mathfrak{Z}$ be the complement of $\mathfrak{U}$ endowed with the reduced substack structure. By Proposition 2.2.6 there exists a non-empty open substack $\mathfrak{Z}'$ of $\mathfrak{Z}$ which is tractable. Let $\mathfrak{X}' = \mathfrak{U} \cup \mathfrak{Z}'$, which is an open substack of $\mathfrak{X}$. If we can prove that the theorem holds for $\mathfrak{X}'$ then we have a contradiction to the maximality of $\mathfrak{U}$, thus proving $\mathfrak{U} = \mathfrak{X}$ and our theorem. So we may assume that $\mathfrak{X} = \mathfrak{X}'$ and hence $\mathfrak{Z} = \mathfrak{Z}'$.

So we have a smooth pair $i : \mathfrak{Z} \to \mathfrak{X}$ of algebraic $\mathbb{F}_q$-stacks of codimension $c = \dim \mathfrak{X} - \dim \mathfrak{Z}$ such that $\mathfrak{Z}$ is tractable and the theorem holds for $\mathfrak{U} = \mathfrak{X} - \mathfrak{Z}$. By Lemma 2.4.4 the theorem also holds for $\mathfrak{Z}$. Now we consider the Gysin sequence (Corollary 2.1.3):

$$\ldots \longrightarrow H^{p-2c}(\overline{\mathfrak{Z}}_{\text{sm}}, \mathbb{Q}_\ell(-c)) \longrightarrow H^p(\overline{\mathfrak{X}}_{\text{sm}}, \mathbb{Q}_\ell) \longrightarrow H^p(\overline{\mathfrak{U}}_{\text{sm}}, \mathbb{Q}_\ell) \longrightarrow \ldots$$

It follows that

$$
\begin{aligned}
& q^{\dim \mathfrak{X}} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{X}}_{\mathrm{sm}}, \mathbb{Q}_\ell) \\
= \; & q^{\dim \mathfrak{Z}+c} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{Z}}_{\mathrm{sm}}, \mathbb{Q}_\ell(-c)) + q^{\dim \mathfrak{U}} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{U}}_{\mathrm{sm}}, \mathbb{Q}_\ell) \\
= \; & q^{\dim \mathfrak{Z}} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{Z}}_{\mathrm{sm}}, \mathbb{Q}_\ell) + q^{\dim \mathfrak{U}} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{U}}_{\mathrm{sm}}, \mathbb{Q}_\ell) \\
= \; & \sum_{\xi \in [\mathfrak{Z}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut}(\xi)} + \sum_{\xi \in [\mathfrak{U}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut}(\xi)} \\
= \; & \sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut}(\xi)}
\end{aligned}
$$

Done. □

3 Artin Stacks

Introduction

In this section we prove the Lefschetz trace formula for a smooth algebraic $\mathbb{F}_q$-stack of finite type $\mathfrak{X}$, provided there exists a Deligne-Mumford stack X and a smooth affine algebraic group G over $\mathbb{F}_q$, acting on X, such that $\mathfrak{X} = [X/G]$. Letting BG be the classifying stack of G, the principal G-bundle $X \to \mathfrak{X}$ induces a morphism $\mathfrak{X} \to BG$. This morphism is the fibering with fiber X associated to the universal principal G-bundle over BG. The main ingredient in the proof of the trace formula is the Leray spectral sequence of this fibering $\mathfrak{X} \to BG$.

The general form of the Leray spectral sequence for morphisms of algebraic stacks is Theorem 3.2.5. It rests heavily on the finiteness theorem (Theorem 3.1.6) proved in Section 3.1. In Section 3.3 we examine the special case that the higher direct images of our morphism of stacks are constant. The result is Theorem 3.3.9. One case in which the higher direct images are constant is that of a fibration with connected structure group. This result is given in Theorem 3.3.12, which is the final form of the Leray spectral sequence we will use.

To prove the trace formula for $\mathfrak{X} = [X/G]$, we may easily reduce to the case that $G = GL_n$, by choosing an embedding $G \to GL_n$ and noting that $[X/G] = [X \times_G GL_n/GL_n]$.

So our proof relies on the spectral sequence of the fribering $\mathfrak{X} \to BGL_n$, which motivates the study of BGL_n in Section 3.4. Even though the dimension of BGL_n is $-n^2$, we call BGL_n the 'infinite dimensional Grassmannian' because of the universal mapping property of BGL_n and the analogue with homotopy theory.

The proof of the trace formula is then carried out in Section 3.5. We have to worry about convergence of the trace in this case. Thus we choose an embedding $\mathbb{Q}_\ell \subset \mathbb{C}$ and consider the trace of the arithmetic Frobenius Φ_q on $H^*(\overline{\mathfrak{X}}_{\mathrm{sm}}, \mathbb{C})$. The result is Theorem 3.5.7.

3.1 A Finiteness Theorem

Throughout this discussion we fix an integer $N > 0$. If A is a noetherian ring, we say that a sheaf of A-modules F on a topos X is *noetherian* if it is a noetherian object in the category of sheaves of A-modules on X. If X is a noetherian scheme, then a sheaf of A-modules on $X_{\text{ét}}$ is noetherian if and only if it is constructible.

Definition 3.1.1 A morphism of topoi $\pi : Y \to X$ is said to *satisfy the finiteness theorem with respect to* N if for any noetherian ring A such that $NA = 0$ and any noetherian sheaf of A-modules F on Y, $R^q \pi_* F$ is a noetherian sheaf of A-modules for all $q \geq 0$.

A topos X satisfies the finiteness theorem with respect to N, if the morphism $X \to pt$ does, where pt is the punctual topos.

Proposition 3.1.2 *If $\pi : Y \to X$ and $\rho : X \to Z$ are morphisms of topoi satisfying the finiteness theorem with respect to N, then $\rho \circ \pi : Y \to Z$ satisfies the finiteness theorem with respect to N also.*

Proof. Let A be a ring annihilated by N. For a noetherian sheaf of A-modules F on Y consider the Leray spectral sequence

$$R^p \rho_* R^q \pi_* F \Rightarrow R^{p+q}(\rho \circ \pi)_* F.$$

The claim follows from the following fact: If

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence in an abelian category (in this case the category of sheaves of A-modules on Z), then F is noetherian if and only if F' and F'' are noetherian. $\square$

Proposition 3.1.3 *Let $\pi : Y \to X$ be a morphism of topoi, where X and Y are quasi-separated and noetherian. Then the property 'π satisfies the finiteness theorem with respect to N' is local on the base X.*

Proof. Let A be a noetherian ring annihilated by N. Then the proposition follows immediately from the fact that the property of being noetherian is local, for a sheaf of A-modules on a quasi-separated and noetherian topos. $\square$

Lemma 3.1.4 *Let $\pi : Y \to X$ be a morphism of topoi, and let Y_0 be an object of Y, covering Y. Assume that for all $p \geq 0$ the induced morphism of topoi $\pi_p : Y_p \to X$ satisfies the finiteness theorem with respect to N, where $Y_p = \underbrace{Y_0 \times_Y \cdots \times_Y Y_0}_{p+1}$. Then $\pi : Y \to X$ satisfies the finiteness theorem with respect to N also.*

Proof. Let A be a noetherian ring annihilated by N. Let F be a noetherian sheaf of A-modules on Y. For any object U of X, $Y_0 \times_Y \pi^* U$ is a one-element cover of $\pi^* U$. Consider the E_1 spectral sequence of this covering:

$$E_1^{p,q} = H^q(Y_p \times_Y \pi^* U, F) \Rightarrow H^{p+q}(\pi^* U, F).$$

If we let U vary over all objects of X, then we can consider this as a spectral sequence in the category of presheaves on X. Applying the associated sheaf functor we get a spectral sequence

$$E_1^{p,q} = R^q {\pi_p}_* (F|Y_p) \Rightarrow R^{p+q} \pi_* F.$$

Using the fact that the restriction of a noetherian sheaf of A-modules is again noetherian, the claim follows. $\square$

Lemma 3.1.5 *Let $\pi : Y \to X$ be a morphism of schemes. Then $\pi_{\text{ét}} : Y_{\text{ét}} \to X_{\text{ét}}$ satisfies the finiteness theorem with respect to N if and only if $\pi_{\text{sm}} : Y_{\text{sm}} \to X_{\text{sm}}$ does.*

Proof. There is a natural morphism of topoi

$$j : X_{\text{sm}} \to X_{\text{ét}}.$$

j_* is the 'restriction' of sheaves from the smooth to the étale site, j^* extends the embedding of the étale site into the smooth site. j_* is exact (in fact j_* even has a right adjoint $j^!$, because j^* is not only continuous, but also cocontinuous).

Let A be a noetherian ring such that $NA = 0$. Since $j_* j^* = \mathrm{id}$ and j_* is faithful, we have for any sheaf of A-modules F on X_{sm}: F is noetherian if and only if $j_* F$ is noetherian. This implies for any sheaf of A-modules G on $X_{\text{ét}}$: G is noetherian if and only if $j^* G$ is noetherian. The same is of course also true for $j : Y_{\mathrm{sm}} \to Y_{\text{ét}}$.

Let F be a sheaf of A-modules on Y_{sm}. For any scheme U, étale over X, we have

$$H^q((\pi^* U)_{\text{ét}}, j_* F) = H^q((\pi^* U)_{\mathrm{sm}}, F).$$

Hence $R^q \pi_{\text{ét}*}(j_* F) = j_*(R^q \pi_{\mathrm{sm}*} F)$, using the fact that j_* commutes with the associated sheaf functor, which follows from the cocontinuity of j^*. So we have

$$R^q \pi_{\text{ét}*} \circ j_* = j_* \circ R^q \pi_{\mathrm{sm}*},$$

which implies also

$$R^q \pi_{\text{ét}*} = j_* \circ R^q \pi_{\mathrm{sm}*} \circ j^*.$$

It follows immediately that $\pi_{\text{ét}}$ satisfies the finiteness theorem with respect to N of π_{sm} does.

For the converse, let F be a noetherian sheaf of A-modules on Y_{sm}. To check whether $R^q \pi_{\mathrm{sm}*} F$ is noetherian, it suffices to check whether $j_* R^q \pi_{\mathrm{sm}*} F$ or equivalently $R^q \pi_{\text{ét}*} j_* F$ is noetherian, which is now clear. □

Theorem 3.1.6 *Let k be a field and let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of algebraic k-stacks of finite type. Then $\pi : \mathfrak{Y}_{\mathrm{sm}} \to \mathfrak{X}_{\mathrm{sm}}$ satisfies the finiteness theorem with respect to N, if the characteristic of k does not divide N.*

Proof. First note that if $\mathfrak{X}$ is an algebraic k-stack of finite type, then $\mathfrak{X}_{\mathrm{sm}}$ is a quasi-separated noetherian topos. So by proposition 3.1.3 we reduce to the case that $X = \mathfrak{X}$ is a scheme of finite type over k. Now choose a presentation $Y \to \mathfrak{Y}$ such that Y is a k-scheme of finite type. Then $Y_p = \underbrace{Y \times_{\mathfrak{Y}} \cdots \times_{\mathfrak{Y}} Y}_{p+1}$ is of finite type for all $p \geq 0$. By lemma 3.1.4 it suffices to show that $(Y_p)_{\mathrm{sm}} \to X_{\mathrm{sm}}$ satisfies the finiteness theorem with respect to N for all $p \geq 0$. From a finiteness theorem of Deligne [4, th. finitude], we know that $(Y_p)_{\text{ét}} \to X_{\text{ét}}$ satisfies the finiteness theorem with respect to N. Our theorem now follows from lemma 3.1.5. □

3.2 The ℓ-adic Leray Spectral sequence

In this section we review the construction of the higher direct images for noetherian ℓ-adic sheaves, and morphisms of topoi satisfying the finiteness theorem. As an application, we prove the existence of the Leray spectral sequence for morphisms of algebraic stacks of finite type.

First recall the definition of ℓ-adic and AR-ℓ-adic sheaves [13, Exp. V]:

Definition 3.2.1 Let E be a topos and ℓ a prime number. A projective system $F = (F_n)_{n \geq 0}$ of abelian sheaves on E is called *AR-null* if there exists an integer $r \geq 0$, such that $F_{n+r} \to F_n$ is the zero map for all $n \geq 0$.

F is called an *ℓ-adic sheaf* (or a *$\mathbb{Z}_\ell$-sheaf*) if the following two conditions are satisfied:

i. For all $n \geq 0$: $\ell^{n+1} F_n = 0$.

ii. For all $m \geq n$ the canonical map $F_m / \ell^{n+1} F_m \to F_n$ is an isomorphism.

F is called an *AR-ℓ-adic sheaf* if condition (i) above is satisfied, and there exists an ℓ-adic sheaf G and a morphism $G \to F$ that is an *AR-isomorphism*, i.e. a morphism whose kernel and cokernel are AR-null.

An ℓ-adic (resp. AR-ℓ-adic) sheaf F is called *noetherian* if for all $n \geq 0$ F_n is a noetherian abelian sheaf on E. The category of noetherian $\mathbb{Z}_\ell$-sheaves on E is denoted by $\mathbb{Z}_\ell$-fn(E). If $E = X_{\text{ét}}$ is the étale topos of a scheme X, we also say *constructible* instead of noetherian.

The following result is essentially proved in [13, Exp. V].

Proposition 3.2.2 *Let $\pi : Y \to X$ be a morphism of topoi, satisfying the finiteness theorem with respect to ℓ^n, for all $n > 0$. Let F be a noetherian ℓ-adic sheaf on Y. Then, for any q, $(R^q\pi_*F_n)_{n\geq 0}$ is a noetherian AR-ℓ-adic sheaf. In particular, there exists a unique ℓ-adic sheaf G^q on X with an AR-isomorphism $G^q \to (R^q\pi_*F_n)_{n\geq 0}$.*

Definition 3.2.3 Let $\pi : Y \to X$ and F be as in the proposition. Then we call the ℓ-adic sheaf G^q from above the q^{th} *higher direct image of F under π* and denote it by $R^q\pi_*F$. If $X = pt$ is the punctual topos, then $R^q\pi_*F$ is denoted $H^q(Y, F)$.

Note that in general, $(R^q\pi_*F)_n \neq R^q\pi_*(F_n)$ and $H^q(Y, F)_n \neq H^q(Y, F_n)$. We have defined an exact δ-functor

$$R^\bullet\pi_* : \mathbb{Z}_\ell\text{-fn}(Y) \to \mathbb{Z}_\ell\text{-fn}(X).$$

Proposition 3.2.4 *Let $\pi : Y \to X$ and $\rho : X \to Z$ be morphisms of topoi satisfying the finiteness theorem with respect to ℓ^n, for all $n > 0$. Then for any noetherian ℓ-adic sheaf F on Y there is a Leray spectral sequence (functorial in F)*

$$R^p\rho_*R^q\pi_*F \Rightarrow R^{p+q}(\rho \circ \pi)_*F.$$

Proof. The proof is completely analogous to the proof of proposition 2.2.4. in [13, Exp. VI]. □

Theorem 3.2.5 *Let k be a separably closed field, $\ell \neq \operatorname{char} k$. Then for any morphism $\pi : \mathfrak{Y} \to \mathfrak{X}$ of algebraic k-stacks of finite type and any noetherian $\mathbb{Z}_\ell$-sheaf F on $\mathfrak{Y}_{\text{sm}}$ we have a spectral sequence*

$$H^p(\mathfrak{X}_{\text{sm}}, R^q\pi_*F) \Rightarrow H^{p+q}(\mathfrak{Y}_{\text{sm}}, F)$$

(in the category of projective systems of finite abelian groups).

Proof. By theorem 3.1.6 $\pi : \mathfrak{Y}_{\text{sm}} \to \mathfrak{X}_{\text{sm}}$ and $\mathfrak{X}_{\text{sm}} \to \operatorname{Spec} k$ satisfy the finiteness theorem with respect to ℓ^n, for all $n > 0$. So by proposition 3.2.4 we have a spectral sequence

$$H^p(\mathfrak{X}_{\text{sm}}, R^q\pi_*F) \Rightarrow H^{p+q}(\mathfrak{Y}_{\text{sm}}, F),$$

using the fact that k is separably closed, when replacing R^q by H^q. □

3.3 The Spectral Sequence of a Fibration of Algebraic Stacks

Constant Higher Direct Images

To prove the next result, we have to pass to $\mathbb{Q}_\ell$-sheaves. Recall that in the category of ℓ-adic sheaves morphisms are simply morphisms of projective systems of abelian sheaves. The category of noetherian ℓ-adic sheaves is a $\mathbb{Z}_\ell$-category, which essentially means that all Hom-spaces are $\mathbb{Z}_\ell$-modules and all Hom-pairings are $\mathbb{Z}_\ell$-bilinear. This justifies the term '$\mathbb{Z}_\ell$-sheaves'.

Definition 3.3.1 Let E be a topos. The *category of noetherian $\mathbb{Q}_\ell$-sheaves on E* is the category whose objects are noetherian ℓ-adic sheaves (i.e. $\mathbb{Z}_\ell$-sheaves) and whose morphisms are defined by

$$\operatorname{Hom}_{\mathbb{Q}_\ell}(F, G) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \operatorname{Hom}_{\mathbb{Z}_\ell}(F, G).$$

We denote this category by $\mathbb{Q}_\ell$-fn(E).

Note that noetherian $\mathbb{Z}_\ell$-sheaves on the punctual topos are just ℓ-adic systems of finite abelian groups, where the definition of 'ℓ-adic system' is as in definition 3.2.1.

Definition 3.3.2 If M is an ℓ-adic system of finite abelian groups, let

$$(M_{\text{tors}})_n = \{x \in M_n \mid \exists r \geq 0\ \forall m \geq n\ \exists y \in M_m : y \mapsto x \text{ and } \ell^r y = 0\}.$$

The ℓ-adic system $M_{\text{tors}} = ((M_{\text{tors}})_n)_{n \geq 0}$ is called the *torsion-subsystem* of M. If $M = M_{\text{tors}}$, M is called *torsion* . If $M_{\text{tors}} = 0$, M is called *torsion-free.*

The following are some basic facts about ℓ-adic systems of finite abelian groups:

Proposition 3.3.3 *Let M be an ℓ-adic system of finite abelian groups.*

i. If M is torsion-free it is free of finite rank.

ii. M/M_{tors} is torsion-free.

iii. If M is torsion, then there exists an $r \geq 0$ such that $\ell^r M = 0$.

iv. there exist integers $r_1, \ldots, r_n$ such that

$$M \cong \mathbb{Z}/\ell^{r_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\ell^{r_n}\mathbb{Z} \oplus \mathbb{Z}_\ell \oplus \cdots \oplus \mathbb{Z}_\ell,$$

where we have written $\mathbb{Z}_\ell$ for the canonical ℓ-adic system given by $(\mathbb{Z}_\ell)_n = \mathbb{Z}/\ell^{n+1}\mathbb{Z}$ for all $n \geq 0$.

Proof. Straightforward. □

Proposition 3.3.4 *Let M be an ℓ-adic system of finite abelian groups. Let E be a topos satisfying the finiteness theorem with respect to ℓ^n, for all $n > 0$. Consider M as a (constant) $\mathbb{Z}_\ell$-sheaf on E. Then for any $p \geq 0$ the canonical homomorphism*

$$H^p(E, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} M \longrightarrow H^p(E, M) \tag{11}$$

is an isomorphism in the category $\mathbb{Q}_\ell$-fn(pt) of 'finite(-ly generated) $\mathbb{Q}_\ell$-abelian groups'. In (11) we abuse notation in a similar fashion as in proposition 3.3.3 with regard to $\mathbb{Z}_\ell$.

Proof. For any $n \geq 0$ we have a canonical homomorphism

$$\begin{array}{rcl} H^p(E, \mathbb{Z}/\ell^{n+1}\mathbb{Z}) \otimes M_n & \longrightarrow & H^p(E, M_n) \\ \xi \otimes x & \longmapsto & H^p(E, x)(\xi) \end{array}$$

where we consider $x \in M_n$ as a homomorphism $x : \mathbb{Z}/\ell^{n+1}\mathbb{Z} \to M_n$. Putting these homomorphisms together, gives a homomorphism of projective systems

$$\left(H^p(E, \mathbb{Z}/\ell^{n+1}\mathbb{Z}) \otimes M_n\right)_{n \geq 0} \longrightarrow (H^p(E, M_n))_{n \geq 0}.$$

Now we have ℓ-adic systems $H^p(E, \mathbb{Z}_\ell) \otimes M$ and $H^p(E, M)$ together with natural AR-isomorphisms

$$H^p(E, \mathbb{Z}_\ell) \otimes M \xrightarrow{\sim} \left(H^p(E, \mathbb{Z}/\ell^{n+1}\mathbb{Z}) \otimes M_n\right)_{n \geq 0}$$

and

$$H^p(E, M) \xrightarrow{\sim} (H^p(E, M_n))_{n \geq 0}.$$

Composing, we get an AR-morphism [13, Exp. V]

$$H^p(E, \mathbb{Z}_\ell) \otimes M \to H^p(E, M),$$

which is actually a homomorphism of projective systems, since $H^p(E,\mathbb{Z}_\ell)\otimes M$ and $H^p(E,M)$ are ℓ-adic. So we have established the existence of the homomorphism (11).

Now we have a short exact sequence of ℓ-adic systems of finite abelian groups

$$0 \to M_{\text{tors}} \to M \to M/M_{\text{tors}} \to 0 \tag{12}$$

where M/M_{tors} is free of finite rank, say r. Hence (12) splits and $M \cong M_{\text{tors}} \oplus (\mathbb{Z}_\ell)^r$. So we have

$$H^p(E,\mathbb{Z}_\ell)\otimes M \cong H^p(E,\mathbb{Z}_\ell)\otimes M_{\text{tors}} \oplus H^p(E,\mathbb{Z}_\ell)^r$$

and

$$H^p(E,M) \cong H^p(E, M_{\text{tors}} \oplus (\mathbb{Z}_\ell)^r).$$

By similar arguments as those proving the existence of (11), we get

$$H^p(E, M\oplus N) = H^p(E,M)\oplus H^p(E,N)$$

for any two ℓ-adic sheaves M,N on E. Using this, our homomorphism is given by

$$H^p(E,\mathbb{Z}_\ell)\otimes M_{\text{tors}} \oplus H^p(E,\mathbb{Z}_\ell)^r \longrightarrow H^p(E,M_{\text{tors}})\oplus H^p(E,\mathbb{Z}_\ell)^r.$$

So it suffices to show that the canonical homomorphism

$$H^p(E,\mathbb{Z}_\ell)\otimes M_{\text{tors}} \longrightarrow H^p(E,M_{\text{tors}})$$

is killed by ℓ^s for some integer $s \geq 0$. But this is clear, because M_{tors} is killed by ℓ^s for some $s \geq 0$. □

Corollary 3.3.5 *Let X be a topos having sufficiently many points and satisfying the finiteness theorem with respect to ℓ^n, for all $n > 0$. Let $\pi : Y \to X$ be a morphism of topoi satisfying the finiteness theorem with respect to ℓ^n, for all $n > 0$. Let F be a noetherian $\mathbb{Z}_\ell$-sheaf on Y such that $(R^q\pi_*F_n)_{n\geq 0}$ is a* constant *projective system of abelian sheaves on X. Then there is a spectral sequence*

$$H^p(X,\mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell} R^q\pi_*F \Rightarrow H^{p+q}(Y,F) \tag{13}$$

*in the category $\mathbb{Q}_\ell$-*fn*(pt) of finite $\mathbb{Q}_\ell$-abelian groups.*

Proof. If X is the empty topos, there is nothing to prove. So we may assume that X has points. Let $\xi : pt \to X$ be one. $(R^q\pi_*F_n)_{n\geq 0}$ is a noetherian AR-ℓ-adic sheaf on X. So there exists a noetherian ℓ-adic sheaf H, and an AR-isomorphism $H \to (R^q\pi_*F_n)_n \geq 0$. Pulling back to pt via the exact functor ξ^* we get an AR-isomorphism $\xi^*H \to \xi^*(R^q\pi_*F_n)_{n\geq 0}$. Since $pt \xrightarrow{\xi} X \to pt$ is isomorphic to the identity, we have proved that there exists an AR-ℓ-adic system of finite abelian groups $N = (N_n)_{n\geq 0}$, such that $\widetilde{N} \cong (R^q\pi_*F_n)_{n\geq 0}$, where we denote by $\widetilde{N}$ the sheaf on X associated to N. Since N is AR-ℓ-adic, there exists en ℓ-adic system of finite abelian groups M together with an AR-isomorphism $M \to N$. Then $\widetilde{M} \to \widetilde{N}$ is an AR-isomorphism of projective systems of abelian sheaves on X. So $\widetilde{M} \cong R^q\pi_*F$, and we have proved that $R^q\pi_*F$ is constant. So we can apply proposition 3.3.4 to conclude that

$$H^p(X,\mathbb{Z}_\ell)\otimes R^q\pi_*F \cong H^p(X, R^q\pi_*F)$$

in the category $\mathbb{Q}_\ell$-fn(pt). From proposition 3.2.4 we now get the spectral sequence

$$H^p(X,\mathbb{Z}_\ell)\otimes R^q\pi_*F \Rightarrow H^{p+q}(Y,F)$$

in the category $\mathbb{Q}_\ell$-fn(pt). □

Definition 3.3.6 For a projective system $M = (M_n)_{n\geq 0}$ of abelian groups such that $\ell^{n+1}M_n = 0$ for all n, we define

$$M \otimes \mathbb{Q}_\ell = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_n M_n .$$

For a topos X and a projective system F of abelian sheaves on X such that $\ell^{n+1}F_n = 0$ for all n, we define

$$H^p(X, F \otimes \mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_n H^p(X, F_n).$$

Note 3.3.7 If X is a topos satisfying the finiteness theorem with respect to ℓ and F a noetherian $\mathbb{Z}_\ell$-sheaf on X, then

$$H^p(X, F \otimes \mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_n H^p(X, F)_n .$$

This is because $(H^p(X,F)_n)_{n\geq 0}$ and $(H^p(X,F_n))_{n\geq 0}$ have the same limit, being AR-isomorphic.

Corollary 3.3.8 *Under the same hypotheses as in corollary 3.3.5 we have a spectral sequence of finite dimensional $\mathbb{Q}_\ell$-vector spaces*

$$H^p(X, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} (R^q\pi_* F \otimes \mathbb{Q}_\ell) \Rightarrow H^{p+q}(Y, F \otimes \mathbb{Q}_\ell).$$

Proof. Simply take the limit of (13) and tensor with $\mathbb{Q}_\ell$. □

Theorem 3.3.9 *Let k be a separably closed field, $\ell \neq \operatorname{char} k$. Let $\pi : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of algebraic k-stacks of finite type and F a noetherian $\mathbb{Z}_\ell$-sheaf on $\mathfrak{Y}_{\mathrm{sm}}$, such that $R^q\pi_{\mathrm{sm}*}F_n$ is constant, for all $n \geq 0$. Then we have a spectral sequence of finite dimensional $\mathbb{Q}_\ell$-vector spaces*

$$H^p(\mathfrak{X}_{\mathrm{sm}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} (R^q\pi_{\mathrm{sm}*}F \otimes \mathbb{Q}_\ell) \Rightarrow H^{p+q}(\mathfrak{Y}_{\mathrm{sm}}, F \otimes \mathbb{Q}_\ell).$$

Proof. $\mathfrak{X}_{\mathrm{sm}}$ has sufficiently many points, because it is quasi-separated and noetherian. It satisfies the finiteness theorem with respect to ℓ^n for all $n > 0$ by theorem 3.1.6, and because k is separably closed. $\pi_{\mathrm{sm}} : \mathfrak{Y}_{\mathrm{sm}} \to \mathfrak{X}_{\mathrm{sm}}$ satisfies the finiteness theorem with respect to ℓ^n for all $n > 0$, also by theorem 3.1.6. So corollary 3.3.8 applies. □

Fibrations With Connected Structure Group

Lemma 3.3.10 *let E be a topos. Let $X \in \operatorname{ob} E$. Let G be a group object in E, acting on X on the left. Let P be a (right) G-torsor and let $Y = P \times_G X$. Let $j_X : E_{/X} \to E$ and $j_Y : E_{/Y} \to E$ be the localization morphisms. Then for any abelian object A of E we have*

$$R^q j_{Y*}(A|Y) \cong P \times_G R^q j_{X*}(A|X)$$

for all $q \geq 0$.

Proof. $R^q j_{Y*}(A|Y)$ is the sheaf associated to the presheaf

$$U \mapsto H^q(Y \times U, A).$$

$P \times_G R^q j_{X*}(A|X)$ is the sheaf associated to the presheaf

$$U \mapsto P(U) \times_{G(U)} H^q(X \times U, A),$$

by exactness of a, the functor that maps a presheaf to the associated sheaf. Note that $g \in G(U)$ induces

$$\begin{aligned} \tilde{g} : X \times U &\longrightarrow X \times U \\ (x, u) &\longmapsto (g(u)x, u). \end{aligned}$$

The operation of g on $H^q(X \times U, A)$ is then given by $g(\xi) = {\tilde{g}^{-1}}^*(\xi)$, for a cohomology class $\xi \in H^q(X \times U, A)$. Note also that $s \in P(U)$ induces an isomorphism

$$\begin{aligned} \tilde{s} : X \times U &\longrightarrow Y \times U \\ (x, u) &\longmapsto ([s(u), x], u). \end{aligned}$$

We denote the isomorphism induced by $\tilde{s}$ on the cohomology level by $s_{\#}$: $s_{\#}(\xi) = {\tilde{s}^{-1}}^*(\xi)$, for a cohomology class $\xi \in H^q(X \times U, A)$. The following facts are easy to check

$$\begin{aligned} g^{-1}\xi &= \tilde{g}^*(\xi) && (14) \\ \widetilde{sg}(x, u) &= (\tilde{s} \circ \tilde{g})(x, u) && (15) \\ (sg)_{\#}(g^{-1}\xi) &= s_{\#}(\xi), && (16) \end{aligned}$$

where (14) and (15) are used to prove (16). We can now define

$$\begin{aligned} \phi(U) : P(U) \times_{G(U)} H^q(X \times U, A) &\longrightarrow H^q(X \times U, A) \\ [s, \xi] &\longmapsto s_{\#}(\xi). \end{aligned}$$

That $\phi(U)$ is well-defined follows from (16). We get an induced map of sheaves

$$\phi : P \times_G R^q {j_X}_*(A|X) \to R^q {j_Y}_*(A|Y).$$

ϕ is obviously an isomorphism. □

Proposition 3.3.11 *Let G be a connected algebraic group over the separably closed field k, acting from the left on an algebraic k-stack X of finite type. Then for every $j, n \geq 0$ the induced action of G on $H^j(X_{\mathrm{sm}}, \mathbb{Z}/\ell^{n+1}\mathbb{Z})$ is trivial.*

Proof. If we give the finite set $H^j(X_{\mathrm{sm}}, \mathbb{Z}/\ell^{n+1}\mathbb{Z})$ the obvious k-scheme structure, then the action of G on $H^j(X_{\mathrm{sm}}, \mathbb{Z}/\ell^{n+1}\mathbb{Z})$ is *morphic*. Hence the stabilizer of any element $\xi \in H^j(X_{\mathrm{sm}}, \mathbb{Z}/\ell^{n+1}\mathbb{Z})$ is a nonempty closed and open subset of G, i.e. G itself. □

Theorem 3.3.12 *Let k be a separably closed field. let $\mathfrak{X}$ be an algebraic k-stack of finite type and let G be a (connected) algebraic group variety over k acting (from the left) on X, a smooth algebraic k-stack of finite type. Let $\mathfrak{P}$ be a principal G-bundle over $\mathfrak{X}$ and let $\mathfrak{Y} = \mathfrak{P} \times_G X$ be the associated bundle with fiber X. Then if $\ell \neq \operatorname{char} k$ we may write the Leray spectral sequence of the morphism $\mathfrak{Y} \to \mathfrak{X}$ as the following spectral sequence of finite dimensional $\mathbb{Q}_\ell$-vector spaces*

$$E_2^{i,j} = H^i(\mathfrak{X}_{\mathrm{sm}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H^j(X_{\mathrm{sm}}, \mathbb{Q}_\ell) \Longrightarrow H^{i+j}(\mathfrak{Y}_{\mathrm{sm}}, \mathbb{Q}_\ell).$$

Proof. Denote by $\widetilde{G}, \widetilde{X}, \widetilde{P}$ and $\widetilde{Y}$ the sheaves on $\mathfrak{X}_{\mathrm{sm}}$ induced by the stacks $G_{\mathfrak{X}}, X_{\mathfrak{X}}, \mathfrak{P}$ and $\mathfrak{Y}$ that are all smooth over $\mathfrak{X}$. Let $\iota_{\widetilde{G}}, \iota_{\widetilde{X}}, \iota_{\widetilde{P}}$ and $\iota_{\widetilde{Y}}$ be the corresponding localization morphisms in $\mathfrak{X}_{\mathrm{sm}}$.

The sheaf $\widetilde{P}$ is a principal $\widetilde{G}$-bundle and $\widetilde{Y} = \widetilde{P} \times_{\widetilde{G}} \widetilde{X}$. Letting $\pi : \mathfrak{Y} \to \mathfrak{X}$ denote the natural morphism and $\pi_{\mathrm{sm}} : \mathfrak{Y}_{\mathrm{sm}} \to \mathfrak{X}_{\mathrm{sm}}$ the induced map of smooth topoi we have

$$R^j {\pi_{\mathrm{sm}}}_*(\mathbb{Z}/\ell^{n+1}\mathbb{Z}) = R^j {\iota_{\widetilde{Y}}}_*(\mathbb{Z}/\ell^{n+1}\mathbb{Z}).$$

By Lemma 3.3.10 we have

$$\begin{aligned} R^j \iota_{\widetilde{Y}*}(\mathbb{Z}/\ell^{n+1}\mathbb{Z}) &\cong \widetilde{P} \times_{\widetilde{G}} R^j \iota_{\widetilde{X}*}(\mathbb{Z}/\ell^{n+1}\mathbb{Z}) \\ &= \widetilde{P} \times_{\widetilde{G}} H^j(X_{\mathrm{sm}}, \mathbb{Z}/\ell^{n+1}\mathbb{Z}). \end{aligned}$$

From Proposition 3.3.11 we get

$$\widetilde{P} \times_{\widetilde{G}} H^j(X_{\mathrm{sm}}, \mathbb{Z}/\ell^{n+1}\mathbb{Z}) \cong H^j(X_{\mathrm{sm}}, \mathbb{Z}/\ell^{n+1}\mathbb{Z})$$

which proves that

$$R^j \pi_{\mathrm{sm}*}(\mathbb{Z}/\ell^{n+1}\mathbb{Z}) = H^j(X_{\mathrm{sm}}, \mathbb{Z}/\ell^{n+1}\mathbb{Z}).$$

Then our theorem follows from Theorem 3.3.9. □

3.4 'The Infinite Dimensional Grassmannian'

Let k be a separably closed field. Consider BGL_n, the classifying stack of GL_n over k. Since BGL_n classifies GL_n-bundles, or equivalently vector bundles of rank n, we may consider BGL_n as an anolog of the infinite dimensional Grassmannian which is used in homotopy theory.

Let $\mathrm{Grass}_n(h)$ be the Grassmannian over k of locally free quotients of rank n of $\mathcal{O}^h$. The universal quotient on $\mathrm{Grass}_n(h)$ induces a natural morphism $\mathrm{Grass}_n(h) \to BGL_n$. The connection between the cohomology of $\mathrm{Grass}_n(h)$ and BGL_n is given by the following lemma.

Lemma 3.4.1 *The natural morphism* $\mathrm{Grass}_n(h) \to BGL_n$ *induces an isomorphism*

$$H^i(BGL_{n\,\mathrm{sm}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(\mathrm{Grass}_n(h)_{\mathrm{sm}}, \mathbb{Q}_\ell)$$

for every $i \leq 2(h-n)$.

Proof. We may consider $\mathrm{Grass}_n(h)$ as the quotient of $M(n\times h)^*$ by the natural action of GL_n. Here we denote by $M(n\times h)^*$ the subvariety of $M(n\times h)$ of non-singular $n\times h$-matrices. Hence we get a natural open immersion of algebraic k-stacks $\mathrm{Grass}_n(h) \to [G_n\backslash M(n\times h)]$. Since $M(n\times h)$ is cohomologically trivial, we get from Theorem 3.3.12 that $[G_n\backslash M(n\times h)]$ has the same cohomology as BGL_n. Now our lemma follows from purity (Corollary 2.1.3) upon estimating the codimension of the singular part of $M(n\times h)$. □

Let V be the vector bundle on BGL_n associated to the universal GL_n-bundle via the standard representation of GL_n. Let $c_i \in H^{2i}(BGL_{n\,\mathrm{sm}}, \mathbb{Q}_\ell(i))$ for $i = 1, \ldots, n$ be the Chern classes of V. We will call $c_1, \ldots, c_n$ the *universal Chern classes.*

Theorem 3.4.2 *We have*

$$H^*(BGL_{n\,\mathrm{sm}}, \mathbb{Q}_\ell([\tfrac{*}{2}])) = \mathbb{Q}_\ell[c_1, \ldots, c_n].$$

The arithmetic Frobenius Φ_q *acts as the identity on* $H^*(BGL_{n\,\mathrm{sm}}, \mathbb{Q}_\ell([\tfrac{*}{2}]))$.

Proof. This may be deduced from the corresponding results on finite dimensional Grassmannians using Lemma 3.4.1. □

Choose an isomorphism $\theta : \mathbb{Q}_\ell(1) \to \mathbb{Q}_\ell$. For any algebraic stack $\mathfrak{X}$ over k we get an induced isomorphism

$$\theta : H^i(\mathfrak{X}_{\mathrm{sm}}, \mathbb{Q}_\ell(\nu)) \longrightarrow H^i(\mathfrak{X}_{\mathrm{sm}}, \mathbb{Q}_\ell)$$

for any pair of integers i, ν. In particular we get an isomorphism

$$\theta : \mathbb{Q}_\ell[c_1, \ldots, c_n] \longrightarrow H^*(BGL_{n\,\mathrm{sm}}, \mathbb{Q}_\ell)$$

of $\mathbb{Q}_\ell$-algebras and the action of the arithmetic Frobenius on the generators $\theta(c_1), \ldots, \theta(c_n)$ is given by

$$\Phi_q(\theta(c_i)) = q^{-i}\theta(c_i),$$

for all $i = 1, \ldots, n$.

3.5 The Trace Formula for the Action of a Linear Algebraic Group on a Deligne-Mumford Stack

Some Preliminary Remarks on Spectral Sequences

Definition 3.5.1 Let $(E, \Phi) = (E^{p,q}, \Phi)_{p,q\in\mathbb{N}_0}$ be a family of $\mathbb{C}$-vector spaces $(E^{p,q})$ with $\Phi : E^{p,q} \to E^{p,q}$ a $\mathbb{C}$-linear endomorphism for every $p, q \in \mathbb{N}_0$. We say that (E, Φ) satisfies condition (*) with respect to N, a nonnegative integer, and the four functions

$$\begin{aligned} a : \{0, \ldots, N\} &\longrightarrow \mathbb{R}_{\geq 0} \\ b : \{0, \ldots, N\} &\longrightarrow \mathbb{N}_0 \\ c : \mathbb{N}_0 &\longrightarrow \mathbb{R}_{\geq 0} \\ d : \mathbb{N}_0 &\longrightarrow \mathbb{N}_0 \end{aligned}$$

if the following conditions are satisfied:

i. For every $q > N$ we have $E^{p,q} = 0$

ii. If λ is an eigenvalue of Φ on $E^{p,q}$ then $|\lambda| \leq a(q)c(p)$

iii. $\dim E^{p,q} \leq b(q)d(p)$ for every $p, q \in \mathbb{N}_0$.

Lemma 3.5.2 *Let $(E, \Phi) = (E^{p,q}, \Phi)_{p,q\in\mathbb{N}_0}$ be a family of $\mathbb{C}$-vector spaces satisfying condition (*) with respect to (N, a, b, c, d). Assume that $\sum_{p\geq 0} c(p)d(p) < \infty$. Then $\sum_{p,q\geq 0}(-1)^{p+q} \operatorname{tr} \Phi|E^{p,q}$ is absolutely convergent.*

Proof. We have

$$\begin{aligned} \sum_{p,q\geq 0} |\operatorname{tr} \Phi|E^{p,q}| &\leq \sum_{q=0}^{N}\sum_{p\geq 0} a(q)c(p)b(q)d(p) \\ &= \sum_{q=0}^{N} a(q)b(q)\sum_{p\geq 0} c(p)d(p) \\ &< \infty \end{aligned}$$

which proves the lemma □

Definition 3.5.3 Let $(E, \Phi) = (E^{p,q}, \Phi)_{p,q\in\mathbb{N}_0}$ be a family of $\mathbb{C}$-vector spaces satisfying condition (*) with respect to (N, a, b, c, d) such that $\sum_{p\geq 0} c(p)d(p) < \infty$. Then we define the trace of Φ on E to be

$$\operatorname{tr} \Phi|E = \sum_{p,q\geq 0} (-1)^{p+q} \operatorname{tr} \Phi|E^{p,q}.$$

Lemma 3.5.4 *Let (E, Φ) be a spectral sequence of $\mathbb{C}$-vector spaces with a compatible endomorphism Φ. Let n be an integer $n \geq 2$. Assume that (E_n, Φ) satisfies condition (*) with respect to (N, a, b, c, d) where $\sum_{p \geq 0} c(p)d(p) < \infty$. Then (E_{n+1}, Φ) also satisfies condition (*) with respect to (N, a, b, c, d). Moreover, we have $\operatorname{tr} \Phi | E_{n+1} = \operatorname{tr} \Phi | E_n$.*

Proof. It is clear that (E_{n+1}, Φ) satisfies condition (*). So $\operatorname{tr} \Phi | E_{n+1}$ is well-defined. We consider E_n as an infinite family $(C_\nu^\bullet)_{\nu \geq 0}$ of finite complexes of finite dimensional $\mathbb{C}$-vector spaces. The next level E_{n+1} is then the family of graded $\mathbb{C}$-vector spaces $(H(C_\nu^\bullet))_{\nu \geq 0}$. So we may compute

$$\begin{aligned} \operatorname{tr} \Phi | E_{n+1} &= \sum_{\nu \geq 0} \operatorname{tr} \Phi | H(C_\nu^\bullet) \\ &= \sum_{\nu \geq 0} \operatorname{tr} \Phi | C_\nu^\bullet \\ &= \operatorname{tr} \Phi | E_n, \end{aligned}$$

where we have to keep track of the signs, of course. □

Lemma 3.5.5 *Let (E, Φ) be an E_2 spectral sequence of $\mathbb{C}$-vector spaces with a compatible endomorphism Φ. Assume that (E_2, Φ) satisfies condition (*) with respect to (N, a, b, c, d) where $\sum_{p \geq 0} c(p)d(p) < \infty$. Let (E, Φ) abut to F:*

$$E_2^{p,q} \Longrightarrow F^{p+q}.$$

Then

$$\operatorname{tr} \Phi | F = \sum_{n=0}^{\infty} (-1)^n \operatorname{tr} \Phi | F^n$$

is absolutely convergent and we have

$$\operatorname{tr} \Phi | F = \operatorname{tr} \Phi | E_2.$$

Proof. First note that our spectral sequence degenerates at E_{N+2}, so that we have $E_\infty = E_{N+2}$. By Lemma 3.5.4 and induction E_∞ satisfies (*) and we have $\operatorname{tr} \Phi | E_2 = \operatorname{tr} \Phi | E_\infty$. So we may estimate as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} | \operatorname{tr} \Phi | F^n | &\leq \sum_{n=0}^{\infty} \sum_{p=0}^{n} | \operatorname{tr} \Phi | E_\infty^{p,n-p} | \\ &\leq \sum_{n=0}^{\infty} \sum_{p=0}^{n} a(n-p)b(n-p)c(p)d(p) \\ &= \sum_{q=0}^{N} a(q)b(q) \sum_{p=0}^{\infty} c(p)d(p) \\ &< \infty. \end{aligned}$$

This proves the absolute convergence of $\operatorname{tr} \Phi | F$. To prove the formula we may calculate

$$\begin{aligned} \operatorname{tr} \Phi | F &= \sum_{n=0}^{\infty} (-1)^n \operatorname{tr} \Phi | F^n \\ &= \sum_{n=0}^{\infty} (-1)^n \sum_{p=0}^{n} \operatorname{tr} \Phi | E_\infty^{p,n-p} \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{p=0}^{n}(-1)^n \operatorname{tr}\Phi|E_\infty^{p,n-p} \\
&= \sum_{p,q\geq 0}(-1)^{p+q}\operatorname{tr}\Phi|E_\infty^{p,q} \\
&= \operatorname{tr}\Phi|E_\infty \\
&= \operatorname{tr}\Phi|E_2,
\end{aligned}$$

finishing the proof. □

The Trace Fromula

Choose an embedding $\mathbb{Q}_\ell \subset \mathbb{C}$. If we are given a topos X we write $H^*(X,\mathbb{C})$ for $H^*(X,\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}\mathbb{C}$.

Let X be a smooth equidimensional Deligne-Mumford stack of finite type over the separably closed field k, together with an action σ of GL_n on X. Let $\mathfrak{X}$ be the quotient stack for this action. We have a 2-cartesian and 2-cocartesian diagram of k-stacks

$$\begin{array}{ccc}
GL_n \times X & \xrightarrow{\sigma} & X \\
{\scriptstyle p_2}\downarrow & \boxed{2} & \downarrow{\scriptstyle \pi} \\
X & \xrightarrow{\pi} & \mathfrak{X}
\end{array}$$

The stack $\mathfrak{X}$ is a smooth algebraic k-stack of finite type. The structure morphism $\pi : X \to \mathfrak{X}$ is a left principal GL_n-bundle and we have another 2-cartesian and 2-cocartesian diagram of k-stacks

$$\begin{array}{ccc}
X & \xrightarrow{\pi} & \mathfrak{X} \\
\downarrow & \boxed{2} & \downarrow{\scriptstyle f} \\
\operatorname{Spec} k & \longrightarrow & BGL_n
\end{array}$$

We will consider the Leray spectral sequence of f, the morphism of associated smooth topoi induced by f. Since f is just the bundle with fiber X associatied to the universal GL_n-bundle we get from Theorem 3.3.12 the following E_2 spectral sequence of finite dimensional $\mathbb{Q}_\ell$-vector spaces.

$$H^i(BGL_{n\,\mathrm{sm}},\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell} H^j(X_{\mathrm{sm}},\mathbb{Q}_\ell) \Longrightarrow H^{i+j}(\mathfrak{X}_{\mathrm{sm}},\mathbb{Q}_\ell)$$

By the remarks following Theorem 3.4.2 the E_2-term of this spectral sequence may be written as $E_2 = H^*(X_{\mathrm{sm}},\mathbb{Q}_\ell)[\theta(c_1),\ldots,\theta(c_n)]$, where $c_1,\ldots,c_n$ are the universal Chern classes and θ denotes an isomorphism $\theta : \mathbb{Q}_\ell(1) \to \mathbb{Q}_\ell$. So we may write this spectral sequence (by slight abuse of notation) as

$$E_2 = H^*(X_{\mathrm{sm}},\mathbb{Q}_\ell)[\theta(c_1),\ldots,\theta(c_n)] \Longrightarrow H^*(\mathfrak{X}_{\mathrm{sm}},\mathbb{Q}_\ell). \tag{17}$$

Now assume that X and σ are defined over the finite field $\mathbb{F}_q$. Then the arithmetic Frobenius Φ_q acts on the spectral sequence (17), and upon tensoring with $\mathbb{C}$ we get a spectral sequence of $\mathbb{C}$-vector spaces with compatible endomorphism Φ_q.

$$H^*(\overline{X}_{\mathrm{sm}},\mathbb{C})[\theta(c_1),\ldots,\theta(c_n)] \Longrightarrow H^*(\overline{\mathfrak{X}}_{\mathrm{sm}},\mathbb{C}). \tag{18}$$

Now define $N = 2\dim X$ and

$$\begin{array}{rcll}
a(j) &=& \max_\lambda|\lambda| & \text{for } j = 0,\ldots,N \\
b(j) &=& \dim H^j(\overline{X}_{\mathrm{sm}},\mathbb{Q}_\ell) & \text{for } j = 0,\ldots,N \\
c(i) &=& q^{-i/2} & \text{for } i \in \mathbb{N}_0 \\
d(i) &=& \dim(\mathbb{Q}_\ell[c_1,\ldots,c_n])_i & \text{for } i \in \mathbb{N}_0.
\end{array}$$

In the definition of $a(j)$ the index λ ranges over the eigenvalues of Φ_q on $H^j(\overline{X}_{\mathrm{sm}}, \mathbb{Q}_\ell)$. In the definition of $d(i)$ the subscript i denotes the homogeneous part of degree i. We define the degree on $\mathbb{Q}_\ell[c_1, \ldots, c_n]$ by declaring the degree of c_ν to be 2ν. Note that $\sum_{i=0}^{\infty} c(i)d(i) < \infty$. Moreover, the family $(H^*(\overline{X}_{\mathrm{sm}}, \mathbb{C})[\theta(c_1), \ldots, \theta(c_n)], \Phi_q)$ satisfies condition (*) with respect to (N, a, b, c, d). Hence by Lemma 3.5.5 we get that $\operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{X}}_{\mathrm{sm}}, \mathbb{C})$ converges absolutely and we have

$$\begin{aligned} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{X}}_{\mathrm{sm}}, \mathbb{C}) &= \operatorname{tr} \Phi_q | H^*(\overline{X}_{\mathrm{sm}}, \mathbb{C})[\theta(c_1), \ldots, \theta(c_n)] \\ &= \operatorname{tr} \Phi_q | H^*(\overline{X}_{\mathrm{sm}}, \mathbb{C}) \operatorname{tr} \Phi_q | \mathbb{C}[\theta(c_1), \ldots, \theta(c_n)] \\ &= \operatorname{tr} \Phi_q | H^*(\overline{X}_{\mathrm{sm}}, \mathbb{Q}_\ell) \prod_{\nu=1}^{n} \frac{1}{1 - 1/q^\nu}. \end{aligned}$$

Now $\dim \mathfrak{X} = \dim X - \dim GL_n$ so we have

$$\begin{aligned} q^{\dim \mathfrak{X}} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{X}}_{\mathrm{sm}}, \mathbb{C}) &= q^{\dim X} \operatorname{tr} \Phi_q | H^*(\overline{X}_{\mathrm{sm}}, \mathbb{Q}_\ell) \Big/ q^{\dim GL_n} \prod_{\nu=1}^{n} (1 - 1/q^\nu) \\ &= q^{\dim X} \operatorname{tr} \Phi_q | H^*(\overline{X}_{\mathrm{sm}}, \mathbb{Q}_\ell) \Big/ \prod_{\nu=1}^{n} (q^n - q^{n-\nu}) \\ &= \sum_{\xi \in [X(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut} \xi} \Big/ \# GL_n(\mathbb{F}_q) \end{aligned}$$

by Theorem 2.4.5.

Lemma 3.5.6 *Let X be a smooth Deligne-Mumford stack of finite type over $\mathbb{F}_q$ and let G be a connected algebraic group over $\mathbb{F}_q$ acting on X. Let $\mathfrak{X}$ be the associated algebraic stack. Then*

$$\sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut} \xi} = \sum_{\xi \in [X(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut} \xi} \Big/ \# G(\mathbb{F}_q).$$

Proof. From a theorem of S. Lang we have $H^1(\mathbb{F}_q, G) = 0$. This implies that $X(\mathbb{F}_q) \to \mathfrak{X}(\mathbb{F}_q)$ is essentially surjective. Hence we may calculate

$$\begin{aligned} \sum_{\xi \in [X(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut} \xi} &= \sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \sum_{\substack{\eta \in [X(\mathbb{F}_q)] \\ \eta \mapsto \xi}} \frac{1}{\# \operatorname{Aut} \eta} \\ &= \sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{\# G(\mathbb{F}_q)}{\# \operatorname{Aut} \xi} \end{aligned}$$

proving the lemma. □

From Lemma 3.5.6 we get finally:

$$q^{\dim \mathfrak{X}} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{X}}_{\mathrm{sm}}, \mathbb{C}) = \sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut} \xi}. \tag{19}$$

Theorem 3.5.7 *Let G be a non-singular linear algebraic group over $\mathbb{F}_q$, acting on a smooth equidimensional Deligne-Mumford stack X of finite type over $\mathbb{F}_q$. Let $\mathfrak{X} = X_G = [X/G]$ be the corresponding smooth algebraic $\mathbb{F}_q$-stack of finite type. Then $\operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{X}}_{\mathrm{sm}}, \mathbb{C})$ is absolutely convergent and we have*

$$q^{\dim \mathfrak{X}} \operatorname{tr} \Phi_q | H^*(\overline{\mathfrak{X}}_{\mathrm{sm}}, \mathbb{C}) = \sum_{\xi \in [\mathfrak{X}(\mathbb{F}_q)]} \frac{1}{\# \operatorname{Aut} \xi}.$$

Proof. Choose an embedding of G into GL_n for some n. Let $Y = X \times_G GL_n$. Then $\mathfrak{X} = Y_{GL_n} = [Y/GL_n]$ and so the theorem follows from (19). $\square$

4 The Algebraic Stack of G-Bundles

Introduction

This section is of a technical nature. We define the stack of G-torsors $\mathfrak{H}^1(X/S, G)$, where G is an algebraic group scheme over the curve X over the base scheme S. (See section 4.3.) In section 4.4 we show that $\mathfrak{H}^1(X/S, G)$ is an algebraic stack, locally of finite type, for the case we are interested in. This is the case that S is the spectrum of a field and G is a reductive group scheme over X. (See Proposition 4.4.6.) In section 4.5 we show that $\mathfrak{H}^1(X/k, G)$ is smooth. (See Corollary 4.5.2.) The calculation of the dimension of $\mathfrak{H}^1(X/k, G)$ will be deferred to section 8.1.

4.1 Some Elementary Non-Abelian Cohomology Theory

Let X be a topos. We want to prove some basic facts about groups in X.

Definition 4.1.1 Let G a group in X and $U \in \operatorname{ob} X$. Then we denote the groupoid $BG(U)$ of (right) G_U-torsors by $\Delta(U, G)$, so that $H^1(U, G)$ is the set of isomorphism classes of $\Delta(U, G)$. If no confusion seems likely to arise, we write $\Delta(G)$ for $\Delta(X, G)$. Note that $\Delta(X, \cdot\,)$ has the same relation to $H^1(X, \cdot\,)$ as $\Gamma(X, \cdot\,)$ has to $H^0(X, \cdot\,)$.

Proposition 4.1.2 *Let G be a group on X and E a G-torsor. Let ${}^E G = E \times_{G,Ad} G = \underline{\mathrm{Aut}}_G\, E$ be the inner form of G, obtained from G by twisting with E. Then we have a natural equivalence of groupoids*

$$\begin{array}{rcl} \Delta(G) & \longrightarrow & \Delta({}^E G) \\ F & \longmapsto & \underline{\mathrm{Hom}}_G(E, F) \end{array}$$

with quasi-inverse

$$\begin{array}{rcl} \Delta({}^E G) & \longrightarrow & \Delta(G) \\ F' & \longmapsto & F' \times_{{}^E G} E. \end{array}$$

Proof. Straightforward. $\square$

Remark 4.1.3 This lemma doesn't necessarily apply to an arbitrary inner form G' of G.

Let $G_1 \to G_2$ be a homomorphism of groups in X. Then there is a natural morphism of groupoids

$$\begin{array}{rcl} \Delta(G_1) & \longrightarrow & \Delta(G_2) \\ E & \longmapsto & E \times_{G_1} G_2. \end{array}$$

Proposition 4.1.4 *If $G' \to G$ is a monomorphism and $E \in \operatorname{ob} \Delta(G)$ then there is a 2-cartesian diagram of groupoids*

$$\begin{array}{ccc} \Gamma(E/G') & \longrightarrow & \{\varnothing\} \\ {\scriptstyle \delta}\downarrow & \boxed{2} & \downarrow{\scriptstyle E} \\ \Delta(G') & \longrightarrow & \Delta(G). \end{array}$$

If $E' \in \operatorname{ob} \Delta(G')$, then we have

$$\begin{array}{ccc} \Gamma(G/G') & \longrightarrow & \{\varnothing\} \\ \downarrow & \boxed{2} & \downarrow{\scriptstyle E' \times_{G'} G} \\ \Delta(G') & \longrightarrow & \Delta(G). \end{array}$$

Proof. The first diagram expresses the fact that the category of reductions of E to G' is a set (since $G' \to G$ is injective) and that this set is equal to the set of global sections of E/G'. The second diagram follows from the first since $(E' \times_{G'} G)/G' = E' \times_{G'} (G/G') = G/G'$. □

Proposition 4.1.5 *If $1 \to G' \to G \to G'' \to 1$ is a short exact sequence of groups in X, and if $E \in \operatorname{ob} \Delta(G)$ then we have a 2-cartesian diagram of groupoids*

$$\begin{tikzcd} \Delta({}^E G') \arrow[r] \arrow[d, "\psi"'] \arrow[dr, phantom, "\boxed{2}"] & \{\varnothing\} \arrow[d, "E \times_G G''"] \\ \Delta(G) \arrow[r] & \Delta(G''). \end{tikzcd}$$

Proof. We have to show that $\Delta({}^E G')$ is equivalent to the category of reductions of the structure group of $E \times_G G''$ to G. Let $F \in \operatorname{ob} \Delta({}^E G')$, i.e. F is an ${}^E G'$-torsor. Under the vertical map ψ, F is mapped to $F \times_{{}^E G'} E$. The action of ${}^E G'$ on E is given by $[e, g'](e) = eg'$. So the induced action on $E \times_G G''$ is given by $[e, g']([e, g'']) = [eg', g''] = [e, g'']$, i.e. it is trivial. So $(F \times_{{}^E G'} E) \times_G G'' = E \times_G G''$. Thus $F \mapsto \psi(F)$ defines a reduction of structure group of $E \times_G G''$ to G.

Conversely, let $\widetilde{E}$ be a G-torsor and $\phi : E \times_G G'' \to \widetilde{E} \times_G G''$ an isomorphism of G''-torsors. Construct F as the following fibered product in the topos X:

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \underline{\mathrm{Hom}}_G(E, \widetilde{E}) \arrow[d] \\ X \arrow[r, "\phi"] & \underline{\mathrm{Hom}}_{G''}(E \times_G G'', \widetilde{E} \times_G G''). \end{tikzcd}$$

Then F is an ${}^E G'$-torsor and $F \times_{{}^E G'} E = \widetilde{E}$. □

Proposition 4.1.6 *Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of groups in X, with G' abelian. If $H^2(X, {}^{E''} G') = 0$ for every $E'' \in \operatorname{ob} \Delta(G'')$, then $\Delta(G) \to \Delta(G'')$ is essentially surjective.*

Proof. Let E'' be a G''-torsor. Let $K(E'')$ be the gerbe of liftings of E'' to G. The *lien* of the gerbe $K(E'')$ is the lien associated to the group ${}^{E''} G'$. The gerbe $K(E'')$ is trivial or neuter if and only if E'' admits a (global) lifting to G (see [10, Chap. IV, Proposition 2.5.8]). Now $K(E'')$ defines a class in $H^2({}^{E''} G')$ in the notation of [10, Chap. IV, Définition 3.1.3]. By [10, Chap. IV, 3.4] $H^2({}^{E''} G') = 0$ so that $K(E'')$ is isomorphic to $B({}^{E''} G')$, the trivial ${}^{E''} G'$-gerbe. Hence $K(E'')$ has a section. □

The above propositions imply the standard facts about the long exact cohomology sequence in non-abelian cohomology theory. Just apply the following lemma to the above diagrams:

Lemma 4.1.7 *Let*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{2}"] & \{\varnothing\} \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

be a 2-cartesian diagram of groupoids. Then we get an induced exact sequence

$$[A] \longrightarrow [B] \longrightarrow [C].$$

Here $[\cdot]$ denotes the associated set of equivalence classes and $[C]$ is made into a pointed set using $\{\varnothing\} \to C$.

Proof. Clear. □

4.2 The Relative Case

Let $\pi : X \to S$ be a morphism of topoi. We want to generalize the results of the previous section to this situation. For a sheaf of groups G on X, we denote the direct image $\pi_* BG$ of the classifying X-stack BG by $\mathfrak{H}^1(X/S, G)$.

Proposition 4.2.1 ('Change of Origin') *Let G be a sheaf of groups on X. Then for any G-torsor E, there is an isomorphism of S-stacks*

$$\mathfrak{H}^1(X/S, G) \xrightarrow{\sim} \mathfrak{H}^1(X/S, {}^E G).$$

Proof. This follows immediately from Proposition 4.1.2. □

Lemma 4.2.2 *Let G be a sheaf of groups on X. Then the natural morphism of S-stacks $\mathfrak{H}^1(S, \pi_* G) \to \mathfrak{H}^1(X/S, G)$ is a monomorphism. So we may consider $\mathfrak{H}^1(S, \pi_* G)$ as an S-substack of $\mathfrak{H}^1(X/S, G)$. Then $\mathfrak{H}^1(S, \pi_* G)$ is the image S-substack of the trivial morphism of S-stacks $S \to \mathfrak{H}^1(X/S, G)$.*

Proof. First let us show that for every $U \in \operatorname{ob} S$ the functor $\mathfrak{H}^1(S, \pi_* G)(U) \to \mathfrak{H}^1(X/S, G)(U)$ is fully faithful. In other words, we need to show that the functor $\Phi_G(U) : \Delta(U, \pi_* G) \to \Delta(\pi^* U, G)$ is fully faithful. Without loss of generality we may assume that $U = S$, so that we need only show that $\Phi_G : \Delta(S, \pi_* G) \to \Delta(X, G)$ is fully faithful.

Let E be a $\pi_* G$-torsor. Let $\widetilde{E} = \Phi_G(E) = \pi^* E \times_{\pi^* \pi_* G} G$. Note that we have a natural morphism of $\pi_* G$-sheaves $E \to \pi_* \widetilde{E}$. Since locally this is obviously an isomorphism, it follows that $E = \pi_* \widetilde{E}$. We also have a natural morphism of S-group sheaves $\pi_*({}^{\widetilde{E}} G) \to {}^E(\pi_* G)$. Since this is also locally an isomorphism, it follows that $\pi_*({}^{\widetilde{E}} G) = {}^E(\pi_* G)$. Therefore, the following diagram is 2-commutative:

$$\begin{array}{ccc} \Delta(S, \pi_* G) & \longrightarrow & \Delta(X, G) \\ \downarrow & & \downarrow \\ \Delta(S, {}^E(\pi_* G)) & \longrightarrow & \Delta(X, {}^{\widetilde{E}} G) \end{array}$$

Here the vertical arrows are the change of origin isomorphisms from proposition 4.1.2, whereas the horizontal arrows are the functors Φ_G and $\Phi_{{}^{\tilde{E}} G}$ of the lemma. so by changing the origin in the indicated way, we reduce to proving that

i. $\Gamma(S, \pi_* G) = \Gamma(X, G)$,

ii. $\Gamma(X, \widetilde{E}) \neq \varnothing \Rightarrow \Gamma(S, E) \neq \varnothing$.

Now (i) is clear, and (ii) likewise, noting that $E = \pi_* \widetilde{E}$, as we already did.

This finishes the proof that $\mathfrak{H}^1(S, \pi_* G) \to \mathfrak{H}^1(X/S, G)$ is a monomorphism. Now $S \to \mathfrak{H}^1(X/S, G)$ obviously factors through $\mathfrak{H}^1(S, \pi_* G)$. Also, $S \to \mathfrak{H}^1(S, \pi_* G)$ is by definition an epimorphism. So we have proved the lemma. □

If $G_1 \to G_2$ is a homomorphism of groups in X, then there is an induced morphism of stacks over S:

$$\mathfrak{H}^1(X/S, G_1) \longrightarrow \mathfrak{H}^1(X/S, G_2).$$

Proposition 4.2.3 *Let $G' \to G$ be a monomorphism of groups in X. Let $E \in \operatorname{ob} \mathfrak{H}^1(X/S, G)(U)$ where $U \in \operatorname{ob} S$. Then we have a 2-cartesian diagram of S-stacks:*

$$\begin{array}{ccc} \pi_{U*}(E/G'_U) & \longrightarrow & U \\ \downarrow & \boxed{2} & \downarrow E \\ \mathfrak{H}^1(X/S, G') & \longrightarrow & \mathfrak{H}^1(X/S, G) \end{array}$$

Let $E' \in \operatorname{ob} \mathfrak{H}^1(X/S, G')(U)$. Then we have

$$\begin{array}{ccc} \pi_{U*}(G/G') \times U & \longrightarrow & U \\ \downarrow & \boxed{2} & \downarrow {\scriptstyle E' \times_{G'} G} \\ \mathfrak{H}^1(X/S, G') & \longrightarrow & \mathfrak{H}^1(X/S, G). \end{array}$$

Proof. Follows immediately from Proposition 4.1.4. □

Proposition 4.2.4 *Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of groups in X. Let $\mathfrak{Y}$ be defined by the diagram below being 2-cartesian. Then we have the following diagram of S-stacks:*

$$\begin{array}{ccccc} \mathfrak{H}^1(X/S, G') & \xrightarrow{\phi} & \mathfrak{Y} & \longrightarrow & \mathfrak{H}^1(S, \pi_* G'') \\ & \searrow & {\scriptstyle \iota} \downarrow & \boxed{2} & \downarrow \\ & & \mathfrak{H}^1(X/S, G) & \longrightarrow & \mathfrak{H}^1(X/S, G'') \end{array}$$

Here $\phi : \mathfrak{H}^1(X/S, G') \to \mathfrak{Y}$ is a principal $\pi_* G''$-bundle, ι is a monomorphism, and ϕ and ι give the decomposition of $\iota \circ \phi$ as an epimorphism followed by a monomorphism.

If $E \in \operatorname{ob} \mathfrak{H}^1(X/S, G)(U)$ then we have a 2-cartesian diagram of S-stacks

$$\begin{array}{ccc} \mathfrak{H}^1(X_U/U, {}^E G'_U) & \longrightarrow & U \\ \downarrow & \boxed{2} & \downarrow {\scriptstyle E \times_G G''} \\ \mathfrak{H}^1(X/S, G) & \longrightarrow & \mathfrak{H}^1(X/S, G''). \end{array}$$

Proof. Clear. □

Proposition 4.2.5 *Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of groups in X and assume that G' is abelian. If $R^2 \pi_{U*}({}^{E''} G'_U) = 0$ for every $E'' \in \operatorname{ob} \mathfrak{H}^1(X/S, G'')(U)$ and for all $U \in \operatorname{ob} S$, then $\mathfrak{H}^1(X/S, G) \to \mathfrak{H}^1(X/S, G'')$ is an epimorphism.*

Proof. Let E'' be a G''-torsor. We need to show that we can locally lift E'' to G. Let $K(E'')$ be as in the proof of Proposition 4.1.6 the ${}^{E''}G'$-gerbe of lifts of E'' to G. The gerbe $K(E'')$ defines an element $d(E'') \in H^2(X, {}^{E''}G')$. It suffices to show that locally over S, this element vanishes. But this follows from the assumption on $R^2 \pi_*({}^{E''}G')$. □

4.3 The Case of Schemes

Let X be a scheme. Endow the category of schemes over X with the fppf-topology and pass to the associated topos. This topos is the *flat topos associated to the scheme* X. We will denote it by X_{fl}. A morphism of schemes $\pi : X \to S$ induces a morphism of the associated topoi, and a group scheme G over X induces a sheaf of groups on the topos X_{fl}.

If G is a group scheme over X, then a *G-torsor* is a torsor under the associated sheaf of groups on X_{fl}. If X is an S-scheme and G an X_{fl}-group (i.e. a sheaf of groups on X_{fl}), then we denote by $\mathfrak{H}^1(X/S, G)$ the S-stack of families of G-torsors. In the notation of section 4.2 the stack $\mathfrak{H}^1(X/S, G)$ is equal to $\mathfrak{H}^1(X_{\mathrm{fl}}/S_{\mathrm{fl}}, G)$.

Sometimes we will also have to consider the étale topos associated to the scheme X. This is the topos associated to the category of étale X-schemes with the étale topology and is denoted by $X_{\text{ét}}$.

Lemma 4.3.1 *Let X be scheme and G/X a group scheme. Then any G-torsor is an algebraic X-space.*

Proof. Let E be a G-torsor. Then there exists a faithfully flat X-scheme U of finite presentation and a cartesian diagram of X-spaces

$$\begin{array}{ccc} U \times_X G & \longrightarrow & U \\ {\scriptstyle s \circ p_1} \downarrow & \square & \downarrow \\ E & \longrightarrow & X \end{array}$$

induced by a section s of E over U. So $s \circ p_1 : U \times_X G \to E$ is a presentation of E. This proves that E is an algebraic X-space. Another way of saying this is, that the property of being an algebraic space is local with respect to the fppf-topology. $\square$

Lemma 4.3.2 *Let S be a scheme and $\pi : E \to X$ a smooth surjective morphism of algebraic S-spaces. Assume that X is quasi-compact. Then there exists an étale morphism of finite type $U \to X$ such that π has a section over U, i.e. such that there exists a morphism of algebraic S-spaces $s : U \to E$ such that $\pi \circ s = \mathrm{id}_X$.*

Proof. This follows immediately from the corresponding result for schemes, which is well-known. $\square$

Corollary 4.3.3 *Let X be a scheme and G/X a smooth group scheme. Then any G-torsor is locally trivial with respect to the étale topology on X.*

Proof. Without loss of generality we may assume that X is quasi-compact. Let E by a G-torsor. By lemma 4.3.1 E is an algebraic X-space. Of course E is smooth over X. So by lemma 4.3.2 $E \to X$ has local sections with respect to the étale topology. $\square$

Lemma 4.3.4 *Let $\pi : Y \to X$ be a finite morphism of schemes. Let G be a group scheme over X.*

i. If π is flat then $G \to \pi_* \pi^* G$ is a monomorphism.

ii. If π is a nilpotent closed immersion and G is smooth over X then $G \to \pi_* \pi^* G$ is an epimorphism, even with respect to the Zariski topology on X.

Proof. First let T be any X-scheme. We have $\pi_* \pi^* G(T) = \pi^* G(Y \times_X T) = G(Y \times_X T)$. So we need to check whether the restriction map $G(T) \to G(Y \times_X T)$ is a monomorphism. But this follows immediately from the faithful flatness of $Y \times_X T \to T$.

For the second part note that if T is an affine X-scheme then the restriction map $G(T) \to G(Y \times_X T)$ is an epimorphism, since G is formally smooth over X. So $G(T) \to \pi_* \pi^* G(T)$ is surjective whenever T is an affine X-scheme. This clearly implies the result. $\square$

4.4 Algebraicity of the Stack of G-Bundles

Representability of Certain Direct Images

Proposition 4.4.1 *Let X be a projective scheme over the field k. Let $\pi : X \to \operatorname{Spec} k$ be the structure morphism. Let S be a k-scheme and $Y \to X_S$ an affine X_S-scheme of finite presentation. Then $\pi_{S*} Y$ is an affine S-scheme of finite presentation.*

Proof. Since the proposition is local in S, we may assume that S is affine. Then using [12, Proposition 8.9.1] we reduce to the case that S is the spectrum of a noetherian ring. Choose a very ample invertible sheaf $\mathcal{O}_X(1)$ on X. We have $Y = \operatorname{Spec} \mathcal{A}$, where $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_{X_S}$-algebras of finite type. Then by [14, 6.9.13] there exists a coherent $\mathcal{O}_{X_S}$-submodule

$\mathcal{E}' \subset \mathcal{A}$ generating $\mathcal{A}$. Then there exists an $\mathcal{O}_{X_S}$-module $\mathcal{E}$ of the type $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{O}_{X_S}(n_i)$ together with an epimorphism $\mathcal{E} \to \mathcal{E}'$. Thus we get a surjective morphism of $\mathcal{O}_{X_S}$-algebras $\mathbf{S}(\mathcal{E}) \to \mathcal{A}$. Let $\mathcal{I}$ be the kernel. Again by [14, 6.9.13] there exists a coherent submodule $\mathcal{F}' \subset \mathcal{I}$ generating $\mathcal{I}$. Let $\mathcal{F}$ be a sheaf of the type $\mathcal{F} = \bigoplus_{i=1}^m \mathcal{O}_{X_S}(m_i)$ mapping onto $\mathcal{F}'$. Then we get a morphism of $\mathcal{O}_{X_S}$-algebras $\mathbf{S}(\mathcal{F}) \to \mathbf{S}(\mathcal{E})$ such that $\mathcal{F}$ generates $\mathcal{I}$. Let $E = \operatorname{Spec} \mathbf{S}(\mathcal{E})$ and $F = \operatorname{Spec} \mathbf{S}(\mathcal{F})$ be the corresponding geometric vector bundles over X_S. We have a cartesian diagram of X_S-schemes

$$\begin{array}{ccc} Y & \longrightarrow & X_S \\ \downarrow & \square & \downarrow 0 \\ E & \longrightarrow & F. \end{array}$$

This follows easily from the exactness of the following sequence of abelian groups:

$$0 \to \operatorname{Hom}_{\mathcal{O}_U\text{-alg.}}(\mathcal{A}_U, \mathcal{O}_U) \to \operatorname{Hom}_{\mathcal{O}_U\text{-mod.}}(\mathcal{E}_U, \mathcal{O}_U) \to \operatorname{Hom}_{\mathcal{O}_U\text{-mod.}}(\mathcal{F}_U, \mathcal{O}_U),$$

where U is any X_S-scheme.

By the formal properties of the functor π_{S*} we get a cartesian diagram of X-sheaves

$$\begin{array}{ccc} \pi_{S*}Y & \longrightarrow & S \\ \downarrow & \square & \downarrow 0 \\ \pi_{S*}E & \longrightarrow & \pi_{S*}F. \end{array}$$

So we have reduced to the case that $Y = \mathbb{V}(\bigoplus_{i=1}^n \mathcal{O}_{X_S}(n_i))$. But then we are also reduced to the case that $S = \operatorname{Spec} k$, for which it is easily seen that $\pi_* \mathbb{V}(\mathcal{E}) = \mathbb{V}(\pi_* \mathcal{E})$, for any locally free coherent $\mathcal{O}_X$-module $\mathcal{E}$. $\square$

Corollary 4.4.2 *Let X be a projective scheme over the field k with structure morphism $\pi : X \to \operatorname{Spec} k$. Let G be an affine group scheme of finite type over X. Let S be a k-scheme and E, F two G_S-torsors over X_S. Then $\pi_{S*} \underline{\operatorname{Isom}}(E, F)$ is an affine S-scheme of finite presentation.*

Proof. The sheaf $\underline{\operatorname{Isom}}(E, F)$ is an ${}^E G_S$-torsor. Since ${}^E G_S$ is a form of G_S and G_S is affine of finite presentation, ${}^E G_S$ is, by faithfully flat descent, again an affine group scheme of finite presentation over X_S. By the same reasoning, $\underline{\operatorname{Isom}}(E, F)$ itself is an affine X_S-scheme of finite presentation. Then the lemma follows from Proposition 4.4.1. $\square$

Passing to Covers

Theorem 4.4.3 *Let X be a projective scheme over the field k. Let $f : Y \to X$ be a projective flat covering of X. Let G be an affine group scheme of finite type over X. Then the natural morphism of k-stacks*

$$\mathfrak{H}^1(X/k, G) \longrightarrow \mathfrak{H}^1(Y/k, f^*G)$$

is representable affine of finite presentation.

Proof. Let U be a k-scheme and $E \in \operatorname{ob} \mathfrak{H}^1(Y/k, f^*G)(U)$, i.e. E is an f^*G_U-torsor over Y_U. Let V be defined so as to make the following diagram 2-cartesian:

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & \boxed{2} & \downarrow E \\ \mathfrak{H}^1(X/k, G) & \longrightarrow & \mathfrak{H}^1(Y/k, f^*G). \end{array}$$

We need to check that V is an affine scheme of finite presentation over U.

Consider the following projections:

$$
\begin{array}{rcll}
p_i : Y \times_X Y & \longrightarrow & Y & \\
(y_1, y_2) & \longmapsto & y_i & \text{for } i = 1,2 \\
q_i : Y \times_X Y \times_X Y & \longrightarrow & Y & \\
(y_1, y_2, y_3) & \longmapsto & y_i & \text{for } i = 1,2,3 \\
\pi_{ij} : Y \times_X Y \times_X Y & \longrightarrow & Y \times_X Y & \\
(y_1, y_2, y_3) & \longmapsto & (y_i, y_j) & \text{for } (i,j) = (1,2),(2,3),(1,3)
\end{array}
$$

For any U-scheme T, $V(T)$ is the set of descent data for E_T. A descent datum is an isomorphism

$$\phi_T : p_1^* E_T \xrightarrow{\sim} p_2^* E_T$$

satisfying the cocycle condition

$$\pi_{23}^* \phi_T \circ \pi_{12}^* \phi_T = \pi_{13}^* \phi_T .$$

In other words, a descent datum is a preimage of the trivial element under the map

$$
\begin{array}{rcl}
\delta_T : \mathrm{Isom}(p_1^* E_T, p_2^* E_T) & \longrightarrow & \mathrm{Aut}(q_1^* E_T) \\
\phi_T & \longmapsto & \pi_{13}^* \phi_T^{-1} \circ \pi_{23}^* \phi_T \circ \pi_{12}^* \phi_T .
\end{array}
$$

This proves that we have a cartesian diagram of sheaves over U_{fl}

$$
\begin{array}{ccc}
V & \longrightarrow & U \\
\downarrow & \square & \downarrow \\
\rho_{2U*} \underline{\mathrm{Isom}}(p_1^* E, p_2^* E) & \xrightarrow{\delta} & \rho_{3U*} \underline{\mathrm{Aut}}(q_1^* E),
\end{array}
$$

where $\rho_i : \underbrace{Y \times_X \ldots \times_X Y}_{i} \to \mathrm{Spec}\, k$ is the structure map, for $i = 0,1,2,3$. So it suffices to prove that δ is an affine morphism of finite presentation. But this follows from Lemma 4.4.2 which says that δ is a morphism between affine U-schemes of finite presentation. $\square$

Application to Reductive Group Schemes

Proposition 4.4.4 *Let X be a projective scheme over the field k. Let V be a vector bundle of rank n over X and $G \subset GL(V)$ a closed subgroup such that the quotient $GL(V)/G$ is an affine scheme over X. Then the natural morphism of k-stacks*

$$\mathfrak{H}^1(X/k, G) \longrightarrow \mathfrak{H}^1(X/k, GL(V))$$

is representable, affine of finite presentation. In particular, $\mathfrak{H}^1(X/k, G)$ is an algebraic k-stack, locally of finite type.

Proof. By Proposition 4.2.3 for any $GL(V)_S$-torsor E, where S is a k-scheme, we have a 2-cartesian diagram of k-stacks:

$$
\begin{array}{ccc}
\pi_{S*}(E/G_S) & \longrightarrow & S \\
\downarrow & \boxed{2} & \downarrow E \\
\mathfrak{H}^1(X/k, G) & \longrightarrow & \mathfrak{H}^1(X/k, GL(V))
\end{array}
$$

where $\pi : X \to \mathrm{Spec}\, k$ is the structure morphism. By Proposition 4.4.1 to prove the first part all we need to do is exhibit E/G_S as an affine X_S-scheme of finite presentation. This question is local with respect to the fppf-topology on X_S, so we may assume that $E = GL(V)_S$. But then $E/G_S = GL(V)_S/G_S = (GL(V)/G)_S$ which is affine of finite presentation by assumption.

For the second part, we will note that $\mathfrak{H}^1(X/k, GL(V))$ is an algebraic k-stack, locally of finite type. By Change of Origin, we have $\mathfrak{H}^1(X/k, GL(V)) \cong \mathfrak{H}^1(X/k, GL_n)$. But $\mathfrak{H}^1(X/k, GL_n)$ is an open substack of $\mathfrak{Coh}_{X/k}$, the k-stack of coherent $\mathcal{O}_X$-modules, which is algebraic and locally of finite type by [19, Théorème 4.14.2.1]. (Note that even though [19, Théorème 4.14.2.1] states that $\mathfrak{Coh}_{X/k}$ is of finite type, all that is true and proved is that $\mathfrak{Coh}_{X/k}$ is *locally* of finite type.) □

Proposition 4.4.5 *Let $\pi : X \to S$ be a projective flat morphism of schemes. Let V be a vector bundle of rank n over X and $G \subset GL(V)$ a closed subgroup. Then the natural morphism of S-stacks*

$$\mathfrak{H}^1(X/S, G) \longrightarrow \mathfrak{H}^1(X/S, GL(V))$$

is representable, locally of finite presentation. In particular, $\mathfrak{H}^1(X/S, G)$ is an algebraic S-stack, locally of finite presentation.

Proof. By Proposition 4.2.3 for any $GL(V)_T$-torsor E, where T is an S-scheme, we have a 2-cartesian diagram of S-stacks:

$$\begin{array}{ccc} \pi_{T*}(E/G_T) & \longrightarrow & T \\ \downarrow & \boxed{2} & \downarrow E \\ \mathfrak{H}^1(X/S, G) & \longrightarrow & \mathfrak{H}^1(X/S, GL(V)) \end{array} .$$

Now since E/G_T is an algebraic X_T-space of finite presentation, we have by [1, 6.] that $\pi_{T*}(E/G_T)$ is an algebraic T-space of finite presentation. So we are reduced to proving that $\mathfrak{H}^1(X/S, GL(V))$ is an algebraic S-stack, locally of finite presentation. This follows as in the proof of Proposition 4.4.4 from [19, Théorème 4.14.2.1]. □

Proposition 4.4.6 *Let X be a curve over the field k and let G be a reductive group scheme over X. Then $\mathfrak{H}^1(X/k, G)$ is an algebraic k-stack, locally of finite type.*

Proof. Let $f : Y \to X$ be a finite cover such that f^*G is rationally trivial. Such an f exists by the beginning of the proof of Theorem 6.4.4. Then by Note 6.3.1 f^*G is an inner form. So by Theorem 4.4.3 we may replace X by Y and thus assume that G is an inner form. Let G_0 be the constant group of the same type as G. The group scheme G_0 is defined over k. So there exists an embedding $G_0 \to GL_n$ making G_0 into a closed subgroup of GL_n for some n. The fact that G is an inner form means that there exists an $\mathrm{Ad}\, G_0$-torsor E such that $G \cong E \times_{\mathrm{Ad}\, G_0} G_0$. So we may assume that $G = E \times_{\mathrm{Ad}\, G_0} G_0$. Via the induced map $\mathrm{Ad}\, G_0 \to \mathrm{Ad}\, GL_n = PGL_n$ we get an associated PGL_n-torsor $E \times_{\mathrm{Ad}\, G_0} PGL_n$. To this PGL_n-torsor we have the associated inner form $E \times_{\mathrm{Ad}\, G_0} PGL_n \times_{PGL_n} GL_n = E \times_{\mathrm{Ad}\, G_0} GL_n$ of GL_n. We also have a homomorphism $G = E \times_{\mathrm{Ad}\, G_0} G_0 \to E \times_{\mathrm{Ad}\, G_0} GL_n$, which identifies G as a closed subgroup scheme of $E \times_{\mathrm{Ad}\, G_0} GL_n$ since it does so locally.

Now consider the exact sequence

$$1 \longrightarrow \mathbb{G}_{\mathrm{m}} \longrightarrow GL_n \longrightarrow PGL_n \longrightarrow 1$$

of group schemes over X. Let $\overline{k}$ be an algebraic closure of k. Then we have $H^2(\overline{X}_{\text{ét}}, \mathbb{G}_{\mathrm{m}}) = 0$ by [4, Arcata, Proposition 3.1]. Hence over $\overline{k}$ any PGL_n-torsor can be lifted to a GL_n-torsor. In particular, our PGL_n-torsor $E \times_{\mathrm{Ad}\, G_0} PGL_n$ can be lifted to a GL_n-torsor F over $\overline{X}$. The torsor F is defined over a finite extension K of k. Now it is easily seen using [19, Théorème 4.1] that to prove the proposition we may replace k by K and thus we may assume that F is defined over k.

So we have a GL_n-torsor F such that $F \times_{GL_n} PGL_n = E \times_{\mathrm{Ad}\, G_0} PGL_n$. Now we have ${}^F GL_n = F \times_{GL_n, Ad} GL_n = F \times_{GL_n} PGL_n \times_{PGL_n} GL_n = E \times_{\mathrm{Ad}\, G_0} PGL_n \times_{PGL_n} GL_n = E \times_{\mathrm{Ad}\, G_0} GL_n$. Hence we have exhibited G as a closed subgroup scheme of ${}^F GL_n = GL(V)$, where V is a vector bundle over X such that F is the bundle of frames of V:

$$G \subset GL(V).$$

This homomorphism is by construction locally isomorphic to $G_0 \subset GL_n$. But by a theorem of Kostant, the quotient GL_n/G_0 over k is an affine k-scheme, since G_0 is reductive. Hence $GL(V)/G$ is an affine X-scheme. So by Proposition 4.4.4 $\mathfrak{H}^1(X/k, G)$ is an algebraic k-stack, locally of finite type, q.e.d. □

4.5 Smoothness of the Stack of G-bundles

Proposition 4.5.1 *Let S be a noetherian affine local scheme, $\pi : X \to S$ a curve and $T \subset S$ a closed subscheme defined by an ideal of square zero. Let G be a smooth group scheme over X. Then the natural morphism*

$$\Delta(X, G) \longrightarrow \Delta(X_T, G_T)$$

is essentially surjective.

Proof. Let $\iota : X_T \to X$ be the natural morphism, which is a closed immersion defined by an ideal sheaf $\mathcal{I}$ of square zero. Since $\mathcal{I}^2 = 0$ we may consider $\mathcal{I}$ as an $\mathcal{O}_{X_T}$-module. By Lemma 4.3.4 we have a natural epimorphism of sheaves of groups on X_{Zar}:

$$G \longrightarrow \iota_* G_T.$$

Let $\mathfrak{g}$ be the Lie algebra of G, which is a locally free coherent $\mathcal{O}_X$-module. If $e : X \to G$ denotes the identity section and $C_{e/G}$ is the conormal sheaf of e, then we have $\mathfrak{g}^\vee = C_{e/G}$. Now from general principles we have for any affine open subset $U \subset X$ an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(C_{e/G}|U, \mathcal{I}|U) \longrightarrow G(U) \longrightarrow G(U_T).$$

This gives rise to an exact sequence of sheaves of groups on X_{Zar}:

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(C_{e/G}, \mathcal{I}) \longrightarrow G \longrightarrow \iota_* G_T \longrightarrow 1.$$

Or, in other notation

$$0 \longrightarrow \mathfrak{g} \otimes \mathcal{I} \longrightarrow G \longrightarrow \iota_* G_T \longrightarrow 1.$$

Now apply Proposition 4.2.5 to this sequence and the morphism of topoi $X_{\mathrm{Zar}} \to S_{\mathrm{Zar}}$. We have $R^2 \pi_{S'*}({}^E(\mathfrak{g} \otimes \mathcal{I})_{S'}) = 0$ for every $S' \subset S$ Zariski-open and any $(\iota_* G)_{S'}$-torsor E, since being a coherent module is local for the fppf-topology and by [17, Chap. III, Corollary 11.2]. Hence S being a local scheme implies that $\Delta(X, G) \to \Delta(X_T, G_T)$ is essentially surjective. □

Corollary 4.5.2 *Let G be a reductive group scheme over the curve X over the field k. Then $\mathfrak{H}^1(X/k, G)$ is a smooth algebraic k-stack.*

Proof. By Proposition 4.4.6 $\mathfrak{H}^1(X/k, G)$ is an algebraic k-stack, locally of finite type. So it suffices to prove that $\mathfrak{H}^1(X/k, G)$ is formally smooth, which is precisely Proposition 4.5.1. □

5 Root Systems and Convex Solids

Introduction

We will define the notion of a root system with complementary convex solid. A root system with complementary convex solid consists of a root system Φ and an additional structure d, the complementary convex solid. If Φ is a root system in the real vector space V, then d is the family of vertices of a solid $F \subset V^*$, where V^* is the dual of V. This solid F gives rise to the name of the structure d. d is a family of points in V^*, parametrized by $\mathfrak{C}$, the set of Weyl chambers of Φ. This family d is subject to two axioms (see definition 5.2.1). The convex hull of d is F, the complementary convex solid. Alternatively, F can be described by its faces. The faces of F are parametrized by the set Λ of *fundamental weights* of Φ (see the all important Lemma 5.2.5).

The terminology 'complementary' is suggested by the fact that there is a bijection between the *facets* of Φ and the solids bounding F. To a facet P there corresponds a solid $F(P) \subset F$, the *dual solid* of P. This correspondence has the property that $\dim F(P) = n - \dim P$, where $n = \dim V$ is the rank of Φ (at least if $F(P)$ is not degenerate).

The terminology 'convex' is used because F is by definition convex, and this convexity is the key point used in the proof of our main result on root systems with complementary convex solids.

The convexity of F is used as follows: Choose a metric on V, identifying V with V^*. Then since F is compact non-empty and convex there exists a unique point $y \in F$ closest to the origin of V. This point y lies in a unique facet P of Φ. This facet is *special* (Proposition 5.3.14). Our main results (Corollaries 5.3.15 and 5.3.17) can be understood as giving different characterizations of this special facet P. (See also Definition 5.3.11.)

5.1 Root Systems

Here we will assemble some elementary facts about root systems that do not usually seem to emphasized in the literature.

We will always assume all root systems to be reduced. For convenience, we briefly review the definition of a (reduced) root system. Let V be a real vector space. A *root system in V* is a subset $\Phi \subset V$ satisfying:

i. Φ is finite and generates V.

ii. For every $\alpha \in \Phi$ there exists $\check{\alpha} \in V^*$ such that $\langle \alpha, \check{\alpha} \rangle = 2$ and such that the map $\sigma_\alpha : V \to V$ defined by $\sigma_\alpha(\xi) = \xi - \langle \xi, \check{\alpha} \rangle \alpha$ maps Φ into Φ.

iii. For every $\alpha \in \Phi$ we have $\check{\alpha}(\Phi) \subset \mathbb{Z}$.

iv. If $\alpha \in \Phi$ then $2\alpha \notin \Phi$.

Because of (i) the element $\check{\alpha}$, whose existence is guaranteed by (ii) is unique. Hence (iii) makes sense.

For the basic facts about root systems, see [3, Chap. VI].

Facets and their Corners

Let V be a finite dimensional $\mathbb{R}$-vector space of dimension n. Let Φ be a root system in V. Let $\mathfrak{C}$ denote the set of Weyl chambers of Φ.

For $\alpha \in \Phi$ let L_α be the hyperplane in V defined by

$$L_\alpha = \{\xi \in V \mid \langle \xi, \check{\alpha} \rangle = 0\}.$$

$(L_\alpha)_{\alpha\in\Phi}$ is a finite set of hyperplanes in V. Hence $(L_\alpha)_{\alpha\in\Phi}$ determines a partition of V into *facets.* We call these the *facets of* Φ. By definition, $\xi, \eta \in V$ are in the same facet if and only if for every $\alpha \in \Phi$ either $\xi \in L_\alpha$ and $\eta \in L_\alpha$ or ξ and η are strictly on the same side of L_α (i.e. $\langle \xi, \check{\alpha}\rangle\langle \eta, \check{\alpha}\rangle > 0$). The facets of dimension n are precisely the Weyl chambers of Φ. The facet of dimension zero is the origin. There is a canonical bijection between the facets of Φ and those of $\check{\Phi}$. For a facet P of Φ, we let $\check{P}$ denote the corresponding facet of $\check{\Phi}$. We partially order the set of facets of Φ by

$$P \geq Q \Longleftrightarrow P \subset \overline{Q}.$$

Definition 5.1.1 For a facet P of Φ let $\mathrm{cham}(P) = \{\mathfrak{c} \in \mathfrak{C} \mid P \geq \mathfrak{c}\}$.

Let Λ be the set of fundamental weights of Φ, i.e.

$$\Lambda = \{\lambda \in V \mid \exists \mathfrak{c} \in \mathfrak{C} : \lambda \text{ is a fundamental dominant weight w.r.t. } \mathfrak{c}\}.$$

Recall that if $\alpha_1, \ldots, \alpha_n$ is the basis of Φ defined by the Weyl chamber $\mathfrak{c}$ then the dual basis $\lambda_1, \ldots, \lambda_n$ of $\check{\alpha}_1, \ldots, \check{\alpha}_n$ is the set of fundamental dominant weights with respect to $\mathfrak{c}$.

For any $\lambda \in \Lambda$ let $\check{\lambda}$ denote the corresponding fundamental weight of $\check{\Phi}$.

Definition 5.1.2 Let P be a facet of Φ. Then $\lambda \in \Lambda$ is called a *corner of* P if $\lambda \in \overline{P}$. The set of corners of P will be denoted by $\mathrm{vert}(P)$. We also define

$$\mathrm{vert}'(P) = \{\lambda \in \Lambda \mid \exists \mathfrak{c} \in \mathrm{cham}(P) : \lambda \in \mathrm{vert}(\mathfrak{c})\}.$$

We have $\mathrm{vert}(P) \subset \mathrm{vert}'(P)$ and $\mathrm{vert}(P) = \mathrm{vert}'(P)$ if and only if P is a chamber. This follows for example from proposition 5.1.4.

Lemma 5.1.3 *Let $\mathfrak{c}$ be a Weyl chamber of Φ. Let $\lambda_1, \ldots, \lambda_n$ be the fundamental dominant weights of Φ with respect to $\mathfrak{c}$. Then* $\mathrm{vert}\,\mathfrak{c} = \{\lambda_1, \ldots, \lambda_n\}$.

Proof. $\lambda_1, \ldots, \lambda_n$ is the dual basis of $\check{\alpha}_1, \ldots, \check{\alpha}_n$, where $\alpha_1, \ldots, \alpha_n$ is the basis of Φ defined by $\mathfrak{c}$. It is well-known that

$$\begin{aligned} \mathfrak{c} &= \{\xi \in V \mid \langle \xi, \check{\alpha}_i\rangle > 0 \text{ for all } i = 1, \ldots, n\} \\ &= \{\xi \in V \mid \xi = \textstyle\sum \xi_i\lambda_i \text{ with } \xi_i > 0 \text{ for all } i = 1, \ldots, n\}. \end{aligned}$$

So $\lambda_i \in \mathrm{vert}\,\mathfrak{c}$ for all $i = 1, \ldots, n$.

Now assume $\lambda \in \mathrm{vert}\,\mathfrak{c}$. Then there exists a Weyl chamber $\mathfrak{d}$ such that λ is a fundamental dominant weight with respect to $\mathfrak{d}$. We have $\mathfrak{d} = \sigma(\mathfrak{c})$ for some $\sigma \in W$, W the Weyl group of Φ. Then $\lambda = \sigma(\lambda_i)$ for some $i = 1, \ldots, n$. Since $\lambda \in \overline{\mathfrak{c}}$ and $\lambda_i \in \overline{\mathfrak{c}}$, we get $\lambda = \lambda_i$ because $\overline{\mathfrak{c}}$ is a fundamental domain for the action of W on V. □

Proposition 5.1.4 *Let P be a facet of Φ and let $\lambda_1, \ldots, \lambda_r$ be the corners of P. Then*

i. $(\lambda_1, \ldots, \lambda_r)$ *is linearly independent,*

ii. The affine support of P is $\mathrm{span}(\lambda_1, \ldots, \lambda_r)$,

iii. $P = \{\xi \in V \mid$ *There exist* $\xi_1, \ldots, \xi_r > 0$ *such that* $\xi = \sum_{i=1}^r \xi_i\lambda_i\}$,

iv. The corners of $\check{P}$ are $\check{\lambda}_1, \ldots, \check{\lambda}_r$.

Proof. In case P is a chamber, the proposition follows from lemma 5.1.3. For the general case, let $\mathfrak{c}$ be a chamber such that $P \geq \mathfrak{c}$. Let $\alpha_1, \ldots, \alpha_n$ be the basis of Φ corresponding to $\mathfrak{c}$, $\lambda_1, \ldots, \lambda_n$ the corners of $\mathfrak{c}$. We may assume that

$$\begin{aligned} P = \{\xi \in V \mid \\ \langle \xi, \check{\alpha}_i \rangle > 0 \text{ for all } i = 1, \ldots, r \text{ and } \langle \xi, \check{\alpha}_i \rangle = 0 \text{ for all } i = r+1, \ldots, n\} \end{aligned}$$

for some $0 \leq r \leq n$. Then $\operatorname{vert} P = \{\lambda_1, \ldots, \lambda_r\}$ and the proposition is obvious. □

Corollary 5.1.5 *Let P be a facet of Φ. Then for every $\lambda \in \operatorname{vert}'(P)$, $\operatorname{vert}(P) \cup \{\lambda\}$ is again the set of corners of a facet of Φ. Similarly, $\operatorname{vert} P - \{\lambda\}$ is the set of corners of a facet for any $\lambda \in \operatorname{vert} P$.*

Parabolic Sets of Roots

Recall that a parabolic set of roots is a subset $R \subset \Phi$ such that for every $\alpha \in \Phi$ we have $\alpha \in R$ or $-\alpha \in R$.

Lemma 5.1.6 *Let P be a facet of Φ with corners $\lambda_1, \ldots, \lambda_r$. Let*

$$R(P) = \{\alpha \in \Phi \mid \langle \alpha, \check{\lambda}_i \rangle \geq 0 \text{ for all } i = 1, \ldots, r\}.$$

Then we have

$$\{\lambda_1, \ldots, \lambda_r\} = \{\lambda \in \Lambda \mid \langle \alpha, \check{\lambda} \rangle \geq 0 \text{ for all } \alpha \in R(P)\}.$$

The set $R(P)$ is called the parabolic set of roots associated to the facet P.

Proof. The inclusion '$\subset$' is obvious. For the other inclusion, choose a chamber $\mathfrak{c} \in \operatorname{cham} P$ giving a basis $\alpha_1, \ldots, \alpha_n$ of Φ with dual basis $\lambda_1, \ldots, \lambda_n$. Let $\lambda \in \Lambda$ satisfy $\langle \alpha, \check{\lambda} \rangle \geq 0$ for all $\alpha \in R(P)$. Since $\alpha_1, \ldots, \alpha_n \in R(P)$ we have $\langle \alpha_i, \check{\lambda} \rangle \geq 0$ and hence $\langle \lambda, \check{\alpha}_i \rangle \geq 0$ for all $i = 1, \ldots, n$. So $\lambda \in \bar{\mathfrak{c}}$, i.e. $\lambda = \lambda_j$ for some $j = 1, \ldots, n$. Assume that $j > r$. Then $\langle \alpha_j, \check{\lambda}_i \rangle = 0$ for all $i = 1, \ldots, r$. Thus $-\alpha_j \in R(P)$ by the definition of $R(P)$ and by the assumption on $\lambda = \lambda_j$ we get $\langle -\alpha_j, \lambda_j \rangle \geq 0$ which is a contradiction. □

Lemma 5.1.7 *Let R be a parabolic set of roots in Φ. Define*

$$\operatorname{vert} R = \{\lambda \in \Lambda \mid \langle \alpha, \check{\lambda} \rangle \geq 0 \text{ for all } \alpha \in R\}.$$

Then there exists a unique facet P of Φ such that $\operatorname{vert} P = \operatorname{vert} R$. Moreover, $R = R(P)$ in the notation of lemma 5.1.6.

Proof. There exists a Weyl chamber $\mathfrak{c}$ such that R contains the set of positive roots with respect to $\mathfrak{c}$. Let $\alpha_1, \ldots, \alpha_n$ be the basis of Φ defined by $\mathfrak{c}$ and $\lambda_1, \ldots, \lambda_n$ the corners of $\mathfrak{c}$. Then there exists a subset, say $\lambda_1, \ldots, \lambda_r$, such that

$$R = \{\alpha \in \Phi \mid \langle \alpha, \lambda_i \rangle \geq 0 \text{ for all } i = 1, \ldots, r\}.$$

By corollary 5.1.5, there exists a facet P such that $\operatorname{vert} P = \{\lambda_1, \ldots, \lambda_r\}$. So $R = R(P)$ and by lemma 5.1.6 we also have $\{\lambda_1, \ldots, \lambda_r\} = \operatorname{vert} R$. □

Corollary 5.1.8 *There is a bijective correspondence*

$$\begin{aligned} \{\text{facets of } \Phi\} &\longrightarrow \{\text{parabolic subsets of } \Phi\} \\ P &\longmapsto R(P) \end{aligned}$$

whose inverse is given by

$$R \longmapsto \text{the unique } P \text{ such that } \operatorname{vert} P = \operatorname{vert} R.$$

This correspondence is order preserving.

Another way of looking at this bijective correspondence is given by the following proposition:

Proposition 5.1.9 *Let P be a facet of Φ, R the corresponding parabolic subset of Φ. Then*

$$\sum_{\alpha \in R} \alpha \in P.$$

Proof. Choose a Weyl chamber $\mathfrak{c} \in \operatorname{cham} P$ with basis $\alpha_1, \ldots, \alpha_n$ and corners $\lambda_1, \ldots, \lambda_n$ such that $\operatorname{vert} P = \{\lambda_1, \ldots, \lambda_r\}$. Let σ_i be the reflection defined by α_i for $i = 1, \ldots, n$. Let $\beta = \sum_{\alpha \in R} \alpha$.

Consider an $i > r$. σ_i permutes R as can easily be seen by looking at the formula $\sigma_i(\alpha) = \alpha - \langle \alpha, \check{\alpha}_i \rangle \alpha_i$. Hence $\sigma_i(\beta) = \beta$ and $\beta \in \check{\alpha}_i^{\perp}$. Therefore, $\beta \in \operatorname{span}(\lambda_1, \ldots, \lambda_r)$. So to prove that $\beta \in P$ it suffices to prove that $\langle \beta, \check{\alpha}_i \rangle > 0$ for all $i = 1, \ldots, r$. To this end, consider for $i = 1, \ldots, r$ the set

$$R_i = \{\alpha \in R \mid \langle \alpha, \check{\alpha}_i \rangle < 0\}.$$

It is easily checked that $\sigma_i(R_i) \subset R$ and that $R_i \cap \sigma_i(R_i) = \varnothing$. So we can decompose R as follows:

$$R = R_i \cup \sigma_i(R_i) \cup S_i \cup \{\alpha_i\},$$

where the union is disjoint. We may now calculate

$$\begin{aligned}
\langle \beta, \check{\alpha}_i \rangle &= \sum_{\alpha \in R_i} \langle \alpha, \check{\alpha}_i \rangle + \sum_{\alpha \in \sigma_i(R_i)} \langle \alpha, \check{\alpha}_i \rangle + \sum_{\alpha \in S_i} \langle \alpha, \check{\alpha}_i \rangle + \langle \alpha_i, \check{\alpha}_i \rangle \\
&\geq \sum_{\alpha \in R_i} \left(\langle \alpha, \check{\alpha}_i \rangle + \langle \sigma_i(\alpha), \check{\alpha}_i \rangle \right) + 2 \\
&= 2
\end{aligned}$$

which finishes the proof. □

Reduction of a Root System to a Facet

Let P be a facet of Φ. P induces direct sum decompositions $V = \operatorname{span} P \oplus \check{P}^{\perp}$ and $V^* = \operatorname{span} \check{P} \oplus P^{\perp}$. We set $V_P = \check{P}^{\perp}$ and $V_P^* = P^{\perp}$. Then we identify V_P^* as the dual of V_P. Let $p : V \to V_P$ and $\check{p} : V^* \to V_P^*$ be the projections given by the above direct sum decompositions. Note that $\check{p}$ is the adjoint of p. $\Phi_P = \Phi \cap V_P$ is a root system in V_P. We call Φ_P the *reduction of Φ to the facet P*. The dual of Φ_P in V_P^* is $\check{\Phi}_P = \check{\Phi} \cap V_P^*$. Let $\mathfrak{C}_P$ denote the set of Weyl chambers of Φ_P, Λ_P the set of fundamental weights of Φ_P.

Lemma 5.1.10 *p induces a bijection*

$$\begin{aligned}
p : \{\text{Facets } Q \text{ of } \Phi \text{ such that } Q \leq P\} &\longrightarrow \{\text{facets of } \Phi_P\} \\
Q &\longmapsto p(Q).
\end{aligned}$$

For a facet Q of Φ such that $Q \leq P$ we have $\dim p(Q) = \dim Q - \dim P$. In particular, we get a bijection

$$p : \operatorname{cham} P \longrightarrow \mathfrak{C}_P.$$

Proof. Straightforward calculation. □

Lemma 5.1.11 *p induces a bijection*

$$\begin{aligned}
p : \operatorname{vert}'(P) - \operatorname{vert}(P) &\longrightarrow \Lambda_P \\
\lambda &\longmapsto p(\lambda).
\end{aligned}$$

Proof. Clear. □

5.2 Root Systems with Complementary Convex Solids

Complementary Convex Solids

Now we are prepared to give the fundamental definition of this section:

Definition 5.2.1 Let $d = (d(\mathfrak{c}))_{\mathfrak{c}\in\mathfrak{C}}$ be a family of vectors in V^*, parametrized by the set of Weyl chambers of Φ. This family d is called a *complementary convex solid for* Φ or, alternatively, the pair (Φ, d) is called a *root system with complementary convex solid* if the following two axioms are satisfied:

(B1) If λ is a corner of both of the Weyl chambers $\mathfrak{c}$ and $\mathfrak{d}$, then

$$\langle \lambda, d(\mathfrak{c})\rangle = \langle \lambda, d(\mathfrak{d})\rangle.$$

(B2) If the Weyl chambers $\mathfrak{c}$ and $\mathfrak{d}$ have precisely $n-1$ corners in common, and if $\alpha \in \Phi$ is the unique root that is positive with respect to $\mathfrak{c}$ and negative with respect to $\mathfrak{d}$, then

$$\langle \alpha, d(\mathfrak{c})\rangle \leq \langle \alpha, d(\mathfrak{d})\rangle.$$

Let P be a facet of Φ. Then we define the *dual solid* $F(P)$ *of* P by

$$F(P) = \operatorname*{conv}_{\mathfrak{c}\in \operatorname{cham} P}(d(\mathfrak{c})).$$

We also define $F = F(\{0\})$. F is the convex solid that gives rise to the name of the structure defined above.

Note 5.2.2 If $\Phi \neq \varnothing$ then $F \neq \varnothing$. If $P \leq Q$ for facets P and Q of Φ, then $F(P) \subset F(Q)$.

For the next definition let Φ be endowed with a complementary convex solid d. Let P be a facet of Φ.

Definition 5.2.3 The *complementary convex solid* d_P *for* Φ_P *induced by* d is defined by

$$d_P(\mathfrak{c}) = \check{p}(d(p^{-1}\mathfrak{c})),$$

for each $\mathfrak{c} \in \mathfrak{C}_P$. Here p^{-1} is the inverse of the map of lemma 5.1.10.

It is easy to check that d_P satisfies axioms (B1) and (B2).

Note 5.2.4 Let Q be a facet of Φ_P and $F_P(Q)$ the dual solid of Q with respect to d_P. Then

$$F_P(Q) = \check{p}(F(p^{-1}Q)).$$

In particular, letting $F_P = F_P(\{0\})$, we have $F_P = \check{p}(F(P))$.

Characterization of F in Terms of Faces

Let $d = (d(\mathfrak{c}))_{\mathfrak{c}\in\mathfrak{C}}$ be a complementary convex solid for Φ. For $\lambda \in \Lambda$ let H_λ be the hyperplane in V^* defined as follows:

$$H_\lambda = \{x \in V^* \mid \langle \lambda, x\rangle = \langle \lambda, d(\mathfrak{c})\rangle\},$$

where $\mathfrak{c}$ is any Weyl chamber such that λ is a corner of $\mathfrak{c}$. By (B1) the definition of H_λ does not depend on the choice of $\mathfrak{c}$. $(H_\lambda)_{\lambda\in\Lambda}$ is a finite set of hyperplanes in V^*.

We define the positive half space associated with λ:

$$E_\lambda^+ = \{x \in V^* \mid \langle \lambda, x\rangle > \langle \lambda, d(\mathfrak{c})\rangle\}.$$

We will also have to consider the closure $\overline{E_\lambda^+}$ of E_λ^+.

Lemma 5.2.5 *Let P be a facet of Φ. Then*

$$F(P) = \bigcap_{\lambda \in \operatorname{vert} P} H_\lambda \cap \bigcap_{\lambda \in \operatorname{vert}' P} \overline{E}^+_\lambda.$$

For example,

$$F = \bigcap_{\lambda \in \Lambda} \overline{E}^+_\lambda.$$

Proof. Let P be a facet of Φ. Consider the projection $\check{p} : V^* \to V_P^*$ and let $\check{p}_1$ be its restriction to $\bigcap_{\lambda \in \operatorname{vert} P} H_\lambda$:

$$\check{p}_1 : \bigcap_{\lambda \in \operatorname{vert} P} H_\lambda \longrightarrow V_P^*.$$

First note that $\check{p}_1$ is a bijection. $\check{p}_1(F(P)) = F_P$ and for any $\mu \in \operatorname{vert}' P - \operatorname{vert} P$ we have

$$\check{p}_1 \left(\bigcap_{\lambda \in \operatorname{vert} P} H_\lambda \cap \overline{E}^+_\mu \right) = \overline{E}^+_{p(\mu)},$$

where $p(\mu)$ is considered as a fundamental weight of Φ_P. Hence

$$\check{p}_1 \left(\bigcap_{\lambda \in \operatorname{vert} P} H_\lambda \cap \bigcap_{\lambda \in \operatorname{vert}' P} \overline{E}^+_\lambda \right) = \bigcap_{\lambda \in \Lambda_P} \overline{E}^+_\lambda,$$

which reduces the proof to the case $P = \{0\}$. so the claim is now

$$\operatorname*{conv}_{\mathfrak{c} \in \mathfrak{C}}(d(\mathfrak{c})) = \bigcap_{\lambda \in \Lambda} \overline{E}^+_\lambda. \tag{20}$$

Let us first prove the inclusion '$\subset$'. To this end define $\eta(\mathfrak{c}) = \sum_{\lambda \in \operatorname{vert} \mathfrak{c}} \lambda$ for every $\mathfrak{c} \in \mathfrak{C}$. We choose a fundamental weight λ and a Weyl chamber $\mathfrak{c}_0$ such that $\lambda \in \operatorname{vert} \mathfrak{c}_0$. Let $\mathfrak{c}$ be an arbitrary Weyl chamber. We need to show that $\langle \lambda, d(\mathfrak{c}) \rangle \geq \langle \lambda, d(\mathfrak{c}_0) \rangle$. Let $\alpha_1, \dots, \alpha_n$ be the simple roots defined by $\mathfrak{c}$ and $\lambda_1, \dots, \lambda_n$ the corners of $\mathfrak{c}$. If for all $i = 1, \dots, n$ we have $\langle \lambda, \check{\alpha}_i \rangle \geq 0$, then $\lambda \in \overline{\mathfrak{c}}$ and we are done by (B1). So we may assume without loss of generality that $\langle \lambda, \check{\alpha}_1 \rangle < 0$. Let σ_1 be the reflection defined by α_1 and let $\mathfrak{d} = \sigma_1(\mathfrak{c})$. By (B2) we get $\langle \alpha_1, d(\mathfrak{c}) \rangle \leq \langle \alpha_1, d(\mathfrak{d}) \rangle$. Moreover, $d(\mathfrak{c}) - d(\mathfrak{d}) \in \mathbb{R}\check{\alpha}_1$ by (B1), because $\lambda_2, \dots, \lambda_n$ are corners of both $\mathfrak{c}$ and $\mathfrak{d}$. So in fact $d(\mathfrak{c}) - d(\mathfrak{d}) = x\check{\alpha}_1$ with $x \leq 0$.

Now we have

$$\begin{aligned} \langle \lambda, d(\mathfrak{c}) \rangle &= \langle \lambda, d(\mathfrak{c}) - d(\mathfrak{d}) \rangle + \langle \lambda, d(\mathfrak{d}) \rangle \\ &= x \langle \lambda, \check{\alpha}_1 \rangle + \langle \lambda, d(\mathfrak{d}) \rangle \\ &\geq \langle \lambda, d(\mathfrak{d}) \rangle. \end{aligned}$$

So it suffices to prove that $\langle \lambda, d(\mathfrak{d}) \rangle \geq \langle \lambda, d(\mathfrak{c}_0) \rangle$. But we have $\eta(\mathfrak{d}) = \eta(\mathfrak{c}) - \alpha_1$ and hence

$$\begin{aligned} \langle \eta(\mathfrak{d}), \check{\lambda} \rangle &= \langle \eta(\mathfrak{c}), \check{\lambda} \rangle - \langle \alpha_1, \check{\lambda} \rangle \\ &> \langle \eta(\mathfrak{c}), \check{\lambda} \rangle, \end{aligned}$$

because $\langle \lambda, \check{\alpha}_1 \rangle < 0$ implies $\langle \alpha_1, \check{\lambda} \rangle < 0$. So since there are only finitely many Weyl chambers we are done.

The other inclusion we will prove by induction on the rank n of Φ. So we assume (20) to hold for root systems of rank $n-1$. Then by the above argument involving $\check{p}_1$, the lemma holds for all one-dimensional facets of Φ. This implies for every $\lambda \in \Lambda$:

$$H_\lambda \cap \bigcap_{\mu\in\Lambda} \overline{E}_\mu^+ \subset \operatorname*{conv}_{\mathfrak{c}\in\mathfrak{C}} d(\mathfrak{c}). \tag{21}$$

to prove (20) choose an element $x \in \bigcap_{\lambda\in\Lambda} \overline{E}_\lambda^+$. Then choose any fundamental weight $\lambda \in \Lambda$ and consider the line $L = (x+t\check{\lambda})_{t\in\mathbb{R}}$. Obviously, $L \cap \bigcap_{\mu\in\Lambda} \overline{E}_\mu^+$ is compact by considering for example $\overline{E}_\lambda^+$ and $\overline{E}_{-\lambda}^+$. It is also clearly convex. Let $[t_0, t_1]$ be the interval such that $L \cap \bigcap_{\mu\in\Lambda} \overline{E}_\mu^+ = \{x + t\check{\lambda} \mid t \in [t_0, t_1]\}$. Then we have $x + t_i\check{\lambda} \in \bigcap_{\mu\in\Lambda} \overline{E}_\mu^+$ for $i = 0, 1$ and $x + t_0\check{\lambda} \in H_{\lambda_0}$ and $x + t_1\check{\lambda} \in H_{\lambda_1}$ for suitable $\lambda_0, \lambda_1 \in \Lambda$. By (21) $x + t_i\check{\lambda} \in \operatorname{conv}_{\mathfrak{c}\in\mathfrak{C}} d(\mathfrak{c})$ for $i = 0, 1$. Hence $x \in \operatorname{conv}_{\mathfrak{c}\in\mathfrak{C}} d(\mathfrak{c})$ also, because $x \in \operatorname{conv}(x + t_0\check{\lambda}, x + t_1\check{\lambda})$. □

By lemma 5.2.5 we can interpret $(d(\mathfrak{c}))_{\mathfrak{c}\in\mathfrak{C}}$ as the family of vertices of the solid F and $(H_\lambda)_{\lambda\in\Lambda}$ as the family of (affine supports of the) faces of F. In particular, d can be recovered from the finite set of hyperplanes $(H_\lambda)_{\lambda\in\Lambda}$.

5.3 Stability of Root Systems with Complementary Convex Solids

Again, let Φ be a root system in the n-dimensional $\mathbb{R}$-vector space V.

Stability

For a facet P of Φ we let

$$R(P) = \{\alpha \in \Phi \mid \text{for all } \lambda \in \operatorname{vert} P\text{: } \langle \alpha, \check{\lambda}\rangle \geq 0\}$$

be the corresponding parabolic set of roots (see Corollary 5.1.8). We also define

$$U(P) = \{\alpha \in \Phi \mid \text{there exists } \lambda \in \operatorname{vert} P\text{: } \langle \alpha, \check{\lambda}\rangle > 0\}.$$

We have $U(P) \subset R(P)$ and in fact $R(P)$ is the disjoint union of $U(P)$ and Φ_P.

Definition 5.3.1 Assume that Φ is endowed with a complementary convex solid d. Let P be a facet of Φ, R the corresponding parabolic set of roots. Then we define the *degree of P (resp. the degree of R) with respect to d* as

$$\deg P = \deg R = \sum_{\alpha\in R} \langle \alpha, d(\mathfrak{c})\rangle = \sum_{\alpha\in U(P)} \langle \alpha, d(\mathfrak{c})\rangle,$$

where $\mathfrak{c}$ is any Weyl chamber such that $P \subset \overline{\mathfrak{c}}$ (or equivalently, $\mathfrak{c}$ is any Weyl chamber such that R contains the roots positive with respect to $\mathfrak{c}$).

By proposition 5.1.9 and property (B1) this definition does not depend on the choice of $\mathfrak{c}$.

Proposition 5.3.2 *If we have a root system with complementary convex solid (Φ, d), then the following are equivalent:*

i. For every facet P of Φ we have $\deg P \leq 0$.

ii. For every one-dimensional facet P of Φ we have $\deg P \leq 0$.

iii. For every pair $(\lambda, \mathfrak{c}) \in \Lambda \times \mathfrak{C}$ *such that* $\lambda \in \operatorname{vert} \mathfrak{c}$ *we have* $\langle \lambda, d(\mathfrak{c}) \rangle \leq 0$.

iv. $0 \in F$.

Proof. (i)⇒(ii) is obvious. In view of proposition 5.1.9 (iii) is just a reformulation of (ii). The equivalence of (iii) and (iv) follows from the characterization 5.2.5 of F. (iii) implies (i) by proposition 5.1.9. □

Definition 5.3.3 We call a root system with complementary convex solid (Φ, d) *semi-stable* if the conditions of proposition 5.3.2 are satisfied.

Remark 5.3.4 Call (Φ, d) *stable* if any of the following equivalent conditions is satisfied:

i. For every positive-dimensional facet P of Φ we have $\deg P < 0$.

ii. For every one-dimensional facet P of Φ we have $\deg P < 0$.

iii. For every pair $(\lambda, \mathfrak{c}) \in \Lambda \times \mathfrak{C}$ such that $\lambda \in \operatorname{vert} \mathfrak{c}$ we have $\langle \lambda, d(\mathfrak{c}) \rangle < 0$.

iv. 0 is an interior point of F.

The behavior of the degree with respect to reduction is described by the following lemma. Choose a complementary convex solid d for Φ.

Lemma 5.3.5 *Let P be a facet of Φ and let $Q \leq P$ be another facet. Let $p : V \to V_P$ be the reduction map. Then we have*

$$\deg Q = \deg p(Q) + \deg P.$$

Proof. Choose a Weyl chamber $\mathfrak{c} \in \operatorname{cham} Q$. Let $\lambda_1, \ldots, \lambda_r$ be the corners of P and $\lambda_1, \ldots, \lambda_s$ the corners of Q. Then

$$\deg Q - \deg P = \sum_{\alpha \in U(Q) - U(P)} \langle \alpha, d(\mathfrak{c}) \rangle,$$

but

$$\begin{aligned}
& U(Q) - U(P) \\
&= \{\alpha \in \Phi \mid \exists i = r+1, \ldots, s : \langle \alpha, \check{\lambda}_i \rangle > 0 \text{ and } \forall i = 1, \ldots, r : \langle \alpha, \check{\lambda}_i \rangle = 0\} \\
&= \{\alpha \in \Phi \mid \exists \lambda \in \operatorname{vert}(p(Q)) : \langle \alpha, \check{\lambda} \rangle > 0\} \\
&= U(p(Q)).
\end{aligned}$$

Hence

$$\begin{aligned}
\deg Q - \deg P &= \sum_{\alpha \in U(p(Q))} \langle \alpha, d(\mathfrak{c}) \rangle \\
&= \sum_{\alpha \in U(p(Q))} \langle \alpha, \check{p}(d(\mathfrak{c})) \rangle \\
&= \deg p(Q),
\end{aligned}$$

so that we are done. □

The Numerical Invariants

For what follows, we will need numerical invariants for facets that are finer than the degree.

Definition 5.3.6 Let P be a facet of Φ. Let λ be a corner of P. Then define

$$\Psi(P,\lambda) = \{\alpha \in \Phi \mid \langle \alpha, \check{\lambda} \rangle = 1 \text{ and } \forall \mu \in \operatorname{vert}(P) - \{\lambda\} : \langle \alpha, \check{\mu} \rangle = 0\}.$$

We call $\Psi(P,\lambda)$ the *elementary set of roots associated to P and λ.*

Lemma 5.3.7 *Let P be a facet of Φ. Then for any corner $\lambda \in \operatorname{vert}(P)$ we have*

$$\sum_{\alpha \in \Psi(P,\lambda)} \alpha \in \operatorname*{span}_{\lambda \in \operatorname{vert} P}(\lambda).$$

Proof. Let $\mathfrak{c} \in \operatorname{cham} P$. Let $\alpha_1, \ldots, \alpha_n$ be the basis associated to $\mathfrak{c}$, $\lambda_1, \ldots, \lambda_n$ the corners of $\mathfrak{c}$ and $\lambda_1, \ldots, \lambda_r$ the corners of P. Let $\sigma_i = \sigma_{\alpha_i}$ for $i = 1, \ldots, n$ be the generators of the Weyl group associated to $\alpha_1, \ldots, \alpha_n$. Let $\alpha \in \Psi(P,\lambda)$. Consider for $i > r$

$$\sigma_i(\alpha) = \alpha - \langle \alpha, \check{\alpha}_i \rangle \alpha_i.$$

For $j \leq r$ we have

$$\langle \sigma_i(\alpha), \check{\lambda}_j \rangle = \langle \alpha, \check{\lambda}_j \rangle$$

so that $\sigma_i(\alpha) \in \Psi(P,\lambda)$ also. So for $i > r$ σ_i permutes $\Psi(P,\lambda)$. Hence $\sum_{\alpha \in \Psi(P,\lambda)} \alpha$ is invariant under σ_i and therefore

$$\langle \sum_{\alpha \in \Psi(P,\lambda)} \alpha, \check{\alpha}_i \rangle = 0$$

for all $i = r+1, \ldots, n$. This proves that $\sum_{\alpha \in \Psi(P,\lambda)} \alpha \in \operatorname{span}(\lambda_1, \ldots, \lambda_r)$. $\square$

Now assume that Φ is endowed with a complementary convex solid d.

Definition 5.3.8 For a facet P and a corner λ of P define

$$n(P,\lambda) = \sum_{\alpha \in \Psi(P,\lambda)} \langle \alpha, d(\mathfrak{c}) \rangle.$$

Here $\mathfrak{c}$ is any Weyl chamber in $\operatorname{cham} P$. We call $n(P,\lambda)$ the *numerical invariant of P with respect to λ and d.*

This definition is justified by lemma 5.3.7 and (B1).

Note 5.3.9 For any $\mathfrak{c} \in \mathfrak{C}$ we have

$$d(\mathfrak{c}) = \sum_{\lambda \in \operatorname{vert} \mathfrak{c}} n(\mathfrak{c},\lambda)\check{\lambda}.$$

So d can be reconstructed from the numerical invariants of the chambers.

The behavior of the numerical invariants with respect to reduction is described by the following lemma.

Lemma 5.3.10 *Let P be a facet of Φ and let $Q \leq P$ be another facet. Let $p : V \to V_P$ be the reduction map. Let $\lambda \in \operatorname{vert} Q - \operatorname{vert} P$. Then*

$$n(Q,\lambda) = n(p(Q), p(\lambda)).$$

Proof. This follows easily from $\Psi(Q,\lambda) = \Psi(p(Q), p(\lambda))$ which is also easy to see. $\square$

Special Facets

Definition 5.3.11 A facet P of Φ is called *special with respect to the complementary convex solid* d, if the following two conditions are satisfied:

(S1) For all $\lambda \in \operatorname{vert} P$ we have $n(P, \lambda) > 0$.

(S2) (Φ_P, d_P) is semi-stable.

Our goal is to prove that there is exactly one special facet for a given root system with complementary convex solid.

Let P be a facet of Φ. Then P determines uniquely a point $y(P) \in V^*$ by:

$$\{y(P)\} = \bigcap_{\lambda \in \operatorname{vert}(P)} H_\lambda \cap \operatorname{span} \check{P}.$$

Lemma 5.3.12 *The facet P of Φ is special if and only if the following two conditions are satisfied:*

(S1′) $y(P) \in \check{P}$,

(S2′) $y(P) \in F(P)$.

Proof. Let $\lambda_1, \ldots, \lambda_r$ be the corners of the facet P. Let β_i be the average over $\Psi(P, \lambda_i)$ for $i = 1, \ldots, r$. Then $\langle \beta_i, \check{\lambda}_j \rangle = \delta_{ij}$ for all $i, j = 1, \ldots, r$. So by lemma 5.3.7 $(\beta_1, \ldots, \beta_r)$ is a basis for $\operatorname{span}(\lambda_1, \ldots, \lambda_r)$.

By the definition of $y(P)$ we have $\langle \lambda_i, y(P) \rangle = \langle \lambda_i, d(\mathfrak{c}) \rangle$ for $i = 1, \ldots, r$. Hence $\langle \beta_i, y(P) \rangle = \langle \beta_i, d(\mathfrak{c}) \rangle$ for $i = 1, \ldots, r$. Now we have

$$\begin{aligned} y(P) &= \sum_{i=1}^{r} \langle \beta_i, y(P) \rangle \check{\lambda}_i \\ &= \sum_{i=1}^{r} \langle \beta_i, d(\mathfrak{c}) \rangle \check{\lambda}_i \\ &= \sum_{i=1}^{r} \frac{n(P, \lambda_i)}{\# \Psi(P, \lambda_i)} \check{\lambda}_i. \end{aligned}$$

This makes the equivalence of (S1) and (S1′) obvious.

Let us prove the equivalence of (S2) and (S2′):

$$\begin{aligned} (\Phi_P, d_P) \text{ is semi-stable} &\Leftrightarrow 0 \in F_P \\ &\Leftrightarrow 0 \in \check{p}(F(P)) \\ &\Leftrightarrow \operatorname{span} \check{P} \cap F(P) \neq \varnothing \\ &\Leftrightarrow y(P) \in F(P) \end{aligned}$$

since $F(P) \subset \bigcap_{i=1}^{r} H_{\lambda_i}$. □

Note 5.3.13 Another way of formulating this result is:

$$P \text{ is special} \Longleftrightarrow \check{P} \cap F(P) \neq \varnothing.$$

In case P is special, $y(P)$ is the unique element of $\check{P} \cap F(P)$.

Proposition 5.3.14 *Endow V with a metric compatible with W, the Weyl group of Φ. If P is a special facet of (Φ, d), then $y(P)$ is the unique point of F closest to the origin of V^*. Conversely, if y is the unique point of F closest to the origin, and P is the facet such that $y \in \check{P}$, then P is special and $y = y(P)$.*

Proof. Let $(,)$ denote a scalar product on V such that W acts through isometries on V. Use $(,)$ to identify V and V^*. Let $\|\cdot\|$ be the associated norm. F being a closed convex and non-empty set in V it contains a unique point of minimal distance from $0 \in V$. Call this point y.

Let P be a special facet with corners $\lambda_1, \ldots, \lambda_r$ and choose a chamber $\mathfrak{c} \in \operatorname{cham} P$. To prove that $y(P) = y$, first note that $y(P) \in F$ by (S2′) and note 5.2.2, then let $x \in F$ be arbitrary and prove that $\|x\| \geq \|y(P)\|$:

Choose $i \in \{1, \ldots, r\}$. x being in F implies that $\langle \lambda_i, x \rangle \geq \langle \lambda_i, d(\mathfrak{c}) \rangle$. By the definition of $y(P)$ we have $\langle \lambda_i, y(P) \rangle = \langle \lambda_i, d(\mathfrak{c}) \rangle$, so we get $\langle \lambda_i, x - y(P) \rangle \geq 0$.

Since $y(P)$ satisfies (S1′) we have $y(P) = \sum_{i=1}^r \eta_i \lambda_i$ with $\eta_i > 0$ for $i = 1, \ldots, r$, because the $\check{\lambda}_i$ are simply positive scalar multiples of the λ_i. Now we have

$$\begin{aligned}
\|x\|^2 &= \|y(P)\|^2 + \|x - y(P)\|^2 + 2(y(P), x - y(P)) \\
&\geq \|y(P)\|^2 + 2(y(P), x - y(P)) \\
&= \|y(P)\|^2 + 2\sum_{i=1}^{r} \eta_i (\lambda_i, x - y(P)) \\
&\geq \|y(P)\|^2
\end{aligned}$$

by the above remarks.

Now let P be the unique facet such that $y \in \check{P}$. Consider the C^∞ function

$$\begin{aligned}
f : [0,1] &\longrightarrow \mathbb{R} \\
t &\longmapsto \|y + t(d(\mathfrak{c}) - y)\|^2.
\end{aligned}$$

For all $t \in [0,1]$ we have $y + t(d(\mathfrak{c}) - y) \in F$ since F is convex. By assumption, f assumes its minimum at $t = 0$. Hence $f'(0) \geq 0$. Differentiating f yields the result $(y, d(\mathfrak{c}) - y) \geq 0$. Since $y \in P$ we have $y = \sum_{i=1}^r \eta_i \lambda_i$ with $\eta_i > 0$ for $i = 1, \ldots, r$. So we have $\sum_{i=1}^r \eta_i (\lambda_i, d(\mathfrak{c}) - y) \geq 0$. On the other hand, $y \in F$ implies $(\lambda_i, d(\mathfrak{c}) - y) \leq 0$ for all $i = 1, \ldots, r$. This certainly forces $(\lambda_i, y) = (\lambda_i, d(\mathfrak{c}))$ for $i = 1, \ldots, r$. So $y \in \bigcap_{\lambda \in \operatorname{vert} P} H_\lambda$. In particular, $y \in F(P) \cap P$ so that P is special and $y = y(P)$. □

Corollary 5.3.15 *(Φ, d) has a unique special facet.*

Proof. Clear. □

Proposition 5.3.16 *If P is a facet that is maximal among the facets of maximal degree of (Φ, d), then P is special.*

Proof. Let $\lambda \in \operatorname{vert} P$. Let Q be the facet with corners $\operatorname{vert} P - \{\lambda\}$. Let $q : V \to V_Q$ be the reduction map. Then $n(P, \lambda) = n(q(P), q(\lambda))$ by lemma 5.3.10. Now both $\sum_{\alpha \in \Psi(q(P), q(\lambda))} \alpha$ and $\sum_{\alpha \in R(q(P))} \alpha$ are positive scalar multiples of $q(\lambda)$. Hence there exists an $\eta > 0$ such that $\eta n(q(P), q(\lambda)) = \deg q(P)$. Now by lemma 5.3.5 we have

$$\begin{aligned}
\deg P &= \deg q(P) + \deg Q \\
&= \eta n(P, \lambda) + \deg Q.
\end{aligned}$$

Since $Q > P$ we have $\deg P > \deg Q$ by the assumptions on P. Hence $n(P, \lambda) > 0$.

Now let $p : V \to V_P$ be the reduction map to P. Let Q be a facet of Φ such that $p(Q)$ is a facet of Φ_P. By lemma 5.3.5 the maximality of $\deg P$ implies $\deg p(Q) \leq 0$. Thus, (Φ_P, d_P) is semi-stable. □

Corollary 5.3.17 *The special facet of* (Φ, d) *is the largest element in the set of facets of maximal degree of* (Φ, d).

Proof. Let P be the special facet of (Φ, d) and let Q be any maximal facet of maximal degree. Then by proposition 5.3.16 and corollary 5.3.15 $P = Q$. □

6 The Canonical Parabolic Subgroup

Introduction

Let k be a field. A *curve* will be a smooth projective geometrically connected k-scheme of dimension one. We will apply the theory of root systems with complementary convex solids to the study of reductive group schemes over curves. If G is a reductive group scheme over the curve X with function field K then for any generic maximal torus $T \subset G_K$ which is split, we get a root system $\Phi = \Phi(G_K, T)$ in $V \subset \mathsf{X}(T) \otimes \mathbb{R}$. The root system Φ is naturally endowed with a complementary convex solid (proposition 6.3.7). This complementary convex solid d is defined in terms of the numerical invariants of the Borel subgroups of G containing T. These numerical invariants were already introduced by Harder in [15]. we review the construction of these invariants in a much more general context in section 6.2. For the proof that d is actually a complementary convex solid for Φ we need two alternative constructions of these invariants (see page 60). Once it is established that a generic split maximal torus gives rise to a root system with complementary convex solid, it is easy to deduce our main result, theorem 6.4.4. (See also definitions 6.1.1 and 6.2.6.) Everything depends on corollary 6.3.11 which summarizes the connection between canonical parabolic subgroups of G and special facets of the various associated root systems with complementary convex solids.

Note that for a Weyl chamber $\mathfrak{c}$ of Φ $d(\mathfrak{c})$ is simply the unique vector in V^* such that

$$\langle \alpha, d(\mathfrak{c}) \rangle = \deg W(B, \alpha)$$

for every $\alpha \in \Phi$. Here B is the Borel subgroup of G corresponding to $\mathfrak{c}$ and $\deg W(B, \alpha)$ is the numerical invariant of B with respect to the root α.

Note also that in the case $G = GL(V)$, where V is a vector bundle on X, our result reduces to the well-known existence of a canonical flag in V, the *Harder-Narasimhan filtration*. (See [16, Proposition 1.3.9.].) Note that the degree of instability (Definition 6.1.4) which plays a central role in our discussion is in this case simply twice the area of the Harder-Narasimhan polygon.

Our results could also be deduced from Proposition 1 in [21]. There Ramanathan states a theorem on reduction of unstable G-bundles to a parabolic subgroup. Unfortunately, it seems that his proof has never been published.

We would like to draw attention to Conjecture 6.4.7. This conjecture, to the effect that the canonical parabolic is isolated, would have many nice consequences.

6.1 Stability of Reductive Groups

Let k be a field and let X be a curve over k, i.e. X is a one-dimensional projective smooth geometrically connected scheme over k.

Definition 6.1.1 Let G be a smooth affine algebraic group scheme over X with connected fibers. Then we define the *degree of* G to be

$$\deg G = \deg \mathfrak{g},$$

where $\mathfrak{g}$ is the scheme of Lie algebras of G, considered simply as a vector bundle on X. (Note that G is of finite type over X, so that $\mathfrak{g}$ is in fact a vector bundle on X.)

We will be concerned with reductive group schemes over X, i.e. smooth affine group schemes over X all of whose geometric fibers are (connected) reductive algebraic groups (see [8, Exp. XIX; 2.7]). For the definition of *parabolic subgroups* see [8, Exp. XXVI; 1.1]. For a parabolic subgroup P of the reductive group scheme G/X we denote its unipotent radical by $R_u(P)$. It is again a smooth affine algebraic group scheme with connected fibers (see [8, Exp. XXVI; 1.6]).

Note 6.1.2 A reductive group scheme has degree zero. In particular, if G/X is a reductive group scheme and $P \subset G$ is a parabolic subgroup, then $\deg P = \deg R_u(P)$.

Proof. By passing to a finite étale cover of X we may assume that G is an inner form. This can be seen as follows:

Let G_0 be the constant group on X of the same type as G. Then G being an inner form means that there exists a G_0-torsor E such that $G \cong E \times_{G_0, Ad} G_0 = \underline{\mathrm{Aut}}_{G_0}(E)$. This is equivalent to $\underline{\mathrm{Isomext}}(G, G_0)$ having a section over X. By [8, Exp. XXIV, th. 1.3.] $\underline{\mathrm{Isomext}}(G, G_0)$ is a constant tordu X-scheme. Now from [8, Exp. XXII, cor. 2.3.] G is quasi-isotrivial. Hence the same is true of $\underline{\mathrm{Isomext}}(G, G_0)$. Now from [7, Exp. X,cor. 5.14.] we get that the connected components of $\underline{\mathrm{Isomext}}(G, G_0)$ are finite (and of course étale) over X. Replacing X by any one of them, G becomes an inner form.

So we may assume that $G \cong E \times_{G_0, Ad} G_0$ for a G_0-torsor E. Then $\mathfrak{g} \cong E \times_{G_0, Ad} \mathfrak{g}_0$. Now the adjoint representation $Ad : G_0 \to GL(\mathfrak{g}_0)$ factors through $SL(\mathfrak{g}_0)$. Hence $\mathfrak{g}$ has a reduction of structure group to $SL(\mathfrak{g}_0)$. So $\det \mathfrak{g} \cong \mathcal{O}_X$. □

Lemma 6.1.3 *Let G/X be a reductive group scheme. There exists an $M > 0$ such that $\deg P \leq M$ for all parabolic subgroups P of G.*

Proof. We have

$$\begin{aligned} \deg P &= \deg \mathfrak{p} \\ &\leq \dim_k H^0(X, \mathfrak{p}) + \mathrm{rk}(\mathfrak{p})\,(g-1) \quad \text{(by Riemann-Roch)} \\ &\leq \dim_k H^0(X, \mathfrak{g}) + \mathrm{rk}(\mathfrak{g})\,g \quad \text{(since } \mathfrak{p} \text{ is a subsheaf of } \mathfrak{g}). \end{aligned}$$

Hence $M = \dim_k H^0(X, \mathfrak{g}) + \mathrm{rk}(\mathfrak{g})\,g$, where g is the genus of X, will do. □

Definition 6.1.4 Let G/X be a reductive group scheme.

i. We call G *semi-stable*, if for every parabolic subgroup P of G we have $\deg P \leq 0$.

ii. We call G *stable*, if for every proper parabolic subgroup P of G we have $\deg P < 0$.

iii. The largest integer d such that there exists a parabolic subgroup P of G with $\deg P = d$ is called the *degree of instability of* G, denoted $\deg_i(G)$.

By Note 6.1.2 we may replace $\deg P$ by $\deg R_u(P)$ in this definition. By Lemma 6.1.3 the degree of instability is finite.

6.2 The Numerical Invariants of a Parabolic Subgroup

Let G/X be a reductive group scheme.

Remark 6.2.1 Let $\mathrm{Dyn}(G)$ be the Dynkin diagram of G ([8, Exp. XXIV, 3.3.]). For a parabolic subgroup $P \subset G$ the *type of* P, $t(P)$, is defined in [8, Exp. XXVI, 3.2.] to be a certain section of $\mathcal{P}(\mathrm{Dyn}(G))$, the 'power scheme' of $\mathrm{Dyn}(G)$. Of course, there is a choice involved here, since one could also take the complement of $t(P)$ as the type of P. (Note that $\mathcal{P}(\mathrm{Dyn}(G))$ indeed has an automorphism given by taking complements.) In fact I believe that the authors of SGA3 made the wrong choice. Here we will adopt the alternative definition. This is justified, for example by lemma 6.2.2 (iii). Our $t(P)$ contains more information about P than the one of SAG3 does. For example, if $P = B$ is a Borel subgroup, the $t(P)$ of SGA3 is empty, whereas our $t(P)$ contains information about how the 'corners' of B are twisted.

Another reason for adopting the complementary definition is to avoid having to talk about the 'complement of the type of P' in what follows.

Lemma 6.2.2 *Let (G, T) be a reductive group scheme G over a connected scheme S, with a maximal torus T which is split. Let $\Phi(G, T)$ be the corresponding root system.*

i. There is a unique bijective correspondence

$$\begin{array}{rcl} \{\text{parabolic subgroups of } G \text{ containing } T\} & \longrightarrow & \{\text{parabolic subsets of } \Phi\} \\ P & \longmapsto & R(P), \end{array}$$

such that for any parabolic P such that $T \subset P$ the Lie algebra $\mathfrak{p}$ is given by

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R(P)} \mathfrak{g}_\alpha .$$

ii. There is a bijective correspondence

$$\begin{array}{rcl} \{\text{parabolic subgroups of } G \text{ containing } T\} & \longrightarrow & \{\text{facets of } \Phi\} \\ P & \longmapsto & P'. \end{array}$$

iii. Let P be a parabolic subgroup of G containing T. There is a natural bijective correspondence

$$\begin{array}{rcl} \pi_0 t(P) & \longrightarrow & \operatorname{vert} P' \\ \mathfrak{v} & \longmapsto & \mathfrak{v}'. \end{array}$$

Proof. i. Since S is connected, T is the group of multiplicative type associated to its character group $\mathsf{X}(T)$ via [7, Exp. X, cor. 5.9.], and there is a unique system of roots $\Phi(G, T)$ of G in $\mathsf{X}(T)$. Now the claim is local with respect to the Zariski topology on S, so we may assume that $\mathfrak{g}_\alpha$ is trivial, for every $\alpha \in \Phi(G, T)$. So in this situation, T defines uniquely a deployment ([8, Exp. XXII, def. 1.13.]) of G. Now if P is a parabolic subgroup of G containing T, then by [8, Exp. XXII, lem. 5.2.7.] there exists a subset $R \subset \Phi$ such that $\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. Clearly, R is uniquely determined by P. By [8, Exp. XXVI, prop. 1.4.] R is a parabolic subset of Φ. The same proposition then also gives the bijectivity of this map.

ii. Follows immediately from corollary 5.1.8.

iii. Without loss of generality assume that G is deployable, i.e. that all $\mathfrak{g}_\alpha$, $\alpha \in \Phi$, are trivial. Choose an épinglage E of G compatible with P. Let B be the Borel subgroup $T \subset B \subset P$ defined by E. Let $\mathfrak{c}$ be the Weyl chamber of Φ such that $\mathfrak{c} = B'$. We have $\mathfrak{c} \leq P'$. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots defined by $\mathfrak{c}$, $\lambda_1, \ldots, \lambda_n$ the corners of $\mathfrak{c}$. Let $\lambda_1, \ldots, \lambda_r$ be the corners of P'. Now, by definition, $\operatorname{Dyn}(G)$ is a constant S-scheme. Its set of components is identified with $\{\alpha_1, \ldots, \alpha_n\}$, since S is connected. $t(P) \subset \operatorname{Dyn}(G)$ is the subscheme defined by the subset $\{\alpha_1, \ldots, \alpha_r\}$. So we define our map simply by $\alpha_i \mapsto \lambda_i$ for $i = 1, \ldots, r$. This is obviously bijective. It remains to show that it is independent of the choice of our épinglage E. So let E_1 be another épinglage of G, compatible with P. The inner automorphism of P mapping E to E_1 induces an element $\sigma \in W$, the Weyl group of Φ. But $\sigma(P') = P'$ which entails $\sigma(\lambda_i) = \lambda_i$ for $i = 1, \ldots, r$. $\square$

Construction. Let $P \subset G$ be a parabolic subgroup and let $t(P)$ be the type of P. $t(P)$ is a finite étale X-scheme. Let $\pi_0 t(P)$ be the set of connected components of $t(P)$ and let $\mathcal{T}$ be the free abelian group on $\pi_0 t(P)$. For simplicity let us write $\pi_0 t(P) = \{\mathfrak{v}_1, \ldots, \mathfrak{v}_s\}$. Let $\mathfrak{v} = \sum_{i=1}^{s} n_i \mathfrak{v}_i$ be a positive element of $\mathcal{T}$. We will associate with $\mathfrak{v}$ a vector bundle on X as follows:

Let $Y \to X$ be an étale X-scheme, which we assume to be connected, for simplicity, and let T be a maximal torus of G_Y such that $T \subset P_Y$ and such that T is split. Let $\Phi = \Phi(G_Y, T)$ be the corresponding root system, and assume that $\mathfrak{g}_\alpha$ is trivial over Y, for $\alpha \in \Phi$ (so that G_Y is uniquely deployable with respect to T). Let P' be the facet of Φ corresponding to P_Y via lemma 6.2.2. The schemes $\mathfrak{v}_{1,Y}, \ldots, \mathfrak{v}_{s,Y}$ form a partition of $t(P)_Y$. So via lemma 6.2.2 we get a partition $\mathfrak{v}'_1, \ldots, \mathfrak{v}'_s$ of vert P'. For $i = 1, \ldots, s$ let $\check{\mu}_i = \sum_{\check{\lambda} \in \mathfrak{v}'_i} \lambda$. Now define

$$\Omega(\mathfrak{v}) = \{\alpha \in \Phi \mid \langle \alpha, \check{\mu}_i \rangle \geq n_i \text{ for all } i = 1, \ldots, s\}.$$

$\Omega(\mathfrak{v})$ is a closed set of roots satisfying $\Omega(\mathfrak{v}) \cap -\Omega(\mathfrak{v}) = \varnothing$. Let

$$U(\mathfrak{v}, Y, T) = \prod_{\alpha \in \Omega(\mathfrak{v})} U_\alpha$$

be the unipotent subgroup scheme of G_Y defined by $\Omega(\mathfrak{v})$. $U(\mathfrak{v}, Y, T)$ is smooth with geometrically connected fibers and independent of any order chosen on $\Omega(\mathfrak{v})$. (See [8, Exp. XXII; 5.6.5, 5.6.1, 5.4.7].)

Proposition 6.2.3 *There exists a unique subgroup scheme $U(\mathfrak{v})$ of G such that for every connected étale $Y \to X$ and every split maximal torus T of G_Y such that $T \subset P_Y$ and such that G_Y is deployable with respect to T, we have $U(\mathfrak{v})_Y = U(\mathfrak{v}, Y, T)$. $U(\mathfrak{v})$ is a closed normal unipotent subgroup of P with geometrically connected fibers. Moreover, $W(P, \mathfrak{v}) = U(\mathfrak{v}) / \prod_{\mathfrak{v}' > \mathfrak{v}} U(\mathfrak{v}')$ is a vector bundle over X in a natural way.*

Proof. We just need to adopt the proof of [8, Exp. XXVI, prop. 2.1.] to our situation: Uniqueness is clear in view of the fact that G can be deployed locally with respect to the étale topology ([8, Exp. XXVI, lem. 1.14.]). For the existence, consider a connected étale Y/X together with an épinglage E of G_Y adapted to P_Y (see [8, Exp. XXVI, def. 1.11.]). Let T be the maximal torus given by E.

Claim: $U(\mathfrak{v}, Y, T)$ is normal in P_Y.

Let $R \subset \Phi = \Phi(G_Y, T)$ be the parabolic set of roots corresponding to P_Y. Then P_Y is generated by T and the various U_α for $\alpha \in R$. T clearly normalizes $U(\mathfrak{v}, Y, T)$. To see that U_α normalizes $U(\mathfrak{v}, Y, T)$, consider the commutation relation of [8, Exp. XXII, cor. 5.5.2.]:

$$p_\alpha(x) p_\beta(y) p_\alpha(-x) = p_\beta(y) \prod_{n,m>0} p_{n\alpha + m\beta}(c_{n,m,\alpha,\beta} x^n y^m). \tag{22}$$

Here $p_\alpha : \mathbb{G}_{aY} \xrightarrow{\sim} U_\alpha : x \mapsto \exp_\alpha(x A_\alpha)$ is the isomorphism given by E, A_α denotes the element of $\Gamma(\mathfrak{g}_\alpha)$ given E. So we have to show that for $\beta \in \Omega(\mathfrak{v})$ and $\alpha \in R$ we have $n\alpha + m\beta \in \Omega(\mathfrak{v})$ if it is in Φ. But this is only a trivial computation.

If we choose another épinglage E' there exists a unique inner automorphism of P_Y mapping E' to E. This inner automorphism will leave $U(\mathfrak{v}, Y, T)$ invariant by the claim. Hence $U(\mathfrak{v}, Y, T)$ does not depend on the choice of T. This proves that the $U(\mathfrak{v}, Y, T)$ indeed glue to give a subgroup scheme $U(\mathfrak{v})$ of G. By the claim, $U(\mathfrak{v})$ is normal in P. Now consider for $Y/X, T, \Phi, \mu_1, \ldots, \mu_s$ as above

$$\Omega'(\mathfrak{v}) = \{\alpha \in \Phi \mid \forall i = 1, \ldots, s : \langle \alpha, \check{\mu}_i \rangle \geq n_i \text{ and } \exists i = 1, \ldots, s : \langle \alpha, \check{\mu}_i \rangle > n_i\}.$$

We can construct a corresponding unipotent normal subgroup scheme of P called $U'(\mathfrak{v})$, just as we did for $\Omega(\mathfrak{v})$. Clearly, $U'(\mathfrak{v}) = \prod_{\mathfrak{v}' > \mathfrak{v}} U(\mathfrak{v}')$. For a given épinglage E of G_Y over Y/X étale we consider the following homomorphism of group schemes over Y:

$$\begin{array}{ccc} \displaystyle\bigoplus_{\alpha \in \Omega(\mathfrak{v}) - \Omega'(\mathfrak{v})} \mathfrak{g}_\alpha & \longrightarrow & U(\mathfrak{v})_Y / U'(\mathfrak{v})_Y \\ \displaystyle\sum_\alpha x_\alpha & \longmapsto & \displaystyle\prod_\alpha \exp_\alpha(x_\alpha). \end{array}$$

By (22) the commutator of two elements in $U(\mathfrak{v})_Y$ is in $U'(\mathfrak{v})_Y$, so that this is indeed a well-defined homomorphism. In fact it is even an isomorphism. Via this isomorphism we introduce a vector bundle structure on $U(\mathfrak{v})_Y / U'(\mathfrak{v})_Y$. Changing the épinglage amounts to applying an inner automorphism coming from P. Let $\alpha \in R$ and $\beta \in \Omega(\mathfrak{v})$. Then we have

$$p_\alpha(x) p_\beta(y) p_\alpha(-x) U'(\mathfrak{v})_Y = p_\beta(y) \prod_{n>0} p_{n\alpha+\beta}(c_{n,1,\alpha,\beta} x^n y) U'(\mathfrak{v})_Y.$$

But $y \mapsto (1 + \sum_{n>0} c_{n,1,\alpha,\beta} x^n) y$ is linear. This shows that P acts linearly on $U(\mathfrak{v})/U'(\mathfrak{v})$ via inner automorphisms. So changing the épinglage doesn't change the structure of vector bundle on $U(\mathfrak{v})/U'(\mathfrak{v})$. □

Proposition 6.2.4 *Let $U_0 \supset U_1 \supset \ldots$ be the filtration of $R_u(P)$ defined in [8, Exp. XXVI, prop. 2.1.]. Then for every $i \geq 0$ we have*

$$U_i / U_{i+1} = \bigoplus_{\mathfrak{v} \in \mathcal{T}_{i+1}} W(P, \mathfrak{v}).$$

Here $\mathcal{T}_i = \{\mathfrak{v} \in \mathcal{T} \mid \text{If } \mathfrak{v} = \sum_{j=1}^s n_j \mathfrak{v}_j \text{ then } \sum_{j=1}^s n_j = i\}$.

Proof. Let $i \geq 0$. Choose a $\mathfrak{v} \in \mathcal{T}_{i+1}$. Then by definition we have $U(\mathfrak{v}) \subset U_i$ and $U(\mathfrak{v}') \subset U_{i+1}$ for every $\mathfrak{v}' > \mathfrak{v}$. This gives a natural map $W(P, \mathfrak{v}) \to U_i / U_{i+1}$. By looking at this map locally, it can easily be seen that it identifies $W(P, \mathfrak{v})$ with a vector subbundle of U_i / U_{i+1}. The rest of the proposition is also easily proved locally, by reducing to questions about the corresponding sets of roots. □

Definition 6.2.5 Let P be a parabolic subgroup of G. Then for any component $\mathfrak{v}$ of the type $t(P)$ of P we define $n(P, \mathfrak{v}) = \deg W(P, \mathfrak{v})$ to be the *numerical invariant of P with respect to the component of the type* $\mathfrak{v}$. We call the collection $(n(P, \mathfrak{v}))_{\mathfrak{v} \in \pi_0 t(P)}$ the *numerical invariants of* P. We call $(W(P, \mathfrak{v}))_{\pi_0 t(P)}$ the *elementary vector bundles* associated with P.

Definition 6.2.6 The parabolic $P \subset G$ is called *canonical* if it satisfies the following conditions:

(C1) The numerical invariants of P are positive, i.e. for every $\mathfrak{v} \in \pi_0 t(P)$ we have $n(P, \mathfrak{v}) > 0$.

(C2) $P / R_u(P)$ is semi-stable.

Our goal is to prove that every reductive group scheme over X has a unique canonical parabolic subgroup (theorem 6.4.4).

6.3 Rationally Trivial Reductive Group Schemes

Let K denote the function field of X. Let G/X be a rationally trivial reductive group scheme over X. This means that there exists a maximal torus $T \subset G_K$ that is split. In this case, every generic maximal torus is split.

Note 6.3.1 G is rationally trivial if and only if G is an inner form.

Proof. Let G_0 be the constant group over X of the same type as G. If G is rationally trivial, then $\underline{\mathrm{Isomext}}(G_0, G)$ has a constant generic fiber. This implies that $\underline{\mathrm{Isomext}}(G_0, G)$ is trivial (see the proof of note 6.1.2). and hence that G is an inner form. Conversely, if G is an inner form any maximal torus of G_K is split by [8, Exp. XXIV, cor. 2.8.]. G_K has maximal tori by [7, Exp. XIV, théorème 1.1.]. □

Let $\mathrm{Dyn}(G)/X$ be the Dynkin diagram of G. $\mathrm{Dyn}(G) \to X$ is a trivial covering, because it is so generically. In particular, for any parabolic $P \subset G$ the type $t(P)$ is a trivial covering of X.

Proposition 6.3.2 *Let T be a generic maximal torus of G. Let $\Phi = \Phi(G_K, T)$ be the corresponding root system.*

i. There exists a natural bijection

$$\begin{array}{rcl} \{\text{parabolic subgroups of } G \text{ containing } T\} & \longrightarrow & \{\text{facets of } \Phi\} \\ P & \longmapsto & P' \end{array}$$

with inverse

$$\begin{array}{rcl} \{\text{facets of } \Phi\} & \longrightarrow & \{\text{parabolic subgroups of } G \text{ containing } T\} \\ Q & \longmapsto & \widetilde{Q}. \end{array}$$

This bijection is order preserving.

ii. Let P be a parabolic subgroup of G containing T. There is a natural bijective correspondence

$$\begin{array}{rcl} \pi_0 t(P) & \longrightarrow & \mathrm{vert}\, P' \\ \mathfrak{v} & \longmapsto & \mathfrak{v}' \end{array}$$

with inverse

$$\begin{array}{rcl} \mathrm{vert}\, P' & \longrightarrow & \pi_0 t(P) \\ \lambda & \longmapsto & \widetilde{\lambda}. \end{array}$$

Proof. i. In view of lemma 6.2.2 it suffices to prove that every parabolic subgroup of G_K extends uniquely to a parabolic subgroup of G. But this follows immediately from the projectivity of $\underline{\mathrm{Par}}(G)/X$ (see [8, Exp. XXVI, cor. 3.5.]).

ii. This is clear from lemma 6.2.2 because $\pi_0 t(P)$ and $\pi_0 t(P_K)$ are in canonical bijection. □

By this proposition the Borel subgroups containing a generic maximal torus $T \subset G_K$ correspond to the Weyl chambers of the corresponding root system $\Phi(G, T)$. The maximal parabolics containing T correspond to the one-dimensional facets of $\Phi(G, T)$, i.e. to the fundamental weights $\lambda \in \Lambda$.

If P is a parabolic subgroup of G containing T, and $\lambda \in \mathrm{vert}\, P'$ we let $W(P, \lambda) = W(P, \widetilde{\lambda})$ and $n(P, \lambda) = n(P, \widetilde{\lambda}) = \deg W(P, \lambda)$.

Alternative Descriptions of the Numerical Invariants

We will give two alternative constructions of the vector bundles $W(P, \mathfrak{v})$. For simplicity we restrict to the case that $P = B$ is a Borel subgroup of G. In this case, all $W(B, \mathfrak{v})$ will be line bundles. We do not insist however on the fact that $\mathfrak{v} \in \mathcal{T}$ be positive. So choose a Borel subgroup $B \subset G$ and fix it throughout this discussion.

First Construction. Let T be a generic maximal torus of G contained in B and $\Phi = \Phi(G_K, T)$ the corresponding root system. Let $\mathfrak{c} = B'$ be the Weyl chamber of Φ defined by B, let $\lambda_1, \ldots, \lambda_n$ be the corners of $\mathfrak{c}$. $\mathfrak{c}$ defines an ordering on Φ by $\alpha < \beta \Leftrightarrow \beta - \alpha$ is positive with respect to $\mathfrak{c}$.

Let $\alpha_0 \in \Phi$ be a root. Then $\mathfrak{v} = \sum_{i=1}^{n} \langle \alpha_0, \check{\lambda}_i \rangle \widetilde{\lambda}_i$ is an element of $\mathcal{T}$. $\mathfrak{v}$ is either positive of negative, according to whether α_0 is positive or negative. We distinguish these two cases:

i. α_0 is positive: Consider

$$\Omega(\alpha_0) = \{\alpha \in \Phi \mid \alpha \geq \alpha_0\} \quad \text{and} \quad \Omega'(\alpha_0) = \{\alpha \in \Phi \mid \alpha > \alpha_0\}. \tag{23}$$

Then define

$$V(\Omega(\alpha_0)) = \bigoplus_{\alpha \in \Omega(\alpha_0)} \mathfrak{g}_{K,\alpha} \quad \text{and} \quad V(\Omega'(\alpha_0)) = \bigoplus_{\alpha \in \Omega'(\alpha_0)} \mathfrak{g}_{K,\alpha}.$$

$V(\Omega(\alpha_0))$ and $V(\Omega'(\alpha_0))$ are vector subspaces of

$$\mathfrak{g}_K = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{K,\alpha}.$$

The spaces $V(\Omega(\alpha_0))$ and $V(\Omega'(\alpha_0))$ extend uniquely to vector subbundles of $\mathfrak{g}$. Call these vector bundles $V(\alpha_0)$ and $V'(\alpha_0)$. Clearly $V(\alpha_0)/V'(\alpha_0)$ is a line bundle on X.

Lemma 6.3.3 *There is a natural isomorphism* $W(B, \mathfrak{v}) = V(\alpha_0)/V'(\alpha_0)$.

Proof. Consider the group schemes $U(\mathfrak{v})$ and $U'(\mathfrak{v}) = \prod_{\mathfrak{v}' > \mathfrak{v}} U(\mathfrak{v}')$ of Proposition 6.2.3. The corresponding Lie algebras $\mathfrak{u}(\mathfrak{v})$ and $\mathfrak{u}'(\mathfrak{v})$ are subbundles of $\mathfrak{g}$. Now by construction we have that $\mathfrak{u}(\mathfrak{v})_K = V(\Omega(\alpha_0))$ and $\mathfrak{u}'(\mathfrak{v})_K = V(\Omega'(\alpha_0))$. Hence $\mathfrak{u}(\mathfrak{v}) = V(\alpha_0)$ and $\mathfrak{u}'(\mathfrak{v}) = V'(\alpha_0)$ and

$$W(B, \mathfrak{v}) = U(\mathfrak{v})/U'(\mathfrak{v}) = \mathfrak{u}(\mathfrak{v})/\mathfrak{u}'(\mathfrak{v}) = V(\alpha_0)/V'(\alpha_0)$$

which finishes the proof. $\square$

ii. α_0 is negative: Consider

$$\Upsilon(\alpha_0) = \{\alpha \in \Phi \mid \alpha \not< \alpha_0\} \quad \text{and} \quad \Upsilon'(\alpha_0) = \{\alpha \in \Phi \mid \alpha \not\leq \alpha_0\}.$$

Then define

$$V(\Upsilon(\alpha_0)) = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Upsilon(\alpha_0)} \mathfrak{g}_{K,\alpha} \quad \text{and} \quad V(\Upsilon'(\alpha_0)) = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Upsilon'(\alpha_0)} \mathfrak{g}_{K,\alpha}.$$

Again, we extend these vector spaces to subbundles of $\mathfrak{g}$. We get vector bundles $V(\alpha_0)$ and $V'(\alpha_0)$. Again, $V(\alpha_0)/V'(\alpha_0)$ is a line bundle on X. In this case, $W(B, \mathfrak{v})$ is not defined yet. So we define $W(B, \mathfrak{v}) = V(\alpha_0)/V'(\alpha_0)$.

Second construction. (See [15, Satz 1.1.1.].) Let T be a generic maximal torus of G contained in B and $\Phi = \Phi(G_K, T)$ the corresponding root system. Let G_0 be a constant reductive group scheme over X of the same type as G. Let $T_0 \subset B_0 \subset G_0$ be a constant maximal torus and a constant Borel subgroup of G_0. Let $\Phi_0 = \Phi(G_0, T_0)$ be the corresponding root system. We identify Φ_0 with $\Phi = (G_K, T)$, making sure that the Weyl chambers corresponding to B_0 and B become equal. Now the pair (G, B) is associated to an $\operatorname{Aut} G_0, B_0$-torsor:

$$\underline{\mathrm{Isom}}(G_0, B_0; G, B) \times_{\operatorname{Aut} G_0, B_0} (G_0, B_0) \cong (G, B).$$

We have that $B_0^{ad} \subset \operatorname{Aut} G_0, B_0$ is a subgroup scheme. A reduction of structure group of $\underline{\mathrm{Isom}}(G_0, B_0; G, B)$ to B_0^{ad} is the same thing as a section of the sheaf $\underline{\mathrm{Isom}}(G_0, B_0; G, B)/B_0^{ad}$. But there is a canonical isomorphism

$$\underline{\mathrm{Isom}}(G_0, B_0; G, B)/B_0^{ad} = \underline{\mathrm{Isomext}}(G_0, G)$$

by [8, Exp. XXIV, cor. 2.2.]. G being an inner form, $\underline{\mathrm{Isomext}}(G_0, G)$ is trivial, so that, indeed, we can reduce the structure group of $\underline{\mathrm{Isom}}(G_0, B_0; G, B)$ to B_0^{ad}. So there exists a B_0^{ad}-torsor E such that $E \times_{B_0^{ad}} (G_0, B_0) \cong (G, B)$. Every $\alpha \in \Phi$ induces a character of B_0^{ad}, which we will again denote α. So for every root $\alpha \in \Phi$ we get a line bundle

$$L(\alpha) = E \times_{B_0^{ad}, \alpha} \mathcal{O}$$

on X.

Lemma 6.3.4 *For every $\alpha \in \Phi$ we have $L(\alpha) \cong V(\alpha)/V'(\alpha)$. In particular, the isomorphism class of $L(\alpha)$ does not depend on the choice of the B_0^{ad}-torsor E such that $E \times_{B_0^{ad}} (G_0, B_0) \cong (G, B)$.*

Proof. Choose an $\alpha_0 \in \Phi$. We distinguish two cases:

i. α_0 is positive. Again consider the subsets $\Omega(\alpha_0)$ and $\Omega'(\alpha_0)$ of Φ defined in (23). Then define

$$V_0(\Omega(\alpha_0)) = \bigoplus_{\alpha \in \Omega(\alpha_0)} \mathfrak{g}_{0,\alpha} \quad \text{and} \quad V(\Omega'(\alpha_0)) = \bigoplus_{\alpha \in \Omega'(\alpha_0)} \mathfrak{g}_{0,\alpha}.$$

These are vector subbundles of $\mathfrak{g}_0$, the Lie algebra of G_0. Clearly, $E \times_{B_0^{ad}} V(\Omega(\alpha_0))$ and $E \times B_0^{ad} V_0(\Omega'(\alpha_0)) = V'(\alpha_0)$.

Claim: B_0 acts on $V_0(\Omega(\alpha_0))$ and $V_0(\Omega'(\alpha_0))$ via Ad. The induced action on $V_0(\Omega(\alpha_0))/V_0(\Omega'(\alpha_0))$ is given by α_0.

We have an isomorphism

$$T_0 \times_X \prod_{\alpha \in \Phi^+} U_{0,\alpha} \longrightarrow B_0$$

(see [8, Exp. XXII, prop. 5.6.1.]). So it suffices to consider the actions of T_0 and $U_{0,\alpha}$ for $\alpha \in \Phi^+$ on $V_0(\Omega(\alpha_0))$ and $V_0(\Omega'(\alpha_0))$. T_0 clearly leaves both bundles invariant. So consider for $\alpha \in \Phi^+$ the morphism

$$\exp_\alpha : \mathfrak{g}_{0,\alpha} \longrightarrow U_{0,\alpha} \subset G_0$$

which maps onto $U_{0,\alpha}$ ([8, Exp. XXII, 1.1.]). So we have to examine the action of $Ad(\exp_\alpha(xA_\alpha))$ on $V_0(\Omega(\alpha_0))$ and $V_0(\Omega'(\alpha_0))$. (Here $A_\alpha \in \Gamma(X, \mathfrak{g}_{0,\alpha})$ is a fixed nowhere vanishing section, for every $\alpha \in \Phi$.) By [8, Exp. XXII, lem. 5.4.9.] we have the following formula

$$Ad(\exp_\alpha(xA_\alpha))A_\beta = A_\beta + \sum_{i \geq 1} c_{\alpha,\beta,i} x^i A_{\beta + i\alpha} \tag{24}$$

for any $\beta \in \Phi$, $\beta \neq -\alpha$. Now $\alpha \in \Phi^+$ and $\beta \in \Omega(\alpha_0)$ implies $\beta + i\alpha \in \Omega(\alpha_0)$ if it is a root at all. The same is true for $\Omega'(\alpha_0)$. So B_0 indeed acts on $V_0(\Omega(\alpha_0))$ and $V_0(\Omega'(\alpha_0))$. Hence B_0 also acts on the quotient $V_0(\Omega(\alpha_0))/V_0(\Omega'(\alpha_0))$. By (24) U_α acts trivially on this quotient. So B_0 acts via α_0 since T_0 does. This finishes the proof of the claim.

So now we have

$$
\begin{aligned}
V(\alpha_0)/V'(\alpha_0) &= \left(E \times_{B_0^{ad}} V_0(\Omega(\alpha_0))\right) / \left(E \times_{B_0^{ad}} V_0(\Omega'(\alpha_0))\right) \\
&= E \times_{B_0^{ad}} \left(V_0(\Omega(\alpha_0))/V_0(\Omega'(\alpha_0))\right) \\
&\cong E \times_{B_0^{ad},\alpha} \mathcal{O}.
\end{aligned}
$$

ii. α_0 is negative. The proof is analogous to the proof in the first case. We use $\Upsilon(\alpha_0)$ and $\Upsilon'(\alpha_0)$ and define

$$V_0(\Upsilon(\alpha_0)) = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Upsilon(\alpha_0)} \mathfrak{g}_{0,\alpha} \quad \text{and} \quad V(\Upsilon'(\alpha_0)) = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Upsilon'(\alpha_0)} \mathfrak{g}_{0,\alpha}.$$

The only complication is that $\alpha \in \Phi^+$ and $\beta \in \Upsilon(\alpha_0)$ does not imply that $\beta \neq -\alpha$. so we have to treat this case separately. But

$$Ad(\exp_\alpha(xA_\alpha))A_{-\alpha} \equiv A_{-\alpha} \mod \mathfrak{t}_0 \oplus \mathfrak{g}_{0,\alpha}$$

by [8, Exp. XX, 2.10.(2)]. So this complication causes no problem. □

Note 6.3.5 If α, β and $\alpha+\beta$ are roots, then $L(\alpha+\beta) \cong L(\alpha) \otimes L(\beta)$. For any root α we have $L(-\alpha) \otimes L(\alpha) \cong \mathcal{O}$.

Proof. Obvious. □

The Root System with Complementary Convex Solid Given by a Generic Maximal Torus of G

Lemma 6.3.6 *Let $P \subset G$ be a parabolic subgroup and $B \subset P \subset G$ a Borel subgroup. Let $\mathfrak{v} \in \mathcal{T}$ be a positive element in the free abelian group on the components of the type of P. Let $U(\mathfrak{v})$ be the corresponding unipotent group defined in proposition 6.2.3.*

Let $T \subset B$ be a maximal torus and Φ the corresponding root system. Let $\Omega(\mathfrak{v}) \subset \Phi$ be the corresponding set of roots. Then we have

$$\deg U(\mathfrak{v}) = \sum_{\alpha\in\Omega(\mathfrak{v})} \deg W(B, \mathfrak{v}(\alpha))$$

where $\mathfrak{v}(\alpha) = \sum_{\lambda\in \operatorname{vert} B'} \langle\alpha, \check{\lambda}\rangle\tilde{\lambda}$.

Proof. Clear. □

Proposition 6.3.7 *Let $T \subset G_K$ be a generic maximal torus and let $\Phi = \Phi(G_K, T)$ be the corresponding root system. For $\mathfrak{c} \in \mathfrak{C}$ let*

$$d(\mathfrak{c}) = \sum_{\lambda\in\operatorname{vert}\mathfrak{c}} n(\tilde{\mathfrak{c}}, \lambda)\check{\lambda}.$$

Then $(d(\mathfrak{c}))_{\mathfrak{c}\in\mathfrak{C}}$ is a complementary convex solid for Φ.

Proof. To check whether (C1) is satisfied let $\lambda \in \Lambda$ be a fundamental weight and $\mathfrak{c} \in \mathfrak{C}$ a Weyl chamber of Φ. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots defined by $\mathfrak{c}$, $\check{\lambda}_1, \ldots, \check{\lambda}_n$ the dual basis. Let $B = \tilde{\mathfrak{c}}$ be the Borel subgroup of G corresponding to $\mathfrak{c}$, let $P = \widetilde{\mathbb{R}_+\lambda}$ be the maximal parabolic in G corresponding to λ and let $\mathfrak{v} = \tilde{\lambda}$ be the component of the type of B corresponding to λ.

Note that the unipotent subgroup $U(\mathfrak{v})$ associated to B and $\mathfrak{v}$ in proposition 6.2.3 is nothing but $R_u(P)$. By lemma 6.3.6 we have

$$\deg U(\mathfrak{v}) = \sum_{\alpha \in \Omega(\mathfrak{v})} \deg W(B, \mathfrak{v}(\alpha)),$$

where $\mathfrak{v}(\alpha) = \sum_{i=1}^{n} \langle \alpha, \check{\lambda}_i \rangle \tilde{\lambda}_i$. By lemmas 6.3.3 and 6.3.4 we have $W(B, \mathfrak{v}(\alpha)) \cong L(\alpha)$. By note 6.3.5 we have $\deg L(\alpha) = \sum_{i=1}^{n} \langle \alpha, \check{\lambda}_i \rangle \deg L(\alpha_i)$. Finally, we have $\deg L(\alpha_i) = n(B, \lambda_i)$. Assembling these various facts we get

$$\begin{aligned}
\deg P &= \deg R_u(P) \\
&= \deg U(\mathfrak{v}) \\
&= \sum_{\alpha \in \Omega(\mathfrak{v})} \deg W(B, \mathfrak{v}(\alpha)) \\
&= \sum_{\alpha \in \Omega(\mathfrak{v})} \deg L(\alpha) \\
&= \sum_{\alpha \in \Omega(\mathfrak{v})} \sum_{i=1}^{n} \langle \alpha, \check{\lambda}_i \rangle \deg L(\alpha_i) \\
&= \sum_{\alpha \in \Omega(\mathfrak{v})} \sum_{i=1}^{n} \langle \alpha, \check{\lambda}_i \rangle n(B, \lambda_i) \\
&= \langle \sum_{\alpha \in \Omega(\mathfrak{v})} \alpha, \sum_{i=1}^{n} n(B, \lambda_i) \check{\lambda}_i \rangle
\end{aligned}$$

Now, by proposition 5.1.9 there exists an $\eta > 0$ such that $\sum_{\alpha \in \Omega(\mathfrak{v})} \alpha = \eta \lambda$. So we get $\eta \langle \lambda, d(\mathfrak{c}) \rangle = \deg P$. $\deg P$ does not depend on the choice of $\mathfrak{c}$, but only on the choice of λ. So (C1) follows.

For the proof of (C2) let $\mathfrak{c}$ and $\mathfrak{d}$ be Weyl chambers having precisely $n-1$ corners in common. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots defined by $\mathfrak{c}$, $\lambda_1, \ldots, \lambda_n$ the corresponding corners of $\mathfrak{c}$. Without loss of generality assume that $\mathfrak{d} = \sigma_1(\mathfrak{c})$ so that the corners of $\mathfrak{d}$ are $\lambda_1 - \alpha_1, \lambda_2, \ldots, \lambda_n$. Then α_1 is the unique root positive with respect to $\mathfrak{c}$ and negative with respect to $\mathfrak{d}$. We have

$$\begin{aligned}
\langle \alpha_1, d(\mathfrak{d}) \rangle &= n(\tilde{\mathfrak{d}}, \lambda_1 - \alpha_1) \langle \alpha_1, \check{\lambda}_1 - \check{\alpha}_1 \rangle + \sum_{i=2}^{n} n(\mathfrak{d}, \lambda_i) \langle \alpha_1, \check{\lambda}_i \rangle \\
&= -n(\tilde{\mathfrak{d}}, \lambda_1 - \alpha_1).
\end{aligned}$$

Now by lemmas 6.3.3 and 6.3.4 (applied to $\tilde{\mathfrak{d}}$) we have that $n(\tilde{\mathfrak{d}}, \lambda_1 - \alpha_1) = \deg L(-\alpha_1)$. By note 6.3.5 we have $\deg L(-\alpha_1) = -\deg L(\alpha_1)$. By lemma 6.3.4 again, $\deg L(\alpha_1) = \deg V_{\mathfrak{d}}(\alpha_1)/V'_{\mathfrak{d}}(\alpha_1)$. Now α_1 is negative with respect to $\mathfrak{d}$, so that $V_{\mathfrak{d}}(\alpha_1)$ and $V'_{\mathfrak{d}}(\alpha_1)$ are defined by $\Upsilon_{\mathfrak{d}}(\alpha_1)$ and $\Upsilon'_{\mathfrak{d}}(\alpha_1)$, respectively. Here the index $\mathfrak{d}$ means that the construction is taking place with respect to $\tilde{\mathfrak{d}}$.

Claim: $\Omega_{\mathfrak{c}}(\alpha_1) \subset \Upsilon_{\mathfrak{d}}(\alpha_1)$ and $\Omega'_{\mathfrak{c}}(\alpha_1) \subset \Upsilon'_{\mathfrak{d}}(\alpha_1)$.

To prove the claim, it suffices to prove the first part. So assume $\alpha \in \Omega_{\mathfrak{c}}(\alpha_1)$. Then $\alpha \geq \alpha_1$ with respect to $\mathfrak{c}$, i.e. for all $i = 1, \ldots, n$ we have $\langle \alpha, \check{\lambda}_i \rangle \geq \delta_{1i}$. Assume $\alpha \notin \Upsilon_{\mathfrak{d}}(\alpha_1)$. Then $\alpha < \alpha_1$ with respect to $\mathfrak{d}$. In particular, $\langle \alpha, \check{\lambda}_i \rangle \leq \langle \alpha_1, \check{\lambda}_i \rangle$ for all $i = 2, \ldots, n$, since $\lambda_2, \ldots, \lambda_n$ are corners of $\mathfrak{d}$. So $\langle \alpha, \check{\lambda}_i \rangle = 0$ for all $i = 2, \ldots, n$ which forces α to be a scalar multiple of α_1. But $\alpha \neq \alpha_1$ because $\alpha \notin \Upsilon_{\mathfrak{d}}(\alpha_1)$ and $\alpha \neq -\alpha_1$ since $\alpha \in \Omega_{\mathfrak{c}}(\alpha_1)$. So we reach a contradiction and prove the claim.

By the claim, we have $V_{\mathfrak{c}}(\alpha_1) \subset V_{\mathfrak{d}}(\alpha_1)$ and $V'_{\mathfrak{c}}(\alpha_1) \subset V'_{\mathfrak{d}}(\alpha_1)$ because this is the case generically. So we have a natural homomorphism of line bundles

$$V_{\mathfrak{c}}(\alpha_1)/V'_{\mathfrak{c}}(\alpha_1) \longrightarrow V_{\mathfrak{d}}(\alpha_1)/V'_{\mathfrak{d}}(\alpha_1)$$

which is non-zero, because generically it is the identity map on $\mathfrak{g}_{K,\alpha_1}$. Hence we have

$$\deg V_{\mathfrak{c}}(\alpha_1)/V'_{\mathfrak{c}}(\alpha_1) \leq \deg V_{\mathfrak{d}}(\alpha_1)/V'_{\mathfrak{d}}(\alpha_1).$$

By lemma 6.3.3 we have

$$\deg V_{\mathfrak{c}}(\alpha_1)/V'_{\mathfrak{c}}(\alpha_1) = n(\tilde{\mathfrak{c}}, \lambda_1).$$

But $n(\tilde{\mathfrak{c}}, \lambda_1) = \langle \alpha_1, d(\mathfrak{c}) \rangle$. This proves (C2). □

Definition 6.3.8 $d(G,T) = (d(\mathfrak{c}))_{\mathfrak{c} \in \mathfrak{C}}$ defined as in proposition 6.3.7 is called the *complementary convex solid of G with respect to T.*

Proposition 6.3.9 *Let $T \subset G$ be a generic maximal torus and (Φ, d) the corresponding root system with complementary convex solid. Let $P \subset G$ be a parabolic subgroup scheme containing T. We have*

$$\deg P = \deg P',$$

where $\deg P'$ is of course taken with respect to d.

Moreover, for every corner λ of P' we have

$$n(P, \lambda) = n(P', \lambda).$$

Proof. These are easy calculations using lemma 6.3.6. □

The following proposition explains the significance of reduction.

Proposition 6.3.10 *Let P be a parabolic subgroup of G, T a generic maximal torus of P. Then T is also a generic maximal torus of G and $P/R_u(P)$. Let (Φ, d) be the root system with complementary convex solid associated with G and T. Let P' be the facet of Φ associated to P via proposition 6.3.2. Then $(\Phi_{P'}, d_{P'})$ is the root system with complementary convex solid associated to $P/R_u(P)$ with the generic maximal torus T.*

Proof. Since a complementary convex solid is completely determined by the numerical invariants of the Weyl chambers (note 5.3.9), it will suffice to check that the numerical invariants of the Borel subgroups behave according to lemma 5.3.10. In other words, we have to show that for any Borel subgroup $B \subset P$ and any corner $\lambda \in \operatorname{vert} B' - \operatorname{vert} P'$ we have $\deg W(B, \lambda) = \deg W(B/R_u(P), p(\lambda))$, where $p(\lambda)$ is the corner of $(B/R_u(P))'$ associated with λ. But this follows immediately from the fact that $R_u(P) \subset U'(\tilde{\lambda})$. (The notation $U'(\lambda)$ is taken from the proof of proposition 6.2.3.) Indeed, we have

$$\begin{aligned} W(B,\lambda) &= U(\tilde{\lambda})/U'(\tilde{\lambda}) \\ &= U(\tilde{\lambda})/R_u(P) \Big/ U'(\tilde{\lambda})/R_u(P) \\ &= W(B/R_u(P), p(\lambda)) \end{aligned}$$

which implies the result. □

Corollary 6.3.11 *Let P be a parabolic subgroup of G. Then P is canonical if and only if for every generic maximal torus of P the corresponding facet P' of the corresponding root system with complementary convex solid is special. In particular, G is semi-stable if and only if for every generic maximal torus of G, the corresponding root system with complementary convex solid is semi-stable.*

Proof. First assume that P is canonical. Let $T \subset P_K$ be a generic maximal torus. Let P' be the corresponding facet of $\Phi = \Phi(G_K, T)$. Then propositions 6.3.9 and 6.3.10 immediately show that P' is special. Conversely, assume that for every $T \subset P_K$ the corresponding P' is special. we need to check that P satisfies (C1) and (C2). (C1) is clear. To prove (C2), assume that $P/R_u(P)$ is not semi-stable. Then there exists a parabolic subgroup Q of G such that $Q \subset P$ and $\deg(Q/R_u(P)) > 0$. Choosing a maximal torus $T \subset Q_K$ this contradicts the fact that the corresponding P' is special. □

Proposition 6.3.12 *G contains a unique canonical parabolic. It is maximal among the parabolics of maximal degree.*

Proof. To prove uniqueness, let P and Q be two canonical parabolics. By [8, Exp. XXVI, 4.1.1.] there exists a maximal torus $T \subset G_K$ such that $T \subset P_K \cap Q_K$. Let (Φ, d) be the corresponding root system with complementary convex solid. Then the corresponding facets P' and Q' of Φ are both special by corollary 6.3.11 and hence equal by corollary 5.3.15. This implies $P = Q$ by proposition 6.3.2.

To prove existence, let P be maximal among the parabolics of maximal degree. We claim that P satisfies (C1) and (C2). Let T be any generic maximal torus of G contained in P. Using the notation of proposition 6.3.2, P' is then maximal among the facets of maximal degree in $(\Phi(G,T), d(G,T))$ by propositions 6.3.9 and 6.3.2. By proposition 5.3.16 P' is special. Hence P is canonical by corollary 6.3.11. □

6.4 The Main Theorem

Lemma 6.4.1 *Let $\pi : Y \to X$ be a finite morphism of curves over k. Assume that Y/X is Galois, i.e. that the corresponding extension of function fields is Galois. Let G/X be a reductive group scheme. Then if $P \subset \pi^*G$ is a parabolic subgroup such that $\sigma^*P = P$ for every $\sigma \in \mathrm{Gal}\, Y/X$ then there exists a unique parabolic $Q \subset G$ such that $\pi^*Q = P$.*

Proof. This follows immediately from the properness of $\underline{\mathrm{Par}}(G) \to X$ ([8, Exp. XXVI, cor. 3.5.]). By this properness, parabolic subgroups are completely characterized by their generic fibers. □

Lemma 6.4.2 *Let $\pi : Y \to X$ be a finite separable morphism of curves over k and G a reductive group scheme over X. Let $P \subset G$ be a parabolic subgroup. The numerical invariants of P are positive if and only if the numerical invariants of π^*P are positive.*

Proof. Let $\mathfrak{v} \in \pi_0 t(P)$ and let $\mathfrak{v}_1, \ldots, \mathfrak{v}_s$ be the elements of $\pi_0\pi^*t(P)$ lying over $\mathfrak{v}$. In other words, $\pi^*\mathfrak{v} = \coprod_{i=1}^s \mathfrak{v}_i$. Clearly we have

$$\pi^* W(P, \mathfrak{v}) = \bigoplus_{i=1}^{s} W(\pi^*P, \mathfrak{v}_i).$$

So if $n(\pi^*P, \mathfrak{v}_i) = \deg W(\pi^*P, \mathfrak{v}_i)$ is positive for every $i = 1, \ldots, s$ then so is $n(P, \mathfrak{v}) = \deg W(P, \mathfrak{v})$. This proves one direction. It also makes it possible to reduce the other direction to the case that Y/X is Galois.

$\mathrm{Gal}\, Y/X$ acts naturally on $\coprod_{i=1}^{s} \mathfrak{v}_i$ and hence on $\{\mathfrak{v}_1, \dots, \mathfrak{v}_s\}$. Now since $\mathfrak{v}$ is connected, the action of $\mathrm{Gal}\, Y/X$ on $\{\mathfrak{v}_1, \dots, \mathfrak{v}_s\}$ is transitive. For any $\sigma \in \mathrm{Gal}\, Y/X$ and $i = 1, \dots, s$ we have $\sigma^* W(\pi^* P, \mathfrak{v}_i) \cong W(\pi^* P, \sigma^{-1}\mathfrak{v}_i)$ and hence

$$\begin{aligned} \deg W(\pi^* P, \mathfrak{v}_i) &= \deg \sigma^* W(\pi^* P, \mathfrak{v}_i) \\ &= \deg W(\pi^* P, \sigma^{-1}\mathfrak{v}_i). \end{aligned}$$

This proves that

$$\deg \pi \deg W(P, \mathfrak{v}) = s \deg W(P, \mathfrak{v}_i)$$

for any $i = 1, \dots, s$ which proves the lemma. □

Lemma 6.4.3 *Let $\pi : Y \to X$ be a finite separable morphism of curves and G a reductive group scheme over X such that $\pi^* G$ is rationally trivial. Then G is semi-stable if and only if $\pi^* G$ is semi-stable.*

Proof. Clearly, $\pi^* G$ semi-stable implies G semi-stable. This allows us to reduce the other direction to the case that Y/X is Galois. So assume that G is semi-stable. By proposition 6.3.12 $\pi^* G$ has a canonical parabolic $P \subset \pi^* G$. For every $\sigma \in \mathrm{Gal}\, Y/X$ $\sigma^* P$ is also a canonical parabolic in $\pi^* G$. Hence $\sigma^* P = P$ and P descends to a parabolic $Q \subset G$ such that $\pi^* Q = P$. Now $\deg Q \leq 0$, so $\deg P \leq 0$ which implies that $P = \pi^* G$ and $\pi^* G$ is semi-stable. □

Theorem 6.4.4 *Let G be a reductive group scheme over the curve X. Then G has a unique canonical parabolic subgroup P. P is the largest element in the set of parabolics of maximal degree in G.*

Proof. Choose a finite Galois morphism of curves $\pi : Y \to X$ such that $\pi^* G$ is rationally trivial. This can be done by choosing a finite Galois extension L of K over which G has a split maximal torus ([8, Exp. XIX, prop. 6.1.]) and then taking Y to be the integral closure of X in L. Now by proposition 6.3.12 $\pi^* G$ has a canonical parabolic P. As in the proof of lemma 6.4.3 P descends to X, so there exists a unique parabolic $Q \subset G$ such that $\pi^* Q = P$. Evidently, Q is canonical. This proves the existence.

Now if Q' is any canonical parabolic in G, then $\pi^* Q'$ is canonical by lemmas 6.4.2 and 6.4.3. This proves $\pi^* Q' = P$ and $Q = Q'$. This proves uniqueness.

For the last part, let $Q \subset G$ be maximal among the parabolics of maximal degree and let $P \subset G$ be the canonical parabolic. Then $\deg P \geq \deg Q$ as can be seen after pulling back via π and using proposition 6.3.12. Hence $\deg P = \deg Q$ and $P \subset Q$ by assumption on Q. This implies $P = Q$, again by pulling back via π and using proposition 6.3.12. □

Further Remarks

Definition 6.4.5 Let G be a reductive group scheme over the curve X. Let $\mathfrak{V}$ be the free abelian group on the connected components of $\mathrm{Dyn}(G)$. Let P be the canonical parabolic subgroup of G. Assume that the type $t(P) = \mathfrak{v}_1 \amalg \dots \amalg \mathfrak{v}_s$. Then we call

$$n(G) = \sum_{i=1}^{s} n(P, \mathfrak{v}_i)\mathfrak{v}_i \in \mathfrak{V}$$

the *type of instability of G*.

Corollary 6.4.6 *The canonical parabolic commutes with pullbacks to separable curves over X. In particular, semi-stability commutes with such pullbacks.*

Proof. Clear. □

Conjecture 6.4.7 *Let G be a reductive group scheme over the curve X. Let P be the canonical parabolic subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of G and P, respectively. Then $H^0(X, \mathfrak{g}/\mathfrak{p}) = 0$.*

The truth of this conjecture would imply that the canonical parabolic subgroup is a geometric invariant, i.e. that even after passing to inseparable extensions of the base field k the canonical parabolic remains canonical. So it would make notions like the *geometric* degree of instability superfluous.

Conjecture 6.4.7 is easily seen to be true if $G = GL(V)$, for a vector bundle V over X, or if P is a Borel subgroup. It is also true if the genus of X is zero or one and G is rationally trivial.

Conjecture 6.4.7 is equivalent to claiming that the canonical parabolic is *isolated* in the sense that the corresponding k-valued point of $\pi_* \operatorname{Par} G$ (here $\pi : X \to \operatorname{Spec} k$ is the structure morphism) is an isolated point of $\pi_* \operatorname{Par} G$ with local ring equal to k.

7 Families of Reductive Group Schemes over Curves

Introduction

In this section we study the behavior of the canonical parabolic subgroup in families of reductive group schemes. For this purpose we consider the following setup. Let S be a locally noetherian scheme and $\pi : X \to S$ a curve over S. This means that π is a smooth projective morphism all of whose geometric fibers are one-dimensional and connected. We will study reductive group schemes G over X.

Our goal is to prove the two fundamental Theorem 7.2.4 and 7.2.5. The first theorem is a semi-continuity theorem. It states that the (geometric) degree of instability is upper semi-continuous in families. Under the assumption that the (geometric) degree of instability is constant, we can *almost* prove that the canonical parabolics P_s in the various members G_s of our family glue together to a parabolic subgroup P of G. The significance of Theorem 7.2.5 is that all we have to do is pass to a cover S' of S that is *universally homeomorphic* to S. Note that the truth of Conjecture 6.4.7 would improve this result. These theorems are nicely summarized in Corollary 7.2.9.

Before considering the general case, we consider the case that S is the spectrum of a discrete valuation ring. This is the contents of Section 7.1.

In Section 7.3 we apply our results to the following situation. We have a reductive group scheme G over a curve X over a field k. We study the algebraic k-stack of G-torsors $\mathfrak{H}^1(X/k, G)$, defined on page 37. We associate three invariants with a G-torsor E.

First, there is the *degree*, $\deg E$, of E, which is the homomorphism $\deg E : \mathsf{X}(G) \to \mathbb{Z}$, where $\mathsf{X}(G)$ is the character group of G, defined by

$$\deg E(\chi) = \deg(E \times_{G,\chi} \mathcal{O}_X).$$

This degree generalizes the degree of a vector bundle.

The second and the third invariant of E are defined in terms of ${}^E G = \underline{\operatorname{Aut}}(E) = E \times_{G,Ad} G$, the twist of G associated to E. The *degree of instability* of E is the degree of instability of ${}^E G$ defined in Definition 6.1.4. The type of instability is defined in terms of the Dynkin diagram $\operatorname{Dyn} G$ of G. If $\mathfrak{V}$ is the free abelian group on the connected components of the scheme $\operatorname{Dyn} G$ then the type of instability is a certain non-negative element of $\mathfrak{V}$. (See Section 7.2.)

For every homomorphism $d : \mathsf{X}(G) \to \mathbb{Z}$ we let $\mathfrak{H}^1_d(X/k, G)$ be the substack of $\mathfrak{H}^1(X/k, G)$ of G-torsors of degree d. It is a closed and open substack. The stack $\mathfrak{H}^1_d(X/k, G)_{\leq m}$ of G-bundles of degree d and degree of instability less that or equal to m is open in $\mathfrak{H}^1_d(X/k, G)$. The family $(\mathfrak{H}^1_d(X/k, G)_{\leq m})_{m\geq 0}$ defines an exhaustion of $\mathfrak{H}^1_d(X/k, G)$. If $\mathfrak{v} \in \mathfrak{V}$ we let $\mathfrak{H}^1_d(X/k, G)_{m,\mathfrak{v}}$ be the stack of G-torsors of degree d, degree of instability m and type of instability $\mathfrak{v}$. The family $(\mathfrak{H}^1_d(X/k, G)_{m,\mathfrak{v}})_{m,\mathfrak{v}}$ defines a stratification of $\mathfrak{H}^1_d(X/k, G)$. This is the analogue of the stratification of Shatz (see [22]).

7.1 The Case of a Discrete Valuation Ring

Lemma 7.1.1 *Let G be a reductive group scheme over the curve X over the field k. Let $P \subset G$ be a parabolic subgroup such that*

i. the numerical invariants of P are positive,

ii. $\deg P = \deg_i(G)$.

Then P is the canonical subgroup.

Proof. Let K be the function field of X. By lemma 6.4.2 and corollary 6.4.6 we may pass to a finite separable cover of X and thus we may assume that G is rationally trivial. Let $T \subset P_K$ be any generic maximal torus. Denote by P also the facet of $\Phi = \Phi(G_K, T)$ corresponding to P. By corollary 6.3.11 it suffices to prove that P is special with respect to the complementary convex solid d on Φ induced by G. For this it suffices to prove that (Φ_P, d_P) is semi-stable. So let $Q \leq P$ be another facet. By lemmas 5.1.10 and 5.3.5 it suffices to show that $\deg Q \leq \deg P$ which is clear. □

Lemma 7.1.2 *Let X be an integral noetherian normal scheme, G over X a reductive group scheme. Let η be the generic point of X. If $P_\eta \subset G_\eta$ is a generic parabolic subgroup such that there exists a vector subbundle $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{p}_\eta$ is the Lie algebra of P_η, then there exists a parabolic subgroup $P \subset G$ such that $\mathfrak{p}$ is the Lie algebra of P.*

Proof. Let $\mathrm{Gr}(\mathfrak{g})$ be the projective X-scheme of vector subbundles of $\mathfrak{g}$. There is a natural X-morphism $f : \mathrm{Par}(G) \to \mathrm{Gr}(\mathfrak{g})$ associating to a parabolic subgroup of G its Lie algebra. We first note that this morphism f is radicial. To prove this, let L be an arbitrary field together with a morphism $x : \operatorname{Spec} L \to X$. We need to show that $\mathrm{Par}(G)(L) \to \mathrm{Gr}(\mathfrak{g})(L)$ is injective. So let $P_1, P_2 \in \mathrm{Par}(G)(L)$ be two parabolic subgroups of $x^*G = G|\operatorname{Spec} L$. By [8, Exp. XXVI, 4.1.1.] there exists a maximal torus T of x^*G such that $T \subset P_1 \cap P_2$. If $f(P_1) = f(P_2)$, then the Lie algebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are equal as subbundles of $x^*\mathfrak{g}$. Thus, by [8, Exp. XXII, Corollaire 5.3.5] we have $P_1 = P_2$. Thus f is indeed radicial.

Now to prove the lemma, note that $\mathfrak{p}$ corresponds to a section of $q : \mathrm{Gr}(\mathfrak{g}) \to X$, which we will also denote by $\mathfrak{p}$. Similarely, P_η is a generic section of $p : \mathrm{Par}(G) \to X$. Let Z be the closure of $P_\eta(\{\eta\})$ in $\mathrm{Par}(G)$, with the reduced subscheme structure. Let $i : Z \to \mathrm{Par}\, G$ be the inclusion morphism. If we can show that $p \circ i$ is an isomorphism we are done, because q is separated.

Towards this goal let us first prove that $p \circ i$ is injective on the underlying topological spaces. If $z_1, z_2 \in Z$ both lie over $x \in X$, then their images $f(z_1)$ and $f(z_2)$ also both lie over x. Now z_1 and z_2 are specializations of $P_\eta(\eta) \in \mathrm{Par}(G)$. Hence $f(z_1)$ and $f(z_2)$ are specializations of $\mathfrak{p}(\eta) \in \mathrm{Gr}(\mathfrak{g})$. But the only specialization of $\mathfrak{p}(\eta)$ lying over x is $\mathfrak{p}(x)$. Hence $f(z_1) = f(z_2) = \mathfrak{p}(x)$. Because f is radicial, this implies $z_1 = z_2$. Thus $p \circ i$ is indeed injective.

Since $p \circ i$ is clearly of finite type, the injectivity implies that $p \circ i$ is quasi-finite. Since $p \circ i$ is also clearly projective, Zariski's main theorem in the formulation of Grothendieck implies that $p \circ i$ is finite, in particular affine. Now since $p \circ i$ is birational and X is normal, $p \circ i$ is indeed an isomorphism. □

Proposition 7.1.3 *Let $\pi : X \to S$ be a curve over the scheme $S = \operatorname{Spec} R$, where R is a discrete valuation ring, and let G be a reductive group scheme over X. Let η be the generic point of S, 0 the special point. Then we have*

i. $\deg_i G_0 \geq \deg_i G_\eta$,

ii. If $\deg_i G_0 = \deg_i G_\eta$, then there exists a parabolic subgroup $P \subset G$ such that P_η is the canonical subgroup of G_η and P_0 is the canonical subgroup of G_0.

Proof. Let P_η be the canonical parabolic subgroup of G_η. Let $\mathfrak{p}_\eta$ be the Lie algebra of P_η. Then we have an exact sequence of vector bundles on X_η, defining $\mathfrak{q}_\eta$:

$$0 \to \mathfrak{p}_\eta \to \mathfrak{g}_\eta \to \mathfrak{q}_\eta \to 0.$$

By the properness of $\mathrm{Quot}(\mathfrak{g}/X/S)$ there exists a coherent $\mathcal{O}_X$-module $\mathfrak{q}$, flat over S, together with an epimorphism $u : \mathfrak{g} \to \mathfrak{q}$ such that $u|X_\eta$ is the map of the above short exact sequence. Let $\mathfrak{p}$ be the kernel:

$$0 \to \mathfrak{p} \to \mathfrak{g} \to \mathfrak{q} \to 0.$$

Then the restriction of $\mathfrak{p}$ to X_η is indeed $\mathfrak{p}_\eta$.

Let y be the generic point of X_0. Then $\mathcal{O}_{X,y}$ is a discrete valuation ring and $R \to \mathcal{O}_{X,y}$ is a local morphism of discrete valuation rings making $\mathcal{O}_{X,y}$ a flat R-module. The $\mathcal{O}_{X,y}$-module $\mathfrak{q}_y$ is flat over R. By some trivial algebra (using the fact that flatness over a discrete valuation ring just means that a local parameter acts faithfully) this implies that $\mathfrak{q}_y$ is flat over $\mathcal{O}_{X,y}$. Since it is also finitely generated, it is free. Thus, letting $U \subset X$ be the maximal open subset over which $\mathfrak{q}$ is locally free, we have $y \in U$. Clearly, we also have $X_\eta \subset U$. By lemma 7.1.2 there exists a parabolic subgroup $P \subset G_U$ such that the corresponding Lie algebra is isomorphic to $\mathfrak{p}_U$ as a vector subbundle of $\mathfrak{g}_U$.

Let $V = U \cap X_0$. Then $P_V = P|V$ is a parabolic subgroup of $G|V$. Since X_0 is a curve, we may extend P_V to a parabolic subgroup $\widetilde{P}_0$ of $G_0 = G|X_0$. Let $\widetilde{\mathfrak{p}}_0$ be the Lie algebra of $\widetilde{P}_0$. Then generically, $\mathfrak{p}_0 = \mathfrak{p}|X_0$ and $\widetilde{\mathfrak{p}}_0$ coincide, so $\widetilde{\mathfrak{p}}_0$ is the vector subbundle of $\mathfrak{g}_0$ generated by the coherent subsheaf $\mathfrak{p}_0$. Hence we have

$$\begin{aligned} \deg \widetilde{P}_0 &= \deg \widetilde{\mathfrak{p}}_0 \\ &\geq \deg \mathfrak{p}_0 \\ &= \deg \mathfrak{p}_\eta \\ &= \deg_i G_\eta. \end{aligned}$$

Clearly, we have $\deg_i G_0 \geq \deg \widetilde{P}_0$. This implies (i).

On the other hand, $\deg_i G_0 = \deg_i G_\eta$ implies that $\deg \widetilde{\mathfrak{p}}_0 = \deg \mathfrak{p}_0$ and that $\mathfrak{p}_0$ is already a vector subbundle of $\mathfrak{g}_0$. By a well-known result this implies that $\mathfrak{q}$ is locally free over all of X and hence that $\mathfrak{p}$ is a vector subbundle of $\mathfrak{g}$. In other words, $U = X$. So P is defined over all of X. To prove (ii) it suffices to prove that P_0 is the canonical parabolic subgroup of G_0.

Let S' be the henselization of S. To prove that P_0 is canonical in G_0 it suffices to prove that P_0' is canonical in G_0'. So by Proposition 6.4.6 we may replace S by S' and assume that S is henselian. By Zariski's connectedness theorem [4, Arcata IV, Prop. 2.1] we have a bijection $\pi_0 \operatorname{Dyn}(G) = \pi_0 \operatorname{Dyn}(G_0)$. This clearly implies that the numerical invariants of P_0 are positive, noting that P_η being canonical, its numerical invariants are positive. This implies that P_0 is canonical by lemma 7.1.1. □

7.2 The Fundamental Theorems

The Scheme of Parabolic Subgroups

Lemma 7.2.1 *Let $\pi : X \to S$ be a flat projective morphism of noetherian schemes. Let Y be a smooth projective X-scheme. Then the functor $\pi_* Y$ defined by*

$$\pi_* Y(T) = Y(X \times_S T)$$

for any S-scheme T, is representable. We denote the S-scheme representing this functor also by $\pi_* Y$. It is locally of finite type over S.

Proof. See [11, 4.c]. □

Let $G/X/S$ be a reductive group scheme over a curve, and let $\pi : X \to S$ be the structure morphism. Let $\operatorname{Par}(G)$ be the scheme of parabolic subgroups of G. It is smooth and projective over X. Consider the S-scheme $\pi_* \operatorname{Par}(G)$. It satisfies the universal mapping property

$$\pi_* \operatorname{Par}(G)(T) = \{\text{parabolic subgroups of } G|X \times_S T\}$$

for any S-scheme T. Also, there is a universal parabolic subgroup P of $G|X \times_S \pi_* \operatorname{Par}(G)$. The Lie algebra $\mathfrak{p}$ of P is a vector bundle on $X \times_S \pi_* \operatorname{Par}(G)$. This vector bundle $\mathfrak{p}$ induces a decomposition

$$\pi_* \operatorname{Par}(G) = \coprod_{(r,d)} (\pi_* \operatorname{Par} G)_{r,d},$$

where $(\pi_* \operatorname{Par} G)_{r,d}$ is the open and closed subscheme of $\pi_* \operatorname{Par} G$ characterized by the fact that $\operatorname{rk}(\mathfrak{p}|X_s) = r$ and $\deg(\mathfrak{p}|X_s) = d$ for all points s of $\pi_* \operatorname{Par} G$. (Note that it does not matter whether we think of points as geometric points, points of the underlying topological space or field-valued points in general.) Similarely, we define $(\pi_* \operatorname{Par} G)_d$ to be the scheme of parabolic subgroups of G that have degree d everywhere over S.

Proposition 7.2.2 *For any integer d the scheme $(\pi_* \operatorname{Par} G)_d$ is of finite type over S. In particular, the same holds for $(\pi_* \operatorname{Par} G)_{r,d}$ for any r.*

Proof. The arguments on pages 124–127 of [15] proving Satz 2.1.1 of [15] can be adapted to prove this proposition. □

The Theorems

Lemma 7.2.3 *Let G be a reductive group scheme over the curve X over the noetherian scheme S. Then there exists an $M > 0$ such that $\deg_i G_s \leq M$ for all $s : \operatorname{Spec} k \to S$, where k is a field.*

Proof. By the proof of Lemma 6.1.3 we have

$$\deg_i G_s \leq \dim_{\kappa(s)} H^0(X_s, \mathfrak{g}_s) + \operatorname{rk}(\mathfrak{g}_s) g(X_s).$$

Now since $\operatorname{rk}(\mathfrak{g}_s)$ and $g(X_s)$ are locally constant on S we might as well assume that they are constant, say equal to r and g respectively. By the semi-continuity theorem $\dim_{\kappa(s)} H^0(X_s, \mathfrak{g}_s)$ is upper semi-continuous on S. Now every increasing sequence of open subsets of S becomes stationary. This finishes the proof. □

Theorem 7.2.4 *Let $\pi : X \to S$ be a curve over the locally noetherian scheme S. Let G be a reductive group scheme over X. Then the geometric degree of instability of G is upper semicontinuous on S.*

Proof. Without loss of generality let S be noetherian. Let d be an integer. Let

$$S_{\leq d} = \{s \in S \mid \text{for every extension } k \text{ of } \kappa(s) \text{ we have } \deg_i G_k \leq d\}.$$

We need to show that $S_{\leq d}$ is open in S. Let M be as in Lemma 7.2.3. Then

$$\coprod_{\delta=d+1}^{M} (\pi_* \operatorname{Par} G)_\delta$$

is by Proposition 7.2.2 an S-scheme of finite type. Its image in S is precisely the complement of $S_{\leq d}$. By Chevalley's theorem this implies that $S_{\leq d}$ is constructible in S. Hence it suffices to prove that $S_{\leq d}$ is stable under generization.

To do this we may assume without loss of generality by [17, II, Exc. 4.11] that S is a discrete valuation ring and that for the generic point $\eta \in S$ we have $\deg_i G_\eta = \deg_i G_k$, for any field extension k of $\kappa(\eta)$. Then we are done by Proposition 7.1.3 i. □

Theorem 7.2.5 *Let $\pi : X \to S$ be a curve over the locally noetherian scheme S. Let G be a reductive group scheme over X, of constant geometric degree of instability d. Then there exists a scheme S', finite radicial and surjective over S, and a parabolic subgroup P of G' such that for every k-valued point s of S', where k is a field, the canonical parabolic subgroup of G'_s is P_s.*

Proof. Without loss of generality assume that S is noetherian. First let us prove that the dimension of the geometric canonical parabolic subgroup is a locally constant function on S. This means that for any r the set

$$\begin{aligned} S_r \quad = \quad & \{s \in S \mid \text{the dimension of the canonical parabolic subgroup} \\ & \text{of } G_{\overline{\kappa(s)}} \text{ is equal to } r\} \end{aligned}$$

is open and closed in S. Here $\overline{\kappa(s)}$ denotes the algebraic closure of $\kappa(s)$. Consider the S-scheme $(\pi_* \operatorname{Par} G)_{r,d}$. Its image in S is contained on $\bigcup_{\rho \geq r} S_\rho$ and contains S_r. By induction we may thus assume that S_r is constructible. So it remains to prove that S_r is stable under specialization and generization. For this, we may assume that S is a discrete valuation ring. We may also assume that the geometric canonical parabolic subgroups are already defined over the two points $0, \eta \in S$. Then the claim follows immediately from Proposition 7.1.3 ii. So we will assume that $S = S_r$ for some r.

Consider the scheme $(\pi_* \operatorname{Par} G)_{r,d}$. By Proposition 7.2.2 it is of finite type over S. Hence by Proposition 7.1.3 $(\pi_* \operatorname{Par} G)_{r,d}$ is proper over S. By the characterization of the canonical parabolic as the unique parabolic subgroup of degree d and dimension r, we get that $(\pi_* \operatorname{Par} G)_{r,d} \to S$ is radicial and surjective, in particular quasi-finite. Now from Zariski's main theorem it follows that a proper quasi-finite morphism is finite. So $(\pi_* \operatorname{Par} G)_{r,d}$ is a finite radicial surjective S-scheme, and taking $S' = (\pi_* \operatorname{Par} G)_{r,d}$ will prove our theorem. □

Note 7.2.6 Let S be a locally noetherian scheme and S' an S-scheme. Then the following are equivalent:

i. The scheme S' is finite, radicial and surjective over S.

ii. The morphism $S' \to S$ is radicial, surjective, universally closed and of finite type.

iii. The morphism $S' \to S$ is locally of finite type and a universal homeomorphism.

Remark 7.2.7 The truth of Conjecture 6.4.7 would imply that (at least if S is integral noetherian) we could take $S = S'$ in Theorem 7.2.5.

The Type of Instability

Let $\pi : X \to S$ be a curve over the locally noetherian scheme S, which we will assume to be connected, for simplicity. Let G be a reductive group scheme over X and let $\mathrm{Dyn}(G)$ be the scheme of Dynkin diagrams of G. This scheme $\mathrm{Dyn}\, G$ is finite and étale over X. Let $\mathfrak{v}_1, \ldots, \mathfrak{v}_n$ be the connected components of $\mathrm{Dyn}\, G$ and let $\mathfrak{V}$ be the free abelian group on the generators $\mathfrak{v}_1, \ldots, \mathfrak{v}_n$. Whenever $T \to S$ is a connected S-scheme, we denote the corresponding free abelian group on the connected components of $\mathrm{Dyn}\, G_T$ by $\mathfrak{V}_T$. The natural morphism $\pi_0(\mathrm{Dyn}\, G_T) \to \pi_0(\mathrm{Dyn}\, G)$ induces a homomorphism $\mathfrak{V}_T \to \mathfrak{V}$.

Now let P be a parabolic subgroup of G. Let $t(P)$ be the type of P (See Remark 6.2.1). It is a closed and open subscheme of $\mathrm{Dyn}\, G$, so by renumbering if necessary, we may assume that $t(P) = \mathfrak{v}_1 \amalg \ldots \amalg \mathfrak{v}_s$. Let $\mathfrak{V}(P) \subset \mathfrak{V}$ be the subgroup generated by $\mathfrak{v}_1, \ldots, \mathfrak{v}_s$. Now we can do the construction of page 57 in this context. We get for every $i = 1, \ldots, s$ a vector bundle $W(P, \mathfrak{v}_i)$ over X. Then the numerical invariant of P with respect to $\mathfrak{v}_i$, denoted $n(P, \mathfrak{v}_i)$, is the degree of $W(P, \mathfrak{v}_i)_s$ for some point $s \in S$. By the connectedness of S this definition is independent of the choice of s. We define

$$n(P) = \sum_{i=1}^{s} n(P, \mathfrak{v}_i)\mathfrak{v}_i$$

which is an element of $\mathfrak{V}(P) \subset \mathfrak{V}$.

More generally, if P is a parabolic subgroup of G_T, where T is a connected S-scheme, we get $n(P) \in \mathfrak{V}_T$. So $n(P)$ has a natural image in $\mathfrak{V}$.

Now let $s \in S$ be a set-theoretic point of S. Choose an algebraically closed field k over $\kappa(s)$. Let P_k be the canonical parabolic subgroup of G_k. As above, we get $n(P_k) \in \mathfrak{V}_k$. The image of $n(P_k)$ in $\mathfrak{V}$ does obviously not depend on the choice of k over $\kappa(s)$. Thus we have constructed a function $n : S \to \mathfrak{V}$.

Definition 7.2.8 We will call this function n the *geometric type of instability* of the reductive group scheme G over the curve X over the connected locally noetherian scheme S.

We will now summarize the results of our considerations in the following corollary.

Corollary 7.2.9 *Let $\pi : X \to S$ be a curve over the connected locally noetherian scheme S. Let G be a reductive group scheme over X. Let $\mathfrak{V}$ be the free abelian group on the connected components of the scheme of Dynkin diagrams of G. Then the geometric degree of instability of G is upper semicontinuous on S. So for any $d \in \mathbb{Z}$ the set $S_d = \{s \in X \mid \deg_i(G_{\overline{\kappa(s)}}) = d\}$ is a locally closed subset of S. Moreover, for any $d \in \mathbb{Z}$ the geometric type of instability $n : S \to \mathfrak{V}$ is continuous when restricted to S_d.*

Proof. The first part of the claim is just Theorem 7.2.4. For the second part, we may replace S by some induced subscheme structure on some connected component of S_d and thus assume that the geometric degree of instability is constant on S. Invoking Theorem 7.2.5 we may even assume that there exists a parabolic subgroup P of G, such that P_s is the canonical parabolic subgroup of G_s for every geometric point s of S. Then n is clearly constant and equal to $n(P)$. □

7.3 Applications to G-bundles

Let S be a connected locally noetherian scheme, $\pi : X \to S$ a curve over S and G a reductive group scheme over X. We will study the S-stack $\mathfrak{H}^1(X/S, G)$ of (families of) G-torsors. For the definition see page 37. By Corollary 4.5.2 $\mathfrak{H}^1(X/S, G)$ is a smooth algebraic S-stack, if S is the spectrum of a field. Whenever we refer to $\mathfrak{H}^1(X/S, G)$ as an *algebraic* S-stack, we will tacitly assume that S is the spectrum of a field.

Let $\mathfrak{V}$ be the free abelian group on the set of connected components of the Dynkin diagram $\mathrm{Dyn}(G)$ of G.

Stability of G-bundles

Let E be a G-torsor. Then ${}^E G = \mathrm{Aut}_G(E) = E \times_{G,Ad} G$ is again a reductive group scheme on X. This follows from descent theory, because ${}^E G$ is a sheaf on X that is locally (with respect to the fppf topology) an *affine* scheme. This argument shows that ${}^E G$ is a scheme over X. Then it is clearly a reductive group scheme, because this property is local with respect to the fppf topology.

Note that there is a canonical isomorphism of X-schemes $\mathrm{Dyn}(G) = \mathrm{Dyn}({}^E G)$. This comes about as the image of the canonical section of $\underline{\mathrm{Isomext}}(G, {}^E G)$ in $\underline{\mathrm{Isom}}(\mathrm{Dyn}(G), \mathrm{Dyn}({}^E G))$. (See [8, Exp. XXIV 3.5].) Hence we can canonically identify the free abelian group on the connected components of $\mathrm{Dyn}({}^E G)$ with $\mathfrak{V}$.

Definition 7.3.1 The G-torsor E is called *geometrically (semi-)stable* if for every geometric point s of S the reductive group scheme ${}^{E_s} G_s = ({}^E G)_s$ on X_s is (semi-)stable. (See Definition 6.1.4.)

The *geometric degree of instability* of E, denoted $\deg_i(E)$, is the function on S associating to every set theoretic point $s \in S$ the degree of instability $\deg_i(({}^E G)_{\overline{\kappa(s)}})$ of $({}^E G)_{\overline{\kappa(s)}}$, where $\overline{\kappa(s)}$ is the algebraic closure of the residue field $\kappa(s)$ of s. By Theorem 7.2.4 $\deg_i(E)$ is upper semicontinuous on S.

The *geometric type of instability* of E, denoted $n(E)$, is simply the geometric type of instability of ${}^E G$ (see Definition 7.2.8). By the above remarks we can consider $n(E)$ as a function $n(E) : S \to \mathfrak{V}$. By Corollary 7.2.9 $n(E)$ is continuous over subsets of S over which $\deg_i(E)$ is constant.

The Natural Stratification of the Stack of G-bundles

Definition 7.3.2 Let $m \geq 0$ be an integer. Then by $\mathfrak{H}^1(X/S, G)_{\leq m}$ we will denote the S-stack of G-torsors of geometric degree of instability less than or equal to m. This stack $\mathfrak{H}^1(X/S, G)_{\leq m}$ is clearly an open substack of $\mathfrak{H}^1(X/S, G)$, in particular also algebraic and smooth over S. We denote the complement of $\mathfrak{H}^1(X/S, G)_{\leq m-1}$ in $\mathfrak{H}^1(X/S, G)_{\leq m}$ by $\mathfrak{H}^1(X/S, G)_m$ and endow it with the reduced induced substack structure (see [19, Lemme 3.9]). So $\mathfrak{H}^1(X/S, G)_m$ is a closed substack of $\mathfrak{H}^1(X/S, G)_{\leq m}$ and hence algebraic and locally of finite type.

For an element $\mathfrak{v} \in \mathfrak{V}$ such that $\mathfrak{v} \geq 0$ we let $\mathfrak{H}^1(X/S, G)_{m,\mathfrak{v}}$ be the substack of $\mathfrak{H}^1(X/S, G)_m$ of G-torsors of geometric type of instability $\mathfrak{v}$. More precisely, let T be any S-scheme, which we may, without loss of generality, assume to be connected. Then a morphism $T \to \mathfrak{H}^1(X/S, G)_m$ defines a G_T-torsor E_T via the induced morphism $T \to \mathfrak{H}^1(X/S, G)$. Noting that the degree of instability of E_T is necessarely constant equal to m, we see that the geometric type of instability $n(E_T) : T \to \mathfrak{V}$ is also constant. We define $T \to \mathfrak{H}^1(X/S, G)_m$ to factor through $\mathfrak{H}^1(X/S, G)_{m,\mathfrak{v}}$ if $n(E_T)$ is equal to $\mathfrak{v}$. Clearly, $\mathfrak{H}^1(X/S, G)_{m,\mathfrak{v}}$ is an open and closed substack of $\mathfrak{H}^1(X/S, G)_m$.

For example, $\mathfrak{H}^1(X/S, G)_0$ is the S-stack of geometrically semi-stable G-torsors. The only element $\mathfrak{v} \in \mathfrak{V}$ such that $\mathfrak{H}^1(X/S, G)_{0,\mathfrak{v}}$ is non-empty is $\mathfrak{v} = 0$. We have $\mathfrak{H}^1(X/S, G)_{0,0} = \mathfrak{H}^1(X/S, G)_0$.

The family $(\mathfrak{H}^1(X/S, G)_m)_{m \geq 0}$ defines a *stratification* of $\mathfrak{H}^1(X/S, G)$. We call it the stratification with respect to the degree of instability. The family $(\mathfrak{H}^1(X/S, G)_{\leq m})_{m \geq 0}$ is an increasing family of open substacks covering the stack $\mathfrak{H}^1(X/S, G)$, i.e. an *exhaustion* of $\mathfrak{H}^1(X/S, G)$. For every $m \geq 0$ we get a decomposition

$$\mathfrak{H}^1(X/S,G)_m = \coprod_{\mathfrak{v} \geq 0} \mathfrak{H}^1(X/S,G)_{m,\mathfrak{v}}.$$

Remark 7.3.3 Let E be a G-torsor. Since for any further G-torsor F the group schemes $\mathrm{Aut}_G(F)$ and $\mathrm{Aut}_{^EG}(\mathrm{Hom}_G(E,F))$ are canonically isomorphic, the Change of Origin morphism $\mathfrak{H}^1(X/S,G) \to \mathfrak{H}^1(X/S,^E G)$ (see Proposition 4.2.1) respects all the above substacks.

The Degree of G-bundles

Let S be a connected locally noetherian scheme and $\pi : X \to S$ a curve over S. Let G be (briefly) more generally a smooth affine group scheme of finite type over X. Let $\mathsf{X}(G) = \mathrm{Hom}(G, \mathbb{G}_\mathrm{m})$ be the character group of G, which we consider simply as an abelian group. If A is an abelian group we denote by $A^\vee = \mathrm{Hom}(A, \mathbb{Z})$ its dual group.

Definition 7.3.4 Let E be a G-torsor. Then the *degree of* E, denoted $\deg E$, is the homomorphism $\deg E : \mathsf{X}(G) \to \mathbb{Z}$ defined by $\deg E(\chi) = \deg(E \times_{G,\chi} \mathcal{O}_X)$. Since $E \times_{G,\chi} \mathcal{O}_X$ is a line bundle on X and S is connected, the degree of $(E \times_{G,\chi} \mathcal{O}_X)_s$ is independent of $s \in S$. So $\deg E(\chi)$ is well-defined. The function $\deg E$ is a homomorphism, since multiplying gluing data for line bundles corresponds to adding degrees.

Definition 7.3.5 Let $d \in \mathsf{X}(G)^\vee$. Then we define $\mathfrak{H}^1_d(X/S,G)$ to be the S-stack of G-torsors of degree d. Here we view the degree of a G_T-torsor E_T over a connected S-scheme T as an element of $\mathsf{X}(G)^\vee$ via the natural homomorphism $\mathsf{X}(G) \to \mathsf{X}(G_T)$. The stack $\mathfrak{H}^1_d(X/S,G)$ is clearly an open and closed substack of $\mathfrak{H}^1(X/S,G)$. We have a decomposition

$$\mathfrak{H}^1(X/S,G) = \coprod_{d \in \mathsf{X}(G)^\vee} \mathfrak{H}^1_d(X/S,G).$$

Definition 7.3.6 Now assume G to be reductive. Then combining the degree, the degree of instability and the type of instability, we get locally closed substacks $\mathfrak{H}^1_d(X/S,G)_m$ of $\mathfrak{H}^1_d(X/S,G)$ and closed and open substacks $\mathfrak{H}^1_d(X/S,G)_{m,\mathfrak{v}}$ of $\mathfrak{H}^1_d(X/S,G)_m$.

Remark 7.3.7 Returning to the case that G is an arbitrary smooth and affine group scheme over X, note that if E is a G-torsor, then there is a canonical isomorphism $\mathsf{X}(G) = \mathsf{X}(^E G)$. So we can compare degrees of G-torsors and $^E G$-torsors. Then the Change of Origin morphism $\mathfrak{H}^1(X/S,G) \to \mathfrak{H}^1(X/S,^E G)$ identifies $\mathfrak{H}^1_d(X/S,G)$ with $\mathfrak{H}^1_{d-\deg E}(X/S,^E G)$, for every $d \in \mathsf{X}(G)^\vee$. Checking this is a straightforward calculation. So for a given $d \in \mathsf{X}(G)^\vee$ any G-torsor E with $\deg E = d$ induces an isomorphism

$$\mathfrak{H}^1_d(X/S,G) \xrightarrow{\sim} \mathfrak{H}^1_0(X/S,^E G).$$

Reduction to a Parabolic Subgroup

Let G be a reductive group scheme over the curve X over the connected locally noetherian scheme S. Let $P \subset G$ be a parabolic subgroup of G. Let $t(P)$ be the type of P (see Remark 6.2.1). This is a closed and open subscheme of $\mathrm{Dyn}(G)$. Hence the components $\mathfrak{v}_1, \ldots, \mathfrak{v}_s$ of $t(P)$ generate a subgroup of $\mathfrak{V}$. This subgroup will be denoted $\mathfrak{V}(P)$. As on page 72 we get for every $i = 1, \ldots, s$ a vector bundle $W(P, \mathfrak{v}_i)$ over X. By construction, P acts on $W(P, \mathfrak{v}_i)$ linearly, via the action induced from inner automorphisms (see the proof of Proposition 6.2.3). Taking the determinant of this representation of P gives a character χ_i of P. This process defines a homomorphism $\mathfrak{V}(P) \to \mathsf{X}(P)$. Identifying $\mathfrak{V}(P)$ with its dual (via the basis $\mathfrak{v}_1, \ldots, \mathfrak{v}_s$) we get a homomorphism

$$\begin{aligned} \mathfrak{v} : \mathsf{X}(P)^\vee &\longrightarrow \mathfrak{V}(P) \\ \delta &\longmapsto \sum_{i=1}^{s} \delta(\chi_i)\mathfrak{v}_i. \end{aligned}$$

Denote the image of this homomorphism $\mathfrak{v}$ by $\mathfrak{V}(P)'$.

Naturally, we have a homomorphism $\mathsf{X}(G) \to \mathsf{X}(P)$ and hence a homomorphism

$$\begin{aligned} d : \mathsf{X}(P)^\vee &\longrightarrow \mathsf{X}(G)^\vee \\ \delta &\longmapsto \delta|\mathsf{X}(G). \end{aligned}$$

Taking the determinant of the action of P on its unipotent radical $R_u(P)$ we get a character χ_0 of P. This makes sense, because $R_u(P)$ has a filtration that is invariant under the action of P and whose factors are vector bundles.

Lemma 7.3.8 *There exist positive rational numbers $y_1, \ldots, y_s$ such that*

$$\chi_0 = \sum_{i=1}^{s} y_i \chi_i.$$

Proof. We may assume that G admits a split maximal torus T. Looking at the corresponding root system, our lemma follows from Proposition 5.1.9. □

Evaluating at χ_0 defines a homomorphism $m : \mathsf{X}(P)^\vee \to \mathbb{Z}$. From Lemma 7.3.8 it easily follows that m factors through $\mathfrak{V}(P)'$. So we may, when the need arises, consider m as a homomorphism $m : \mathfrak{V}(P)' \to \mathbb{Z}$. We have

$$m\left(\sum_{i=1}^{s} n_i \mathfrak{v}_i\right) = \sum_{i=1}^{s} n_i y_i.$$

Note 7.3.9 If E is a P-torsor of degree δ, then the associated G-torsor $E \times_P G$ has degree $d(\delta)$. If $\mathfrak{v}(\delta) = \sum_{i=1}^{s} n_i \mathfrak{v}_i$, then $n_i = n(E \times_{P,Ad} P, \mathfrak{v}_i)$, where we consider $E \times_{P,Ad} P$ as parabolic subgroup of $E \times_{P,Ad} G$. Also, $\deg(E \times_{P,Ad} P) = m(\delta)$.

Let $R_u(P)$ be the unipotent radical of P and $H = P/R_u(P)$. We have a natural morphism

$$\mathfrak{H}^1(X/S, P) \longrightarrow \mathfrak{H}^1(X/S, H).$$

Let $\mathfrak{H}^1(X/S, P)_0$ be the preimage of $\mathfrak{H}^1(X/S, H)_0$ under this morphism. In other words, we call a P-torsor semi-stable if its associated H-torsor is.

Definition 7.3.10 We call an element $\delta \in \mathsf{X}(P)^\vee$ *positive* if $\delta(\chi_i) > 0$ for all $i = 1, \ldots, s$. We denote the set of positive elements of $\mathsf{X}(P)^\vee$ by $\mathsf{X}(P)^\vee_+$.

Proposition 7.3.11 *The natural homomorphism*

$$d \times \mathfrak{v} : \mathsf{X}(P)^\vee \longrightarrow \mathsf{X}(G)^\vee \times \mathfrak{V}(P)$$

is an injection with finite cokernel.

Proof. Let $R(G)$ and $R(P)$ be the radicals of G and P respectively. We have $\mathsf{X}(G) \subset \mathsf{X}(R(G))$ and $\mathsf{X}(P) \subset (R(P))$. So since $R(G) \subset P$ it makes sense to consider the sequence of abelian groups

$$0 \longrightarrow \mathfrak{V}(P) \xrightarrow{\iota} \mathsf{X}(P) \xrightarrow{\rho} \mathsf{X}(R(G)) \longrightarrow 0. \tag{25}$$

Note that we do not claim that it is exact.

By construction, the characters $\chi_1, \ldots, \chi_s$ vanish on the center of G. After all, they are induced by inner automorphisms of G. Since $R(G)$ is contained in the center of G we see that $\rho \circ \iota = 0$.

Let $U \to X$ be an étale morphism with U connected and T_U a split maximal torus of G_U, contained in P_U. Let $\Phi = \Phi(G_U, T_U)$ be the corresponding root system. Let P also denote the facet of Φ corresponding to P_U via Lemma 6.2.2. For a corner λ of P let $\Psi(P, \lambda)$ be the corresponding elementary set of roots (see Definition 5.3.6). Let $\gamma(\lambda) = \sum_{\alpha \in \Psi(P,\lambda)} \alpha$ for $\lambda \in \operatorname{vert} P$. There exists a natural map $f_U : \operatorname{vert} P \to \{\mathfrak{v}_1, \ldots, \mathfrak{v}_s\}$. Restricting the characters $\chi_1, \ldots, \chi_s$ to T_U we get

$$\chi_i = \sum_{\lambda \in f_U^{-1}(\mathfrak{v}_i)} \gamma(\lambda)$$

for every $i = 1, \ldots, s$.

Now let $\chi = \sum_{i=1}^{s} \nu_i \chi_i = 0$ be a relation among the characters $\chi_1, \ldots, \chi_s$. We have

$$\begin{aligned} \chi &= \sum_{i=1}^{s} \nu_i \sum_{\lambda \in f_U^{-1}(\mathfrak{v}_i)} \gamma(\lambda_i) \\ &= \sum_{\lambda \in \operatorname{vert} P} \nu_{f_U(\lambda)} \gamma(\lambda). \end{aligned}$$

By abuse of notation, we have written $n_{\mathfrak{v}_i}$ instead of n_i here. Now since $(\gamma(\lambda))_{\lambda \in \operatorname{vert} P}$ is linearly independent, we get $n_{f_U(\lambda)} = 0$ for all $\lambda \in \operatorname{vert} P$ and hence $\chi = 0$. This proves the injectivity of the homomorphism ι in the sequence (25).

Now let $\chi \in \mathsf{X}(P)$ and assume that $\rho(\chi) = 0$. Considering again U and T_U as above, this implies that, restricting χ to T_U, we get $\chi \in V$, where $V = \operatorname{span} \Phi \subset \mathsf{X}(T_U) \otimes \mathbb{R}$ is the vector space in which our root system Φ lives. In fact, by considering the action of the Weyl group on χ, it is easily seen that $\chi \in \operatorname{span}_{\lambda \in \operatorname{vert} P}(\lambda)$. We can write

$$\chi = \sum_{\lambda \in \operatorname{vert} P} \nu_\lambda \gamma(\lambda),$$

since $\gamma(\lambda))_{\lambda \in \operatorname{vert} P}$ is a basis for $\operatorname{span}_{\lambda \in \operatorname{vert} P}(\lambda)$. Replacing χ by some integer multiple, we may assume that all ν_λ $(\lambda \in \operatorname{vert} P)$ are integers.

Claim: For all $\lambda, \mu \in \operatorname{vert} P$ we have: $f_U(\lambda) = f_U(\mu) \Rightarrow \nu_\lambda = \nu_\mu$.

To prove the claim, choose an épinglage E_U of G_U compatible with T_U. Then $\operatorname{Dyn}(G)_U$ is constant, equal to the Dynkin diagram of (E_U, G_U). The type $t(P)$ of P is a subscheme of $\operatorname{Dyn}(G)_U$ whose connected components are in canonical bijection with the corners of P. So we may consider $(\nu_\lambda)_{\lambda \in \operatorname{vert} P}$ as a continuous function ν_U on $t(P)_U \subset \operatorname{Dyn}(G)_U$. Choosing a different maximal torus and a different épinglage, does not change this function ν_U. Hence the ν_U, for U connected and étale over X such that G is épinglable over U, glue together and define a continuous function ν on $t(P) \subset \operatorname{Dyn} G$. This function ν is then necessarily constant on every connected component $\mathfrak{v}_i$, for $i = 1, \ldots, s$. This proves the claim.

Now consider the element

$$\mathfrak{v} = \sum_{i=1}^{s} \nu_{f_U^{-1}(\mathfrak{v}_i)} \mathfrak{v}_i$$

of $\mathfrak{V}(P)$. Here we mean by $\nu_{f_U^{-1}(\mathfrak{v}_i)}$ a number ν_λ for some $\lambda \in f_U^{-1}(\mathfrak{v}_i)$. By the claim it does not matter which λ we choose. This element $\mathfrak{v}$ maps to χ under ι. Thus we have proven that $\iota(\mathfrak{v}(P))$ has finite index in $\ker\rho$.

It is well-known that $\mathsf{X}(G)$ has finite index in $\mathsf{X}(R(G))$. this implies that $\operatorname{im}\rho$ has finite index. So (25) is a complex of finitely generated free $\mathbb{Z}$-modules. At the first place it is exact, and at the other two places it has finite cohomology. Taking the dual of (25) gives a complex with the same properties

$$0 \longrightarrow \mathsf{X}(R(G))^\vee \longrightarrow \mathsf{X}(P)^\vee \longrightarrow \mathfrak{V}(P) \longrightarrow 0.$$

Taking the dual of the injective map with finite cokernel $\mathsf{X}(G) \to \mathsf{X}(R(G))$ we get another such, i.e. $\mathsf{X}(R(G))^\vee$ is a subgroup of finite index in $\mathsf{X}(G)^\vee$.

Together, these facts imply that $\mathsf{X}(P)^\vee$ injects into the direct product. We also see that $\mathsf{X}(P)^\vee \to \mathfrak{V}(P)$ and $\mathsf{X}(P)^\vee \to \mathsf{X}(G)^\vee$ have finite cokernel. $\square$

Proposition 7.3.12 *Let $\delta \in \mathsf{X}(P)^\vee_+$. If E is a geometrically semi-stable P-torsor of degree δ, then the associated G-torsor $E \times_P G$ has degree $d(\delta)$ and geometric type of instability $\mathfrak{v}(\delta)$. Moreover, its geometric degree of instability is equal to $m(\delta)$. Hence we get a natural morphism of S-stacks*

$$\mathfrak{H}^1_\delta(X/S, P)_0 \longrightarrow \mathfrak{H}^1_{d(\delta)}(X/S, G)_{m(\delta),\mathfrak{v}(\delta)}.$$

Under the assumption that the connected components of $\mathrm{Dyn}(G)$ *are geometrically connected, this morphism is finite and a universal homeomorphism.*

Proof. Directly from the definitions (and Note 7.3.9) we get the existence of a morphism

$$\mathfrak{H}^1_\delta(X/S, P)_0 \longrightarrow \mathfrak{H}^1_{d(\delta)}(X/S, G)_{\leq m(\delta)}.$$

This morphism actually factors through $\mathfrak{H}^1_{d(\delta)}(X/S, G)_{m(\delta)}$ because $\mathfrak{H}^1_\delta(X/S, P)_0$ is reduced, being smooth. Thus we have the existence of our morphism.

To prove the second claim we choose a connected S-scheme T and a morphism

$$E : T \to \mathfrak{H}^1_{d(\delta)}(X/S, G)_{m(\delta),\mathfrak{v}(\delta)}$$

which we interpret as a G_T-torsor. By Proposition 4.2.3 we have a 2-cartesian diagram

$$\begin{tikzcd} \pi_{T*}(E/P_T) \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{2}"] & T \arrow[d, "E"] \\ \mathfrak{H}^1(X/S, P) \arrow[r] & \mathfrak{H}^1(X/S, G). \end{tikzcd}$$

Let $\pi_{T*}(E/P_T)_{\mathfrak{v}(\delta)}$ be the open and closed subscheme of $\pi_{T*}(E/P_T)$ defined by the following cartesian diagram

$$\begin{tikzcd} \pi_{T*}(E/P_T)_{\mathfrak{v}(\delta)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{2}"] & \pi_{T*}(E/P_T) \arrow[d] \\ \coprod_{\substack{\delta' \in \mathsf{X}(P)^\vee \\ \mathfrak{v}(\delta') = \mathfrak{v}(\delta)}} \mathfrak{H}^1_{\delta'}(X/S, P) \arrow[r] & \mathfrak{H}^1(X/S, P). \end{tikzcd}$$

Then we have the diagram

$$\begin{tikzcd} \pi_{T*}(E/P_T)_{\mathfrak{v}(\delta)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{2}"] & \pi_{T*}(E/P_T) \arrow[d] \\ \mathfrak{H}^1_\delta(X/S, P)_0 \arrow[r] & \mathfrak{H}^1(X/S, P). \end{tikzcd}$$

This follows from Proposition 7.3.11 and the fact that E has degree $d(\delta)$. We also use the fact that a reduction of structure group of E_t to P_t of degree δ' with $\mathfrak{v}(\delta') = \mathfrak{v}(\delta)$ is necessarily semi-stable, for any geometric point t of T. Here we take advantage of the fact that the components of $\mathrm{Dyn}(G)$ correspond bijectively to the components of $\mathrm{Dyn}(G_t)$.

Now by [8, Exp. XXVI, Lemme 3.20] we have a canonical isomorphism

$$E/P_T = \mathrm{Par}_{t(P_T)}({}^E G_T).$$

Applying π_* we get

$$\pi_*(E/P_T) = \pi_* \mathrm{Par}_{t(P_T)}({}^E G_T).$$

Letting $(\pi_* \mathrm{Par}_{t(P_T)}({}^E G_T))_{\mathfrak{v}(\delta)}$ be the image of $\pi_*(E/P_T)_{\mathfrak{v}(\delta)}$ under this canonical isomorphism, we get a 2-cartesian diagram

$$\begin{array}{ccc} \left(\pi_* \mathrm{Par}_{t(P_T)}({}^E G_T)\right)_{\mathfrak{v}(\delta)} & \longrightarrow & T \\ \downarrow & \boxed{2} & \downarrow E \\ \mathfrak{H}^1_\delta(X/S, P)_0 & \longrightarrow & \mathfrak{H}^1_{d(\delta)}(X/S, G)_{m(\delta), \mathfrak{v}(\delta)}. \end{array}$$

We know that the geometric degree of instability of E is equal to $m = m(\delta)$. The geometric dimension of the canonical parabolic associated to E is also constant, say equal to r. Then from the proof of Theorem 7.2.5 we know that $\pi_{T*} \mathrm{Par}({}^E G_T)_{r,m}$ is finite radicial and surjective over T. Now it is easy to see that

$$\pi_{T*} \mathrm{Par}({}^E G_T)_{r,m} = \left(\pi_* \mathrm{Par}_{t(P_T)}({}^E G_T)\right)_{\mathfrak{v}(\delta)}.$$

This shows that our morphism is representable and finite radicial and surjective. In view of Note 7.2.6 this finishes the proof. □

Remark 7.3.13 If Conjecture 6.4.7 is true, the morphism of Proposition 7.3.12 can be shown to be an *isomorphism.*

8 The Trace Formula for the Stack of G-Bundles

Introduction

In this section we search for open substacks of $\mathfrak{H}^1(X/k, G)$ that are of finite type. Our main result is, that $\mathfrak{H}^1_d(X/k, G)_{\leq m}$, for $d \in \mathsf{X}(G)^\vee$ and $m \geq 0$ is of finite type (see Theorem 8.2.6).

In section 8.3 we prove that $\mathfrak{H}^1_d(X/k, G)_{\leq m}$ is the quotient of a Deligne-Mumford stack by an affine algebraic group. We do this by studying level-D-structures, where D is an effective divisor on X. For a G-torsor E, a level-D-structure s is just a section of E over D: $s \in E(D)$. The stack of G-torsors with level-D-structure is denoted by $\mathfrak{H}^1(X/k, G; D)$ and the natural morphism

$$\mathfrak{H}^1(X/k, G; D) \longrightarrow \mathfrak{H}^1(X/k, G)$$

given by forgetting the level-D-structure is a principal fiber bundle with structure group $\rho_*(G_D)$, where $\rho : D \to \operatorname{Spec} k$ is the structure morphism. (See Corollary 8.3.8.) If we choose D large enough, $\mathfrak{H}^1_d(X/k, G; D)_{\leq m}$ is a Deligne-Mumford stack. So we may apply the results of Section 3 to $\mathfrak{H}^1_d(X/k, G)_{\leq m}$ and get the trace formula Corollary 8.3.13.

Finally, in Section 8.4 we prove the trace formula for $\mathfrak{H}^1_d(X/\mathbb{F}_q, G)$, where we have to make an assumption on the group G. We call this assumption (§) (see Definition 8.4.7). This assumption amounts to the truth of Conjecture 6.4.7 for G and all its twists ${}^E G$, where E is a G-torsor. Since $\mathfrak{H}^1_d(X/\mathbb{F}_q, G)$ is in general not of finite type, both sides of the trace formula are infinite sums.

8.1 The Case of Vector Groups

Proposition 8.1.1 *Let $\pi : X \to S$ be a curve of genus g over the locally noetherian scheme S. Let V be a vector group over X, of constant degree d and rank r. Assume that $R^1\pi_*V$ is a vector group and commutes with base change. (The same is then automatically true for π_*V.) Then the natural morphism*

$$\alpha : \mathfrak{H}^1(X/S, V) \longrightarrow R^1\pi_*V$$

makes $\mathfrak{H}^1(X/S, V)$ a gerbe over $R^1\pi_*V$. If T is an affine $R^1\pi_*V$-scheme, then the T-gerbe $\mathfrak{H}^1(X/S, V)_T$ is trivial, isomorphic to $(B\pi_*V)_T$. In particular, $\mathfrak{H}^1(X/S, V)$ is an algebraic S-stack of finite type and smooth of relative dimension $r(g-1) - d$ over S.

Clear.

Proof. First we will define the morphism α. For any S-scheme T we have $\mathfrak{H}^1(X/S, V)(T) = \Delta(X_T, V_T)$ and $R^1\pi_*V(T) = R^1\pi_{T*}V_T(T)$. We also have natural maps $\Delta(X_T, V_T) \to H^1(X_T, V_T)$ and $H^1(X_T, V_T) \to R^1\pi_{T*}V_T(T)$. The map $\alpha(T)$ is taken to be the composition.

Claim: The morphism α makes $\mathfrak{H}^1(X/S, V)$ a gerbe over $R^1\pi_*V$.

To prove the claim, let first T be an S-scheme and let $E, F \in \operatorname{ob}\Delta(X_T, V_T)$ such that $\alpha(E) = \alpha(F)$ in $R^1\pi_{T*}V_T(T)$. We have to prove that E and F are locally isomorphic (with respect to the flat topology on T). We may assume that T is affine. Then $R^1\pi_{T*}V_T(T) = H^1(X_T, V_T)$ and $\alpha(E) = \alpha(F)$ implies that E and F are isomorphic.

Secondly, let T be an S-scheme and $\xi \in R^1\pi_{T*}V_T(T)$. We have to prove that ξ may locally be lifted. So again we may assume hat T is affine and hence that $\xi \in H^1(X_T, V_T)$. Taking a V_T-torsor E representing the cohomology class ξ we get an object $E \in \operatorname{ob}\Delta(X_T, V_T)$ mapping to ξ under α. This proves the claim.

Now let T be an affine scheme with a morphism $t : T \to R^1\pi_*V$. Considering T as an S-scheme we have $R^1\pi_*V(T) = H^1(X_T, V_T)$, so that t induces a cohomology class $\xi \in H^1(X_T, V_T)$. Choosing a V_T-torsor E representing ξ we get an element $E \in \Delta(X_T, V_T)$ which gives rise to a morphism $\tilde{t} : T \to \mathfrak{H}^1(X/S, V)$. Clearly, we have $\alpha(\tilde{t}) = t$. So $\mathfrak{H}^1(X/S, V)$ is trivial over the $R^1\pi_*V$-scheme T. Now we have $\pi_{T*}\underline{\mathrm{Aut}}(E) = (\pi_*V)_T$ so that the T-gerbe $\mathfrak{H}^1(X/S, V)_T$ is isomorphic to $(B\pi_*V)_T$.

The last part of the proposition is local in S. So we may assume that S is affine. Then $R^1\pi_*V$ is an affine scheme. Hence by what we just proved, we have $\mathfrak{H}^1(X/S, V) \cong B\pi_*V \times_S R^1\pi_*V$. Another way of writing this is

$$\mathfrak{H}^1(X/S, V) \cong [R^1\pi_*V/R^0\pi_*V], \tag{26}$$

where $R^0\pi_*V$ acts on $R^1\pi_*V$ trivially. The rest of the proposition is then obvious, using Riemann-Roch to compute the dimension. □

Corollary 8.1.2 *Let X be a curve over the field k and V a vector group over X. Then we have*

$$\mathfrak{H}^1(X/k, V) = [H^1(X, V)/H^0(X, V)],$$

where $H^i(X, V)$ for $i = 0, 1$ is considered as a vector group over k, and $H^0(X, V)$ acts trivially on $H^1(X, V)$. In particular, $\mathfrak{H}^1(X/k, V)$ is a smooth algebraic k-stack of finite type and of dimension $\operatorname{rk} V(g-1) - \deg V$.

Proof. Since the base scheme is a field, the hypotheses of Proposition 8.1.1 are satisfied. Note in particular formula (26). □

Corollary 8.1.3 *Let X be a curve over the locally noetheraian scheme S and V a vector group on X. Then $\mathfrak{H}^1(X/S,V)$ is a smooth algebraic S-stack of finite type and of relative dimension* $\operatorname{rk} V(g-1) - \deg V$.

Proof. Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module such that $V \cong \mathbb{V}(\mathcal{E}^\vee)$. Then we consider the homomorphism of group schemes over X

$$\begin{array}{ccc} V & \longrightarrow & GL(\mathcal{E} \oplus \mathcal{O}_X) \\ v & \longmapsto & 1+v \end{array}$$

where 1 is the identity on $\mathcal{E} \oplus \mathcal{O}_X$ and v acts on $\mathcal{E} \oplus \mathcal{O}_X$ via $v(e,\alpha) = (\alpha v, 0)$. In this way, V becomes a closed subgroup of $GL(\mathcal{E} \oplus \mathcal{O}_X)$ and by Proposition 4.4.5 we have that $\mathfrak{H}^1(X/S,V)$ is an algebraic S-stack, locally of finite presentation. The smoothness of $\mathfrak{H}^1(X/S,V)$ follows from Proposition 4.5.1. The calculation of the dimension can now be done fiberwise, i.e. we may assume that $X = \operatorname{Spec} k$ is a field and apply Corollary 8.1.2. □

Applications to Parabolic Subgroups

Lemma 8.1.4 *Let X be a curve over the field k and*

$$0 \longrightarrow V \longrightarrow G \longrightarrow H \longrightarrow 1$$

a short exact sequence of group schemes on X, where V is a vector group. Then the natural morphism $\mathfrak{H}^1(X/k,G) \to \mathfrak{H}^1(X/k,H)$ is a smooth epimorphism of finite type and relative dimension $\operatorname{rk} V(g-1) - \deg V$.

Proof. Since any pullback and any twist of a vector group is again a vector group and $R^2\pi_*$ applied to a vector group is always zero, Proposition 4.2.5 implies that $\mathfrak{H}^1(X/k,G) \to \mathfrak{H}^1(X/k,H)$ is an epimorphism. Then the other properties are local on the base and the second half of Proposition 4.2.4 implies the result, making use of Corollary 8.1.3. □

Corollary 8.1.5 *Let X be a curve over the field k and V a group scheme over X having a filtration $V = V_0 \supset \ldots \supset V_n = 0$ such that all factors V_i/V_{i+1} for $i = 0,\ldots,n-1$ are vector groups. Then $\mathfrak{H}^1(X/k,V)$ is a smooth algebraic k-stack of finite type and dimension* $\dim_X V(g-1) - \deg(V)$.

Proof. By induction on n. Let $i < n$ be an index such that $V_{i+1} = 0$. Then V_i is a vector group. We consider the short exact sequence of groups on X

$$0 \longrightarrow V_i \longrightarrow V \longrightarrow V/V_i \longrightarrow 0.$$

By Lemma 8.1.4 $\mathfrak{H}^1(X/k,V) \to \mathfrak{H}^1(X/k,V/V_i)$ is smooth of finite type and relative dimension $\operatorname{rk} V_i(g-1) - \deg V_i$. By induction hypothesis we have $\mathfrak{H}^1(X/k,V/V_i)$ smooth of finite type and relative dimension $\dim_X V/V_i(g-1) - \deg V/V_i$. The corollary follows. □

Corollary 8.1.6 *Let X be a curve over the field k and*

$$0 \longrightarrow V \longrightarrow G \longrightarrow H \longrightarrow 1$$

a short exact sequence of group schemes on X, where V a group scheme over X having a filtration $V = V_0 \supset \ldots \supset V_n = 0$ such that all factors V_i/V_{i+1} for $i = 0,\ldots,n-1$ are vector groups. Then the natural morphism $\mathfrak{H}^1(X/k,G) \to \mathfrak{H}^1(X/k,H)$ is a smooth epimorphism of finite type and relative dimension $\dim_X V(g-1) - \deg^E V$, where E is the universal G-torsor.

Proof. Use Propositions 4.2.4 and 4.2.5. Note also, that the relative dimension of this morphism is not necessarily constant. $\square$

Proposition 8.1.7 *Let G be a reductive group scheme over the curve X over the field k. Let $P \subset G$ be a parabolic subgroup and let $H = P/R_u(P)$. Then the natural morphism of k-stacks $\mathfrak{H}^1(X/k, P) \to \mathfrak{H}^1(X/k, H)$ is a smooth epimorphism of finite type and of relative dimension $\dim_X(R_u(P))(g-1) - \deg({}^E P)$, where E is the universal G-torsor.*

Proof. By [8, Exp. XXVI, Prop. 2.1] $R_u(P)$ has a filtration such that the quotients are vector groups. So we may apply Corollary 8.1.6. $\square$

Calculating the Dimension of the Stack of G-Bundles

Lemma 8.1.8 *Let $\mathfrak{X}$ be a smooth algebraic k-stack of dimension n. Then $\mathbb{T}\mathfrak{X} \to \mathfrak{X}$ is smooth of relative dimension n.*

Proof. $\square$

Corollary 8.1.9 *Let G be a reductive algebraic group scheme over the curve X over the field k. Then $\mathfrak{H}^1(X/k, G)$ is a smooth algebraic k-stack of dimension $\dim_X G(g-1)$.*

Proof. Note that by Corollary 4.5.2 all we have to check is that the formula for the dimension is correct. Consider the short exact sequence of group schemes on X:

$$0 \longrightarrow \mathfrak{g} \longrightarrow \mathbb{T}G \longrightarrow G \longrightarrow 1.$$

Then Lemma 8.1.4 implies that $\mathfrak{H}^1(X/k, \mathbb{T}G) \to \mathfrak{H}^1(X/k, G)$ is smooth of relative dimension $\operatorname{rk}\mathfrak{g}(g-1)$. (Note that $\deg\mathfrak{g} = 0$ by Note 6.1.2.) But directly from the definitions we have $\mathfrak{H}^1(X/k, \mathbb{T}G) = \mathbb{T}\mathfrak{H}^1(X/k, G)$. So the corollary follows from Lemma 8.1.8. $\square$

8.2 Algebraic Stacks of Finite Type

Quasi Compact Stacks

Lemma 8.2.1 *Let $f : X \to Y$ be a continuous map of topological spaces X, Y. If X is quasi compact and f is surjective then Y is quasi-compact.*

Proof. Clear. $\square$

Lemma 8.2.2 *Let $\mathfrak{X}$ be an algebraic S-stack, where S is an affine scheme. Then $\mathfrak{X}$ is quasi compact if and only if $|\mathfrak{X}|$, the set of points of $\mathfrak{X}$ with the Zariski topology, is a quasi compact topological space.*

Proof. If $\mathfrak{X}$ is quasi compact there exists a presentation $p : X \to \mathfrak{X}$ such that X is quasi compact over S. The morphism p induces a continuous and surjective map $p : |X| \to |\mathfrak{X}|$ of the corresponding Zariski topologies. Since S is affine $|X|$ is quasi compact and by lemma 8.2.1 $|\mathfrak{X}|$ is quasi compact.

Conversely, assume that $|\mathfrak{X}|$ is quasi compact. Choose a presentation $p : X \to \mathfrak{X}$ of $\mathfrak{X}$. Let $(V_i)_{i \in I}$ be a family parametrizing all affine open subsets of X. For every $i \in I$ let $\mathfrak{U}_i = p(V_i)$, which is an open substack of $\mathfrak{X}$, and let U_i be the open subset of $|\mathfrak{X}|$ corresponding to $\mathfrak{U}_i$. Then $(U_i)_{i \in I}$ is a covering, so that there exist $1, \ldots, n \in I$ such that $U_1, \ldots, U_n$ cover $|\mathfrak{X}|$. Now let $X' = \bigcup_{i=1}^n V_i$ and let $p' = p|X'$. Then $p' : X' \to \mathfrak{X}$ is a presentation of $\mathfrak{X}$ and X' is quasi compact, as a finite union of affine schemes. $\square$

Corollary 8.2.3 *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of S-stacks where S is quasi compact. If $\mathfrak{X}$ is quasi compact and f is surjective, then $\mathfrak{Y}$ is quasi compact.*

Proof. Since S is quasi compact, the question of $\mathfrak{Y}$ being quasi compact is local in S, so we may assume that S is affine. Then we consider the induced map $f : |\mathfrak{X}| \to |\mathfrak{Y}|$ of the associated sets of points with the respective Zariski topologies. The corollary then follows immediatly from Lemmas 8.2.1 and 8.2.2. □

Passing to Galois Covers

Let X be a curve over the field k. Let $\pi : Y \to X$ be a finite Galois extension of X. This means that Y is another curve over k, and π is a finite morphism of curves making $K(Y)$ a Galois extension of $K(X)$. Let $\Gamma = \mathrm{Gal}(Y/X)$.

Let G be a reductive group scheme over X, with character group $\mathsf{X}(G)$. Then $\mathsf{X}(\pi^* G)$ is a Γ-module and $\mathsf{X}(G) = \mathsf{X}(\pi^* G)^\Gamma$. We have the trace map $\mathrm{tr}_{Y/X} : \mathsf{X}(\pi^* G) \to \mathsf{X}(G)$ defined by $\mathrm{tr}_{Y/X}(\chi) = \sum_{\sigma \in \Gamma} \sigma\chi$. Taking the dual, we get $\mathrm{tr}^\vee : \mathsf{X}(G)^\vee \to \mathsf{X}(\pi^* G)^\vee$. With these notations, we have the following proposition.

Proposition 8.2.4 *Let G be a reductive group scheme over the curve X over the field k. Let $\pi : Y \to X$ be a Galois cover of X. Then we have for every $d \in \mathsf{X}(G)^\vee$ a natural morphism of algebraic k-stacks*

$$\mathfrak{H}^1_d(X/k, G) \longrightarrow \mathfrak{H}^1_{\mathrm{tr}^\vee(d)}(Y/k, \pi^* G),$$

induced by pulling back via π^*. This morphism is affine of finite type.

Proof. Let $n = \#\Gamma$. Let $d \in \mathsf{X}(G)^\vee$. If $\tilde{d}$ is a Γ-invariant extension of $nd \in \mathsf{X}(G)^\vee$ to $\mathsf{X}(\pi^* G)$, then for any $\chi \in \mathsf{X}(\pi^* G)$ we have

$$\begin{aligned} n\tilde{d} &= \sum_{\sigma \in \Gamma} \tilde{d}(\chi) \\ &= \sum_{\sigma \in \Gamma} \tilde{d}(\sigma\chi) \\ &= \tilde{d}\left(\sum_{\sigma \in \Gamma} \sigma\chi\right) \\ &= nd\left(\sum_{\sigma \in \Gamma} \sigma\chi\right) \end{aligned}$$

and hence $\tilde{d} = \mathrm{tr}^\vee(d)$. In particular, $\tilde{d}$ is uniquely determined by d. If E is a G-torsor of degree $\deg E$ we have for $\chi \in \mathsf{X}(G)$:

$$\begin{aligned} \deg \pi^* E(\pi^* \chi) &= \deg(\pi^* E \times_{\pi^* G, \pi^* \chi} \mathcal{O}_Y) \\ &= \deg(\pi^*(E \times_{G,\chi} \mathcal{O}_X)) \\ &= n \deg(E \times_{G,\chi} \mathcal{O}_X) \\ &= n \deg E(\chi), \end{aligned}$$

so that $\deg \pi^* E$ extends $n \deg E$. We also have for $\sigma \in \Gamma$ and $\chi \in \mathsf{X}(\pi^* G)$:

$$\begin{aligned} \deg \pi^* E(\sigma\chi) &= \deg \pi^* E(\sigma^* \chi) \\ &= \deg \sigma^* \pi^* E(\sigma^* \chi) \\ &= \deg \pi^* E(\chi), \end{aligned}$$

so that $\deg \pi^* E$ is Γ-invariant. Hence $\deg \pi^* E = \operatorname{tr}^\vee(\deg E)$, by the above remarks. This proves that we get a natural morphism

$$\mathfrak{H}^1_d(X/k, G) \longrightarrow \mathfrak{H}^1_{\operatorname{tr}^\vee(d)}(Y/k, \pi^* G).$$

It is affine by Theorem 4.4.3. □

Borel Subgroups

Proposition 8.2.5 *Let G be a rationally trivial reductive group scheme over the curve X over the field k. Assume that X has a k-valued point. Let $B \subset G$ be a Borel subgroup and let $\delta \in \mathsf{X}(B)^\vee$. Then $\mathfrak{H}^1_\delta(X/k, B)$ is an algebraic k-stack of finite type.*

Proof. The quotient group $T = B/R_u(B)$ is a torus over X. The torus T is split, since G is rationally trivial. By Proposition 8.1.7 we only need to prove that $\mathfrak{H}^1_d(X/k, T)$ is of finite type, where d is the image of δ under the natural map $\mathsf{X}(B)^\vee \to \mathsf{X}(T)^\vee$. Write T as $T = {\mathbb{G}_{\mathrm{m}}}^n$. Then $\mathsf{X}(T)^\vee = \mathbb{Z}^n$. Let $d = (d_1, \ldots, d_n)$. Then we have

$$\mathfrak{H}^1_d(X/k, {\mathbb{G}_{\mathrm{m}}}^n) = \mathfrak{H}^1_{d_1}(X/k, \mathbb{G}_{\mathrm{m}}) \times \ldots \times \mathfrak{H}^1_{d_n}(X/k, \mathbb{G}_{\mathrm{m}}).$$

Thus we are reduced to the case $T = \mathbb{G}_{\mathrm{m}}$. By Remark 7.3.7 a line bundle L of degree d induces an isomorphism $\mathfrak{H}^1_d(X/k, \mathbb{G}_{\mathrm{m}}) \cong \mathfrak{H}^1_0(X/k, \mathbb{G}_{\mathrm{m}})$ since any inner form of $\mathbb{G}_{\mathrm{m}}$ is equal fo $\mathbb{G}_{\mathrm{m}}$. Now choosing a k-valued point P of X we have $\mathfrak{H}^1_0(X/k, \mathbb{G}_{\mathrm{m}}(P)) = \operatorname{Pic}^0(X) = \operatorname{Jac} X$. Hence we have $\mathfrak{H}^1_0(X/k, \mathbb{G}_{\mathrm{m}}) = [\operatorname{Jac} X/\mathbb{G}_{\mathrm{m}}]$, where $\mathbb{G}_{\mathrm{m}}$ acts trivially on $\operatorname{Jac} X$. □

The Theorem

Theorem 8.2.6 *Let G be a reductive group scheme over the curve X over the field k. Let $m \geq 0$ be an integer and $d : \mathsf{X}(G) \to \mathbb{Z}$ a homomorphism. Then $\mathfrak{H}^1_d(X/k, G)_{\leq m}$ is a smooth algebraic k-stack of finite type.*

Proof. By Corollary 4.5.2 all we have to prove is that $\mathfrak{H}^1_d(X/k, G)_{\leq m}$ is of finite type. If $\pi : Y \to X$ is a Galois covering of X, then by Proposition 8.2.4 and Corollary 6.4.6 we have a finite type morphism

$$\mathfrak{H}^1_d(X/k, G)_{\leq m} \longrightarrow \mathfrak{H}^1_{\operatorname{tr}^\vee(d)}(Y/k, \pi^* G)_{\leq m}.$$

So for the purpose of proving our theorem we may pass to a Galois cover of X an assume that G is rationally trivial and that X admits k-valued points. By Corollary 8.2.3 we need only find an algebraic k-stack of finite type $\mathfrak{Y}$, together with a surjective morphism

$$\mathfrak{Y} \longrightarrow \mathfrak{H}^1_d(X/k, G)_{\leq m}.$$

Choose a Borel subgroup $B \subset G$. Let $\chi_1, \ldots, \chi_s$ be the characters of B given by the action of B on its associated elementary line bundles. Consider the natural homomorphisms $d : \mathsf{X}(B)^\vee \to \mathsf{X}(G)^\vee$ and $m : \mathsf{X}(B)^\vee \to \mathbb{Z}$ (see pages 75 and following). Consider finally the morphism

$$\coprod_\delta \mathfrak{H}^1_\delta(X/k, B) \longrightarrow \mathfrak{H}^1_d(X/k, G)_{\leq m}, \tag{27}$$

where the disjoint sum is taken over all $\delta \in \mathsf{X}(B)^\vee$ satisfying the following three conditions:

i. $d(\delta) = d$

ii. $m(\delta) \leq m$

iii. $\delta(\chi_i) \geq -2g$ for $i = 1, \ldots, s$.

By Note 7.3.9 this morphism indeed exists. Also, if E is a B-torsor of degree δ, then $\delta(\chi_1), \ldots, \delta(\chi_s)$ are the numerical invariants of the Borel subgroup $E \times_{B,Ad} B$ of $E \times_{B,Ad} G$.

Claim: The number of $\delta \in \mathsf{X}(B)^\vee$ satisfying the above three conditions is finite.

By Proposition 7.3.11 all we need to do is check that

$$\{(n_1, \ldots, n_s) \in \mathbb{Z}^s \mid n_i \geq -2g \text{ for } i = 1, \ldots, s \text{ and } \textstyle\sum_{i=1}^s n_i y_i \leq m\}$$

is finite. Here $y_1, \ldots, y_s \in \mathbb{Q}_{>0}$ are given as in Lemma 7.3.8. But this is clear.

Hence by Proposition 8.2.5 the stack on the left hand side of (27) is of finite type. So it only remains to prove the surjectivity of the morphism in (27). Letting $E : \operatorname{Spec} K \to \mathfrak{H}^1_d(X/k, G)_{\leq m}$ represent a point of $\mathfrak{H}^1_d(X/k, G)_{\leq m}$ we may replace k by K and thus assume that E is a G-torsor over X of degree d such that $\deg_i({}^E G) \leq m$. By [15, Satz 2.2.6] there exists a Borel subgroup $B' \subset {}^E G$ such that the numerical invariants $n_1(B'), \ldots, n_s(B')$ satisfy $n_i(B') \geq -2g$ for $i = 1, \ldots, s$. This Borel subgroup B' will define a reduction of structure group E' of E to B, i.e. E' is a B-torsor such that $E' \times_B G = E$ and $E' \times_{B,Ad} B = B'$. Let $\delta = \deg E'$ be the degree of E'. Then by Note 7.3.9 we have $d(\delta) = \deg E = d$, $m(\delta) = \deg(B') \leq \deg_i({}^E G) \leq m$ and $\delta(\chi_i) = n_i(B') \geq -2g$ for $i = 1, \ldots, s$. Thus E' defines a point of the stack on the left hand side of (27) mapping to E. This proves the surjectivity of the morphism in (27) and thus the theorem. □

8.3 Level Structures

Let X be a scheme and $\iota : D \to X$ a scheme over X. Let G/X be a group scheme.

Definition 8.3.1 We define the *groupoid of G-torsors with level-D-structure*, denoted $\Delta(X, G; D)$, to be the category whose objects are pairs (E, s), where E is a (right) G-torsor (over X) and s is a section of $E_D = \iota^* E$. The set of morphisms from (E, s) to (E', s'), for $(E, s), (E', s') \in \operatorname{ob} \Delta(X, G; D)$, is defined to be the set of all isomorphisms of G-torsors $\phi : E \to E'$ such that $\phi_D(s) = s'$.

Note 8.3.2 Clearly, we have a morphism of groupoids $f : \Delta(X, G; D) \to \Delta(X, G)$, given by forgetting the level-D-structure. We also have a natural action σ of $\Gamma(D, G)$ on the right of $\Delta(X, G; D)$ making the following diagram a 2-cartesian and 2-cocartesian diagram of groupoids:

$$\begin{array}{ccc} \Delta(X, G; D) \times \Gamma(D, G) & \xrightarrow{\sigma} & \Delta(X, G; D) \\ {\scriptstyle p_1} \downarrow & \boxed{2} & \downarrow {\scriptstyle f} \\ \Delta(X, G; D) & \xrightarrow{f} & \Delta(X, G). \end{array}$$

Proof. Straightforward checking. □

Note 8.3.3 If E is a G-torsor then we have a 2-cartesian diagram of groupoids

$$\begin{array}{ccc} \Gamma(D, E) & \longrightarrow & \{\varnothing\} \\ {\scriptstyle \tau} \downarrow & \boxed{2} & \downarrow {\scriptstyle E} \\ \Delta(X, G; D) & \xrightarrow{f} & \Delta(X, G) \end{array}$$

where τ is given by $\tau(s) = (E, s)$ for every $s \in \Gamma(D, E)$.

Proof. Clear. □

Let X be a scheme over the base scheme S. Let $\iota : D \to X$ be an X-scheme and $\rho : D \to S$ the structure morphism. Let G be a group scheme over X.

Definition 8.3.4 We define the *S-stack of G-torsors with level-D-structure*, denoted $\mathfrak{H}^1(X/S, G; D)$, by setting

$$\mathfrak{H}^1(X/S, G; D)(U) = \Delta(X_U, G_U; D_U)$$

for every S-scheme U.

Note 8.3.5 We have a natural morphism of S-stacks

$$f : \mathfrak{H}^1(X/S, G; D) \longrightarrow \mathfrak{H}^1(X/S, G)$$

and a natural action σ of the sheaf of groups $\rho_* \iota^* G$ on $\mathfrak{H}^1(X/S, G; D)$ yielding a 2-cartesian diagram of S-stacks

$$\begin{array}{ccc} \mathfrak{H}^1(X/S, G; D) \times \rho_* \iota^* G & \xrightarrow{\sigma} & \mathfrak{H}^1(X/S, G; D) \\ {\scriptstyle p_1} \downarrow & \boxed{2} & \downarrow {\scriptstyle f} \\ \mathfrak{H}^1(X/S, G; D) & \xrightarrow{f} & \mathfrak{H}^1(X/S, G). \end{array}$$

The question of whether this diagram is 2-cocartesian (i.e. whether f is an epimorphism) is a little more subtle.

Proof. Follows immediatly form Note 8.3.2. □

Lemma 8.3.6 *Let $\pi : Y \to X$ be a finite flat morphism of schemes. Let $U \to Y$ be a faithfully smooth (étale) Y-scheme of finite presentation. Then $\pi_* U$ is a faithfully smooth (étale) X-scheme, locally of finite presentation.*

Proof. This follows easily using formal smoothness and the formal properties of the functor π_*. □

Note 8.3.7 Let U be an S-scheme and E a G_U-torsor. Then we have a 2-cartesian diagram of S-stacks

$$\begin{array}{ccc} \rho_{U*} \iota_U^* E & \longrightarrow & U \\ {\scriptstyle \tau} \downarrow & \boxed{2} & \downarrow {\scriptstyle E} \\ \mathfrak{H}^1(X/S, G; D) & \xrightarrow{f} & \mathfrak{H}^1(X/S, G) \end{array}$$

where τ is given by $\tau(s) = (E_U, s)$ for any $s \in E(D_U)$. So if G is of finite presentation over X and D is proper and flat over S, then f is representable and locally of finite presentation. If, in addition, G is smooth over X and D is finite over S then f is in addition smooth and surjective. So in this case f is an epimorphism and the diagram of Note 8.3.5 is 2-cocartesian.

Proof. The fact that the diagram is 2-cartesian follows immediatly from Note 8.3.3. If G is of finite presentation over X, then E is an algebraic space of finite presentation over X_U. Hence the same is true for $\iota_U^* E$ over D_U. Now $\rho_U : D_U \to U$ is proper and flat, so by [1, 6.] $\rho_{U*} \iota_U^* E$ is an algebraic space, locally of finite presentation (over U). This proves that f is representable, locally of finite presentation. Now if G is smooth over X, then $\iota_U^* E$ is smooth over D_U. Hence the claim follows form Lemma 8.3.6. □

Corollary 8.3.8 *If G is a smooth group scheme over X and D is finite and flat over S, then $\rho_* \iota^* G$ is a smooth group scheme over S and*

$$f : \mathfrak{H}^1(X/S, G; D) \longrightarrow \mathfrak{H}^1(X/S, G)$$

is a principal homogeneous $\rho_* \iota^* G$-bundle.

Proof. Immediate from Notes 8.3.5 and 8.3.7. □

Applications to the Stack of G-Bundles

Lemma 8.3.9 *Let k be field and X a curve over k. Let $\mathfrak{H}$ be a k-stack of finite type. Let $\mathfrak{g}$ be a locally free coherent sheaf on $X \times \mathfrak{H}$. Then there exists a finite set of closed points $D = \{x_1, \ldots, x_n\} \subset X$ such that $\pi_* \mathfrak{g}(-D) = 0$, where $\pi : X \times \mathfrak{H} \to \mathfrak{H}$ is the structure morphism.*

Proof. We will use noetherian induction. If $\mathfrak{H} = \varnothing$ there is nothing to prove. Otherwise choose a point ξ of $\mathfrak{H}$ and let $\xi : \operatorname{Spec} K \to \mathfrak{H}$ be a represetative of ξ. Let $\mathfrak{g}_K$ be the pullback of $\mathfrak{g}$ to X_K. Then we choose finitely many closed points $D_1 = \{x_1, \ldots, x_n\}$ of X such that the vector bundle $\mathfrak{g}(-D_1)_K$ on the curve X_K over the field K has no globel sections. By cohomology and base change ([17, Chap. III, Theorem 12.11]), there exists an open substack $\mathfrak{U} \subset \mathfrak{H}$, containing ξ, such that $\pi_* \mathfrak{g}(-D)|\mathfrak{U} = 0$. Let $\mathfrak{H}' = \mathfrak{H} - \mathfrak{U}$ with the reduced substack structure. Then $\mathfrak{H}'$ is a proper closed substack of $\mathfrak{H}$. So by the induction hypothesis, there exists $D_2 = \{y_1, \ldots, y_m\} \subset X$ such that $\pi_* \mathfrak{g}_{\mathfrak{H}'}(-D_2) = 0$. Let $D = D_1 \cup D_2$. Then, again by cohomology and base change, we have $\pi_* \mathfrak{g}(-D) = 0$ as required. $\square$

We will now calculate the Zariski tangent space to the scheme of automorphisms of a G-torsor with level-D-structure (E, s). We will need the following lemma.

Lemma 8.3.10 *Let S be a locally noetherian scheme and $\pi : X \to S$ a projective flat S-scheme. Let G be an affine smooth group scheme over X. Let $\iota : D \to X$ be a closed subscheme of X that is finite and flat over S and $\rho = \pi \circ \iota$. Let E be a G-torsor and $s : D \to E$ a section of E over D. Then we have a short exact sequence of group schemes of finite type over S:*

$$1 \longrightarrow \pi_* \underline{\mathrm{Aut}}(E, s) \longrightarrow \pi_*({}^E G) \xrightarrow{\phi_s} \rho_* \iota^* G. \tag{28}$$

Proof. First note that ${}^E G$ is an affine X-scheme, so that $\pi_*({}^E G)$ is a group scheme of finite over S by Proposition 4.4.1. For the same reason, $\rho_* \iota^* G$ is a finite type group scheme over S. We will now construct the homomorphism

$$\phi_s : \pi_*({}^E G) \longrightarrow \rho_* \iota^* G.$$

Let $g \in \pi_*({}^E G)(S)$. We may consider g as an automorphism $g : E \to E$ of the G-torsor E. Then let $\phi_s(g) \in \rho_* \iota^* G(S) = G(D)$ be the unique element such that $s\phi_s(g) = g(s)$. Since $s\phi_s(gh) = gh(s) = g(s\phi_s(h)) = g(s)\phi_s(h) = s\phi_s(g)\phi_s(h)$ we see that we have defined a homomorphism of group schemes. Directly from the definition we have

$$\begin{aligned} \phi_s(g) = 1 \quad &\iff \quad g(s) = s \\ &\iff \quad g \in \mathrm{Aut}(E, s). \end{aligned}$$

This proves the exactness of the above sequence. Hence $\pi_* \underline{\mathrm{Aut}}(E, s)$ is a closed subgroup scheme of $\pi_*({}^E G)$, in particular of finite type over S. $\square$

Under the same hypotheses as in the Lemma, let us now assume that $S = \operatorname{Spec} k$, where k is a field. Let $\phi \in \mathrm{Aut}(E, s)$ be an automorphism of (E, s). The element ϕ itself induces an isomorphism of the Zariski tangent space of $\pi_* \underline{\mathrm{Aut}}(E, s)$ at ϕ with the Zariski tangent space of $\underline{\mathrm{Aut}}(E, s)$ at 1. Denote these Zariski tangent spaces by $\mathbb{T}\pi_* \underline{\mathrm{Aut}}(E, s)(\phi)$ and $\mathbb{T}\pi_* \underline{\mathrm{Aut}}(E, s)(1)$, respectively. Now directly from (28) we have an exact sequence of k-vector spaces

$$0 \longrightarrow \mathbb{T}\pi_* \underline{\mathrm{Aut}}(E, s)(1) \longrightarrow H^0(X, {}^E\mathfrak{g}) \longrightarrow H^0(D, \mathfrak{g}).$$

So we have

$$\mathbb{T}\pi_* \underline{\mathrm{Aut}}(E, s)(\phi) = H^0(X, {}^E\mathfrak{g}(-D)). \tag{29}$$

Lemma 8.3.11 *Let G be a reductive group scheme over the curve X over the field k. Let $p : X \times \mathfrak{H}^1(X/k, G) \longrightarrow \mathfrak{H}^1(X/k, G)$ be the second projection. Let E be the universal G-torsor on $X \times \mathfrak{H}^1(X/k, G)$. Let $m \geq 0$ be an integer, and $d : \mathsf{X}(G) \to \mathbb{Z}$ a homomorphism. Let $D = \{x_1, \ldots, x_n\} \subset X$ be closed points such that $p_* {}^E\mathfrak{g}(-D)|\mathfrak{H}^1_d(X/k, G)_{\leq m} = 0$. (Such points exist by Lemma 8.3.9.) Then $\mathfrak{H}^1_d(X/k, G; D)_{\leq m}$ is a Deligne-Mumford stack over k.*

Proof. Let
$$(E, s) : \operatorname{Spec} K \longrightarrow \mathfrak{H}^1_d(X/k, G; D)_{\leq m}$$
represent a point of $\mathfrak{H}^1_d(X/k, G; D)_{\leq m}$. So E is a G_K-torsor on X_K of degree d and geometric degree of istability less than or equal to m, and s is a section of E over D_K. We need to show that the K-scheme $\pi_{K*} \underline{\mathrm{Aut}}(E, s)$, where $\pi : X \to \operatorname{Spec} k$ is the structure morphism, is unramified over $\operatorname{Spec} K$. Without loss of generality, we may assume that $K = k$. By (29) we have $\mathbb{T}\pi_* \underline{\mathrm{Aut}}(E, s)(\phi) = H^0(X, {}^E\mathfrak{g}(-D))$ which is equal to zero by assumption. □

Proposition 8.3.12 *Let G be a reductive group scheme over the curve X over the field k. Let $m \geq 0$ and $d \in \mathsf{X}(G)^\vee$ be given. Then there exists a Deligne-Mumford stack $\mathfrak{X}$ over k and a non-singular affine algebraic group Γ over k acting on $\mathfrak{X}$ such that*
$$\mathfrak{H}^1_d(X/k, G)_{\leq m} \cong [\mathfrak{X}/\Gamma].$$

Proof. Choose D as in Lemma 8.3.11 and $\mathfrak{X} = \mathfrak{H}^1_d(X/k, G; D)_{\leq m}$. Take $\Gamma = \rho_* \iota^* G$ where $\iota : D \to X$ is the inclusion and $\rho : D \to \operatorname{Spec} k$ the structure morphism. Then Γ is an affine algebraic k-group by Proposition 4.4.1 and non-singular by Lemma 8.3.6. By Corollary 8.3.8 we have
$$\mathfrak{H}^1_d(X/k, G)_{\leq m} \cong [\mathfrak{X}/\Gamma].$$
Done. □

Corollary 8.3.13 *Let G be a reductive group scheme over the curve X over the field k. Let $m \geq 0$ and $d \in \mathsf{X}(G)^\vee$ be given. Then $\mathfrak{H}^1_d(X/k, G)_{\leq m}$ is a smooth algebraic k-stack of finite type and dimension $\dim_X G(g-1)$ and if $k = \mathbb{F}_q$ is finite, the Lefschetz trace formula holds for this $\mathbb{F}_q$-stack:*
$$q^{\dim_X G(g-1)} \operatorname{tr} \Phi_q | H^*(\mathfrak{H}^1_d(\overline{X}/\overline{\mathbb{F}}_q, \overline{G})_{\leq m_{\mathrm{sm}}}, \mathbb{C}) = \sum_{E \in [\Delta_d(X,G)_{\leq m}]} \frac{1}{\# \operatorname{Aut}(E)} \tag{30}$$
where the sum is taken over all isomorphism classes of G-torsors E such that $\deg E = d$ and $\deg_i(E) \leq m$.

Proof. Combine Theorem 8.2.6 with the dimension calculation Corollary 8.1.9. Then apply Theorem 3.5.7, which is possible because of Proposition 8.3.12. Note also that $\mathbb{F}_q$ is perfect and Corollary 6.4.6. □

8.4 Passing to the Limit

Exhausting Topoi

Definition 8.4.1 Let E be a topos and X an object of E. A covering $\mathfrak{U} = (U_i)_{i \in \mathbb{N}}$ of X is called an *exhaustion of* X, if U_i is a subobject of X for all $i \in \mathbb{N}$ and $U_i \subset U_{i+1}$ for all $i \in \mathbb{N}$.

Lemma 8.4.2 *Let E be a topos, X an object of E and $\mathfrak{U} = (U_i)_{i \in \mathbb{N}}$ an exhaustion of X. Let M be a presheaf of abelian groups on E, such that $M(U_i)$ is finite for all $i \in \mathbb{N}$. Then $H^p(\mathfrak{U}, M) = 0$ for all $p > 0$.*

Proof. $\mathbb{N}$ is an ordered set, so we may consider $\mathbb{N}$ as a category. We endow $\mathbb{N}$ with the chaotic topology. Then $\widehat{\mathbb{N}}$, the category of presheaves (of sets) on $\mathbb{N}$, is the topos of sheaves on the site $\mathbb{N}$. The category $\mathrm{ab}\,\widehat{\mathbb{N}}$ of abelian sheaves on the topos $\widehat{\mathbb{N}}$ is the category of projective systems of abelian groups indexed by $\mathbb{N}$. For an abelian sheaf F on $\widehat{\mathbb{N}}$ we have the Cartan-Leray spectral sequence corresponding to the covering $\mathfrak{N} = (n)_{n\in\mathbb{N}}$ of the final object of $\widehat{\mathbb{N}}$:

$$H^p(\mathfrak{N}, \mathcal{H}^q(F)) \Longrightarrow H^{p+q}(\widehat{\mathbb{N}}, F). \tag{31}$$

For any $n \in \mathbb{N}$ we have $H^q(n, F) = 0$ for $q > 0$, because the functor $F \mapsto F(n)$ is exact, $\mathbb{N}$ having the chaotic topology. So since $\mathbb{N}$ is closed under direct products, the Čech complex $C^\bullet(\mathfrak{N}, {}^q(F))$ is zero if $q > 0$:

$$C^p(\mathfrak{N}, \mathcal{H}^q(F)) = \prod_{(n_1,\ldots,n_p)\in\mathbb{N}^p} H^q(\min(n_1, \ldots, n_p), F)$$

So from the spectral sequence (31) we get $H^p(\widehat{\mathbb{N}}, F) = H^p(\mathfrak{N}, F)$. Now the global section functor $\Gamma(\widehat{\mathbb{N}}, \cdot)$ is nothing but the inverse limit functor. So we have $H^p(\widehat{\mathbb{N}}, F) = \varprojlim_{\mathbb{N}}{}^p F$, and hence $\varprojlim_{\mathbb{N}}{}^p F = H^p(\mathfrak{N}, F)$. Since $\widetilde{M} = (M(U_i))_{i\in\mathbb{N}}$ is a projective system of abelian groups, we can apply this result and get:

$$\varprojlim_{\mathbb{N}}{}^p \widetilde{M} = H^p(\mathfrak{N}, \widetilde{M})$$

Obviously, $H^p(\mathfrak{N}, M) = H^p(\mathfrak{U}, M)$, by looking at the Čech complexes. So we have

$$H^p(\mathfrak{U}, M) = \varprojlim_{i}{}^p M(U_i).$$

Since our indexing category is $\mathbb{N}$, $\varprojlim^p$ always vanishes for $p \geq 2$.Since $M(U_i)$ is finite for all $i \in \mathbb{N}$, $\widetilde{M}$ satisfies the Mittag-Leffler condition, and hence $\varprojlim^1 \widetilde{M} = 0$. □

Proposition 8.4.3 *Let E be a topos that can be exhausted by noetherian open subtopoi that satisfy the finiteness theorem with respect to N. Then for any constructible abelian sheaf F on E with $NF = 0$ we have*

$$H^p(E, F) = \varprojlim_{i} H^p(U_i, F),$$

where $(U_i)_{i\in\mathbb{N}}$ is an exhaustion of E, such that, for all $i \in \mathbb{N}$, $E_{/U_i}$ is a noetherian topos satisfying the finiteness theorem with respect to N.

Proof. Let $\mathfrak{U} = (U_i)_{i\in\mathbb{N}}$ be an exhaustion of E, such that, for all $i \in \mathbb{N}$, $E_{/U_i}$ is a noetherian topos satisfying the finiteness theorem with respect to N. Let F be a constructible abelian sheaf on E with $NF = 0$. Then for any $i \in \mathbb{N}$ the sheaf $F|U_i$ is noetherian, and so $H^q(U_i, F)$ is finite. So we can apply Lemma 8.4.2 to the presheaf $\mathcal{H}^q(F)$ on E, obtaining $H^p(\mathfrak{U}, \mathcal{H}^q(F)) = 0$ for $p > 0$. So the Cartan-Leray spectral sequence of the covering $\mathfrak{U}$ gives us

$$H^q(E, F) = H^0(\mathfrak{U}, \mathcal{H}^q(F)).$$

But this is what we wanted to prove. □

Corollary 8.4.4 *Let k be a separably closed field and $\ell \neq \operatorname{char} k$. If $\mathfrak{X}$ is an algebraic k-stack, that can be exhausted by open substacks $(\mathfrak{X}_i)_{i\in\mathbb{N}}$ of finite type, then*

$$H^p(\mathfrak{X}_{\mathrm{sm}}, F) = \varprojlim_{i} H^p(\mathfrak{X}_{i\,\mathrm{sm}}, F)$$

for any constructible abelian sheaf F on $\mathfrak{X}$ with $\ell^n F = 0$ for some $n \geq 0$.

Proof. By Theorem 3.1.6 the topos $\mathfrak{X}_{i_{\mathrm{sm}}}$ satisfies the finiteness theorem with respect to ℓ^n for every $n > 0$. So the Corollary follows form Proposition 8.4.3. □

The Condition (§)

If X is a curve over an algebracically closed field k, then the following lemma shows that the index m in the notation $\mathfrak{H}^1_d(X,G)_{m,\mathfrak{v}}$ is redundant, so we may write $\mathfrak{H}^1_d(X,G)_{\mathfrak{v}}$ instead.

Lemma 8.4.5 *Let G be a reductive group scheme over the curve X over the algebraically closed field k. Let $\mathfrak{V}$ be the free abelian group on the connected components of $\operatorname{Dyn} G$. Then there exists a function $m : \mathfrak{V} \to \mathbb{Z}$ such that if E is a G-torsor of type of instability $\mathfrak{v} \in \mathfrak{V}$, then $m(\mathfrak{v}) = \deg_i({}^E G)$. If P is a parabolic subgroup of G, then we have a commutative diagram*

$$\begin{array}{ccc} \mathsf{X}(P)^\vee & \xrightarrow{\mathfrak{v}} & \mathfrak{V} \\ & m\searrow & \downarrow m \\ & & \mathbb{Z} \end{array}$$

where the horizontal map $\mathfrak{v}$ and the diagonal map m are defined as on page 75.

Proof. If there exists no G-torsor of type of instability $\mathfrak{v}$, then we may assign an arbitrary value to $m(\mathfrak{v})$. So let E and F be G-torsors of type of instability $\mathfrak{v}$. We need to show that $\deg_i({}^E G) = \deg_i({}^F G)$. Without loss of generality we may assume that G contains a parabolic subgroup P of type η, where

$$\eta = \coprod_{\substack{i \\ n_i \neq 0}} \mathfrak{v}_i$$

and $\mathfrak{v} = \sum_{i=1}^s n_i \mathfrak{v}_i$. Let E' and F' be the canonical reductions of E and F to P. Then $\mathfrak{v}(\deg E') = \mathfrak{v} = \mathfrak{v}(\deg F')$, so our lemma follows from the fact that m factors through $\mathfrak{V}(P)'$. □

Lemma 8.4.6 *Let G be a reductive group scheme over the curve X over the algebraically closed field k. Then there exists a function $r : \mathfrak{V} \to \mathbb{Z}$ such that if E is a G-torsor of type of instability $\mathfrak{v} \in \mathfrak{V}$, then $r(\mathfrak{v}) = \dim_X R_u(P)$, where P is the canonical parabolic subgroup of ${}^E G$.*

Proof. This follows from the fact that parabolic subgroups of the same type are forms of each other. □

Definition 8.4.7 Let G be a reductive group scheme over the curve X over the field k. We will say that G satisfies condition (§) if the following is satisfied. Whenever K is a field extension of k and E is a G_K-torsor, we have

$$H^0(X_K, {}^E\mathfrak{g}/\mathfrak{p}) = 0,$$

where $\mathfrak{p}$ denotes the Lie algebra of the canonical parabolic subgroup of ${}^E G_K$.

Note 8.4.8 If Conjecture 6.4.7 is true, then every reductive group scheme over a curve satisfies condition (§).

Proposition 8.4.9 *Let G be a reductive group scheme over the curve X over the algebraically closed field k. Assume that G satisties Condition (§). Let P be a parabolic subgroup of G and let $\delta \in \mathsf{X}(P)^\vee_+$. Let $H = P/R_u(P)$. Then we have a natural morphism of algebraic k-stacks*

$$\rho_{P,\delta} : \mathfrak{H}^1_{d(\delta)}(X/G)_{\mathfrak{v}(\delta)} \longrightarrow \mathfrak{H}^1_\delta(X/H)_0.$$

The morphism $\rho_{P,\delta}$ is a smooth epimorphism of fintie type and relative dimension $\dim_X R_u(P)(g-1) - m(\delta)$. It induces isomorphisms

$$H^i(\mathfrak{H}^1_{d(\delta)}(X/G)_{\mathfrak{v}(\delta)}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(\mathfrak{H}^1_\delta(X/H)_0, \mathbb{Q}_\ell),$$

for every $i \geq 0$.

Proof. By Proposition 7.3.12 we have a natural morphism

$$\mathfrak{H}^1_\delta(X,P)_0 \longrightarrow \mathfrak{H}^1_{d(\delta)}(X,G)_{\mathfrak{v}(\delta)}. \tag{32}$$

Using condition (§) this can be shown to be an isomorphism. Now composing the inverse of (32) with the morphism of Proposition 8.1.7 we get $\rho_{P,\delta}$. The condition (§) implies that we can apply Proposition 8.1.1 to the factors of the filtration of $R_u(P)$. This implies that $\rho_{P,\delta}$ induces isomorphisms on ℓ-adic cohomology. □

Lemma 8.4.10 *Let G be a reductive group scheme over the curve X of genus g over the algebraically closed field k. Assume that G satisfies Condition (§). Let $d \in \mathsf{X}(G)^\vee$ be fixed. Then for every $\mathfrak{v} \in \mathfrak{V}$ the natural closed immersion*

$$\mathfrak{H}^1_d(X,G)_{\mathfrak{v}} \longrightarrow \mathfrak{H}^1_d(X,G)_{\leq m(\mathfrak{v})}$$

is a smooth pair of algebraic k-stacks of codimension $c(\mathfrak{v}) = r(\mathfrak{v})(g-1)+m(\mathfrak{v})$. For the definition of $r(\mathfrak{v})$ and $m(\mathfrak{v})$ see Lemmas 8.4.5 and 8.4.6.

Proof. If $\mathfrak{H}^1_d(X,G)_{\mathfrak{v}} = \varnothing$ there is nothing to prove. Otherwise, there exists (maybe only after a change of origin) a parabolic subgroup P of G and an element $\delta \in \mathsf{X}(P)^\vee$ such that $d(\delta) = d$ and $\mathfrak{v}(\delta) = \mathfrak{v}$. Then, as we already noted in the proof of Proposition 8.4.9, we have $\mathfrak{H}^1_d(X,G)_{\mathfrak{v}} \cong \mathfrak{H}^1_\delta(X,P)_0$. This proves that $\mathfrak{H}^1_d(X,G)_{\mathfrak{v}}$ is smooth of dimension $\dim_X(P)(g-1) - m(\delta)$. So the codimention is

$$\begin{aligned} c(\mathfrak{v}) &= \dim_X G(g-1) - (\dim_X P(g-1) - m(\delta)) \\ &= \dim R_u(P)(g-1) + m(\delta), \end{aligned}$$

which finishes our proof. □

Definition 8.4.11 Let G be a reductive group scheme over the curve X of genus g over the algebraically closed field k. Let $\#\Phi$ be the number of roots of G. Then we define for every integer $i \geq 0$ the number $\gamma(i)$ to be the smallest integer satisfying

$$\gamma(i) \geq \begin{cases} 1 + \frac{i}{2} & \text{if } g > 0 \\ 1 + \frac{i}{2} + \frac{\#\Phi}{2} & \text{if } g = 0. \end{cases}$$

Proposition 8.4.12 *Let G be a reductive group scheme over the curve X of genus g over the algebraically closed field k. Assume that G satisfies Condition (§). Let $i \geq 0$ be an integer. Then if $m \geq \gamma(i)$ the canonical homomorphism*

$$H^i(\mathfrak{H}^1_d(X,G), \mathbb{Q}_\ell) \longrightarrow H^i(\mathfrak{H}^1_d(X,G)_{\leq m}, \mathbb{Q}_\ell)$$

is an isomorphism.

Proof. By Lemma 8.4.10 we get for every $m \geq \gamma(i)$ an isomorphism

$$H^i(\mathfrak{H}^1_d(X,G)_{\leq m}, \mathbb{Q}_\ell) \longrightarrow H^i(\mathfrak{H}^1_d(X,G)_{\leq m-1}, \mathbb{Q}_\ell),$$

since we may estimate the codimension c of $\mathfrak{H}^1_d(X,G)_{\mathfrak{v}}$ in $\mathfrak{H}^1_d(X,G)_{\leq m}$ (for $\mathfrak{v}$ such that $m(\mathfrak{v}) = m$) as follows:

$$c = r(\mathfrak{v})(g-1) + m \geq \begin{cases} m & \text{if } g > 0 \\ m - \frac{\#\Phi}{2} & \text{if } g = 0, \end{cases}$$

so that for $m \geq \gamma(i)$ we have $i \leq 2c - 2$. By Corollary 8.4.4 this implies our proposition. □

Eigenvalues of Frobenius

Lemma 8.4.13 *Let G be a non-singular linear algebraic group over $\mathbb{F}_q$, acting on a smooth equidimensional Deligne-Mumford stack X of finite type over $\mathbb{F}_q$. Let $\mathfrak{X} = [X/G]$ be the corresponding smooth algebraic $\mathbb{F}_q$-stack of finite type. Then for any embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ and any eigenvalue λ of Φ_q on $H^i(\overline{\mathfrak{X}}_{\mathrm{sm}}, \mathbb{C})$ we have $|\lambda| \leq q^{-i/2}$.*

Proof. As in the proof of Theorem 3.5.7 we reduce to the case that $G = GL_n$. Then using the spectral sequence (18) on page 32 we reduce to proving the lemma for the Deligne-Mumford stack X. Then using the Gysin sequence (Corollary 2.1.3) and Proposition 2.2.6, we reduce to the case that X is 'nice'. Then by Corollary 2.2.8 we reduce to the case that X is a smooth variety over $\mathbb{F}_q$. Again using the Gysin sequence, we may even reduce to the case that X is an affine smooth $\mathbb{F}_q$-variety. Now using Poincaré duality as in the proof of Proposition 2.4.3 our lemma translates to the following statement. If λ is an eigenvalue of the geometric Frobenius F_q on $H^i_c(\overline{X}_{\text{ét}}, \mathbb{C})$ then we have $|\lambda| \leq q^{i/2}$. But this follows from [5, Théoréme 3.3.1]. □

Corollary 8.4.14 *Let G be a reductive group scheme over the curve X over the field $\mathbb{F}_q$. Let $d \in \mathsf{X}(G)^\vee$ be given. Then if λ is an eigenvalue of Φ_q on $H^i(\mathfrak{H}^1_d(\overline{X}, \overline{G})_{\leq m}, \mathbb{C})$, for some $m \geq 0$, or on $H^i(\mathfrak{H}^1_d(\overline{X}, \overline{G}), \mathbb{C})$, then we have $|\lambda| \leq q^{-i/2}$.*

Proof. The statement for $\mathfrak{H}^1_d(\overline{X}, \overline{G})_{\leq m}$ follows from Lemma 8.4.13 using Proposition 8.3.12. Now the statement for $\mathfrak{H}^1_d(\overline{X}, \overline{G})$ follows from Proposition 8.4.12. □

Torsors under the Radical of G

Lemma 8.4.15 *Let X be a curve over the field k and F an étale abelian group scheme over X. Then $H^1(X, F)$ is torsion.*

Proof. Let E be an F-torsor. Clearly, there exists a Galois extension of curves $Y \to X$ such that E has a section s over Y. Let $\Gamma = \mathrm{Gal}(Y/X)$ and let $n = \#\Gamma$. Since F is abelian, we may form the n-fold twist of E with itself:

$$E^{\otimes n} = \underbrace{E \times_F \ldots \times_F E}_{n}.$$

Letting $\sigma_1, \ldots, \sigma_n$ be the elements of Γ, it is obvious that $[\sigma_1(s), \ldots, \sigma_n(s)]$ is a section of $E^{\otimes n}$ over X. Thus $E^{\otimes n}$ is trivial. □

Lemma 8.4.16 *Let T be a torus over the curve X over the algebraically closed field k. Let*

$$M = \{\delta \in \mathsf{X}(T)^\vee \mid \mathfrak{H}^1_\delta(X, T) \neq \varnothing\}.$$

Then M is a subgroup of finite index in $\mathsf{X}(T)^\vee$.

Proof. Let $\delta \in \mathsf{X}(T)^\vee$. We need to show that there exists a T-torsor E such that $\deg E = n\delta$ for some $n \neq 0$. Let $\widetilde{\mathsf{X}}(T)$ be the character sheaf of T and consider the character group $\mathsf{X}(T)$ of T as a constant sheaf on X. Then we have a short exact sequence of étale abelian group schemes over X:

$$0 \longrightarrow \mathsf{X}(T) \longrightarrow \widetilde{\mathsf{X}}(T) \longrightarrow F \longrightarrow 0, \tag{33}$$

where F is defined so as to make this sequence exact. It es easily seen that $\mathsf{X}(T)$ is locally a direct summand of $\widetilde{\mathsf{X}}(T)$, so that F is locally free. Hence $\mathrm{Ext}^1(F, \mathsf{X}(T)) = H^1(X, \underline{\mathrm{Hom}}(F, \mathsf{X}(T)))$, which is torsion by Lemma 8.4.15. So the sequence (33) is quasi split. The dual of (33) is a sequence of tori over X:

$$1 \longrightarrow S \longrightarrow T \longrightarrow {\mathbb{G}_{\mathrm{m}}}^r \longrightarrow 1,$$

for some integer $r \geq 0$. By the above arguments, $T \to {\mathbb{G}_{\mathrm{m}}}^r$ has a quasi-section. We may identify $\mathsf{X}(T)^\vee$ with $\mathsf{X}({\mathbb{G}_{\mathrm{m}}}^r)^\vee$ and it is clear that there exists a ${\mathbb{G}_{\mathrm{m}}}^r$-torsor of degree δ. Taking the associated T-torsor (via a quasi-section of $T \to {\mathbb{G}_{\mathrm{m}}}^r$) we get a T-torsor E such that $\deg E = n\delta$ for some $n \neq 0$. □

Lemma 8.4.17 *Let G be a reductive group scheme over the curve X over the algebraically closed field k. Let R be the radical of G. Then there exist finitely many elements $d_1, \ldots, d_n \in \mathsf{X}(G)^\vee$ such that for every $d \in \mathsf{X}(G)^\vee$ there exists an $i \in \{1, \ldots, n\}$ and an R-torsor E of degree $d_i - d$.*

Proof. Apply Lemma 8.4.16 to R. so we have $M \subset \mathsf{X}(R)^\vee$ of finite index. Now as noted in the proof of Proposition 7.3.11, $\mathsf{X}(R)^\vee$ has finite index in $\mathsf{X}(G)^\vee$, so M has finite index in $\mathsf{X}(G)^\vee$. Let $d_1, \ldots, d_n$ be a set of representatives for $\mathsf{X}(G)^\vee / M$. So if $d \in \mathsf{X}(G)^\vee$, there exists an $i \in \{1, \ldots, n\}$ such that $d - d_i \in M$. □

Passing to the Limit

From now on, we let G be a reductive group scheme over the curve X over the algebraically closed field k. Assume that G satisfies Condition (§). This is for example the case if G is rationally trivial and $g \in \{0, 1\}$, if $G = GL_n$ or if $\operatorname{Dyn} G$ is connected. Let $\mathfrak{V}$ be the free abelian group on the components of the Dynkin diagram of G.

For a closed and open subscheme $\eta \subset \operatorname{Dyn} G$ and an integer $\mu \geq 0$ we let

$$C(\eta, \mu) = \#\{\mathfrak{v} \in \mathfrak{V}(\eta)^+ \mid m(\mathfrak{v}) = \mu\}.$$

Here m is defined as in Lemma 8.4.5 and $\mathfrak{V}(\eta)^+$ is the set of linear combinations of components contained in η, all of whose coefficients are positive.

Lemma 8.4.18 *We have $C(\eta, \mu) = O(\mu^s)$, where s is the number of connected components of* $\operatorname{Dyn}(G)$.

Proof. Use Lemma 7.3.8. □

Let A be the set of closed and open subschemes $\eta \subset \operatorname{Dyn} G$ such that there exists a G-torsor E such that the canonical parabolic subgroup of ${}^E G$ has type η. For every $\eta \in A$ choose such a G-torsor and call it E_η. Let $P_\eta \subset {}^{(E_\eta)}G$ be the canonical parabolic subgroup (which is of type η) and let $H_\eta = P_\eta / R_u(P_\eta)$. For each $\eta \in A$ choose $\delta(\eta, 1), \ldots, \delta(\eta, n) \in \mathsf{X}(H_\eta)^\vee$ according to Lemma 8.4.17. We get a finite family of smooth algebraic k-stacks

$$\left(\mathfrak{H}^1_{\delta(\eta,j)}(X, H_\eta)_0\right)_{\eta,j},$$

parametrized by $A \times \{1, \ldots, n\}$. By the finiteness theorem (Theorem 3.1.6) we have that $H^i(\mathfrak{H}^1_{\delta(\eta,j)}(X, H_\eta)_0, \mathbb{Q}_\ell)$ is finite dimensional over $\mathbb{Q}_\ell$ for every pair $(\eta, j) \in A \times \{1, \ldots, n\}$ and for every $i \in \mathbb{Z}$. So if we set

$$b_i(\eta, j) = \dim_{\mathbb{Q}_\ell} H^i(\mathfrak{H}^1_{\delta(\eta,j)}(X, H_\eta)_0, \mathbb{Q}_\ell),$$

then $b_i(\eta, j)$ is a family of non-negative integers. Let $B(i)$, for $i \in \mathbb{Z}$, be chosen such that $B(i) \geq b_i(\eta, j)$ for every $(\eta, j) \in A \times \{1, \ldots, n\}$, and such that $B(i) = 0$ for $i < 0$.

Lemma 8.4.19 *There exists an integer $N > 0$ such that $B(i) = O(i^N)$.*

Proof. This depends essentially on the spectral sequence (17) on page 32. $\square$

We now define for $i \geq 0$:

$$D(i) = \sum_{\eta \in A} \sum_{\mu=0}^{\gamma(i)} C(\eta, \mu) B(i - 2\mu - 2r(\eta)(g-1)),$$

where $r(\eta)$ is defined as in Lemma 8.4.6, noting that $r(\mathfrak{v})$, for $\mathfrak{v} \in \mathfrak{V}$, depends only on the components of $\operatorname{Dyn} G$ with respect to which $\mathfrak{v}$ has non-zero coefficients.

Lemma 8.4.20 *There exists an integer $M > 0$ such that $D(i) = O(i^M)$.*

Proof. Follows form Lemmas 8.4.18 and 8.4.19. $\square$

Lemma 8.4.21 *Let $i \geq 0$ be given. Then we have*

$$\dim_{\mathbb{Q}_\ell} H^i(\mathfrak{H}^1_d(X, G), \mathbb{Q}_\ell) \leq D(i).$$

Proof. By Proposition 8.4.12 we have

$$\dim_{\mathbb{Q}_\ell} H^i(\mathfrak{H}^1_d(X, G), \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell} H^i(\mathfrak{H}^1_d(X, G)_{\leq \gamma(i)}, \mathbb{Q}_\ell).$$

By Lemma 8.4.10 and the Gysin sequence we have

$$\begin{aligned} &\dim_{\mathbb{Q}_\ell} H^i(\mathfrak{H}^1_d(X, G)_{\leq \gamma(i)}, \mathbb{Q}_\ell) \\ \leq &\sum_{\substack{\mathfrak{v} \in \mathfrak{V} \\ \mathfrak{v} \geq 0 \\ m(\mathfrak{v}) \leq \gamma(i)}} \dim_{\mathbb{Q}_\ell} H^{i - 2(r(\mathfrak{v})(g-1) + m(\mathfrak{v}))}(\mathfrak{H}^1_d(X, G)_{\mathfrak{v}}, \mathbb{Q}_\ell) \\ = &\sum_{\eta \in A} \sum_{\mu=0}^{\gamma(i)} \sum_{\substack{\mathfrak{v} \in \mathfrak{V}(\eta)^+ \\ m(\mathfrak{v}) = \mu}} \dim_{\mathbb{Q}_\ell} H^{i - 2(r(\mathfrak{v})(g-1) + \mu)}(\mathfrak{H}^1_d(X, G)_{\mathfrak{v}}, \mathbb{Q}_\ell). \end{aligned}$$

Now if $\mathfrak{v} \in \mathfrak{V}(\eta)^+$, for $\eta \in A$, we have

$$\mathfrak{H}^1_d(X, G)_{\mathfrak{v}} = \mathfrak{H}^1_{d - \deg E_\eta}(X, {}^{(E_\eta)}G)_{\mathfrak{v}},$$

by Remark 7.3.7. Let $\delta \in \mathsf{X}(P_\eta)^\vee$ be the unique element such that $d(\delta) = d - \deg E_\eta$ and $\mathfrak{v}(\delta) = \mathfrak{v}$. Then by Proposition 8.4.9 we have

$$H^i(\mathfrak{H}^1_{d - \deg E_\eta}(X, {}^{(E_\eta)}G)_{\mathfrak{v}}, \mathbb{Q}_\ell) = H^i(\mathfrak{H}^1_\delta(X, H_\eta)_0, \mathbb{Q}_\ell).$$

(Note that if no such δ exists, then $\mathfrak{H}^1_d(X, G)_{\mathfrak{v}} = \varnothing$.) Now by construction (and Lemma 8.4.17) there exists a $j \in \{1, \ldots, n\}$ such that

$$\mathfrak{H}^1_\delta(X, H_\eta)_0 \cong \mathfrak{H}^1_{\delta(\eta, j)}(X, H_\eta)_0.$$

So assembling these remarks together, we see that

$$H^i(\mathfrak{H}^1_d(X, G)_{\mathfrak{v}}, \mathbb{Q}_\ell) = H^i(\mathfrak{H}^1_{\delta(\eta, j)}(X, H_\eta)_0, \mathbb{Q}_\ell)$$

and hence

$$\dim_{\mathbb{Q}_\ell} H^i(\mathfrak{H}^1_d(X,G)_{\mathfrak{v}},\mathbb{Q}_\ell) = b_i(\eta,j) \leq B(i).$$

So we have

$$\begin{aligned}
\dim_{\mathbb{Q}_\ell} H^i(\mathfrak{H}^1_d(X,G),\mathbb{Q}_\ell) &\leq \sum_{\eta\in A}\sum_{\mu=0}^{\gamma(i)} \sum_{\substack{\mathfrak{v}\in\mathfrak{V}(\eta)^+\\ m(\mathfrak{v})=\mu}} B(i-2r(\eta)(g-1)-2\mu)\\
&= \sum_{\eta\in A}\sum_{\mu=0}^{\gamma(i)} C(\eta,\mu)B(i-2r(\eta)(g-1)-2\mu)\\
&= D(i),
\end{aligned}$$

what was to be proven. □

Theorem 8.4.22 *Let G be a reductive group scheme over the curve X over the field $\mathbb{F}_q$. Assume that G satisfies Condition (§). Then $\operatorname{tr}\Phi_q|H^*(\mathfrak{H}^1_d(\overline{X},\overline{G}),\mathbb{C})$ converges absolutely, and we have*

$$\operatorname{tr}\Phi_q|H^*(\mathfrak{H}^1_d(\overline{X},\overline{G}),\mathbb{C}) = \sum_{E\in H^1_d(X,G)} \frac{1}{\#\operatorname{Aut}E}.$$

Proof. The fact that $\operatorname{tr}\Phi_q|H^*(\mathfrak{H}^1_d(\overline{X},\overline{G}),\mathbb{C})$ converges absolutely follows from Lemma 8.4.21, Corollary 8.4.14 and Proposition 8.4.12, because

$$\sum_{i=0}^{\infty} D(i)q^{-i/2}$$

converges absolutely. This follows for example from the quotient criterion and Lemma 8.4.20.

We will now prove the convergence of

$$\lim_{m\to\infty}\sum_{E\in H^1_d(X,G)_{\leq m}} \frac{1}{\#\operatorname{Aut}E}$$

and calculate its limit.

So let $\epsilon>0$ be given, and choose i'_0 is such a way that

$$\sum_{i=i'_0}^{\infty} D(i)q^{-i/2} < \epsilon/2.$$

Then let $m'_0=\gamma(i'_0)$. Choose $m_0\geq m'_0$ and $i_0\geq i'_0$ such that $m_0\leq\gamma(i_0)$. Then we have

$$\sum_{i=i_0}^{\infty} D(i)q^{-i/2} < \epsilon/2.$$

For every $i\geq i_0$ we have $\gamma(i)\geq m_0$. So we may calculate as in the proof of Lemma 8.4.21:

$$\begin{aligned}
\dim H^i(\mathfrak{H}^1_d(X,G)_{\leq m_0},\mathbb{Q}_\ell) &\leq \sum_{\eta\in A}\sum_{\mu=0}^{m_0} C(\eta,\mu)B(i-2r(\eta)(g-1)-2\mu)\\
&\leq \sum_{\eta\in A}\sum_{\mu=0}^{\gamma(i)} C(\eta,\mu)B(i-2r(\eta)(g-1)-2\mu)\\
&= D(i).
\end{aligned}$$

Hence we have

$$\sum_{i=i_0}^{\infty} \dim H^i(\mathfrak{H}^1_d(X,G)_{\leq m_0}, \mathbb{Q}_\ell) q^{-i/2} \leq \sum_{i=i_0}^{\infty} D(i) q^{-i/2} < \epsilon/2. \tag{34}$$

So we may now estimate

$$\begin{aligned}
&\left| \operatorname{tr} \Phi_q | H^*(\mathfrak{H}^1_d(\overline{X},\overline{G}),\mathbb{C}) - \sum_{E \in H^1_d(X,G)_{\leq m_0}} \frac{1}{\#\operatorname{Aut} E} \right| \\
\leq\ & \epsilon/2 + \left| \sum_{i=0}^{i_0} (-1)^i \operatorname{tr} \Phi_q | H^i(\mathfrak{H}^1_d(\overline{X},\overline{G}),\mathbb{C}) - \sum_{E \in H^1_d(X,G)_{\leq m_0}} \frac{1}{\#\operatorname{Aut} E} \right| \\
& \text{(by Lemma 8.4.21)} \\
=\ & \epsilon/2 + \left| \sum_{i=0}^{i_0} (-1)^i \operatorname{tr} \Phi_q | H^i(\mathfrak{H}^1_d(\overline{X},\overline{G})_{\leq m_0},\mathbb{C}) - \sum_{E \in H^1_d(X,G)_{\leq m_0}} \frac{1}{\#\operatorname{Aut} E} \right| \\
& \text{(by Proposition 8.4.12)} \\
\leq\ & \epsilon + \left| \operatorname{tr} \Phi_q | H^*(\mathfrak{H}^1_d(\overline{X},\overline{G})_{\leq m_0},\mathbb{C}) - \sum_{E \in H^1_d(X,G)_{\leq m_0}} \frac{1}{\#\operatorname{Aut} E} \right| \\
& \text{(by (34))} \\
=\ & \epsilon
\end{aligned}$$

by Corollary 8.3.13. □

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