Gromov-Witten Invariants in Algebraic Geometry

K. Behrend

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Abstract

Gromov-Witten invariants for arbitrary projective varieties and arbitrary genus are constructed using the techniques from [K. Behrend, B. Fantechi. *The Intrinsic Normal Cone.*]

Introduction

In [2] the problem of constructing the Gromov-Witten invariants of a smooth projective variety V was reduced to defining a 'virtual fundamental class'

$$[\overline{M}_{g,n}(V,\beta)]^{\mathrm{virt}} \in A_{(1-g)(\dim V-3)-\beta(\omega_V)+n}(\overline{M}_{g,n}(V,\beta))$$

in the Chow group of the algebraic stack

$$\overline{M}_{g,n}(V,\beta)$$

of stable maps of class $\beta \in H_2(V)$ from an n-marked prestable curve of genus g to V.

If $g = 0$ and V is convex (i.e. $H^1(\mathbb{P}^1, f^*T_V) = 0$, for all $f : \mathbb{P}^1 \to V$), then $\overline{M}_{0,n}(V,\beta)$ is smooth of the expected dimension $\dim V - 3 - \beta(\omega_V) + n$ and the usual fundamental class

$$[\overline{M}_{g,n}(V,\beta)]$$

will work. This was proved in [2].

In this paper we treat the general case using the construction from [1]. Recall from [ibid.] that virtual fundamental classes are constructed using an *obstruction theory*, and the *intrinsic normal cone*. The obstruction theory serves to give rise to a vector bundle stack $\mathfrak{E}$, into which the intrinsic normal cone $\mathfrak{C}$ can be embedded as a closed subcone stack. The virtual fundamental class is then obtained by intersecting $\mathfrak{C}$ with the zero section of $\mathfrak{E}$.

In our context, this process works as follows. Let $\mathfrak{M}_{g,n}$ be the algebraic stack of n-marked prestable curves of genus g. This is an algebraic stack, not of Deligne-Mumford (or even finite) type, but smooth of dimension $3(g-1)+n$. There is a canonical morphism

$$\overline{M}_{g,n}(V,\beta) \to \mathfrak{M}_{g,n},$$

given by forgetting the map, retaining the curve (but not stabilizing). Then $\overline{M}_{g,n}(V,\beta) \to \mathfrak{M}_{g,n}$ is an open substack of a stack of morphisms, and as such has a relative obstruction theory, which in this case is $(R\pi_* f^* T_V)^\vee$, where $\pi : C \to \overline{M}_{g,n}(V,\beta)$ is the universal curve and $f : C \to V$ is the universal stable map. Saying that $(R\pi_* f^* T_V)^\vee$is a relative obstruction theory means that there is a homomorphism

$$\phi : (R\pi_* f^* T_V)^\vee \longrightarrow L^\bullet_{\overline{M}_{g,n}(V,\beta)/\mathfrak{M}_{g,n}},$$

(where $L^\bullet$ is the cotangent complex) such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.

The homomorphism ϕ induces a closed immersion

$$\phi^\vee : \mathfrak{N}_{\overline{M}_{g,n}(V,\beta)/\mathfrak{M}_{g,n}} \longrightarrow h^1/h^0(R\pi_* f^* T_V)$$

of abelian cone stacks (see [1]) over $\overline{M}_{g,n}(V,\beta)$, where $\mathfrak{N}$ is the relative intrinsic normal sheaf. The relative intrinsic normal cone $\mathfrak{C}_{\overline{M}_{g,n}(V,\beta)/\mathfrak{M}_{g,n}}$ is a closed subcone stack of $\mathfrak{N}_{\overline{M}_{g,n}(V,\beta)/\mathfrak{M}_{g,n}}$, and so we get a closed immersion of cone stacks

$$\mathfrak{C}_{\overline{M}_{g,n}(V,\beta)/\mathfrak{M}_{g,n}} \longrightarrow h^1/h^0(R\pi_* f^* T_V).$$

Now since $R\pi_* f^* T_V$ has global resolutions (see Proposition 5), we may intersect $\mathfrak{C}_{\overline{M}_{g,n}(V,\beta)/\mathfrak{M}_{g,n}}$ with the zero section of the vector bundle stack $h^1/h^0(R\pi_* f^* T_V)$ to get the virtual fundamental class $[\overline{M}_{g,n}(V,\beta)]^{\text{virt}}$.

The fundamental axioms (see [5]) Gromov-Witten invariants need to satisfy to deserve their name are reduced in [2] to five basic compatibilities between the virtual fundamental classes. These follow from the basic properties proved in [1]. The dimension axiom, for example, follows from the basic fact that the intrinsic normal cone always has dimension zero.

We also show that if $V = G/P$, for a reductive group G and a parabolic subgroup P, there is an alternative construction of the virtual fundamental classes avoiding the intrinsic normal cone. We construct a cone C in the vector bundle $R^1\pi_*\mathcal{O}\otimes\mathfrak{g}$ on $\overline{M}_{g,n}(V,\beta)$, which may then be intersected with the zero section of $R^1\pi_*\mathcal{O}\otimes\mathfrak{g}$ to obtain the virtual fundamental class. This cone C is constructed as the normal cone of an embedding of $\overline{M}_{g,n}(V,\beta)$ into a certain stack of principal P-bundles (which is smooth, but not of Deligne-Mumford type).

A construction of Gromov-Witten invariants using a cone inside a vector bundle has also been announced by J. Li and G. Tian. Their methods differ from ours in that they use analytic methods, including the Kuranishi map.

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Preliminaries on Prestable Curves

Let k be a field. We shall work over the category of locally noetherian k-schemes (with the fppf-topology). For a modular graph τ (see [2], Definition 1.5) let $\mathfrak{M}(\tau)$ denote the k-stack of τ-marked prestable curves (which are defined in [2], Definition 2.6).

Lemma 1 *The algebraic k-stack $\overline{M}(\tau)$ of stable τ-marked curves is an open substack of $\mathfrak{M}(\tau)$.*

Proof. Let $\mathcal{C}_v\to\mathfrak{M}(\tau)$ be the universal curve corresponding to the vertex $v\in V_\tau$. Let $\widetilde{\mathcal{C}}_v$ be the stabilization. Then $\overline{M}(\tau)$ is the substack of $\mathfrak{M}(\tau)$ over which all $p_v:\mathcal{C}_v\to\widetilde{\mathcal{C}}_v$ are isomorphisms. This is open because the $\mathcal{C}_v$ are proper over $\mathfrak{M}(\tau)$. □

Now consider a modular graph τ' obtained from τ by adding some tails. We get an induced morphism of k-stacks $\mathfrak{M}(\tau')\to\mathfrak{M}(\tau)$ which simply forgets the markings corresponding to the tails $S_{\tau'}-S_\tau$. If $S_{\tau'}-S_\tau$ has cardinality 1, then $\mathfrak{M}(\tau')\to\mathfrak{M}(\tau)$ is a smooth curve, hence representable and smooth of relative dimension 1. So by induction, $\mathfrak{M}(\tau')\to\mathfrak{M}(\tau)$ is representable and smooth of relative dimension $\#(S_{\tau'}-S_\tau)$. By Lemma 1 the same is true for $\overline{M}(\tau')\to\mathfrak{M}(\tau)$.

Proposition 2 *The stack $\mathfrak{M}(\tau)$ is a smooth algebraic k-stack of dimension*

$$\dim(\tau)=\#S_\tau-\#E_\tau-3\chi(\tau).$$

Proof. For the definition of $\dim(\tau)$ and $\chi(\tau)$ see [2], Definitions 6.1 and 6.2.

Note that for every point of $\mathfrak{M}(\tau)$ there exists a τ' as above such that the induced morphism $\overline{M}(\tau') \to \mathfrak{M}(\tau)$ contains this given point in its image. Thus $\coprod_{\tau'} \overline{M}(\tau')$ is a presentation of $\mathfrak{M}(\tau)$ showing that $\mathfrak{M}(\tau)$ is algebraic. □

Now let τ^s be the stabilization of τ. Stabilization defines a morphism of algebraic k-stacks

$$s : \mathfrak{M}(\tau) \longrightarrow \overline{M}(\tau^s).$$

If τ' is obtained as above by adjoining tails to τ such that τ' is stable, we have a commutative diagram

$$\begin{array}{ccc} \overline{M}(\tau') & & \\ \downarrow & \searrow \overline{M}(\phi) & \\ \mathfrak{M}(\tau) & \xrightarrow{s} & \overline{M}(\tau^s). \end{array}$$

Here $\phi : \tau' \to \tau^s$ is the canonical morphism of stable modular graphs. In fact, one may define s locally by using such diagrams.

Proposition 3 *The morphism $s : \mathfrak{M}(\tau) \to \overline{M}(\tau^s)$ is flat.*

Proof. This follows by descent since the morphism $\overline{M}(\phi)$ for various $\phi : \tau' \to \tau^s$ as above are flat. □

The Virtual Fundamental Classes

Over $\mathfrak{M}(\tau)$ there is a family $(\mathcal{C}_v)_{v \in V_\tau}$ of universal curves, with sections $x_i : \mathfrak{M}(\tau) \to \mathcal{C}_{\partial_\tau(i)}$. Let $\mathcal{C}(\tau) \to \mathfrak{M}(\tau)$ be the curve obtained from $\coprod_{v \in V_\tau} \mathcal{C}_v$ by identifying x_i and x_j, for every edge $\{i,j\} \in E_\tau$. The curve $\mathcal{C}(\tau)$ has markings $x_i : \mathfrak{M}(\tau) \to \mathcal{C}(\tau)$, for each $i \in S_\tau$. In fact, $\mathcal{C}(\tau)$ is a $\widetilde{\tau}$-marked prestable curve, where $\widetilde{\tau}$ is the graph obtained from τ by contracting all edges of τ. Let us denote the structure morphism by

$$\pi : \mathcal{C}(\tau) \longrightarrow \mathfrak{M}(\tau).$$

We shall also denote any base change of π by π.

Now let V be a smooth projective k-variety, (τ, β) a stable V-graph and let $\mathrm{Mor}_{\mathfrak{M}(\tau)}(\tau, V)$ be the $\mathfrak{M}(\tau)$-space of morphisms from $\mathcal{C}(\tau)$ to V. Denote the universal morphism by

$$f : \mathcal{C}(\tau) \times \mathrm{Mor}_{\mathfrak{M}(\tau)}(\tau, V) \longrightarrow V.$$

By [3] the stack $\mathrm{Mor}_{\mathfrak{M}(\tau)}(\tau, V)$ is an algebraic k-stack and the structure morphism

$$\mathrm{Mor}_{\mathfrak{M}(\tau)}(\tau, V) \longrightarrow \mathfrak{M}(\tau)$$

is representable.

Proposition 4 *The proper Deligne-Mumford stack $\overline{M}(V,\tau,\beta)$ of stable maps is an open substack of* $\mathrm{Mor}_{\mathfrak{M}(\tau)}(\tau, V)$.

PROOF. The set of points where stabilization is an isomorphism is open. □

To define the virtual fundamental class on $\overline{M}(V,\tau,\beta)$ we consider the morphism $\overline{M}(V,\tau,\beta) \to \mathfrak{M}(\tau)$ and denote the relative intrinsic normal cone (see [1]) by

$$\mathfrak{C}(V,\tau,\beta) = \mathfrak{C}_{\overline{M}(V,\tau,\beta)/\mathfrak{M}(\tau)}$$

The intrinsic normal sheaf [ibid.] of $\overline{M}(V,\tau,\beta)$ over $\mathfrak{M}(\tau)$ we shall denote by $\mathfrak{N}(V,\tau,\beta)$.

By the relative version of [1] Proposition 6.2 we have a perfect relative obstruction theory [ibid.]

$$\pi_*(e^\vee)^\vee : R\pi_*(f^*T_V)^\vee \longrightarrow L^\bullet_{\mathrm{Mor}_{\mathfrak{M}(\tau)}(\tau,V)/\mathfrak{M}(\tau)}.$$

Restricting to the open substack $\overline{M}(V,\tau,\beta)$ we get a perfect relative obstruction theory

$$\pi_*(e^\vee)^\vee : R\pi_*(f^*T_V)^\vee \longrightarrow L^\bullet_{\overline{M}(V,\tau,\beta)/\mathfrak{M}(\tau)},$$

which we shall also denote by $E^\bullet(V,\tau,\beta)$. Thus $\mathfrak{C}(V,\tau,\beta)$ is embedded as a closed subcone stack in the vector bundle stack

$$\mathfrak{E}(V,\tau,\beta) = h^1/h^0(R\pi_* f^*T_V).$$

Note that the relative virtual dimension of $\overline{M}(V,\tau,\beta)$ over $\mathfrak{M}(\tau)$ with respect to the obstruction theory $R\pi_*(f^*T_V)^\vee$ is equal to

$$\begin{aligned}
\mathrm{rk}\, R\pi_*(f^*T_V)^\vee &= \chi(f^*T_V) \\
&= \deg f^*T_V + \dim V \cdot \chi(\mathcal{C}(\tau)) \\
&= \chi(\tau)\dim V - \beta(\tau)(\omega_V).
\end{aligned}$$

Essential is the following result.

Proposition 5 *Let (C, x, f) be a stable map over T to V, where T is a finite type algebraic k-stack. Let E be a vector bundle on C. Then $R\pi_* E$ has global resolutions, where $\pi : C \to T$ is the structure map.*

PROOF. Let M be an ample invertible sheaf on V and let

$$L = \omega_{C/T}(x_1 + \ldots + x_n) \otimes f^* M^{\otimes 3}.$$

By Proposition 3.9 of [2] the sheaf L is ample on the fibers of π. So for sufficiently large N we have that

1. $\pi^* \pi_* (E \otimes L^{\otimes N}) \to E \otimes L^{\otimes N}$ is surjective,
2. $R^1 \pi_* (E \otimes L^{\otimes N}) = 0$,
3. for all $t \in T$ we have that $H^0(C_t, L_t^{\otimes -N}) = 0$.

Let

$$F = \pi^* \pi_* (E \otimes L^{\otimes N}) \otimes L^{\otimes -N}$$

and let H be the kernel of the map $F \to E$. Thus we have a short exact sequence

$$0 \longrightarrow H \longrightarrow F \longrightarrow E \longrightarrow 0$$

of vector bundles on C. Note that for every $t \in T$ we have

$$\begin{aligned} H^0(C_t, F) &= H^0(C_t, \pi_*(E \otimes L^{\otimes N})_t \otimes L_t^{\otimes -N}) \\ &= H^0(C_t, L_t^{\otimes -N}) \otimes \pi_*(E \otimes L^{\otimes N})_t \\ &= 0 \end{aligned}$$

and hence $H^0(C_t, H) = 0$, also. Therefore, $\pi_* H$ and $\pi_* F$ are zero and $R^1\pi_* H$ and $R^1 \pi_* F$ are locally free. This implies that

$$R\pi_* E \cong [R^1 \pi_* H \to R^1 \pi_* F].$$

□

As shown in [1], by Proposition 5 the obstruction theory $R\pi_*(f^* T_V)^\vee$ gives rise to a virtual fundamental class

$$[\overline{M}(V, \tau, \beta), R\pi_*(f^* T_V)^\vee] \in A_{\dim(V,\tau,\beta)}(\overline{M}(V, \tau, \beta)),$$

since

$$\begin{aligned}
&\dim \mathfrak{M}(\tau) + \operatorname{rk} R\pi_*(f^*T_V)^\vee \\
&\quad= \chi(\tau)(\dim V - 3) - \beta(\tau)(\omega_V) + \#S_\tau - \#E_\tau \\
&\quad= \dim(V, \tau, \beta).
\end{aligned}$$

(See Definition 6.2 in [2] for the definition of $\dim(V, \tau, \beta)$.)

Theorem 6 *The system of virtual fundamental classes*

$$J(V, \tau, \beta) = [\overline{M}(V, \tau, \beta), R\pi_*(f^*T_V)^\vee]$$

is an orientation of $\overline{M}$ *over* $\mathfrak{G}_s(V)$. *If* V *is convex, on the tree level subcategory* $\mathfrak{T}_s(V)$, *we get back the orientation of [2], Theorem 7.5.*

Proof. If V is convex and τ a forest, then $R^1\pi_*(f^*T_V) = 0$, so that the virtual fundamental class is the usual fundamental class by [1] Proposition 7.3. Thus the virtual fundamental class agrees with the orientation of [2], Theorem 7.5. To check that J is an orientation, we need to check the five axioms listed in [2], Definition 7.1. This shall be done in the next Section. □

Remark As shown in [2], we get an associated system of Gromov-Witten classes for V.

Checking the Axioms

Axiom I. Mapping to a point

Let τ be a stable V-graph of class zero such that $|\tau|$ is non-empty and connected. As noted in [2] Section 7 we have

$$\overline{M}(V, \tau, 0) = V \times \overline{M}(\tau)$$

which is obviously smooth over $\mathfrak{M}(\tau)$. In fact, the morphism $\overline{M}(V, \tau, 0) \to \mathfrak{M}(\tau)$ is just the composition

$$V \times \overline{M}(\tau) \longrightarrow \overline{M}(\tau) \longrightarrow \mathfrak{M}(\tau)$$

of projection followed by inclusion. If $\widetilde{\pi} : \mathcal{C}(\tau) \to \overline{M}\tau$ is the universal curve over $\overline{M}(\tau)$, then $\mathcal{C}(V, \tau, 0) = V \times \mathcal{C}(\tau)$ and $\pi : \mathcal{C}(V, \tau, 0) \to \overline{M}(V, \tau, 0)$ is identified with $\mathrm{id} \times \widetilde{\pi} : V \times \mathcal{C}(\tau) \to V \times \overline{M}(\tau)$. Hence

$$\begin{aligned}
R^1\pi_*f^*T_V &= T_V \boxtimes R^1\widetilde{\pi}_*\mathcal{O}_{\mathcal{C}(\tau)} \\
&= \mathcal{T}^{(1)}
\end{aligned}$$

is locally free. So by [1] Proposition 7.3 we have

$$\begin{aligned} J(V,\tau,0) &= c_{\operatorname{rk} R^1\pi_* f^* T_V}(R^1\pi_* f^* T_V)\cdot[\overline{M}(V,\tau,0)] \\ &= c_{g(\tau)\dim V}(\mathcal{T}^{(1)})\cdot[\overline{M}(V,\tau,0)], \end{aligned}$$

which is Axiom I.

Axiom II. **Products**

Let (σ,α) and (τ,β) be stable V-graphs and denote the 'product' by $(\sigma\times\tau,\alpha\times\beta)$. Note that

$$E^\bullet(V,\sigma\times\tau,\alpha\times\beta)=E^\bullet(V,\sigma,\alpha)\boxplus E^\bullet(V,\tau,\beta),$$

so by [1] Proposition 7.4 we have

$$\begin{aligned} J(V,\sigma\times\tau,\alpha\times\beta) &= [\overline{M}(V,\sigma\times\tau,\alpha\times\beta),E^\bullet(V,\sigma,\alpha)\boxplus E^\bullet(V,\tau,\beta)] \\ &= [\overline{M}(V,\sigma,\alpha),E^\bullet(V,\sigma,\alpha)]\times[\overline{M}(V,\tau,\beta),E^\bullet(V,\tau,\beta)] \\ &= J(V,\sigma,\alpha)\times J(V,\tau,\beta), \end{aligned}$$

which is the product axiom.

Axiom III. **Cutting Edges**

Use notation as in [2], Section 7, modified as necessary to avoid confusion. Let β denote the $H_2(V)^+$-structure on both σ and τ. Write $\mathfrak{M}=\mathfrak{M}(\tau)=\mathfrak{M}(\sigma)$. Consider the cartesian diagram

$$\begin{array}{ccc} \overline{M}(V,\sigma,\beta) & \xrightarrow{\overline{M}(\Phi)} & \overline{M}(V,\tau,\beta) \\ {\scriptstyle g}\downarrow & & \downarrow \\ \mathfrak{M}\times V & \xrightarrow{\Delta} & \mathfrak{M}\times V\times V \end{array}$$

of stacks over $\mathfrak{M}$. Let us show that the obstruction theories $E^\bullet(V,\tau,\beta)$ and $E^\bullet(V,\sigma,\beta)$ are compatible over Δ (see [1]).

Over $\overline{M}(V,\sigma,\beta)$ let us consider the following two curves. First the curve $\mathcal{C}=\mathcal{C}(V,\sigma,\beta)$ obtained from the universal curves $(C_v)_{v\in V_\sigma}$ by gluing according to the edges of σ. Secondly, we have the curve $\mathcal{C}'$, which we obtain from $(C_v)_{v\in V_\sigma}$ by gluing according to the edges of τ. In other words, $\mathcal{C}'=\overline{M}(\Phi)^*\mathcal{C}(V,\tau,\beta)$. Moreover, $\mathcal{C}$ is obtained from $\mathcal{C}'$ by identifying the two sections x_1 and x_2 of $\mathcal{C}'$, corresponding to the edge $\{i_1,i_2\}$ of σ which is cut by Φ. Thus there is a structure morphism $p : \mathcal{C}' \to \mathcal{C}$ fitting into the commutative diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{p} & \mathcal{C} \\ & {\scriptstyle \pi'} \searrow & \downarrow {\scriptstyle \pi} \\ & & \overline{M}(V, \sigma, \beta). \end{array}$$

We shall also use the diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{p} & \mathcal{C} \\ & {\scriptstyle f'} \searrow & \downarrow {\scriptstyle f} \\ & & V, \end{array}$$

where $f : \mathcal{C} \to V$ is the universal map. Let $x = p \circ x_1 = p \circ x_2$.

If E is any locally free sheaf on $\mathcal{C}$, then for $i = 1, 2$ we have the evaluation homomorphism

$$u_i : p^* E \longrightarrow x_{i*} x_i{}^* p^* E = x_{i*} x^* E.$$

Applying p_* we get

$$p_*(u_i) : p_* p^* E \longrightarrow x_* x^* E.$$

Letting $u = p_*(u_2) - p_*(u_1)$ we have a short exact sequence

$$0 \longrightarrow E \longrightarrow p_* p^* E \xrightarrow{u} x_* x^* E \longrightarrow 0$$

of coherent sheaves on $\mathcal{C}$. Applying $R\pi_*$ we get a distinguished triangle

$$R\pi_* E \longrightarrow R\pi'_* p^* E \stackrel{R\pi_*(u)}{\longrightarrow} x^* E \longrightarrow R\pi_* E[1]$$

in $D(\mathcal{O}_{\overline{M}(V,\sigma,\beta)})$. Taking $E = f^* T_V$ we get the distinguished triangle

$$R\pi_* f^* T_V \longrightarrow R\pi'_* f'^* T_V \stackrel{R\pi_*(u)}{\longrightarrow} x^* f^* T_V \longrightarrow R\pi_* f^* T_V[1],$$

or dually,

$$x^* f^* \Omega_V \stackrel{R\pi_*(u)^\vee}{\longrightarrow} (R\pi'_* f'^* T_V)^\vee \longrightarrow (R\pi_* f^* T_V)^\vee \longrightarrow x^* f^* \Omega_V[1]. \quad (1)$$

Note that we have $E^\bullet(V, \sigma, \beta) = (R\pi_* f^* T_V)^\vee$ and $\overline{M}(\Phi)^*(E^\bullet(V, \tau, \beta)) = (R\pi'_* f'^* T_V)^\vee$. Moreover, $L^\bullet_\Delta = \Omega_V[1] | \mathfrak{M} \times V$, so that $g^* L_\Delta = x^* f^* \Omega_V[1]$, since $f \circ x = p_V \circ g$. So (1) gives the distinguished triangle

$$g^* L_\Delta[-1] \stackrel{R\pi_*(u)^\vee}{\longrightarrow} \overline{M}(\Phi)^* E^\bullet(V, \tau, \beta) \longrightarrow E^\bullet(V, \sigma, \beta) \longrightarrow g^* L_\Delta,$$

which we may shuffle around to give

$$\overline{M}(\Phi)^*E^\bullet(V,\tau,\beta) \longrightarrow E^\bullet(V,\sigma,\beta) \longrightarrow g^*L_\Delta \stackrel{R\pi_*(-u)^\vee}{\longrightarrow} \overline{M}(\Phi)^*E^\bullet(V,\tau,\beta)[1].$$

Now we have the obstruction morphisms $E^\bullet(V,\tau,\beta) \to L^\bullet_{\overline{M}(V,\tau,\beta)/\mathfrak{M}}$ and $E^\bullet(V,\sigma,\beta) \to L^\bullet_{\overline{M}(V,\sigma,\beta)/\mathfrak{M}}$. Moreover, we have the natural homomorphism $g^*L_\Delta \to L^\bullet_{\overline{M}(\Phi)}$. These give rise to a homomorphism of distinguished triangles

$$\begin{array}{ccccccc}
\overline{M}(\Phi)^*E^\bullet(V,\tau,\beta) & \longrightarrow & E^\bullet(V,\sigma,\beta) & \longrightarrow & g^*L_\Delta & \stackrel{R\pi_*(-u)^\vee}{\longrightarrow} & \overline{M}(\Phi)^*E^\bullet(V,\tau,\beta)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{M}(\Phi)^*L^\bullet_{\overline{M}(V,\tau,\beta)/\mathfrak{M}} & \longrightarrow & L^\bullet_{\overline{M}(V,\sigma,\beta)/\mathfrak{M}} & \longrightarrow & L^\bullet_{\overline{M}(\Phi)} & \longrightarrow & \overline{M}(\Phi)^*L^\bullet_{\overline{M}(V,\tau,\beta)/\mathfrak{M}}[1],
\end{array}$$

showing that $E^\bullet(V,\tau,\beta)$ and $E^\bullet(V,\sigma,\beta)$ are compatible over Δ. Hence by [1] Proposition 7.5 we have

$$\Delta^! J(V,\tau,\beta) = J(V,\sigma,\beta)$$

which is Axiom III.

Axiom IV. **Forgetting Tails**

Let us deal with the incomplete case, leaving the tripod losing cases to the reader. Letting $\mathcal{C} \to \mathfrak{M}(\tau)$ be the universal curve corresponding to the vertex $w \in V_\tau$ (notation from [2], Section 7). We have a cartesian diagram of algebraic k-stacks

$$\begin{array}{ccc}
\overline{M}(V,\sigma,\beta) & \stackrel{\overline{M}(\Phi)}{\longrightarrow} & \overline{M}(V,\tau,\beta) \\
{\scriptstyle d}\downarrow & & \downarrow \\
\mathcal{C} & \longrightarrow & \mathfrak{M}(\tau).
\end{array}$$

By [1] Proposition 7.2 we have

$$\overline{M}(\Phi)^*J(V,\tau,\beta) = [\overline{M}(V,\sigma,\beta), \overline{M}(\Phi)^*E^\bullet(V,\tau,\beta)].$$

Here the class on the right hand side is the virtual fundamental class defined by the relative intrinsic normal cone of the morphism d and the relative obstruction theory $\overline{M}(\Phi)^*E^\bullet(V,\tau,\beta)$. Note that the structure morphism $\overline{M}(V,\sigma,\beta) \to \mathfrak{M}(\sigma)$ factors through $d : \overline{M}(V,\sigma,\beta) \to \mathcal{C}$.

$$\begin{array}{ccc}
\overline{M}(V,\sigma,\beta) & \stackrel{d}{\longrightarrow} & \mathcal{C} \\
& \searrow & \downarrow \\
& & \mathfrak{M}(\sigma)
\end{array}$$

The morphism $d : \overline{M}(V,\sigma,\beta) \to \mathcal{C}$ associates to the stable map (C,x,h) the pair $((C',x'),y)$, where (C',x',h') is the image of (C,x,h) under $\overline{M}(\Phi)$ and (C',x') the underlying τ-marked prestable curve. Letting x_f be the section of C_v corresponding to the flag f, we obtain (C',x',h') by forgetting x_f and stabilizing. Moreover, y is the image of the forgotten section x_f in C'_w.

The morphism $\mathcal{C} \to \mathfrak{M}(\sigma)$ associates to the pair $((C,x),y)$, where (C,x) is a τ-marked prestable curve and y a section of C_w, the σ-marked prestable curve $(\widetilde{C},\widetilde{x})$ obtained as follows. For $v' \neq v$ we have $\widetilde{C}_{v'} = C_{w'}$, where w' is the vertex of τ corresponding to v'. The curve $(\widetilde{C}_v,(\widetilde{x}_j)_{j\in F_\sigma(v)})$ is obtained from $((C_w,(x_j)_{j\in F_\tau(w)}),y)$ by 'prestabilizing' (i.e. separating the special points) as in [4], Definition 2.3.

Lemma 7 *The morphism $\mathcal{C} \to \mathfrak{M}(\sigma)$ is étale.*

Proof. We will use the formal criterion for étaleness. Without loss of generality assume that w is the only vertex of τ. So let $((C,x),y)$ be a τ-marked prestable curve with section over the scheme T, $T \to T'$ a square zero extension and (C',x') a σ-marked prestable curve over T' such that $(C',x')|T$ is the prestabilization of $((C,x),y)$. We may assume that we may choose additional sections s of C over T, making (C,x,s) a stable marked curve. Then we extend the sections s to sections s' of C' over T'. Taking the stabilization of (C',x',s') after forgetting the section x'_f gives an extension of $((C,x),y)$ to T' whose prestabilization is (C',x'). $\square$

Consider the natural morphism $p : \mathcal{C}(V,\sigma,\beta) \to \overline{M}(\Phi)^*\mathcal{C}(V,\tau)$, which fits into the two commutative diagrams

$$\begin{array}{ccc} \mathcal{C}(V,\sigma,\beta) & \xrightarrow{p} & \overline{M}(\Phi)^*\mathcal{C}(V,\tau,\beta) \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle\pi'} \\ & & \overline{M}(V,\sigma,\beta) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{C}(V,\sigma,\beta) & \xrightarrow{p} & \overline{M}(\Phi)^*\mathcal{C}(V,\tau,\beta) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f'} \\ & & V \quad . \end{array}$$

Whenever E is a locally free sheaf on $\overline{M}(\Phi)^*\mathcal{C}(V,\tau,\beta)$ the canonical homomorphism $E \to p_*p^*E$ is an isomorphism. Applying this principle to $E = f'^*T_V$ we get an isomorphism

$$f'^*T_V \longrightarrow p_*f^*T_V.$$

Applying $R\pi'_*$ to this, gives an isomorphism

$$R\pi'_* f'^* T_V \longrightarrow R\pi_* f^* T_V.$$

Noting that $R\pi'_* f'^* T_V = \overline{M}(\Phi)^* E^\bullet(V,\tau,\beta)$ we get an isomorphism

$$\overline{M}(\Phi)^* E^\bullet(V,\tau,\beta) \longrightarrow E^\bullet(V,\sigma,\beta)$$

and whence an isomorphism

$$\mathfrak{C}(V,\sigma,\beta) \longrightarrow \overline{M}(\Phi)^* \mathfrak{C}(V,\tau,\beta).$$

By [1] Proposition 7.1 there is a natural isomorphism

$$\mathfrak{C}_{\overline{M}(V,\sigma,\beta)/\mathcal{C}} \longrightarrow \overline{M}(\Phi)^* \mathfrak{C}_{\overline{M}(V,\tau,\beta)/\mathfrak{M}(\tau)}.$$

By Lemma 7 we have a canonical isomorphism

$$\mathfrak{C}_{\overline{M}(V,\sigma,\beta)/\mathcal{C}} \longrightarrow \mathfrak{C}_{\overline{M}(V,\sigma,\beta)/\mathfrak{M}(\sigma)},$$

such that the diagram

$$\begin{array}{ccc}
\mathfrak{C}_{\overline{M}(V,\sigma,\beta)/\mathfrak{M}(\sigma)} & \stackrel{\sim}{\longleftarrow} & \mathfrak{C}_{\overline{M}(V,\sigma,\beta)/\mathcal{C}} \\
\cap & & \cap \\
\mathfrak{C}(V,\sigma,\beta) & \stackrel{\sim}{\longrightarrow} & \overline{M}(\Phi)^* \mathfrak{C}(V,\tau,\beta)
\end{array}$$

commutes. So finally, we have

$$\begin{aligned}
\overline{M}(\Phi)^* J(V,\tau,\beta) &= [\overline{M}(V,\sigma,\beta), \overline{M}(\Phi)^* E^\bullet(V,\tau,\beta)] \\
&= [\overline{M}(V,\sigma,\beta), E^\bullet(V,\sigma,\beta)] \\
&= J(V,\sigma,\beta),
\end{aligned}$$

which is Axiom IV.

Axiom V. **Isogenies**

Before we start with the proof, some general remarks. Let $\Phi : \tau \to \sigma$ be an elementary contraction of stable modular graphs, contracting the edge $\{f, \overline{f}\}$ of τ. Let $a : \tau \to \tau'$ and $b : \sigma \to \sigma'$ be combinatorial morphisms of modular graphs identifying τ and σ as the stabilizations of τ' and σ', respectively. Finally, let $\Phi' : \tau' \to \sigma'$ be as follows. We require $\{a(f), a(\overline{f})\}$ to be an edge of τ' and $\Phi' : \tau' \to \sigma'$ to be the elementary contraction contracting the edge

$\{a(f), a(\overline{f})\}$. Moreover, we require Φ to be the stabilization of Φ'. To fix notation, denote the vertex onto which Φ' contracts the edge $\{a(f), a(\overline{f})\}$ by $v_0 \in V_{\sigma'}$ and let $v_1 = \partial_{\tau'}(a(f))$ and $v_2 = \partial_{\tau'}(a(\overline{f}))$.

In this situation we get a commutative diagram of algebraic stacks

$$\begin{array}{ccc} \mathfrak{M}(\tau') & \stackrel{\mathfrak{M}(\Phi')}{\longrightarrow} & \mathfrak{M}(\sigma') \\ s \downarrow & & \downarrow s \\ \overline{M}(\tau) & \stackrel{\overline{M}(\Phi)}{\longrightarrow} & \overline{M}(\sigma). \end{array}$$

Define $\mathfrak{P}$ to be the fibered product

$$\begin{array}{ccc} \mathfrak{P} & \longrightarrow & \mathfrak{M}(\sigma') \\ \downarrow & & \downarrow s \\ \overline{M}(\tau) & \stackrel{\overline{M}(\Phi)}{\longrightarrow} & \overline{M}(\sigma). \end{array}$$

Consider the induced morphism $l : \mathfrak{M}(\tau') \to \mathfrak{P}$.

Proposition 8 *We have* $l_*[\mathfrak{M}(\Phi')] = s^*[\overline{M}(\Phi)]$.

PROOF. First note that $\mathfrak{M}(\tau')$ is irreducible, since $\mathfrak{M}(\tau')$ is a product of stacks of the form $\mathfrak{M}_{g,n}$, which are irreducible since the stacks $\overline{M}_{g,n}$ are. Moreover, $\mathfrak{M}(\tau') \to \mathfrak{P}$ is surjective, so that $\mathfrak{P}$ is irreducible, too.

Secondly, let us remark that there exist non-empty (hence dense) open substacks $\mathfrak{M}(\tau')^0 \subset \mathfrak{M}(\tau')$ and $\mathfrak{P}^0 \subset \mathfrak{P}$ such that l induces an isomorphism $l^0 : \mathfrak{M}(\tau')^0 \xrightarrow{\sim} \mathfrak{P}^0$. In fact, let $\mathfrak{M}(\tau')^0$ be the open substack of $\mathfrak{M}(\tau')$ characterized by the requirement that the marked curves C_{v_1} and C_{v_2} be stable. To construct $\mathfrak{P}^0$, let $\mathfrak{M}(\sigma')^0$ be the open substack of $\mathfrak{M}(\sigma')$ where the marked curve C_{v_0} is stable. Then set

$$\mathfrak{P}^0 = \overline{M}(\tau) \times_{\overline{M}(\sigma)} \mathfrak{M}(\sigma')^0.$$

These facts imply the claim. □

Now let $(\Phi, m) : \tau \to \sigma$ be an elementary isogeny of type forgetting a tail. Let $f \in F_\tau$ be the forgotten tail. Let $a : \tau \to \tau'$ and $b : \sigma \to \sigma'$ be as above. Finally, let $\Phi' : \tau' \to \sigma'$ be the 'adjoint' of a combinatorial morphism of graphs, such that there exists a tail map m', a semigroup A and A-structures on τ' and σ' making (Φ', m') the elementary isogeny of stable A-graphs forgetting the tail $a(f)$. Moreover, we require Φ to be the stabilization of Φ'.

Let $\mathfrak{P}$ be the fibered product

$$\begin{array}{ccc} \mathfrak{P} & \longrightarrow & \mathfrak{M}(\sigma') \\ \downarrow & & \downarrow s \\ \overline{M}(\tau) & \longrightarrow & \overline{M}(\sigma) \end{array}$$

and $\mathcal{C}$ the universal curve over $\mathfrak{M}(\sigma')$ corresponding to $w \in V_{\sigma'}$, where w is the vertex of the forgotten tail. (If w does not exist, i.e. if Φ' is complete, then $\mathcal{C} = \mathfrak{M}(\sigma')$.) As in the proof of Axiom IV we have a morphism $\mathcal{C} \to \mathfrak{M}(\tau')$ giving rise to a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \stackrel{\pi'}{\longrightarrow} & \mathfrak{M}(\sigma') \\ \downarrow & & \\ \mathfrak{M}(\tau') & & \downarrow s \\ s \downarrow & & \\ \overline{M}(\tau) & \stackrel{\overline{M}(\Phi)}{\longrightarrow} & \overline{M}(\sigma) \end{array}$$

and hence to a morphism $l : \mathcal{C} \to \mathfrak{P}$.

Proposition 9 *We have $l_*[\pi'] = s^*[\overline{M}(\Phi)]$.*

PROOF. Again, $\mathcal{C}$ and $\mathfrak{P}$ are irreducible and l induces an isomorphism $l^0 : \mathcal{C}^0 \to \mathfrak{P}^0$, where $\mathcal{C}^0$ is the restriction of $\mathcal{C}$ to $\mathfrak{M}(\sigma')^0$ and $\mathfrak{P}^0 = \overline{M} \times_{\overline{M}(\sigma)} \mathfrak{M}(\sigma')^0$. Here $\mathfrak{M}(\sigma')^0 \subset \mathfrak{M}(\sigma')$ is the open substack where C_w is stable. $\Box$

Now let us prove Axiom V. According to [2], Remark 7.2, it suffices to do this for the case that $\Phi : \tau \to \sigma$ is an elementary isogeny, $\#J = 1$ and $(a_i, \tau_i, \Phi_i)_{i \in I}$ a pullback. So we shall use notation as in the Definition of pullback ([2], Definition 6.10). We shall include the $H_2(V)^+$-structures on σ' and τ_i ($i \in I$) in the notation. They shall be denoted by β' and β_i ($i \in I$), respectively. The underlying graph of (τ_i, β_i) is the same for all $i \in I$. Let us call it simply τ'.

Let us first consider the case where Φ is a contraction.

Lemma 10 *We have a cartesian diagram*

$$\begin{array}{ccc} \coprod_{i \in I} \overline{M}(V, \tau', \beta_i) & \longrightarrow & \overline{M}(V, \sigma', \beta') \\ \downarrow & & \downarrow \\ \mathfrak{M}(\tau') & \stackrel{\mathfrak{M}(\Phi')}{\longrightarrow} & \mathfrak{M}(\sigma') \end{array}$$

of algebraic k-stacks. Moreover,

$$\mathfrak{M}(\Phi')^! J(V,\sigma',\beta') = \sum_{i\in I} J(V,\tau',\beta').$$

PROOF. The first fact follows immediately from the definitions. The second fact is [1] Proposition 7.2. □

Axiom V will follow by putting Lemma 10 and Proposition 8 together as follows. By Lemma 10 all squares in the following diagram are cartesian.

$$\begin{array}{ccccc}
\coprod_{i\in I}\overline{M}(V,\tau',\beta_i) & \xrightarrow{h} & \overline{M}(\tau)\times_{\overline{M}(\sigma)}\overline{M}(V,\sigma',\beta') & \longrightarrow & \overline{M}(V,\sigma',\beta') \\
\downarrow & & \downarrow & & \downarrow a \\
\mathfrak{M}(\tau') & \xrightarrow{l} & \overline{M}(\tau)\times_{\overline{M}(\sigma)}\mathfrak{M}(\sigma') & \longrightarrow & \mathfrak{M}(\sigma') \\
& s \searrow & \downarrow & & \downarrow s \\
& & \overline{M}(\tau) & \xrightarrow{\overline{M}(\Phi)} & \overline{M}(\sigma)
\end{array}$$

So we may calculate as follows.

$$\begin{aligned}
\overline{M}(\Phi)^! J(V,\sigma',\beta') &= a^* s^* [\overline{M}(\Phi)] \cdot J(V,\sigma',\beta') \\
&= a^* l_* [\mathfrak{M}(\Phi')] \cdot J(V,\sigma',\beta') \\
\text{(by Proposition 8)} & \\
&= h_* \mathfrak{M}(\Phi')^! J(V,\sigma',\beta') \\
&= h_* \sum_{i\in I} J(V,\tau',\beta_i)
\end{aligned}$$

by Lemma 10. This is the context of Axiom V.

The case that Φ is of type forgetting a tail is similar. Instead of Lemma 10 one uses Axiom IV, and Proposition 8 is replaced by Proposition 9.

This finishes the proof of Axiom V and hence the proof of Theorem 6.

Homogeneous Spaces

In the case where V is a generalized flag variety, we can give a more explicit construction of Gromov-Witten invariants as follows.

Curves and Principal Bundles

For a smooth algebraic k-group G with Lie algebra $\mathfrak{g}$, we denote by

$$\mathfrak{H}^1(\tau, G)$$

the k-stack of G-torsors on τ-marked prestable curves. More precisely, for a k-scheme T, the category $\mathfrak{H}^1(\tau, G)(T)$ is the category of pairs (C, E), where $C = (C_v)_{v \in V_\tau}$ is a τ-marked prestable curve over T, giving rise to a morphism $f : T \to \mathfrak{M}(\tau)$, and E is a G-torsor on $f^*\mathcal{C}(\tau)$.

Let (C, E) be such a pair. Denote by E_v, for $v \in V_\tau$, the G-bundle induced by E on C_v. We call

$$\deg_v(E) = \deg(E_v) = \deg(E_v \times_{G,Ad} \mathfrak{g})$$

the *degree* of E at the vertex $v \in V_\tau$. The degree thus defines a $\mathbb{Z}_{\geq 0}$-structure on τ, which is locally constant on T. (See [2], Definition 1.6, for $\mathbb{Z}_{\geq 0}$-structures.)

In this way, we get for every $\mathbb{Z}_{\geq 0}$-structure α on τ an open and closed substack $\mathfrak{H}^1_\alpha(\tau, G) \subset \mathfrak{H}^1(\tau, G)$, the substack of G-torsors of degree α.

Proposition 11 *For every $\mathbb{Z}_{\geq 0}$-structure α on τ the stack $\mathfrak{H}^1_\alpha(\tau, G)$ is an algebraic k-stack. The canonical morphism*

$$\mathfrak{H}^1_\alpha(\tau, G) \longrightarrow \mathfrak{M}(\tau)$$

is smooth of relative dimension

$$-\chi(\tau) \dim G - \alpha(\tau),$$

where $\alpha(\tau) = \sum_{v \in V_\tau} \alpha(v)$.

Proof. To prove that $\mathfrak{H}^1(\tau, G)$ is algebraic, choose a suitable embedding $G \hookrightarrow GL_n$ to reduce the case of G-bundles to the case of vector bundles, for which it is well-known. The smoothness of $\mathfrak{H}^1(\tau, G)$ follows from the fact that $H^2(C, E \times_{G,Ad} \mathfrak{g}) = 0$ for any G-torsor E on a τ-marked prestable curve C. The dimension of $\mathfrak{H}^1(\tau, G)$ is equal to

$$\begin{aligned} -\chi(E \times_{G,Ad} \mathfrak{g}) &= -\deg(E \times_{G,Ad} \mathfrak{g}) - \chi(\mathcal{O}_C) \operatorname{rk}(E \times_{G,Ad} \mathfrak{g}) \\ &= -\alpha(\tau) - \chi(\tau) \dim G \end{aligned}$$

by Riemann-Roch. □

Maps to G/P

Now let G be a reductive algebraic group over k and P a parabolic subgroup of G. Then G/P is a smooth projective variety over k. Let us assume for simplicity that G is split over k. The morphism $G \to G/P$ is a principal P-bundle, which we shall denote by F.

Let $U_1, \ldots, U_r$ be the elementary representations of P over k, $V_1, \ldots, V_r$ the corresponding vector bundles on G/P and $L_1, \ldots, L_r$ their determinants. For every $i = 1, \ldots, r$ we have

$$V_i = F \times_P U_i.$$

Note that $\operatorname{Pic}(G/P) \otimes \mathbb{Q}$ is spanned by $L_1, \ldots, L_r$ and that $L_1^{-1} \otimes \ldots \times L_r^{-1}$ is ample.

Let $H_2(G/P)^+$ be the set of homomorphisms of abelian groups $\psi : \operatorname{Pic}(G/P) \to \mathbb{Z}$, which are non-negative on ample line bundles. Then we get a canonical injection

$$\begin{aligned} H_2(G/P)^+ &\longrightarrow (\mathbb{Z}_{\geq 0})^r \\ \psi &\longmapsto (\psi(L_1^{-1}), \ldots, \psi(L_r^{-1})). \end{aligned}$$

Using this injection we shall think of classes in $H_2(G/P)^+$ as r-tuples of non-negative integers.

Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of G and P, respectively. We will consider these only as adjoint representations, ignoring the Lie algebra structure. Denote by $\mathfrak{p}$ also the induced vector bundle

$$F \times_{P,Ad} \mathfrak{p}$$

on G/P. Evaluating on the inverse of its determinant defines a morphism

$$\begin{aligned} \deg : H_2(G/P)^+ &\longrightarrow \mathbb{Z}_{\geq 0} \\ \psi &\longmapsto \psi(\det(\mathfrak{p})^{-1}). \end{aligned}$$

This morphism has the property that $\deg(\psi) = 0$ implies $\psi = 0$.

Remark We have $\det \mathfrak{p} \cong \omega_{G/P}$. In particular, $\deg \psi = -\psi(\omega_{G/P})$.

Now fix an $H_2(G/P)^+$-graph (τ, β), with underlying modular graph τ. Let $(\tilde{\tau}, \widetilde{\beta})$ be the $H_2(G/P)^+$-graph obtained by contracting all edges of τ.

Consider the algebraic k-stacks $\mathfrak{H}^1(\tau, G)$ and $\mathfrak{H}^1(\tau, P)$. Since G is reductive, any G-torsor on a curve has degree zero, and thus

$$\mathfrak{H}^1(\tau, G) \longrightarrow \mathfrak{M}(\tau)$$

is smooth of relative dimension

$$-\chi(\tau)\dim G.$$

If E is a P-torsor, then associated to $U_1, \ldots, U_r$ we have associated vector bundles $E_i = E \times_P U_i$, for $i = 1, \ldots, r$, and thus we may associate to E the *multi-degree*

$$\text{mult-deg}(E) = (-\deg(E_1), \ldots, -\deg(E_r)).$$

Let $\mathfrak{H}^1_\beta(\tau, P)$ be the open and closed substack of $\mathfrak{H}^1(\tau, P)$ of P-torsors whose multi-degree is equal to β.

Let $\alpha = \deg \beta$ be the $\mathbb{Z}_{\geq 0}$-structure on τ associated to β. Then we have

$$\mathfrak{H}^1_\beta(\tau, P) \subset \mathfrak{H}^1_{-\alpha}(\tau, P),$$

so that by Proposition 11 the stack $\mathfrak{H}^1_\beta(\tau, P)$ is smooth of relative dimension

$$-\chi(\tau)\dim P - \beta(\tau)(\omega_{G/P})$$

over $\mathfrak{M}(\tau)$.

Now let $\mathfrak{M}(G/P, \tau, \beta)$ be the stack of maps from τ-marked prestable curves to G/P of class β. More precisely, for a k-scheme T, the objects of $\mathfrak{M}(G/P, \tau, \beta)(T)$ are triples (C, x, f), where (C, x) is a τ-marked prestable curve over T and $f = (f_v)_{v \in V_\tau}$ is a family of k-morphisms $f_v : C_v \to G/P$ such that

1. for all $i \in F_\tau$ we have $f_{\partial(i)}(x_i) = f_{\partial(j_\tau(i))}(x_{j_\tau(i)})$,
2. for all $v \in V_\tau$ we have $f_{v*}[C_v] = \beta(v)$.

Remark If (τ, β) is stable, then $\overline{M}(G/P, \tau, \beta)$ is an open substack of $\mathfrak{M}(G/P, \tau, \beta)$.

Note that $G^{V_{\widetilde{\tau}}}$ acts on $\mathfrak{M}(G/P, \tau, \beta)$ as follows. An element $(g_w)_{w \in V_{\widetilde{\tau}}}$ of $G^{V_{\widetilde{\tau}}}$ takes $(C, x, (f_v)_{v \in V_\tau})$ to $(C, x, (g_{\phi(v)} \circ f_v)_{v \in V_\tau})$, where $\phi : \tau \to \widetilde{\tau}$ is the structure contraction. Let

$$\mathfrak{M}(G/P, \tau, \beta)/G^{V_{\widetilde{\tau}}}$$

be the stack-theoretic quotient of this action. This is an abuse of notation, since this is a left and not a right action.

We shall let $G^{V_{\widetilde{\tau}}}$ act trivially on $\mathfrak{M}(\tau)$ and denote by

$$\mathfrak{M}(\tau)/G^{V_{\widetilde{\tau}}}$$

the quotient.

Proposition 12 *There is a natural cartesian diagram of algebraic k-stacks*

$$\begin{array}{ccc} \mathfrak{M}(G/P,\tau,\beta)/G^{V_{\tilde{\tau}}} & \xrightarrow{\kappa} & \mathfrak{H}^1_\beta(\tau,P) \\ \eta\downarrow & & \downarrow \\ \mathfrak{M}(\tau)/G^{V_{\tilde{\tau}}} & \xrightarrow{\iota} & \mathfrak{H}^1(\tau,G). \end{array}$$

The vertical maps are representable, the horizontal maps are local immersions.

Proof. This is essentially the fact that a map to G/P is the same as a principal P-bundle with a trivialization of the associated G-bundle. $\square$

The morphism ι is a local regular immersion with normal bundle $R^1\pi_*\mathcal{O}\otimes\mathfrak{g}$. Thus the normal cone $C(\tau,\beta)$ of $\mathfrak{M}(G/P,\tau,\beta)/G^{V_{\tilde{\tau}}}$ in $\mathfrak{H}^1_\beta(\tau,P)$ is a cone in

$$\mathfrak{n}(\tau,\beta)=\eta^*R^1\pi_*\mathcal{O}\otimes\mathfrak{g}.$$

Pulling back to $\mathfrak{M}(G/P,\tau,\beta)$ and, if (τ,β) is stable, to $\overline{M}(G/P,\tau,\beta)$ defines $G^{V_{\tilde{\tau}}}$-equivariant cones, which we shall still denote $C(\tau,\beta)$, inside equivariant vector bundles, which we shall still denote by $\mathfrak{n}(\tau,\beta)$.

Let us now assume that (τ,β) is stable. Then we may intersect the cone $C(\tau,\beta)$ over $\overline{M}(G/P,\tau,\beta)$ with the zero section of the vector bundle $\mathfrak{n}(\tau,\beta)$, to define a cycle class

$$J(\tau,\beta)\in A_{\dim(G/P,\tau,\beta)}(\overline{M}(G/P,\tau,\beta))$$

with rational coefficients. Note that $C(\tau,\beta)$ is pure of the correct dimension, since it is constructed as a normal cone inside a smooth stack of the correct dimension.

Proposition 13 *The collection of cycle classes $J(\tau,\beta)$ is the orientation of $\overline{M}$ over $\mathfrak{G}_s(G/P)$ defined using the intrinsic normal cone.*

Proof. This follows from [1] Example 7.6, since

$$(R\pi_*f^*T_{G/P})^\vee=\kappa^*L^\bullet_{\mathfrak{H}^1_\beta(\tau,P)/\mathfrak{H}^1(\tau,G)}.$$

$\square$

Remark As a corollary we get that the orientation classes $J(\tau,\beta)$ are $G^{V_{\tilde{\tau}}}$-invariant. The same is then true for the Gromov-Witten invariants.

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behrend@math.ubc.ca