Introduction to Algebraic Stacks

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December 17, 2012

Abstract

These are lecture notes based on a short course on stacks given at the Newton Institute in Cambridge in January 2011. They form a self-contained introduction to some of the basic ideas of stack theory.

Contents

Introduction

Stacks and algebraic stacks were invented by the Grothendieck school of algebraic geometry in the 1960s. One purpose (see [11]), was to give geometric meaning to higher cohomology classes. The other (see [9] and [2]), was to develop are more general framework for studying moduli problems. It is the latter aspect that interests us in these notes. Since the 1980s, stacks have become an increasingly important tool in geometry, topology and theoretical physics.

Stack theory examines how mathematical objects can vary in families. For our examples, the mathematical objects will be the triangles familiar from Euclidean geometry, and closely related concepts. At least to begin with, we will let these vary in continuous families, parametrized by topological spaces.

A surprising number of stacky phenomena can be seen in such simple cases. (In fact, one of the founders of the theory of algebraic stacks, M. Artin, is famously reputed to have said that one need only understand the stack of triangles to understand stacks.)

These lecture notes are divided into three parts. The first is a very leisurely and elementary introduction to stacks, introducing the main ideas by considering a few elementary examples of topological stacks. The only prerequisites for this part are basic undergraduate courses in abstract algebra (groups and group actions) and topology (topological spaces, covering spaces, the fundamental group).

The second part introduces the basic formalism of stacks. The prerequisites are the same, although this part is more demanding than the previous.

The third part introduces algebraic stacks, culminating in the Riemann-Roch theorem for stacky curves. The prerequisite here is some basic scheme theory.

We do not cover much of the 'algebraic geometry' of algebraic stacks, but we hope that these notes will prepare the reader for the study of more advanced texts, such as [17] or the forthcoming book [23].

The following outline uses terminology which will be explained in the body of the text.

The first fundamental notion is that of *symmetry groupoid of a family of objects*. This is introduced first for discrete families of triangles, and then for continuous families of triangles.

In Sections 1.1 through 1.3, we consider Euclidean triangles up to similarity (the stack of such triangles is called $\mathfrak{M}$). We define what a fine moduli space is, and show how the symmetries of the isosceles triangles and the equilateral triangle prevent a fine moduli space from existing. We study the coarse moduli space of triangles, and discover that it parametrizes a *modular family*, even though this family is, of course, not universal.

Sections 1.4 through 1.6, introduce other examples of moduli problems. In 1.4, we encounter a fine moduli space (the fine moduli space of scalene triangles), in 1.5, where we restrict attention to isosceles triangles, we encounter a coarse moduli space supporting several non-isomorphic modular families. Restricting attention entirely to the equilateral triangle, in 1.6, we come across a coarse moduli space which parametrizes a modular family, which is versal, but not universal.

In Section 1.7, we finally exhibit an example of a coarse moduli space which does not admit m any modular family at all. We start studying *oriented triangles*. We will eventually prefer working with oriented triangles, because they are more closely related to algebraic geometry. The stack of oriented triangles is called $\widetilde{\mathfrak{M}}$.

In Section 1.8, we make a first few general and informal remarks about stacks, and their role in the study of moduli problems.

The second fundamental concept is that of *versal family*. Versal families replace universal families, where the latter do not exist. Stacks that admit versal families are called *geometric*, which means *topological* in this first two chapters, but will mean *algebraic* in Chapter 3.

We introduce versal families in Section 1.9, and give several examples. We explain how a stack which admits a versal family is essentially equal to the stack of 'generalized moduli maps' (or torsors, in more advanced terminology).

In Section 1.10, we start including degenerate triangles in our examinations: triangles whose three vertices are collinear. The main reason to do this is to provide examples of compactifications of moduli stacks. There are several different natural ways to compactify the stack of triangles: there is a naïve point of view, which we dismiss rather quickly. We then explain a more interesting and natural, but also more complicated point of view: in this, the stack of degenerate triangles turns out to be the quotient stack of a bipyramid modulo its symmetries, which form a group of order 12. This stack of degenerate triangles is called $\overline{\mathfrak{M}}$.

We encounter a very useful construction along the way: the construction of a stack by *stackification*, which means by first describing families only locally, then constructing a versal family, and then giving the stack as the stack of generalized moduli maps to the universal family (or torsors for the symmetry groupoid of the versal family).

We then consider oriented degenerate triangles, and introduce the *Legendre family* of triangles which is parametrized by the Riemann sphere. It exhibits the stack of oriented degenerate triangles as the quotient stack of the Riemann sphere by the action of the dihedral group with 6 elements. (In particular, it endows the stack of oriented degenerate triangles with the structure of an algebraic, not just topological, stack.) We call this stack $\mathfrak{L}$, and refer to it as the *Legendre compactification* of the stack of oriented triangles $\widetilde{\mathfrak{M}}$.

The Legendre family provides the following illustration of the concept of *generalized moduli map* (or groupoid torsor). We try to characterize, i.e., completely describe the similarity type of an (oriented, maybe degenerate) triangle, by specifying the complex cross-ratio of its three vertices together with the point at infinity. However, the cross-ratio is not a single valued invariant, but rather a multi-valued one: the six possible values of the cross-ratio are acted upon by the group S_3. Thus the stack $\mathfrak{L}$ of (oriented, maybe degenerate) triangles is the quotient stack of the Riemann sphere divided by S_3.

In Section 1.11, we explain how to relate different versal families for the same stack with one another, and how to recognize two stacks as being essentially the same, by exhibiting a bitorsor for the respective symmetry groupoids of respective versal families. We apply this both ways: we exhibit two different versal families for 'non-pinched' triangles, and how a bitorsor intertwines them. Then we construct a bitorsor intertwining two potentially different moduli problems, namely two potentially different ways to treat families containing 'pinched' triangles, thus showing that the two moduli problems are equivalent.

In Section 1.12, we introduce another compactification of the moduli stack of oriented triangles, which we call the *Weierstrass compactification*, because we construct it from the family of degree 3 polynomials in Weierstrass normal form. We denote this stack by $\mathfrak{W}$. We encounter our first example of a non-trivial morphism of stacks, namely the natural morphism $\mathfrak{L} \to \mathfrak{W}$. We also introduce a holomorphic coordinate on the coarse moduli space of oriented triangles known as the j-invariant.

In Part 2, we introduce the formalism of stacks. This will allow us to discuss topological stacks in general, without reference to specific objects such as triangles.

In Sections 2.1 to 2.4 we discuss the standard notions. We start with categories fibered in groupoids, which formalize what a moduli problem is. Then come the prestacks, which have well-behaved isomorphism spaces, and allow for the general definition of versal family. After a brief discussion of stacks, we define topological stacks to be stacks which admit a versal family. We discuss the basic fact that every topological stack is isomorphic to the stack of torsors for the symmetry groupoid of a versal family. This also formalizes our approach to stackification: start with a prestack, find a versal family, and then replace the given prestack by the stack of torsors for the symmetry groupoid of the versal family.

In Section 2.5 we discuss a new idea: namely that symmetry groupoids of versal families should be considered as gluing data for topological stacks, in analogy to atlases for topological manifolds. This also leads to the requirement that the parameter space of a versal family should reflect the local topological structure of a stack faithfully, and conversely, that a topo-

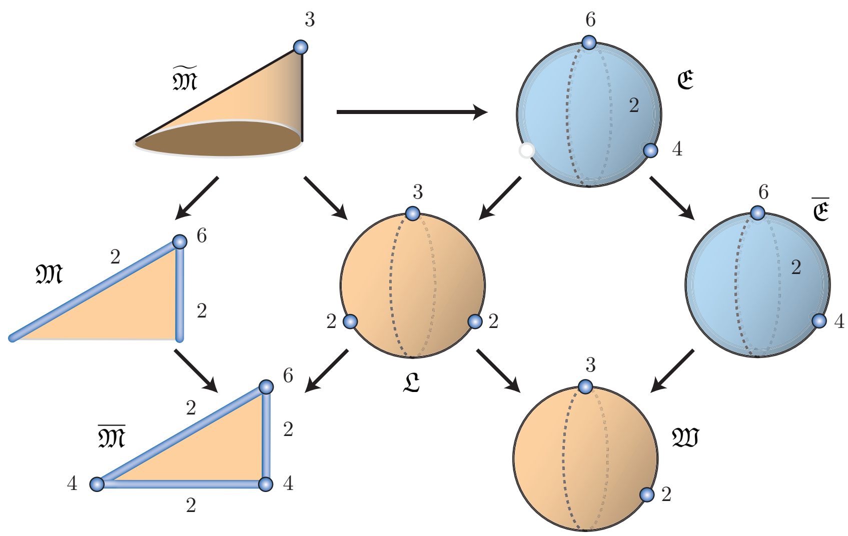


Figure 1: Some of the stacks we encounter in these notes, and the morphisms between them. Stacky points (coloured blue) are labelled with the order of their isotropy groups.

logical stack should locally behave in a manner controlled by the parameter space of a versal family, in order that we can 'do geometry' on the stack.

This idea leads to the introduction of *étale versal families*, and the associated stacks, which we call *Deligne-Mumford topological stacks*, in analogy with the algebraic case. We prove a structure theorem, that says that every separated Deligne-Mumford topological stack has an open cover by finite group quotient stacks.

This shows that all 'well-behaved' moduli problems with discrete symmetry groups are locally described by finite group quotients. Therefore, the seemingly simple examples we start out with, in fact turn out be quite typical of the general case.

We also encounter examples of moduli problems without symmetries, that nevertheless do not admit fine moduli spaces. For sufficiently badly behaved equivalence relations (when the quotient map does not admit local sections), the quotient space is not a fine moduli space.

In Section 2.6, we continue our series of examples of moduli problems related to triangles, by considering lattices up to homothety. This leads to the stack of elliptic curves, which we call $\mathfrak{E}$, and its compactification $\overline{\mathfrak{E}}$. We see another example of a morphism of stacks, namely $\mathfrak{E} \to \mathfrak{W}$, which maps a lattice to the triangle of values of the Weierstrass $\wp$-function at the half periods. This is an example of a $\mathbb{Z}_2$-gerbe.

As an illustration of some simple 'topology with stacks', we introduce the fundamental group of a topological stack in Section 2.7, and compute it for some of our examples.

The third part of these notes is a brief introduction to algebraic stacks. The algebraic theory requires more background than the topological one: we need, for example, the theory of cohomology and base change. We will therefore assume that the reader has a certain familiarity with scheme theory as covered in [16].

We limit our attention to algebraic stacks with affine diagonal. This avoids the need for algebraic spaces as a prerequisite. For many applications, this is not a serious limitation. As typical examples we discuss the stack of elliptic curves $\mathfrak{E}$ and its compactification $\overline{\mathfrak{E}}$, as well as the stack of vector bundles on a curve.

Our definition of algebraic stack avoids reference to Grothendieck topologies, algebraic spaces, and descent theory. Essentially, a category fibered in groupoids is an algebraic stack, if it is equivalent to the stack of torsors for an algebraic groupoid. Sometimes, for example for $\overline{\mathfrak{E}}$, we can verify this condition directly. We discuss a useful theorem, which reduces the verification that a given groupoid fibration is an algebraic stack to the existence of a versal family, with sufficiently well-behaved symmetry groupoid, and the gluing property in the étale topology.

We include a discussion of the coarse moduli space in the algebraic context: the theory is much more involved than in the topological case. We introduce algebraic spaces as algebraic stacks 'without stackiness'. We sketch the proof that separated Deligne-Mumford stacks admit coarse moduli spaces, which are separated algebraic stacks. As a by-product, we show that separated Deligne-Mumford stacks are locally, in the étale topology of the coarse moduli space, finite group quotients.

We then define what vector bundles and coherent sheaves on stacks are, giving the bundle of modular forms on $\overline{\mathfrak{E}}$ as an example. In a final section 3.6, we study stacky curves, and as an example of some algebraic geometry over stacks, we prove the Riemann-Roch theorem for orbifold curves. As an illustration, we compute the well-known dimensions of the spaces of modular forms.

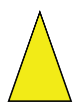
1 Topological stacks: Triangles

This first part is directed at the student of mathematics who has taken an introduction to topology (covering spaces and the fundamental group) and an introduction to abstract algebra (group actions). Most of the formal mathematics has been relegated to exercises, which can be skipped by the reader who lacks the requisite background. The end of these exercises is marked with the symbol '$\Box$'.

We are interested in two ideas: *symmetry* and *form*, and their role in *classification*.

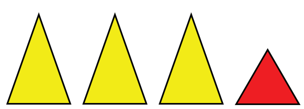
1.1 Families and their symmetry groupoids

Consider a mathematical concept, for example *triangle*, together with a notion of isomorphism, for example *similarity*. This leads to the idea of *symmetry*. Given an object (for example an isosceles triangle)



a *symmetry* is an isomorphism of the object with itself (for example, the reflection across the 'axis of symmetry'). All the symmetries of an object form a *group*, the *symmetry group* of the object. (The symmetry group of our isosceles triangle is $\{\mathrm{id}, \mathrm{refl}\}$.)

To capture the essence of *form*, in particular how form may vary, we consider *families* of objects, rather than single objects (for example, the family of 4 triangles,



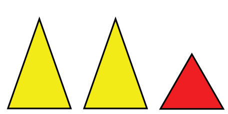
(1)

consisting of three congruent isosceles triangles and one equilateral triangle).

1.1 Definition. A **symmetry** of a family of objects is an isomorphism of one member of the family with another member of the family.

1.2 Example. The family (1) of 4 triangles has 24 symmetries: there are 2 symmetries from each of the isosceles triangles to every other (including itself), adding up to 18, plus 6 symmetries of the equilateral triangle.

If we restrict the family to contain only the latter 2 isosceles triangles and the equilateral triangle,

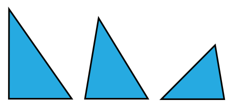


the family has 14 symmetries.

Various types of symmetry groupoids

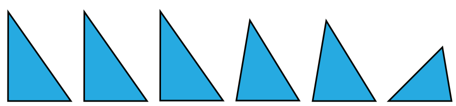
1.3 Definition. The collection of all symmetries of a given family is called the **symmetry groupoid** of the family.

1.4 Example (Set)**.** The symmetry groupoid of a family of non-isomorphic asymmetric objects



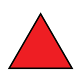
consists of only the trivial symmetries, one for each object. Such a groupoid is essentially the same thing as the set of objects in the family (or, more precisely, the indexing set of the family).

1.5 Example (Equivalence relation)**.** The symmetry groupoid of a family of asymmetric objects



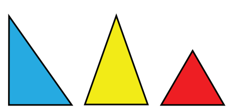
is *rigid*. From any object to another there is at most one symmetry. A rigid groupoid is essentially the same thing as an equivalence relation on the set of objects (or the indexing set of the family).

1.6 Example (Group)**.** The symmetry groupoid of a single object



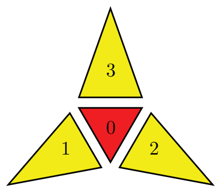
is a group.

1.7 Example (Family of groups)**.** The symmetry groupoid of a family of non-isomorphic objects



is a family of groups.

1.8 Example (Transformation groupoid)**.** Consider again the family of triangles (1) above, but now rearranged like this:



This figure has dihedral symmetry, and so the dihedral group with 6 elements, i.e., the symmetric group on 3 letters S_3, acts on this figure. Each element of S_3 defines 4 symmetries of the family, because it defines a symmetry originating at each of the 4 triangles.

For example, the rotation by $\frac{2\pi}{3}$ (or the permutation $1 \mapsto 3$, $3 \mapsto 2$, $2 \mapsto 1$), gives rise to the $\frac{2\pi}{3}$-rotational symmetry of the equilateral triangle in the centre of the figure, as well as 3 isomorphisms, each from one isosceles triangle to another.

The reflection across a vertical line (or the permutation $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 3$) gives rise to reflectional symmetries of the triangles labelled 0 and 3, as well as an isomorphism and its inverse between the two isosceles triangles labelled 1 and 2.

The family (1) is in fact so symmetric, that every one of its symmetries comes from an element of the dihedral group acting on the figure (1).

More formally, let $\mathscr{F} = (\mathscr{F}_i)_{i=0,1,2,3}$ be the family of triangles, and Γ its symmetry groupoid. Then we have a bijection

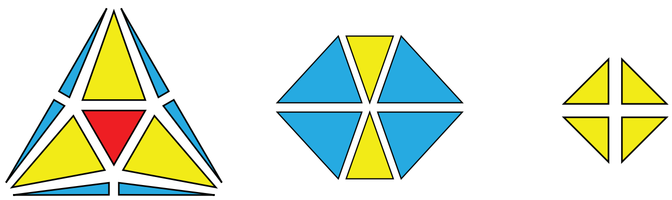
$$\begin{aligned} \{0,1,2,3\} \times S_3 &\longrightarrow \Gamma \\ (i,\sigma) &\longmapsto \phi_{i,\sigma} \end{aligned} \tag{2}$$

where $\phi_{i,\sigma} : \mathscr{F}_i \to \mathscr{F}_{\sigma(i)}$ is the symmetry from the triangle $\mathscr{F}_i$ to the triangle $\mathscr{F}_{\sigma(i)}$ induced by the geometric transformation of the whole figure defined by σ.

The action of S_3 on the figure induces an action on the indexing set $\{0,1,2,3\}$, and the symmetry groupoid is completely described by this group action. The bijection (2) is an *isomorphism of groupoids*.

Whenever we have an arbitrary group G acting on a set X, we get an associated transformation groupoid $\Gamma = X \times G$.

1.9 Example. Here are three more examples of families of triangles whose symmetry groupoids are transformation groupoids:



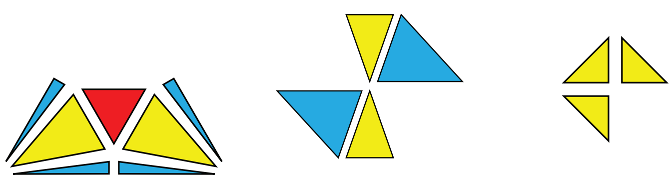
In the first, we have added 6 scalene (i.e., completely asymmetric) triangles to the family (1). The indexing set of the family has 10 elements, and the group S_3 acts on this set of 10 elements, in a way induced by the symmetries of the figure.

In the second case, the family consists of 2 isosceles and 4 scalene triangles. The symmetry group of the figure is the dihedral group with 4 elements (which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$).

In the last case, the family consists of 4 isosceles triangles, and the symmetry group of the figure is the dihedral group with 8 elements, D_4.

In each of the three case, the symmetry groupoid of the family of triangles is equal to the transformation groupoid given by the action of the symmetry group of the figure on the indexing set of the family. Note that in each case, the number of times a certain triangle appears in the family is equal to the number of elements in the symmetry group of the figure, divided by the number of symmetries of the triangle.

Removing triangles breaks the symmetry, and leads to families whose symmetry groupoids are not transformation groupoids any longer:



1.10 Exercise. A **groupoid** Γ consists of two sets: the set of objects Γ_0, and the set of arrows Γ_1. Also part of Γ are the *source* and *target* maps $s, t : \Gamma_1 \to \Gamma_0$, as well as the *groupoid operation* $\mu : \Gamma_2 \to \Gamma_1$, where Γ_2 is the set of *composable pairs*, which is the fibered product

$$\begin{tikzcd} \Gamma_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & \Gamma_1 \arrow[d, "s"] \\ \Gamma_1 \arrow[r, "t"] & \Gamma_0 \,. \end{tikzcd}$$

We write $\mu(\alpha, \beta) = \alpha * \beta$, for composable pairs of arrows $(\alpha, \beta) \in \Gamma_2$, where

$t(\alpha) = s(\beta)$.

$$\begin{tikzcd} \bullet \arrow[r, "\alpha"] \arrow[rr, bend right, "\alpha*\beta"'] & \bullet \arrow[r, "\beta"] & \bullet \end{tikzcd}$$

Three properties are required to hold:

(i) (identities) For every object $x \in \Gamma_0$, there exists an arrow $e_x \in \Gamma_1$, whose source and target are x, and such that $e_x * \alpha = \alpha$, for all α with source x and $\beta * e_x = \beta$, for all β with target x.

(ii) (inverses) For every arrow $\alpha \in \Gamma_1$, there exists an arrow α^{-1}, such that $\alpha * \alpha^{-1} = e_{s(\alpha)}$ and $\alpha^{-1} * \alpha = e_{t(\alpha)}$.

(iii) (associativity) For every triple (α, β, γ) of composable arrows, we have $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$.

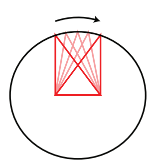
Using the language of categories, we notice that a groupoid is nothing but a small category, all of whose arrows are invertible. Often it is more natural to use categorical notation for the groupoid operation: $\beta \circ \alpha = \alpha * \beta$.

The symmetry groupoid of a family parametrized be the set T has $\Gamma_0 = T$.

An **isomorphism of groupoids** consists of two bijections: $\Gamma_0 \to \Gamma_0'$ and $\Gamma_1 \to \Gamma_1'$, compatible with the composition (and hence identities and inverses). $\square$

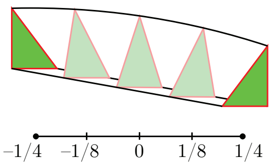
1.2 Continuous families

So far, we have considered discrete families. More interesting are continuous families. For example, suppose given a piece of string of length 2 and two pins, distanced $\frac{1}{2}$ from each other, and draw a part of an ellipse:



(3)

We start with a 3:4:5 right triangle, whose sides have lengths $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$, and we end up with a congruent 3:4:5 triangle. This is a family of triangles parametrized by an interval. To make this more explicit, suppose that we are in the plane $\mathbb{R}^2$, and the two pins have coordinates $(-\frac{1}{4}, 0)$ and $(\frac{1}{4}, 0)$. Let us take the interval $[-\frac{1}{4}, \frac{1}{4}]$ as parameter space, and let us denote the family of triangles by $\mathscr{F}$. Then every parameter value $t \in [-\frac{1}{4}, \frac{1}{4}]$ corresponds to a triangle $\mathscr{F}_t$, where $\mathscr{F}_t$ is the triangle subtended by the string when the x-coordinate of the pen point is t.



(4)

In (4) we see another view of this family, 'lying over' the parameter space. The five family members $\mathscr{F}_{-\frac{1}{4}}$, $\mathscr{F}_{-\frac{1}{8}}$, $\mathscr{F}_0$, $\mathscr{F}_{\frac{1}{8}}$, $\mathscr{F}_{\frac{1}{4}}$ are highlighted (but, of course, the mind's eye is supposed to fill in the other family members).

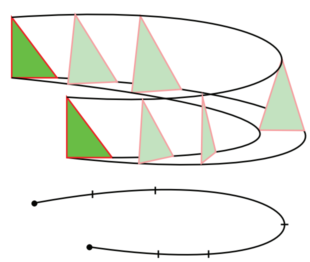
The group with two elements $\mathbb{Z}_2$ acts on the picture (3) by reflection across the y-axis. This action induces all symmetries of the family $\mathscr{F}$. The symmetry groupoid of $\mathscr{F}$ is given by the induced action of $\mathbb{Z}_2$ on the parameter space $[-\frac{1}{4}, \frac{1}{4}]$ (where the non-identity element of $\mathbb{Z}_2$ acts by multiplication by -1). The symmetry groupoid of the family (4) is the transformation groupoid $[-\frac{1}{4}, \frac{1}{4}] \times \mathbb{Z}_2$.

Gluing families

One essential feature of continuous families is that they can be 'glued'. The first and last members of the family $\mathscr{F}$ of (4) are similar to each other, and we can therefore glue the two endpoints of the parameter interval to obtain a circle, and glue the two corresponding triangles to obtain a family of triangles parametrized by the circle.

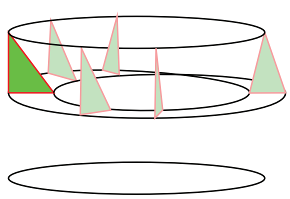
First we bend the second half of the family around:

$\mathscr{F}$:



and then we glue the parameter interval and the two end triangles:

$\widetilde{\mathscr{F}}$:



(5)

(These figures are not to scale.) The new parameter space is the circle S^1. Let us call the family of triangles we obtain in this way $\widetilde{\mathscr{F}}$ over S^1, or $\widetilde{\mathscr{F}}/S^1$. This family exhibits an interesting feature: the shortest sides of all the triangles in the family put together form a Moebius band. The perimeters of all triangles in the family put together form a Klein bottle. There is no way to label the vertices of the family members in a consistent way with labels A, B, C, say. Equivalently, it is impossible to label the sides of the triangles in the family consistently with labels a, b, c. (On the other hand, such a consistent labelling is always possible if we only vary the parameter values inside the circle S^1 a little bit, but not too much. We say that *locally* we can label the vertices of the family member triangles consistently.)

1.11 Exercise. As we are studying triangles up to similarity, let us fix the perimeter of the triangles we consider. We can take any value, but for aesthetic reasons we will take the value 2.

Formally, let us then define a continuous family of triangles parametrized by the topological space T to consist of a degree 3 covering map $T' \to T$ and a continuous map $a : T' \to \mathbb{R}_{>0}$. The data $\mathscr{F} = (T', a)$ has to satisfy the triangle inequalities for all $t \in T$. More precisely, for $t \in T$, there are three points of T' lying over t, call them t'_1, t'_2, t'_3, and three positive real numbers $a(t'_1)$, $a(t'_2)$, $a(t'_3)$. The latter have to satisfy the three triangle inequalities: $a(t'_1) + a(t'_2) > a(t'_3)$, $a(t'_2) + a(t'_3) > a(t'_1)$, $a(t'_3) + a(t'_1) > a(t'_2)$. Moreover, for every $t \in T$, we require $a(t'_1) + a(t'_2) + a(t'_3) = 2$. The triangle $\mathscr{F}_t$ corresponding to the parameter value $t \in T$ is then the triangle whose sides have lengths $a(t'_1)$, $a(t'_2)$ and $a(t'_3)$.

Suppose that $\mathscr{F}/T$ and $\mathscr{G}/T$ are two families of triangles parametrized by the same space T, where $\mathscr{F} = (T', a)$ and $\mathscr{G} = (T'', b)$. Define an **isomorphism** of families of triangles $\phi : \mathscr{F} \to \mathscr{G}$ to consist of a homeomorphism of covering spaces $f : T' \to T''$ (f has to commute with the projections to T), such that $a = b \circ f$.

Prove that the family $\widetilde{\mathscr{F}}$ we constructed above (5) is, indeed, a continuous family of triangles, according to this formal definition. Prove that families of triangles can be glued, i.e., that they satisfy the following *gluing axiom*:

Suppose $T = A \cup B$ is a topological space with closed subsets $A \subset T$, $B \subset T$, and $\mathscr{F}/A$ and $\mathscr{G}/B$ are continuous families of triangles, and $\phi : \mathscr{F}|_{A\cap B} \to \mathscr{G}|_{A\cap B}$ is an isomorphism. Then there exists, in an essentially unique way, a continuous family of triangles $\mathscr{H}/T$, and isomorphisms $\psi : \mathscr{H}|_A \cong \mathscr{F}$ and $\chi : \mathscr{H}|_B \cong \mathscr{G}$, such that $\phi \circ \psi|_{A\cap B} = \chi|_{A\cap B}$. □

1.12 Exercise. A formulation of the gluing principle which has wider applicability than the one alluded to in the previous exercise, is the following.

Suppose the space T is the union of a family of open subsets $U_i \subset T$, and over each U_i we have a continuous family of triangles (or other mathematical objects) $\mathscr{F}_i$, parametrized by U_i. Assume that over each intersection

$U_{ij} = U_i \cap U_j$ we are given an isomorphism of families $\phi_{ij} : \mathscr{F}_i|_{U_{ij}} \to \mathscr{F}_j|_{U_{ij}}$, and that over all triple overlaps the compatibility condition (the *cocycle* condition) $\phi_{ik}|_{U_{ijk}} = \phi_{jk}|_{U_{ijk}} \circ \phi_{ij}|_{U_{ijk}}$ holds. (The data $(\{\mathscr{F}_i\}, \{\phi_{ij}\})$ are called *gluing data* for a continuous family.)

Then there exists a continuous family $\mathscr{F}$, parametrized by T, together with isomorphisms of families $\phi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$, such that over the overlaps we have $\phi_j|_{U_{ij}} = \phi_{ij} \circ \phi_i|_{U_{ij}}$. (The pair $(\mathscr{F}, \{\phi_i\})$ is said to be *obtained by gluing* from the above gluing data.)

The pair $(\mathscr{F}, \{\phi_i\})$ is unique, in the following sense: given $(\mathscr{G}, \{\psi_i\})$, solving the same gluing problem, there exists an isomorphism of families $\chi : \mathscr{F} \to \mathscr{G}$, such that on each open U_i we have $\psi_i \circ \chi|_{U_i} = \phi_i$.

If our notion of continuous family of some type of mathematical object has this gluing property (i.e. for every space T, and for every gluing data over T, the solution exists, and is essentially unique in the described way), then we say that these types of families *can be glued.*

Prove that families of triangles can be glued. $\square$

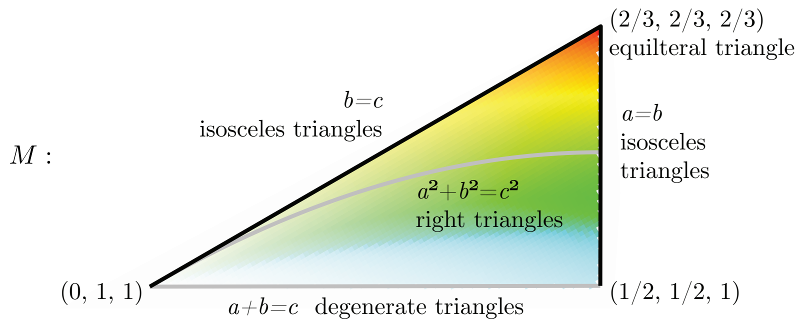
1.3 Classification

Our goal is to describe the totality of our mathematical objects as a space: in our example of triangles up to similarity, we would like a space whose points correspond in a one-to-one fashion to similarity classes of triangles. Such a space would be called a *moduli space* of triangles up to similarity, and it would be said to solve the *moduli problem* posed by triangles up to similarity.

In fact, such a space is easily constructed. Every triangle is similar to a triangle of perimeter 2, say, and every triangle of perimeter 2 is given (up to congruence) by the lengths of its sides, which we can label a, b, c, where $a \leq b \leq c$. Thus an example of a space whose points correspond to similarity classes of triangles is

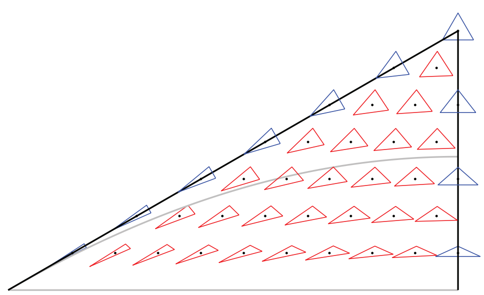
$$M = \{(a, b, c) \in \mathbb{R}^3 \mid a \leq b \leq c\,,\ a + b + c = 2\,,\ c < a + b\}\,. \tag{6}$$

The space M is a subspace of $\mathbb{R}^3$. Every point (a, b, c) in M defines the triangle whose sides have lengths a, b and c. Every triangle is similar to one of these. Different points in M give rise to non-similar triangles. This is a pictorial representation of M:



The space M contains two boundary lines of isosceles triangles, it does not contain its third boundary line, where the triangles degenerate. The curve defined by $a^2 + b^2 = c^2$ is indicated, which is the locus of right triangles. Above this curve are the triangles with three acute angles, below this curve are the triangles with one obtuse angle. (The shading corresponds to the size of the angle opposing the side c.)

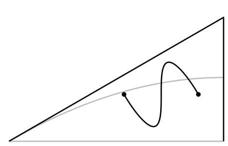
In the following sketch of M we have marked a few representative points, and displayed the corresponding triangles:



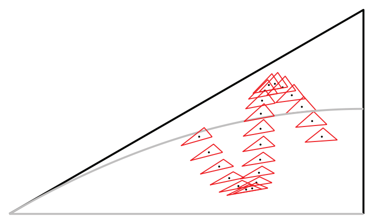
(7)

The triangles are displayed with c as base, b as left edge and a as right edge. Isosceles triangles are highlighted.

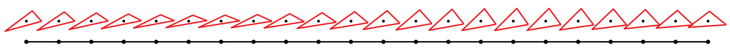
This is already quite a satisfying picture: it gives us an overview of all triangles up to similarity. But now we notice that it does much more: it also describes *continuous families* of triangles. A path in M



gives rise to family of triangles:



This is a family parametrized by an interval:



(8)

The shape of the triangles in the family is determined completely by the path in M.

Pulling back families

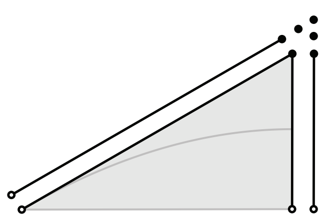
This is an example of *pullback of families*. It is a basic property of families of mathematical objects that they can be pulled back, via any map to the parameter space.

The space M itself parametrizes a continuous family of triangles, which is sketched in (7), and which we shall denote by $\mathscr{M}$. Our path in M is a continuous map $\gamma : [0,1] \to M$. Via the path γ, we pull back the family $\mathscr{M}/M$, to obtain a family parametrized by $[0,1]$, which is denoted by $\gamma^*\mathscr{M}$. This family is defined in such a way that

$$(\gamma^*\mathscr{M})_t = \mathscr{M}_{\gamma(t)}\,, \qquad \text{for all } t \in [0,1].$$

The family $\gamma^*\mathscr{M}/[0,1]$ is displayed in (8).

1.13 Exercise. The symmetry groupoid Γ of the family $\mathscr{M}$ is a family of groups, as no distinct family members are isomorphic. As a topological space, Γ looks like this:

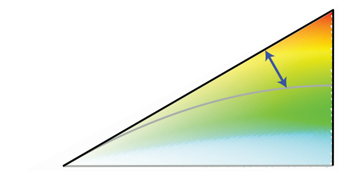


Over the isosceles but not equilateral locus, the fibres of Γ are groups with 2 elements, over the equilateral locus the fibre of Γ is isomorphic to S_3. $\square$

The moduli map of a family

Conversely, every continuous family of triangles $\mathscr{F}/T$, parametrized by a space T, gives rise to a continuous map $T \to M$, the **moduli map** of $\mathscr{F}$. The moduli map takes the point $t \in T$ to the point $(a, b, c) \in M$, where the triangle $\mathscr{F}_t$ has side lengths $a \le b \le c$.

1.14 Example. For example, the moduli map of the family $\mathscr{F}$ of (4) is the path in M, which starts at the 3:4:5 triangle on the curve of right triangles, follows the line orthogonal to the $b = c$ isosceles edge (this is the line $a = \frac{1}{2}$) until it reaches this edge, and then retraces itself until it comes back to the curve of right triangles.



1.15 Example. The moduli map of the family $\gamma^*\mathscr{M}$ over $[0, 1]$, displayed in (8), is, of course, the path $\gamma : [0, 1] \to M$, which gave rise to it.

1.16 Exercise. Prove that the moduli map of a family given by T'/T and $a : T' \to \mathbb{R}_{>0}$ is continuous. Continuity of the moduli map $T \to M$ is a local property, so you can assume that the cover $T' \to T$ is trivial, and the triangle is given by 3 continuous functions f, g, h on T, representing the lengths of the sides of the triangles in your family. Then $f \le g \le h$ defines a closed subspace of T on which the moduli map is identified with (f, g, h), and is therefore continuous. Other conditions, such as $g \le f \le h$, give rise to other moduli maps, such as (g, f, h), which are also continuous. The pasting lemma finishes the proof. $\square$

Fine moduli spaces

One may be tempted to think therefore, that M does not just classify triangles, but also that M classifies *families* of triangles, in the sense that families parametrized by T are in one-to-one correspondence, via their moduli map, with continuous maps $T \to M$. This leads to the following definition.

1.17 Definition. A **fine moduli space** is a space M, such that

(i) the points of M are in one-to-one correspondence with isomorphism classes of the objects we are studying,

(ii) (technical condition) for every family $\mathscr{F}/T$, the associated moduli map $T \to M$ (which maps the point $t \in T$ to the isomorphism class of the family member $\mathscr{F}_t$) is continuous,

(iii) every continuous map from a space T to M is the moduli map of some family parametrized by T (equivalently, M parametrizes a family $\mathscr{M}$, whose moduli map is the identity id_M),

(iv) if two families have the same moduli map, they are isomorphic families.

Is our space M from (6) a fine moduli space for triangles up to similarity? We have constructed M so that (i) would be satisfied. The technical condition (ii) can be checked if one agrees on a formal mathematical definition of continuous family of triangles (see Exercise 1.16).

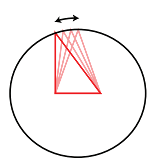
Above, we saw how the continuous map $\gamma : [0,1] \to M$ gives rise to a family over $[0,1]$, whose moduli map is γ. We can do the same thing for any map $f : T \to M$, from an arbitrary space T to M. We can use f to pull back the family $\mathscr{M}$ to a family $f^*\mathscr{M}$, and this family has moduli map f. Thus Condition (iii) is satisfied.

Note that applying Condition (iii) to $T = M$ and the identity map id_M, says that the space M parametrizes a family $\mathscr{M}$, such that $\mathscr{M}_m$ represents the isomorphism class corresponding to m by Condition (i), for all $m \in M$. Let us call such a family a **modular family**. Our family of triangles (7) is a modular family.

What about Condition (iv)?

Consider, again, the family $\mathscr{F}/I$, where I is the interval $[-\frac{1}{4}, \frac{1}{4}]$, from (4).

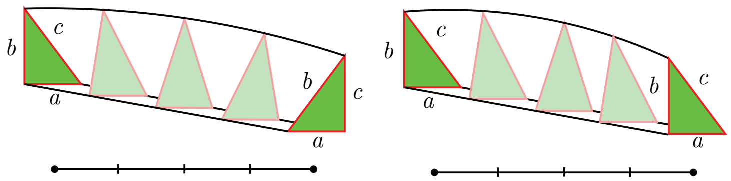
We will now construct another family $\mathscr{G}/I$, parametrized by the same interval I. The family $\mathscr{G}$ is equal to the family $\mathscr{F}$ over the first half of the interval $[-\frac{1}{4}, 0]$, but then goes back to the starting point, rather than continuing on beyond the isosceles triangle in the middle:



For a value $t \in [0, \frac{1}{4}]$, the corresponding triangle is subtended by the string when the pen point has x-coordinate $-t$.

Now note that the family $\mathscr{G}$ has the same moduli map as the family $\mathscr{F}$, see Example 1.14.

On the other hand, the two families $\mathscr{F}$ and $\mathscr{G}$ are not isomorphic. Here is another representations of the two families, with $\mathscr{F}$ on the left and $\mathscr{G}$ on the right:



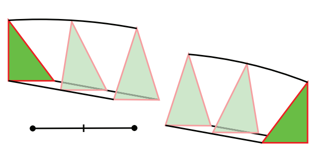
(9)

If we label the sides of the initial triangles with a, b and c, in such a way that $a < b < c$, and then label all the following triangles in the respective families in a *continuous* way, then the two end triangles, which are congruent, are labelled differently: one in such a way that $a < b < c$, the other such that $a < c < b$. This shows that the two families are essentially different. We can look at it another way: if you try to construct a continuous isomorphism between the two families, in the first half of the interval, $[-\frac{1}{4}, 0]$, this isomorphism would simply translate a triangle from $\mathscr{F}$ over to the corresponding triangle in $\mathscr{G}$. In the second half of the interval, $[0, \frac{1}{4}]$, an isomorphism would have to translate the triangle from $\mathscr{F}$ over, and then reflect it, to map it onto the corresponding triangle in $\mathscr{G}$. In the middle, at the isosceles triangle, we get two contradicting requirements: continuity from the left requires us to translate the isosceles triangle over, continuity from the right requires us to reflect this isosceles triangle across. There cannot be a continuous isomorphism between the two families $\mathscr{F}$ and $\mathscr{G}$.

We conclude that M is not a fine moduli space, because two non-isomorphic families of triangles have the same moduli map. In fact, there does not exist any fine moduli space of triangles. Our two families are pointwise the same, so define the same moduli map to *any* potential fine moduli space.

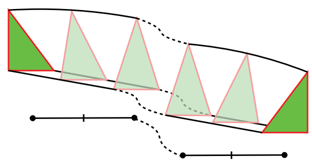
The family $\mathscr{G}$ is obtained by pulling back the modular family $\mathscr{M}/M$ via the common moduli map of $\mathscr{F}$ and $\mathscr{G}$. The family $\mathscr{F}$ cannot be obtained by pulling back $\mathscr{M}$: any family pulled back from M can be labelled compatibly and continuously with a, b and c, such that $a < b < c$ everywhere, because $\mathscr{M}$ has this property.

Of course, it is easy to see who the culprit is: it is the isosceles triangle, which has a non-trivial symmetry, which allows us to cut the family $\mathscr{F}$ in the middle,

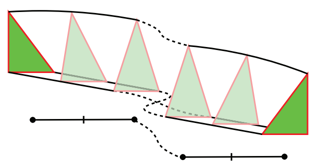


and reassemble it in two different ways. Just gluing it back together we get

$\mathscr{F}$ back:



But flipping the isosceles triangle in the middle while gluing



gives us the family $\mathscr{G}$.

Coarse moduli spaces

As property (iv) is violated, M is not a fine moduli space of triangles, but it does satisfy the following definition.

1.18 Definition. A **coarse moduli space** is a space M, such that the first two conditions of Definition 1.17 are satisfied, and moreover, (technical condition) M carries the finest topology making Condition (ii) true.

1.19 Remark. There is essentially only *one* coarse moduli space for any mathematical notion. It can be constructed as follows: take the set of isomorphism classes of objects under consideration as points of M, and then endow M with the finest topology such that all moduli maps of all continuous families are continuous. This gives a coarse moduli space. Any other coarse moduli space is necessarily homeomorphic to this one. Thus it is customary to speak of *the* coarse moduli space. (This requires the class of objects to be small enough for isomorphism classes to form a set.)

1.20 Exercise. Any space satisfying the first three conditions of Definition 1.17 is a coarse moduli space. (The converse is not true, see Section 1.7 for an example.) □

1.21 Exercise. A subset $U \subset M$ of the coarse moduli space is open if and only if it defines an *open condition* on continuous families. This means that for every continuous family $\mathscr{F}/T$, the set of $t \in T$, such that $[\mathscr{F}_t] \in U$ is open in T.

1.22 Remark. It is possible for a coarse moduli space to carry several non-isomorphic modular families. (For an example, see Section 1.5.)

1.23 Remark. The existence of a fine moduli space implies the existence of pullbacks of families (they correspond to composition of maps).

1.24 Remark. If M is a fine moduli space, and $\mathscr{M}$ is a modular family, then every continuous family $\mathscr{F}/T$ is the pullback of $\mathscr{M}$ via its moduli map, and therefore $\mathscr{M}$ is called a **universal family**. Any other modular family is isomorphic to the pullback of $\mathscr{M}$ via the identity, in other words isomorphic to $\mathscr{M}$. So there is essentially only one universal family, and one speaks of *the* universal family.

To conclude: the coarse moduli space of triangles up to similarity is isomorphic to the space M from (6), it admits a modular family, but no universal family, there is no fine moduli space of triangles. Moreover, the coarse moduli space of triangles is a 2-dimensional manifold with boundary.

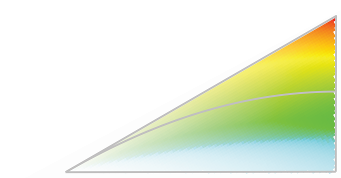
Let us consider a few related classification problems.

1.4 Scalene triangles

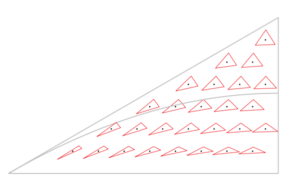
Here we provide an example of a fine moduli space.

Recall that a triangle is *scalene*, if all three sides have different length. Scalene triangles are completely asymmetric: they each have a trivial symmetry group. There exists a fine moduli space for scalene triangles. In fact, remove the boundary from M, to obtain

$$M' = \{(a, b, c) \in \mathbb{R}^3 \mid a < b < c,\ a + b + c = 2,\ c < a + b\}\,.$$



Let us denote the family parametrized by M' by $\mathscr{M}'/M'$.



We claim that $\mathscr{M}'$ is a universal family for scalene triangles.

To see this, we have to show that every continuous family of scalene triangles $\mathscr{F}/T$ is isomorphic to the pullback of $\mathscr{M}'$ via the moduli map of $\mathscr{F}$.

The key observation is that in a scalene triangle there is never any ambiguity as to which side is the shortest, and which side is the longest. So in a continuous family of scalene triangles we can unambiguously label the sides with a, b and c, where the shortest side is labelled a, and the longest is labelled c. (So it is impossible to construct families of scalene triangles such as $\mathscr{F}$ from (4), where the longest side jumps.)

In the family $\mathscr{M}'$, the sides are already labelled in this way. Therefore, in the pullback $f^*\mathscr{M}'$, where $f : T \to M'$ is the moduli map of $\mathscr{F}$, the sides are again labelled in this way.

We can now use this labelling of the sides to define an isomorphism

$$\mathscr{F} \xrightarrow{\sim} f^*\mathscr{M}' , \tag{10}$$

by sending the side labelled a to the side labelled a, the side labelled b to the side labelled b, and the side labelled c to the side labelled c. Note that without the canonical labelling of the sides of $\mathscr{F}$ it is impossible to define (10).

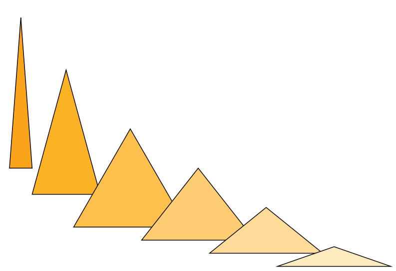
1.25 Exercise. This defines an isomorphism of families, because for every $t \in T$, the lengths of the sides of $\mathscr{F}_t$ are given by the triple of real numbers $f(t)$ (by definition of the moduli map f), and this triple of real numbers gives the lengths of the sides of $(f^*\mathscr{M}')_t = \mathscr{M}'_{f(t)}$, by the definition of the modular family $\mathscr{M}'$. Prove that (10) is a continuous isomorphism of families (i.e., an isomorphism of continuous families). $\square$

We conclude that scalene triangles admit a fine moduli space (and a universal family), which is a 2-dimensional manifold.

1.5 Isosceles triangles

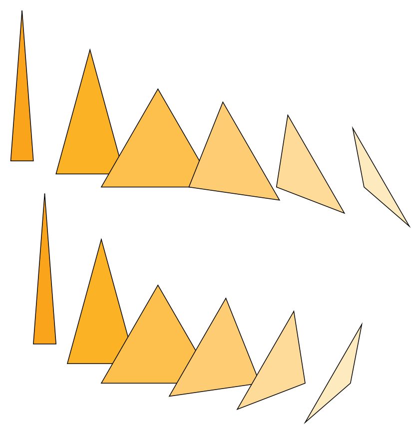
Next we shall see a coarse moduli space with several modular families.

Let us consider all isosceles triangles. These are classified, up to similarity, by the angle which subtends the two equal sides. This angle can take any value between 0 and π. If $\mathscr{I}/T$ is a continuous family of isosceles triangles, then this angle defines a continuous function $T \to (0, \pi)$, the moduli map of $\mathscr{I}$. Of course, the interval $(0, \pi)$ parametrizes a continuous family of triangles



which is a modular family, i.e., the triangle over the point $\gamma \in (0, \pi)$ has angle γ subtending the two equal sides. Therefore, we see that the interval $(0, \pi)$ is a coarse moduli space for isosceles triangles.

But note that there are two further continuous families of isosceles triangles parametrized by $(0, \pi)$, which are modular:

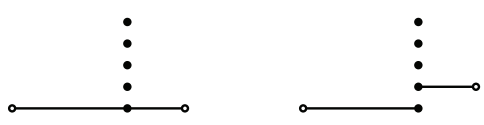


In fact, these latter two families are isomorphic, via reflection across the vertical. But they are not isomorphic to the first family, above. For the case of the first family, the isosceles angles all fit together continuously (they are always at the top). For the latter two families, the isosceles angle changes position at the equilateral triangle. This is essentially different behaviour.

The two essentially different families are competing for the title of 'universal family of isosceles triangles'. Of course, only one family can carry this title, so there is no universal family. Each of these two families describes one possible way a one-parameter family of isosceles triangles can 'pass through' the equilateral triangle.

We conclude that the coarse moduli space of isosceles triangles is a 1-dimensional connected manifold, and it admits two non-isomorphic modular families.

1.26 Exercise. The symmetry groupoids of the two modular families look like this:



□

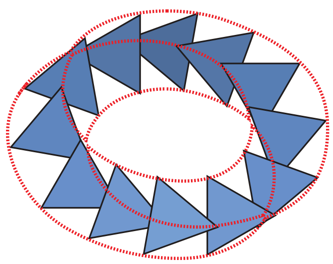
1.6 Equilateral triangles

Let us restrict attention entirely to equilateral triangles. These are, of course, all similar to each other, which seems to indicate that the classification should be quite trivial.

In fact, the one-point space $*$ is, of course, a coarse moduli space for equilateral triangles up to similarity. Let us pick an equilateral triangle, and call it δ. Then the single triangle δ is a continuous family of equilateral triangles parametrized by $*$, and it is a modular family.

Every family pulled back from $\delta/*$ is *trivial*, or *constant*.

On the other hand, there are families of equilateral triangles which are not at all trivial. For example, the following family is parametrized by the circle. It was obtained by taking the trivial family over a closed interval, and gluing the first and last triangle with a $\frac{2\pi}{3}$ twist.



(11)

The dotted line indicates an attempt to continuously label one (and only one) vertex in each triangle with A. This is impossible. Instead, the dotted line defines a degree 3 cyclic cover of the parameter circle.

So $\delta/*$ is not a universal family, and there exists no fine moduli space for equilateral triangles.

The symmetry groupoid of $\delta/*$ is the group S_3, the symmetric group on three letters a, b, c.

1.27 Exercise. In fact, the vertices of any continuous family of equilateral triangles over a topological space T form a degree 3 covering space of T, and, conversely, every degree 3 covering space of T defines a continuous family of equilateral triangles. $\square$

1.7 Oriented triangles

We get an interesting variation on our moduli problem by considering oriented triangles. This means that similarity transformations between triangles are only rotations, translations, and scalings, but not reflections. In this context, the equilateral triangle is the only triangle with non-trivial symmetries, all other isosceles triangles have lost their symmetry. The symmetry group of the oriented equilateral triangle is the cyclic group with 3 elements.

For now, let us agree that an *oriented triangle* is a triangle with a cyclic ordering of its the edges (or its vertices). Any isomorphism of triangles has to preserve this cyclic ordering. (There are two ways to cyclically order the edges of a triangle.)

All scalene triangles have two oriented incarnations, which are transformed into each other by a reflection. For example, the two incarnations of the 3:4:5 right triangle are the following:

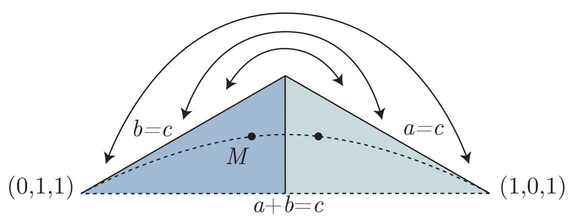
For the one on the left, the cyclic ordering of the edges according to ascending length is counterclockwise, for the one on the right it is clockwise. There is no oriented similarity transformation of the plane which makes these two triangles equal. On the other hand, isosceles triangles have only one oriented version: if two isosceles triangles are similar, they can be made equal by an oriented similarity transformation, not involving any reflections.

The coarse moduli space for oriented triangles is therefore 'twice as big' as the one for unoriented triangles, which we called M. We can construct it by starting with the space

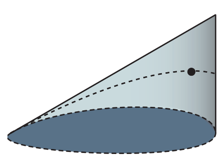
$$\widetilde{M}^{\text{pre}} = \{(a, b, c) \in \mathbb{R}^3 \mid a \leq c,\, b \leq c,\, a + b + c = 2,\, c < a + b\},$$

and gluing together the two boundary lines $b = c$ and $a = c$ of isosceles triangles, as indicated in the sketch (with the locations of the 3:4:5 and the 4:3:5 triangle marked):

$\widetilde{M}^{\text{pre}}$:



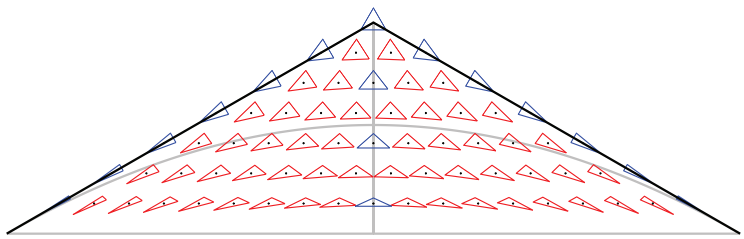
Let us call the resulting space $\widetilde{M}$. Thus $\widetilde{M}$ can be pictured as the surface of a cone with solid angle $\frac{2\pi}{3}$ steradians at its vertex.

$\widetilde{M}$:  (12)

Every oriented triangle corresponds to a unique point in $\widetilde{M}$: given an oriented triangle, label its sides by a, b, c, in such a way that a, b and c

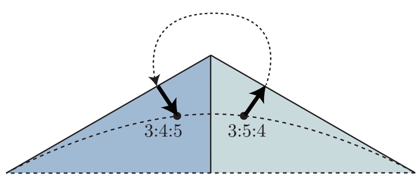
appear in alphabetical order, when going around the triangle counterclockwise, and such that the longest side is labelled with c. Then rescale the triangle until it has perimeter 2. The resulting side lengths define a unique point in $\widetilde{M}$.

Here is the family of triangles parametrized by $\widetilde{M}^{\mathrm{pre}}$, before gluing:



(13)

We can consider the family $\mathscr{F}$ of (4), obtained by drawing an ellipse as in (3), also as a family of oriented triangles. It is a family starting at the 3:5:4 right triangle and ending at the 3:4:5 right triangle. The corresponding path in $\widetilde{M}^{\mathrm{pre}}$ looks like this:



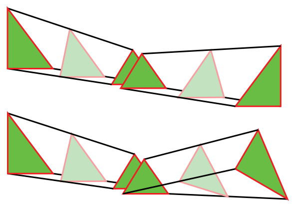
We want the oriented family $\mathscr{F}$ to define a continuous moduli map $[-\frac{1}{4}, \frac{1}{4}] \to \widetilde{M}$. That is why we have to glue the two boundaries of $\widetilde{M}^{\mathrm{pre}}$ together. We cannot simply remove one of the edges $b = c$ or $a = c$, in defining $\widetilde{M}$, because that would make the moduli map of the oriented family $\mathscr{F}$ discontinuous at the 'break point', seen in the sketch.

1.28 Exercise. Formally, define a continuous family of oriented triangles parametrized by the topological space T to consist of a *cyclic* degree 3 covering $T' \to T$, and a continuous map $a : T' \to T$, satisfying the conditions of Exercise 1.11. (A cyclic cover of degree 3 is a covering space $T' \to T$, together with a given deck transformation $\sigma : T' \to T'$, which induces a degree 3 permutation in each fibre of $T' \to T$.) Isomorphisms of families of oriented triangles are defined as in Exercise 1.11, with the additional requirement that the isomorphism of covering spaces has to commute with the respective deck transformations σ.

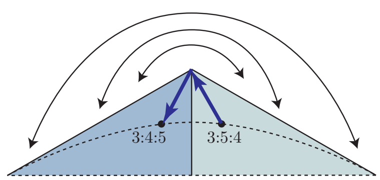
Prove that $\widetilde{M}$ is a coarse moduli space of oriented triangles. $\square$

Does there exist a modular family over $\widetilde{M}$? Can different families have the same moduli map?

The answer to the second question is 'yes'. Non-isomorphic families with the same moduli map can be constructed the same way as before: take a family $\mathscr{F}$, parametrized by $[-\epsilon, \epsilon]$, such that $\mathscr{F}_t$ is non-equilateral for $t \neq 0$, and equilateral for $t = 0$. Then create a new family $\mathscr{F}'$, by gluing $\mathscr{F}|_{[-\epsilon,0]}$ and $\mathscr{F}|_{[0,\epsilon]}$, together at $t = 0$ by using a non-trivial symmetry of the equilateral triangle.



In this sketch, the triangles in the second half of the lower family are obtained by rotating the triangles in the second half of the upper family by $\frac{2\pi}{3}$ clockwise. Then the families are glued together along the central equilateral triangles. The two families have the same moduli map, which is a path in $\widetilde{M}$ connecting the 4:3:5 right triangle with the 3:4:5 right triangle, passing through the 'vertex' of $\widetilde{M}$, at the equilateral triangle:

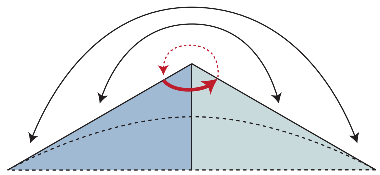


Yet, the two families are essentially different: in one of them, the longest side 'jumps', in the other it does not. Any isomorphism between the two families would consist of a family of translations for $t \in [-\epsilon, 0]$, and a family of translation-rotations for $t \in [0, \epsilon]$. This family of isomorphisms is not continuous at $t = 0$, and so our two families are different as continuous families (as discrete families, they would, of course, be isomorphic, because they are isomorphic pointwise at each parameter value).

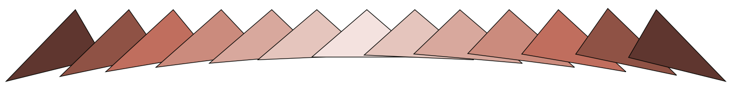
One reason to introduce oriented triangles at this point is that they provide an example where there exists *no* modular family over the coarse moduli space.

To see this, we proceed by contradiction. Suppose that there exists a continuous modular family of oriented triangles $\widetilde{\mathscr{M}}/\widetilde{M}$. Then, very close to the vertex point of $\widetilde{M}$ (i.e., in some, maybe very small, open neighbourhood of this point), we can consistently label the vertices of the family $\widetilde{\mathscr{M}}$ in some way. Hence, when we restrict $\widetilde{\mathscr{M}}$ to small enough loops around the vertex of $\widetilde{M}$, these restricted families can also be consistently labelled.

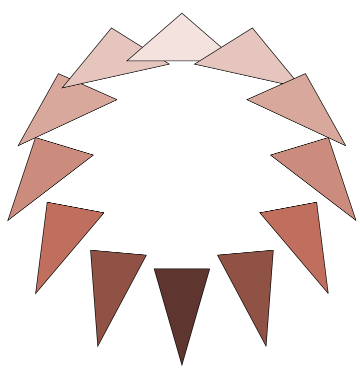
Consider such a loop around the vertex of $\widetilde{M}$:



Because this loop avoids all oriented triangles with symmetries, it corresponds to a unique continuous family parametrized by the circle S^1. The family parametrized by the open circle in $\widetilde{M}^{\mathrm{pre}}$ would look something like this:



Of course, if the path in $\widetilde{M}^{\mathrm{pre}}$ is very close to the vertex, the triangles in the family will be very close to equilateral. For clarity, we have depicted the family corresponding to a path further form the vertex. Notice how following along a path in (13) from the left equilateral edge to the right equilateral edge gives rise to a family as displayed here. The induced loop in $\widetilde{M}$ parametrizes the family obtained by gluing together the two triangles at the end (here implemented by bending the two ends downward):



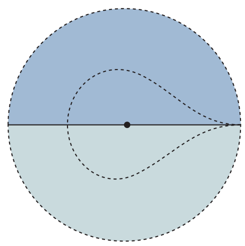
(14)

Examining this family, we see that it does not admit an unambiguous labelling. Rather, any attempt at such a labelling will run up against a cyclic degree 3 cover of the parameter circle, just like for (11).

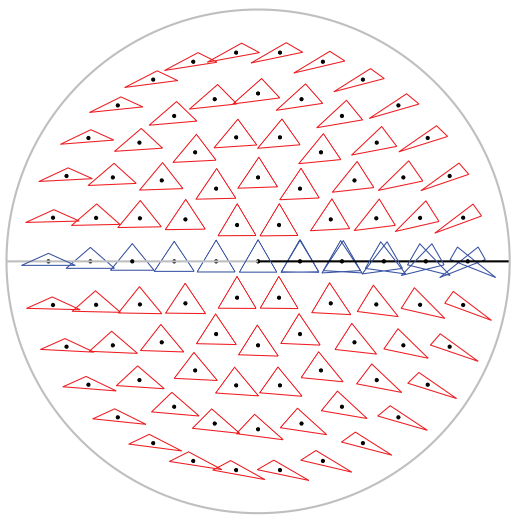
No matter how close the loop in $\widetilde{M}$ is to the vertex, the corresponding family will always have this feature. So there is no way that any putative modular family $\widetilde{\mathscr{M}}$ could have a consistent vertex labelling, even in a tiny neighbourhood of the equilateral vertex. So $\widetilde{\mathscr{M}}$ cannot exist.

Non-equilateral oriented triangles

To look at $\widetilde{M}$ another way, we can flatten out the cone, until $\widetilde{M}$ becomes a disc. Alternatively, we can bend the two isosceles edges of $\widetilde{M}^{\text{pre}}$ around, shortening them, and then glue them to get this view of $\widetilde{M}$:

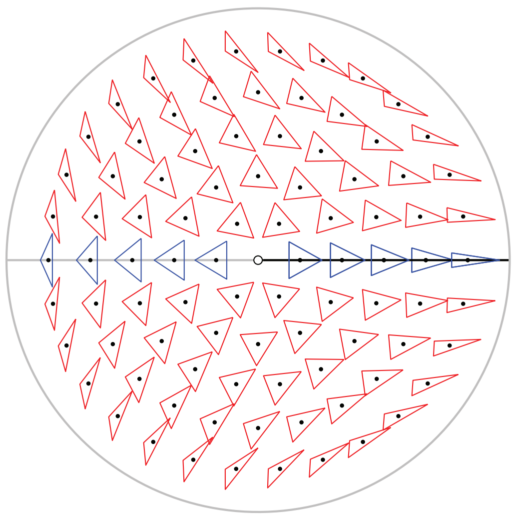


If we follow along with the family parametrized by $\widetilde{M}^{\text{pre}}$, see (13), we get



and we see that it is impossible to glue this family together in a consistent way. The equilateral triangle in the middle forces the acute isosceles triangles into this incompatible position.

If we remove the central point and the equilateral triangle, we can rotate all the remaining triangles a little, and then we can glue successfully:



$$(15)$$

There is no way to put the equilateral triangle back into the centre of this picture, in a way compatible with the neighbouring triangles, because the neighbouring triangles exhibit rotation behaviour when going around the centre in small loops, as we saw above (14).

Notice that (15) is the universal family of non-equilateral oriented triangles.

Recall that the universal family (and therefore *every* family) of scalene unoriented triangles admits a global consistent labelling of vertices. The universal family of non-equilateral oriented triangles does not admit a global labelling, as there are families of such triangles which contain twists. Notice how even after removing the symmetric object, the universal family still retains some properties of this object: the universal family contains a twist by the symmetry group of the central object.

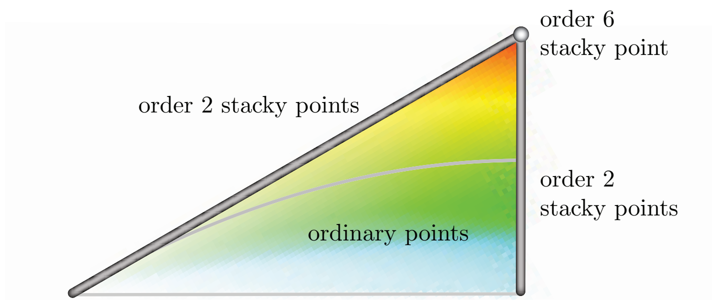
1.8 Stacks

Stacks are mathematical constructs invented to solve the various problems we encountered when studying moduli problems. Stacks are more general than spaces, but every space is a stack.

There exists no fine moduli space for triangles, but there does exist a fine moduli *stack* of triangles (although in the stack context the word 'fine' is usually omitted). The moduli stack of triangles, let us call it $\mathfrak{M}$, parametrizes a universal family $\mathscr{U}/\mathfrak{M}$ of triangles. Every family of triangles $\mathscr{F}/T$ is isomorphic to the pullback of $\mathscr{U}/\mathfrak{M}$ via a continuous map $T \to \mathfrak{M}$, which is *essentially* unique. The word 'essential' is key. An important difference between stacks and spaces is that for a stack such as $\mathfrak{M}$, the continuous maps $T \to \mathfrak{M}$ do not form a set, but rather a groupoid, and, in fact, the groupoid of maps $T \to \mathfrak{M}$ is equivalent to the groupoid of families over T (for all T, in a way compatible with pullbacks of families).

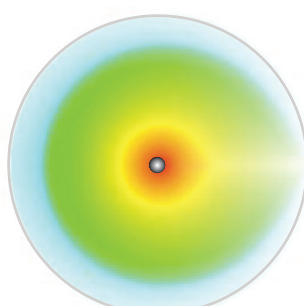
Over the locus of scalene triangles, there is no difference between the coarse moduli space M and the fine moduli stack $\mathfrak{M}$, because over the scalene locus, M is a fine moduli space. But the isosceles locus consists of so-called 'stacky points' of $\mathfrak{M}$. There are two ways a path can pass through a stacky point representing an isosceles triangle, and six ways it can pass through the stacky point representing the equilateral triangle.

$\mathfrak{M}$:



Let us call the stack of oriented triangles $\widetilde{\mathfrak{M}}$. It has one stacky point of order 3 in the centre:

$\widetilde{\mathfrak{M}}$:



(16)

The mathematical definition of the notion of stack is a stroke of genius, or a cheap copout, depending on your point of view: one simply declares the problem to be its own solution!

The problem we had set ourselves was to describe all continuous families of triangles. We saw that this problem would be solved quite nicely by a universal family, if there was one. But there isn't one. So instead of trying to single out one family to rule all others, we consider *all* families over *all* parameter spaces to *be* the moduli stack of triangles.

Thus, the notions of *moduli problem* and *stack* become synonymous.

The challenge is then to develop techniques for dealing with such a stack as a geometric object, as if it were a space. For this to be successful, we will need the existence of *versal families*. These fulfill a dual purpose: they allow us to do geometry with the moduli stack, and we can describe all families explicitly in terms of a versal family.

There are two difficulties with this:

(i) the description of the stack of all triangles in terms of a versal family is more complicated than the description in terms of a universal family,

(ii) there are many versal families, and so we have to also study how different versal families relate to each other.

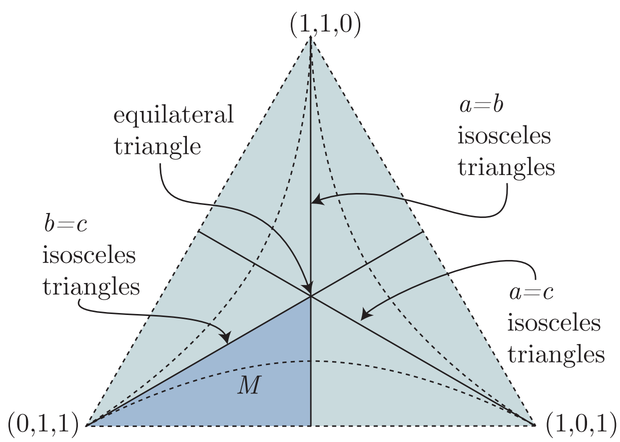
But these problems cannot be avoided if the objects we are studying are symmetric.

1.9 Versal families

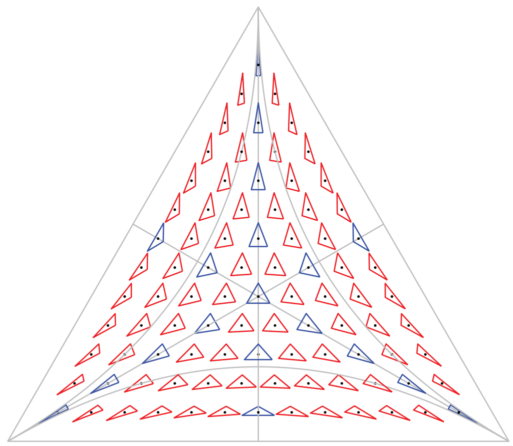
Consider, again, the family $\mathscr{F}/I$ from (4). The behaviour of this family near the isosceles triangle is not modelled anywhere by the modular family $\mathscr{M}$ over the course moduli space M. So to describe all possible families (even locally) we need to enlarge M. Here is a better family. The parameter space is

$$N = \{(a,b,c) \in \mathbb{R}^3 \mid a+b+c = 2,\ a,b,c < 1\}.$$

N :



The members of the family $\mathscr{N}/N$ have their vertices labelled with the letters A, B, and C, in a consistent way, and the side lengths of $\mathscr{N}_{(a,b,c)}$, for $(a,b,c) \in N$, are such that the side opposite the vertex A has length a, the side opposite the vertex B has length b, and the side opposite C has length c. (It is easy to check that the conditions on a, b and c imply the three triangle inequalities.)

$\mathscr{N}$:  (17)

Note that the most symmetric of all triangles, the equilateral one, appears only once in $\mathscr{N}$, the isosceles triangles (except the equilateral one) appear three times each, and the scalene triangles appear six times each. In fact, the number of times a given triangle appears is inversely proportional to the number of its symmetries.

The key feature of the family $\mathscr{N}/N$ is that it models all possible local behaviours of continuous families of triangles, as we shall see below.

Let us examine the symmetry groupoid of the family $\mathscr{N}/N$. Notice that (17) looks very much like a more elaborate (in fact continuous) version of Example 1.8. The symmetry group of Figure (17) is the dihedral group with 6 elements, i.e., S_3. Every element $\sigma \in S_3$ defines a transformation $\sigma : N \to N$, as well as, for every $n \in N$, an isomorphism $\mathscr{N}_n \to \mathscr{N}_{\sigma(n)}$. (In fact, these isomorphisms combine into an isomorphism of families $\mathscr{N} \to \sigma^*\mathscr{N}$.) Every similarity of some triangle $\mathscr{N}_n$ with another triangle $\mathscr{N}_{n'}$ in (17) comes about in this way. Therefore, the symmetry groupoid of the family $\mathscr{N}/N$ is equal to the transformation groupoid $N \times S_3$, given by the action of S_3 on the parameter space N. (The geometric action via rotations and reflections on Figure (17) induces the permutation action on the components of the points of N.)

1.29 Exercise. Define the *canonical topology* on the symmetry groupoid $\Gamma \rightrightarrows T$ of a continuous family $\mathscr{F}$, parametrized by the space T, to be the finest topology on Γ with the following property: for any space S, and any triple (f, ϕ, g), where $f : S \to T$ and $g : S \to T$ are continuos maps, and $\phi : f^*\mathscr{F} \to g^*\mathscr{F}$ is a continuous isomorphism of families, the induced map

$S \to \Gamma$, is continuous. (The induced map $S \to \Gamma$ maps $s \in S$ to the element $\phi_s : \mathscr{F}_{f(s)} \to \mathscr{F}_{g(s)}$ in Γ.)

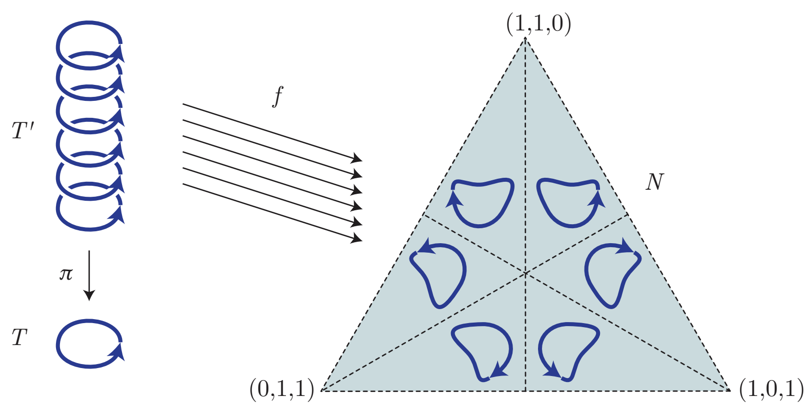
Define the *tautological isomorphism* over Γ to be the isomorphism $\phi : s^*\mathscr{F} \to t^*\mathscr{F}$, such that for every $\gamma \in \Gamma$, the isomorphism $\phi_\gamma : \mathscr{F}_{s(\gamma)} \to \mathscr{F}_{t(\gamma)}$ is the isomorphism given by γ itself.

Prove that the canonical topology on the symmetry groupoid of the family $\mathscr{N}/N$ is the product topology on $N \times S_3$. Prove that the tautological isomorphism over Γ is a continuous isomorphism of families (i.e., an isomorphism of continuous families). $\square$

1.30 Exercise. Prove that N is a fine moduli space of *labelled triangles.* A labelled triangle is a triangle together with a labelling of the edges, with the labels a, b, c. A scalene triangle such as the 3:4:5 right triangle has six different labellings. An isosceles but not equilateral triangle has three essentially different labellings, but the equilateral triangle has only one. $\square$

Generalized moduli maps

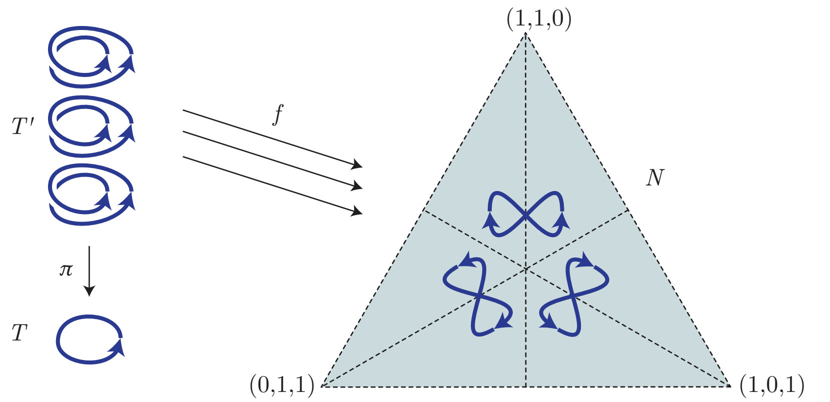
As individual triangles appear multiple times in $\mathscr{N}$, the moduli map of a family of triangles is multi-valued. For example, a loop of scalene triangles will have a 6-valued moduli map, which may look something like this:



The group S_3 acts on the six image loops in N, and it therefore also acts on T', which is, in this case, a disjoint union of 6 copies of the parameter space $T = S^1$. Each component of T' corresponds to one way of labelling the triangles in the family.

More interesting is the moduli map of a Moebius family such as $\widetilde{\mathscr{F}}/S^1$

from (5):



$$(18)$$

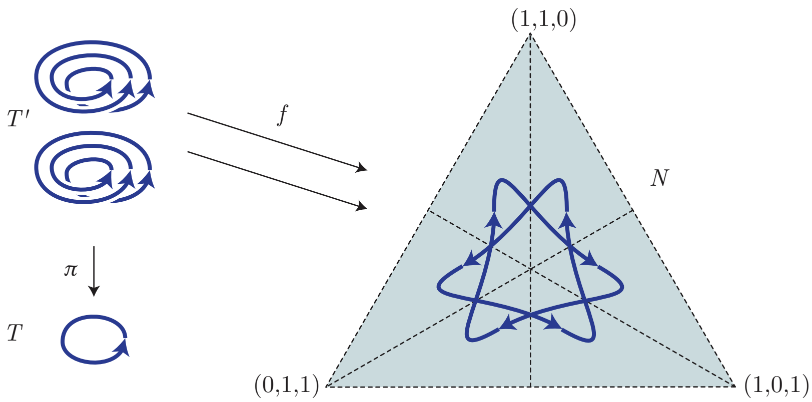
For this figure, we have deformed the family a little bit: in the part before the equilateral triangle, we have made the triangles a little more acute (lengthened the string), afterwards a little more obtuse (shortened the string). This is to avoid the moduli map collapsing to three lines in N, rather than three figure eights.

The image of the moduli map in N consist of six paths, which are joined head to tail in pairs. Technically, the 6-valued moduli map $T \to N$ consists of a degree 6 covering $\pi : T' \to T$ and a continuous map $f : T' \to N$. The space T' can be viewed as the space of all labellings of the triangles in our family: for $t \in T$, the preimage $\pi^{-1}(t)$ consists of the six different ways the vertices of the triangle $\mathscr{F}_t$ can be labelled with the letters A, B, and C. The map $f : T' \to N$, then maps a labelled triangle to the triple of lengths of its sides (because the vertices are labelled, the side lengths form an ordered triple, which is a well-defined point in N).

The covering $\pi : T' \to T$ decomposes into three components: one component consists of those labellings where A is opposite the shortest side (recall that only the shortest side is globally consistent in $\mathscr{F}/T$, the two others swap). Another component consists of labellings where B is opposite the shortest side, and the last components consists of labellings where C is opposite the shortest side. In this way, the three components of T' are labelled with A, B, and C, too.

An important part of the structure of the moduli map $(T'/T, f)$ is the action of the group S_3 on T' and N, and the fact that f respects these actions (f is S_3-*equivariant*). On T' the group S_3 acts by changing the labelling. For example, the permutation which transposes A and B swaps the two components of T' called A and B, and induces the branch swap on the component labelled C. The same permutation acts on N by swapping a and b (it is the reflection across the vertical axis), and exactly mirrors the action on T'.

As another example, consider a family with a $\frac{2\pi}{3}$-twist, such as (14). Its moduli map $(T'/T, f)$ might look something like this:



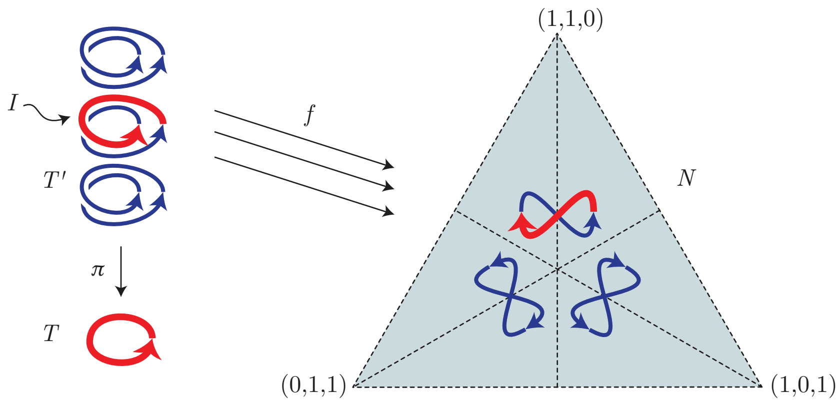
This procedure is general: every continuous family of triangles $\mathscr{F}$ over some parameter space T gives rise to an S_3-covering $T' \to T$, namely the space of labellings of $\mathscr{F}$, and an S_3-equivariant map $f : T' \to N$, given by the triple of the lengths of the labelled sides. The structure $(T'/T, f)$ replaces the concept of moduli map.

1.31 Exercise. Using the formal definition of continuous family of triangles from Exercise 1.11, we construct the generalized moduli map as follows. Let $T' \to T$ be a degree 3 covering map *without* structure group, and $a : T' \to \mathbb{R}_{>0}$ a continuous map. Then define T'' to be the space of all maps $\ell : \{A, B, C\} \to T'$, which are bijections onto a fibre of $T' \to T$. Then $T'' \to T$ is a covering space of degree 6, endowed with a right S_3-action, hence a covering map *with* structure group. The map $f : T'' \to N$ defined by $f(\ell) = \big(a\ell(A), a\ell(B), a\ell(C)\big)$ is S_3-equivariant, and (T'', f) is the generalized moduli map of the continuous family of triangles (T', a). $\square$

Reconstructing a family from its generalized moduli map

One of the problems with the coarse moduli space M was, that we were not able to reconstruct a family from its moduli map. This problem we have now solved! Any family of triangles $\mathscr{F}/T$ is, in fact, completely determined by its generalized moduli map $(T'/T, f)$.

Consider, for example, a family $\widetilde{\mathscr{F}}/S^1$ with a Moebius twist, whose moduli map is displayed in (18). The way to obtain this family by gluing as in (5), is completely encoded by $(T'/T, f)$. To see this, notice that you can pick a closed interval $I \subset T'$, which maps down to $T = S^1$ in a one-to-one fashion, except that the two endpoints t_0, t_1 get glued together by the map π.



The two points $f(t_0)$ and $f(t_1)$ in N are related by a unique element $\sigma \in S_3$, (in this case the reflection across the vertical) and this σ defines an isomorphism between the triangles over these two points, which are $\mathscr{N}_{f(t_0)}$ and $\mathscr{N}_{f(t_1)}$. Pull back the family $\mathscr{N}$ to obtain $f^*\mathscr{N}|_I$. This family on I can be glued together to a family over $T = S^1$, using the isomorphism $\sigma : \mathscr{N}_{f(t_0)} \to \mathscr{N}_{f(t_1)}$ of the two end triangles. The family obtained in this way is isomorphic to the family $\widetilde{\mathscr{F}}$ which gave rise to the generalized moduli map $(T'/T, f)$ in the first place.

1.32 Exercise. Using the formal definition of family of triangles from Exercise 1.11, we construct a family of triangles from a generalized moduli map as follows. Let $T'' \to T$ be a covering map with structure group S_3 and $f : T'' \to N$ an S_3-equivariant continuous map. Construct the degree 3 cover *without* structure group $T' \to T$ as $T' = T'' \times_{S_3} \{A, B, C\}$. This notation means that T' is the quotient of $T'' \times \{A, B, C\}$ by the equivalence relation $(\ell\sigma, L) \sim (\ell, \sigma L)$, for all $\sigma \in S_3$, and $(\ell, L) \in T'' \times \{A, B, C\}$ (which amounts to a quotient of $T'' \times \{A, B, C\}$ by an action of S_3). Then define the continuous map $a : T' \to \mathbb{R}_{>0}$ by $a[\ell, L] = \mathrm{pr}_L\, f(\ell)$. Here $\mathrm{pr}_L : N \to \mathbb{R}_{>0}$ denotes the projection onto the L-th component, and $[\ell, L] \in T'$ is the equivalence class of $(\ell, L) \in T'' \times \{A, B, C\}$.

Prove that the procedures described here and in Exercise 1.31 are inverses of each other. □

Versal families: definition

The features of the family $\mathscr{N}/N$, which allow for generalized moduli maps to exist, and for us to reconstruct any family from its generalized moduli map are listed in the following definition:

1.33 Definition. A family $\mathscr{N}/N$ is a **versal family**, if it satisfies the following conditions:

(i) every family $\mathscr{F}/T$ is *locally* induced from $\mathscr{N}/N$ via pullback. This means that for every $t \in T$, there exists a neighbourhood U of t in T, and a continuous map $f : U \to N$, such that the restricted family $\mathscr{F}|_U$ is isomorphic to the pullback family $f^*\mathscr{N}$.

(ii) (technical condition) endowing the symmetry groupoid $\Gamma \rightrightarrows N$ of the family $\mathscr{N}/N$ with its canonical topology (see Exercise 1.29), the source and target maps $s : \Gamma \to N$ and $t : \Gamma \to N$ are continuous, and the tautological isomorphism of families $\phi : s^*\mathscr{N} \to t^*\mathscr{N}$ is a continuous isomorphism of families (of whatever objects we are studying).

Our family $\mathscr{N}$ of triangles is a versal family. To see that Condition (i) is satisfied, it is enough to remark that in a small enough neighbourhood U of any parameter value $t \in T$, the family of triangles $\mathscr{F}$ can be consistently labelled. And once a family over U is labelled, it is completely determined by the three continuous functions $a, b, c : U \to \mathbb{R}_{>0}$ giving the lengths of the three sides. (To even talk about these three functions, we need labels on the edges of the triangles.) The three functions a, b, c define a continuous map $f : U \to N$, making the family over U isomorphic to $f^*\mathscr{N}$.

Condition (ii) was checked in Exercise 1.29.

1.34 Exercise. Suppose you have a moduli problem satisfying the gluing axiom (see Exercise 1.12), and $\mathscr{N}/N$ is a versal family for this moduli problem, whose symmetry groupoid is a transformation groupoid $N \times G$, for a (discrete) group G.

Associate to a family $\mathscr{F}/T$ a generalized moduli map by endowing the set of isomorphisms

$$T' = \{(n, \phi, t) \mid n \in N,\, t \in T, \text{ and } \phi : \mathscr{N}_n \to \mathscr{F}_t \text{ is an isomorphism}\}$$

with the structure of a covering space $T' \to T$ and a G-action, and then defining a continuous G-equivariant map $f : T' \to N$ by $f(n, \phi, t) = n$.

Conversely, assume given a G-cover $T' \to T$ and a G-equivariant map $f : T \to N$, construct a family $\mathscr{F}/T$ whose generalized moduli map is (T', f), as follows: choose local sections of T'/T, i.e., choose an open covering $T = \bigcup_i U_i$, and continuous sections $s_i : U_i \to T'$, so that $\pi \circ s_i$ is equal to the inclusion $U_i \to T$. Pull back the family $\mathscr{N}$ via each $f \circ s_i$, to a family $\mathscr{F}_i$ over U_i. Over the intersections $U_{ij} = U_i \cap U_j$, you have two sections of π, namely $s_i|_{U_{ij}}$, and $s_j|_{U_{ij}}$, so there is a continuous map $\sigma_{ij} : U_{ij} \to G$, such that $s_i = s_j \sigma_{ij}$, because $T' \to T$ is a G-covering. Then σ_{ij} defines an isomorphism of restricted families $\sigma_{ij} : \mathscr{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathscr{F}_j|_{U_{ij}}$, because G acts by symmetries on the family $\mathscr{N}$. Then $(\{\mathscr{F}_i\}, \{\sigma_{ij}\})$ is gluing data for the family $\mathscr{F}/T$. $\square$

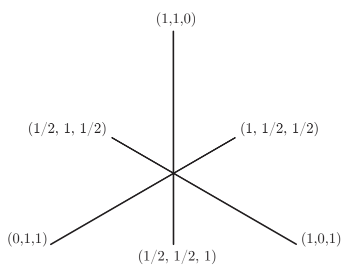
1.35 Exercise. (For families of triangles.) Prove that the coarse moduli spaces M and $\widetilde{M}$ are quotient spaces $M = N/S_3$ and $\widetilde{M} = N/\mathbb{Z}_3$.

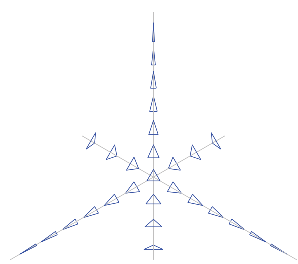
(For families of arbitrary mathematical objects.) Prove that if $\mathscr{N}/N$ is a versal family whose symmetry groupoid is the transformation groupoid $N \times G \rightrightarrows N$, for a (discrete) group G, then the quotient space N/G is a coarse moduli space. $\square$

Isosceles triangles

We exhibit versal families for the other moduli problems related to triangles, that we studied above.

Recall that the course moduli space of isosceles triangles, the interval $(0, \pi)$, admitted two distinct modular families. They both exhibit essentially different behaviour at the equilateral triangle, so neither of them is versal. To obtain a versal family of isosceles triangles, we can restrict the versal family of triangles $\mathscr{N}/N$ to the isosceles locus in N:



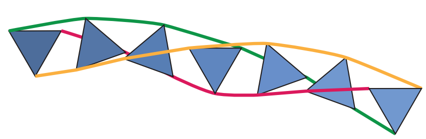


The description of generalized moduli maps is essentially the same as in the case of general triangles.

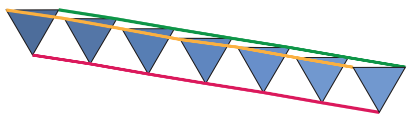
This versal family of isosceles triangles is not a manifold, it has a singularity at the equilateral triangle point. (In fact, the stack of isosceles triangles is singular.)

Equilateral triangles

The modular family consisting of one equilateral triangle $\delta/*$ is, in fact, a versal family. To see this, first let us recall that a family is constant if it is obtained by pullback from a family with one member, such as $\delta/*$. Then note that if a family of equilateral triangles can be consistently labelled, (here indicated by consistent colouring of the vertices)



then it is isomorphic to a constant family:



(In this example, we can imagine the three coloured strings being pulled taught, which will rotate the individual equilateral triangles, and rescale them, if they have different size. We get an isomorphic family this way.)

So because every family of equilateral triangles can be locally consistently labelled, every family of equilateral triangles is *locally constant.* Thus $\delta/*$ satisfies the first property required of a versal family.

1.36 Exercise. Complete the proof that $\delta/*$ is a versal family by proving that the canonical topology on the symmetry group S_3 of δ is the discrete topology. $\square$

For a family of equilateral triangles, the generalized moduli map $(T'/T, f)$ consists only of the S_3-covering T'/T, there is no information contained in $f : T' \to *$, because there is always a unique map to the one-point set from any space.

For a family of equilateral triangles $\mathscr{F}/T$, the associated S_3-cover is the space of all labellings of $\mathscr{F}$. Conversely, an S_3-covering T'/T of a space T, encodes gluing data for a family of equilateral triangles, because the symmetry group of the equilateral triangle is S_3.

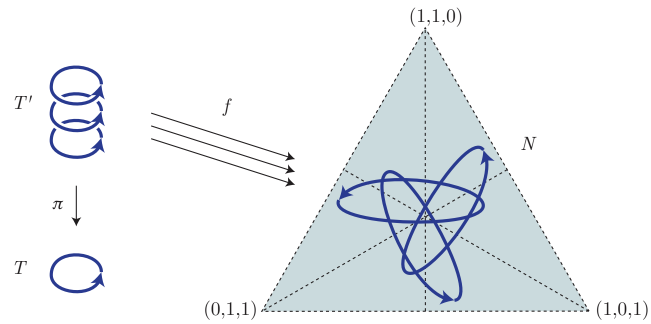
Note that $\delta/*$ is a unique modular family (up to isomorphism), which is nevertheless not universal.

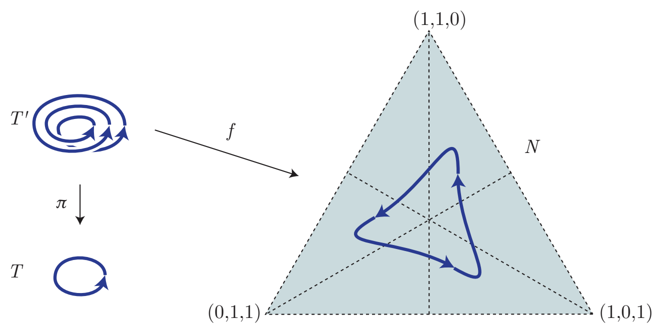
Oriented triangles

The same family $\mathscr{N}/N$ from above (17) is also a versal family for oriented triangles. Simply declare all triangles in $\mathscr{N}/N$ to have the alphabetical cyclic ordering on their edges. This makes $\mathscr{N}/N$ a family of oriented triangles. Any family of oriented triangles can locally be labelled alphabetically, and is therefore locally a pullback from the oriented $\mathscr{N}$.

The difference to the unoriented case is that the symmetry groupoid of the oriented $\mathscr{N}$ consists only of the cyclic subgroup with three elements $\mathbb{Z}_3 \cong A_3 \subset S_3$ acting on N. Therefore, generalized moduli maps of oriented triangles consist of pairs $(T'/T, f)$, where T'/T is a cyclic cover of degree

3, and $f : T \to N$ is an A_3-equivariant continuous map. Examples:



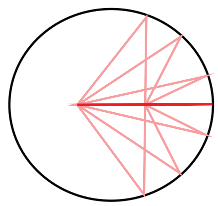


1.10 Degenerate triangles

For many reasons it is nice to have a complete (i.e. compact) moduli space. (Completeness is needed, for example, for the Riemann-Roch theorem, and intersection theory in general.) The coarse moduli spaces M of triangles and $\widetilde{M}$ of oriented triangles, are not compact, because they do not contain the boundary line of degenerate triangles.

So let us consider degenerate triangles. Things get very interesting now, because there are many ways to think of degenerate triangles, all giving rise to different compactifications of the moduli stack $\mathfrak{M}$.

Let us return to our very first continuous family of triangles (4), which was obtained by drawing part of an ellipse. When the pen hits the horizontal line through the two pin points, the vertices of the triangle become collinear, and the triangle becomes *degenerate*. (The triangle inequality becomes an equality.) So drawing the ellipse to this point or beyond does *not* define a family of triangles.

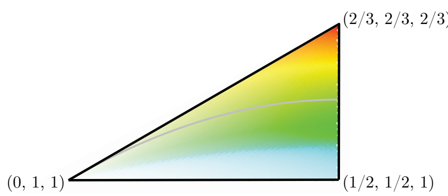


Therefore, we will broaden our point of view, and include such degenerate triangles in our moduli problem. We would like to do this in such a way that drawing the complete ellipse leads to a continuous family of degenerate triangles.

On the level of the course moduli space, not much happens: we are just adding the boundary. In fact, one representation of the coarse moduli space of degenerate triangles is

$$\overline{M} = \{(a, b, c) \in \mathbb{R}^3 \mid a \leq b \leq c,\ a + b + c = 2,\ c \leq a + b\}$$

$\overline{M}$:



Note that this also adds the triangle $(0, 1, 1)$, which is not only collinear, but also has two of its vertices coinciding. Let us call this the *pinched triangle*. The symmetric degenerate triangle $(\frac{1}{2}, \frac{1}{2}, 1)$ we call the *bisected line segment*. The coarse moduli space $\overline{M}$ is compact.

There are several natural stack structures over this coarse moduli space. They depend on what we mean exactly, by a continuous family of degenerate triangles. We will cover three approaches.

Lengths of sides viewpoint

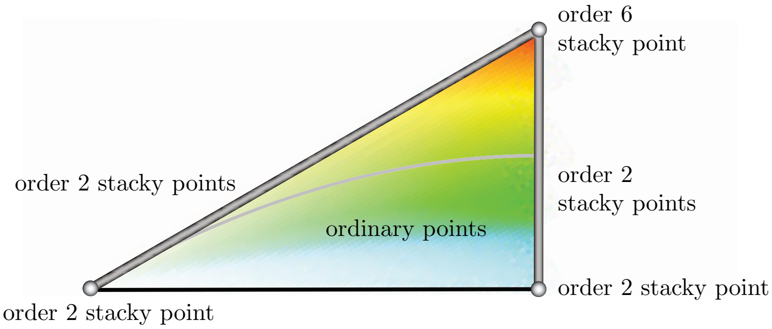
In this scenario, a degenerate triangle is a set of three real numbers $\{a, b, c\}$, which are not required to be distinct or strictly positive, and which satisfy weak versions of the triangle inequality. More precisely,

(i) $a \geq 0$, $b \geq 0$, $c \geq 0$, but not all three are equal to 0,

(ii) $a + b \geq c$, $a + c \geq b$, and $b + c \geq a$.

We exclude the degenerate triangle whose sides all have length 0, because it is not represented in the coarse moduli space $\overline{M}$ (it cannot be rescaled to have perimeter 2). (It also has infinitely many symmetries, as all rescalings, rotations and reflections preserve it, and we do not need it to compactify $\mathfrak{M}$. See, however, Exercise 1.59) Let us denote the stack of degenerate triangles obtained in this way by $\overline{\mathfrak{M}}^{\text{naïve}}$.

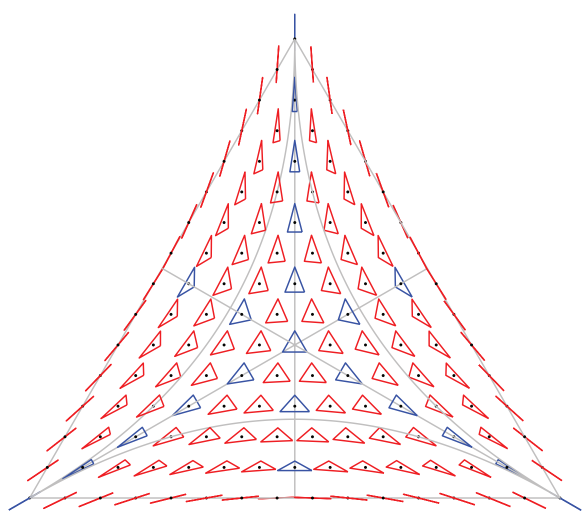
$\overline{\mathfrak{M}}^{\text{naïve}}$:



Up to similarity, we have added triangles with $(a, b, c) = (a, 1-a, 1)$, with $0 \leq a \leq \frac{1}{2}$. None of these have any symmetries, except the two isosceles ones, $(0, 1, 1)$ and $(\frac{1}{2}, \frac{1}{2}, 1)$, which have two symmetries each, namely the swap of the two equal sides.

The closure $\overline{N}$ of N inside $\mathbb{R}^3$ supports a versal family, $\overline{\mathscr{N}}$, whose symmetry groupoid is $\overline{N} \times S_3$, so the behaviour of $\overline{\mathfrak{M}}^{\text{naïve}}$ is quite similar to that of $\mathfrak{M}$: families of degenerate triangles are essentially the same thing as S_3-covering spaces together with S_3-equivariant continuous maps to $\overline{N}$.

$\overline{\mathscr{N}}$:

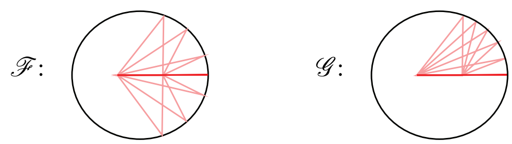


(19)

The oriented version is very similar, we only replace S_3 by $\mathbb{Z}_3$. (In this scenario, an oriented triangle is a set of three numbers satisfying weak triangle inequalities with a cyclic ordering on the three numbers.)

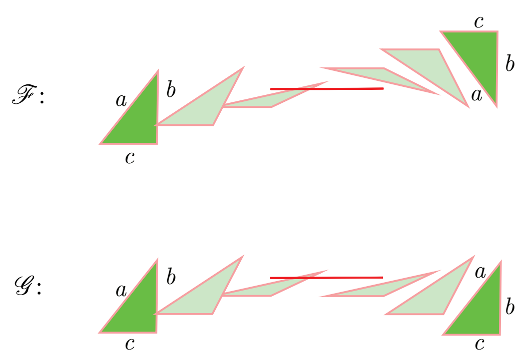
This 'naïve' point of view on the stack of degenerate triangles has the advantage of being no more complicated than the stack of non-degenerate triangles, and it leads to a 'compact' moduli stack. Disadvantages are: the stack is a stacky version of *manifold with boundary*, and it also does not capture correctly the geometric nature of families of triangles:

Let us consider two (unoriented) families obtained by drawing part of the ellipse:



Both are parametrized by an interval, start and end at the 3:4:5 right triangle, and have a degenerate triangle in the middle. It is instructive, to compare these two families with the two families we considered above, (9), which had an isosceles triangle in the middle, rather than a degenerate one. (It is of no consequence for what we are saying here, that this degenerate triangle is also isosceles. Everything would be the same if we shortened or lengthened the string giving rise to our two families a little.)

These two families are completely identical families if all we take into account are the lengths of the three sides. If we label their edges consistently, the two labelled families have the same moduli map to $\overline{N}$. The generalized moduli maps of the unlabelled families to $\overline{N}$ are identical. (All this is in stark contrast to the two families from (9) crossing the equilateral locus, rather than the degenerate locus.)



(20)

Yet, the two families are essentially different: an isomorphism between the two families would have to be the identity on the first part of the parameter interval and the reflection across the horizontal for the second half of the parameter interval. So the two families are not continuously isomorphic, if we require isomorphisms to be geometric similarity transformations of the plane the triangles are embedded inside, rather than just permutations and rescalings of the three lengths of the sides.

The difference between the degenerate triangle and the isosceles triangle is that the reflectional symmetry permutes the labels in the latter case, but cannot be detected by labels in the former case. Therefore, the point of view of a triangle as a collection of three numbers giving the lengths of the sides is inadequate for degenerate triangles. This naïve point of view cannot capture the reflectional symmetry of the degenerate triangles, which is present as a *consequence* of the fact that the two families $\mathscr{F}$ and $\mathscr{G}$ of (20) are different.

If we consider the two families $\mathscr{F}$ and $\mathscr{G}$ as families of oriented triangles, they are plainly different: $\mathscr{F}$ connects the anticlockwise 3:4:5 triangle with the clockwise one, whereas $\mathscr{G}$ has the anticlockwise one on both ends. But for these statements to hold true, we need 'orientation' to be a structure on the ambient plane, not any structure defined on the unordered triple of the sides of the triangles.

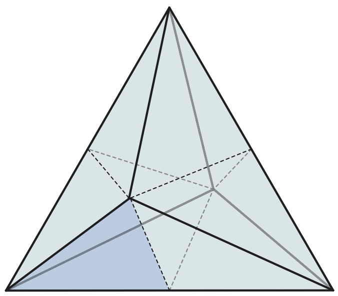
We therefore have two notions of *orientation* on a degenerate triangle: a cyclic ordering on the sides (or vertices), or an orientation on the ambient plane. We call the latter an *embedded orientation.* In the case of non-degenerate triangles, these two kinds of orientations determine each other. In the case of degenerate triangles, neither determines the other. (An orientation on the plane can be specified by a cyclic ordering of the edges of an embedded non-degenerate triangle, or by a parametrized circle in the plane.)

Embedded viewpoint

We will now describe the second compactification of the stack of triangles, $\overline{\mathfrak{M}}$, which we call the stack of (degenerate) *embedded* triangles, if it needs emphasizing. We will do this by agreeing on what such families of triangles look like *locally*, and exhibiting a versal family. Then the theory of generalized moduli maps gives an explicit description of all continuous families *globally.*

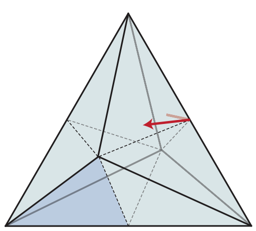
To tell the difference between the two families $\mathscr{F}$ and $\mathscr{G}$ from (9), it was sufficient to label the sides of the families consistently. Labels on the sides are not sufficient to tell the difference between the two families $\mathscr{F}$ and $\mathscr{G}$ from (20). Therefore, we introduce extra structure: This extra information is an orientation on the plane the triangles are contained in. Once we fix such an orientation (in addition to the labels), it is easy to tell the difference between the two families. For $\mathscr{G}$, the alphabetical orientation of the labels on the sides of the triangle and the given orientation on the plane disagree everywhere. For $\mathscr{F}$, the alphabetical orientation on the edges disagrees up to the degenerate triangle, but agrees afterwards.

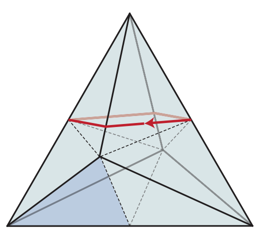
We obtain a versal family by considering triangles with labelling and orientation on the ambient plane. Since there are two orientations on the plane (counterclockwise and clockwise), this means to take two copies of $\overline{N}$, which we shall call $\overline{N}^+$ and $\overline{N}^-$, and glue them together along their boundary. The result, which we shall denote $\overline{N}^\pm$ is here sketched as the surface of a bipyramid ($\overline{N}^+$ is in the front, $\overline{N}^-$ in the back):

$\overline{N}^\pm$:  (21)

We agree that over $\overline{N}^+$ the plane is oriented in such a way that the alphabetical order on the triangles is counterclockwise, and over $\overline{N}^-$, clockwise.

Consider the family $\mathscr{F}$ with marking and orientation as in (20). Its moduli map to $\overline{N}^\pm$ is displayed on the left hand side in the sketch below. The first half is in $\overline{N}^-$, the second half in $\overline{N}^+$. (The corresponding moduli map for $\mathscr{G}$ would coincide with the one for $\mathscr{F}$ for the first half of the parameter interval, and then would retrace itself for the second half of the parameter interval.) On the right hand side, we have displayed the moduli map of a (marked and oriented) family obtained by drawing the complete ellipse.





We see how the requirement that these moduli maps be continuous, tells us how to glue the two copies of $\overline{N}$ together.

To construct a versal family $\overline{\mathscr{N}}^\pm$ parametrized by the bipyramid $\overline{N}^\pm$, we glue together the two families of marked and oriented triangles $\overline{\mathscr{N}}^+$ parametrized by $\overline{N}^+$, and $\overline{\mathscr{N}}^-$, parametrized by $\overline{N}^-$. Here $\overline{\mathscr{N}}^+$ is the labelled family $\overline{\mathscr{N}}$ of (19), endowed with the embedded orientation making the labels counterclockwise, and $\overline{\mathscr{N}}^-$ is the labelled family $\overline{\mathscr{N}}$ endowed with the clockwise embedded orientation. When gluing, we use the unique isomorphism of labelled embedded-oriented triangles which exists over the locus of degenerate triangles. (If we think of both $\overline{\mathscr{N}}^+$ and $\overline{\mathscr{N}}^-$ as identical copies of $\overline{\mathscr{N}}$, this amounts to gluing the two copies of $\overline{\mathscr{N}}$ using the reflectional symmetry of the degenerate triangles, which swaps the ambient orientation, but preserves the labels.) See Figure 2 for an attempt at depicting $\overline{\mathscr{N}}^{\pm}$.

1.37 Exercise. (More advanced.) The family $\overline{\mathscr{N}}^{\pm}$ is a universal family of labelled degenerate embedded-oriented triangles. (Define a family of labelled degenerate embedded-oriented triangles, parametrized by the topological space T, to consist of a complex line bundle $\mathscr{L}/T$, together with three sections $A, B, C \in \Gamma(T, \mathscr{L})$, such that $A + B + C = 0$, and no more than two sections ever agree anywhere in T. Isomorphisms consist of isomorphisms of line bundles preserving the three sections.) $\square$

1.38 Exercise. The symmetry groupoid of the family $\overline{\mathscr{N}}^{\pm}$ is the transformation groupoid given by the group of symmetries of the bipyramid, which consists of the identity, two rotations by $\frac{2\pi}{3}$, three rotations by π, four reflections and two rotation-reflections. This group of symmetries of the bipyramid is isomorphic to $S_3 \times \mathbb{Z}_2$, where S_3 acts as before, preserving $\overline{N}^+$ and $\overline{N}^-$, and $\mathbb{Z}_2$ acts via the reflection across the common base of the two pyramids making up the bipyramid. $\square$

1.39 Exercise. Retaining of the family $\overline{\mathscr{N}}^{\pm}$ only its orientation circle bundle, we obtain a Hopf fibration. More precisely, for every family member of $\overline{\mathscr{N}}^{\pm}$ construct a circle whose centre is at the centroid of the triangle, and whose circumference is equal to the perimeter of the triangle (e.g. 2, in our conventions). Consider these circles together with the action of the circle group S^1: it acts via rotations. The embedded orientations of the triangles tell us in which direction S^1 acts on these circles. The union of all these circles forms a topological space $P \to \overline{N}^{\pm}$, together with an action of the group S^1 on P, in such a way that $\overline{N}^{\pm}$ is the quotient of P by this S^1-action. In other words, we have constructed a *principal homogeneous S^1-bundle* P over $\overline{N}^{\pm}$.

In Figure 2 we have indicated this circle bundle. Above is the family $\overline{\mathscr{N}}^+$, below the family $\overline{\mathscr{N}}^-$. Both are displayed such that the orientation on the circles is counterclockwise. The labels are indicated by different colours on the vertices of the family members. The two halves of $\overline{N}^{\pm}$ are displayed as discs, rather than triangles. When gluing together these two half families, the circles on the boundary complete a rotation by 2π. Another way to say this is that gluing data for the principal bundle P is given by a map $S^1 \to S^1$, which has winding number ± 1.

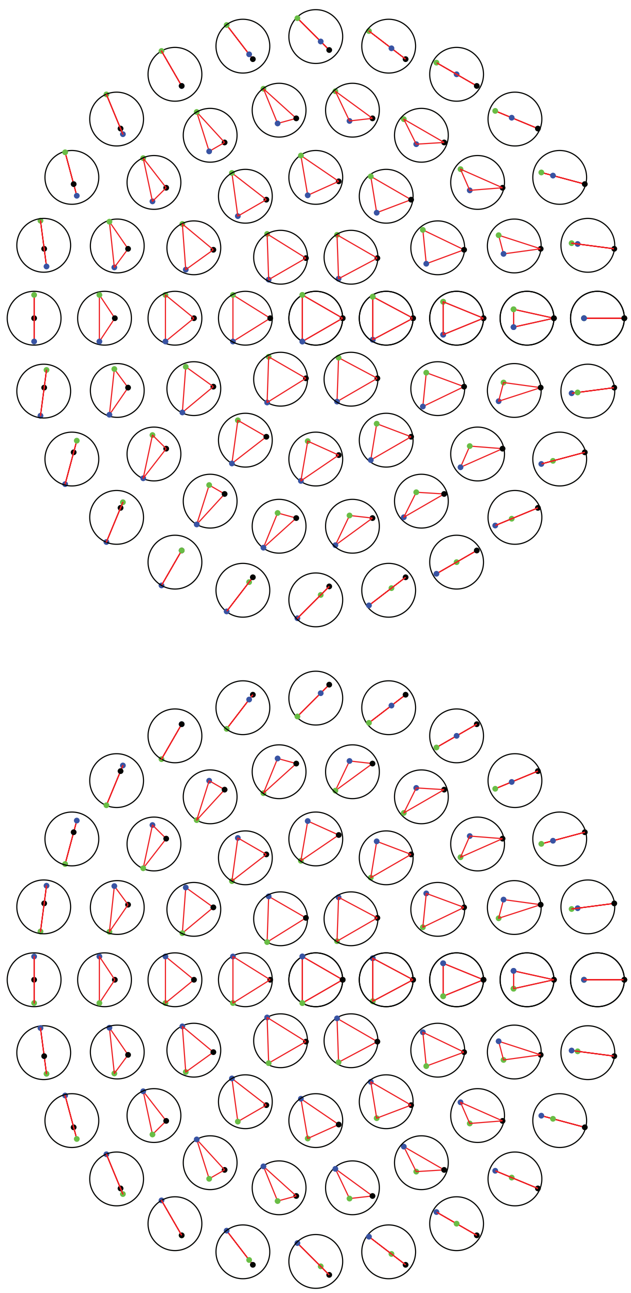
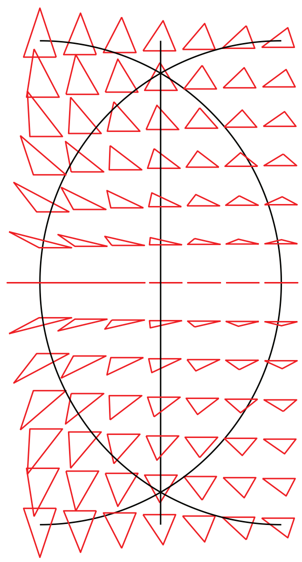


Figure 2: The Hopf fibration over $\overline{N}^{\pm}$

This shows that $P \to \overline{N}^{\pm}$ is homeomorphic to the Hopf fibration $S^3 \to S^2$. Since the Hopf fibration is topologically non-trivial, there is no way to embed all the triangles in $\overline{\mathscr{N}}^{\pm}$ into the plane in a compatible way, just as it is impossible to consistently label the Moebius family (5). $\square$

The following is a qualitative sketch of a small neighbourhood of the pinched triangle in the family $\overline{\mathscr{N}}^{\pm}$. Note how a line of isosceles triangles and a line of degenerate triangles intersect at the pinched triangle.



(22)

We claim that $\overline{\mathscr{N}}^{\pm}$ is a versal family of degenerate triangles. To prove this, all we need is an understanding of what a family of degenerate triangles looks like locally. Thus, let us agree that locally, every family of degenerate triangles can be labelled consistently and embedded into one fixed oriented plane, which we will take to be $\mathbb{C}$ (recall that the complex number plane is oriented).

1.40 Definition. If $\mathscr{F}$ is a **continuous family of degenerate embedded triangles** parametrized by the topological space T, then for every point $t_0 \in T$, there exists an open neighbourhood $t_0 \in U \subset T$ of t_0, and three continuous functions $A, B, C : U \to \mathbb{C}$, determining the restriction $\mathscr{F}|_U$. At every point of U, no more than two of the three functions A, B, C are allowed to agree. For $t \in U$, the points $A(t), B(t), C(t) \in \mathbb{C}$ are the three vertices of the triangle $\mathscr{F}_t$.

If two such families are given over all of T by $(A, B, C) : T \to \mathbb{C}$ and $(A', B', C') : T \to \mathbb{C}$, then they are **isomorphic** if, after relabelling of the three functions, there exist continuous functions $R : T \to \mathbb{C}$ and

$S : T \to \mathbb{C}^*$, such that $(A', B', C') = (SA + R, SB + R, SC + R)$, or $(A', B', C') = (\overline{SA + R}, \overline{SB + R}, \overline{SC + R})$.

1.41 Exercise. Deduce from the result of Exercise 1.37 that $\overline{\mathscr{N}}^{\pm}$ is a versal family of degenerate embedded triangles. An approach avoiding Exercise 1.37, follows below. $\square$

To prove that $\overline{\mathscr{N}}^{\pm}$ is a versal family, assume that a family of (degenerate, embedded) triangles is given over the whole parameter space T by three functions $A, B, C : T \to \mathbb{C}$. We extract from A, B, C the functions $a, b, c : T \to \mathbb{R}_{\geq 0}$, and $\epsilon : T \to \{+1, 0, -1\}$, where $a = |B - C|$, $b = |C - A|$, $c = |A - B|$, and $\epsilon = 0$ where A, B, C are collinear, $\epsilon = +1$, where A, B, C are in counterclockwise position, and $\epsilon = -1$, where A, B, C are in clockwise position. We define a map

$$\begin{aligned} f : T &\longrightarrow \overline{N}^{\pm} \\ t &\longmapsto \begin{cases} \frac{2}{a+b+c}\big(a(t), b(t), c(t)\big) \in \overline{N}^{+} & \text{if } \epsilon(t) \geq 0 \\ \frac{2}{a+b+c}\big(a(t), b(t), c(t)\big) \in \overline{N}^{-} & \text{if } \epsilon(t) \leq 0 \end{cases} \end{aligned}$$

The map f is continuous by the pasting lemma. We have to prove that the family given by $A, B, C : T \to \mathbb{C}$ is, at least in a neighbourhood of a base point $t_0 \in T$, isomorphic to the one given by $f^*\overline{\mathscr{N}}^{\pm}$.

Let us explain how $f^*\overline{\mathscr{N}}^{\pm}$ gives rise, at least locally, to a family of embedded-oriented triangles given by three continuous functions, say A', B', C', to $\mathbb{C}$. In the case of $f^*\overline{\mathscr{N}}^{\pm}$, the family of triangles is described by three continuous functions $a, b, c : T \to [0, 1]$, and two closed subsets T^+ and T^-, where $T^+ \cap T^- = \{a = 1\} \cup \{b = 1\} \cup \{c = 1\}$. To this data, we associate A', B', C' as follows: over the locus where $c \neq 0$, define $A'(t) = 0$, $B'(t) = c(t)$, and $C'(t)$ in such a way that $|C'(t)| = b(t)$, $|C'(t) - B'(t)| = a(t)$, and $C'(t)$ is in the upper half plane if $t \in T^+$, and in the lower half plane if $t \in T^-$. Similar constructions can be made over the locus where $a \neq 0$, or $b \neq 0$, respectively.

Now it is not hard to see that, after relabelling, if necessary, the functions A', B', C' are related to the functions A, B, C by a similarity transformation of $\mathbb{C}$, which depends continuously on $t \in T$. This proves that, indeed, all families of triangles can locally be described as pullbacks from $\overline{\mathscr{N}}^{\pm}$.

1.42 Exercise. Prove that the technical condition on the symmetry groupoid of $\overline{\mathscr{N}}^{\pm}$ is satisfied, i.e., verify that with our current local description of families of triangles, a continuous isomorphism $\phi : (A, B, C) \to (A', B', C')$ gives rise to a continuous map $T \to \overline{N}^{\pm} \times S_3 \times \mathbb{Z}_2$, and that the tautological isomorphism is continuous.

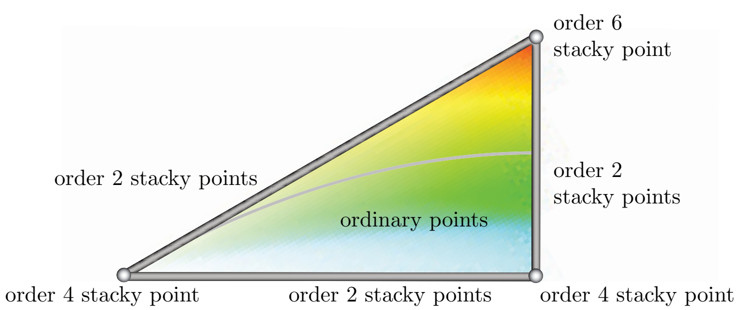
As we now have a versal family, and we know its symmetry groupoid, we know what families of degenerate triangles look like globally. The relevant result was stated in Exercise 1.34.

As the symmetry group $S_3 \times \mathbb{Z}_2$ is a product, an $S_3 \times \mathbb{Z}_2$-cover is the same thing as a pair consisting of an S_3-cover and a $\mathbb{Z}_2$-cover. Thus we have:

1.43 Proposition. *Globally, a family of degenerate embedded triangles over T is a pair (T', f), (T'', g), where $T' \to T$ is an S_3-cover and $T'' \to T$ is a degree 2 cover, and $f : T' \to \overline{N}^{\pm}$ and $g : T'' \to \overline{N}^{\pm}$ are equivariant maps. The S_3-cover T' gives all labellings on the family and the degree 2 cover T'' gives all embedded orientations.*

The stack of embedded triangles, which we shall call $\overline{\mathfrak{M}}$, looks like this:

$\overline{\mathfrak{M}}$:



(23)

As embedded triangles, the pinched triangle, as well as the bisected line segment have an order 4 symmetry group. In the case of the bisected line segment, this group consists of the rotation by π, and two reflections. In the case of the pinched triangle, there is the reflectional symmetry, the swap of the two coinciding points, and the composition of these two symmetries.

Complex viewpoint

There is a canonical family of degenerate embedded triangles parametrized by the complex plane $\mathbb{C}$. It is customary to write the parameter as $\lambda \in \mathbb{C}$, in honour of Legendre. Figure 3 shows a picture of the λ-plane. The family of triangles is given by the three functions $A(\lambda) = 0$, $B(\lambda) = 1$, and $C(\lambda) = \lambda$. The 12 locations of the 3:4:5 triangle have been marked. The locus of acute triangles is shaded. The family is shown in the lower part of Figure 3. As it is labelled and oriented, it induces a map $f : \mathbb{C} \to \overline{N}^{\pm}$, which is given by

$$
\begin{aligned}
f : \mathbb{C} &\longrightarrow \overline{N}^{\pm} \\
\lambda &\longmapsto \begin{cases} \left(\frac{2|\lambda-1|}{1+|\lambda|+|\lambda-1|}, \frac{2|\lambda|}{1+|\lambda|+|\lambda-1|}, \frac{2}{1+|\lambda|+|\lambda-1|}\right) \in \overline{N}^{+} & \text{if } \operatorname{Im}\lambda \geq 0 \\ \left(\frac{2|\lambda-1|}{1+|\lambda|+|\lambda-1|}, \frac{2|\lambda|}{1+|\lambda|+|\lambda-1|}, \frac{2}{1+|\lambda|+|\lambda-1|}\right) \in \overline{N}^{-} & \text{if } \operatorname{Im}\lambda \leq 0\,. \end{cases}
\end{aligned}
$$

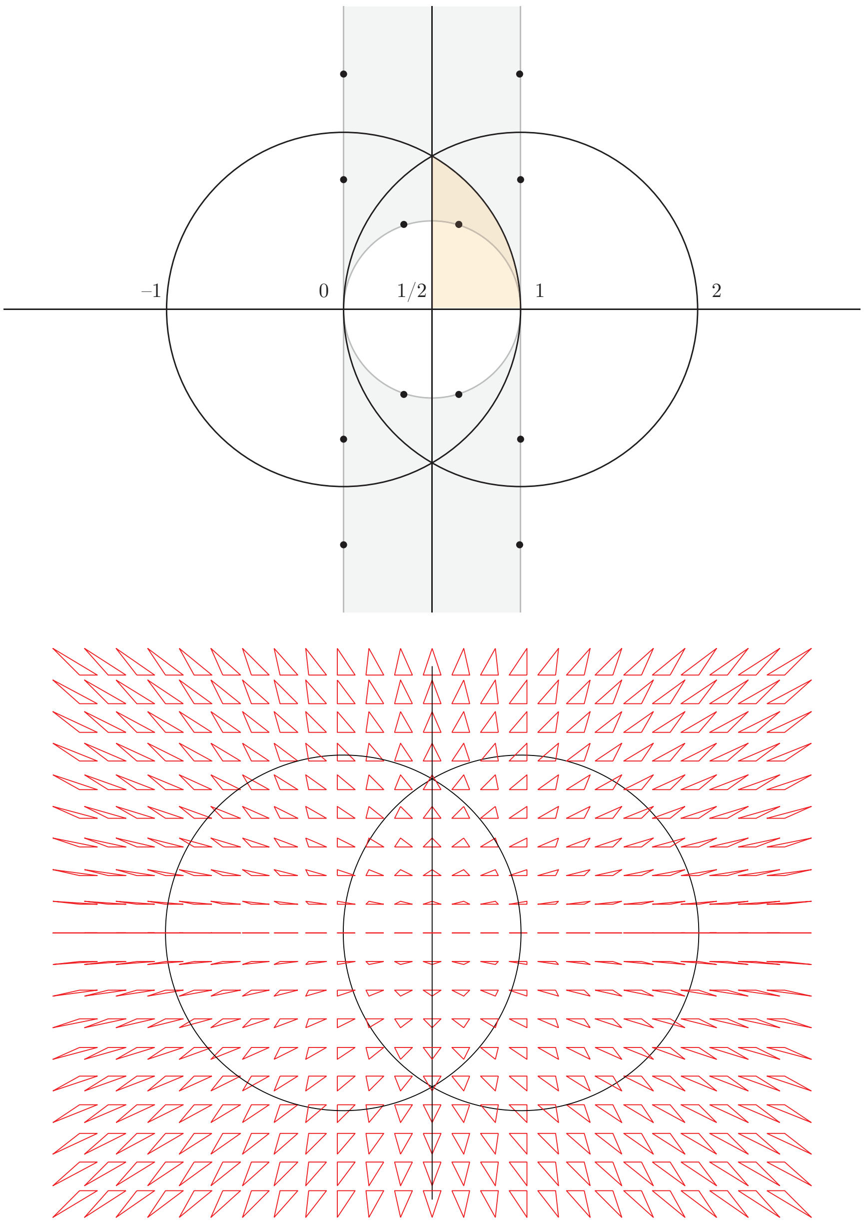
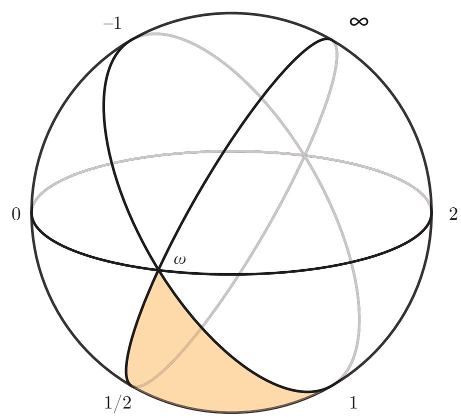


Figure 3: The λ-plane (above), and the versal family over it (below)

This is a homeomorphism onto the complement of the point $(1,1,0)$ in $\overline{N}^{\pm}$, so it identifies $\overline{N}^{\pm}$ with the one-point compactification $\mathbb{C}\cup\{\infty\}$ of $\mathbb{C}$.

We also see that this family parametrized by $\mathbb{C}$ extends to a family parametrized by the Riemann sphere $\mathbb{C}\cup\{\infty\}$, by pulling back the family $\overline{\mathscr{N}}^{\pm}$ via the homeomorphism $\mathbb{C}\cup\{\infty\}\to\overline{N}^{\pm}$.

So $\mathbb{C}\cup\{\infty\}$ is just as good a versal parameter space as $\overline{N}^{\pm}$. In fact, it is better: it has the structure of a Riemann surface! (It endows the stack $\overline{\mathfrak{M}}$ with the structure of *real analytic stack*.)



(24)

In this viewpoint, the action of S_3 is given by the reflections in the three great circles passing through the sixth root of unity ω, and rotations by multiples of $\frac{2\pi}{3}$ around the axis through ω. The group $\mathbb{Z}_2$ acts by reflection across the great circle representing the real values of λ.

Notice how Figure 3 exhibits, in a neighbourhood of the locus where λ is real, the behaviour of the family $\overline{\mathscr{N}}^{\pm}$ near where it has been glued. At the centre of Figure 3, for example, is a bisected line segment, and close by, isosceles triangles, as well as degenerate triangles appear twice, whereas scalene triangles appear four times, reflecting the fact that the bisected line segment has 4 symmetries.

Oriented degenerate triangles

An oriented (degenerate, embedded) triangle is a set of three points in the plane (or rather an unordered triple of points in the plane), no more than two coinciding, together with an orientation on the plane, i.e., a notion of counterclockwise and clockwise. Of course, for non-collinear triangles, such an orientation is the same thing as a cyclic ordering on the vertices, but we have seen that for degenerate triangles the current notion is superior.

Isomorphisms, or similarity transformations, are now required to preserve the orientation, i.e., they are not allowed to be reflections or glide reflections.

Of course, a continuos family of oriented triangles can, at least locally, be endowed with a labelling, if we do not make any requirements about the alphabetical ordering of the labelling agreeing with the counterclockwise orientation on the ambient plane. In other words, just as families of unoriented triangles could be described locally by three complex valued functions on the parameter space, the same is true for oriented families. The difference is only in when families are considered isomorphic. (In fact, we can use Definition 1.40 verbatim, adding only the word 'oriented', and deleting the last 'or' involving complex conjugation.)

Therefore, $\overline{N}^{\pm}$ is a versal parameter space, and $\overline{\mathscr{N}}^{\pm}$ is a versal family. The symmetry groupoid of $\overline{\mathscr{N}}^{\pm}$ as a family of *oriented* triangles is a subgroupoid of the symmetry groupoid of $\overline{\mathscr{N}}^{\pm}$ as family of triangles. In fact, it is the transformation groupoid of the subgroup of *oriented* symmetries of the bipyramid. This group consists of the rotations by $\frac{2\pi}{3}$ about the axis through the two pyramid vertices, and the three rotations about the axes through the vertices of the common base of the two pyramids. This group is, again, isomorphic to S_3, although it is a subgroup of $S_3 \times \mathbb{Z}_2$ in a different way than the copy of S_3 which acts on $\overline{N}^{+}$ and $\overline{N}^{-}$.

In the Legendre picture, where we have identified $\overline{N}^{\pm}$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$, this symmetry groupoid acts by the six linear fractional transformations

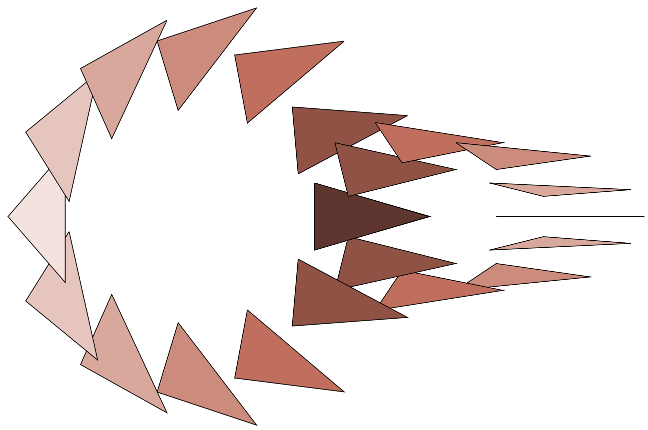
$$\lambda \longmapsto \qquad \lambda\,, \qquad \frac{1}{\lambda}\,, \qquad 1-\lambda\,, \qquad \frac{\lambda}{\lambda-1}\,, \qquad \frac{1}{1-\lambda}\,, \qquad \frac{\lambda-1}{\lambda}\,. \tag{25}$$

We will write Λ for the Riemann sphere, endowed with the action of S_3 by these six transformations.

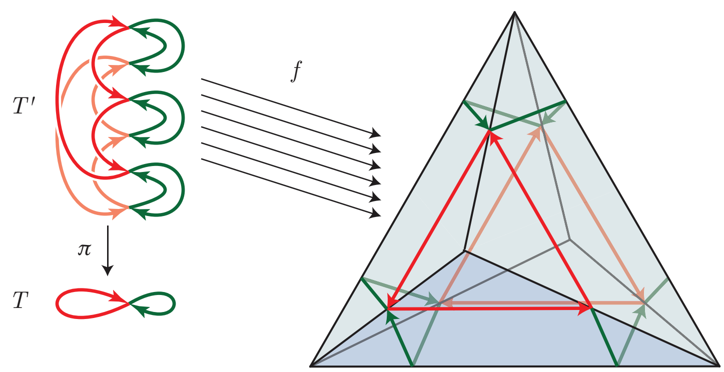
1.44 Exercise. Deduce from Exercise 1.34, that a family of degenerate oriented embedded triangles parametrized by the space T is given by an S_3-covering space $T' \to T$ and an S_3-equivariant map $T' \to \Lambda = \mathbb{C} \cup \{\infty\}$.

1.45 Exercise. Prove that for every family of oriented triangles $\mathscr{F}/T$, the corresponding generalized moduli map to $\Lambda = \mathbb{C} \cup \{\infty\}$ is given by the cross-ratio of the three vertices and ∞. More precisely, the six-valued function $T' \to \mathbb{C} \cup \{\infty\}$ is given by the six cross-ratios of the vertices and ∞. Moreover, the action by S_3 on T' is compatible with the action of S_3 on the six cross-ratios.

For example, consider a family parametrized by the figure eight, such as the following: over the left loop of the figure eight, this family contains a degree 3 cyclic cover, such as the family (14). Over the right loop of the figure eight, the family contains a degree 2 cover (a Moebius band).

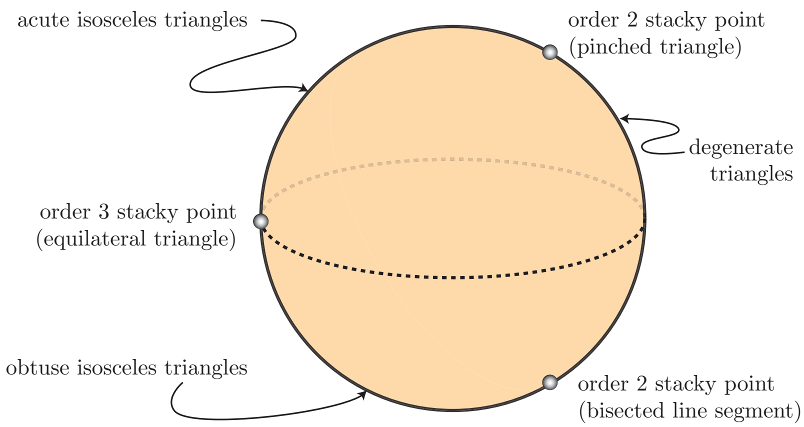


The generalized moduli map of such a family will look like this:



In fact, the image of the moduli map is the Cayley graph of the group S_3. The two cycles of length 3 cover the left loop of the figure eight, and the three cycles of length 2 cover the right loop of the figure eight.

Let us denote the stack of degenerate embedded oriented triangles by $\mathfrak{L}$. In the sketch of $\mathfrak{L}$, below, the front represents triangles oriented according to ascending length of sides, the back represents triangles oriented according to descending length of sides. Identifying front and back, making the sphere flat, we get the picture of $\overline{\mathfrak{M}}$ from (23).



1.11 Change of versal family

There can be only one universal family, but there are many versal families. To see examples, let us restrict to *oriented* triangles. So far we have seen one versal family, with parameter space $\mathbb{C} \cup \{\infty\} \cong \overline{N}^{\pm}$, and symmetry groupoid given the action of S_3 on this parameter space, by oriented symmetries of a bipyramid.

Oriented triangles by projecting equilateral ones

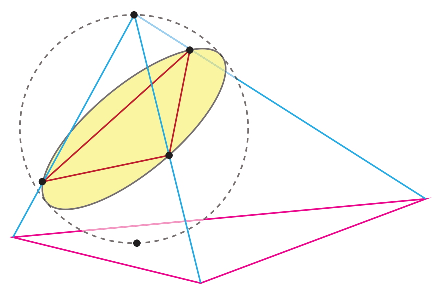
Let us construct another versal family of oriented degenerate triangles. Fix a sphere S positioned on a horizontal plane, such that the South pole of the sphere coincides with the origin of the plane. The parameter space will be

$$E = \{\text{great circle equilateral triangles on } S\}.$$

(The vertices or edges of the elements of E are *not* labelled.)

1.46 Exercise. Prove that E is a 3-dimensional manifold. (In fact, it is the quotient of the 3-dimensional Lie group SO_3 of 3×3 orthogonal matrices with determinant 1, by a discrete subgroup isomorphic to S_3.) □

The family of degenerate triangles parametrized by E, which we shall call $\mathscr{E}$, is given by stereographic projection from the North pole of S onto the plane. In other words, $\mathscr{E}_t = p(t)$, for all $t \in E$, where p is the stereographic projection. In the following sketch, a great circle triangle is displayed, as well as its stereographic projection into the plane. Thus, this sketch displays just one family member of $\mathscr{E}$ (as the parameter space is 3-dimensional, it is hard to sketch the the entire family).



Note that this never produces any triangles whose vertices coincide, instead it produces triangles with a vertex at ∞ (when one of the vertices of the great circle equilateral triangle is at the North pole). Thus, this family takes a different point of view on the 'pinched triangles', than the one parametrized by $\mathbb{C} \cup \{\infty\} = \overline{N}^{\pm}$.

The group of oriented similarity transformations, which consists of translations, rotations and scalings, is isomorphic to the semi-direct product $\mathbb{C}^+ \rtimes \mathbb{C}^*$. The subgroup of translations, $\mathbb{C}^+$, is a normal subgroup, and the subgroup of scaling-rotations with centre the origin, is isomorphic to $\mathbb{C}^*$. The conjugation action of the scaling-rotations on the translations is the multiplication action of $\mathbb{C}^*$ on $\mathbb{C}^+$.

Using the stereographic projection, we can translate any geometric statement about the plane into a statement about the sphere, and conversely. It is, in fact, more convenient to work on the sphere, rather than the plane, because we can describe everything we need in terms of the group of conformal symmetries of the sphere. Conformal transformations of the sphere are transformations that preserve angles, as well as the orientation.

The group of orientation and angle preserving (i.e., conformal) transformations of the sphere is known as $PSL_2(\mathbb{C})$. If (A, B, C) and (A', B', C') are ordered triples of distinct points on the sphere, then there exists a unique $P \in PSL_2(\mathbb{C})$, such that $PA = A'$, $PB = B'$, and $PC = C'$. The oriented similarity transformations of the plane $\mathbb{C}^+ \rtimes \mathbb{C}^*$ correspond via stereographic projection to the subgroup of $PSL_2(\mathbb{C})$ fixing the north pole ∞. The subgroup of $PSL_2(\mathbb{C})$ which fixes lengths, as well as angles and orientation, is the group of rotations of the sphere, $SO_3 \subset PSL_2(\mathbb{C})$. The intersection of these two groups is the group of rotations about the axis through the north and south pole. This group is isomorphic to the circle group S^1.

$$\begin{tikzcd}
& PSL_2(\mathbb{C}) & \\
\mathbb{C}^+ \rtimes \mathbb{C}^* \arrow[ur, hook] & & SO_3 \arrow[ul, hook'] \\
& S^1 \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

To explain why the family $\mathscr{E}$ is versal, let us work on the sphere S, rather than the plane. Let us agree, therefore, that a family of degenerate triangles is locally given by three continuous functions $A, B, C : T \to S$, no two of which agree anywhere in T. Isomorphisms are locally given by relabellings and by continuously varying elements of the group $\mathbb{C}^+ \times \mathbb{C}^* \subset PSL_2(\mathbb{C})$.

To prove that every such family is induced from $\mathscr{E}$, locally, we will prove that given such a family $(A, B, C)/T$, we can apply a continuous family of elements of $\mathbb{C}^+ \rtimes \mathbb{C}^*$, making it a family of great circle equilateral triangles.

Let e_0 be a fixed great circle equilateral triangle, which we may as well assume to be labelled, as it is fixed. Firstly, there is a unique continuous map $P : T \to PSL_2(\mathbb{C})$, such that $P(t)e_0 = \big(A(t), B(t), C(t)\big)$, for all $t \in T$ (as labelled families of triangles). This follows from the fact that A, B, C never coincide, and the fact that $PSL_2(\mathbb{C})$ acts simply transitively on the set of ordered triples of distinct points in S. Secondly, there exists a continuous family of rotations $R(t)$, such that $R\infty = P^{-1}\infty$ (at least locally in T). Then $R^{-1}P^{-1}(A, B, C) = R^{-1}e_0$, and $R^{-1}P^{-1}\infty = \infty$. Because of the latter property, $R^{-1}P^{-1}$ is a continuous family of elements of $\mathbb{C}^+ \rtimes \mathbb{C}^*$, and, of course $R^{-1}e_0$ is a continuous family of great circle equilateral triangles. This proves what we needed.

To determine the symmetry groupoid of the family $\mathscr{E}/E$, assume that $\{A, B, C\} \subset \mathbb{C} \cup \{\infty\}$ is a member of the family $\mathscr{E}$, in other words a great circle equilateral triangle.

Let $P \in PSL_2(\mathbb{C})$ such that $P\infty = \infty$ be an arbitrary oriented similarity transformation. Assume that $P\{A, B, C\}$ is another member of $\mathscr{E}$, i.e., another great circle equilateral triangle. As the rotation group $SO_3 \subset PGL_2(\mathbb{C})$ acts transitively on the great circle equilateral triangles, there exists a rotation $R \in SO_3$ such that $R\{A, B, C\} = P\{A, B, C\}$. Hence $R^{-1}P$ is in the stabilizer subgroup of $\{A, B, C\}$ inside $PSL_2(\mathbb{C})$. This stabilizer subgroup consists of 6 rotations, and is isomorphic to S_3. As $R^{-1}P$ is a rotation, it follows that P itself is a rotation. As P fixes the north-south axis it is in the subgroup $S^1 \subset SO_3 \subset PSL_2(\mathbb{C})$.

Thus, every symmetry of the family $\mathscr{E}$ is induced by an element of S^1, and we see that the symmetry groupoid of the family $\mathscr{E}$ is the transformation groupoid $E \times S^1$, where S^1 acts by rotations about the North-South axis of S.

The following exercise finishes the proof of the versality of $\mathscr{E}$, by proving that the technical conditions on the symmetry groupoid are satisfied.

1.47 Exercise. Prove that with the manifold structure on E from Exercise 1.46, and with the local notions of continuous family and continuous isomorphism described above, the canonical topology on the symmetry groupoid of the family $\mathscr{E}/E$ is the product topology $E \times S^1$.

Prove that the tautological isomorphism of families over the symmetry groupoid is continuous. $\square$

1.48 Exercise. Explain why we cannot define a global notion of family of triangles to be given by a degree 3 cover $T' \to T$ together with a continuous map $T' \to S$. $\square$

Now that we have a versal family, and we know its symmetry groupoid, we can describe all families globally, in terms of generalized moduli maps. We need a generalization of Exercise 1.34, because the group in our symmetry groupoid is not discrete. Therefore, we have to consider *principal bundles*, instead of G-covering spaces.

1.49 Definition. Let G be a topological group and T a topological space. A **principal homogeneous bundle** over T with structure group G is given by a topological space T', endowed with a continuous map $\pi : T' \to T$, and a continuous action by the topological group G. The following two conditions are required to hold:

(i) $\pi(tg) = \pi(t)$, for all $t \in T'$ and $g \in G$,

(ii) (local triviality) for every $t \in T$, there exists a neighbourhood U of t in T, and a continuous section $\sigma : U \to T'$ of $T' \to T$ over U, such that the induced map $U \times G \to \pi^{-1}(U)$, given by $(u, g) \mapsto \sigma(u)g$ is a homeomorphism.

Instead of 'principal homogeneous bundle with structure group G', we often say 'principal G-bundle', or simply 'G-bundle'. The terminology 'G-torsor' is also common.

1.50 Exercise. Using the concept of principal bundle, generalize both the statement and the results of Exercise 1.34 to the case of a versal family whose symmetry groupoid is a transformation groupoid $T \times G$ with a general topological group G.

In the case of $G = S^1$, we call principal S^1-bundles **circle bundles**. Thus, in the present case, a generalized moduli map is given by a circle bundle $T' \to T$, together with an S^1-equivariant continuous map $T' \to E$.

Given a family of degenerate triangles $\mathscr{F}/T$, we should think of the circle bundle $T' \to T$ as the space of all ways the members of $\mathscr{F}$ can be obtained by stereographic projection from great circle equilateral triangles on S.

Comparison

Let us now compare our two versal families of degenerate triangles: the one parametrized by E, and the one parametrized by $\Lambda = \mathbb{C} \cup \{\infty\} \cong \overline{N}^{\pm}$. (We use the notation Λ for the Riemann sphere, to emphasize its role as parameter space for the Legendre family of triangles.)

Let us restrict to unpinched triangles, because for those, our two notions of degenerate triangles agree. Thus no vertices are allowed to coincide, no vertices are allowed to 'escape to infinity'. Therefore, we restrict to

$$\Lambda^0 = \mathbb{C} - \{0,1\} \cong \overline{N}^{\pm} - \{(0,1,1),(1,0,1),(1,1,0)\}\,,$$

and

$$E^0 = E - \{\text{great circle triangles in } S \text{ with a vertex at the North pole}\}\,.$$

Now the family $\mathscr{E}|_{E^0}$ has a generalized moduli map to Λ^0, and the family $\overline{\mathscr{N}}^{\pm}|_{\Lambda^0}$ has a generalized moduli map to E^0. The key is that these generalized moduli map are *'the same'*. In fact, consider the space $Q^0 = \underline{\mathrm{Isom}}(\overline{\mathscr{N}}^{\pm}|_{\Lambda^0}, \mathscr{E}|_{E^0})$, of isomorphisms between the two families. It fits into the diagram:

$$\begin{tikzcd} Q^0 \arrow[rr, "\substack{S^1\text{-equivariant}\\ S_3\text{-bundle}}"] \arrow[d, "\substack{S_3\text{-equivariant}\\ S^1\text{-bundle}}"'] & & E^0 \\ \Lambda^0 & & \end{tikzcd} \tag{26}$$

There are two group actions on Q^0, one by S_3, one by S^1, and these actions commute with each other. This makes $Q^0 \to \Lambda^0$ an S^1-bundle, and $Q^0 \to E^0$ an S_3-bundle. The map $Q^0 \to E^0$ is S^1-equivariant, the map $Q^0 \to \Lambda^0$ is S_3-equivariant.

Thus, Diagram (26) displays both generalized moduli maps at the same time. We see that different versal families for the same moduli problem will always have a common generalized moduli map intertwining the two parameter spaces in this way.

A key fact is that the converse is true: if you have two a priori different moduli problems, and a versal family for each, and if you can intertwine the two symmetry groupoids in this way, the two moduli problems are actually equivalent. We will discuss this next.

The comparison theorem

Suppose Q is a space with commuting actions by two topological groups, G and H. Suppose further that the quotient maps $\pi : Q \to X = Q/G$

and $\rho : Q \to Y = Q/H$ are principal bundles. Then the quotient maps are equivariant:

$$\begin{tikzcd} Q \arrow[r, "\substack{G\text{-equivariant}\\ H\text{-bundle}}", "\rho"'] \arrow[d, "\pi", "\substack{H\text{-equivariant}\\ G\text{-bundle}}"'] & Y \\ X \end{tikzcd} \tag{27}$$

1.51 Theorem. *In this situation, a G-bundle T'/T and a G-equivariant map $f : T' \to Y$, is essentially the same thing as a $G \times H$-bundle $\widetilde{T}' \to T$ and a $G \times H$-equivariant map $F : \widetilde{T}' \to Q$.*

Proof. Given T'/T and $f : T' \to Y$, we let $\widetilde{T}' = Q \times_Y T'$ be the fibered product, and $F : \widetilde{T}' \to Q$ the first projection:

$$\begin{tikzcd} & \widetilde{T}' \arrow[r] \arrow[dl, "F"'] & T' \arrow[dl, "f"'] \arrow[d] \\ Q \arrow[r] \arrow[d] & Y & T \\ X \end{tikzcd}$$

Then $\widetilde{T}' \to T$ is a $G \times H$-bundle, because the G and H-actions on $\widetilde{T}'$ commute, and $\widetilde{T}' \to T'$ is an H-bundle.

Conversely, Given $\widetilde{T}'/T$ and $F : \widetilde{T}' \to Q$, define $T' = \widetilde{T}'/H$ to be the quotient space and $f : T' \to Y$ the map induced by F on quotient spaces, noting that $Y = Q/H$.

One checks that these two processes are essentially inverses of one another. □

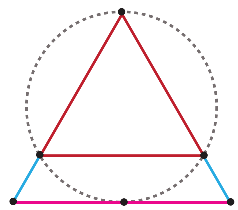
1.52 Corollary. *In this situation, a G-bundle T'/T and a G-equivariant map $f : T' \to Y$, is essentially the same thing as an H-bundle $\widetilde{T} \to T$ and an H-equivariant map $g : \widetilde{T} \to X$.*

Proof. This follows from the theorem by the symmetry of the situation. □

This corollary allows us to prove that moduli problems are equivalent, by comparing the symmetry groupoids of versal families.

We apply this principle to the two notions of family of degenerate triangles.

Let us fix some more data. Let the sphere S have diameter 1, so that under the stereographic projection the equator of S corresponds to the unit circle in $\mathbb{C}$. Fix e_0 to be the great circle equilateral triangle on S, which lies on the real great circle and has a vertex at the North pole ∞. The stereographic projection of e_0 is the pinched triangle with vertices at $-\frac{1}{2}$, $\frac{1}{2}$ and ∞.

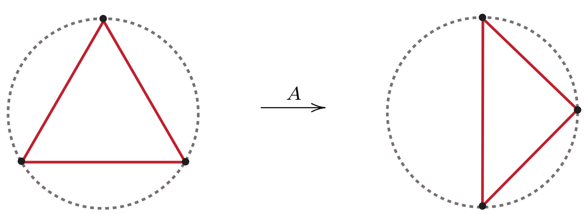


We use the stereographic projection to identify Λ with S.

Now construct the diagram

$$\begin{tikzcd} SO_3 \arrow[r, "\rho"] \arrow[d, "\pi"'] & E \\ \Lambda & \end{tikzcd} \tag{28}$$

The map ρ is defined by $\rho(R) = Re_0$, for every rotation $R \in SO_3$. The map π is defined by $\pi(R) = AR^{-1}\infty$, where $A : S \to S$ is the similarity transformation of S, defined by



In the complex plane, A corresponds to the translation $z \mapsto z + \frac{1}{2}$, for $z \in \mathbb{C} \cup \{\infty\}$. Then we claim that

(i) Diagram (28) is a $(S_3 \times \Lambda, S^1 \times E)$-bibundle,

(ii) restricting to Λ^0 and E^0, we get Diagram (26).

The first claim says that the two moduli problems are equivalent, the second that this equivalence of moduli problems is compatible with the obvious one for non-pinched triangles. We conclude that E supports a versal family of degenerate triangles in the coincident sense, and that its symmetry groupoid is equal to the transformation groupoid $S^1 \times E$.

To see the second claim, note that, by Exercise 1.45, the generalized moduli map $\rho^{-1}(E_0) \to \Lambda$ of $\mathscr{E}|_{E_0}$ is given by $R \mapsto cr(Re_0, \infty) = cr(e_0, R^{-1}\infty) = cr(Ae_0, AR^{-1}\infty) = AR^{-1}\infty$.

1.53 Exercise. Prove that (28) is isomorphic to

$$\begin{tikzcd} SO_3 \arrow[r] \arrow[d] & S_3\backslash SO_3 \\ SO_3/S_1 & \end{tikzcd}$$

The stack of triangles is a stacky version of the double quotient $S_3\backslash SO_3/S^1$, or equivalently, $S_3\backslash PSL_2\mathbb{C}/(\mathbb{C}^+ \rtimes \mathbb{C}^\times)$. □

1.12 Weierstrass compactification

We will now discuss a different way to compactify the stack of oriented triangles. The notion of family containing a pinched triangle will be different.

This new point of view comes about by viewing an oriented degenerate triangle as the zero locus (in $\mathbb{C}$) of a degree 3 polynomial with complex coefficients. If $a_0z^3 + a_1z^2 + a_2z + a_3$, with $a_0, a_1, a_2, a_3 \in \mathbb{C}$, and $a_0 \neq 0$, is a degree 3 polynomial, factor

$$a_0z^3 + a_1z^2 + a_2z + a_3 = a_0(z - e_1)(z - e_2)(z - e_3)\,,$$

and the corresponding triangle has vertices at $e_1, e_2, e_3 \in \mathbb{C}$. We ask of the polynomial that not all three roots coincide.

Two polynomials give rise to the same triangle if they differ by an overall multiplication by an element of $\mathbb{C}^*$, they give rise to similar triangles if one can be transformed into the other by substituting z with $\alpha z+\beta$, for $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$.

To simplify, we can put degree 3 polynomials into *Weierstrass normal form*

$$4z^3 - g_2z - g_3\,, \tag{29}$$

and consider only polynomials of this form. Factoring this polynomial

$$4z^3 - g_2z - g_3 = 4(z - e_1)(z - e_2)(z - e_3)\,,$$

we see that $e_1 + e_2 + e_3 = 0$, and so the centroid (or centre of mass) of the triangle is at the origin.

The family of all degree 3 polynomials in Weierstrass normal form is parametrized by $W = \mathbb{C}^2 - \{(0, 0)\}$, where the coordinates in W are named g_2 and g_3. The polynomial

$$4z^3 - g_2z - g_3 \in \mathbb{C}[z, g_2, g_3]$$

can be thought of as a family of degree 3 polynomials parametrized by W.

1.54 Exercise. We claim that this is a versal family of polynomials. To prove this, we have to agree on what a family of polynomials is locally, and what an isomorphism of local families of polynomials is.

Let us agree that a family of degree 3 polynomials without triple root is locally given by four continuous functions $a_0, a_1, a_2, a_3 : T \to \mathbb{C}$, where a_0 does not vanish anywhere in T, and the polynomial $a_0(t)z^3 + a_1(t)z^2 + a_2(t)z + a_3(t)$ does not have a triple root, for any value $t \in T$.

Prove that locally, you can make continuous coordinate changes and continuous rescalings of the coefficients, to put the polynomial $a_0z^3+a_1z^2+a_2z + a_3$ into *Weierstrass normal form* $4z^3 - g_2z - g_3$. This takes care of Definition 1.33 (i).

Now suppose given two continuous families g, f of polynomials in Weierstrass normal form, both parametrized by the space T. Let us agree that an isomorphism $g \to f$ is given by a continuous map $(a, b) : T \to \mathbb{C}^* \times \mathbb{C}$, such that $4z^3 - g_2z - g_3$ and $4(az + b)^3 - f_2(az + b) - f_3$ differ by an overall rescaling of the coefficients. Show that this implies that $b = 0$, and that $(f_2, f_3) = (a^2g_2, a^3g_3)$. Conclude that the symmetry groupoid of the Weierstrass family, with its canonical topology, is isomorphic to the transformation groupoid $W \times \mathbb{C}^*$, where $\mathbb{C}^*$ acts on W with weights 2 and 3. This takes care of Definition 1.33 (ii). □

1.55 Corollary. *From this exercise, as well as Exercise 1.50, it follows that a continuous family of degree 3 polynomials, up to linear coordinate changes, without triple root, is a pair (P, g), where P is a principal $\mathbb{C}^*$-bundle, and $g : P \to W$ is a $\mathbb{C}^*$-equivariant map. Equivalently, it is given by a complex line bundle $\mathscr{L}/T$ with two sections $g_2 \in \mathscr{L}^{\otimes 2}$ and $g_3 \in \mathscr{L}^{\otimes 3}$, that do not vanish simultaneously.*

The question now arises whether or not this point of view using normalized degree 3 polynomials up to linear substitutions is equivalent to the previous point of view on triangles as a set of three points together with a map to $\mathbb{C}$, see Definition 1.40 and Exercise 1.44.

In the polynomial picture, the only two symmetric oriented triangles are the equilateral one, given by $4z^3 - g_3$, and the bisected line segment, given by $4z^3 - g_2z$. They have 3 and 2 symmetries, respectively. All other triangles are completely asymmetric in this picture, because $\mathbb{C}^*$ has nontrivial stabilizers on W only if one of the coordinates vanishes. In particular, the pinched triangle, given by any polynomial with $g_2^3 = 27g_3^2$, is completely asymmetric. (Recall that previously, the pinched triangle had a non-trivial symmetry given by swapping the two points which map to identical points in $\mathbb{C}$.)

Thus, the stack $\mathfrak{W}$ of degree 3 polynomials cannot be isomorphic to the stack $\mathfrak{L}$ of oriented degenerate triangles. Let us see if there is at least a morphism in either direction.

To define a morphism $\mathfrak{W} \to \mathfrak{L}$ would mean to turn any family of degree 3 polynomials into a family of oriented triangles. In particular, we would have to convert the Weierstrass family of polynomials parametrized by W into a family of oriented degenerate triangles in the sense of Definition 1.40 and Exercise 1.44. This almost works:

The subspace $W' \subset \mathbb{C} \times W$ defined by

$$W' = \{(z, g_2, g_3) \in \mathbb{C} \times W \mid 4z^3 - g_2 z - g_3 = 0\}$$

has a projection map $\pi : W' \to W$, and also a map $W' \to \mathbb{C}$. In general, over every point $(g_2, g_3) \in W$, there are 3 points e_1, e_2, e_3 in W' lying over it, and the images of these points in $\mathbb{C}$ form the vertices of an oriented triangle in $\mathbb{C}$. If $\pi : W' \to W$ were a topological covering map, this would be a family of oriented triangles (because in that case we could, locally, label the three roots of the polynomial by e_1, e_2, e_3). But over the discriminant locus of W, where

$$\Delta = g_2^3 - 27g_3^2$$

vanishes, the covering $W' \to W$ is ramified, and this is impossible.

1.56 Exercise. Prove that, in fact, W does not support any family of degenerate oriented triangles in the sense of Exercise 1.44, which restricts to the Weierstrass family over $W \setminus \{\Delta = 0\}$.

So let us try to define a morphism $\mathfrak{L} \to \mathfrak{W}$. This would amount to converting any family of triangles into a family of polynomials. This is, in fact, possible: a family of oriented triangles is locally given by three continuous functions $A, B, C : T \to \mathbb{C}$, and to these we can associate the continuous family of polynomials $(z-A)(z-B)(z-C)$, whose (continuous!) coefficient functions are given by $a_0 = 1$, $a_1 = -(A + B + C)$, $a_2 = (AB + AC + BC)$, $a_3 = -ABC$. We also have to check compatibility with isomorphisms of families. Indeed, if the family of triangles given by A, B, C is isomorphic to the one given by $A', B'C'$, via $r : T \to \mathbb{C}$ and $s : T \to \mathbb{C}^*$, as in Definition 1.40, then

$$s^3(z - A)(z - B)(z - C) = (sz + r - A')(sz + r - B')(sz + r - C')\,,$$

and so, indeed the corresponding families of polynomials are isomorphic.

We have defined a morphism of stacks $\mathfrak{L} \to \mathfrak{W}$. We can also get a more global geometric picture of this morphism by considering the space

$$Q = \mathbb{C}^2 \setminus \{(0, 0)\}\,,$$

whose coordinates we write as (μ, λ), together with two commuting actions by the groups S_3 and $\mathbb{C}^*$. The group S_3 acts by the six substitutions

$$(\mu, \lambda) \longmapsto$$
$$(\mu, \lambda)\,, \quad (\lambda, \mu)\,, \quad (-\mu, \lambda-\mu)\,, \quad (\lambda-\mu, -\mu)\,, \quad (-\lambda, \mu-\lambda)\,, \quad (\mu-\lambda, -\lambda)\,, \tag{30}$$

and the group $\mathbb{C}^*$ acts by rescaling: $(\mu,\lambda)\cdot\alpha = (\mu\alpha,\lambda\alpha)$, for $\alpha\in\mathbb{C}^*$. We have a diagram of continuous maps

$$\begin{tikzcd} Q \arrow[r, "{g_2(\mu,\lambda),g_3(\mu,\lambda)}", "\substack{\mathbb{C}^*\text{-equivariant}\\ S_3\text{-invariant map}}"'] \arrow[d, "\substack{S_3\text{-equivariant}\\ \mathbb{C}^*\text{-bundle}}"'] & W \\ \Lambda & \end{tikzcd} \tag{31}$$

Here the vertical map $Q\to\Lambda$ is the map $(\mu,\lambda)\mapsto\lambda/\mu$, and the horizontal map $Q\to W$ is given by the formulas

$$g_2 = 2(\lambda^2+\mu^2)-\frac{2}{3}(\lambda+\mu)^2\,, \qquad g_3 = \frac{4}{9}(\lambda^3+\mu^3)-\frac{4}{27}(\lambda+\mu)^3\,. \tag{32}$$

The vertical map $Q\to\Lambda$ is a principal homogeneous $\mathbb{C}^*$-bundle. (It is the tautological $\mathbb{C}^*$-bundle of the Riemann sphere considered as the complex projective line.) Moreover, the map $Q\to\Lambda$ commutes with the S_3-actions on Q and Λ, which one sees by comparing (30) with (25).

The horizontal map is invariant under the S_3-action on Q: the formulas (32) are invariant under the substitutions (30). It is also equivariant with respect to the $\mathbb{C}^*$-actions on Q and W, because in (32), g_2 is quadratic in μ and λ, whereas g_3 is cubic in μ and λ.

1.57 Exercise. Suppose that G and H are topological groups, and that Q is a space with commuting actions by G and H. Suppose that $Q\to X$ is an H-equivariant G-bundle and $Q\to Y$ is a G-equivariant map which is H-invariant. Then there is a natural construction which associates to every pair (P,f), where P is an H-bundle and $f:P\to X$ an H-equivariant map, a pair (P',f'), where P' is a G-bundle and $f':P'\to Y$ is a G-equivariant map. □

So we see that we can associate to every global family of triangles in the sense of Exercise 1.44, a global family of polynomials in the sense of Corollary 1.55. So we get another construction of a morphism of stacks $\mathfrak{L}\to\mathfrak{W}$. (Of course, it is the same morphism, because substituting (32) into (29), and dividing by μ we get $z(z-1)(z-\lambda)$, up to normalization.)

In fact, every morphism of stacks comes about in this way:

1.58 Theorem. *Suppose that X parametrizes a versal family $\mathscr{F}$ for the stack $\mathfrak{X}$, with symmetry groupoid $X\times H$, and Y parametrizes a versal family $\mathscr{G}$ for the stack $\mathfrak{Y}$ with symmetry groupoid $G\times Y$. Any diagram*

$$\begin{tikzcd} Q \arrow[r, "\substack{G\text{-equivariant}\\ H\text{-invariant map}}"] \arrow[d, "\substack{H\text{-equivariant}\\ G\text{-bundle}}"'] & Y \\ X & \end{tikzcd}$$

gives rise to a morphism of stacks $\mathfrak{X} \to \mathfrak{Y}$. (It is an isomorphism, if and only if the H-invariant map $Q \to Y$ is a principal H-bundle.) Conversely, every morphism of stacks $\mathfrak{X} \to \mathfrak{Y}$ comes from such a diagram.

PROOF. That such a space Q gives rise to a morphism of stacks was proved in Exercise 1.57. For the converse, we would need the formal definition of stacks, see Section 2, so let us just remark for now, that given the morphism of stacks $F : \mathfrak{X} \to \mathfrak{Y}$, we can define Q to be the space of triples (x, ϕ, y), where $x \in X$, $y \in Y$, and $\phi : F(\mathscr{F}|_x) \to \mathscr{G}|_y$ is an isomorphism in $\mathfrak{Y}$. The space Q can be endowed with a canonical topology. The H-action on Q is given by $(x, \phi, y) \cdot h = \big(xh, F(h^{-1} : \mathscr{F}|_{xh} \to \mathscr{F}|_x) * \phi, y\big)$, the G-action is given by $(x, \phi, y) \cdot g = \big(x, \phi * (g : \mathscr{G}|_y \to \mathscr{G}|_{yg}), yg\big)$. □

This theorem only applies to the case where there exist versal families whose symmetry groupoids are transformation groupoids. For the general case, see Exercise 2.36.

We can informally write our morphism $\mathfrak{L} \to \mathfrak{W}$ as

$$g_2 = 2(\lambda^2 + 1) - \frac{2}{3}(\lambda + 1)^2 , \qquad g_3 = \frac{4}{9}(\lambda^3 + 1) - \frac{4}{27}(\lambda + 1)^3 ,$$

but we should keep in mind that it is really defined by the diagram (31).

The coarse moduli spaces of $\mathfrak{L}$ and $\mathfrak{W}$ are isomorphic. This common moduli space is another copy of the Riemann sphere, denoted J, and it is customary to write the coordinate as j, and normalize j in such a way that

$$j = 1728 \frac{g_2^3}{g_2^3 - 27 g_3^2} = 256 \frac{(\lambda^2 - \lambda + 1)^2}{\lambda^2 (\lambda - 1)^2} .$$

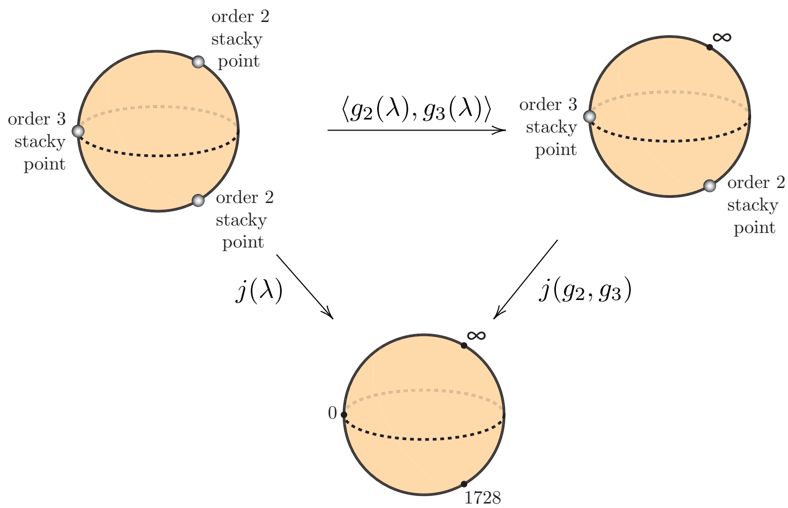
The point $j = 0$ gives the equilateral triangle, the point $j = 1728$ gives the bisected line segment, and the point $j = \infty$ corresponds to the pinched triangle. The real axis in the j-plane contains both the isosceles triangles (for $j < 1728$) and the properly degenerate triangles (for $j \geq 1728$).

We have seen that both $\mathfrak{L}$, the stack of 3 points on the Riemann sphere, up to affine linear transformations, and $\mathfrak{W}$, the stack of degree 3 polynomials up to affine linear substitutions are compactifications of the stack of oriented triangles. The morphism $\mathfrak{L} \to \mathfrak{W}$ is an isomorphism away from the point corresponding to the pinched triangle, or $j = \infty$.

A pictorial representation of the commutative diagram

$$\begin{tikzcd} \mathfrak{L} \arrow[rr] \arrow[dr] & & \mathfrak{W} \arrow[dl] \\ & J & \end{tikzcd}$$

follows:



If we remove from $\mathfrak{L}$ or $\mathfrak{W}$ the line segment corresponding in the j-sphere to the real segment $[1728, \infty]$, we obtain the moduli stack $\widetilde{\mathfrak{M}}$ with a single order 3 stacky point in the middle, which we obtained from the length of sides point of view, see (16).

The sphere J can also be viewed as obtained by sewing together the edge of the cone $\widetilde{M}$ of (12). It is the topological quotient of the bipyramid $\overline{N}^{\pm}$ of (21) or the sphere (24) by S_3.

The j-plane

We have seen that the Riemann sphere is a course moduli space for oriented triangles. There does not exist a modular family parametrized by the j-sphere. That is why there are discontinuities in the triangles corresponding to various j-values of Figure 4.

To see the behaviour near $j = 0$, we pass to a neighbourhood of $j = 0$ by removing $j = 1728$. This corresponds to setting $g_3 = 1$, and going up to the Riemann surface defined by solving the equation

$$j = 1728\frac{g_2^3}{g_2^3 - 27}$$

for g_2, which is of degree 3 over the j-plane. We obtain Figure 5. The pinched triangle $j = \infty$ appears three times in this picture, namely at $g_2 = 3e^{2\pi i n/3}$, $n = 0, 1, 2$. No oriented triangle appears more than once near these points. This corresponds to the fact that the pinched triangle is asymmetric in this picture.

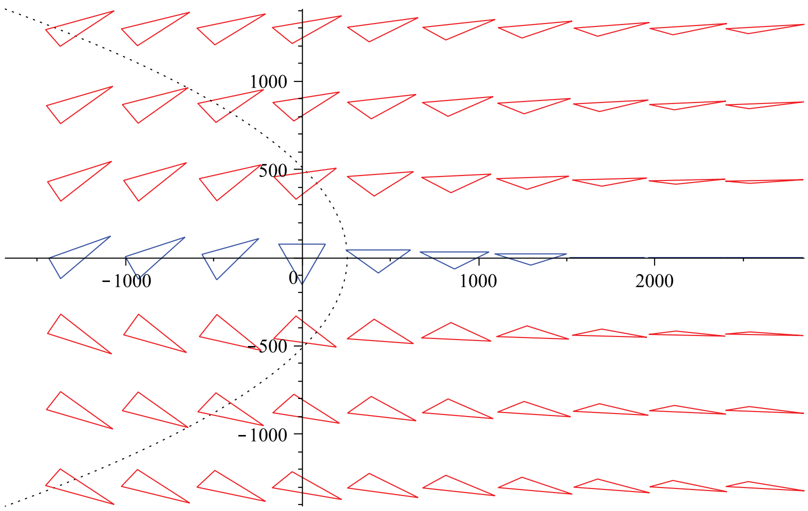


Figure 4: The j-plane. The locus of right triangles is a parabola.

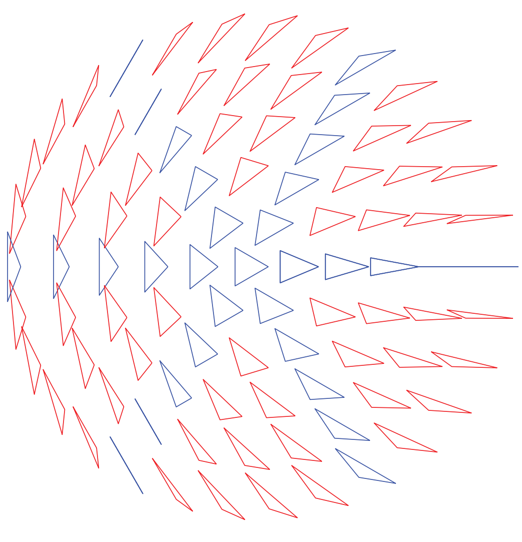


Figure 5: The g_2-plane

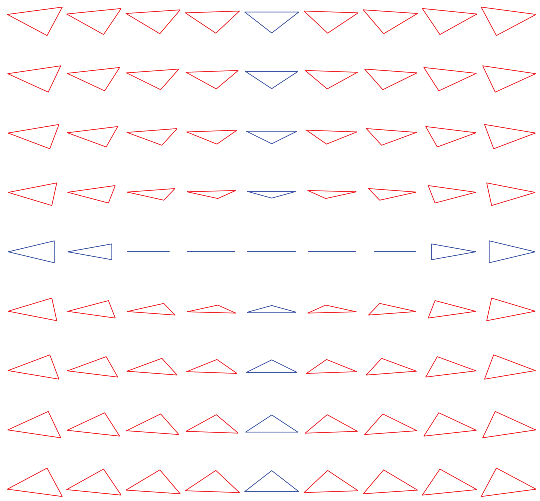


Figure 6: The g_3-plane

To see the behaviour near $j = 1728$, we pass to a neighbourhood of $j = 1728$ by setting $g_2 = 1$. This means going up to the degree 2 Riemann surface defined by solving

$$j = 1728 \frac{1}{1 - 27g_3^2}$$

for g_3. We get Figure 6. This time there are two pinched triangles, at $g_3 = \pm\frac{1}{9}\sqrt{3}$. At the centre is the bisected line segment which has order 2 symmetry group, it appears once in the picture. All other triangles appear twice in the picture, they are asymmetric.

For further discussion of the g_2-and the g_3-planes, see Example 2.50.

Note the difference between the neighbourhoods of the pinched triangle in these pictures, and in the $\mathfrak{L}$-picture, (22). In the latter, the three vertices can be consistently labelled near $\lambda = \infty$, but in the current ones, this is impossible: small loops around the pinched triangles give rise to Moebius strips and Klein bottles as in (5). So in the polynomial point of view, there is no local labelling of vertices.

It is a matter of taste, which of the two completions $\mathfrak{L}$ or $\mathfrak{W}$ of $\widetilde{\mathfrak{M}}$ one considers to be the 'right one'. It also depends on applications, which one of the two could be more useful.

1.59 Exercise. We can enlarge the stack $\mathfrak{W}$ to include the triangle whose 3 vertices coincide. In Corollary 1.55, replace W by $\mathbb{C}^2$, or equivalently drop the requirement that the sections g_2 and g_3 cannot vanish simultaneously.

The Weierstrass family of polynomials extends to $\mathbb{C}^2$, and is versal. It has symmetry groupoid given by the action of $\mathbb{C}^*$ on $\mathbb{C}^2$, rather than $W = \mathbb{C}^2 \setminus \{(0,0)\}$.

The coarse moduli space of this triangle has one more point than the j-sphere. Every point in the j-sphere is in the closure of this additional point. Thus, the coarse moduli space is not Hausdorff any longer. This is the main reason for excluding the triangle reduced to a point. $\square$

2 Formalism

Let us now make the theory we have developed completely rigorous. This will require some formalism.

2.1 Objects in continuous families: Categories fibered in groupoids

We will first make precise what we mean by mathematical objects that can vary in continuous families.

We start with the category $\mathscr{T}$ of topological spaces. This category consists of all topological spaces and all continuous maps. The class of topological spaces forms the class of **objects** of $\mathscr{T}$, and for every two topological spaces S, T, the set of continuous maps from S to T forms the set of **morphisms** from the object S to the object T. Every object has an identity morphism (the identity map, which is continuous) and composition of morphisms (i.e., composition of continuous maps) is associative.

2.1 Definition. A **groupoid fibration** (or a *category fibered in groupoids*) over $\mathscr{T}$ is another category $\mathfrak{X}$, together with a functor $\mathfrak{X} \to \mathscr{T}$, such that two axioms, specified below, are satisfied. If the functor $\mathfrak{X} \to \mathscr{T}$ maps the object x of $\mathfrak{X}$ to the topological space T, we say that x *lies over* T, or that x is an $\mathfrak{X}$-*family parametrized* by T, and we write x/T. If the morphism $\eta : x \to y$ in $\mathfrak{X}$ maps to the continuous map $f : T \to S$, we say that η *lies over* f, or *covers* f. The two groupoid fibration axioms are

(i) for every continuous map $T' \to T$, and every $\mathfrak{X}$-family x/T, there exists an $\mathfrak{X}$-family x'/T' and an $\mathfrak{X}$-morphism $x' \to x$ covering $T' \to T$,

$$\begin{tikzcd} x' \arrow[r] \arrow[d, no head] & x \arrow[d, no head] \\ T' \arrow[r] & T \end{tikzcd} \tag{33}$$

(ii) the object x'/T' together with the morphism $x' \to x$ is unique up to a unique isomorphism,

$$\begin{tikzcd} x'' \arrow[rr] \arrow[dr, dotted] \arrow[dd, no head] & & x \arrow[dd, no head] \\ & x' \arrow[ur] \arrow[dl, no head] & \\ T' \arrow[rr] & & T \end{tikzcd}$$

which means that if $x'' \to x$ is another $\mathfrak{X}$-morphism covering $T' \to T$, there exists a unique $\mathfrak{X}$-morphism $x' \to x''$, covering the identity of

T', and making the diagram

$$\begin{tikzcd} x' \arrow[r] \arrow[dr] & x'' \arrow[d] \\ & x \end{tikzcd}$$

in $\mathfrak{X}$ commute.

If we have a diagram (33), then the family x'/T' is said to be the *pullback*, or *restriction* of the family x/T, via the continuous map $f : T' \to T$. Using the definite article is justified by the fact that x' is, up to isomorphism, completely determined by x/T and $T' \to T$. We use notation $x' = f^*x$, or $x' = x|_{T'}$. Sometimes the word restriction is reserved for the case that $T' \to T$ is the inclusion map of a subspace.

The notion of groupoid fibration over $\mathscr{T}$ captures two notions at once: isomorphisms of families, and pullbacks of families. The first axiom says that restriction/pullback always exists, and the second that restriction/pullback is essentially unique. Note that the axioms also imply that pullback is associative: $f^*g^*x = (gf)^*x$. The equality sign stands for *canonically isomorphic*.

For isomorphisms of families, see the following exercise:

2.2 Exercise. Let $\mathfrak{X} \to \mathscr{T}$ be a groupoid fibration. Let T be a topological space. The *fibre* of $\mathfrak{X}$ over T, notation $\mathfrak{X}(T)$, consists of all objects of $\mathfrak{X}$ lying over T, and all morphisms of $\mathfrak{X}$ lying over the identity map of T. Prove that $\mathfrak{X}(T)$ is a groupoid, i.e., a category in which all morphisms are invertible.

Suppose x/T is an $\mathfrak{X}$-family parametrized by T. Let $t \in T$ be a point of t. If we think of t as a continuous map $t : * \to T$, from the one point space $*$ to T, we see that we have a pullback object $x_t = t^*x$ in the category $\mathfrak{X}(*)$. As we vary $t \in T$, the various x_t form the *family members* of x.

2.3 Exercise. For $\mathfrak{M}$, the stack of triangles, as formalized in Exercise 1.11, the corresponding category fibered in groupoids has objects (T, T', a), where T is a topological space, $T' \to T$ is a degree 3 covering and $a : T' \to \mathbb{R}_{>0}$ is continuous (such that the triangle inequality is satisfied). A morphism in $\mathfrak{M}$ from (S, S', b) to (T, T', a) is a pair (f, ϕ), where f and ϕ are continuous maps making the triangle in (34) commute,

$$\begin{tikzcd} S' \arrow[rr, "b"] \arrow[d] \arrow[dr, "\phi"] & & \mathbb{R}_{>0} \\ S \arrow[dr, "f"] & T' \arrow[ur, "a"] \arrow[d] & \\ & T & \end{tikzcd} \tag{34}$$

and the parallelogram in (34) a pullback diagram. Composition in the category $\mathfrak{M}$ is defined in a straightforward manner, and the functor $\mathfrak{M} \to \mathscr{T}$ is defined by projecting onto the first component: $(T, T', a) \mapsto T$, and $(f, \phi) \mapsto f$.

The fact that $\mathfrak{M}$ is a groupoid fibration follows from the requirement that the morphisms in $\mathfrak{M}$ define pullback diagrams. □

2.4 Exercise. It is sometimes convenient to choose, for every x/T and for every $T' \to T$ a pullback. (This could be done, for example, by specifying a particular construction of the pullback family, but in general requires the axiom of choice for classes.) The chosen pullbacks give rise to a pullback functor $f^* : \mathfrak{X}(T) \to \mathfrak{X}(T')$, for every $f : T' \to T$. They also give rise, for every composition of continuous maps $T'' \xrightarrow{f} T' \xrightarrow{g} T$, to a natural transformation $\theta_{fg} : (gf)^* \Rightarrow f^* \circ g^*$. The θ have to satisfy an obvious compatibility condition, with respect to composition of continuous maps. So a groupoid fibration with chosen pullbacks gives rise to a *lax functor* from $\mathscr{T}$ to the 2-category of groupoids.

2.5 Exercise. Every topological space X gives rise to a tautological groupoid fibration $\underline{X}$, in such a way that X is the fine moduli space of $\underline{X}$. So $\underline{X}$-families parametrized by the topological space T are continuous maps $T \to X$, and morphisms are commutative triangles

$$\begin{tikzcd} T' \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & T & \end{tikzcd}$$

The structure functor $\underline{X} \to \mathscr{T}$ maps $T \to X$ to T.

2.6 Definition. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be groupoid fibrations over $\mathscr{T}$. A **morphism** of groupoid fibrations $F : \mathfrak{X} \to \mathfrak{Y}$ is a functor F, compatible with the structure functors to $\mathscr{T}$. This means that

(i) for every object x of $\mathfrak{X}$, lying over the topological space T, the object $F(x)$ of $\mathfrak{Y}$ also lies over T,

(ii) for every morphism $\eta : x' \to x$ of $\mathfrak{X}$ lying over $f : T' \to T$, the morphism $F(\eta) : F(x') \to F(x)$ also lies over f.

Therefore a morphism of groupoid fibrations turns $\mathfrak{X}$-families into $\mathfrak{Y}$-families, in a way compatible with pullback of families: $F(f^*x) = f^*F(x)$. (Again, the equality stands for canonically isomorphic).

2.7 Definition. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be groupoid fibrations over $\mathscr{T}$, and $F, G : \mathfrak{X} \to \mathfrak{Y}$ morphisms. Then a **2-isomorphism** from F to G

$$\begin{tikzcd} \mathfrak{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathfrak{Y} \end{tikzcd}$$

is a natural transformation $\theta : F \to G$, such that for every object $x/T \in \mathfrak{X}$ the morphism $\theta(x) : F(x) \to G(x)$ in $\mathfrak{Y}$ lies over the identity of T.

The hierarchy (groupoid fibrations,morphisms,2-isomorphisms) forms a 2-category, formally identical to the hierarchy (categories,functors,natural transformations), with the added benefit that all 2-isomorphisms are invertible.

Two groupoid fibrations $\mathfrak{X}$ and $\mathfrak{Y}$ are called **isomorphic**, or **equivalent**, if there exist morphisms $F : \mathfrak{X} \to \mathfrak{Y}$ and $G : \mathfrak{Y} \to \mathfrak{X}$, and 2-isomorphisms $\theta : G \circ F \Rightarrow \mathrm{id}_{\mathfrak{X}}$ and $\eta : F \circ G \Rightarrow \mathrm{id}_{\mathfrak{Y}}$. In this case, both F and G are called **isomorphisms** or *equivalences* of groupoid fibrations.

2.8 Exercise. Prove that a morphism of groupoid fibrations $F : \mathfrak{X} \to \mathfrak{Y}$ is an isomorphism if it is an equivalence of categories. This is the case if F is *fully faithful* and *essentially surjective*.

2.9 Exercise. Let $\mathfrak{X}$ be a groupoid fibration and T a topological space. Show that a morphism $\underline{T} \to \mathfrak{X}$ is the same thing as an $\mathfrak{X}$-family x, parametrized by T, together with a chosen pullback family f^*x, for every continuous map $f : T' \to T$. So if $\mathfrak{X}$ is endowed with chosen pullbacks, as in Exercise 2.4, then a morphism $\underline{T} \to \mathfrak{X}$ is the same thing as an $\mathfrak{X}$-family over T. Moreover, a 2-isomorphism

$$\begin{tikzcd} \underline{T} \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathfrak{X} \end{tikzcd}$$

is the same thing as an isomorphism of $\mathfrak{X}$-families $\theta : x \to y$.

One should think of the morphism $\underline{T} \to \mathfrak{X}$ as the moduli map corresponding to the family x/T. Viewing an $\mathfrak{X}$-family over T as a morphism $\underline{T} \to \mathfrak{X}$ is a very powerful way of thinking, but all arguments can always be formulated purely in the language of groupoid fibrations and $\mathfrak{X}$-families, no result depends on choices of pullbacks existing.

Fibered products of groupoid fibrations

Let $F : \mathfrak{X} \to \mathfrak{Z}$ and $G : \mathfrak{Y} \to \mathfrak{Z}$ be morphisms of groupoid fibrations over $\mathscr{T}$. The **fibered product** of $\mathfrak{X}$ and $\mathfrak{Y}$ over $\mathfrak{Z}$ is the groupoid fibration $\mathfrak{W}$ defined as follows: $\mathfrak{W}$-families parametrized by T are triples (x, ϕ, y), where x/T is an $\mathfrak{X}$-family, y/T is a $\mathfrak{Y}$-family, and $\phi : F(x) \to G(y)$ is an isomorphism of $\mathfrak{Z}$-families. A morphism from (x', ϕ', y') over T' to (x, ϕ, y) over T, covering the continuous map $f : T' \to T$, is a pair (α, β), where $\alpha : x' \to x$ is a morphism in $\mathfrak{X}$ covering f and $\beta : y' \to y$ is a morphism in $\mathfrak{Y}$ covering f, such that

$$\begin{tikzcd} F(x') \arrow[r, "\phi'"] \arrow[d, "F(\alpha)"'] & G(y') \arrow[d, "G(\beta)"] \\ F(x) \arrow[r, "\phi"] & G(y) \end{tikzcd}$$

commutes in $\mathfrak{Z}$.

There is a 2-commutative diagram

$$\begin{tikzcd} \mathfrak{W} \arrow[r, "\mathrm{pr}_{\mathfrak{Y}}"] \arrow[d, "\mathrm{pr}_{\mathfrak{X}}"'] \arrow[dr, phantom, "\overset{\phi}{\nearrow\!\!\!\nearrow}"] & \mathfrak{Y} \arrow[d, "G"] \\ \mathfrak{X} \arrow[r, "F"] & \mathfrak{Z} \end{tikzcd}$$

This means that ϕ is a 2-isomorphism from $F \circ \mathrm{pr}_{\mathfrak{X}}$ to $G \circ \mathrm{pr}_{\mathfrak{Y}}$. It is defined by $(x, \phi, y) \mapsto \phi$.

Given an arbitrary 2-commutative diagram

$$\begin{tikzcd} \mathfrak{U} \arrow[r, "Q"] \arrow[d, "P"'] \arrow[dr, phantom, "\overset{\psi}{\nearrow\!\!\!\nearrow}"] & \mathfrak{Y} \arrow[d, "G"] \\ \mathfrak{X} \arrow[r, "F"] & \mathfrak{Z} \end{tikzcd} \tag{35}$$

there is an induced morphism $\mathfrak{U} \to \mathfrak{W}$ to the fibered product, given on objects by $u \mapsto \big(P(u), \psi(u), Q(u)\big)$. The diagram (35) is called **2-cartesian**, if $\mathfrak{U} \to \mathfrak{W}$ is an equivalence of groupoid fibrations.

2.10 Example. The main example is the symmetry groupoid of a family: let x/T be an $\mathfrak{X}$-family parametrized by T, and assume that its symmetry groupoid $\Gamma \rightrightarrows T$ satisfies the technical condition of Definition 1.33. Think of x as a morphism $\underline{T} \to \mathfrak{X}$ as in Exercise 2.9. Then there is a 2-cartesian diagram of groupoid fibrations

$$
\begin{tikzcd}
\underline{\Gamma} \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\overset{\phi}{\nearrow}"] & \underline{T} \arrow[d, "x"] \\
\underline{T} \arrow[r, "x"] & \mathfrak{X}
\end{tikzcd}
\tag{36}
$$

where $\phi : s^*x \to t^*x$ is the tautological isomorphism.

2.2 Families characterized locally: Prestacks

We had several examples where we characterized a moduli problem by specifying what continuous families looked like locally, and what isomorphisms between families looked like. These were examples of prestacks. Also, it is in the context of prestacks that symmetry groupoids behave well.

2.11 Remark. For experts we should note that our definition of prestack is stronger than the usual one: we define a prestack to be a groupoid fibration with representable diagonal: this implies that all isomorphism functors are sheaves. In practice, the stronger condition is often verified, so our non-standard terminology seems justified.

We need some terminology. Let $\mathfrak{X}$ be a groupoid fibration over $\mathscr{T}$. Let x/T and y/S be $\mathfrak{X}$-families. The *space of isomorphisms* $\underline{\mathrm{Isom}}(x, y)$ is the set of all triples (t, ϕ, s), where $t \in T$, $s \in S$, and $\phi : x_t \to y_s$ is an isomorphism in $\mathfrak{X}(*)$. The topology on this space of isomorphisms is, by definition, the finest topology such that for every pair (U, ϕ), where U is a topological space, endowed with maps $U \to T$ and $U \to S$, and $\phi : x|_U \to y|_U$ is an isomorphism in the category $\mathfrak{X}(U)$, the induced map $U \to \underline{\mathrm{Isom}}(x, y)$, defined by $u \mapsto \phi_u$, is continuous.

2.12 Definition. A **prestack** is a groupoid fibration $\mathfrak{X}$ over $\mathscr{T}$, such that for any two objects x/T and y/S, the following conditions are satisfied:

(i) the canonical maps $\underline{\mathrm{Isom}}(x, y) \to T$ and $\underline{\mathrm{Isom}}(x, y) \to S$ are continuous,

(ii) for any continuous map $\alpha : U \to \underline{\mathrm{Isom}}(x, y)$, there exists a unique isomorphism of families $\phi : x|_U \to y|_U$ giving rise to α.

For the second condition, it suffices that the tautological isomorphism over $\underline{\mathrm{Isom}}(x, y)$ is continuous (i.e., occurs in the groupoid fibration $\mathfrak{X}$).

2.13 Exercise. If the prestack $\mathfrak{X}$ has chosen pullbacks, we can think of

x/T and y/S as morphisms, as in Exercise 2.9. Then the diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \underline{S} \arrow[d, "y"] \\ \underline{T} \arrow[r, "x"] & \mathfrak{X} \end{tikzcd}$$

is 2-cartesian.

2.14 Exercise. Conversely, suppose $\mathfrak{X}$ is a prestack and $F : \underline{U} \to \mathfrak{X}$ a morphism, where U is a topological space. Then there exists a topological space R, and a 2-cartesian diagram

$$\begin{tikzcd} \underline{R} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \underline{U} \arrow[d, "F"] \\ \underline{U} \arrow[r, "F"] & \mathfrak{X} \end{tikzcd}$$

Moreover $R \rightrightarrows U$ is a topological groupoid. (It is isomorphic to the symmetry groupoid of the $\mathfrak{X}$-family $F(\mathrm{id}_U)$.)

Versal families

The following repeats Definition 1.33:

2.15 Definition. Let $\mathfrak{X}$ be a groupoid fibration. An $\mathfrak{X}$-family x/T is called **versal**, if every family can be locally pulled back from x/T, and if $\underline{\mathrm{Isom}}(x,x)$ satisfies the conditions of Definition 2.12.

So, if $\mathfrak{X}$ is a prestack, then a family x/T is versal if every family can be locally pulled back from x/T. The following converse is more useful:

2.16 Lemma. *Suppose that a groupoid fibration admits a versal family. Then it is a prestack.*

Proof. Let x/T and y/S be $\mathfrak{X}$-families. We have to prove that there exists a topological space I, and a 2-cartesian diagram

$$\begin{tikzcd} \underline{I} \arrow[r] \arrow[d] & \underline{T} \times \underline{S} \arrow[d, "x \times y"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

Because we can glue topological spaces along open subspaces, it is enough to cover T and S with open subspaces $T = \bigcup U_i$, and $S = \bigcup V_j$, and prove that there exists a topological space J_{ij} and a 2-cartesian diagram

$$\begin{tikzcd}
\underline{J}_{ij} \arrow[r] \arrow[d] & \underline{U}_i \times \underline{V}_j \arrow[d, "x|_{U_i} \times y|_{V_j}"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

for all i, j. Now we know that there is a 2-cartesian diagram

$$\begin{tikzcd}
\underline{\Gamma}_1 \arrow[r] \arrow[d] & \underline{\Gamma}_0 \times \underline{\Gamma}_0 \arrow[d] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

where $\Gamma_1 \rightrightarrows \Gamma_0$ is the symmetry groupoid of the given versal family. By the first property of versal family, we can cover $T = \bigcup U_i$ and $S = \bigcup_j V_j$, and find 2-commutative diagrams

$$\begin{tikzcd}
U_i \arrow[r, "f_i"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \Gamma_0 \arrow[d] \\
T \arrow[r, "x"] & \mathfrak{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
V_j \arrow[r, "g_j"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \Gamma_0 \arrow[d] \\
S \arrow[r, "y"] & \mathfrak{X}
\end{tikzcd}$$

Then we define J_{ij} to be the fibered product

$$\begin{tikzcd}
J_{ij} \arrow[r] \arrow[d] & U_i \times V_j \arrow[d, "f_i \times g_j"] \\
\Gamma_1 \arrow[r] & \Gamma_0 \times \Gamma_0
\end{tikzcd}$$

and glue the J_{ij} to obtain I. $\square$

2.17 Exercise. For a topological space X, a versal family for the groupoid fibration $\underline{X}$ is the same thing as a continuous map $f : T \to X$ which admits local sections, i.e., for every $x \in X$, there exists an open neighbourhood $x \in U \subset X$, and a continuous map $s : U \to T$, such that $f \circ s$ is equal to the inclusion map $U \to X$.

The symmetry groupoid of $T \to X$ is the fibered product groupoid $T \times_X T \rightrightarrows T$. Such groupoids are called **banal groupoids**. Banal groupoids are equivalence relations.

2.3 Families which can be glued: Stacks

2.18 Definition. A prestack is called a **stack**, if it satisfies the gluing axiom of Exercise 1.12.

2.19 Example. In Section 1.10, (see also Definition 1.40) we defined a groupoid fibration which we shall call $\mathfrak{L}^{\mathrm{pre}}$. Objects of $\mathfrak{L}^{\mathrm{pre}}$ are quadruples (T, A_1, A_2, A_3), where T is the parameter space and $A_i : T \to \mathbb{C}$ are continuous functions (no more than two of which are ever allowed to coincide). A morphism from (T', A_1', A_2', A_3') to (T, A_1, A_2, A_3) is a quadruple (f, σ, R, S), where $f : T' \to T$ is a continuous map between the parameter spaces, $\sigma \in S_3$ is a permutation of $\{1, 2, 3\}$, and $R : T' \to \mathbb{C}$ and $S : T' \to \mathbb{C}^*$ are continuous maps, such that $A'_{\sigma(i)} = S \cdot (A_i \circ f) + R$, for $i = 1, 2, 3$.

This groupoid fibration is a prestack, but not a stack. It is a prestack, because it admits a versal family, as we had seen in Section 1.10. But it is not a stack: it is possible to specify gluing data in $\mathfrak{L}^{\mathrm{pre}}$ which give rise to families with a twist, even though all $\mathfrak{L}^{\mathrm{pre}}$-families are untwisted.

2.4 Topological stacks

2.20 Definition. A stack $\mathfrak{X}$, which admits a versal family, is called a **topological stack**.

If $\Gamma_1 \rightrightarrows \Gamma_0$ is the symmetry groupoid of a versal family for $\mathfrak{X}$, we say that $\Gamma_1 \rightrightarrows \Gamma_0$ is a **presentation** of $\mathfrak{X}$.

All our examples $\mathfrak{M}$, $\widetilde{\mathfrak{M}}$, $\overline{\mathfrak{M}}$, $\mathfrak{L}$, $\mathfrak{W}$, etc., are topological stacks.

In practice the current definition is not strong enough: to be able to 'do topology' on a topological stack, we have to put conditions on the spaces Γ_0 and Γ_1, or on the maps $s, t : \Gamma_1 \to \Gamma_0$ of the symmetry groupoid of a versal family. For example, to do homotopy theory, we need that s and t are *topological submersions*, see Section 2.7.

The nicest topological stacks, which are closest to topological spaces are those of Deligne-Mumford type, see Definition 2.45. In this case s and t are required to be local homeomorphisms. In particular, all symmetry groups are discrete. If they are *separated* (see Exercise 2.40), these stacks admit the structure of an *orbispace*, see Theorem 2.49.

Properties of $\Gamma_1 \to \Gamma_0 \times \Gamma_0$ are *separation properties* of $\mathfrak{X}$. See for example Exercise 2.40 or Proposition 2.51.

2.21 Exercise. Note that Definition 1.18 applies to any groupoid fibration. Prove that if $\Gamma_1 \rightrightarrows \Gamma_0$ is a presentation of a topological stack $\mathfrak{X}$, then the image of Γ_1 in $\Gamma_0 \times \Gamma_0$ defines an equivalence relation on Γ_0, and the topological quotient of Γ_0 by this equivalence relation is a coarse moduli space for $\mathfrak{X}$.

Topological groupoids

The main general example of a topological stack is the stack of Γ-torsors, for a topological groupoid Γ.

2.22 Definition. A **topological groupoid** is a groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, as in Definition 1.10, where Γ_1 and Γ_0 are also topological spaces, and all structure maps s, t, e, μ, ϕ are continuous. Here $e : \Gamma_0 \to \Gamma_1$ is the identity map, $\mu : \Gamma_2 \to \Gamma_1$ is the composition map, and $\phi : \Gamma_1 \to \Gamma_1$ is the inverse map. Often we abbreviate the notation to $\Gamma_\bullet$ or simply Γ.

A **continuous morphism** of topological groupoids is a functor $\phi : \Gamma \to \Gamma'$, such that the two maps $\phi_0 : \Gamma_0 \to \Gamma'_0$ and $\phi_1 : \Gamma_1 \to \Gamma'_1$ are continuous.

2.23 Exercise. The symmetry groupoid of a family in a prestack is a topological groupoid.

2.24 Exercise. Suppose Γ is the symmetry groupoid of the family x/T in a prestack, and that $f : T' \to T$ is a continuous map. Form the fibered product of topological spaces

$$\begin{tikzcd} \Gamma' \arrow[r] \arrow[d] & T' \times T' \arrow[d, "f \times f"] \\ \Gamma \arrow[r, "s \times t"] & T \times T \end{tikzcd}$$

Prove that Γ' is a topological groupoid, and that it is isomorphic to the symmetry groupoid of the pullback family f^*x over T'.

This exercise leads to the following definition.

2.25 Definition. Let $\Gamma_1 \rightrightarrows \Gamma_0$ be a topological groupoid and $\Gamma'_0 \to \Gamma_0$ a continuous map. The fibered product

$$\begin{tikzcd} \Gamma'_1 \arrow[r] \arrow[d] & \Gamma'_0 \times \Gamma'_0 \arrow[d] \\ \Gamma_1 \arrow[r] & \Gamma_0 \times \Gamma_0 \end{tikzcd}$$

defines another topological groupoid $\Gamma'_1 \rightrightarrows \Gamma'_0$, called the **restriction** of the groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ via the map $\Gamma'_0 \to \Gamma_0$. It comes with a continuous morphism of groupoids $\Gamma' \to \Gamma$ which is *fully faithful*, in categorical terms.

2.26 Exercise. If G is a topological group acting continuously on the topological space X, then the transformation groupoid $X \times G \rightrightarrows X$ is a topological groupoid. Note the special cases $G = \{e\}$ or $X = \{*\}$.

Generalized moduli maps: Groupoid Torsors

2.27 Definition. Let $\Gamma_\bullet$ be a topological groupoid. A **$\Gamma_\bullet$-torsor** over the topological space T is a pair (P_0, ϕ), where P_0 is a topological space, endowed with a continuous map $\pi : P_0 \to T$, and $\phi : P_\bullet \to \Gamma_\bullet$ is a continuous morphism of topological groupoids. Here $P_\bullet$ is the banal groupoid associated to $P_0 \to T$ (Exercise 2.17). Moreover, it is required that

(i) the diagram

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d] & \Gamma_1 \arrow[d] \\ P_0 \arrow[r] & \Gamma_0 \end{tikzcd} \tag{37}$$

is a pullback diagram of topological spaces,

(ii) the map $P_0 \to T$ admits local sections, as in Exercise 2.17.

A **morphism** of $\Gamma_\bullet$-torsors from (P_0', ϕ') over T' to (P, ϕ) over T, consists of a pullback diagram of topological spaces

$$\begin{tikzcd} P_0' \arrow[r] \arrow[d] & P_0 \arrow[d] \\ T' \arrow[r] & T \end{tikzcd}$$

such that the induced diagram

$$\begin{tikzcd} P_\bullet' \arrow[d] \arrow[dr] & \\ P_\bullet \arrow[r] & \Gamma_\bullet \end{tikzcd}$$

is a commutative diagram of topological groupoids.

2.28 Exercise. The Γ-torsors form a stack. It is called the **stack associated to** the topological groupoid Γ.

2.29 Exercise. Show that if the topological groupoid Γ is a transformation groupoid $X \times G \rightrightarrows X$, then a Γ-torsor over T is the same thing as a principal homogeneous G-bundle $P \to T$, together with a G-equivariant map $P \to X$. Note the special cases $G = \{e\}$ and $X = \{*\}$.

2.30 Exercise. Let Γ be a topological groupoid. Prove that the stack of Γ-torsors $\mathfrak{X}$ is a topological stack, by proving that there is a tautological Γ-torsor over the topological space Γ_0. The symmetry groupoid of the tautological Γ-torsor is the groupoid Γ itself. Thus Γ itself is a presentation of $\mathfrak{X}$. There is a 2-cartesian diagram

$$\begin{tikzcd} \underline{\Gamma}_1 \arrow[r] \arrow[d] & \underline{\Gamma}_0 \arrow[d] \\ \underline{\Gamma}_0 \arrow[r] \arrow[ur, phantom, "\nearrow\!\!\!\!\nearrow"] & \mathfrak{X} \end{tikzcd} \tag{38}$$

The following theorem generalizes Exercise 1.34, and Exercise 1.50.

2.31 Theorem. *If $\mathfrak{X}$ is a topological stack, and $\Gamma_1 \rightrightarrows \Gamma_0$ is the symmetry groupoid of a versal family x/X_0, then $\mathfrak{X}$ is isomorphic to the stack of $\Gamma_\bullet$-torsors.*

Proof. As we have studied the proof in detail in special cases, we will only say that to define the morphism from $\mathfrak{X}$ to $\Gamma_\bullet$-torsors, we associate to an $\mathfrak{X}$-family y/T the $\Gamma_\bullet$-torsor $\underline{\mathrm{Isom}}(x, y)$ (which is the generalized moduli map of y/T). $\square$

In our examples, we often specified a moduli problem by giving a prestack, then constructing a versal family for the prestack, and finally replacing the prestack by the stack of torsors for the symmetry groupoid of the prestack. This process is known as **stackification**. We followed it, for example, when passing from Definition 1.40 to Proposition 1.43, or when going from Exercise 1.54 to Corollary 1.55.

2.32 Example. If $V/*$ is a versal family with only one family member, so that the symmetry groupoid of $V/*$ is just a topological group G, then families are the same thing as *twisted forms* of V, i.e., locally constant families all of whose family members are isomorphic to V. The stack of twisted forms of V is equivalent to the stack of G-torsors. Often, we say simply *forms*, instead of twisted forms.

Change of versal family: Morita equivalence of groupoids

Suppose x/Γ_0 is a versal $\mathfrak{X}$-family with symmetry groupoid $\Gamma_\bullet$, and let y/S be an arbitrary $\mathfrak{X}$-family. There is a 2-cartesian diagram (we have stopped underlining)

$$\begin{tikzcd} P_1 \arrow[r, "\phi_1"] \arrow[d, shift left] \arrow[d, shift right] & \Gamma_1 \arrow[d, shift left] \arrow[d, shift right] \\ P_0 \arrow[r, "\phi_0"] \arrow[d] & \Gamma_0 \arrow[d, "x"] \\ S \arrow[r, "y"] \arrow[ur, phantom, "\Nearrow"] & \mathfrak{X} \end{tikzcd} \tag{39}$$

in the shape of a cube (all 6 sides of the cube are cartesian). The right hand edge of the diagram abbreviates (36), and $(P_\bullet, \phi_\bullet)$ is the generalized moduli map of y. Because this diagram is 2-cartesian, we have $P_0 = \underline{\mathrm{Isom}}(y, x)$.

Suppose x/Γ_0 is a versal family with symmetry groupoid $\Gamma_\bullet$, and y/Γ_0' is a second versal family with symmetry groupoid $\Gamma_\bullet'$. Then we can form a larger 2-cartesian diagram

$$\begin{tikzcd}
P_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & Q \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \Gamma_1 \arrow[d, shift left] \arrow[d, shift right] \\
Q' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \Gamma_0 \arrow[d, "x"] \\
\Gamma'_1 \arrow[r, shift left] \arrow[r, shift right] & \Gamma'_0 \arrow[r, "y"] & \mathfrak{X}
\end{tikzcd}$$

We see that P_0 is at the same time a $\Gamma_\bullet$-torsor over Γ'_0 and a $\Gamma'_\bullet$-torsor over Γ_0. We say that P_0 is a $\Gamma_\bullet$-$\Gamma'_\bullet$-bitorsor.

2.33 Exercise. Conversely, if there exists a $\Gamma_\bullet$-$\Gamma'_\bullet$-bitorsor, then the stack of $\Gamma_\bullet$-torsors and the stack of $\Gamma'_\bullet$-torsors are isomorphic. This is the general case of Corollary 1.52.

2.34 Definition. Two topological groupoids Γ and Γ' are called **Morita equivalent** if there exists a Γ-Γ'-bitorsor.

Thus we can say that stacks 'are' *groupoids up to Morita equivalence*.

2.35 Exercise. Prove that two topological groupoids Γ and Γ' are Morita equivalent if and only if there exists a third topological groupoid Γ'' and two morphisms $\Gamma'' \to \Gamma$ and $\Gamma'' \to \Gamma'$ which are *topological equivalences*. Here, a morphism $\Gamma'' \to \Gamma$ of topological groupoids is a **topological equivalence** if

(i) (topological full faithfulness) the diagram

$$\begin{tikzcd}
\Gamma''_1 \arrow[r] \arrow[d] & \Gamma''_0 \times \Gamma''_0 \arrow[d] \\
\Gamma_1 \arrow[r] & \Gamma_0 \times \Gamma_0
\end{tikzcd}$$

is a pullback diagram of topological spaces,

(ii) (topological essential surjectivity) the morphism

$$\Gamma''_0 \times_{\Gamma_0, s} \Gamma_1 \xrightarrow{t} \Gamma_0$$

admits local sections (see Exercise 2.17).

In fact, the 2-category of topological stacks is a localization of the category of topological groupoids at the topological equivalences.

More generally, let $\mathfrak{X} \to \mathfrak{Y}$ be a morphism of topological stacks, let $X_\bullet$ be a groupoid presentation of $\mathfrak{X}$ and $Y_\bullet$ one of $\mathfrak{Y}$. Form the larger 2-cartesian diagram:

$$\begin{tikzcd}
P_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & Q \arrow[rr] \arrow[d, shift left] \arrow[d, shift right] & & Y_1 \arrow[d, shift left] \arrow[d, shift right] \\
Q' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P_0 \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Nearrow"] & & Y_0 \arrow[d] \\
X_1 \arrow[r, shift left] \arrow[r, shift right] & X_0 \arrow[r] & \mathfrak{X} \arrow[r] & \mathfrak{Y}
\end{tikzcd}$$

Then P_0 is a $Y_\bullet$-torsor over X_0, and an $X_\bullet$-equivariant map to Y_0.

2.36 Exercise. State and prove the general case of Theorem 1.58.

Quotient stacks

Suppose the topological group G acts on the topological space X. The associated stack of pairs (P, ϕ), where P is a G-bundle and $\phi : P \to X$ an equivariant continuous map, is usually denoted by $[X/G]$ and called the **quotient stack** of X by G. There is a 2-cartesian diagram of groupoid fibrations (38)

$$\begin{tikzcd}
\underline{X} \times \underline{G} \arrow[r] \arrow[d] & \underline{X} \arrow[d] \\
\underline{X} \arrow[r] & {[X/G]}
\end{tikzcd}$$

We have seen that if a topological stack $\mathfrak{X}$ admits a versal family whose symmetry groupoid is the transformation groupoid $X \times G \rightrightarrows X$, then $\mathfrak{X}$ is isomorphic to the quotient stack $[X/G]$.

For every (P, ϕ) as above, parametrized by T, there is a 2-cartesian diagram (39)

$$\begin{tikzcd}
\underline{P} \arrow[r] \arrow[d] & \underline{X} \arrow[d] \\
\underline{T} \arrow[r] & {[X/G]}
\end{tikzcd}$$

If $X = *$ is the one-point space, the quotient stack $[*/G]$ is denoted by BG, and is called the *classifying stack* of G. There is a 2-cartesian diagram

$$\begin{tikzcd}
\underline{G} \arrow[r] \arrow[d] & \underline{*} \arrow[d] \\
\underline{*} \arrow[r] & BG
\end{tikzcd}$$

and for every principal G-bundle G/T a 2-cartesian diagram

$$\begin{array}{ccc} \underline{P} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \underline{T} & \longrightarrow & BG \end{array}$$

Therefore $\underline{*} \to BG$ is known as the *universal principal G-bundle*.

2.37 Exercise. Let G be a topological group acting on the topological space X, and let $P \to T$ be a G-bundle. Then there is a 2-cartesian diagram

$$\begin{array}{ccc} \underline{P \times_G X} & \longrightarrow & [X/G] \\ \downarrow & & \downarrow \\ \underline{T} & \longrightarrow & BG \end{array}$$

Therefore, $[X/G] \to BG$ is called the *universal fibre bundle with fibre X*. For example, there is always a 2-cartesian diagram

$$\begin{array}{ccc} \underline{X} & \longrightarrow & \underline{*} \\ \downarrow & & \downarrow \\ [X/G] & \longrightarrow & BG \end{array}$$

2.38 Exercise. The quotient space X/G admits a morphism $[X/G] \to \underline{X/G}$, which turns X/G into the coarse moduli space of $[X/G]$.

2.39 Exercise. Suppose that G acts trivially on X. Then $[X/G] = X \times BG$.

Separated topological stacks

Many properties of topological stacks can be defined in terms of presenting groupoids, if these properties are invariant under Morita equivalence. The following exercise treats an example.

2.40 Exercise. We call a topological groupoid Γ **separated**, if the map $\Gamma_1 \to \Gamma_0 \times \Gamma_0$ is *universally closed*, i.e., *proper* in the sense of Bourbaki [5]. Prove that if Γ' is Morita equivalent to Γ, then Γ' is separated if and only if Γ is. Therefore, we call a topological stack **separated**, if any groupoid presentation of it is separated. Being separated is the analogue of the Hausdorff property for stacks. Prove that separated topological stacks have Hausdorff coarse moduli spaces.

When working with separated topological stacks, additional assumptions (such as the parameter space of a versal family being Hausdorff or at least locally Hausdorff) may be necessary. See, for example, Theorem 2.49.

2.5 Deligne-Mumford topological stacks

We now introduce the important idea that the parameter space of a versal family should be thought of as a local model for a topological stack. For this to hold true, the versal family has to have additional properties. We introduce the most basic of these in this section. It comes about in analogy to gluing data for manifolds.

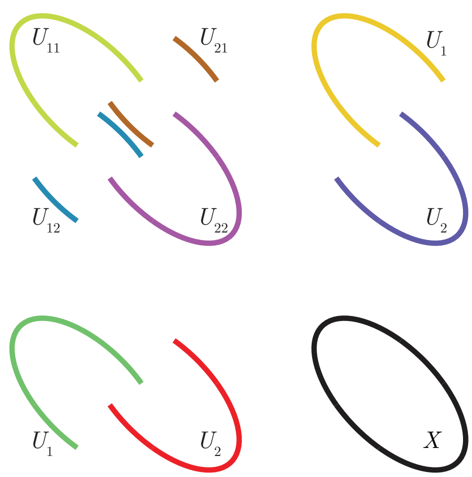
Suppose X is a topological manifold, with an atlas $\{U_i\}_{i\in I}$ of local charts $U_i \to X$. The atlas gives rise to a versal family for the groupoid fibration $\underline{X}$ (see Exercise 2.17). The parameter space

$$\Gamma_0 = \coprod_{i\in I} U_i$$

is the disjoint union of the charts in the atlas, and the versal family is the induced continuous map $\Gamma_0 \to X$. The symmetry groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ has morphism space

$$\Gamma_1 = \coprod_{(i,j)\in I\times I} U_i \cap U_j \,.$$

We write $U_{ij} = U_i \cap U_j$. This symmetry groupoid is an equivalence relation.



2.41 Exercise. The groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ is the restriction (Definition 2.25) of the trivial groupoid $X \rightrightarrows X$ via the map $\coprod U_i \to X$. The morphism from $\Gamma_1 \rightrightarrows \Gamma_0$ to $X \rightrightarrows X$ is a topological equivalence of topological groupoids (Exercise 2.35). This expresses the fact that $\{U_i\}$ is an atlas of the manifold X in groupoid language. The groupoid $\coprod U_{ij} \rightrightarrows \coprod U_i$ encodes the way that X is obtained by gluing the U_i. Morita equivalence encodes the way different atlases for the same manifold relate to one another.

By analogy, a general topological groupoid Γ, presenting a topological stack $\mathfrak{X}$ should be thought of as an atlas for $\mathfrak{X}$, in fact, **atlas** is a commonly used synonym for 'presentation'.

There are many topological equivalence relations giving rise to $\underline{X}$ as associated topological stack: the banal groupoid associated to any continuous map $Y \to X$ admitting local sections will do. For example, we could take a point $P : * \to X$ and pass to the equivalence relation $\Gamma_1' \rightrightarrows \Gamma_0'$, with $\Gamma_0' = \Gamma_0 \amalg *$. But unless X is a manifold of dimension 0, this equivalence relation Γ' does not reflect the local structure of X faithfully any more.

The morphism $\Gamma_0 \to X$ is a local homeomorphism (every point of Γ_0 has an open neighbourhood which maps homeomorphically to an open neighbourhood of the image point in X). Because $\Gamma_0 \to X$ has local sections, this is equivalent to source and/or target maps $\Gamma_1 \to \Gamma_0$ being local homeomorphisms.

As another example, consider a discrete group G acting on a topological space Y in such a way that every point of Y has an open neighbourhood U such that all Ug, $g \in G$, are disjoint. The quotient map $Y \to X$ is a local homeomorphism, and $Y \to X$ is a versal family for $\underline{X}$.

The property that $\Gamma_0 \to X$ is a local homeomorphism also makes sense for the morphism $\Gamma_0 \to \mathfrak{X}$ of a groupoid presentation for a topological stack, and gives rise to the notion of étale versal family:

2.42 Definition. A family x/T is **étale at the point** $t \in T$, if for every family y/S, point $s \in S$, and isomorphism $\phi : y_s \to x_t$,

(i) there exists an open neighbourhood U of s in S, a continuous map $f : U \to T$, and an isomorphism of continuous families $\Phi : y|_U \to f^*x$, such that $\Phi_s = \phi$,

(ii) Given (U, f, Φ) and (U', f', Φ') as in (i), there exists a third open neighbourhood $V \subset U \cap U'$ of s, such that $f|_V = f'|_V$ and $\Phi|_V = \Phi'|_V$.

The family x/T is **étale**, if it is étale at every point of T.

A topological groupoid is called *étale*, if source and target maps are local homeomorphisms.

2.43 Exercise. Every étale family has an étale symmetry groupoid. Every versal family with étale symmetry groupoid is étale. (So most of the versal families we constructed are étale. Exceptions are the versal family of oriented triangles parametrized by the space of great circle equilateral triangles, and the versal family of degree three polynomials parametrized by W.) Having an étale symmetry groupoid is by itself not sufficient for being an étale family.

2.44 Exercise. If an $\mathfrak{X}$-family x/T is étale, and every object of $\mathfrak{X}(*)$ is isomorphic to x_t, for some point $t \in T$, then x/T is versal.

2.45 Definition. If the topological stack $\mathfrak{X}$ admits an étale versal family, it is called a **Deligne-Mumford topological stack**.

Thus Deligne-Mumford topological stacks 'look like' topological spaces, locally. If Γ is an étale groupoid presentation for $\mathfrak{X}$, then $\mathfrak{X}$ looks locally like Γ_0 (and also Γ_1).

2.46 Exercise. Let us make this statement precise.

Surjective local homeomorphisms are *local in the base.* This means that if $X \to Y$ is a continuous map of topological spaces, and $Y' \to Y$ is a continuos map admitting local sections, then $X \to Y$ is a surjective local homeomorphism if and only if the base change $X' \to Y'$, defined by the pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a surjective local homeomorphism.

The morphism $\Gamma_0 \to \mathfrak{X}$ given by an étale versal family for $\mathfrak{X}$, is considered to admit local sections, because for every morphism $T \to \mathfrak{X}$, the base change $T \times_{\mathfrak{X}} \Gamma_0 \to T$ admits local sections.

Therefore, the morphism $\Gamma_0 \to \mathfrak{X}$ is considered to be a surjective local homeomorphism:

$$\begin{tikzcd} \Gamma_1 \arrow[rr] \arrow[d, "\text{surjective local homeomorphism}"'] & & \Gamma_0 \arrow[d, "\therefore\ \text{surjective local homeomorphism}"] \\ \Gamma_0 \arrow[rr, "\text{admits local sections}"] & & \mathfrak{X} \end{tikzcd}$$

2.47 Example. The topological stacks $\mathfrak{M}$, $\widetilde{\mathfrak{M}}$, $\overline{\mathfrak{M}}$ and $\mathfrak{L}$ are topological Deligne-Mumford stacks. We will see below (Example 2.50), that $\mathfrak{W}$ is of Deligne-Mumford type, too.

2.48 Example. If $X_1 \subset X_0 \times X_0$ is an étale equivalence relation, where X_1 has the subspace topology of the product topology, then the associated Deligne-Mumford topological stack is equal to $\underline{X}$, where X is the quotient of X_0 by X_1 with the quotient topology.

For example, consider the equivalence relation on $\mathbb{R}$, defined by the action of $\mathbb{Q}$ by translation. If we endow $\mathbb{Q}$ with the discrete topology, the equivalence relation is étale and we obtain a Deligne-Mumford topological quotient stack $[\mathbb{R}/\mathbb{Q}]$. If we endow $\mathbb{Q}$ with the subspace topology, we obtain a topological stack not of Deligne-Mumford type $[\mathbb{R}/\mathbb{Q}]'$. There are morphisms

$$[\mathbb{R}/\mathbb{Q}] \longrightarrow [\mathbb{R}/\mathbb{Q}]' \longrightarrow \underline{\mathbb{R}/\mathbb{Q}}\,,$$

neither of which are an isomorphism.

In particular, $[\mathbb{R}/\mathbb{Q}]$ and $[\mathbb{R}/\mathbb{Q}]'$ are examples of moduli problems without symmetries, that still do not admit fine moduli spaces.

Structure theorem

Let us call a topological Deligne-Mumford stack *separated*, if it separated according to Definition 2.40, and admits an étale versal family with Hausdorff parameter space.

2.49 Theorem. *Every separated Deligne-Mumford topological stack is locally a quotient stack by a finite group.*

Proof. Let Γ be an étale groupoid presenting the stack $\mathfrak{X}$. We may assume that Γ_0 is Hausdorff, and that $s \times t : \Gamma_1 \to \Gamma_0 \times \Gamma_0$ is proper. Then Γ_1 is Hausdorff, as well. Let $P_0 \in \Gamma_0$ be a point, and let G be its automorphism group. Then G is a compact subspace of the discrete space $s^{-1}(P_0) \subset \Gamma_1$, and is therefore finite.

We start by choosing disjoint open neighbourhoods of the points of $G \subset \Gamma_1$, which, via s, map homeomorphically to an open neighbourhood U_0 of P_0 in Γ_0. (This is possible because s is a local homeomorphism, and G is finite.) This identifies $U_0 \times G$ with an open neighbourhood of G in Γ_1. Hence, we have a commutative diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & U_0 \times G \arrow[r] \arrow[d, "p_1"'] & \Gamma_1 \arrow[d, "s"] \\
P_0 \arrow[r] & U_0 \arrow[r] & \Gamma_0
\end{tikzcd}$$

Now, using the closedness of $s \times t : \Gamma_1 \to \Gamma_0 \times \Gamma_0$, we choose an open neighbourhood V_0 of P_0 in U_0, such that $V_1 = (s \times t)^{-1}(V_0 \times V_0) \subset U_0 \times G \subset \Gamma_1$. Then $V_1 \rightrightarrows V_0$ is a subgroupoid of $\Gamma_1 \rightrightarrows \Gamma_0$, and the arrows in V_1 are pairs (u, g), with $u \in V_0 \subset U_0$, and $g \in G$.

Consider the diagram

$$\begin{tikzcd}
G \times G \arrow[r] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "\mu"] & V_2 \arrow[r] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "\mu"] & U_0 \times G \times G \arrow[r] \arrow[d, shift right, "\mathrm{id} \times p_1"'] \arrow[d, shift left, "\mathrm{id} \times \mu"] & \Gamma_2 \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "\mu"] \\
G \arrow[r] \arrow[d] & V_1 \arrow[r] \arrow[d, "s"'] & U_0 \times G \arrow[r] \arrow[d, "\mathrm{pr}"'] & \Gamma_1 \arrow[d, "s"'] \\
P_0 \arrow[r] & V_0 \arrow[r] & U_0 \arrow[r] & \Gamma_0
\end{tikzcd}$$

There are four vertical cartesian squares in this diagram, and the upper horizontal arrows are uniquely determined by the lower commutative diagrams. There is, a priori, no reason for

$$\begin{tikzcd} V_2 \arrow[rr] \arrow[d, "p_2"'] & & U_0 \times G \times G \arrow[d, "t\times \mathrm{id}"] \\ V_1 \arrow[r] & U_0 \times G \arrow[r] & \Gamma_0 \times G \end{tikzcd} \tag{40}$$

to commute, although, after projecting onto Γ_0, the two induced maps $V_2 \to \Gamma_0$ are equal to the projection onto the 'middle' object. Therefore, the locus in V_2, where the two maps to $\Gamma_0 \times G$ are equal, is an open neighbourhood $V_2' \subset V_2$ of $G \times G$. Now, using the closedness of $\Gamma_2 \to \Gamma_0 \times \Gamma_0 \times \Gamma_0$, we can find an open neighbourhood V_0' of P in V_0, such that the preimage of $V_0 \times V_0 \times V_0$ is contained in V_2'. Restricting the groupoid Γ further to V_0', we get a subgroupoid $V' \subset V \subset \Gamma$, and replacing V with V', we may assume that, in fact, Diagram (40) does commute. This means that, for all $(u,g,h) \in V_2$, we have $p_1(u,g,h) = (u,g)$, $\mu(u,g,h) = (u,gh)$, and $p_2(u,g,h) = \big(ug,h\big)$, where we have written ug for $t(u,g)$. In other words, we have

$$(u,g) * (ug,h) = (u,gh)\,, \qquad \text{for all } (u,g,h) \in V_2\,.$$

For $u \in V_0$, and $g \in G$, define $ug = t(u,g)$. Then let

$$W_0 = \{u \in V_0 \mid \forall g \in G\colon ug \in V_0\}\,.$$

Then W_0 is an open neighbourhood of P_0 in V_0, and for $u \in W_0$, we have $(ug)h = u(gh)$. Restricting our groupoid to $W_0 \subset V_0$, we see that W is the transformation groupoid of the G-action on W_0, defined above. We have a morphism of groupoids

$$\begin{tikzcd} W_0 \times G \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \Gamma_1 \arrow[d, shift left] \arrow[d, shift right] \\ W_0 \arrow[r] & \Gamma_1 \end{tikzcd}$$

which induces an open immersion of topological stacks $[W_0/G] \to \mathfrak{X}$. $\square$

2.50 Example. The Weierstrass stack $\mathfrak{W}$ is a separated Deligne-Mumford topological stack. An étale versal family is parametrized by the disjoint union of two copies of $\mathbb{C}$. This family is the union of the two families $4z^3 - g_2 z - 1$ and $4z^3 - z - g_3$. A picture of this family is the union of (5) and (6). The symmetry groupoid of this étale family is the restriction of the transformation groupoid $W \times \mathbb{C}^*$ via the map $\mathbb{C} \amalg \mathbb{C} \to W$, which is $g_2 \mapsto (g_2, 1)$ on one copy of $\mathbb{C}$, and $g_3 \mapsto (1, g_3)$ on the other copy of $\mathbb{C}$. It is not a transformation groupoid.

Thus, removing the bisected line segment we get $\mathfrak{W} \setminus B\mathbb{Z}_2 \cong [\mathbb{C}/\mathbb{Z}_3]$, and removing the equilateral triangle we get $\mathfrak{W} \setminus B\mathbb{Z}_3 \cong [\mathbb{C}/\mathbb{Z}_2]$. We have $\mathfrak{W} = [\mathbb{C}/\mathbb{Z}_3] \cup [\mathbb{C}/\mathbb{Z}_2]$, a union of two open substacks.

On the other hand, $\mathfrak{W}$ is not globally a finite group quotient, because it is simply connected, see Example 2.64.

Orbifolds

Working in the category of differentiable manifolds and differentiable maps gives rise to the notion of *differentiable stack*. Care needs to be taken, because not all fibered products exist in this category (although pullbacks via differentiable submersions exist, which is sufficient).

In this context, a stack over the category of differentiable manifolds is said to be *differentiable*, if it admits a versal family whose symmetry groupoid is a *Lie groupoid*. A Lie groupoid is a topological groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, where both Γ_1 and Γ_0 are endowed with the structure of differentiable manifold, s and t are differentiable submersions, and all structure maps are differentiable.

Different presentations of a differentiable stack give rise to Lie groupoids which are differentiably Morita equivalent (this means that the structure maps of a bitorsor have to be differentiable submersions). Lie groupoids form a classical subject in differential geometry, see for example [18].

By abuse of terminology, we call a differentiable stack an **orbifold**, if it admits a presentation by an étale Lie groupoid. (A Lie groupoid is *étale*, if its source and target maps are local diffeomorphisms.) An analogue of Theorem 2.49 (with the same proof) shows that every orbifold is locally the quotient of a finite group acting by diffeomorphisms on an open subset of $\mathbb{R}^n$. All the examples of topological Deligne-Mumford stacks we encountered are naturally orbifolds. There is a vast literature on orbifolds, see for example [24], [7].

If $U \subset \mathbb{R}^n$ is open, and endowed with an action by a finite group G, and if $[U/G]$ is an open substack of an orbifold $\mathfrak{X}$, then it is called a local *orbifold chart* of $\mathfrak{X}$. In the literature, the term orbifold is usually reserved for those $\mathfrak{X}$, which admit orbifold charts $[U/G]$, where G acts effectively on U.

The language of group actions is not well-suited for the global description of orbifolds. The way different orbifold charts are glued together is described by a groupoid presentation as in Example 2.50. Moreover, Morita equivalence of groupoids encodes what happens when two different orbifold atlases describe the same orbifold.

The following result shows that the orbifold property for differentiable stacks is a property of the diagonal, hence a separation property. We say that a Lie groupoid $X_1 \rightrightarrows X_0$ has *immersive diagonal*, if $s \times t : X_1 \to X_0 \times X_0$ is injective on tangent spaces. This property is invariant under differentiable Morita equivalence, and hence gives rise to a separation property of differentiable stacks.

2.51 Proposition. *Every differentiable stack with immersive diagonal is an orbifold.*

PROOF. We have to show that every Lie groupoid with immersive diagonal is Morita equivalent to an étale Lie groupoid. The fact that $X_\bullet$ has immersive diagonal allows us to construct a foliation $T_{X_0/\mathfrak{X}} \hookrightarrow T_{X_0}$, by taking $T_{X_0/\mathfrak{X}}$ to be equal to the normal bundle N_{X_0/X_1} of the identity section $X_0 \to X_1$, and embedding it into T_{X_0} via the difference of the two maps $Ds, Dt : T_{X_1}|_{X_0} \to T_{X_0}$. Then we take $U_0 \to X_0$ to be transverse to the foliation $T_{X_0/\mathfrak{X}}$ and containing each isomorphism class in $\mathfrak{X}(*)$ at least once. Restricting the groupoid X_1 via $U_0 \to X_0$ gives the Morita equivalent groupoid $U_1 \rightrightarrows U_0$, which is an étale groupoid presenting $\mathfrak{X}$.

There is a theory of foliations using étale groupoids, see [19]. □

2.52 Corollary. *Every separated differentiable stack with immersive diagonal is locally a quotient of $\mathbb{R}^n$ by a finite group.*

This result explains that finite group actions are, in fact, quite typical for stacks, and justifies the heavy reliance on them in our examples.

2.6 Lattices up to homothety

We very briefly cover the classical moduli problems of lattices and elliptic curves, and see how they are related to the moduli problem of oriented triangles. For a more detailed account, see [15].

A *lattice* is a subgroup of $\mathbb{C}^+$, which is a free abelian group of rank 2, and which generates $\mathbb{C}$ as $\mathbb{R}$-vector space. Two lattices $\Lambda_1 \subset \mathbb{C}$ and $\Lambda_2 \subset \mathbb{C}$ are *homothetic*, if there exits a non-zero complex number ϕ, such that $\Lambda_2 = \phi \cdot \Lambda_1$.

A *local* continuous family of lattices, parametrized by the topological space T is given by two continuous functions $\omega_1, \omega_2 : T \to \mathbb{C}^*$, which are not real multiples of one-another, anywhere in T. The corresponding family of lattices is $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset T \times \mathbb{C}$. A *homothety* between two local families $\phi : (\omega_1, \omega_2) \to (\tau_1, \tau_2)$ is a continuous map $\phi : T \to \mathbb{C}^*$, such that $\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2 = \phi \cdot (\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$. This defines the prestack of lattices up to homothety.

Obviously, $S = \{(\omega_1, \omega_2) \in (\mathbb{C}^*)^2 \mid \mathbb{R}\omega_1 \neq \mathbb{R}\omega_2\}$ parametrizes a local continuous family of lattices, in a tautological fashion: the two functions ω_1, ω_2, are simply the coordinate projections. Just as obviously, every local family of lattices is pulled back from this tautological one. Thus the family parametrized by S is versal. The symmetry groupoid of this family is the transformation groupoid of $\mathbb{C}^* \times GL_2(\mathbb{Z})$ acting on S.

By Theorem 2.31, i.e., stackifying our prestack, a (global) continuous family of lattices parametrized by T is a complex line bundle L/T together with a rank 2 local system $\Lambda \subset L$. We will call the stack of lattices up to homothety $\mathfrak{E}$.

To every compact Riemann surface E of genus 1, we associate the lattice

$$\begin{aligned} H_1(E, \mathbb{Z}) &\longrightarrow \Gamma(E, \Omega_E)^* \\ \gamma &\longmapsto \int_\gamma \end{aligned}$$

It is a lattice in the one-dimensional complex vector space dual to $\Gamma(E, \Omega_E)$, the space of holomorphic 1-forms on E. It is known as the *period lattice*.

Conversely, to a lattice $\Lambda \in \mathbb{C}$ we associate the compact Riemann surface $\mathbb{C}/\Lambda$. These two processes define an equivalence of groupoids between elliptic curves (compact Riemann surfaces of genus 1 with a choice of base point serving as zero for the group law) and lattices up to homothety. We are therefore justified to declare a *continuous family of elliptic curves* to be a continuous family of lattices. Thus we refer to $\mathfrak{E}$ also as the (topological) stack of elliptic curves.

2.53 Exercise. The upper half plane parametrizes a versal family of lattices with symmetry groupoid given by $SL_2(\mathbb{Z})$ acting by linear fractional transformations. The lattice at the point $\tau \in \mathbb{H}$ is $\mathbb{Z} + \tau\mathbb{Z}$. The τ-value of an elliptic curve is the quotient of its two periods. The corresponding elliptic curve is $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

Compactification

Let D denote the open disc in $\mathbb{C}$ of radius $e^{-2\pi}$ centred at the origin. Let $D^* \subset D$ be the pointed disc. Then D^* parametrizes a continuous family of lattices: over $q \in D^*$, the corresponding lattice is $\Lambda_q = \mathbb{Z} + \tau\mathbb{Z}$, where $\tau \in \mathbb{C}$ is any complex number such that $e^{2\pi i \tau} = q$. The corresponding family of elliptic curves can also be written as $\mathbb{C}/\Lambda_q = \mathbb{C}^*/q^{\mathbb{Z}}$.

2.54 Exercise. Prove that this is, indeed, a continuous family of lattices. Prove that for different points in D^*, the corresponding lattices are not homothetic. Conclude that the the symmetry groupoid of this family of lattices over D^* is the family of groups $D^* \times \mathbb{Z}_2$ over D^*, or, in other words, the transformation groupoid $D^* \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts trivially on D^*. This uses the fact that we have restricted to $|q| < e^{-2\pi}$, and can be deduced from Exercise 2.53.

We therefore have a morphism of topological stacks $D^* \times B\mathbb{Z}_2 \to \mathfrak{E}$. This is, in fact, an open substack. We will compactify $\mathfrak{E}$ by gluing in a copy of $D \times B\mathbb{Z}_2$, along $D^* \times B\mathbb{Z}_2 \subset \mathfrak{E}$.

To make this rigorous, we construct a groupoid as follows: start with the symmetry groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ of the family of lattices parametrized by the disjoint union $\mathbb{H} \amalg D^*$. This has, as subgroupoid, the symmetry groupoid $D^* \times \mathbb{Z}_2 \rightrightarrows D^*$ of the family over D^*. To construct $\overline{\Gamma}_1 \rightrightarrows \overline{\Gamma}_0$ from $\Gamma_1 \rightrightarrows \Gamma_0$, take out $D^* \times \mathbb{Z}_2 \rightrightarrows D^*$, and replace it by $D \times \mathbb{Z}_2 \rightrightarrows D$, the groupoid given by the trivial action of $\mathbb{Z}_2$ on D.

2.55 Exercise. The object space $\overline{\Gamma}_0$ is the disjoint union $\mathbb{H} \amalg D$, and the morphism space $\overline{\Gamma}_1$ is the disjoint union of $\mathbb{H} \times SL_2\mathbb{Z}$, $D \times \mathbb{Z}_2$, and four more components, which are all homeomorphic to D^* (or the part of $\mathbb{H}$ with imaginary part larger than 1).

Then we let $\overline{\mathfrak{E}}$ be the stack associated to $\overline{\Gamma}_1 \rightrightarrows \overline{\Gamma}_0$. This is the stack of degenerate elliptic curves.

2.56 Exercise. A family of degenerate elliptic curves over T is therefore given by

(i) a cover of T by two open subsets, U and V,
(ii) over U, a family of lattices, $\Lambda \subset L$,
(iii) over V, a continuous map $q : V \to D$, and a degree 2 covering space $V' \to V$,
(iv) over $U \cap V$, an isomorphism of families of lattices $\Lambda|_{U\cap V} \cong V' \times_{\mathbb{Z}_2} \Lambda_q$,

with a natural notion of isomorphism.

2.57 Exercise. The disc D parametrizes a family of groups: the quotient of $D \times \mathbb{C}^*$ by the subgroup of all $(q, q^n) \in D \times \mathbb{C}^*$, for $q \in D^*$, $n \in \mathbb{Z}$. (The fibre of this family of groups over the origin is $\mathbb{C}^*$.) The groupoid $D \times \mathbb{Z}_2$ is a groupoid of symmetries of this family of groups. The stack $\overline{\mathfrak{E}}$ supports a family of groups, the *universal degenerate elliptic curve*, denoted $\overline{\mathfrak{F}}$.

2.58 Exercise. There is a morphism of stacks $\overline{\mathfrak{E}} \to \mathfrak{W}$, defined by mapping a lattice $\Lambda \subset \mathbb{C}$ to the triangle $\wp(\frac{1}{2}\Lambda)$, where $\wp$ is the Weierstrass $\wp$-function corresponding to the lattice Λ. This morphism of stacks induces a homeomorphism on coarse moduli spaces. The fibres of this morphism are all isomorphic to $B\mathbb{Z}_2$. This means that for every triangle δ, there is a 2-cartesian diagram

$$\begin{tikzcd} B\mathbb{Z}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow"] & * \arrow[d, "\delta"] \\ \overline{\mathfrak{E}} \arrow[r] & \mathfrak{W} \end{tikzcd}$$

We say that $\overline{\mathfrak{E}}$ is a $\mathbb{Z}_2$**-gerbe** over $\mathfrak{W}$.

2.59 Exercise. Every $\overline{\mathfrak{E}}$-family over T comes with a complex line bundle L/T. In the notation of Exercise 2.56, this line bundle is equal to L over U, and equal to $V' \times_{\mathbb{Z}_2} \mathbb{C}$, over V, where $\mathbb{Z}_2$ acts by multiplication by -1 on $\mathbb{C}$. These line bundles assemble to a line bundle $\mathscr{L}$ over $\overline{\mathfrak{E}}$. The bundle $\mathscr{L}^{\otimes -k}$

is called the *bundle of modular forms* of weight k. Global sections are called *continuous modular forms* of weight k. (The term 'modular form' is usually reserved for holomorphic or algebraic modular forms, see Example 3.73.)

Prove that a modular form of weight k is the same thing as a continuous map $f : \mathbb{H} \to \mathbb{C}$ which satisfies the functional equation

$$f\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^k f(\tau)\,,$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and which is continuous at $\mathrm{Re}(\tau) = \infty$.

Pulling back a modular form via $D \to \overline{\mathfrak{E}}$ gives rise to its *q-expansion.*

2.7 Fundamental groups of topological stacks

As an example of the topology of topological stacks, we give a brief introduction to the fundamental group. For details, see [22].

Let $\mathfrak{X}$ be a topological stack that admits a versal family whose symmetry groupoid $X_1 \rightrightarrows X_0$ has the property that both source and target maps are topological submersions (locally in X_1, the map $X_1 \to X_0$ is homeomorphic to a product of the base times another topological space). This property will ensure that $\mathfrak{X}$ has the gluing property along closed subsets, Exercise 1.11.

Let ξ be an object of the groupoid $\mathfrak{X}(*)$, where $*$ is the one-point space. The fundamental group of $\mathfrak{X}$ with respect to the base point ξ is defined as follows. Denote the base point of S^1 by e.

A *loop* in $\mathfrak{X}$, based at ξ, is an $\mathfrak{X}$-family x/S^1, parametrized by the circle S^1, together with an isomorphism $\xi \to x_e$, where x_e is the family member at the base point $e \in S^1$. Equivalently, a loop is a diagram

$$\begin{tikzcd} \xi \arrow[r, "\alpha"] \arrow[d, dash] & x \arrow[d, dash] \\ * \arrow[r, "e"] & S^1 \end{tikzcd}$$

Imagine a 'film', or a 'movie', of a an $\mathfrak{X}$-loop as it changes over time. The film shows the loop varying continuously, as time passes. Throughout the duration of the film, the family member over the base point $e \in S^1$ is always ξ. Such a film is called a *homotopy* between the loop depicted on the first frame of the movie and the loop depicted on the last frame. The first and last loop shown in the movie are then called *homotopic.*

Formally, a homotopy from the $\mathfrak{X}$-loop (x, α) to the $\mathfrak{X}$-loop (y, β), is a quadruple (h, η, ϕ, ψ). Here h is an $\mathfrak{X}$-family parametrized by $I \times S^1$, where $I = [0, 1]$ is the unit interval in $\mathbb{R}$. Moreover, η is an isomorphism $\xi_I \to (\mathrm{id}_I, e)^* h$, where ξ_I is the constant family over I obtained by pulling back ξ via $I \to *$

$$\begin{tikzcd}
\xi_I \arrow[r, "\eta"] \arrow[d, dash] & h \arrow[d, dash] \\
I \arrow[r, "\mathrm{id}\times e"] & I \times S^1
\end{tikzcd}$$

and ϕ, ψ are isomorphisms $x \to (0 \times \mathrm{id}_{S^1})^* h$ and $y \to (1 \times \mathrm{id}_{S^1})^* h$

$$\begin{tikzcd}
x \arrow[r, "\phi"] \arrow[d, dash] & h \arrow[d, dash] & y \arrow[l, "\psi"'] \arrow[d, dash] \\
S^1 \arrow[r, "0\times\mathrm{id}"] & I \times S^1 & S^1 \arrow[l, "1\times\mathrm{id}"']
\end{tikzcd}$$

The two diagrams

$$\begin{tikzcd}
\xi \arrow[r, "\alpha"] \arrow[d] & x \arrow[d, "\phi"] \\
\xi_I \arrow[r, "\eta"] & h
\end{tikzcd}
\qquad
\begin{tikzcd}
\xi \arrow[r, "\beta"] \arrow[d] & y \arrow[d, "\psi"] \\
\xi_I \arrow[r, "\eta"] & h
\end{tikzcd}$$

are required to commute.

The set of homotopy classes of $\mathfrak{X}$-loops based at ξ is denoted $\pi_1(\mathfrak{X}, \xi)$ and called the *fundamental group* of $\mathfrak{X}$, based at ξ. This is, in fact, a group: loops can be concatenated, by the gluing property, and homotopies can be constructed, which prove well-definedness, associativity, and existence of units and inverses.

The fundamental group of the stack of triangles

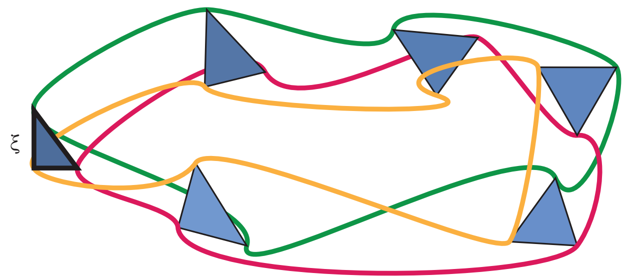
Let us compute the fundamental group of the stack $\mathfrak{M}$ of non-degenerate non-oriented triangles. Let us take the 3:4:5 right triangle as base point ξ. Let us label the edges of the base triangle with 3, 4, and 5, according to their length.

Define a map

$$p : \pi_1(\mathfrak{M}, \xi)^{\mathrm{op}} \longrightarrow S_3\,,$$

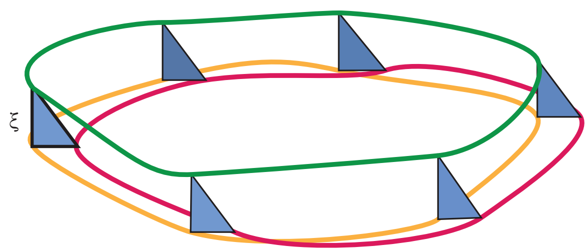
where we think of S_3 as the group of permutations of the set $\{3, 4, 5\}$. For a given loop, which starts and ends at the 3:4:5 triangle, we define the corresponding permutation of $\{3, 4, 5\}$, by following the labels around the loop, in the counterclockwise direction. For example, the loop (5) gives rise to the permutation (45), in cycle notation. Because concatenation of loops $x * y$ means that x is traversed before y, but composition of permutations $\pi \circ \sigma$ means that π is applied after σ, the map $p : \pi_1(\mathfrak{M}, \xi) \to S_3$ reverses the group operation, and is therefore a homomorphism of groups $\pi_1(\mathfrak{M}, \xi)^{\mathrm{op}} \to S_3$, where $\pi_1(\mathfrak{M}, \xi)^{\mathrm{op}}$ is the opposite group of $\pi_1(\mathfrak{M}, \xi)$.

We claim that p is an isomorphism of groups. To prove injectivity, assume that x/S^1 is a loop leading to the trivial permutation of $\{3, 4, 5\}$.



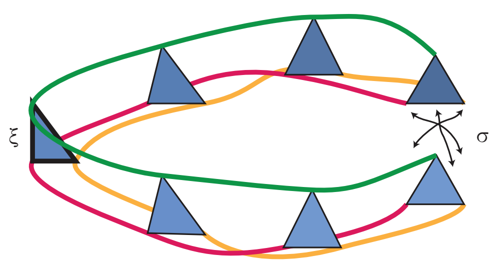
(41)

This means that the edges can be consistently labelled, by the labels $3, 4, 5$. To make a movie transforming (41) into the trivial family ξ_{S^1} (representing the identity element in $\pi_1(\mathfrak{X}, \xi)$), simply deform the triangles continuously until each side has length equal to its label.



Then, at the end of the movie, all triangles in the family are 3:4:5 right triangles, and the family is isomorphic to the trivial family ξ_{S^1}, as there cannot be any non-trivial families of 3:4:5 triangles, the 3:4:5 right triangle being scalene.

To prove surjectivity of p, suppose σ is a given permutation of $\{3, 4, 5\}$. To construct a loop of triangles, based at ξ, giving rise to this permutation, take a family parametrized by an interval, which deforms the 3:4:5 triangle in the middle to two equilateral triangles on either end



(42)

and then glue according to σ.

As any group is isomorphic to its opposite group, we see that the fundamental group of $\mathfrak{M}$ is S_3.

2.60 Exercise. Prove that the fundamental group of the stack of equilateral triangles is S_3. (More generally, the stack of forms of a single object with discrete symmetry group has as fundamental group the symmetry group of the object.)

2.61 Exercise. The stack $\widetilde{\mathfrak{M}}$ of oriented non-degenerate triangles has cyclic fundamental group with 3 elements.

More examples

The computation of the fundamental group of $\mathfrak{M}$ can be generalized to the following statement:

2.62 Theorem. *Suppose that $\mathfrak{X}$ admits a versal family whose symmetry groupoid is a transformation groupoid $X \times G$. Suppose that X is connected and simply connected and that G is locally path connected. Then the fundamental group of $\mathfrak{X}$ is isomorphic to $\pi_0(G)$, the group of connected components of G.*

Proof. Given an $\mathfrak{X}$-loop, its generalized moduli map is a G-bundle $P \to S^1$, together with a G-equivariant continuous map $f : P \to X$. Divide P by G^0, the connected component of the identity, to obtain a $\pi_0(G) = G/G^0$-cover $\overline{P} \to S^1$. Then going once around the loop inside $\overline{P}$ gives rise to an element of $\pi_0(G)$.

This process defines the homomorphism $\pi_1(\mathfrak{X})^{\mathrm{op}} \to \pi_0(G)$.

To prove injectivity, suppose that the element of $\pi_0(G)$ obtained from $\overline{P}$ is trivial. This means that the $\pi_0(G)$-cover $\overline{P}$ is trivial. Choosing a trivialization, the space P splits up into components indexed by $\pi_0(G)$. The component P^0, corresponding to the identity element is then a G^0-bundle over S^1.

Note that any H-bundle Q over S^1, for a topological group H, can be obtained by gluing the trivial bundle over an interval with an element h of H, similarly to (42). If the group H is path connected, then choosing a path connecting h to the identity element gives us a homotopy between Q and the trivial bundle.

Applying this to the above G^0-bundle P^0, we get a homotopy between P and the trivial G-bundle. So we may assume, without loss of generality, that the G-bundle P is trivial.

So then our map f is an equivariant map $f : S^1 \times G \to X$. Such a map is completely determined by a continuous map $S^1 \to X$, i.e., a loop in X. Contracting this loop in X gives rise to a second homotopy turning f into a trivial map $S^1 \times G \to X$, given by $(s, g) \mapsto x_0 g$, for a point $x_0 \in X$. Now our loop in $\mathfrak{X}$ is trivial.

We leave the surjectivity to the reader. $\square$

2.63 Example. The stack of degenerate triangles $\overline{\mathfrak{M}}$ has fundamental group $S_3 \times \mathbb{Z}_2$. The stack $\mathfrak{L}$ of oriented, degenerate triangles in the Legendre compactification has fundamental group S_3.

2.64 Example. The stack $\mathfrak{W}$ of oriented degenerate triangles in the Weierstrass compactification is simply connected. The stack of degenerate elliptic curves $\overline{\mathfrak{E}}$ is simply connected.

2.65 Example. The stack of non-degenerate lattices has fundamental group $SL_2(\mathbb{Z})$. The stack of non-pinched oriented triangles has fundamental group $PSL_2(\mathbb{Z})$.

3 Algebraic stacks

For algebraic stacks, the parameter spaces are not topological spaces, but rather algebraic varieties, or other algebro-geometric objects, such as schemes or algebraic spaces.

Let us work over a fixed base field k. The reader may assume that k is algebraically closed, or that $k = \mathbb{C}$. Let us take as category of parameter spaces $\mathscr{S}$ the category of k-schemes with affine diagonal. (Group schemes are affine group schemes over k, and we will always tacitly assume that they are smooth.)

3.1 Groupoid fibrations

A groupoid fibration will now be a groupoid fibration $\mathfrak{X} \to \mathscr{S}$. The definition is the same as Definition 2.1, replacing $\mathscr{T}$ by $\mathscr{S}$, 'topological space' by 'k-scheme', and 'continuous map' by 'morphism of k-schemes'. Morphisms are defined, *mutatis mutandis* as in Definition 2.6

3.1 Example. As an example, let k be a field of characteristic neither 2 nor 3, and consider $\mathfrak{E}$, the groupoid fibration of *elliptic curves*. An object of $\mathfrak{E}$ is a triple (T, E, P), where T is a k-scheme, E is a scheme endowed with a structure morphism $\pi : E \to T$, and $P : T \to E$ is a section of π, i.e., a morphism such that $\pi \circ P = \mathrm{id}_T$. Moreover $\pi : E \to T$ is required to satisfy

(i) π is a smooth and proper morphism of finite presentation,

(ii) every geometric fibre of π is a curve of genus 1. This means that for any algebraically closed field K, and any morphism $t : \operatorname{Spec} K \to T$, the pullback E_t defined by the cartesian diagram

$$\begin{tikzcd} E_t \arrow[r] \arrow[d] & E \arrow[d] \\ \operatorname{Spec} K \arrow[r, "t"] & T \end{tikzcd}$$

is a one-dimensional irreducible (complete and non-singular by the first property) variety of genus 1, i.e., $\dim \Gamma(E_t, \Omega_{E_t}) = \dim H^1(E_t, \mathscr{O}_{E_t}) = 1$.

A morphism in $\mathfrak{E}$, from (T', E', P') to (T, E, P) is a cartesian diagram of k-schemes

$$\begin{tikzcd} E' \arrow[r, "\phi"] \arrow[d] & E \arrow[d] \\ T' \arrow[r, "f"] & T \end{tikzcd} \tag{43}$$

such that $\phi \circ P' = P \circ f$.

The structure functor $\mathfrak{E} \to \mathscr{S}$ maps the object (T, E, P) to the k-scheme T, and the morphism (f, ϕ) to the morphism of k-schemes f.

3.2 Example. To define the groupoid fibration of *degenerate elliptic curves* (more precisely: with multiplicative reduction) $\overline{\mathfrak{E}}$, replace in Example 3.1 Condition (i) by 'π is a flat an proper morphism of finite presentation', and Condition (ii) by 'every geometric fibre of π is of one of two types: either a smooth curve of genus 1 as in Example 3.1, or an irreducible one-dimensional scheme, non-singular except for a single node, whose arithmetic genus is 1, i.e., $\dim H^1(E_t, \mathscr{O}_{E_t}) = 1$. In addition, one needs to require that P avoids any of the nodes in any of the fibres of π.

3.3 Example. A still larger groupoid fibration is $\widetilde{\mathfrak{E}}$, where the fibres are only required to be reduced and irreducible curves of arithmetic genus 1. This groupoid fibration will also include an elliptic curve with additive reduction, i.e., a genus 1 curve with a cusp. (This groupoid fibration is analogous to the stack of degenerate triangles including the one-point triangle, see Exercise 1.59.)

For more details on $\mathfrak{E}$, $\overline{\mathfrak{E}}$ and $\widetilde{\mathfrak{E}}$, see [8].

3.4 Example. Let X be a fixed smooth projective curve over k. A *family of vector bundles of rank r and degree d over X*, parametrized by the k-scheme T, is a vector bundle V of rank r over $X \times T$, such that, for every $t \in T$, the pullback of V to X_t has degree d. A morphism of families of vector bundles from V'/T' to V/T is a pair (f, ϕ) which fits into a cartesian diagram

$$\begin{tikzcd}
V' \arrow[r, "\phi"] \arrow[d] & V \arrow[d] \\
X \times T' \arrow[r, "\mathrm{id} \times f"] \arrow[d] & X \times T \arrow[d] \\
T' \arrow[r, "f"] & T
\end{tikzcd}$$

Let us denote this groupoid fibration by $\mathfrak{V}_X^{r,d}$.

3.5 Exercise. Let Γ be a groupoid. The *mass* of Γ is

$$\#\Gamma = \sum_{x \in \operatorname{ob}\Gamma/\cong} \frac{1}{\#Aut(x)}\,.$$

The sum is taken over the set of isomorphism classes of objects of Γ. Consider Example 3.4 with $k = \mathbb{F}_q$, the finite field with q elements, and $X = \mathbb{P}^1$. Prove that $\#\mathfrak{V}_{\mathbb{P}^1}^{r,d}(\operatorname{Spec}\mathbb{F}_q)$ converges and find its value.

Representable morphisms

Every k-scheme X defines a groupoid fibration $\underline{X}$, as follows. An $\underline{X}$-family over T is a k-morphism $T \to X$, and pullbacks are defined by composition. This groupoid fibration is special: all fibres $\underline{X}(T)$ are sets (not groupoids) and pullbacks are unique (not unique up to unique isomorphism). We may think of $\underline{X}$ as the functor represented by X.

This construction makes a groupoid fibration out of every scheme, and a morphism of groupoid fibrations out of every morphism of schemes. As one can reconstruct X from $\underline{X}$ (Yoneda's lemma), we lose no information when passing from X to $\underline{X}$, and, in fact, we usually identify X with $\underline{X}$, and omit the underscore from the notation.

If a groupoid fibration $\mathfrak{X}$ is equivalent to $\underline{X}$, for a scheme X, via an equivalence $F : \underline{X} \to \mathfrak{X}$, then X is called the *fine moduli scheme* of $\mathfrak{X}$, and $F(\mathrm{id}_X)$, which is an $\mathfrak{X}$-family parametrized by X, is called the *universal family.* In this case, we also say that $\mathfrak{X}$ is *representable* by the scheme X.

3.6 Definition. A morphism of groupoid fibrations $F : \mathfrak{Y} \to \mathfrak{X}$ is **representable** (more precisely: *representable by schemes*), if for every $\mathfrak{X}$-family x/T, the groupoid fibration of liftings of x to a $\mathfrak{Y}$-family admits a fine moduli scheme. Thus, there exists a scheme $U \to T$, with a $\mathfrak{Y}$-family y/U, and an isomorphism $\theta : x|_U \to F(y)$ of $\mathfrak{X}$-families over U, such that (U, y, θ) is universal, for liftings of x. The universal mapping property can be succinctly specified by saying that the diagram of groupoid fibrations

$$\begin{tikzcd} U \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "\theta \nearrow"] & \mathfrak{Y} \arrow[d, "F"] \\ T \arrow[r, "x"] & \mathfrak{X} \end{tikzcd}$$

is 2-cartesian.

3.7 Definition. A representable morphism of groupoid fibrations is *affine* or *proper* or *smooth* or *flat* or *unramified* or *étale* or *of finite presentation* or *finite* or *an open immersion* or *a closed immersion* (or any other property of morphisms of schemes, stable under base extension) if the morphism $U \to T$ has this property, for all x/T as in Definition 3.6

3.8 Example. Let $\mathfrak{F}$ be the groupoid fibration of quadruples (T, E, P, s), where (T, E, P) is a family of elliptic curves parametrized by T, and $s : T \to E$ is another section of $E \to T$. Forgetting s defines a morphism of groupoid fibrations $\pi : \mathfrak{F} \to \mathfrak{E}$.

Let (E, P) be an elliptic curve parametrized by T. Then the groupoid fibration of liftings of (E, P) to $\mathfrak{F}$ is represented by $E \to T$ (this is more or less a tautology). Thus π is representable. It is also smooth and proper and of finite presentation. Since for every family of elliptic curves (E, P) the diagram

$$\begin{tikzcd}
E \arrow[r, "{(E\times_T E, \mathrm{id}\times P, \Delta)}"] \arrow[d] & \mathfrak{F} \arrow[d, "\pi"] \\
T \arrow[r, "{(E,P)}"] & \mathfrak{E}
\end{tikzcd}$$

is 2-cartesian, we call $\mathfrak{F} \to \mathfrak{E}$ the *universal elliptic curve*. (The base-point section of $\pi : \mathfrak{F} \to \mathfrak{E}$ is given by $(E, P) \mapsto (E, P, P)$.)

Similarly, we can define the universal degenerate elliptic curve $\overline{\mathfrak{F}} \to \overline{\mathfrak{E}}$. The morphism $\overline{\mathfrak{F}} \to \overline{\mathfrak{E}}$ is representable, flat, proper and of finite presentation.

3.2 Prestacks

As we have seen, prestacks are groupoid fibrations where isomorphism spaces are well-behaved. In the algebraic context there are several natural conditions which we have to consider.

One of the stronger conditions is the following:

3.9 Definition. The groupoid fibration $\mathfrak{X} \to \mathscr{S}$ has **scheme-representable diagonal**, if for any two objects x/T and y/U, the groupoid fibration of isomorphisms from x to y admits a fine moduli scheme. In other words, there exists a scheme I, with structure maps $I \to T$ and $I \to U$, and an isomorphism of $\mathfrak{X}$-families $\phi : x|_I \to y|_I$, such that (I, ϕ) satisfies the following universal mapping property:

$$\begin{tikzcd}
\psi \arrow[dr, dotted] \arrow[drrr, bend left] \arrow[dd] \arrow[ddrr, bend right] & & & \\
 & \phi \arrow[rr] \arrow[dr] \arrow[dd] & & y \arrow[dd] \\
J \arrow[dr, dotted] \arrow[drrr, bend left] \arrow[ddrr, bend right] & & x \arrow[dd] & \\
 & I \arrow[rr] \arrow[dr] & & U \\
 & & T &
\end{tikzcd}$$

For any scheme J with given morphisms $J \to T$ and $J \to U$, and any isomorphism of $\mathfrak{X}$-families $\psi : x|_J \to y|_J$, there exists a unique morphism of schemes $J \to I$, such that $\phi|_J = \psi$.

The scheme I is called the *scheme of isomorphisms* from x to y, and is also denoted by $\underline{\mathrm{Isom}}(x, y)$. The isomorphism ϕ is called the *universal isomorphism* from x to y.

3.10 Exercise. Prove that there are 2-cartesian diagrams

$$\begin{tikzcd}
\underline{\mathrm{Isom}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\phi\,\Nearrow"] & U \arrow[d, "y"] \\
T \arrow[r, "x"] & \mathfrak{X}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\underline{\mathrm{Isom}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\phi\,\Nearrow"] & T \times U \arrow[d, "x\times y"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

If x and y are parametrized by the same scheme T, we can define the scheme $\underline{\mathrm{Isom}}_T(x,y)$, if it exists. It represents, for every $T' \to T$, the isomorphisms from x to y in the fibre $\mathfrak{X}(T)$. Alternatively, it is the pullback

$$\begin{tikzcd}
\underline{\mathrm{Isom}}_T(x,y) \arrow[r] \arrow[d] & \underline{\mathrm{Isom}}(x,y) \arrow[d] \\
T \arrow[r, "\Delta"] & T \times T
\end{tikzcd}$$

Prove that $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable by schemes according to Definition 3.6, if and only if for any two families parametrized by the same scheme T, the groupoid fibration $\underline{\mathrm{Isom}}_T(x,y)$ admits a fine moduli scheme. Prove that this is equivalent to $\mathfrak{X}$ having scheme-representable diagonal according to Definition 3.9.

We can strengthen the condition by requiring the schemes $\underline{\mathrm{Isom}}(x,y)$ or the morphisms $\underline{\mathrm{Isom}}(x,y) \to T \times U$ to satisfy additional conditions. For example, we can require $\underline{\mathrm{Isom}}(x,y) \to T \times U$ to be an *affine morphism* of schemes, or a *finite* morphism of schemes. This leads to the notion of $\mathfrak{X}$ having *affine diagonal* or *finite diagonal*, respectively. A very common requirement is that the diagonal be of finite presentation.

We can also weaken the condition to require only that $\underline{\mathrm{Isom}}(x,y)$ be an algebraic space.

The weakest possible condition is that $\underline{\mathrm{Isom}}(x,y)$ is only a sheaf in the étale topology. This leads to the notion of **prestack in the étale topology**. This is the 'usual' notion of prestack.

3.11 Exercise. Let us prove that $\overline{\mathfrak{E}}$ is a prestack with finite (hence affine) diagonal. We will present a proof which applies more generally, to explain a commonly used method. For a more direct proof see Exercise 3.30 which uses Lemma 3.15. So let (T,E,P) and (U,F,Q) be families of degenerate elliptic curves. By passing to the product $T \times U$, we may assume that $T = U$, and that we want to construct the relative isomorphism scheme $\underline{\mathrm{Isom}}_T\big((E,P),(F,Q)\big)$. Because we can glue schemes along open subschemes, the claim that $\underline{\mathrm{Isom}}_T\big((E,P),(F,Q)\big)$ is representable is local in the Zariski topology on T. Families of curves can always embedded into projective space at least locally, so we may assume that $E \to T$ and $F \to T$ are projective morphisms.

We can quote a general theorem: Let $X \to T$ and $Y \to T$ be flat and projective morphisms of schemes. Then $\underline{\mathrm{Isom}}_T(X,Y)$ is represented by a k-scheme, which is a (potentially countably infinite) disjoint union of quasi-projective k-schemes. In fact, the scheme $\underline{\mathrm{Isom}}_T(X,Y)$ is an open subscheme of the *Hilbert scheme* of closed subschemes of $X \times_T Y$, via identifying an isomorphism with its graph. For Hilbert schemes, see [13] or [10].

Using this fact, and the fact that $\underline{\mathrm{Isom}}_T\big((E,P),(F,Q)\big)$ is a closed subscheme of $\underline{\mathrm{Isom}}_T(E,F)$, we see that $\mathfrak{E}$ has scheme-representable diagonal.

In fact $\mathfrak{E}$ has *finite* diagonal. To prove this, we can exploit that fact that E and F are curves: in this case, the condition on a subscheme of $E \times F$ to define an isomorphism is a condition on the Hilbert polynomial of the subscheme, because it is just a condition on the degrees. Therefore, our scheme of isomorphisms is projective over T. One checks that fibre-wise there are only finitely many isomorphisms, and then uses the fact that a projective morphism with finite fibres is finite.

3.12 Exercise. The groupoid fibration $\mathfrak{V}_X^{r,d}$ does not have finite type diagonal. For every vector bundle E over X, the automorphism group $\mathrm{Aut}(E)$ is a linear algebraic k-group, but there is no bound on the dimension of $\mathrm{Aut}(E)$, as E varies in $\mathfrak{V}_X^{r,d}(k)$.

On the other hand, $\mathfrak{V}_X^{r,d}$ can be covered by open subfibrations $\mathfrak{V}_X^{r,d,N}$, which are prestacks with affine diagonals of finite presentation. Here $\mathfrak{V}_X^{r,d,N}$ consists of bundles which are Castelnuovo-Mumford N-regular (see[21]). A family of N-regular bundles E over $X \times T$ admits (at least locally in T) a resolution

$$P_1 \longrightarrow P_0 \longrightarrow E \longrightarrow 0$$

where the P_i are direct sums of $\mathscr{O}(n)$, for $n \ll 0$, and $\mathscr{O}(1)$ is a very ample invertible sheaf on X. If F is another family of N-regular bundles over $X \times T$, then we have an exact sequence

$$0 \longrightarrow \pi_* \mathscr{H}\!om(E,F) \longrightarrow \pi_* \mathscr{H}\!om(P_0,F) \longrightarrow \pi_* \mathscr{H}\!om(P_1,F) \tag{44}$$

where $\pi : X \times T \to T$ is the projection. As $\pi_* \mathscr{H}\!om(P_i,F)$ commutes with arbitrary base change, and is a vector bundle over T, we see that $\pi_* \mathscr{H}\!om(E,F)$ is representable by an affine T-scheme, namely the fibered product (44). Similarly, $\pi_* \mathscr{H}\!om(F,E)$, $\pi_* \mathscr{E}\!nd(E)$ and $\pi_* \mathscr{E}\!nd(F)$ are affine

T-schemes. Finally, $\underline{\mathrm{Isom}}(E,F)$ is a fibered product of affine T-schemes

$$\begin{tikzcd}
\underline{\mathrm{Isom}}(E,F) \arrow[r] \arrow[d] & T \arrow[d] \\
\pi_* \mathscr{H}om(E,F) \times_T \pi_* \mathscr{H}om(F,E) \arrow[r] & \pi_* \mathscr{E}nd(E) \times_T \pi_* \mathscr{E}nd(F)
\end{tikzcd}$$

and is therefore an affine T-scheme itself. It is also of finite presentation.

3.13 Example. Let X be a k-variety, and G an algebraic k-group acting on X. Define a groupoid fibration $[X/G]^{\mathrm{pre}}$ as follows: families parametrized by the scheme T are morphisms $x : T \to X$. Morphisms in $[X/G]^{\mathrm{pre}}$ are pairs (f,ϕ), where $f : T' \to T$ is a morphism of parameter schemes, and $\phi : T' \to G$ is a morphism, such that $x' = x(f) \cdot \phi$. Hence the fibre $[X/G]^{\mathrm{pre}}(T)$ is the transformation groupoid of the group $G(T)$ acting on the set $X(T)$. Then $[X/G]^{\mathrm{pre}}$ is a prestack with scheme-representable diagonal, because for $x : T \to X$ and $y : U \to X$ we have that

$$\begin{tikzcd}
\underline{\mathrm{Isom}}(x,y) \arrow[r] \arrow[d] & T \times U \arrow[d, "x\times y"] \\
X \times G \arrow[r, "\mathrm{pr}\times\sigma"] & X \times X
\end{tikzcd}$$

is a cartesian diagram. We note that the properties of the diagonal of $[X/G]^{\mathrm{pre}}$ are the properties of the morphism $X \times G \to X \times X$.

Versal families and their symmetry groupoids

The definition of versal family uses étale covers. If we were to use only Zariski covers, there would not be enough versal families to make the theory interesting.

3.14 Definition. Suppose that $\mathfrak{X}$ is a groupoid fibration. A **versal family** for $\mathfrak{X}$ is a family x/Γ_0 such that

(i) for every $\mathfrak{X}$-family y/T, there exist étale morphisms $U_i \to T$, whose images cover T, and morphisms of k-schemes $f_i : U_i \to \Gamma_0$, such that $y|_{U_i} \cong f_i^* x$,

(ii) the symmetry groupoid $\Gamma_1 = \underline{\mathrm{Isom}}(x,x)$ of x is representable.

A useful analogue of Lemma 2.16 in this context is the following.

3.15 Lemma. *If a groupoid fibration $\mathfrak{X}$ admits a versal family whose symmetry groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ has the property that $\Gamma_1 \to \Gamma_0 \times \Gamma_0$ is affine, then $\mathfrak{X}$ is a prestack with affine diagonal.*

Proof. The proof is analogous to the proof of Lemma 2.16. The morphisms $U_i \to T$ and $V_j \to S$ will be étale, and the morphisms $J_{ij} \to U_i \times V_j$ will be affine. By étale descent of affine schemes, it follows that $I \to T \times S$ is affine.

The theory of descent is about generalizing the construction of schemes by gluing along open subschemes to gluing over an étale cover (or more general types of flat covers), as in this proof. For the result needed here, see Théoreme 2 in [12]. See also [10]. $\square$

3.16 Example. Consider the groupoid fibration of degree 2 unramified covers. A family parametrized by the scheme T is a degree 2 finite étale covering $\widetilde{T} \to T$. The one point scheme $\operatorname{Spec} k$ parametrizes a trivial family, which is versal. If we were to insist on Zariski open covers in Definition 3.14, this would not be the case.

We adapt Definition 2.27 to the present context.

3.17 Definition. An **algebraic groupoid** $\Gamma_1 \rightrightarrows \Gamma_0$ is a groupoid in $\mathscr{S}$, which means that Γ_0 and Γ_1 are k-schemes, and that all structure morphisms are morphisms of k-schemes. We will always assume that our algebraic groupoids also satisfy

(i) the diagonal $s \times t : \Gamma_1 \to \Gamma_0 \times \Gamma_0$ is affine,

(ii) the source and target maps $s, t : \Gamma_1 \to \Gamma_0$ are smooth.

The notion of *Morita equivalence* (see Definition 2.34) carries over *mutatis mutandis*.

3.18 Definition. Let $\Gamma_1 \rightrightarrows \Gamma_0$ be an algebraic groupoid. A **Γ-torsor** over the k-scheme T is a pair (P_0, ϕ), where P_0 is a k-scheme, endowed with a smooth surjective morphism $\pi : P_0 \to T$, and $\phi : P_\bullet \to \Gamma_\bullet$ is a morphism of algebraic groupoids, such that (37) is a pullback diagram in $\mathscr{S}$, where $P_\bullet$ is the banal groupoid associated to $\pi : P_0 \to T$. The second axiom in Definition 2.27 is not necessary: every smooth surjective morphism admits étale local sections. (If we were to insist on Zariski local sections, we would get a different notion of torsor.)

If $\Gamma_1 \rightrightarrows \Gamma_0$ is an algebraic group $G \rightrightarrows \operatorname{Spec} k$, then a torsor is also called a **principal homogeneous G-bundle**, or **G-bundle**, for short.

3.19 Exercise. Given an algebraic groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, and a smooth morphism of schemes $U_0 \to \Gamma_0$. Then the fibered product

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & U_0 \times U_0 \arrow[d] \\ \Gamma_1 \arrow[r] & \Gamma_0 \times \Gamma_0 \end{tikzcd}$$

defines an algebraic groupoid $U_1 \rightrightarrows U_0$, the **restriction** of $\Gamma_\bullet$ via the morphism $U_0 \to \Gamma_0$. If $U_0 \to \Gamma_0$ is surjective, then $U_\bullet$ is Morita equivalent to $\Gamma_\bullet$.

3.20 Exercise. Every algebraic groupoid Γ with Γ_0 quasi-compact, is Morita equivalent to an algebraic groupoid Γ', with Γ'_0 and Γ'_1 affine, i.e., an *affine groupoid*. This is proved by restricting Γ via $\coprod U_i \to \Gamma_0$, where U_i is a finite affine open cover of Γ_0 and so $\coprod U_i$ is an affine scheme and $\coprod U_i \to \Gamma_0$ is an étale surjection.

Because of this, we could work entirely with affine schemes and affine groupoids to develop the theory of (quasi-compact) algebraic stacks with affine diagonal. We do not do this, because many interesting versal families have non-affine parameter space.

3.21 Exercise. Construct the *tautological* Γ-torsor. It is parametrized by Γ_0 and has Γ itself as symmetry groupoid. It is versal for the groupoid fibration of Γ-torsors over $\mathscr{S}$.

3.22 Exercise. Let G be an algebraic group acting on the scheme X. Then a torsor for the algebraic transformation groupoid $X \times G \rightrightarrows X$ is the same thing as a G-bundle, together with an equivariant morphism to X.

The analogue of the gluing property (Exercise 1.12) in the algebraic context is expressed in terms of the étale topology on $\mathscr{S}$ and gives rise to the notion of *stack in the étale topology*, in analogy to Definition 2.18.

3.23 Proposition. *Let Γ be an algebraic groupoid as in Definition 3.17. Then the groupoid fibration of Γ-torsors is a prestack with affine diagonal, and it satisfies the gluing axiom with respect to the étale topology (hence it is a stack in the étale topology).*

Proof. The part about the affine diagonal follows from Exercise 3.21 and Lemma 3.15, thus ultimately from descent for affine schemes. The gluing axiom is similar to the topological case, and uses again descent for affine schemes. The point is that the scheme $P_0 \to T \times \Gamma_0$, which is to be constructed by gluing, is going to be affine over $T \times \Gamma_0$. □

3.3 Algebraic stacks

We will only consider algebraic stacks with affine diagonal. Most algebraic stacks that occur in the literature have this property.

The following definition of algebraic stack avoids explicit reference to Grothendieck topologies, algebraic spaces, or descent theory.

3.24 Definition. A groupoid fibration $\mathfrak{X}$ over $\mathscr{S}$ is an **algebraic stack**, if

(i) $\mathfrak{X}$ admits a versal family x/Γ_0, whose symmetry groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ is an algebraic groupoid in the sense of Definition 3.17,

(ii) the tautological morphism of groupoid fibrations

$$\begin{aligned} \mathfrak{X} &\longrightarrow (\Gamma\text{-torsors}) \\ y &\longmapsto \underline{\mathrm{Isom}}(y, x) \end{aligned}$$

is an equivalence.

The groupoid fibration of Γ-torsors, for an algebraic groupoid Γ, is an algebraic stack, see Example 3.21.

Every algebraic stack is a prestack with affine diagonal, because of Lemma 3.15.

3.25 Theorem. *Suppose a groupoid fibration $\mathfrak{X}$ satisfies (i) in Definition 3.24. If $\mathfrak{X}$ satisfies the gluing axiom with respect to the étale topology in $\mathscr{S}$, then the tautological morphism $\mathfrak{X} \to (\Gamma\text{-torsors})$ is an equivalence, and hence $\mathfrak{X}$ is an algebraic stack.*

Proof. (See also Exercise 1.34.) We have to associate to every Γ-torsor P/T an $\mathfrak{X}$-family over T, whose generalized moduli map is the given torsor P/T. Over P_0 we have an $\mathfrak{X}$-family, and between the two pullbacks to P_1 we have an isomorphism of $\mathfrak{X}$-families, and the cocycle condition is satisfied. In other words, we have smooth gluing data for an $\mathfrak{X}$-family. To obtain étale gluing data for the same family, choose an étale surjection $U_0 \to T$ and a section $\sigma : U_0 \to P_0$. We get an induced morphism of banal groupoids $U_\bullet \to P_\bullet$, via which we can pull back our gluing data. $\square$

3.26 Definition. Suppose a groupoid fibration $\mathfrak{X}$ satisfies (i) in Definition 3.24. Then $\widetilde{\mathfrak{X}} = (\Gamma\text{-torsors})$ is the **stack associated to** $\mathfrak{X}$. It is an algebraic stack.

3.27 Exercise. Prove that the tautological functor $\mathfrak{X} \to \widetilde{\mathfrak{X}}$ is fully faithful.

3.28 Example. Let G be a linear algebraic group acting on a scheme X in such a way that $X \times G \to X \times X$ is affine. The groupoid fibration $[X/G]^{\mathrm{pre}}$ of Example 3.13 satisfies (i) in Definition 3.24. The stack associated to $[X/G]^{\mathrm{pre}}$ is $[X/G]$, the stack of torsors for the algebraic transformation groupoid $X \times G \rightrightarrows X$, Example 3.22.

3.29 Exercise. The groupoid fibration $\overline{\mathfrak{E}}$ of degenerate elliptic curves is an algebraic stack. We have already proved that $\overline{\mathfrak{E}}$ is a prestack with affine diagonal in Exercise 3.11. For the fact that $\overline{\mathfrak{E}}$ satisfies the gluing axiom with respect to the étale topology on $\mathscr{S}$, we can quote descent for projective schemes, see [10]. Finally, we need a versal family. The general way to produce versal families uses Hilbert schemes again, see [10] and [13].

For details, and the proof that moduli stacks of curves and marked curves of genus other than 1 are algebraic, see [1].

We can also prove that $\overline{\mathfrak{E}}$ is algebraic directly, avoiding descent theory and the use of Theorem 3.25, see 3.55.

If $k = \mathbb{C}$, the topological stack underlying the algebraic stack $\overline{\mathfrak{E}}$ is the stack of degenerate lattices of Section 2.6.

3.30 Exercise. Denote the affine plane over k, with the origin removed, by W. Denote the two coordinates by g_2 and g_3. The affine equation

$$y^2 = 4x^3 - g_2 x - g_3$$

with homogenization

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3$$

defines a family of projective plane curves $E \subset \mathbb{P}^2_W$. Together with the section P at infinity, (E, P) is a family of generalized elliptic curves parametrized by W. Show that it is a versal family for $\overline{\mathfrak{E}}$, and that its symmetry groupoid is the transformation groupoid of the multiplicative group $\mathbb{G}_m$ acting on W with weights 4 and 6. (Exercise 3.54 will be useful, to prove that the usual procedure for embedding an abstract genus 1 curve into the plane, and putting it into Weierstrass normal form, works in families, at least locally.)

Conclude that $\overline{\mathfrak{E}} \cong [W/\mathbb{G}_m]$. The quotient stack $[W/\mathbb{G}_m]$ is known as the *weighted projective line* with weights 4 and 6, notation $\mathbb{P}(4, 6)$.

Given the symmetry groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ of a versal family x of an algebraic stack, we obtain a 2-cartesian diagram

$$\begin{tikzcd} \Gamma_1 \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow"] & \Gamma_0 \arrow[d, "x"] \\ \Gamma_0 \arrow[r, "x"] & \mathfrak{X} \end{tikzcd}$$

The morphism $x : \Gamma_0 \to \mathfrak{X}$ given by the versal family is representable and smooth. We say that $\Gamma_0 \to \mathfrak{X}$ is a **smooth presentation** of $\mathfrak{X}$. The scheme Γ_0 should be thought of as a *smooth cover* of $\mathfrak{X}$. The stack $\mathfrak{X}$ 'looks like' Γ_0, locally (in the smooth topology). The fact that $\Gamma_0 \to \mathfrak{X}$ is smooth (which comes from the requirement that s and t be smooth), is essential for 'doing geometry' over $\mathfrak{X}$. For example, it makes it possible to decide when $\mathfrak{X}$ is smooth:

3.31 Definition. The algebraic stack $\mathfrak{X}$ is **smooth** (non-singular), if there exists a smooth presentation $\Gamma_0 \to \mathfrak{X}$ where Γ_0 (hence also Γ_1) is non-singular.

This definition is sensible, because for schemes, given a smooth surjective morphisms $Y \to X$ where Y is smooth, then X is smooth.

See Exercise 3.56, for another result that requires smoothness (or at least flatness) of the structure morphism $\Gamma_0 \to \mathfrak{X}$ of a presentation.

3.32 Definition. An algebraic stack is called a **separated Deligne-Mumford stack**, if its diagonal is finite and unramified.

Of course, $\overline{\mathfrak{E}}$ is a smooth separated Deligne-Mumford stack.

3.33 Exercise. An algebraic groupoid is *étale*, if its source and target morphism are étale morphisms of schemes. Prove that every smooth separated Deligne-Mumford stack $\mathfrak{X}$ admits a versal family whose symmetry groupoid is étale, by imitating the proof of Proposition 2.51. Let us remark that using sheaves of differentials rather than tangent bundles, it can be shown that the assumption that $\mathfrak{X}$ be smooth is not necessary, see [17].

3.34 Exercise. Consider the groupoid fibration of 'triangles with centroid at the origin'. This is the groupoid fibration of triples $(T', A, \mathscr{L})$, where T'/T is a degree three finite étale cover of the parameter scheme T, and $\mathscr{L}$ is a line bundle over the parameter space T. Moreover, $A : T' \to \mathscr{L}$ is a morphism, with the property that for every geometric point $t \to T$, the three points $A(T'_t) \subset \mathscr{L}_t$ add to 0, and no more than two of them coincide. Prove that this is an algebraic stack in two ways:

(i) Use descent for coherent sheaves ([12] Théoreme 1), to prove that Theorem 3.25 applies. Then prove that the family parametrized by $\mathbb{P}^1$ (with homogeneous coordinates x, y), where $\mathscr{L} = \mathscr{O}(1)$ and A is given by the three sections $x, y, -x-y \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(1)\big)$, is a versal family. Prove also that the symmetry groupoid of this family is given by the standard action of S_3 on $\mathbb{P}^1$, see (30). Conclude that this stack of triangles is isomorphic to $[\mathbb{P}^1/S_3]$.

(ii) Prove directly that this stack is isomorphic to $[\mathbb{P}^1/S_3]$ by doing an algebraic analogue of Exercise 1.45.

Now assume that $k = \mathbb{C}$ and conclude that the topological stack associated to this algebraic stack is isomorphic to $\mathfrak{L}$. Thus, we have endowed $\mathfrak{L}$ with the structure of a smooth separated Deligne-Mumford stack.

For general k, we will now denote the algebraic stack of triangles by $\mathfrak{L}$.

3.35 Exercise. Prove that the stack of triangles $\mathfrak{L}$ from Exercise 3.34 is isomorphic to the stack of triple sections of a rank 1 projective bundle with a marked section, ∞, where no points are allowed to come together.

3.36 Exercise. Suppose that G is a linear k-group acting on the k-variety X. If $X \times G \to X \times X$ is finite, then $[X/G]$ is a separated Deligne-Mumford stack. If X is smooth, so is the quotient stack $[X/G]$.

3.37 Exercise. The open substack $\mathfrak{V}_X^{r,d,N} \subset \mathfrak{V}_X^{r,d}$ of N-regular bundles is algebraic: it satisfies the gluing property with respect to the étale topology because of descent for coherent sheaves. It has affine diagonal by Exercise 3.12. So all that is left to do is exhibit a versal family with smooth source and target maps. This is provided by the universal family of the quot-scheme of quotients of $\mathscr{O}_X(-N)^{\oplus h(N)}$ of Hilbert polynomial h, where h is the Hilbert polynomial of bundles on X of rank r and degree d. The fact that this family is versal follows directly from properties of regularity. Once we restrict to the open subscheme of the quot-scheme where the quotient is N-regular, source and target map of the symmetry groupoid become smooth, because then the symmetry groupoid is a transformation groupoid for the group $GL_{h(N)}$. We conclude that $\mathfrak{V}_X^{r,d,N}$ is a quotient stack.

3.38 Exercise. Prove that any 2-fibered product of algebraic stacks is algebraic.

3.4 The coarse moduli space

3.39 Definition. Let $\mathfrak{X}$ be an algebraic stack. A **coarse moduli scheme** for $\mathfrak{X}$ is a scheme X, together with a morphism $\mathfrak{X} \to X$, which satisfies the following properties:

(i) $\mathfrak{X} \to X$ is universal for morphisms to schemes, in the sense that any morphism $\mathfrak{X} \to Y$, where Y is a scheme, factors uniquely through X:

$$\begin{tikzcd} \mathfrak{X} \arrow[d] \arrow[dr] & \\ X \arrow[r, dotted] & Y \end{tikzcd}$$

(ii) for every flat morphism of schemes $Y \to X$, form the 2-fibered product

$$\begin{tikzcd} \mathfrak{Y} \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathfrak{X} \arrow[r] & X \end{tikzcd}$$

then $\mathfrak{Y} \to Y$ satisfies property (i).

A course moduli scheme is unique up to unique isomorphism, if it exists. There is no reason why a coarse moduli scheme should exist in general, or why, even if it exists, we should be able to prove anything useful about it.

3.40 Example. Suppose G is a finite group acting on an affine k-scheme of finite type $X = \operatorname{Spec} A$. Then a *universal categorical quotient* (see [20]) exists. It is given by $X = \operatorname{Spec} A^G$. It is a coarse moduli scheme for $[X/G]$. We use notation X/G for this moduli scheme.

3.41 Example. Suppose $\mathfrak{X}$ is a quotient stack $\mathfrak{X} = [X/G]$, where G is a reductive k-group, and X is a finite type k-scheme. Suppose further, that all points of X are semi-stable with respect to some linearization of the G-action. Then the Geometric Invariant Theory quotient $X/\!\!/G$ is a coarse moduli scheme, see [20].

3.42 Example. Suppose that $\mathfrak{X}$ is a one-dimensional smooth separated Deligne-Mumford stack. Then we construct a coarse moduli scheme as follows: Choose an étale morphism from an affine curve $U_0 \to \mathfrak{X}$ (Exercise 3.33), and let $U_1 \rightrightarrows U_0$ be the corresponding étale groupoid. Restrict this groupoid to the function field L of U_0. The restricted groupoid corresponds to a diagram $L \rightrightarrows B$, where B is a finite étale L-algebra in two ways. Let $K \subset L$ be the equalizer of the two maps $L \rightrightarrows B$, and let $\overline{X}$ be the complete non-singular curve with function field K.

Then given any étale groupoid presentation $Y_1 \rightrightarrows Y_0$ of $\mathfrak{X}$, both Y_1 and Y_0 are smooth curves (not connected), and there is a unique morphism of groupoids from $Y_1 \rightrightarrows Y_0$ to $\overline{X} \rightrightarrows \overline{X}$, and hence a morphism $\mathfrak{X} \to \overline{X}$. Let $X \subset \overline{X}$ be the image of $\mathfrak{X} \to \overline{X}$, which is an open subcurve of $\overline{X}$. Then $\mathfrak{X} \to X$ is a coarse moduli space.

For higher dimensional separated Deligne-Mumford stacks, we have to enlarge our class of spaces to include separated algebraic spaces.

3.43 Definition. A **separated algebraic space** is a separated Deligne-Mumford stack for which every object in every fibre $\mathfrak{X}(T)$ over every scheme T is completely asymmetric.

3.44 Proposition. *An algebraic stack is a separated algebraic space if and only if its diagonal is a closed immersion.*

Proof. This follows from the fact that a finite unramified morphism, which is universally injective on points is a closed immersion. $\square$

Algebraic spaces are a generalization of schemes. They behave a lot like schemes, except that they are not locally affine in the Zariski topology, but only in the étale topology.

3.45 Definition. Let $\mathfrak{X}$ be a separated Deligne-Mumford stack. A **coarse moduli space** for $\mathfrak{X}$ is a separated algebraic space X, together with a morphism $\mathfrak{X} \to X$, which satisfies the two properties of Definition 3.39, where Y denotes separated algebraic spaces, rather than schemes.

3.46 Proposition. *Every finite type separated Deligne-Mumford stack $\mathfrak{X}$ admits a coarse moduli space X, which is a finite type separated algebraic space. Moreover, if $\bar{k}$ is the algebraic closure of k, then $\mathfrak{X}(\bar{k})/{\sim} \to X(\bar{k})$ is bijective.*

Proof. We briefly sketch the proof, because the proof shows that $\mathfrak{X}$ is *étale locally in* X a quotient stack. We try to imitate the poof of Theorem 2.49, of course, but the main reason why the proof does not carry over is that the Zariski topology on $\Gamma_0 \times \Gamma_0$ is not generated by boxes, like the product topology, which we used twice in the other proof.

Let $\Gamma_1 \rightrightarrows \Gamma_0$ be an étale presentation of $\mathfrak{X}$, and let $P_0 \in \Gamma_0$ be a point with automorphism group G. We pass to a different presentation of $\mathfrak{X}$. In fact, let $\Gamma_1' \rightrightarrows \Gamma_0'$ be the groupoid of 'stars with $\#G - 1$ rays' in $\Gamma_1 \rightrightarrows \Gamma_0$. Elements of Γ_0' are triples (x, ϕ, y), where $x \in \Gamma_0$, and $y = (y_g)_{g \in G, g \neq 1}$ is a family of elements of Γ_0, and $\phi = (\phi_g)_{g \in G, g \neq 1}$ is a family of elements of Γ_1, where for $g \in G$, $g \neq 1$, we have $\phi_g : x \to y_g$, in the groupoid $\Gamma_1 \rightrightarrows \Gamma_0$. Then $\Gamma_1' \rightrightarrows \Gamma_0'$ is another étale presentation of $\mathfrak{X}$. Replacing $\Gamma_1 \rightrightarrows \Gamma_0$ by $\Gamma_1' \rightrightarrows \Gamma_0'$, we may assume that there is an embedding $\Gamma_0 \times G \to \Gamma_1$, which identifies $\{P_0\} \times G$ with $G \subset \Gamma_1$, and makes the diagram

$$\begin{tikzcd} \Gamma_0 \times G \arrow[r] \arrow[d, "\mathrm{pr}"'] & \Gamma_1 \arrow[d, "s"'] \\ \Gamma_0 \arrow[r, "\mathrm{id}"] & \Gamma_0 \end{tikzcd}$$

commute. Replacing Γ_0 by the connected component containing P_0, and passing to the restricted groupoid, we assume that Γ_0 is connected. (Of course, this may result in a presentation for an open substack of $\mathfrak{X}$, which is fine, as we only need to construct the moduli space locally, by the claimed compatibility with flat base change.)

Now, arguing as in the proof of Theorem 2.49, we prove that we obtain a commutative diagram

$$\begin{tikzcd} \Gamma_0 \times G \times G \arrow[r] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \Gamma_2 \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\ \Gamma_0 \times G \arrow[r] \arrow[d, shift left=1] \arrow[d, shift right=1] & \Gamma_1 \arrow[d, shift left=1] \arrow[d, shift right=1] \\ \Gamma_0 \arrow[r] & \Gamma_0 \end{tikzcd}$$

This means that we have constructed a group action of G on Γ_0 and a morphism of groupoids from the transformation groupoid of G on Γ_0 to $\Gamma_1 \rightrightarrows \Gamma_0$. Finally we restrict to an affine open neighbourhood U_0 of P_0 in Γ_0, such that

$$(s \times t)^{-1}\big(\Delta(U_0)\big) \subset \Gamma_0 \times G \subset \Gamma_1 \,.$$

We obtain an étale morphism of stacks $[U_0/G] \to \mathfrak{X}$. The main point is that this morphism preserves automorphism groups.

Finally, the various coarse moduli spaces U_0/G give an étale presentation for the coarse moduli space of $\mathfrak{X}$. So if X is the coarse moduli space for $\mathfrak{X}$, we obtain a 2-cartesian diagram

$$\begin{tikzcd} [U_0/G] \arrow[r, "\text{étale}"] \arrow[d] & \mathfrak{X} \arrow[d] \\ U_0/G \arrow[r, "\text{étale}"] & X \end{tikzcd}$$

which explains that $\mathfrak{X}$ is, locally in the étale topology on X, a quotient by a finite group. $\square$

3.5 Bundles on stacks

3.47 Definition. A **coherent sheaf** $\mathscr{F}$ over the algebraic stack $\mathfrak{X}$ consists of the following data:

(i) for every $\mathfrak{X}$-family x/T, a coherent sheaf $x^*\mathscr{F}$ over the scheme T,

(ii) for every morphism of $\mathfrak{X}$-families $x'/T' \to x/T$ an isomorphism of coherent sheaves $(x^*\mathscr{F})|_{T'} \to (x')^*\mathscr{F}$.

The isomorphisms in (ii) have to be compatible with each other in the obvious way.

If we think of the family x/T as giving a morphism of stacks $T \xrightarrow{x} \mathfrak{X}$, then $x^*\mathscr{F}$ is the pullback of $\mathscr{F}$ over $\mathfrak{X}$ along x. This explains the notation.

The sections of $x^*\mathscr{F}$ over T are called the *sections of $\mathscr{F}$ over T*. A *global section* of $\mathscr{F}$ is a rule that assigns to every x/T a section of $x^*\mathscr{F}$ over T, compatible with pullbacks. We write $\Gamma(\mathfrak{X}, \mathscr{F})$ for the space of global sections of $\mathscr{F}$.

3.48 Example. The line bundle ω over $\overline{\mathfrak{E}}$ is defined by

$$E^*\omega = P^*\Omega_{E/T}\,,$$

for any family of generalized elliptic curves E/T with identity section $P : T \to E$. Let us denote the dual of ω by $\mathfrak{t}$. If $k = \mathbb{C}$, then the line bundle over the underlying topological stack of $\overline{\mathfrak{E}}$ corresponding to $\mathfrak{t}$ is the ambient bundle of the tautological degenerate lattice, denoted $\mathscr{L}$ in Exercise 2.59.

3.49 Example. The structure sheaf $\mathscr{O}_{\mathfrak{X}}$ of any stack $\mathfrak{X}$ is defined by $x^*\mathscr{O}_{\mathfrak{X}} = \mathscr{O}_T$, for every $\mathfrak{X}$-family x/T.

3.50 Example. For any representable morphism $\mathfrak{Y} \to \mathfrak{X}$ of stacks, the sheaf of relative differentials $\Omega_{\mathfrak{Y}/\mathfrak{X}}$ on $\mathfrak{Y}$ is defined by

$$y^*\Omega_{\mathfrak{Y}/\mathfrak{X}} = z^*\Omega_{Z/T}\,,$$

for any $\mathfrak{Y}$-family y/T. Here Z and z are defined by the cartesian diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \mathfrak{Y} \arrow[d] \\ T \arrow[r] \arrow[ur, "y"] \arrow[u, bend left, "z"] & \mathfrak{X} \end{tikzcd}$$

3.51 Example. For any morphism of stacks $F : \mathfrak{X} \to \mathfrak{Y}$, and coherent sheaf $\mathscr{F}$ on $\mathfrak{Y}$, the pullback $F^*\mathscr{F}$ on $\mathfrak{X}$ is defined by

$$x^* F^* \mathscr{F} = F(x)^* \mathscr{F} \,,$$

for any $\mathfrak{X}$-family x/T.

3.52 Example. The line bundle ω on $\overline{\mathfrak{E}}$ can also be defined as $P^*\Omega_{\overline{\mathfrak{F}}/\overline{\mathfrak{E}}}$, where $P : \overline{\mathfrak{E}} \to \overline{\mathfrak{F}}$ is the universal section of the universal generalized elliptic curve.

3.53 Example. For $n > 0$, a vector bundle V_n of rank n on $\overline{\mathfrak{E}}$ is defined by

$$E^* V_n = \pi_* \mathscr{O}_E(nP) \,,$$

for any family of generalized elliptic curves E/T with structure map $\pi : E \to T$ and section $P : T \to E$. To prove this, first notice that P defines an effective Cartier divisor on E, so that $\mathscr{O}_E(nP)$ is a well-defined line bundle on E. Then $\pi_*\mathscr{O}_E(nP)$ and $R^1\pi_*\mathscr{O}_E(nP)$ are coherent sheaves on T. Using cohomology and base change (and Nakayama's lemma), deduce from the fact that $H^1\big(E_t, \mathscr{O}(nP)\big) = 0$, for any fibre E_t of E, that $R^1\pi_*\mathscr{O}_E(nP) = 0$. Then apply cohomology and base change again, to deduce that the formation of $\pi_*\mathscr{O}_E(nP)$ commutes with arbitrary base change. Finally, apply cohomology and base change a third time, to deduce from the fact that $H^0\big(E_t, \mathscr{O}(nP)\big)$ has dimension n, for every fibre E_t of E, that $\pi_*\mathscr{O}_E(nP)$ is locally free of rank n.

3.54 Exercise. Prove that $V_1 = \mathscr{O}_{\overline{\mathfrak{E}}}$. Prove that for every $n \geq 2$, there is an exact sequence of vector bundles on $\overline{\mathfrak{E}}$:

$$0 \longrightarrow V_{n-1} \longrightarrow V_n \longrightarrow \mathfrak{t}^{\otimes n} \longrightarrow 0 \;.$$

Conclude that, if the characteristic of k is neither 2 nor 3, then for every nowhere vanishing section θ of $\mathfrak{t}$ over a scheme T, there exist unique sections x of V_2 and y of V_3 over T, mapping to θ^2, and $2\theta^3$, respectively, and satisfying an equation of the form $y^2 = 4x^3 - g_2 x - g_3$ in V_6 over T. Here g_2 and g_3 are regular functions on T, not vanishing simultaneously.

The induced morphism $\theta \mapsto (g_2, g_3)$, from the total space of $\mathfrak{t}$ with its zero section removed to $\mathbb{A}^2 \setminus 0$ is $\mathbb{G}_m$-equivariant, if $\mathbb{G}_m$ acts with weights 4 and 6 on $\mathbb{A}^2$. Thus, g_2 and g_3 naturally give rise to global sections $g_2 \in \omega^{\otimes 4}$, and $g_3 \in \omega^{\otimes 6}$.

Deduce that every degenerate elliptic curve (E, P) over a parameter scheme T naturally embeds into $\mathbb{P}(\mathscr{O}_T \oplus \omega^{\otimes 2} \oplus \omega^{\otimes 3})$. Recall that $\mathbb{P}(\mathscr{O}_T \oplus \omega^{\otimes 2} \oplus \omega^{\otimes 3}) = \mathbb{P}\mathrm{roj}\,\big(\mathrm{Sym}_{\mathscr{O}_T}(\mathscr{O}_T \oplus \mathfrak{t}^{\otimes 2} \oplus \mathfrak{t}^{\otimes 3})\big)$. There is a natural homomorphism $\mathfrak{t}^{\otimes 6} \to \mathrm{Sym}^3_{\mathscr{O}_T}(\mathscr{O}_T \oplus \mathfrak{t}^{\otimes 2} \oplus \mathfrak{t}^{\otimes 3})$ whose image generates a homogeneous sheaf of ideals which cuts out E_T inside $\mathbb{P}^2(\mathscr{O}_T \oplus \omega^{\otimes 2} \oplus \omega^{\otimes 3})$. The word 'natural' in this context means *commutes with pullback of families.* Naturality implies that the construction is universal, in other words, the universal degenerate elliptic curve $\overline{\mathfrak{F}}$ embeds into $\mathbb{P}_{\overline{\mathfrak{E}}}(\mathscr{O}_{\overline{\mathfrak{E}}} \oplus \omega^{\otimes 2} \oplus \omega^{\otimes 3})$.

3.55 Exercise. Prove that the morphism $\theta \mapsto (g_2, g_3)$ from the total space of the line bundle $\mathfrak{t}$ over $\overline{\mathfrak{E}}$ minus its zero section to $W = \mathbb{A}^2 \setminus \{(0,0)\}$ is an isomorphism.

Deduce that W is a fine moduli space for the stack of triples (E, P, θ), where (E, P) is a degenerate elliptic curve parametrized by T, say, and θ is a global invertible section of $E^*\mathfrak{t}$ over T.

The morphism of groupoid fibrations $\overline{\mathfrak{E}} \to [W/\mathbb{G}_m]$ given by (g_2, g_3) is an equivalence. An inverse is given by mapping a line bundle $\mathscr{L}$, with sections $g_2 \in \mathscr{L}^{\otimes 4}$ and $g_3 \in \mathscr{L}^{\otimes 6}$ to the elliptic curve in $\mathbb{P}(\mathscr{O} \oplus \mathscr{L}^{\otimes 2} \oplus \mathscr{L}^{\otimes 3})$ with equation $y^2 = 4x^3 - g_2 x - g_3$.

This gives another proof that $\overline{\mathfrak{E}}$ is the weighted projective line $\overline{\mathfrak{E}} = \mathbb{P}(4, 6)$, avoiding Hilbert schemes (for representability of $\underline{\mathrm{Isom}}$-spaces), and descent (for the étale gluing property).

3.56 Exercise. If x/X is a versal family for $\mathfrak{X}$, with symmetry groupoid $X_1 \rightrightarrows X_0$, then a coherent sheaf over $\mathfrak{X}$ may be specified by the following data: a coherent sheaf $\mathscr{F}_0$ over X_0, and an isomorphism $\phi : s^*\mathscr{F} \to t^*\mathscr{F}$, such that $\mu^*\phi = p_2^*\phi \circ p_1^*\phi$, where $p_1, \mu, p_2 : X_2 \to X_1$ are the first projection, the groupoid multiplication, and the second projection, respectively. If $\mathscr{F}$ is the corresponding coherent sheaf on $\mathfrak{X}$, prove that we have an exact sequence

$$0 \longrightarrow \Gamma(\mathfrak{X}, \mathscr{F}) \longrightarrow \Gamma(X_0, \mathscr{F}) \xrightarrow{t^* - \phi \circ s^*} \Gamma(X_1, t^*\mathscr{F}) \,.$$

For example, a coherent sheaf on the quotient stack $[X/G]$ is the same thing as a G-equivariant coherent sheaf on X, and global sections over $[X/G]$ are invariant global sections over X.

3.57 Exercise. A global section of $\omega^{\otimes n}$ is called a **modular form of weight** n. Prove that the ring of modular forms is a polynomial ring over the field k, generated by $g_2 \in \Gamma(\overline{\mathfrak{E}}, \omega^{\otimes 4})$ and $g_3 \in \Gamma(\overline{\mathfrak{E}}, \omega^{\otimes 6})$. Thus, we have an isomorphism of graded rings

$$\bigoplus_{n=0}^{\infty} \Gamma(\overline{\mathfrak{E}}, \omega^{\otimes n}) = k[g_2, g_3] \,,$$

where the degrees of g_2 and g_3 are 4 and 6, respectively.

3.58 Exercise. Let $X_1 \rightrightarrows X_0$ be an étale presentation of a smooth separated Deligne-Mumford stack $\mathfrak{X}$. We get an induced groupoid by passing to the total spaces of the tangent bundles $TX_1 \rightrightarrows TX_0$. Exercise 3.56 shows that this data gives rise to a vector bundle on $\mathfrak{X}$, the **tangent bundle** of $\mathfrak{X}$, notation $T_{\mathfrak{X}}$.

3.59 Exercise. Let $\mathfrak{X}$ be an orbifold, i.e., a smooth separated Deligne-Mumford stack whose coarse moduli space is connected, and which has a non-empty open substack which is a scheme. Prove that the frame bundle of $T_{\mathfrak{X}}$ is an algebraic space, and deduce that $\mathfrak{X}$ is a global quotient of an algebraic space by GL_n, where $n = \dim \mathfrak{X}$.

3.60 Exercise. Find k, such that $T_{\overline{\mathfrak{E}}} = \omega^{\otimes k}$.

3.6 Stacky curves: the Riemann-Roch theorem

As an example of algebraic geometry over stacks, we shall briefly discuss the Riemann-Roch theorem for line bundles over stacky curves. For an account of some of the basics of stacky curves in the analytic category, see [4].

We assume that the ground field k has characteristic 0, to avoid issues with non-separable morphisms and wild ramification. We also assume k to be algebraically closed. A *curve* is a one-dimensional smooth connected scheme over k. Curves are quasi-projective. A *complete curve* is a projective curve.

3.61 Definition. A **stacky curve** is a one-dimensional smooth separated Deligne-Mumford stack $\mathfrak{X}$, whose coarse moduli space X (which is a curve) is irreducible. The stacky curve $\mathfrak{X}$ is **complete** if X is complete. An **orbifold curve** is a stacky curve $\mathfrak{X}$ which is generically a scheme, i.e., there is a non-empty open subscheme $U \subset X$ of the coarse moduli space, such that $\mathfrak{X} \times_X U \to U$ is an isomorphism.

Every dominant morphism $Y \to \mathfrak{X}$ from a curve to a stacky curve is representable and has a relative sheaf of differentials $\Omega_{Y/\mathfrak{X}}$, and therefore a ramification divisor $R_{Y/\mathfrak{X}}$. The closed points of Y have ramification indices relative $\mathfrak{X}$.

Orbifold curves and root stacks

3.62 Definition. Let X be a curve, P a closed point of X and $r > 0$ an integer. The associated **root stack** $\mathfrak{X} = X[\sqrt[r]{P}]$ is defined such that an $\mathfrak{X}$-family parametrized by T consists of

(i) a morphism $T \to X$,

(ii) a line bundle L over T

(iii) an isomorphism $\phi : L^{\otimes r} \xrightarrow{\sim} \mathscr{O}_X(P)|_T$,

(iv) a section s of L over T, such that $\phi(s^r)$ is the canonical section 1 of $\mathscr{O}_X(P)|_T$.

Isomorphisms of $\mathfrak{X}$-families are given by isomorphisms of line bundles, respecting ψ and s.

To construct a versal family for $X[\sqrt[r]{P}]$, choose an affine open neighbourhood U of P in X, and a uniformizing parameter π at P, and assume that the order of π at all points of $U - \{P\}$ is 0. Then let $V \to U$ be the Riemann surface of $\sqrt[r]{\pi}$, its affine coordinate ring is $\mathscr{O}(V) = \mathscr{O}(U)[\sqrt[r]{\pi}]$. There is a unique point Q of V lying over P, and the ramification index $e(Q/P)$ is equal to r.

The parameter space of our versal family is $V \amalg (X - \{P\})$. The line bundle is $\mathscr{O}_V(Q)$ on V and the structure sheaf on $X - \{P\}$. We see that the root stack is isomorphic to X away from P, and so we may as well assume that $X = U$. Then we can take the family parametrized by V alone as a versal family. The symmetry groupoid of this family over V is the transformation groupoid of μ_r, the group or r-th roots of unity, acting on V by Galois transformations given by the natural action of μ_r on the r-th roots of π. We see that $U[\sqrt[r]{P}] \cong [V/\mu_r]$, and that $U = V/\mu_r$.

We conclude that the root stack $\mathfrak{X} = X[\sqrt[r]{P}]$ is an orbifold curve. There is a morphism $\mathfrak{X} \to X$, which makes X the coarse moduli space of $\mathfrak{X}$, and is an isomorphism away from P. The fibre of $\mathfrak{X} \to X$ over P is isomorphic to $B\mu_r$. We can view $\mathfrak{X}$ as obtained by inserting a stacky point of order $\frac{1}{r}$ into the curve X at the point P.

The stack $\mathfrak{X}$ comes with a canonical line bundle over it, we will write it as $\mathscr{O}_{\mathfrak{X}}(\frac{1}{r}P)$. This line bundle has a canonical section, which we will write as 1.

Suppose $Y \to X$ is a non-constant morphism of curves. Then there exists a morphism $Y \to X[\sqrt[r]{P}]$ such that

$$\begin{tikzcd} Y \arrow[r, dotted] \arrow[dr] & X[\sqrt[r]{P}] \arrow[d] \\ & X \end{tikzcd}$$

commutes (such a lift is unique up to unique isomorphism), if and only if the ramification indices $e(Q/P)$ of all $Q \in Y$ lying over P are divisible by r. In this case, the ramification index of such a point Q in Y relative to $X[\sqrt[r]{P}]$ is $\frac{1}{r}e(Q/P)$. In particular, if the ramification index $e(Q/P)$ is equal to r, then the morphism $Y \to X[\sqrt[r]{P}]$ is unramified at Q. Because ramification indices are multiplicative with respect to composition of morphisms, it makes sense to say that $X[\sqrt[r]{P}]$ is ramified of order r over X.

Suppose that $\mathfrak{X} \to X$ is an orbifold curve together with its coarse moduli space, and assume that $\mathfrak{X} \to X$ is an isomorphism away from $P \in X$. We define the **ramification index** of $\mathfrak{X} \to X$ over P to be the integer r, such that for any étale presentation $Y \to \mathfrak{X}$, and any point $Q \in Y$ mapping to P in X the ramification index $e(Q/P)$ is equal to r. Then there is a canonical X-morphism $\mathfrak{X} \to X[\sqrt[r]{P}]$, because over any such Y there is a canonical r-th root of the line bundle $\mathscr{O}_X(P)$. This morphism $\mathfrak{X} \to X[\sqrt[r]{P}]$ is an isomorphism, because any such $Y \to X$ factors (at least locally) through the curve obtained from X by adjoining the r-th root of a uniformizing parameter at P.

We conclude that any orbifold curve with only one stacky point is a root stack.

By considering n-tuples of line bundles with sections, we can also glue in stacky points of orders $\frac{1}{r_1}, \ldots, \frac{1}{r_n}$ at points $P_1, \ldots, P_n$ into X. If $\mathfrak{X}$ is obtained in this way from X, we say that $\mathfrak{X}$ is the **root stack** associated to the effective divisor $\sum_i^n (r_i - 1)P_i$ on X, notation $\mathfrak{X} = X[\sqrt[r_1]{P_1}, \ldots, \sqrt[r_n]{P_n}]$. (The divisor $\sum_i^n (r_i - 1)P_i$ should be thought of as the discriminant of $\mathfrak{X}/X$.) Over this stack we have line bundles $\mathscr{O}_{\mathfrak{X}}(\frac{1}{r_i}P_i)$, each with a canonical section 1. For a (not necessarily effective) divisor $\sum_i s_i P_i$ on X, we can form the tensor product

$$\mathscr{O}_{\mathfrak{X}}\Big(\sum_{i=1}^n \frac{s_i}{r_i} P_i\Big) = \mathscr{O}_{\mathfrak{X}}\big(\frac{1}{r_1}P_1\big)^{\otimes s_1} \otimes \ldots \otimes \mathscr{O}_{\mathfrak{X}}\big(\frac{1}{r_n}P_n\big)^{\otimes s_n} .$$

In this way, any $\mathbb{Q}$-divisor $\sum_i q_i P_i$ on X, with $q_i \in \mathbb{Q}$ such that $r_i q_i \in \mathbb{Z}$, for all i, can be thought of as a proper (i.e., integral) divisor on $\mathfrak{X}$. All divisors on $\mathfrak{X}$ come about in this way. Therefore, we get a canonical bijection between divisors on $\mathfrak{X}$ and $\mathbb{Q}$-divisors on X, which become integral when multiplied by the r_i.

We have proved:

3.63 Theorem. *Let $\mathfrak{X}$ be an orbifold curve with coarse moduli curve X.*

(i) There is a unique effective divisor $\sum_{i=1}^n (r_i - 1)P_i$ on X, such that $\mathfrak{X} \cong X[\sqrt[r_1]{P_1}, \ldots, \sqrt[r_n]{P_n}]$.

(ii) The ramification index $e(P)$ of $\mathfrak{X}$ at $P \in X$ is equal to 1, unless $P = P_i$, for some $i = 1, \ldots, n$, in which case it is $e(P) = r_i$.

(iii) Every divisor on $\mathfrak{X}$ is the pullback from X of a unique $\mathbb{Q}$-divisor $\sum_{j=1}^m q_j P_j$ on X, such that $e(P_j)q_j \in \mathbb{Z}$, for all $j = 1, \ldots, m$.

We will always identify every divisor on $\mathfrak{X}$ with the corresponding $\mathbb{Q}$-divisor on X.

3.64 Theorem. *Let $\mathfrak{X}$ be an orbifold curve with coarse moduli curve X. Denote by π the structure morphism $\pi : \mathfrak{X} \to X$. Let D be a divisor on $\mathfrak{X}$,* *also considered as a $\mathbb{Q}$-divisor on X. Then*

$$\pi_*\mathscr{O}_{\mathfrak{X}}(D) = \mathscr{O}_X(\lfloor D \rfloor)\,,$$

and $R^i\pi_*\mathscr{O}_{\mathfrak{X}}(D) = 0$, for all $i > 0$. Here $\lfloor D \rfloor$ is the integral divisor on X, obtained by rounding down all coefficients of D. In particular, we have

$$H^i\big(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}(D)\big) = H^i\big(X, \mathscr{O}_X(\lfloor D \rfloor)\big)\,,$$

for all $i \geq 0$.

Proof. The claim about π_* is easily deduced from the fact that $\mathfrak{X}$ is a root stack. The claim about $R^i\pi_*$, for $i > 0$, requires some basic cohomology theory for stacks, see, for example, [3]. The main point is that group cohomology of a finite group with coefficients in k vanishes. $\square$

3.65 Corollary (Orbifold Riemann-Roch)**.** *Assume that $\mathfrak{X}$ is a complete orbifold curve. Let g be the genus of the coarse moduli curve X. We have*

$$\chi\big(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}(D)\big) = \deg\lfloor D \rfloor + 1 - g\,,$$

and

$$\dim \Gamma\big(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}(D)\big) = \deg\lfloor D \rfloor + 1 - g\,,$$

if $\lfloor D \rfloor$ is non-special, in particular, if $\deg\lfloor D \rfloor > 2g - 2$.

3.66 Example. Consider, for example, the stack $\mathfrak{W}$ of triangles with the Weierstrass compactification, and its canonical line bundle $\mathscr{L}$ (which contains the universal triangle). This line bundle comes with two sections $g_2 \in \mathscr{L}^{\otimes 2}$ and $g_3 \in \mathscr{L}^{\otimes 3}$. These sections do not vanish simultaneously. The quotient $\frac{g_3}{g_2}$ is then a meromorphic section of $\mathscr{L}$. Using the coordinate j on the coarse moduli space $\mathbb{P}^1$ of $\mathfrak{W}$, we see that the divisor of zeroes of $\frac{g_3}{g_2}$ is equal to

$$D = \frac{1}{2}(1728) - \frac{1}{3}(0)\,,$$

because g_2 vanishes to order 3 at $j = 0$, and g_3 vanishes to order 2 at $j = 1728$. We conclude that

$$\mathscr{L} \cong \mathscr{O}_{\mathfrak{W}}\Big(\frac{1}{2}(1728) - \frac{1}{3}(0)\Big)\,,$$

and $\deg \mathscr{L} = \frac{1}{6}$. We conclude that all $\mathscr{L}^{\otimes n}$, for $n \geq 0$ are non-special, and hence

$$\dim \Gamma(\mathfrak{W}, \mathscr{L}^{\otimes n}) = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{-n}{3} \rfloor + 1 = \begin{cases} \lfloor \frac{n}{6} \rfloor & \text{if } n \equiv 1 \mod 6 \\ \lfloor \frac{n}{6} \rfloor + 1 & \text{otherwise} \end{cases}$$

for all $n \geq 0$.

3.67 Example. Now consider the stack $\mathfrak{L}$ of triangles, Exercise 3.34, with its tautological line bundle $\mathscr{L}$. The canonical morphism $(g_2, g_3) : \mathfrak{L} \to \mathfrak{W}$ is given by mapping the universal triple section of $\mathscr{L}$ to its second symmetric polynomial $g_2 \in \mathscr{L}^{\otimes 2}$ and its third symmetric polynomial $g_3 \in \mathscr{L}^{\otimes 3}$. Hence $(g_2, g_3)^* \mathscr{L} = \mathscr{L}$. Therefore, we have $\Gamma(\mathfrak{L}, \mathscr{L}) = \Gamma(\mathfrak{W}, \mathscr{L})$. Note that $\mathscr{O}_{\mathfrak{L}}(\frac{1}{2}\infty) \cong \mathscr{O}_{\mathfrak{L}}(\frac{1}{2}1728)$.

Stacky curves

3.68 Definition. For a finite group G, a G**-gerbe** is a morphism of algebraic stacks $\mathfrak{X} \to \mathfrak{Y}$, such that there exists a smooth presentation $Y \to \mathfrak{Y}$, such that the pullback $\mathfrak{X} \times_{\mathfrak{Y}} Y \to Y$ is isomorphic to $Y \times BG$.

3.69 Remark. Usually, the term G-gerbe means something stronger: namely a G-gerbe together with a trivialization of an associated $\mathrm{Out}(G)$-torsor, the *band* of G. For details, see for example [6].

3.70 Example. The morphism $\overline{\mathfrak{E}} \to \mathfrak{W}$ (see Exercise 2.58), is described algebraically as follows: a line bundle ω, with sections $g_2 \in \omega^{\otimes 4}$ and $g_3 \in \omega^{\otimes 6}$ (Exercise 3.55) is mapped to the line bundle $\mathscr{L} = \omega^{\otimes 2}$, with the same sections $g_2 \in \mathscr{L}^{\otimes 2}$ and $g_3 \in \mathscr{L}^{\otimes 3}$. Alternatively, it is given by the morphism of transformation groupoids $W \times \mathbb{G}_m \to W \times \mathbb{G}_m$, $(g_2, g_3, u) \mapsto (g_2, g_3, u^2)$.

The morphism $\overline{\mathfrak{E}} \to \mathfrak{W}$ is a $\mathbb{Z}_2$-gerbe. In fact, one way to think of $\overline{\mathfrak{E}}$ is as the stack of square roots $\omega^{\otimes 2} = \mathscr{L}$ of the tautological line bundle $\mathscr{L}$ on $\mathfrak{W}$. Pulling back via the smooth presentation $W \to \mathfrak{W}$, this turns into the stack of square roots of the trivial bundle on W. This is $W \times B\mathbb{Z}_2$.

Let $\mathfrak{X}$ be a stacky curve and $X_1 \rightrightarrows X_0$ an étale groupoid presentation of $\mathfrak{X}$. Then all connected components of both X_1 and X_0 are curves. Let $\overline{X}_1$ be the normalization of the image of the morphism $X_1 \to X_0 \times X_0$. Then the connected components of $\overline{X}_1$ are also curves. Moreover, we have a factorization $X_1 \to \overline{X}_1 \to X_0 \times X_0$, where $X_1 \to \overline{X}_1$ is finite and hence surjective. By the functorial properties of normalization, $\overline{X}_1 \rightrightarrows X_0$ is an algebraic groupoid. Moreover, source and target maps of this groupoid are unramified and hence étale. We have a morphism of groupoids $X_1 \to \overline{X}_1$, which is also unramified and hence étale, and and the kernel of this morphism is a finite étale group scheme $G \to X_0$. In fact, we have what is known as a *central extension of groupoids*:

$$\begin{tikzcd}
G \arrow[r] \arrow[dr] & X_1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \overline{X}_1 \arrow[dl, shift left] \arrow[dl, shift right] \\
& X_0 &
\end{tikzcd}$$

All three groupoids in this sequence have the same space of objects. Because $\mathfrak{X}$ is connected, it follows that G is a twisted form of a single finite group G_0. Let $\overline{\mathfrak{X}}$ be the stack associated to $\overline{X}_1 \rightrightarrows X_0$, it is an orbifold curve. The morphism of groupoids $X_1 \to \overline{X}_1$ defines a morphism of stacks $\mathfrak{X} \to \overline{\mathfrak{X}}$, and this morphism is a G_0-gerbe. We have proved the following:

3.71 Theorem. *Every stacky curve $\mathfrak{X}$ is isomorphic to a gerbe over an orbifold curve $\overline{\mathfrak{X}}$, hence a gerbe over a root stack. The orbifold $\overline{\mathfrak{X}}$ is uniquely determined by $\mathfrak{X}$, and is called the* **underlying orbifold** *of $\mathfrak{X}$.*

3.72 Theorem. *Let $\mathfrak{X}$ be a stacky curve, and $\pi : \mathfrak{X} \to \overline{\mathfrak{X}}$ the morphism to the underlying orbifold curve. If $\mathscr{L}$ is a line bundle on $\mathfrak{X}$, then either $\mathscr{L}$ comes from a line bundle $\overline{\mathscr{L}}$ on $\overline{\mathfrak{X}}$ via pullback along π, or $\pi_*\mathscr{L} = 0$. Moreover, $R^i\pi_*\mathscr{L} = 0$, for all $i > 0$. In the first case, we have, for all $i \geq 0$*

$$H^i(\mathfrak{X}, \mathscr{L}) = H^i(\overline{\mathfrak{X}}, \overline{\mathscr{L}}) \,,$$

in the second case, we have, for all $i \geq 0$

$$H^i(\mathfrak{X}, \mathscr{L}) = 0 \,.$$

Proof. The first claim about π_* can be checked generically, where it follows from the fact that a one-dimensional representation of a finite group is either trivial or has no invariant subspace. The rest is not hard using some basic cohomology theory of stacks. Just as for Theorem 3.64, the main point is that group cohomology of a finite group with coefficients in k vanishes. □

Together with Corollary 3.65, this theorem determines the cohomology of stacky curves with values in line bundles, i.e., the stacky Riemann-Roch theorem. It shows that the result is not much different than the one for orbifold curves, so instead of formulating the theorem, we finish with an example.

3.73 Example. Consider the bundle ω of modular forms over $\overline{\mathfrak{E}}$, the stack of generalized elliptic curves. The corresponding orbifold curve is $\mathfrak{W}$, the stack of triangles in the Weierstrass compactification. Let $\mathscr{L}$ denote the tautological bundle on $\mathfrak{W}$. If n is even, then $\omega^{\otimes n} = \pi^*\mathscr{L}^{\otimes \frac{n}{2}}$, if n is odd, then $\pi_*\omega^{\otimes n} = 0$. It follows that, for all $n \geq 0$

$$H^i(\overline{\mathfrak{E}}, \omega^{\otimes n}) = 0 \,, \qquad \text{for all } i > 0 \,,$$

$$\dim \Gamma(\overline{\mathfrak{E}}, \omega^{\otimes n}) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \lfloor \frac{n}{12} \rfloor & \text{if } n \equiv 2 \mod 12 \\ \lfloor \frac{n}{12} \rfloor + 1 & \text{otherwise} \,. \end{cases}$$

Note that these dimensions agree with the dimensions that one can read off from Exercise 3.57

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