# Differential Graded Schemes II:
## The 2-category of differential graded schemes

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**Abstract**

We construct a 2-category of differential graded schemes. The local affine models in this theory are differential graded algebras, which are graded commutative with unit over a field of characteristic zero, are concentrated in non-positive degrees and have perfect cotangent complex. Quasi-isomorphic differential graded algebras give rise to 2-isomorphic differential graded schemes and a differential graded algebra can be recovered up to quasi-isomorphism from the differential graded scheme it defines. Differential graded schemes can be glued with respect to an étale topology and fibered products of differential graded schemes correspond on the algebra level to derived tensor products.

# Contents

# Introduction

The goal of this paper is to define a useful notion of differential graded scheme. This is done with the following criteria in mind:

(i) Differential graded schemes can be glued from local data. Quasi-isomorphisms are considered to be isomorphisms for the purposes of gluing.

(ii) Every differential graded scheme locally determines a differential graded algebra up to quasi-isomorphism, the local affine coordinate ring.

(iii) Fibered products of differential graded schemes exist and are given locally on the level of affine coordinate rings as derived tensor products.

(iv) Differential graded schemes form a category. Thus moduli spaces such as the ones constructed in [5] solve universal mapping problems in this category and so their differential graded structure is determined entirely by a universal mapping property.

It turns out that these requirements cannot be met by a usual category. Some higher categorical structure is needed. Our central message is that the simplest of all higher categorical structures, namely that of *2-category*, is sufficient for a satisfying theory.

This is somewhat of a surprise, because differential graded algebras form a *simplicial category*, which is a version of (weak) infinity category. Passing to 2-categories is achieved by a process of *truncation*, which leads, by its nature, to loss of information. The fact that the lost information was not necessary for the purposes of geometry is rather subtle. It is the content of the results we present under the heading of *Descent Theory*.

### Overview of the construction

We start with a suitable 2-category of differential graded algebras. This is the 2-category of *perfect resolving algebras*, which we denote by $\mathfrak{R}_{\mathrm{pf}}$. A perfect resolving algebra is a differential graded algebra concentrated in non-positive degrees (the differential has degree $+1$), such that the underlying graded algebra is free (commutative with 1, over a field $k$ of characteristic 0) on finitely many generators in each degree, and such that its complex of differentials is *perfect*. For a detailed study of perfect resolving algebras, see [1].

The perfect resolving algebras form a full subcategory of the differential graded algebras, which form a simplicial closed model category. Thus, given any two perfect resolving algebras $B$, $A$, the set of morphisms from $B$ to $A$ is a simplicial set $\mathrm{Hom}^{\Delta}(B, A)$. Because perfect resolving algebras are both fibrant and cofibrant, the simplicial set $\mathrm{Hom}^{\Delta}(B, A)$ is fibrant, i.e., it has the Kan extension property. Thus the fundamental groupoid of $\mathrm{Hom}^{\Delta}(B, A)$ exists and we define

$$\mathbf{Hom}(B, A) = \Pi_1 \mathrm{Hom}^{\Delta}(B, A).$$

With this definition of hom-groupoid, the perfect resolving algebras form the 2-category $\mathfrak{R}_{\mathrm{pf}}$. Let us note that two perfect resolving algebras are isomorphic in $\mathfrak{R}_{\mathrm{pf}}$ (in the 'relaxed', 2-categorical sense), if and only if they are quasi-

isomorphic. This is because quasi-isomorphisms are the *weak equivalences* in the closed model category of differential graded algebras.

The 2-category $\mathfrak{R}_{\mathrm{pf}}$ serves as the category of affine coordinate rings of affine differential graded schemes. To construct differential graded schemes over $\mathfrak{R}_{\mathrm{pf}}$, we imitate the usual construction of algebraic spaces over the category of $k$-algebras (of finite type, to keep the analogy with $\mathfrak{R}_{\mathrm{pf}}$).

Thus, the first step is to pass to the opposite category of $\mathfrak{R}_{\mathrm{pf}}$, which we denote $\mathfrak{S}$. Then we introduce a Grothendieck topology on the 2-category $\mathfrak{S}$. The usual étale topology on the category of affine $k$-schemes of finite type has an analogue in $\mathfrak{S}$, called, not surprisingly, the *étale* topology on $\mathfrak{S}$. The necessary facts about étale morphisms between perfect resolving algebras are proved in [1].

As soon as we have a 2-category with a Grothendieck topology, we have the category of *sheaves* over it. We call a sheaf over $\mathfrak{S}$ a *differential graded sheaf*. *Differential graded schemes* are then defined to be differential graded sheaves satisfying an extra condition (see below).

Since we are working over a 2-category, the notion of sheaf resembles more closely the usual concept of *stack*, rather than the usual concept of sheaf. In fact, a sheaf over $\mathfrak{S}$ is defined to be a *2-category fibered in groupoids* over $\mathfrak{S}$. It is required to satisfy sheaf axioms, which are direct adaptations of the usual stack axioms. Thus, on a certain formal level, our theory of differential graded schemes resembles the usual theory of algebraic stacks.

There is one essential difference: there is no 1-category which generates the 2-category of differential graded schemes in the same way that the 1-category of usual schemes generates the 2-category of algebraic stacks. The local affine models for differential graded schemes already form a 2-category, in contrast to the local affine models for algebraic stacks, which form only a 1-category.

A key ingredient in the construction of usual algebraic spaces is *descent theory*. By this we mean two results: descent for morphisms and descent for algebras. Descent for morphisms says that the contravariant functor

$$\mathrm{Spec}(R) : \text{(finite type affine } k\text{-schemes)} \longrightarrow \text{(sets)} \tag{1}$$

represented by the finite type $k$-algebra $R$, is a sheaf. Thus we obtain a contravariant functor

$$\mathrm{Spec} : \text{(finite type } k\text{-algebras)} \longrightarrow \big(\text{sheaves on (finite type affine } k\text{-schemes)}\big).$$

By Yoneda's lemma, it is fully faithful. Without descent theory, we would have to pass from $\mathrm{Spec}\,R$ to the associated sheaf, which would destroy the fully faithful property of Spec. As a consequence, we would not be able to reconstruct the finite type $k$-algebra $R$ from the sheaf (and hence the algebraic space) associated to it.

Thus, in view of our requirement (ii) on differential graded schemes, a result on descent for morphisms in $\mathfrak{S}$ is essential. It says that the 2-category fibered in groupoids over $\mathfrak{S}$

$$\mathbb{S}\mathrm{pec}(B)$$

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represented by the perfect resolving algebra $B$ is a sheaf. From the 'lax functor' point of view, this 2-category fibered in groupoids $\mathbb{S}\mathrm{pec}(B)$ may be considered as a contravariant 2-functor

$$\mathbb{S}\mathrm{pec}(B) : \mathfrak{S} \longrightarrow (\mathrm{groupoids}) ,$$

making the analogy with (1) more apparent.

We obtain a contravariant 2-functor

$$\mathbb{S}\mathrm{pec} : \mathfrak{R}_{\mathrm{pf}} \longrightarrow (\mathrm{sheaves\ on\ } \mathfrak{S}) .$$

By Yoneda's lemma for 2-categories, it is fully faithful. Again, the key point is that there is no need to pass to an associated sheaf, and so there is no information loss when passing from a perfect resolving algebra to the sheaf on $\mathfrak{S}$ it gives rise to. This means that our requirement (ii), above, is fulfilled.

Usual descent for algebras can be formulated as follows. Call a morphism of sheaves $f : X \to Y$ on the category of finite type $k$-algebras *affine*, if for every morphism $\mathrm{Spec}\, R \to Y$, the fibered product $X \times_Y \mathrm{Spec}\, R$ is isomorphic to $\mathrm{Spec}\, S$, for some finite type $k$-algebra $S$. Descent for algebras says that for $f : X \to Y$ to be affine it suffices to have an étale cover $\mathrm{Spec}\, R_i \to Y$ of $Y$, such that for every $i$, the fibered product $X \times_Y \mathrm{Spec}\, R_i$ is isomorphic to $\mathrm{Spec}\, S_i$, for some finite type $k$-algebra $S_i$. We abbreviate this property by saying that 'affine' is a *local* property for morphisms between sheaves. When developing the theory of algebraic spaces from the theory of affine schemes, this fact is essential. Without it, it would be impossible to ever check that any given morphism is affine. By extension, it would be impossible to ever prove that a given sheaf is an algebraic space.

Thus we prove an analogue of descent for algebras in the 2-category $\mathfrak{S}$. Once this is done, we define differential graded schemes in three steps:

• An *affine* differential graded scheme is a differential graded sheaf, 2-isomorphic to $\mathbb{S}\mathrm{pec}\, B$, for some perfect resolving algebra $B$.

• An *affine morphism* of differential graded sheaves is a morphism, whose base change to an affine differential graded scheme always gives rise to an affine differential graded scheme. For affine morphisms, the property of being *étale* makes sense.

• A *differential graded scheme* is a differential graded sheaf $\mathfrak{X}$, which can be covered by affine étale morphisms $\mathbb{S}\mathrm{pec}\, B_i \to \mathfrak{X}$. Thus, a differential graded scheme is étale locally affine.

By the local nature of this definition, it is clear that it satisfies our criterion (i), above. A more detailed study of the gluing properties of differential graded schemes is the content of [2]. There we will prove, for example, that every local complete intersection scheme can be considered as a differential graded scheme. Requirement (iv) is also clearly fulfilled: morphisms between differential graded schemes are just morphism of differential graded sheaves. Because of the truncation procedure involved in our construction, Property (iii) is somewhat non-trivial. It will be dealt with in [3]. For the present purposes is sufficient to have base changes by étale morphisms (see Proposition 1.39).

Recall (Theorem 3.8 in [1]), that every perfect resolving algebra is locally finite. Thus every differential graded scheme can be glued using only finite resolving algebras. In other words, the 2-category of affine differential graded schemes associated to finite resolving algebras generates the 2-category of differential graded schemes. Thus we could base our theory entirely on finite resolving algebras instead of perfect resolving algebras. On the other hand, the descent result for algebras fails in the context of finite resolving algebras. Hence finite resolving algebras give rise differential graded schemes which are somewhat too local, to be considered as the class of *all* affine differential graded schemes.

Because of these observations, one might speculate that it should be possible to develop the theory of differential graded schemes without descent for algebras. On the other hand, the development is greatly simplified by its use.

## Outline of the paper

In Section 1, we start by reviewing a few basic facts about 2-categories. Then we define presheaves over 2-categories. We introduce the notion of Grothendieck topology on a 2-category and define sheaves on a 2-category endowed with a Grothendieck topology.

We proceed to introduce the 2-category of resolving algebras $\mathfrak{R}$, together with its subcategories of *quasi-finite*, *perfect* and *finite* resolving algebras, $\mathfrak{R}_{qf}$, $\mathfrak{R}_{pf}$ and $\mathfrak{R}_f$. We obtain a base 2-category $\mathfrak{S}$ by passing to the opposite 2-category of any of $\mathfrak{R}_{qf}$, $\mathfrak{R}_{pf}$ or $\mathfrak{R}_f$. We prove that base changes by étale morphisms exist in $\mathfrak{S}$.

Finally, we introduce the étale topology on $\mathfrak{S}$ and define differential graded sheaves as sheaves on $\mathfrak{S}$.

Section 2 contains our results on descent theory. There is, first of all, a theorem on descent for morphisms: Theorem 2.1 and its Corollary 2.3. It holds for both finite and perfect resolving algebras. This result is really the technical heart of the whole theory, because it justifies using 2-categories for differential graded schemes. Without it, one would have to consider some type of infinity category (as is done in [10]). To prove our descent theorem, we use the main result of [1], on 'linearization of homotopy groups'. It says that for every $\ell > 0$ there exists a canonical bijection

$$\Xi_\ell : h^{-\ell} \underline{\mathrm{Der}}(B, A) \longrightarrow \pi_\ell \mathrm{Hom}^\Delta(B, A) \,,$$

where $\underline{\mathrm{Der}}(B, A)$ is the differential graded $A$-module of (internal) derivations $D : B \to A$. Thus $\pi_\ell \mathrm{Hom}^\Delta(B, A)$ can be given the structure of $h^0(A)$-module, which suffices to reduce descent for morphisms in $\mathfrak{S}$ to usual descent for morphisms in the category of finite type $k$-algebras.

To formulate our theorem on descent for algebras, we need to introduce a special class of *gluing data* in $\mathfrak{S}$. We leave the general theory of gluing data in $\mathfrak{S}$ to [2]. Here we only require *relative* gluing data, which have a much simpler structure. These gluing data can be conveniently pictured as diagrams in the shape of truncated hypercubes in $\mathfrak{S}$. We prove that relative gluing data can be

*strictified*, which means that 2-arrows appearing in the squares of the hypercube can be replaced by identity 2-arrows.

Once these preliminaries are dispensed with, we proceed to prove our theorem on descent for algebras, Theorem 2.11. As mentioned above, it is the basis for the theory of affine morphisms of differential graded sheaves and is therefore also an integral part of the definition of differential graded scheme.

Section 3 contains this definition, as outlined above. At this point, we refrain from going much further than the bare definition of differential graded scheme.

Instead, in Section 4, we proceed to construct the basic '1-categorical invariants' of a differential graded scheme $\mathfrak{X}$. All of these are sheaves (or complexes of sheaves) on the 1-category associated to the 2-category underlying $\mathfrak{X}$. An object of this 1-category may be thought of as an isomorphism class of morphisms

$$\mathbb{S}\mathrm{pec}\, A \to \mathfrak{X}\,.$$

There are first of all the *higher structure sheaves*. These associate to $\mathbb{S}\mathrm{pec}\, A \to \mathfrak{X}$ the $h^0(A)$-module $h^i(A)$, for $i \leq 0$.

Then there are the *higher tangent sheaves* $h^\ell(\Theta_\mathfrak{X})$. If $\mathfrak{X} = \mathbb{S}\mathrm{pec}\, B$ is affine, they associate to $\mathbb{S}\mathrm{pec}\, A \to \mathbb{S}\mathrm{pec}\, B$ the $h^0(A)$-module $h^\ell \underline{\mathrm{Der}}(B, A)$, for various $\ell$.

Next there are the *homotopy sheaves* $\pi_\ell(\mathfrak{X})$, for $\ell > 0$. Again, let us just say here that in the affine case $\mathfrak{X} = \mathbb{S}\mathrm{pec}\, B$, they are given by associating to $\mathbb{S}\mathrm{pec}\, A \to \mathbb{S}\mathrm{pec}\, B$ the group $\pi_\ell \mathrm{Hom}^\Delta(B, A)$. These are, in fact, sheaves, by our results on descent theory. Moreover, they coincide with certain of the higher tangent sheaves: there exists a canonical isomorphism of sheaves

$$\Xi_\ell : h^{-\ell}(\Theta_\mathfrak{X}) \xrightarrow{\sim} \pi_\ell(\mathfrak{X})\,,$$

for all $\ell > 0$.

To put the homotopy sheaves $\pi_\ell(\mathfrak{X})$ into context, let us make a few general remarks. Let $\mathcal{C}$ be a simplicial closed model category with homotopy category $Ho(\mathcal{C})$, say in its incarnation as category of fibrant-cofibrant objects with simplicial homotopy classes of maps as morphisms. For every $\ell \geq 0$ and every morphism $f : X \to Y$ in $Ho(\mathcal{C})$ we let $\pi_\ell(X/Y)$ be the presheaf on $Ho(\mathcal{C})/X$ defined by $\pi_\ell(X/Y)(U) = \pi_\ell \mathrm{Hom}_Y^\Delta(U, X)$, for all $U \to X$ in $Ho(\mathcal{C})/X$. Here $\mathrm{Hom}_Y^\Delta(U, X)$ denotes the fiber of $\mathrm{Hom}^\Delta(U, X) \to \mathrm{Hom}^\Delta(U, Y)$. For $Y = *$, we obtain the presheaf $\pi_\ell(X)$. There is a long exact sequence of presheaves of pointed sets on $Ho(X)$

$$\ldots \longrightarrow \pi_\ell(X/Y) \longrightarrow \pi_\ell(X) \longrightarrow f^{-1}\pi_\ell(Y) \xrightarrow{\partial} \pi_{\ell-1}(X/Y) \longrightarrow \ldots$$
$$\ldots \longrightarrow f^{-1}\pi_1(Y) \xrightarrow{\partial} \pi_0(X/Y) \longrightarrow \pi_0(X) \longrightarrow \pi_0(Y)\,.$$

Formally, our homotopy sheaves $\pi_\ell(\mathfrak{X})$, of which there also exist relative versions $\pi_\ell(\mathfrak{X}/\mathfrak{Y})$, behave somewhat as if they were obtained, as above, from a simplicial closed model category structure underlying the 2-category of differential graded schemes. In particular, they also fit into a long exact sequence.

We do not know if there exists such a simplicial closed model category structure underlying the 2-category of simplicial schemes, but we find it quite likely.

There is a special feature in our case of differential graded schemes: the analogue of the homotopy category $Ho(\mathcal{C})/X$ is the 1-category associated to the differential graded scheme $\mathfrak{X}$. Thus in our case this homotopy category is endowed with a Grothendieck topology, with respect to which all the $\pi_\ell(X)$, for $\ell > 0$, are sheaves. This property does not seem to have a meaningful analogue, for example, in the simplicial closed model categories of topological spaces or simplicial sets.

Finally, we define the cotangent complex of a morphism of differential graded schemes and the algebraic space associated to a differential graded scheme. The associated algebraic space is obtained by gluing the truncations $\operatorname{Spec} h^0(B_i)$ of the affine differential graded schemes $\mathbb{S}\mathrm{pec}\, B_i$, which cover a differential graded scheme $\mathfrak{X}$. The cotangent complex of a differential graded scheme gives rise to an *obstruction theory* in the sense of [4] on the associated algebraic space. Thus in the case of perfect amplitude 1, it defines a *virtual fundamental class* on the associated algebraic space.

### Notation

References of the form I.0.0, refer to Result 0.0 of [1].

### Acknowledgements

look up the exact name and date of the conference

# 1 The étale topology

## 1.1 2-Categories

All our 2-categories will have invertible 2-morphisms. Thus they are categories enriched over groupoids. To fix notation, let us recall the definition. (See also [9]).

**Definition 1.1** A **2-category** $\mathfrak{S}$ consists of

    (i) a set of objects ob $\mathfrak{S}$,

    (ii) for every pair $U$, $V$ of objects of $\mathfrak{S}$ a groupoid $\mathbf{Hom}(U, V)$,

    (iii) for every triple $U$, $V$, $W$ of objects of $\mathfrak{S}$ a functor

$$\circ : \mathbf{Hom}(V, W) \times \mathbf{Hom}(U, V) \longrightarrow \mathbf{Hom}(U, W) \qquad (2)$$
$$(f, g) \longmapsto f \circ g$$

    (iv) for every object $U$ of $\mathfrak{S}$ an object $\mathrm{id}_U$ of $\mathbf{Hom}(U, U)$, or rather a functor

$$* \xrightarrow{\mathrm{id}_U} \mathbf{Hom}(U, U),$$

    such that

    (i) the composition $\circ$ is associative, i.e., for four objects $U$, $V$, $W$, $Z$ of $\mathfrak{S}$ the diagram of functors

$$
\begin{array}{ccc}
\mathbf{Hom}(W, Z) \times \mathbf{Hom}(W, V) \times \mathbf{Hom}(V, U) & \longrightarrow & \mathbf{Hom}(V, Z) \times \mathbf{Hom}(U, V) \\
\downarrow & & \downarrow \\
\mathbf{Hom}(W, Z) \times \mathbf{Hom}(U, W) & \longrightarrow & \mathbf{Hom}(U, Z)
\end{array}
$$

commutes (strictly),

    (ii) $\mathrm{id}_U$ acts as identity, i.e., the induced diagrams

$$
\begin{array}{cc}
\mathbf{Hom}(U, V) & \quad \mathbf{Hom}(U, V) \rightarrow \mathbf{Hom}(V, V) \times \mathbf{Hom}(U, V) \\
\downarrow \quad\searrow & \qquad\qquad\qquad \searrow \quad\downarrow \\
\mathbf{Hom}(U, V) \times \mathbf{Hom}(U, U) \rightarrow \mathbf{Hom}(U, V) & \qquad\qquad \mathbf{Hom}(U, V)
\end{array}
$$

commute (strictly), for any two objects $U$, $V$.

    Composition in $\mathbf{Hom}(U, V)$ is called **vertical** composition, the operation $\circ$ of (2) is called **horizontal** composition. Vertical composition we shall denote by $\alpha \cdot \beta$.

    Objects of $\mathbf{Hom}(U, V)$ are called **1-morphisms** (of $\mathfrak{S}$), morphisms in $\mathbf{Hom}(U, V)$ are called **2-morphisms** (of $\mathfrak{S}$). We also use the words *2-isomorphism* and *2-arrow* instead of 2-morphism. An identity 2-arrow is also called a *strictly commutative diagram.*

Every set $X$ is a category by taking the elements of $X$ as objects and admitting only identity morphisms. Every category $\mathcal{C}$ is a 2-category, by considering the Hom-set $\mathrm{Hom}(A, B)$ as a category, for any two objects $A$, $B$ of $\mathcal{C}$.

Given a 2-category $\mathfrak{S}$, the objects of $\mathfrak{S}$ together with the 1-morphisms and horizontal composition form a 1-category, the **underlying** 1-category of $\mathfrak{S}$. Replacing $\mathbf{Hom}(A, B)$ by its set of isomorphism classes, we obtain another 1-category, the 1-category **associated** to $\mathfrak{S}$, which we denote by $\overline{\mathfrak{S}}$. There is a canonical functor from the underlying 1-category of $\mathfrak{S}$ to $\overline{\mathfrak{S}}$.

Compatibilities between 2-morphisms can often be phrased conveniently by saying that certain '2-spheres' commute. This means that the objects involved should be considered as vertices, 1-morphisms as edges and 2-morphisms as faces of a 'triangulation' of a topological 2-sphere. There should always be one 'source object', having only 1-morphisms emanating from it, and one 'target object', which has no 1-morphism emanating from it. Then all the different directed paths from the source to the target object can be considered as vertices of a commutative polygon of 2-arrows, i.e., a commutative polygon for the vertical composition. Often we project such a 2-sphere stereographically onto the plane, so that we get a flat diagram, whose exterior should be considered as a 2-cell, even if it is not labelled as such.

If the distinction is important, we say that such a 2-sphere *2-commutes*. This is contrast to a 2-sphere all of whose faces are identity 2-arrows (in other words, are commutative diagrams in the underlying 1-category), which we call *strictly* commutative. If all faces are strictly commutative, a 2-sphere is automatically 2-commutative.

### Isomorphisms and fibered products in 2-categories

**Definition 1.2** For lack of better terminology, we will call a morphism of groupoids $f : X \to Y$ **categorically étale**, if for every object $x$ of $X$ the induced group homomorphism $\mathrm{Aut}_X(x) \to \mathrm{Aut}_Y\big(f(x)\big)$ is bijective.

Let $\mathfrak{S}$ be a 2-category.

**Definition 1.3** We call a 1-morphism $A \to B$ in $\mathfrak{S}$ **faithful** (**categorically étale**, a **monomorphism**), if for every object $S$ the induced morphism of groupoids $\mathbf{Hom}(S, A) \to \mathbf{Hom}(S, B)$ is faithful (categorically étale, fully faithful).

Thus we have the implications:

$$\text{monomorphism} \Longrightarrow \text{categorically étale} \Longrightarrow \text{faithful}\,.$$

**Definition 1.4** Let $f : A \to B$ be a 1-morphism in a 2-category $\mathfrak{S}$. A **2-inverse** of $f$ is given by the data $(g, \phi, \psi)$, where $g : B \to A$ is a 1-morphism and $\phi : \mathrm{id}_A \Rightarrow g \circ f$ and $\psi : f \circ g \Rightarrow \mathrm{id}_B$ are 2-arrows, such that the two diagrams
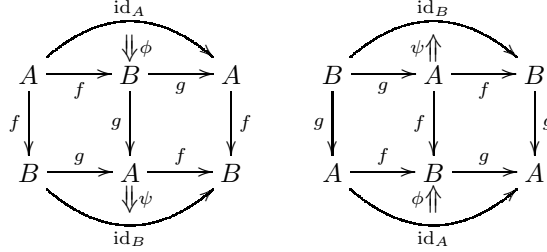
commute.

The inverse of $f : A \to B$ is unique up to a unique 2-isomorphism in the following sense. If $(g', \phi', \psi')$ is another inverse to $f$, then there exists a unique 2-isomorphism $\theta : g \to g'$, such that the two diagrams
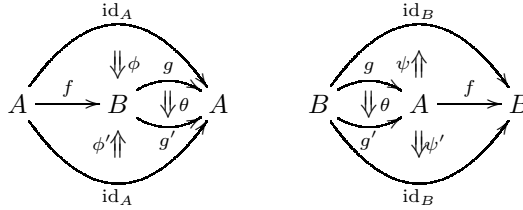
commute.

**Definition 1.5** We call a 1-morphism $f : A \to B$ in a 2-category **2-invertible**, an **equivalence** or even an **isomorphism**, if it admits a 2-inverse. We hope that there will be no confusion with the term *2-isomorphism*, which stands for the 2-arrows in $\mathfrak{S}$.

By the same token, we call two objects $A$ and $B$ **isomorphic**, if there exists a 2-invertible morphism $f : A \to B$.

The following is a very useful criterion by which to recognize equivalences:

**Proposition 1.6** *Let $f : A \to B$ be a 1-morphism in $\mathfrak{S}$. If there exists a 1-morphism $g : B \to A$ such that $\mathrm{id}_A \cong g \circ f$ and $\mathrm{id}_B \cong f \circ g$, then $f$ is an equivalence.* $\square$

**Corollary 1.7** *A 1-morphism $A \to B$ in $\mathfrak{S}$ is an isomorphism, if and only if $\mathbf{Hom}(S, A) \to \mathbf{Hom}(S, B)$ is an equivalence of groupoids, for all objects $S$.*

**Definition 1.8** A diagram

$$
\begin{array}{ccc}
W & \longrightarrow & B \\
\downarrow & \nearrow & \downarrow \\
A & \longrightarrow & Z
\end{array}
\tag{3}
$$

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in a 2-category $\mathfrak{S}$ is called a **fibered product**, if for any object $S$ of $\mathfrak{S}$ the functor

$$\mathbf{Hom}(S,W) \longrightarrow \mathbf{Hom}(S,A) \times_{\mathbf{Hom}(S,Z)} \mathbf{Hom}(S,B) \qquad (4)$$

is an equivalence of groupoids. Here, the fibered product in (4) is the usual fibered product of groupoids.

We also say that Diagram (3) is **2-cartesian**.

A *1-cartesian* diagram in the 2-category $\mathfrak{S}$ is a strictly commutative square, which is cartesian in the underlying 1-category.

### 2-Functors and natural 2-transformations

**Definition 1.9** Let $\mathfrak{S}$ and $\mathfrak{T}$ be 2-categories. A **2-functor** $f : \mathfrak{S} \to \mathfrak{T}$ consists of

(i) a map $f : \mathrm{ob}\,\mathfrak{S} \to \mathrm{ob}\,\mathfrak{T}$,

(ii) for every pair $U$, $V$ of objects of $\mathfrak{S}$ a functor

$$f : \mathbf{Hom}(U,V) \longrightarrow \mathbf{Hom}(f(U),f(V)),$$

such that

(iii) for every object $U$ of $\mathfrak{S}$ the diagram of functors



commutes (strictly),

(iv) for every triple $U$, $V$, $W$ of objects of $\mathfrak{S}$



is a (strictly) commutative diagram of functors.

**Definition 1.10** Consider a 2-functor $f : \mathfrak{S} \to \mathfrak{T}$.

(i) $f$ is called **fully faithful**, if for any two objects $U$, $V$ of $\mathfrak{S}$ the functor $\mathbf{Hom}(U,V) \to \mathbf{Hom}\big(f(U),f(V)\big)$ is an equivalence of groupoids.

(ii) $f$ is called an **equivalence of 2-categories**, if it is fully faithful, and for every object $T$ of $\mathfrak{T}$, there exists an object $S$ of $\mathfrak{S}$, such that $f(S) \cong T$.

**Definition 1.11** Let $f : \mathfrak{S} \to \mathfrak{T}$ and $g : \mathfrak{S} \to \mathfrak{T}$ be two 2-functors between the 2-categories $\mathfrak{S}$ and $\mathfrak{T}$. A **natural 2-transformation** $\theta : f \to g$ is given by a functor

$$* \xrightarrow{\theta(U)} \mathbf{Hom}(f(U),g(U)),$$

for every object $U$ of $\mathfrak{S}$, such that

$$\mathbf{Hom}(U,V) \xrightarrow{\theta(V)\times f} \mathbf{Hom}(f(V),g(V)) \times \mathbf{Hom}(f(U),f(V))$$

with vertical maps $g\times\theta(U)$ and $\circ$, and bottom row

$$\mathbf{Hom}(g(U),g(V)) \times \mathbf{Hom}(f(U),g(U)) \xrightarrow{\ \circ\ } \mathbf{Hom}(f(U),g(V))$$

is a commutative diagram of functors, for any two objects $U$, $V$ in $\mathfrak{S}$.

**Remark** 2-categories, 2-functors and natural 2-transformations form a 2-category.

**Definition 1.12** Let $\mathfrak{S}$ be a 2-category. We define the **opposite** 2-category $\mathfrak{S}^{\mathrm{op}}$ as follows:
    (i) objects of $\mathfrak{S}^{\mathrm{op}}$ are the same as the objects of $\mathfrak{S}$,
    (ii) given objects $U$ and $V$ of $\mathfrak{S}^{\mathrm{op}}$, we set

$$\mathbf{Hom}_{\mathfrak{S}^{\mathrm{op}}}(U,V) = \mathbf{Hom}_{\mathfrak{S}}(V,U)^{\mathrm{op}},$$

    (iii) horizontal composition in $\mathfrak{S}^{\mathrm{op}}$ is defined to be the 'opposite' of horizontal composition in $\mathfrak{S}$, up to changing the order of the arguments,
    (iv) identity objects in $\mathfrak{S}^{\mathrm{op}}$ are the same as in $\mathfrak{S}$.
    A **contravariant** 2-functor $\mathfrak{S} \to \mathfrak{T}$ is a 2-functor $\mathfrak{S}^{\mathrm{op}} \to \mathfrak{T}$.

Of course, the opposite 2-category is again a 2-category.

## 1.2 Presheaves

We shall now introduce the notion of a *presheaf* over a 2-category. This generalizes the notion of *category fibered in groupoids*, known from the theory of 1-categories (and used in the theory of stacks). If the base category is a 2-category, it is natural to think of this structure as generalizing the notion of presheaf over a 1-category.

**Definition 1.13** A 2-functor $\pi : \mathfrak{F} \to \mathfrak{S}$ of 2-categories is called a **presheaf**, if
    (i) for every 1-morphism

$$V \longrightarrow U$$

in $\mathfrak{S}$, and every object $x$ of $\mathfrak{F}$ lying over $U$, there exists a 1-morphism

$$y \longrightarrow x$$

in $\mathfrak{F}$, lying over $V \to U$,
    (ii) for every 2-commutative triangle

$$\begin{array}{ccc} V' & \longrightarrow & V \\ & {}_{f'}\searrow \ \nearrow & \downarrow f \\ & & U \end{array} \qquad (5)$$

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in $\mathfrak{S}$, and every diagram of 1-morphisms

$$
\begin{array}{ccc}
y' & & y \\
& \searrow^{e'} & \downarrow^{e} \\
& & x
\end{array}
$$

in $\mathfrak{F}$, where $e'$ lies over $f'$ and $e$ lies over $f$, there exists a unique pair $(r, \gamma)$ in $\mathfrak{F}$ such that the triangle

$$
\begin{array}{ccc}
y' & \xrightarrow{\ r\ } & y \\
& {}_{\gamma}\nearrow & \\
& \searrow^{e'} & \downarrow^{e} \\
& & x
\end{array}
\tag{6}
$$

2-commutes in $\mathfrak{F}$ and the triangle (6) lies over the triangle (5).

The following special case of Condition (ii) is worth pointing out:

**Lemma 1.14** *Let $\pi : \mathfrak{F} \to \mathfrak{S}$ be a presheaf. Then for every 2-morphism*

$$
V \underset{f}{\overset{f'}{\rightrightarrows}} {}_{\Downarrow \alpha}\, U
$$

*in $\mathfrak{S}$, whose target $f : V \to U$ has been lifted to a 1-morphism*

$$
y \underset{e}{\longrightarrow} x
$$

*in $\mathfrak{F}$, there exists a unique 2-morphism*

$$
y \underset{e}{\overset{e'}{\rightrightarrows}} {}_{\Downarrow \gamma}\, x
$$

*in $\mathfrak{F}$ with target $e$, which lies over $\alpha$.*

PROOF. Apply the case that $y = x$ and $e$ the identity in Condition (ii) of Definition 1.13 to the inverse of $\alpha$. $\square$

Note that the fibers of a presheaf are groupoids. We denote the fiber of $\mathfrak{F} \to \mathfrak{S}$ over the object $U$ of $\mathfrak{S}$ by $\mathfrak{F}_U$.

**Definition 1.15** If $\pi : \mathfrak{F} \to \mathfrak{S}$ is a presheaf, and the lifting required in Condition (i) of Definition 1.13 is always *unique*, then we call $\mathfrak{F}$ a **presheaf of sets**. Note that this makes Condition (ii) empty. Moreover, it implies that all fibers of $\pi$ are sets.

**Definition 1.16** Let $\mathfrak{F}$ and $\mathfrak{G}$ be presheaves over $\mathfrak{S}$. A **morphism** of presheaves is a 2-functor $\mathfrak{F} \to \mathfrak{G}$ commuting with the projections to $\mathfrak{S}$. A **2-morphism** of morphisms of presheaves is a natural 2-transformations lying over the identity natural 2-transformation.

The presheaf morphisms from $\mathfrak{F}$ to $\mathfrak{G}$ form a groupoid $\mathbf{Hom}_{\mathfrak{S}}(\mathfrak{F}, \mathfrak{G})$, the *groupoid of presheaf morphisms* from $\mathfrak{F}$ to $\mathfrak{G}$.

There is an obvious way to define a composition functor

$$\mathbf{Hom}_{\mathfrak{S}}(\mathfrak{G}, \mathfrak{H}) \times \mathbf{Hom}_{\mathfrak{S}}(\mathfrak{F}, \mathfrak{G}) \longrightarrow \mathbf{Hom}_{\mathfrak{S}}(\mathfrak{F}, \mathfrak{H}),$$

thus turning the presheaves over $\mathfrak{S}$ into a 2-category.

Since the fibers of a presheaf over $\mathfrak{S}$ are groupoids, presheaves behave more like categories than like 2-categories:

**Lemma 1.17** *Let $\mathfrak{F}$ and $\mathfrak{G}$ be presheaves over $\mathfrak{S}$. A morphism $F : \mathfrak{F} \to \mathfrak{G}$ is a 1-functor of the underlying 1-categories of $\mathfrak{F}$ and $\mathfrak{G}$, commuting with the underlying 1-functors to the underlying 1-category $\mathfrak{S}$, with the following property: if*

$$y \overset{f}{\underset{g}{\Rightarrow\theta}} x$$

*is a 2-morphism in $\mathfrak{F}$, lying over*

$$V \Rightarrow U, \tag{7}$$

*then there exists a 2-morphism*

$$F(y) \overset{F(f)}{\underset{F(g)}{\Rightarrow\eta}} F(x)$$

*lying over (7), also.*

Let $F, G : \mathfrak{F} \to \mathfrak{G}$ be 1-morphisms of presheaves over $\mathfrak{S}$. A 2-isomorphism $\theta : F \Rightarrow G$ is a natural 1-transformation between the underlying 1-functors $F$ and $G$, which maps every object $x$ of $\mathfrak{F}$ to a morphism of $\mathfrak{G}$ lying over $\mathrm{id}_U$ (where $x$ lies over $U$). The compatibility with 2-isomorphisms is then automatic. $\square$

**Proposition 1.18** *Let $F : \mathfrak{F} \to \mathfrak{G}$ be a morphism of presheaves over $\mathfrak{S}$. The following are equivalent:*

*(i) $F$ is faithful (categorically étale, a monomorphism, an isomorphism) as a 1-morphism in the 2-category of presheaves over $\mathfrak{S}$,*

*(ii) for every object $U$ of $\mathfrak{S}$, the fiber functor $F_U : \mathfrak{F}_U \to \mathfrak{G}_U$ is faithful (categorically étale, fully faithful, an equivalence).* $\square$

**Relative 2-categories, Yoneda Theory**

Given an object $U$ of a 2-category $\mathfrak{S}$, we define the *relative 2-category* $\mathfrak{S}/U$ as follows.

Objects of $\mathfrak{S}/U$ are 1-morphisms $V \to U$ in $\mathfrak{S}$ with target $U$. Given two such relative objects $f' : V' \to U$ and $f : V \to U$, we define the groupoid of morphisms $\mathbf{Hom}_{\mathfrak{S}/U}(V', V)$ to have as objects the 2-commutative diagrams

$$
\begin{array}{ccc}
V' & \xrightarrow{\ e\ } & V \\
& {\scriptstyle \alpha}\nearrow & \\
{\scriptstyle f'}\searrow & & \downarrow {\scriptstyle f} \\
& U, &
\end{array}
$$

and to have as 2-morphisms from $(e, \alpha)$ to $(g, \beta)$ the 2-morphisms in $\mathfrak{S}$

$$
V' \underset{g}{\overset{e}{\rightrightarrows}}\; {\Downarrow \theta}\; V,
$$

such that $(f \circ \theta) \cdot \alpha = \beta$, i.e., the diagram of 2-arrows in $\mathfrak{S}$

$$
\begin{array}{ccc}
f' & \xRightarrow{\ \alpha\ } & f \circ e \\
& {\scriptstyle \beta}\searrow & \Downarrow {\scriptstyle f \circ \theta} \\
& & f \circ g
\end{array}
$$

commutes.

Composition in $\mathbf{Hom}_{\mathfrak{S}/U}(V', V)$ is induced from vertical composition in $\mathfrak{S}$. We define horizontal composition $\mathfrak{S}/U$ by the formula

$$
(e, \alpha) \circ (g, \beta) = \big(e \circ g, (\alpha \circ g) \cdot \beta\big).
$$

By projecting onto the source, more precisely, mapping $V \to U$ to $V$, $(e, \alpha)$ to $e$ and $\theta$ to $\theta$, we get a 2-functor $\mathfrak{S}/U \to \mathfrak{S}$.

**Lemma 1.19** *The 2-functor $\mathfrak{S}/U \to \mathfrak{S}$ is a presheaf.* $\square$

Abbreviate for an object $U$ of $\mathfrak{S}$ the relative 2-category $\mathfrak{S}/U$ by $\underline{U}$.

The association $U \to \underline{U}$ defines a 2-functor from $\mathfrak{S}$ to the 2-category of presheaves over $\mathfrak{S}$. The analogue of Yoneda's lemma in this context is that this 2-functor is fully faithful:

**Proposition 1.20 (Yoneda's lemma for 2-categories)** *The 2-functor*

$$
\begin{aligned}
\mathfrak{S} &\longrightarrow (\text{presheaves}/\mathfrak{S}) \\
U &\longmapsto \underline{U}
\end{aligned}
$$

*is fully faithful.* $\square$

**Remark** A 1-morphism $f : U \rightarrow V$ in $\mathfrak{S}$ is faithful (categorically étale, a monomorphism, an isomorphism) if and only if the induced 1-morphism $\underline{U} \rightarrow \underline{V}$ of presheaves over $\mathfrak{S}$ is faithful (categorically étale, a monomorphism, and isomorphism).

**Remark** For every presheaf $\mathfrak{F}$ over $\mathfrak{S}$ and every object $U$ of $\mathfrak{S}$, there is a canonical morphism of groupoids

$$\mathbf{Hom}(\underline{U}, \mathfrak{F}) \longrightarrow \mathfrak{F}_U \,, \tag{8}$$

given by evaluation at $\mathrm{id}_U$. It is always an equivalence of groupoids. Given an object $x$ of $\mathfrak{F}$ lying over $U$, any choice of pullbacks for $x$ defines a morphism $\underline{U} \rightarrow \mathfrak{F}$ mapping to $x$ under (8).

Thus, it is justified to write $\mathfrak{F}(U)$ instead of $\mathfrak{F}_U$.

### Fibered products of presheaves

Let $\mathfrak{S}$ be a 2-category.

**Proposition 1.21** *Fibered products exist in the 2-category of presheaves over $\mathfrak{S}$. The Yoneda functor $\mathfrak{S} \rightarrow (\text{presheaves}/\mathfrak{S})$ commutes with any fibered products which exist in $\mathfrak{S}$.*

PROOF. The construction is analogous to the proof of the fact that fibered products exist in the 2-category of categories fibered in groupoids over a 1-category. $\square$

## 1.3 Topologies and sheaves

### Sieves

**Definition 1.22** Let $\mathfrak{S}$ be a 2-category and $U$ an object of $\mathfrak{S}$. A **sieve** for $U$ is given by a collection $R$ of objects $V \rightarrow U$ of $\underline{U}$ such that if $V \rightarrow U$ is in $R$ and

$$\begin{array}{ccc} W & \longrightarrow & V \\ & \searrow \nearrow & \downarrow \\ & & U \end{array}$$

is a 1-morphism in $\underline{U}$, then $W \rightarrow U$ is in $R$, also.

A sieve $R$ for $U$ defines a sub-2-category of $\underline{U}$ by

$$\mathbf{Hom}_R(W, V) = \mathbf{Hom}_{\underline{U}}(W, V),$$

whenever $V$ and $W$ are objects in $R$. We shall always identify a sieve $R$ for $U$ with this sub-2-category of $\underline{U}$ it generates.

Thus, given a sieve $R$ for $\underline{U}$, we have the canonical inclusion 2-functor $R \rightarrow \underline{U}$ and by composing with $\underline{U} \rightarrow \mathfrak{S}$ a canonical 2-functor $R \rightarrow \mathfrak{S}$.

**Proposition 1.23** *If $R$ is a sieve for $\underline{U}$, then $R \to \mathfrak{S}$ is a presheaf. Conversely, a sub-2-category $R$ of $\underline{U}$, such that $R \to \mathfrak{S}$ is a presheaf comes from a unique sieve for $U$.* $\square$

Note that a morphism $V \to U$ is an object of the sieve $R \subset \underline{U}$ if and only if the induced morphism of $\mathfrak{S}$-presheaves $\underline{V} \to \underline{U}$ factors through $R \subset \underline{U}$.

More precisely, if $V \to U$ partakes in the sieve $R \subset \underline{U}$, then there exists a unique strictly commutative diagram

$$\begin{array}{ccc} \underline{V} & \longrightarrow & R \\ & \searrow & \downarrow \\ & & \underline{U} \end{array}$$

of presheaves over $\mathfrak{S}$. If there exists a 2-commutative diagram

$$\begin{array}{ccc} \underline{V} & \longrightarrow & R \\ & \searrow \nearrow & \downarrow \\ & & \underline{U} \end{array}$$

of presheaves over $\mathfrak{S}$, then $V \to U$ partakes in $R$.

**Lemma 1.24** *Finite intersections and arbitrary unions of sieves for $U$ are sieves for $U$.* $\square$

**Construction 1.25 (pullback sieve)** Consider a morphism $f : V \to U$ in $\mathfrak{S}$. If $R$ is a sieve for $U$, define $f^{-1}R \subset \underline{V}$ to consist of all $W \to V$ such that the composition $W \to V \to U$ is in $R$. Of course $f^{-1}R$ is a sieve for $V$. Note that

$$\begin{array}{ccc} f^{-1}R & \lhook\joinrel\longrightarrow & \underline{V} \\ \downarrow & & \downarrow \\ R & \lhook\joinrel\longrightarrow & \underline{U} \end{array}$$

is a 2-cartesian diagram of presheaves over $\mathfrak{S}$.

### Topologies

We shall now define topologies on 2-categories. A topology is characterized by its collection of *covering sieves*.

**Definition 1.26** Let $\mathfrak{S}$ be a 2-category. A **topology** on $\mathfrak{S}$ is given by the data
  • for every object $U$ of $\mathfrak{S}$ a collection of sieves for $U$, called the *covering sieves* of $U$,
subject to the constraints:

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(i) (pullbacks) for all morphisms $f : V \to U$ in $\mathfrak{S}$ and all covering sieves $R \subset \underline{U}$ of $U$, the pullback $f^{-1}R \subset \underline{V}$ is a covering sieve for $V$,

(ii) (local nature) if $R \subset \underline{U}$ is a covering sieve for $U$ and $R' \subset \underline{U}$ is another sieve, which *covers $R$-locally*, the sieve $R'$ is also a covering sieve of $U$. Here we say that $R'$ covers $R$-locally, if for all $f : V \to U$ in $R$ the pullback $f^{-1}R'$ is a covering sieve of $V$,

(iii) (identities) for every object $U$ of $\mathfrak{S}$ the sieve $\underline{U}$ is a covering sieve for $U$.
A 2-category which has been endowed with a topology is called a **2-site**.

Sometimes, a topology can be defined in terms of a pretopology. A pretopology is given by its collection of *covering families*.

**Definition 1.27** Let $\mathfrak{S}$ be a 2-category. A **pretopology** on $\mathfrak{S}$ is given by the data

• for every object $U$ of $\mathfrak{S}$ a collection of families of $U$-objects, called the *covering families* of $U$,

subject to the constraints:

(i) (pullbacks) if $(U_i \to U)_{i \in I}$ is a covering family of $U$ and $V \to U$ is a morphism, then there exists a 2-pullback $V_i \to V$ of $U_i \to U$, for all $i \in I$, such that $(V_i \to V)_{i \in I}$ is a covering family of $V$;

(ii) (composition) if $(U_i \to U)_{i \in I}$ is a covering family of $U$ and for every $i \in I$ we have a covering family $(V_{ij} \to U_i)_{j \in J_i}$ of $U_i$, then the total family $(V_{ij} \to U)_{i \in I, j \in J_i}$, obtained by composition, is a covering family of $U$;

(iii) (identity) the one-member family (id : $U \to U$) is a covering family of $U$, for all objects $U$ of $\mathfrak{S}$.

**Remark**  Note that in the pullback condition we do not require that *every* pullback of a covering family is a covering family. We only demand that there exists at least one pullback family which is covering.

In particular, a family isomorphic to a covering family need not be a covering family.

We can associate a topology to a given pretopology as follows. Call a sieve $R \subset \underline{U}$ *covering* if there exists a covering family (for the given pretopology) $(U_i \to U)$ such that for all $i$ we have that $U_i$ is in $R$.

**Lemma 1.28** *This defines a topology on $\mathfrak{S}$. This topology is called the topology* **associated** *to the given pretopology.*

PROOF. This proof is very similar to the corresponding proof in the 1-category setting. $\square$

**Sheaves**

**Definition 1.29** A presheaf $\mathfrak{F} \to \mathfrak{S}$ is called a **sheaf** if, for every object $U$ of $\mathfrak{S}$ and every covering sieve $R \subset \underline{U}$, the canonical restriction functor

$$\mathbf{Hom}_{\mathfrak{S}}(\underline{U}, \mathfrak{F}) \longrightarrow \mathbf{Hom}_{\mathfrak{S}}(R, \mathfrak{F})$$

is an equivalence of groupoids.

Suppose that $(U_i \to U)$ is a covering family for a pretopology. We denote by $U_{i_0 \ldots i_p}$ the fibered product $U_{i_0} \times_U \ldots \times_U U_{i_p}$. For a presheaf $\mathfrak{F}$, and an object $x$ of $\mathfrak{F}_U$, we write $x \,|\, U_{i_0 \ldots i_p}$ for a chosen pullback of $x$. Given a morphism $\alpha : x \to y$ in $\mathfrak{F}(U)$, it induces a unique morphism $\alpha \,|\, U_{i_0 \ldots i_p} : x \,|\, U_{i_0 \ldots i_p} \to y \,|\, U_{i_0 \ldots i_p}$.

**Lemma 1.30** *Assume that the topology on $\mathfrak{S}$ is defined by a pretopology. Then the presheaf $\mathfrak{F} \to \mathfrak{S}$ is a sheaf if and only if for every object $U$ of $\mathfrak{S}$ and every covering family $\mathcal{U} = (U_i \to U)$ the following three conditions are satisfied:*

*(i) Assume given two objects $x, y \in \mathfrak{F}(U)$ and two morphisms $\alpha, \beta : x \to y$. If $\alpha|U_i = \beta|U_i$, for all $i$, then $\alpha = \beta$.*

*(ii) Assume given two objects $x, y \in \mathfrak{F}(U)$ and for every $i$ a morphism $\alpha_i : x|U_i \to y|U_i$, such that $\alpha_i|U_{ij} = \alpha_j|U_{ij}$, for all $i, j$. Then there exists a morphism $\alpha : x \to y$ such that $\alpha|U_i = \alpha_i$, for all $i$.*

*(iii) Given, for every $i$, an object $x_i$ of $\mathfrak{F}(U_i)$ and for all $i, j$ a morphism $\alpha_{ij} : x_i \,|\, U_{ij} \to x_j \,|\, U_{ij}$ in $\mathfrak{F}(U_{ij})$, such that for all $i, j, k$ we have $\alpha_{jk} \,|\, U_{ijk} \circ \alpha_{ij} \,|\, U_{ijk} = \alpha_{ik} \,|\, U_{ijk}$, then there exists an object $x$ of $\mathfrak{F}(U)$, and morphisms $\alpha_i : x \,|\, U_i \to x_i$, such that $\alpha_{ij} \circ \alpha_i \,|\, U_{ij} = \alpha_j \,|\, U_{ij}$, for all $i, j$.* $\square$

**Definition 1.31** Given two sheaves $\mathfrak{F}$, $\mathfrak{G}$ over $\mathfrak{S}$, the groupoid of sheaf morphisms from $\mathfrak{F}$ to $\mathfrak{G}$ is defined to be the groupoid $\mathbf{Hom}_{\mathfrak{S}}(\mathfrak{F}, \mathfrak{G})$ of presheaf morphisms from $\mathfrak{F}$ to $\mathfrak{G}$.

Thus the sheaves over $\mathfrak{S}$ are a 2-category. There is the canonical fully faithful inclusion functor

$$i : (\text{sheaves}/\mathfrak{S}) \longrightarrow (\text{presheaves}/\mathfrak{S}) \,.$$

**Proposition 1.32** *Fibered products of sheaves exist and $i$ commutes with them. A morphism of sheaves is faithful (categorically étale, a monomorphism, an isomorphism) if and only if it is faithful (categorically étale, a monomorphism, an isomorphism) considered as a morphism of presheaves (cf. Proposition 1.18).* $\square$

**Definition 1.33** A family $\mathfrak{F}_i \to \mathfrak{F}$ of morphisms of sheaves is called **epimorphic**, if for every object $U$ of $\mathfrak{S}$ and every morphism $\underline{U} \to \mathfrak{F}$, the sieve for $U$, consisting of all $V \to U$ admitting a 2-commutative diagram



for some $i$, is a covering sieve.

**Remark** If the topology is given by a pretopology, then $\mathfrak{F}_i \to \mathfrak{F}$ is an epimorphic family of sheaf morphisms, if and only if for every $\underline{U} \to \mathfrak{F}$, there exists a

covering family $V_j \to U$ for the pretopology, such that for every $j$ there exists an $i$ and a 2-commutative diagram

$$
\begin{array}{ccc}
\underline{V_j} & \longrightarrow & \underline{U} \\
\downarrow & \nearrow & \downarrow \\
\mathfrak{F}_i & \longrightarrow & \mathfrak{F}
\end{array}
$$

## 1.4 From simplicial categories to 2-categories

The 2-category we use to construct differential graded schemes comes from a simplicial category of differential graded algebras. See [1] for our conventions concerning simplicial categories and differential graded algebras.

We may consider every groupoid as a simplicial set, by passing to the simplicial nerve. Every groupoid becomes a Kan (i.e. fibrant) simplicial set in this way. Thus we may consider every 2-category as a simplicial category.

Recall (see, for example, Page 36 of [6]) that to a Kan simplicial set $X$ we can associate the fundamental groupoid $\Pi_1 X$ as follows. Objects of $\Pi_1 X$ are the vertices $\Delta^0 \to X$. Morphisms in $\Pi_1 X$ are homotopy classes (relative $\partial \Delta^1$) of 'paths' $\Delta^1 \to X$. Composition is defined by using the Kan property: Any two composable paths give rise to a horn. Filling this horn with a 2-simplex yields the composition as the third edge.

**Proposition 1.34** *Let $\mathfrak{S}$ be a simplicial category all of whose hom-spaces are fibrant. If $U$ and $V$ are objects of $\mathfrak{S}$, define*

$$\mathbf{Hom}(U,V) = \Pi_1 \operatorname{Hom}^\Delta(U,V).$$

*This definition endows $\mathfrak{S}$ with the structure of a 2-category $\widetilde{\mathfrak{S}}$ in such a way that $\mathfrak{S} \to \widetilde{\mathfrak{S}}$ is a simplicial functor.* $\square$

Note that the condition that $\mathfrak{S} \to \widetilde{\mathfrak{S}}$ is a simplicial functor determines that structure of 2-category on $\widetilde{\mathfrak{S}}$ uniquely. It is called the 2-category **associated** to the simplicial category $\mathfrak{S}$.

**Example 1.35** Let $\mathfrak{S}$ be a simplicial closed model category and $\mathfrak{U} \subset \mathfrak{S}$ a full subcateogry all of whose objects are fibrant and cofibrant. Then all hom-spaces in $\mathfrak{U}$ are fibrant and so there is an associated 2-category $\widetilde{\mathfrak{U}}$. This is how our 2-categories arise.

For the following result concerning the compatability of homotopy fibered products with fibered products in the associated 2-category, we need an additional property of the simplicial closed model category $\mathfrak{S}$. We say that $\mathfrak{S}$ *admits finite tensors*, if for every object $X$ of $\mathfrak{S}$ and every finite simplicial set $K$, there exists an object $K \otimes X$ in $\mathfrak{S}$, and a simplicial map $K \to \operatorname{Hom}^\Delta(X, K \otimes X)$, such that for every object $Y$ of $\mathfrak{S}$, the induced map

$$\operatorname{Hom}(K \otimes X, Y) \longrightarrow \operatorname{Hom}\big(K, \operatorname{Hom}^\Delta(X,Y)\big)$$

is bijective.

**Lemma 1.36** *Let $\mathfrak{S}$ be a simplicial clsoed model category admitting finite tensors and $\mathfrak{U}$ a full subcategory of fibrant-cofibrant objects. Let $V \to U$ be a fibration in $\mathfrak{U}$, and $U' \to U$ an arbitrary morphism in $\mathfrak{U}$. Consider the (strict) fibered product in $\mathfrak{S}$*

$$\begin{array}{ccc} V' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ V & \longrightarrow & U. \end{array} \qquad (9)$$

*Assume that $V'$ is in $\mathfrak{U}$. Moreover, let at least one of the two conditions*

*(i) for all $Z \in \mathfrak{U}$, we have that $\pi_2 \operatorname{Hom}^\Delta(Z, U) = 0$,*

*(ii) for all $Z \in \mathfrak{U}$, we have that $\pi_1 \operatorname{Hom}^\Delta(Z, V') \to \pi_1 \operatorname{Hom}^\Delta(Z, U')$ is injective,*

*be satisfied. Then (9) is a fibered product in the associated 2-category $\widetilde{\mathfrak{U}}$.*

PROOF. Because $\mathfrak{S}$ admits finite tensors, the diagram

$$\begin{array}{ccc} \operatorname{Hom}^\Delta(Z, V') & \longrightarrow & \operatorname{Hom}^\Delta(Z, U') \\ \downarrow & & \downarrow \\ \operatorname{Hom}^\Delta(Z, V) & \longrightarrow & \operatorname{Hom}^\Delta(Z, U) \end{array}$$

is a cartesian diagram of simplicial sets, for every object $Z$ of $\mathfrak{U}$. Moreover, since $V \to U$ is a fibration, by the simplicial model category axiom, $\operatorname{Hom}^\Delta(Z, V) \to \operatorname{Hom}^\Delta(Z, U)$ is a fibration of (fibrant) simplicial sets. Using these two facts, it is easy to prove that under either of the two assumptions (i) or (ii), the induced diagram

$$\begin{array}{ccc} \Pi_1 \operatorname{Hom}^\Delta(Z, V') & \longrightarrow & \Pi_1 \operatorname{Hom}^\Delta(Z, U') \\ \downarrow & & \downarrow \\ \Pi_1 \operatorname{Hom}^\Delta(Z, V) & \longrightarrow & \Pi_1 \operatorname{Hom}^\Delta(Z, U) \end{array}$$

is a cartesian diagram of groupoids. $\square$

### Resolving algebras

Recall (see [1]), that a *differential graded algebra* is always graded commutative with unit, over a field $k$ of characteristic zero. A differential graded algebra is a *resolving algebra* if is free as a graded commutative algebra with unit, on generators in non-positive degrees. If finitely many generators in each degree suffice, we call a resolving algebra *quasi-finite*, if in total finitely many generators suffice we speak of a *finite* resolving algebra. A quasi-finite resolving algebra with perfect complex of differentials is called a *perfect* resolving algebra (see Definition I.3.1).

We proved (see Corollary I.1.18), that the resolving algebras are fibrant-cofibrant objects in the simplicial closed model category $\mathfrak{A}$ of all differential graded algebras. Thus, by Example 1.35, the category of all resolving algebras admits an associated 2-category, as all hom-spaces between such algebras are fibrant.

**Definition 1.37** This 2-category of resolving algebras is called $\mathfrak{R}$.

The full sub-2-categories of resolving algebras which are quasi-finite, perfect or finite are denoted by $\mathfrak{R}_{\mathrm{qf}}$, $\mathfrak{R}_{\mathrm{pf}}$ and $\mathfrak{R}_{\mathrm{f}}$, respectively.

**Remark** By the results of Section 1.4 in [1], a morphism of resolving algebras $A \to B$ is a quasi-isomorphism if and only if it is 2-invertible in $\mathfrak{R}$. If two morphisms $f, g : A \to B$ are 2-isomorphic, they induce identical homomorphisms $h^*(A) \to h^*(B)$ on cohomology.

### Fibered products

Since, for the purposes of doing geometry, we will pass to the opposite category $\mathfrak{R}^{\mathrm{op}}$ of $\mathfrak{R}$, we will state our results here in terms of $\mathfrak{R}^{\mathrm{op}}$. Note that by Remark I.1.15, the opposite of $\mathfrak{A}$ admits finite tensors.

**Proposition 1.38** *Absolute products exist in the opposite categories of* $\mathfrak{R}$, $\mathfrak{R}_{\mathrm{qf}}$, $\mathfrak{R}_{\mathrm{pf}}$ *and* $\mathfrak{R}_{\mathrm{f}}$. *The inclusions* $\mathfrak{R}_{\mathrm{f}} \subset \mathfrak{R}_{\mathrm{pf}} \subset \mathfrak{R}_{\mathrm{qf}} \subset \mathfrak{R}$ *commute with them.*

PROOF. Use tensor products over $k$. Note that $\mathrm{Hom}^\Delta(k, A) = *$, for all differential graded algebras $A$, so that $\pi_2 \mathrm{Hom}^\Delta(k, A) = 0$, and we can apply Lemma 1.36. $\square$

Recall the definitions of *étale* morphism and *standard étale* morphism of quasi-finite resolving algebras, Definitions I.2.8 and I.2.16.

**Proposition 1.39** *Let* $A \to B$ *be an étale morphism of finite (perfect, quasi-finite) resolving algebras. Let* $A \to A'$ *be an arbitrary morphism of finite (perfect, quasi-finite) resolving algebras.*

$$
\begin{array}{ccc}
& & A' \\
& & \uparrow \\
B & \xleftarrow{\text{étale}} & A
\end{array}
$$

*The induced fibered product in* $\mathfrak{R}_{\mathrm{f}}^{\mathrm{op}}$ *(*$\mathfrak{R}_{\mathrm{pf}}^{\mathrm{op}}$*,* $\mathfrak{R}_{\mathrm{qf}}^{\mathrm{op}}$*) exists. If* $B'$ *is this fibered product, then* $A' \to B'$ *is again étale.*

*Moreover, if* $A \to B$ *is standard étale, then we may choose* $A' \to B'$ *to be standard étale, too.*

PROOF. By the results of Section I.5, we can choose a finite (quasi-finite) resolution of $A \to B$. Thus, we may assume without loss of generality that $A \to B$ is itself a finite (quasi-finite) resolving morphism. We let $B' = B \otimes_A A'$, which represents a strict fibered product in the opposite of the category of all resolving algebras. Moreover, $A' \to B'$ is again a finite (quasi-finite) resolving morphism, and so $B'$ is a finite (perfect, quasi-finite) resolving algebra. Clearly, $A' \to B'$ is again étale. By Proposition I.4.18, Lemma 1.36 applies, and so $B'$ provides us with a fibered product in $\mathfrak{R}_{\mathrm{f}}^{\mathrm{op}}$ ($\mathfrak{R}_{\mathrm{pf}}^{\mathrm{op}}$, $\mathfrak{R}_{\mathrm{qf}}^{\mathrm{op}}$). □

## 1.5 The étale topology

We need a base category over which to do geometry. There are various choices, all leading to the same notion of differential graded scheme. This base category will be the opposite category of a suitable category of resolving algebras, somewhere between $\mathfrak{R}_{\mathrm{f}}$ and $\mathfrak{R}_{\mathrm{qf}}$.

**Definition 1.40** Let $\mathfrak{S}$ be a full sub-2-category of $\mathfrak{R}^{\mathrm{op}}$ satisfying
(i) every object of $\mathfrak{S}$ is quasi-finite,
(ii) if $A \to B$ is a finite resolving morphism in $\mathfrak{R}$ and $A$ belongs to $\mathfrak{S}$, then so does $B$. The ground field $k$ belongs to $\mathfrak{S}$.

In particular, all finite resolving algebras are contained in $\mathfrak{S}$.

**Example 1.41** We could let $\mathfrak{S}$ consist of any of the following:
(i) all quasi-finite resolving algebras,
(ii) all perfect resolving algebras,
(iii) all finite resolving algebras.
If we use one of these categories for $\mathfrak{S}$, then we write $\mathfrak{S}_{\mathrm{qf}}$, $\mathfrak{S}_{\mathrm{pf}}$ or $\mathfrak{S}_{\mathrm{f}}$, respectively.

We call a morphism $V \to U$ in $\mathfrak{S}$ **étale** or **standard étale**, respectively, if the corresponding morphism of quasi-finite resolving algebras is étale or standard étale.

We define a functor

$$\mathfrak{S} \longrightarrow (\text{finite type } k\text{-schemes})$$
$$U \longmapsto h^0(U) \, ,$$

by associating to a differential graded algebra $A$ the spectrum of $h^0(A)$. This functor maps étale morphisms to étale morphisms.

**Definition 1.42** The **étale topology** on $\mathfrak{S}$ is defined by calling, for an object $U$ of $\mathfrak{S}$, a sieve $R \subset \underline{U}$ *covering*, if there exists a family of étale morphisms $U_i \to U$ in $R$ such that $\coprod_i h^0(U_i) \to h^0(U)$ is a surjective morphism of schemes.

We will show that this notion of covering sieve defines a topology on $\mathfrak{S}$ by proving that there exists a pretopology on $\mathfrak{S}$, whose associated topology is given by Definition 1.42.

**Definition 1.43** The **étale pretopology** on $\mathfrak{S}$ is defined by calling a family $(U_i \to U)$ a *covering family* if

    (i) every $U_i \to U$ is standard étale,

    (ii) $\coprod_i h^0(U_i) \to h^0(U)$ is a surjective morphism of schemes.

**Proposition 1.44** *Definition 1.43 defines a pretopology on $\mathfrak{S}$.*

PROOF. We need to check the three properties of Definition 1.27.

    (i) (pullbacks) Let $(U_i \to U)$ be a covering family for the étale pretopology. Thus every $U_i \to U$ is standard étale. By Proposition 1.39, the base change $V_i \to V$ exists in $\mathfrak{S}$ and may be chosen to be standard étale, again. Note that $h^0$ commutes with pullback. Hence $(V_i \to V)$ is a covering family for the étale pretopology.

    (ii) (composition) This property is satisfied because a composition of standard étale morphisms is standard étale.

    (iii) the identity property is trivially verified. $\square$

**Lemma 1.45** *Let $X \to Y$ be an étale morphism in $\mathfrak{S}$. Then there exists a family of 2-commutative diagrams in $\mathfrak{S}$*

$$
\begin{array}{ccc}
X_i & & \\
\downarrow & \overset{\scriptstyle /\!\!/}{\searrow} & \\
X & \longrightarrow & Y
\end{array}
\qquad (10)
$$

*such that*

    *(i) Every $X_i \to X$ is an open immersion,*

    *(ii) $\coprod h^0(X_i) \to h^0(X)$ is onto,*

    *(iii) every $X_i \to Y$ is standard étale.*

PROOF. Translating the Main Lemma I.2.19 into the opposite category $\mathfrak{S}$, we get diagrams

$$
\begin{array}{ccc}
X_i' & \longrightarrow & X_i \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

with

    (i) every $X_i' \to X$ is an elementary open immersion,

    (ii) $\coprod h^0(X_i') \to h^0(X)$ is onto,

    (iii) every $X_i' \to X_i$ is an isomorphism in $\mathfrak{R}$,

    (iv) every $X_i \to Y$ is standard étale.

Choosing a 2-inverse for $X_i' \to X_i$ we obtain (10). $\square$

**Theorem 1.46** *A sieve $R \subset \underline{U}$ is a covering sieve for the topology induced by the étale pretopology if and only if it satisfies the condition of Definition 1.42.*

PROOF. If $R \subset \underline{U}$ is a covering sieve for the associated topology it satisfies Definition 1.42 trivially. Let us prove the converse. Thus assume that $R \subset \underline{U}$ is a sieve and that $(U_i \to U)_{i \in I}$ is a family of étale morphisms in $R$ such that $\coprod_i h^0(U_i) \to h^0(U)$ is surjective. We have to show that $R$ contains a covering family for the étale pretopology.

For given $i \in I$, choose a family of 2-commutative diagrams

$$
\begin{array}{ccc}
V_{ij} & & \\
\downarrow & \searrow & \\
U_i & \longrightarrow & U,
\end{array}
$$

for $j \in J_i$, as in Lemma 1.45. Then the total family $(V_{ij} \to U)_{i \in I, j \in J_i}$ is in $R$ and is a covering family for the étale pretopology. $\square$

**Corollary 1.47** *Definition 1.42 defines a topology on $\mathfrak{S}$.*

**Corollary 1.48** *The étale topology is the topology associated to the étale pretopology.*

We end this section with a definition:

**Definition 1.49** A **differential graded sheaf** is a sheaf on $\mathfrak{S}$ with the étale topology.

# 2 Descent theory

## 2.1 Descent for morphisms

Let us fix a morphism of quasi-finite resolving algebras $C \to A$ and a family of étale quasi-finite resolving morphisms $A \to A_i$, such that $\coprod_i \operatorname{Spec} h^0(A_i) \to \operatorname{Spec} h^0(A)$ is surjective. We think of the family $A \to A_i$ as giving rise to a covering family $U_i \to U$ for the étale topology on $\mathfrak{S}$.

Let

$$A_{i_0 \ldots i_p} = A_{i_0} \otimes_A \ldots \otimes_A A_{i_p}$$

and denote the corresponding object of $\mathfrak{S}$ by $U_{i_0 \ldots i_p}$. For every $\lambda : \{0, \ldots, q\} \to \{0, \ldots, p\}$ we have a canonical morphism of differential graded algebras

$$A_{i_{\lambda(0)} \ldots i_{\lambda(q)}} \longrightarrow A_{i_0 \ldots i_p} \tag{11}$$

$$a_0 \otimes \ldots \otimes a_q \longmapsto \bigotimes_{\lambda(\kappa)=0} a_\kappa \otimes \ldots \otimes \bigotimes_{\lambda(\kappa)=p} a_\kappa.$$

Let $C \to B$ be quasi-finite resolving morphism and let $\sigma = (\sigma_{i_0 \ldots i_q})_{i_0, \ldots, i_q}$ be a family of $\ell$-simplices $\sigma_{i_0 \ldots i_q} \in \operatorname{Hom}_C^\Delta(B, A_{i_0 \ldots i_q})$. We denote by

$$\sigma_{i_{\lambda(0)} \ldots i_{\lambda(q)}} \,|\, U_{i_0 \ldots i_p}$$

the image of $\sigma_{i_{\lambda(0)} \ldots i_{\lambda(q)}}$ under the map

$$\operatorname{Hom}_C^\Delta(B, A_{i_{\lambda(0)} \ldots i_{\lambda(q)}}) \longrightarrow \operatorname{Hom}_C^\Delta(B, A_{i_0 \ldots i_p})$$

induced by (11).

For a composition of maps $\lambda : \{0, \ldots, q\} \to \{0, \ldots, p\}$ and $\mu : \{0, \ldots, p\} \to \{0, \ldots, r\}$ we have

$$\left( \sigma_{i_{\mu\lambda(0)} \ldots i_{\mu\lambda(q)}} \,|\, U_{i_{\mu(0)} \ldots i_{\mu(p)}} \right) \,|\, U_{i_0 \ldots i_r} = \sigma_{i_{\mu\lambda(0)} \ldots i_{\mu\lambda(q)}} \,|\, U_{i_0 \ldots i_r} \,.$$

This means that we have a cosimplicial space

$$\prod_i \operatorname{Hom}_C^\Delta(B, A_i) \rightrightarrows \prod_{i,j} \operatorname{Hom}_C^\Delta(B, A_{ij}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \ldots$$

For every $\ell \geq 0$, we also get a cosimplicial set

$$\prod_i \pi_\ell \operatorname{Hom}_C^\Delta(B, A_i) \rightrightarrows \prod_{i,j} \pi_\ell \operatorname{Hom}_C^\Delta(B, A_{ij}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \ldots, \tag{12}$$

where for $\ell \geq 1$ this assumes that we have chosen a base point $P : B \to A$.

We set

$$H^0\big(\mathfrak{U}, \pi_\ell(B/C)\big) = \ker\Big( \prod_i \pi_\ell \operatorname{Hom}_C^\Delta(B, A_i) \rightrightarrows \prod_{i,j} \pi_\ell \operatorname{Hom}_C^\Delta(B, A_{ij}) \Big).$$

For $\ell \geq 1$, having chosen a base point $P : B \to A$, we also define the pointed set

$$H^1\big(\mathfrak{U}, \pi_\ell(B/C)\big)$$

in analogy to non-abelian first Čech cohomology. More precisely, we set

$$Z^1\big(\mathfrak{U}, \pi_\ell(B/C)\big) =$$
$$\big\{(\alpha_{ij}) \in \prod_{ij} \pi_\ell \operatorname{Hom}^{\Delta}_C(B, A_{ij}) \mid \forall i, j, k : \alpha_{ik} \,|\, U_{ijk} = \alpha_{jk} \,|\, U_{ijk} * \alpha_{ij} \,|\, U_{ijk}\big\} \,.$$

Then we let
$$C^0\big(\mathfrak{U}, \pi_\ell(B/C)\big) = \prod_i \pi_\ell \operatorname{Hom}^{\Delta}_C(B, A_i)$$

act (from the left) on $Z^1$ by $(\gamma_i)_i * (\alpha_{ij})_{ij} = (\gamma_j \alpha_{ij} \gamma_i^{-1})_{ij}$ and let $H^1\big(\mathfrak{U}, \pi_\ell(B/C)\big)$ by the quotient of $Z^1$ by this action.

Finally, for $\ell \geq 2$, we associate to (12) the cochain complex obtained by setting the coboundary map equal to $\partial = \sum_i (-1)^i \partial_i$. We denote the associated cohomology groups by
$$H^i\big(\mathfrak{U}, \pi_\ell(B/C)\big) \,.$$

More notation: $V = \operatorname{Spec} h^0(A)$, $V_i = \operatorname{Spec} h^0(A_i)$ and $\mathfrak{V}$ denotes the étale covering family $V_i \to V$ of affine schemes. Let

$$V_{i_0 \ldots i_p} = V_{i_0} \times_V \ldots \times_V V_{i_p} \,.$$

Note that $h^0(U_{i_0 \ldots i_p}) = V_{i_0 \ldots i_p}$.

**Theorem 2.1 (Descent)** *Assume that $B$ is finite over $C$. Then*
  *(i) $H^0\big(\mathfrak{U}, \pi_\ell(B/C)\big) = \pi_\ell \operatorname{Hom}^{\Delta}_C(B, A)$, for every $\ell \geq 0$,*
  *(ii) $H^1\big(\mathfrak{U}, \pi_\ell(B/C)\big) = 0$, for every $\ell \geq 1$,*
  *(iii) $H^i\big(\mathfrak{U}, \pi_\ell(B/C)\big) = 0$, for all $i \geq 2$ and for every $\ell \geq 2$.*

PROOF. Induction on the number $n$ of elements in a basis for $B$ over $C$. Choose a subalgebra $B' \subset B$ such that $B'$ has a $C$-basis of $n - 1$ elements and $B$ has a $B'$-basis consisting of one element $x$ of degree $r$. By induction, we can assume the theorem to hold for $B'$.

Let us start by considering the case $\ell = 0$. Recall (Section I.4.3), that for a given morphism of $C$-algebras $P : B' \to A$ we have defined the homomorphism of $h^0(A)$-modules

$$\xi_P : h^{-1} \underline{\operatorname{Der}}_C(B', A) \longrightarrow h^r(A)$$
$$D \longmapsto D(dx) \,.$$

The cokernel $\operatorname{cok} \xi_P$ is hence a finitely generated $h^0(A)$-module. Thus we may consider $\operatorname{cok} \xi_P$ as a coherent sheaf on the affine scheme $V$.

Note that the cokernel of

$$h^{-1}\underline{\mathrm{Der}}_C(B', A_{i_0\dots i_p}) \longrightarrow h^r(A_{i_0\dots i_p})$$
$$D \longmapsto D(dx)$$

is equal to $\mathrm{cok}\,\xi_P \otimes_{h^0(A)} h^0(A_{i_0\dots i_p})$, by Corollary I.2.12.

Consider the commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{cok}\,\xi_P & \longrightarrow & \pi_0\,\mathrm{Hom}_C^\Delta(B,A) & \longrightarrow & \pi_0\,\mathrm{Hom}_C^\Delta(B',A) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^0(\mathfrak{V}, \mathrm{cok}\,\xi_P) & \longrightarrow & \prod_i \pi_0\,\mathrm{Hom}_C^\Delta(B,A_i) & \longrightarrow & \prod_i \pi_0\,\mathrm{Hom}_C^\Delta(B',A_i)
\end{array}
$$
$$(13)$$

The rows are exact, by Corollary I.4.14. The left vertical arrow is injective by usual étale descent theory for coherent modules. The right vertical arrow is injective by induction hypothesis.

To prove injectivity of $\pi_0\,\mathrm{Hom}_C^\Delta(B,A) \to \prod_i \pi_0\,\mathrm{Hom}_C^\Delta(B,A_i)$, we may choose a base point of $\pi_0\,\mathrm{Hom}_C^\Delta(B,A)$ and thus a base point $P : B' \to A$, as above. Thus we have diagram (13) at our disposal, and a simple chase around the diagram proves the required injectivity.

Now let us prove surjectivity of

$$\pi_0\,\mathrm{Hom}_C^\Delta(B,A) \longrightarrow \ker\Big(\ \prod_i \pi_0\,\mathrm{Hom}_C^\Delta(B,A_i) \rightrightarrows \prod_{i,j} \pi_0\,\mathrm{Hom}_C^\Delta(B,A_{ij})\ \Big)$$

$$(14)$$

For this we start with the diagram

$$
\begin{array}{ccccc}
\pi_0\,\mathrm{Hom}_C^\Delta(B,A) & \longrightarrow & \pi_0\,\mathrm{Hom}_C^\Delta(B',A) & \longrightarrow & h^{r+1}(A) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_i \pi_0\,\mathrm{Hom}_C^\Delta(B,A_i) & \longrightarrow & \prod_i \pi_0\,\mathrm{Hom}_C^\Delta(B',A_i) & \longrightarrow & C^0\big(\mathfrak{V}, h^{r+1}(A)\big) \\
\downdownarrows & & \downdownarrows & & \\
\prod_{ij} \pi_0\,\mathrm{Hom}_C^\Delta(B,A_{ij}) & \longrightarrow & \prod_{ij} \pi_0\,\mathrm{Hom}_C^\Delta(B',A_{ij}) & &
\end{array}
$$

$$(15)$$

whose right horizontal arrows are defined by evaluation at $dx$. The rows are exact in the middle by Proposition I.4.15. The middle column is exact by the induction hypothesis and the rightmost column is injective by usual étale descent theory applied to the coherent sheaf $h^{r+1}(A)$ on $V$.

Let $(\alpha_i) \in \prod_i \pi_0\,\mathrm{Hom}_C^\Delta(B,A_i)$, such that $\alpha_i\,|\,U_{ij} = \alpha_j\,|\,U_{ij}$, for all $i,j$. Chasing $(\alpha_i)$ around Diagram (15), we obtain $\alpha \in \pi_0\,\mathrm{Hom}_C^\Delta(B,A)$, such that $(\alpha_i)$ and $(\alpha\,|\,U_i)$ map to the same element of $\prod_i \pi_0\,\mathrm{Hom}_C^\Delta(B',A)$. We also obtain

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a base point $P : B' \to A$ (to which $\alpha$ maps), supplying us with the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{cok}\xi_P & \longrightarrow & \pi_0 \operatorname{Hom}_C^\Delta(B, A) & \longrightarrow & \pi_0 \operatorname{Hom}_C^\Delta(B', A) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^0(\mathfrak{V}, \operatorname{cok}\xi_P) & \longrightarrow & \prod_i \pi_0 \operatorname{Hom}_C^\Delta(B, A_i) & \longrightarrow & \prod_i \pi_0 \operatorname{Hom}_C^\Delta(B', A_i) \\
& & \downarrow & & \Vert & & \\
0 & \longrightarrow & C^1(\mathfrak{V}, \operatorname{cok}\xi_P) & \longrightarrow & \prod_{ij} \pi_0 \operatorname{Hom}_C^\Delta(B, A_{ij}) & &
\end{array}
$$

$$(16)$$

Again, the rows are exact by Corollary I.4.14. The leftmost column is exact by étale descent for the coherent sheaf $\operatorname{cok}\xi_P$ on $V$. Now chasing $(\alpha_i)$ and $(\alpha\,|\,U_i)$ around Diagram 16, we obtain $\beta \in \operatorname{cok}\xi_P$ such that $\beta * \alpha\,|\,U_i = \alpha_i$, for all $i$. This proves surjectivity of (14) and finishes the proof of the theorem in the case $\ell = 0$.

Let us now consider the case $\ell = 1$ and prove that $H^1\big(\mathfrak{U}, \pi_1(B/C)\big) = 0$. The advantage over the previous case is that we now have a fixed base point $P : B \to A$ for all spaces we consider. We have a commutative diagram

$$
\begin{array}{ccccc}
0 & \longrightarrow & C^0(\mathfrak{V}, \operatorname{cok}\delta) & \longrightarrow & \prod_i \pi_1 \operatorname{Hom}_C^\Delta(B, A_i) \longrightarrow \\
& & \downarrow & & \Vert \\
0 & \longrightarrow & C^1(\mathfrak{V}, \operatorname{cok}\delta) & \longrightarrow & \prod_{ij} \pi_1 \operatorname{Hom}_C^\Delta(B, A_{ij}) \longrightarrow \\
& & \downarrow & & \Vert\Vert \\
0 & \longrightarrow & C^2(\mathfrak{V}, \operatorname{cok}\delta) & \longrightarrow & \prod_{ijk} \pi_1 \operatorname{Hom}_C^\Delta(B, A_{ijk}) \longrightarrow \\
\end{array}
$$

$$
\begin{array}{ccccc}
& \pi_1 \operatorname{Hom}_C^\Delta(B', A) & \longrightarrow & \operatorname{im}\delta & \longrightarrow 0 \\
& \downarrow & & \downarrow \\
\longrightarrow & \prod_i \pi_1 \operatorname{Hom}_C^\Delta(B', A_i) & \longrightarrow & C^0(\mathfrak{V}, \operatorname{im}\delta) & \longrightarrow 0 \\
& \Vert & & \downarrow \\
\longrightarrow & \prod_{ij} \pi_1 \operatorname{Hom}_C^\Delta(B', A_{ij}) & \longrightarrow & C^1(\mathfrak{V}, \operatorname{im}\delta) & \longrightarrow 0 \\
& \Vert\Vert & & \\
\longrightarrow & \prod_{ijk} \pi_1 \operatorname{Hom}_C^\Delta(B', A_{ijk}) & &
\end{array}
$$

$$(17)$$

Here $\operatorname{cok}\delta$ is the cokernel of the boundary map

$$
\delta : h^{-2}\,\underline{\operatorname{Der}}_C(B', A) \longrightarrow h^{-1}\,\underline{\operatorname{Der}}_{B'}(B, A) = h^{r-1}(A)\,,
$$

which is given by $\delta(D) = -D(dx)$ (see Proposition I.4.10). Moreover, $\operatorname{im}\delta$ is

the image of the boundary map

$$\delta : h^{-1} \underline{\mathrm{Der}}_C(B', A) \longrightarrow h^0 \underline{\mathrm{Der}}_{B'}(B, A) = h^r(A) \,,$$

which is given by $\delta(D) = D(dx)$. Both $\mathrm{cok}\,\delta$ and $\mathrm{im}\,\delta$ are coherent sheaves on $V$ and so the first and last columns of (17) are exact. The $B'$-column is exact by induction hypothesis. The rows of (17) are exact by Lemma I.4.8 and Theorem I.4.11. We can now prove that the $B$-column is exact in the middle by a diagram chase around (17).

For all other cases of the theorem, note that

$$H^i\big(\mathfrak{U}, \pi_\ell(B/C)\big) = H^i\big(\mathfrak{V}, h^{-\ell} \underline{\mathrm{Der}}_C(B, A)\big) \,,$$

by Theorem I.4.11 and Corollary I.2.12. Thus we are reduced to usual étale descent for the coherent sheaf $h^{-\ell} \underline{\mathrm{Der}}_C(B, A)$ over $V$. $\square$

**Corollary 2.2** *The same holds if we assume only that $B$ is perfect over $C$.*

PROOF. This follows immediately by passing to the limit over the various truncations $B_{(n)}$. This is permitted, because of Corollary I.4.12 and also Equation (I.18), which features in its proof. We should remark that for $\ell = 0$, we are only claiming the left exactness of a certain sequence, which is preserved by taking limits.

For $\ell = 1$, we wish to see that the sequence

$$0 \longrightarrow \pi_1 \mathrm{Hom}_C^\Delta(B, A) \longrightarrow C^0\big(\mathfrak{U}, \pi_1(B/C)\big) \longrightarrow Z^1\big(\mathfrak{U}, \pi_1(B/C)\big) \longrightarrow 0$$

is exact, in the sense that the group in the middle acts transitively on the pointed set on the right, in such a way that the stabilizer of the distinguished point is the group on the left. This exactness follows from

check that this it true!

$$\varprojlim_{n}{}^1 \pi_1 \mathrm{Hom}_C^\Delta(B_{(n)}, A) = 0 \,,$$

which is true, by Equation (I.18). $\square$

Notation: if $x, y \in X$ are points of a fibrant simplicial set $X$ and $\alpha, \beta : x \to y$ are paths in $X$, then we write $\alpha \sim \beta$ if there exists a homotopy between $\alpha$ and $\beta$, which fixes the endpoints $x$ and $y$. In other words, $\alpha \sim \beta$ if and only if $\alpha$ and $\beta$ define the same arrow inside the fundamental groupoid $\Pi_1 X$.

**Corollary 2.3** *Let $B$ be a perfect resolving algebra.*

*(i) Given two points $x, y \in \mathrm{Hom}^\Delta(B, A)$ and two paths $\alpha, \beta : x \to y$ in $\mathrm{Hom}^\Delta(B, A)$, such that for every $i$, we have $\alpha \,|\, U_i \sim \beta \,|\, U_i$, then $\alpha \sim \beta$.*

*(ii) Given two points $x, y \in \mathrm{Hom}^\Delta(B, A)$ and for every $i$ a path $\alpha_i : x \,|\, U_i \to y \,|\, U_i$, such that $\alpha_i \,|\, U_{ij} \sim \alpha_j \,|\, U_{ij}$, for all $i, j$, there exists a path $\alpha : x \to y$ such that $\alpha \,|\, U_i \sim \alpha_i$, for all $i$.*

*(iii) Given, for every $i$, a point $x_i \in \mathrm{Hom}^\Delta(B, A_i)$, and for all $i, j$ a path $\alpha_{ij} : x_i \,|\, U_{ij} \to x_j \,|\, U_{ij}$, such that for all $i, j, k$ we have $\alpha_{jk} \,|\, U_{ijk} \circ \alpha_{ij} \,|\, U_{ijk} \sim \alpha_{ik} \,|\, U_{ijk}$, there exists a point $x \in \mathrm{Hom}^\Delta(B, A)$ and paths $\alpha_i : x \,|\, U_i \to x_i$, such that $\alpha_{ij} \circ \alpha_i \,|\, U_{ij} \sim \alpha_j \,|\, U_{ij}$, for all $i, j$.*

PROOF. This is easy to prove using Theorem 2.1. □

## 2.2 Hypercubes

We need a few definitions to make the following more efficient.

**Definition 2.4** Let $\mathfrak{S}$ be a 2-category and $I$ a set. A **truncated hypercube** in $\mathfrak{S}$ with indexing set $I$ is given by the following data:
- four families of objects of $\mathfrak{S}$:

$$(U_i)_{i \in I} \quad (U_{ij})_{(i,j) \in I^2} \quad (U_{ijk})_{(i,j,k) \in I^3} \quad (U_{ijkl})_{(i,j,k,l) \in I^4}$$

- nine families of 1-morphisms as follows:

$$t : U_{ij} \to U_i, \quad s : U_{ij} \to U_j$$

$$p_1 : U_{ijk} \to U_{ij}, \quad m : U_{ijk} \to U_{ik}, \quad p_2 : U_{ijk} \to U_{jk}$$

$$a : U_{ijkl} \to U_{ijk}, \quad b : U_{ijkl} \to U_{ijl}, \quad c : U_{ijkl} \to U_{ikl}, \quad d : U_{ijkl} \to U_{jkl}$$
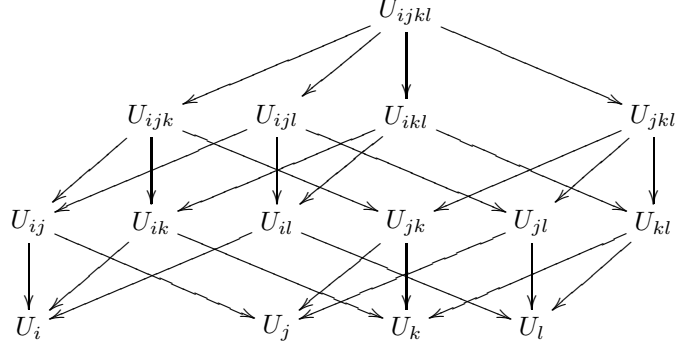
- nine families of 2-isomorphisms as follows:
- three families of 2-isomorphisms fitting into the truncated cube



$$(18)$$

- six families of 2-isomorphisms fitting into half of a hypercube



$$(19)$$

subject to the constraint

• all four cubes in the truncated hypercube

$$
\begin{array}{c}
U_{ijkl} \\
U_{ijk} \quad U_{ijl} \quad U_{ikl} \quad U_{jkl} \\
U_{ij} \quad U_{ik} \quad U_{il} \quad U_{jk} \quad U_{jl} \quad U_{kl} \\
U_i \quad U_j \quad U_k \quad U_l
\end{array}
$$

are 2-commutative, for all $(i, j, k, l) \in I^4$.

We call a truncated hypercube $U_\bullet$ **2-cartesian** (**strict**), if every one of the 2-commutative squares appearing in the definition is 2-cartesian (strictly commutative).

**Definition 2.5** Let $U_\bullet$ and $V_\bullet$ be truncated hypercubes in $\mathfrak{S}$, both with indexing set $I$. Then a **morphism** of truncated hypercubes $\phi_\bullet : U_\bullet \to V_\bullet$ consists of

• four families of 1-morphisms in $\mathfrak{S}$:

$$\phi_i : U_i \to V_i, \quad \phi_{ij} : U_{ij} \to V_{ij}, \quad \phi_{ijk} : U_{ijk} \to V_{ijk}, \quad \phi_{ijkl} : U_{ijkl} \to V_{ijkl},$$

• nine families of 2-morphisms fitting into the diagrams:

$$
\begin{array}{ccc}
U_{ij} & \xrightarrow{\phi_{ij}} & \\
U_i \quad U_j & & V_{ij} \\
\xrightarrow{\phi_i} V_i \quad \xrightarrow{\phi_j} V_j
\end{array}
\tag{20}
$$

$$
\begin{array}{ccc}
U_{ijk} & \xrightarrow{\phi_{ijk}} & \\
U_{ij} \quad U_{ik} \quad U_{jk} & & V_{ijk} \\
\xrightarrow{\phi_{ij}} V_{ij} \quad V_{ik} \quad V_{jk},
\end{array}
\tag{21}
$$

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$$
\begin{array}{c}
U_{ijkl} \\
\swarrow \;\; \downarrow \;\; \searrow \xrightarrow{\phi_{ijkl}} V_{ijkl} \\
U_{ijk} \quad U_{ijl} \quad U_{ikl} \quad U_{jkl} \qquad V_{ijkl} \\
\xrightarrow{\phi_{ijk}} \\
V_{ijk} \quad V_{ijl} \quad V_{ikl} \quad V_{jkl},
\end{array}
\tag{22}
$$

subject to the condition that the obvious nine cubes, built over the nine squares (18) and (19), 2-commute, for every $(i,j,k) \in I^3$ and every $(i,j,k,l) \in I^4$.

A morphism of truncated hypercubes is **2-cartesian** (**strict**), if every one of the 2-commutative squares appearing in (20), (21) and (22) is 2-cartesian (strictly commutative).

**Definition 2.6** Given two morphisms $\phi_\bullet$ and $\psi_\bullet$ from $U_\bullet$ to $V_\bullet$, then a **2-morphism** of morphisms of truncated hypercubes $\theta_\bullet : \phi_\bullet \Rightarrow \psi_\bullet$ is given by four families of 2-morphisms in $\mathfrak{S}$:

$$
U_i \underset{\psi_i}{\overset{\phi_i}{\Downarrow \theta_i}} V_i \qquad
U_{ij} \underset{\psi_{ij}}{\overset{\phi_{ij}}{\Downarrow \theta_{ij}}} V_{ij} \qquad
U_{ijk} \underset{\psi_{ijk}}{\overset{\phi_{ijk}}{\Downarrow \theta_{ijk}}} V_{ijk} \qquad
U_{ijkl} \underset{\psi_{ijkl}}{\overset{\phi_{ijkl}}{\Downarrow \theta_{ijkl}}} V_{ijkl}
$$

such that the nine families of '2-cylinders' built over (20), (21) and (22) all 2-commute.

It is clear that the truncated hypercubes in $\mathfrak{S}$ form a 2-category.

**Definition 2.7** Let $U_\bullet$ be a truncated hypercube in $\mathfrak{S}$. An **augmentation** of $U_\bullet$ consists of
- an object $X$ of $\mathfrak{S}$,
- a family of 1-morphisms:

$$
\iota : U_i \longrightarrow X
$$

- a family of 2-morphisms:

$$
\begin{array}{ccc}
 & U_{ij} & \\
\swarrow & & \searrow \\
U_i & \Rightarrow & U_j \\
\searrow & & \swarrow \\
 & X &
\end{array}
\tag{23}
$$

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subject to the constraint that the cube

$$
\begin{array}{ccccc}
 & & U_{ijk} & & \\
 & \swarrow & \downarrow & \searrow & \\
U_{ij} & & U_{ik} & & U_{jk} \\
\downarrow & \searrow & & \swarrow & \downarrow \\
U_i & & U_j & & U_k \\
 & \searrow & \downarrow & \swarrow & \\
 & & X & &
\end{array}
$$
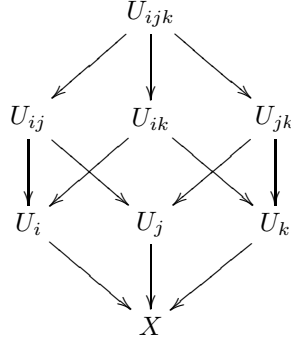
2-commutes, for all $(i, j, k) \in I^3$.

An augmentation is called **2-cartesian** (**strict**), if Diagram (23) is 2-cartesian (strictly commutative), for every $(i, j) \in I^2$.

A truncated hypercube endowed with an augmentation is also called a **hypercube**. A hypercube is **2-cartesian** (**strict**) if its underlying truncated hypercube and its augmentation are both 2-cartesian (strict).

**Definition 2.8** Let $U_\bullet \to V_\bullet$ be a morphism of truncated hypercubes. Let $U_\bullet \to X$ and $V_\bullet \to Y$ be augmentations. Then a **morphism** of augmentations from $X$ to $Y$ consists of

- a morphism $f : X \to Y$,
- a family of 2-morphisms:

$$
\begin{array}{ccc}
U_i & \xrightarrow{\phi_i} & V_i \\
\downarrow & \nearrow & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\tag{24}
$$

such that for every $(i, j) \in I^2$ the cube



2-commutes.

A morphism of augmentations is **2-cartesian** (**strict**), if Diagram (24) is 2-cartesian (strictly commutative) for every $(i, j) \in I^2$.

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A **morphism** of hypercubes is a morphism of truncated hypercubes together with a morphism of augmentations. A morphism of hypercubes is **2-cartesian** (**strict**) if both its underlying morphism of hypercubes and its underlying morphism of augmentations are 2-cartesian (strict).

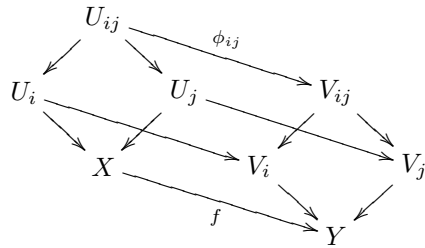**Definition 2.9** Let $(\phi_\bullet, f)$ and $(\psi_\bullet, g)$ be two morphisms of hypercubes. Then a **2-isomorphism** of morphisms of augmentations is a 2-isomorphism

$$X \underset{g}{\overset{f}{\underset{\Downarrow \eta}{\rightrightarrows}}} Y \ ,$$

such that for every $i \in I$ the '2-cylinder'



2-commutes.

Of course, the hypercubes in $\mathfrak{S}$ also form a 2-category. Moreover, all the augmentations of a fixed truncated hypercube form a 2-category.

## 2.3  Descent for algebras

We begin by describing our setup:

Let $A$ be a quasi-finite resolving algebra and $A \to A_i$ a family of étale quasi-finite resolving morphisms such that $\coprod_i \operatorname{Spec} h^0(A_i) \to \operatorname{Spec} h^0(A)$ is surjective. Define $A_{ij} = A_i \otimes_A A_j$, $A_{ijk} = A_i \otimes_A A_j \otimes_A A_k$ and $A_{ijkl} = A_i \otimes_A A_j \otimes_A A_k \otimes_A A_l$. Note that the induced strict hypercube $A \to A_\bullet$ in $\mathfrak{R}_{\mathrm{qf}}^{\mathrm{op}}$ is 2-cartesian, because of Proposition 1.39.

Now suppose given perfect resolving morphisms $A_i \to B_i$, $A_{ij} \to B_{ij}$, $A_{ijk} \to B_{ijk}$ and $A_{ijkl} \to B_{ijkl}$. Moreover, assume that these 1-morphisms form part of the data for a strict, 2-cartesian (*sic!*) morphism of truncated hypercubes $A_\bullet \to B_\bullet$ in $\mathfrak{R}_{\mathrm{qf}}^{\mathrm{op}}$. Finally, assume that the truncated hypercube $B_\bullet$ itself is strict (it is necessarily 2-cartesian).

In this situation, we wish to construct
(i) a perfect resolving morphism $A \to B$,
(ii) for every $i$, a morphisms of $A$-algebras $f_i : B \to B_i$,
(iii) for all $i, j$, a morphism of $A$-algebras $f_{ij} : B \to B_{ij} \otimes \Omega_1$, such that

$$f_{ij}(0) = f_i, \quad f_{ij}(1) = f_j,$$

in such a way that the following conditions are met:

(iv) for all $i, j, k$ there exists a morphism of $A$-algebras $f_{ijk} : B \to B_{ijk} \otimes \Omega_2$, such that

$$f_{ijk}(t, 0) = f_{ij}(t), \quad f_{ijk}(0, t) = f_{ik}(t), \quad f_{ijk}(1 - t, t) = f_{jk}(t),$$

(in other words, the above data (i), (ii) and (iii) give rise to an augmentation of the truncated hypercube $B_\bullet$)

(v) for every $i$, the (strictly commutative) square

$$\begin{array}{ccc} A & \longrightarrow & A_i \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_i \end{array} \tag{25}$$

is 2-cartesian.

Our first aim is to reformulate this 2-cartesian requirement. For this, consider the finite type affine $k$-scheme defined by $h^0(A)$. It is endowed with an étale cover $h^0(A_i)$. The induced cartesian cube in the category of finite type $k$-schemes is given by

$$h^0(A) \longrightarrow h^0(A_i) \rightrightarrows h^0(A_{ij}) \rrightarrows h^0(A_{ijk}) . \tag{26}$$

We also have a cartesian truncated cube

$$h^0(B_i) \rightrightarrows h^0(B_{ij}) \rrightarrows h^0(B_{ijk}) \tag{27}$$

and a cartesian morphism of truncated cubes from (26) to (27). In other words, (27) is gluing data for a finite type $h^0(A)$-algebra $R$:

$$R \longrightarrow h^0(B_i) \rightrightarrows h^0(B_{ij}) \rrightarrows h^0(B_{ijk}) .$$

Let us denote $h^0(B_*)$ by $R_*$, for every multi-index $*$.

Furthermore, for every $n \leq 0$, the truncated cube

$$h^n(B_i) \rightrightarrows h^n(B_{ij}) \rrightarrows h^n(B_{ijk})$$

is gluing data for a finitely generated $R$-module $M^n$, because it is in a certain sense cartesian over (26):

$$M^n \longrightarrow h^n(B_i) \rightrightarrows h^n(B_{ij}) \rrightarrows h^n(B_{ijk}) .$$

Of course, $M^0 = R$.

Now, given the data (i), (ii) and (iii), or $f_\bullet : B \to B_\bullet$, we get an induced morphism of $h^0(A)$-algebras $h^0(B) \to R$ and induced homomorphisms of $h^0(B)$-modules $h^n(B) \to M^n$ such that for every $i$ the diagram

$$\begin{array}{ccc} h^n(B) & & \\ \downarrow & \searrow & \\ M^n & \longrightarrow & h^n(B_i) \end{array}$$

commutes.

Now we can say that (25) is 2-cartesian, if and only if $h^n(B) \to M^n$ is bijective, for all $n \leq 0$.

If we are given data (i), (ii) and (iii), we say that $f_\bullet : B \to B_\bullet$ is a **homotopy square**. If condition (iv) is satisfied for $f_\bullet : B \to B_\bullet$, we say that this homotopy square **defines an augmentation**.

We will build up $B$ by an inductive procedure. The following lemma will be useful:

**Lemma 2.10** *Suppose given a homotopy square $f_\bullet : B \to B_\bullet$ and a finite resolving morphism $B \to \widetilde{B}$ together with a homotopy square $F_\bullet : \widetilde{B} \to B_\bullet$, such that $F_\bullet \,|\, B$ is equal to $f_\bullet$. Suppose that $\widetilde{B}$ has a B-basis consisting of finitely many elements, all in the same degree $r$.*

*Assume that $f_\bullet$ defines an augmentation. Then we may replace the $F_{ij}$ by other morphisms $F'_{ij} : \widetilde{B} \to B_{ij} \otimes \Omega_1$, in such a way that the modified homotopy square $F'_\bullet : \widetilde{B} \to B_\bullet$, with $F'_i = F_i$, still restricts to $f_\bullet$, i.e., $F'_\bullet \,|\, B = f_\bullet$, but now also defines an augmentation.*

PROOF. Choose $f_{ijk}$ as in Condition (iv).

For purposes of abbreviation, let us introduce the notation $X_* = \mathrm{Hom}_A^\Delta(B, B_*)$ and $\widetilde{X}_* = \mathrm{Hom}_A^\Delta(\widetilde{B}, B_*)$, for every multi-index $*$. Then we have a commutative diagram of spaces

$$\prod_i \widetilde{X}_i \rightrightarrows \prod_{i,j} \widetilde{X}_{ij} \Rrightarrow \prod_{i,j,k} \widetilde{X}_{ijk} \;\substack{\longrightarrow\\[-0.3em]\longrightarrow\\[-0.3em]\longrightarrow\\[-0.3em]\longrightarrow}\; \prod_{i,j,k,l} \widetilde{X}_{ijkl}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\prod_i X_i \rightrightarrows \prod_{i,j} X_{ij} \Rrightarrow \prod_{i,j,k} X_{ijk} \;\substack{\longrightarrow\\[-0.3em]\longrightarrow\\[-0.3em]\longrightarrow\\[-0.3em]\longrightarrow}\; \prod_{i,j,k,l} X_{ijkl}$$

All vertical maps in this diagram are fibrations.

The space $X_i$ has a canonical base point, given by $f_i$. The space $X_{ij}$ has two canonical base points, given by $f_i$ and $f_j$. Use the notation $X_{ij}^0$ to denote the space $X_{ij}$ endowed with the base point $f_i$ and $X_{ij}^1$ for $X_{ij}$ endowed with the base point $f_j$. These two base points are connected by the path $f_{ij}$, which gives an isomorphism

$$\pi_\ell X_{ij}^0 \xrightarrow{\sim} \pi_\ell X_{ij}^1\,, \tag{28}$$

for all $\ell \geq 0$, which we consider to be *canonical*. The space $X_{ijk}$ has three base points, giving rise to three pointed spaces $X_{ijk}^0$, $X_{ijk}^1$ and $X_{ijk}^2$, with base points $f_i$, $f_j$ and $f_k$, respectively. We have canonical isomorphisms

$$\pi_\ell X_{ijk}^0 = \pi_\ell X_{ijk}^1 = \pi_\ell X_{ijk}^2\,,$$

because, by existence of $f_{ijk}$, it is irrelevant which of the canonical paths we take between the three different base points.

By Theorem I.4.11 and its Corollary I.4.12, for $\ell \geq 2$, the homotopy groups $\pi_\ell X_*^\epsilon$ are finitely generated $R_*$-modules and by Remark I.4.5(iii), all the canonical maps between them are $R_*$-linear. Moreover, we have

$$\pi_\ell X_i \otimes_{R_i} R_{ij} = \pi_\ell X_{ij}^0$$

and

$$\pi_\ell X_j \otimes_{R_j} R_{ij} = \pi_\ell X_{ij}^1 \,.$$

Thus, for $\ell \geq 2$, the canonical isomorphisms (28) define a gluing datum for a finitely generated $R$-module $N_\ell$, which comes endowed with homomorphisms of $R$-modules

$$N_\ell \longrightarrow \pi_\ell X_i \,,$$

inducing isomorphisms $N_\ell \otimes_R R_i \to \pi_\ell X_i$, and which make the diagrams

$$\begin{array}{ccc}
N_\ell \longrightarrow \pi_\ell X_i \longrightarrow \pi_\ell X_{ij}^0 \\
\searrow \qquad \qquad \downarrow \cong \\
\pi_\ell X_j \longrightarrow \pi_\ell X_{ij}^1
\end{array}$$

commute. We will only use $N_2$, in what follows.

The $F_i$ induce in a similar way various base points for the $\widetilde{X}_*$. We use notation $\widetilde{X}_*^\epsilon$ in a way compatible with $X_*^\epsilon$, to denote the induced pointed spaces. Let us denote the fiber of the fibration of pointed spaces $\widetilde{X}_*^\epsilon \to X_*^\epsilon$ by $Y_*^\epsilon$.

The $F_{ij}$ induce a commutative diagram

$$\begin{array}{ccc}
\pi_1 Y_{ijk}^0 \longrightarrow \pi_1 Y_{ijk}^1 \\
\searrow \qquad \swarrow \\
\pi_1 Y_{ijk}^2
\end{array} \qquad (29)$$

because these homotopy groups are abelian (see Lemma I.4.8) and the closed path $\eta_{ijk} = F_{ik}^{-1} * F_{jk} * F_{ij}$ representing the obstruction to commutativity of (29) maps to the boundary of a 2-simplex in $X_{ijk}$, and hence can be brought into any fiber $Y_{ijk}^\epsilon$.

Thus, by gluing, we obtain another $R$-module $P_1$ which is locally isomorphic to $\pi_1 Y_i$. There is a canonical homomorphism of $R$-modules $N_2 \to P_1$, which makes the diagrams

$$\begin{array}{ccc}
N_2 & \longrightarrow & \pi_2 X_i \\
\downarrow & & \downarrow \delta \\
P_1 & \longrightarrow & \pi_1 Y_i
\end{array}$$

commute.

Of particular importance to us will be the $R$-module

$$Q = \mathrm{cok}(N_2 \to P_1).$$

Note that $F_{ij}$, $F_{ik}$ and $F_{jk}$ are paths in the space $\widetilde{X}_{ijk}$, which fit together so as to form the circumference of a triangle and thus give rise to an element of $\pi_1 \widetilde{X}_{ijk}^0$, which we shall denote by $\eta_{ijk}$. Note that under the fibration $\widetilde{X}_{ijk} \to X_{ijk}$, the homotopy class $\eta_{ijk}$ maps to zero, because its image forms the boundary of the 2-simplex $f_{ijk}$. Thus, via our above identifications, we may think of $\eta_{ijk}$ as an element of $Q \otimes_R h^0(B_{ijk})$.

Now the key observation is that $\eta_{ijk}$ is a Čech 2-cocycle of $\mathrm{Spec}\,R$ with respect to the étale covering $\mathrm{Spec}\,h^0(B_i)$ and with values in the coherent $R$-module $Q$. This can be checked by considering the 1-skeleton of a tetrahedron defined by the $F_{ij}$ inside the space $\widetilde{X}_{ijkl}$.

Since this Čech cohomology group vanishes, there exist

$$\theta_{ij} \in Q \otimes_R h^0(B_{ij}) = \mathrm{cok}(\pi_2 X_{ij}^0 \to \pi_1 Y_{ij}^0) = \ker(\pi_1 \widetilde{X}_{ij}^0 \to \pi_1 X_{ij}^0),$$

such that

$$\eta_{ijk} = \theta_{ik}^{-1} * \theta_{jk} * \theta_{ij}$$

in $\pi_1 \widetilde{X}_{ijk}^0$. We are careful to choose representatives $\theta_{ij}$ which are contained in the fiber $Y_{ij}^0$.

Now we define

$$F_{ij}' = F_{ij} * \theta_{ij}^{-1}.$$

More precisely, we choose $F_{ij}' : \widetilde{B} \to B_{ij} \otimes \Omega_1$ in such a way that $F_{ij}'$ is homotopic to $F_{ij} * \theta_{ij}^{-1}$ and such that $F_{ij}' \mid B = f_{ij}$. This is possible because $\theta_{ij}$ is contained in the fiber of the fibration $\widetilde{X}_{ij} \to X_{ij}$.

Now $F_{ij}'$, $F_{ik}'$ and $F_{jk}'$ again form the circumference of a triangle in $\widetilde{X}_{ijk}$. The homotopy class of this triangle is

$$\eta_{ijk}' = \theta_{ik} * \eta_{ijk} * \theta_{jk}^{-1} * \theta_{ij}^{-1},$$

which is zero in $\pi_1 \widetilde{X}_{ijk}^0$, because the kernel of the homomorphism from $\pi_1 \widetilde{X}_{ijk}^0$ to $\pi_1 X_{ijk}^0$ is abelian. Thus we can find a 2-simplex $F_{ijk}'$ in $\widetilde{X}_{ijk}$, whose boundary consists of $F_{ij}'$, $F_{ik}'$ and $F_{jk}'$.

There is no reason why we should be able to make $F_{ijk}'$ restrict to $f_{ijk}$. For this, $\eta_{ijk}'$ would have to represent zero in $\pi_1 Y_{ijk}^0$, and not just in $\mathrm{im}(\pi_1 Y_{ijk}^0 \to \pi_1 \widetilde{X}_{ijk}^0)$. $\square$

**Theorem 2.11** *There exists a perfect resolving morphism $A \to B$, together with the structure of a 2-cartesian augmentation $f_\bullet : B \to B_\bullet$, such that $A \to B$ becomes a strict, 2-cartesian morphism of augmentations (in $\mathfrak{R}_{\mathrm{qf}}^{\mathrm{op}}$).*

Note: this is the only place we actually need the $B_{ijkl}$

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PROOF. As mentioned, we will build up $B$ and $f_\bullet$ by an inductive procedure.

Suppose given an integer $n \geq -1$ and a finite resolving morphism $A \to B_{(n)}$, together with morphisms of $A$-algebras $f_i : B_{(n)} \to B_i$ and $f_{ij} : B_{(n)} \to B_{ij} \otimes \Omega_1$, such that $f_{ij}(0) = f_i$ and $f_{ij}(1) = f_j$. Suppose also that there exists $f_{ijk} : B_{(n)} \to B_{ijk} \otimes \Omega_2$, satisfying Condition (iv), above.

Moreover, assume that the induced homomorphism of $h^0(A)$-modules

$$h^\ell(B_{(n)}) \longrightarrow M^\ell$$

is bijective, for all $\ell > -n$, and surjective for $\ell = -n$.

We will construct a resolving morphism $B_{(n)} \to B_{(n+1)}$ and extensions $F_i$ and $F_{ij}$ of the $f_i$ and the $f_{ij}$ to $B_{(n+1)}$ in such a way that this extended homotopy square also defines an augmentation and

$$h^\ell(B_{(n+1)}) \longrightarrow M^\ell$$

is bijective, for all $\ell > -n - 1$, and surjective for $\ell = -n - 1$.

For the construction, let us choose $f_{ijk} : B_{(n)} \to B_{ijk} \otimes \Omega_2$, satisfying Condition (iv).

We start by choosing elements $b^\nu \in Z^{-n}(B_{(n)})$ whose classes $[b^\nu]$ in $h^{-n}(B_{(n)})$ generate the kernel of the epimorphism of $R$-modules $h^{-n}(B_{(n)}) \to M^{-n}$ (if $n = -1$, we do not choose any $b^\nu$, if $n = 0$, we choose elements $b^\nu \in B_{(0)}$ generating the kernel of the morphism $h^0(B_{(0)})$-algebras $h^0(B_{(0)}) \to R$). We also choose generators $m^\mu$ for the $R$-module $M^{-n-1}$ (if $n = -1$, we take generators for the $h^0(A)$-algebra $R$). Then we choose $\gamma_i^\mu \in Z^{-n-1}(B_i)$ such that $m^\mu$ maps to $[\gamma_i^\mu]$ under $M^{-n-1} \to h^{-n-1}(B_i)$.

Let $B_{(n+1)} = B_{(n)}[x^\nu, y^\mu]$, where $x^\nu$ and $y^\mu$ are formal variables in degree $-n - 1$. Set $dx^\nu = b^\nu$ and $dy^\mu = 0$. We construct the $F_i$ and $F_{ij}$ by specifying where they send the $x^\nu$ and the $y^\mu$.

Let us start with $x^\nu$. Since $b^\nu$ maps to zero in $h^{-n}(B_i)$, the image $b_i^\nu = f_i(b^\nu)$ of $b^\nu$ in $B_i$ is a coboundary. Choose $\beta_i^\nu \in B_i^{-n-1}$, such that $d\beta_i^\nu = b_i^\nu$. Similarly, let $b_{ij}^\nu = f_{ij}(b^\nu)$. Since $d(b_{ij}^\nu - b_i^\nu) = 0$ and $(b_{ij}^\nu - b_i^\nu)(0) = 0$, by Lemma I.4.2, there exists $\beta_{ij}^\nu \in (B_{ij} \otimes \Omega_1)^{-n-1}$, such that

$$d\beta_{ij}^\nu = b_{ij}^\nu - b_i^\nu \quad \text{and} \quad \beta_{ij}^\nu(0) = 0 \,.$$

Again, using Lemma I.4.2, we choose $\beta_{ijk}^\nu \in (B_{ijk} \otimes \Omega_2)^{-n-1}$, such that

$$d\beta_{ijk}^\nu(s, t) = b_{ijk}^\nu(s, t) - b_{ij}^\nu(s) - b_{ik}^\nu(t) + b_i^\nu$$

and

$$\beta_{ijk}^\nu(t, 0) = \beta_{ijk}^\nu(0, t) = 0 \,,$$

where $b_{ijk}^\nu = f_{ijk}(b^\nu)$.

Let us prove that $\beta_{ij}^\nu(1) - \beta_{ik}^\nu(1) + \beta_{jk}^\nu(1)$ is a coboundary. For this we consider the expression

$$\delta_{ijk}^\nu(t) = \beta_{ijk}^\nu(1 - t, t) + \beta_{ij}^\nu(1 - t) + \beta_{ik}^\nu(t) - \beta_{jk}^\nu(t) - \beta_{ij}^\nu(1) \,,$$

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which is an element of $B_{ijk} \otimes \Omega_1$. We have that $d\delta^\nu_{ijk} = 0$ and $\delta^\nu_{ijk}(0) = 0$, so using Lemma I.4.2 once again, we find $\Delta^\nu_{ijk}(t)$, such that $d\Delta^\nu_{ijk} = \delta^\nu_{ijk}$ and $\Delta^\nu_{ijk}(0) = 0$. Evaluating $\Delta^\nu_{ijk}$ at $t = 1$, reveals that $\beta^\nu_{ij}(1) - \beta^\nu_{ik}(1) + \beta^\nu_{jk}(1)$ is, indeed, a coboundary.

Thus we have proved that $[\beta^\nu_j - \beta^\nu_i - \beta^\nu_{ij}(1)]$ defines a Čech 1-cocycle of the affine scheme $\operatorname{Spec} R$ and the étale cover $\operatorname{Spec} h^0(B_i)$ with values in the coherent sheaf $M^{-n-1}$. Since this Čech cohomology group vanishes, we can bound the cocycle $[\beta^\nu_j - \beta^\nu_j - \beta^\nu_{ij}(1)]$. So by changing the $\beta^\nu_i$, we may assume that $[\beta^\nu_j - \beta^\nu_i - \beta^\nu_{ij}(1)] = 0$ in $h^{-n-1}(B_{ij})$. Therefore, there exist $\theta^\nu_{ij} \in B^{-n-2}_{ij}$, such that $d\theta^\nu_{ij} = \beta^\nu_j - \beta^\nu_i - \beta^\nu_{ij}(1)$, for all $i, j$.

We now define

$$F_i(x^\nu) = \beta^\nu_i$$

and

$$F_{ij}(x^\nu)(t) = (1 - t)\beta^\nu_i + t\big(\beta^\nu_j - \beta^\nu_{ij}(1)\big) + \beta^\nu_{ij}(t) + (-1)^n \theta^\nu_{ij}\, dt$$

Let us deal with $y^\mu$. In this case, $[\gamma^\mu_j - \gamma^\mu_i]$ is directly seen to be zero in $h^{-n-1}(B_{ij})$, and there is no need (and no freedom anyway) to change the $\gamma^\mu_i$ to find $\theta^\mu_{ij} \in B^{-n-2}_{ij}$, such that $d\theta^\mu_{ij} = \gamma^\mu_j - \gamma^\mu_i$, for all $i, j$. In this case we set

$$F_i(y^\mu) = \gamma^\mu_i \quad \text{and} \quad F_{ij}(y^\mu) = (1 - t)\gamma^\mu_i + t\gamma^\mu_j + (-1)^n \theta^\mu_{ij}\, dt\,.$$

We see that this does extend the morphisms of $A$-algebras $f_i$ and $f_{ij}$ to $B_{(n+1)} = B_{(n)}[x^\nu, y^\mu]$, and that the relations

$$F_{ij}(0) = F_i \quad \text{and} \quad F_{ij}(1) = F_j$$

are satisfied. Moreover, by construction, $h^\ell(B_{(n+1)}) \longrightarrow M^\ell$ is bijective, for all $\ell > -n - 1$, and surjective, for all $\ell \geq -n - 1$.

The only thing left to worry about is if the homotopy square $F_\bullet : B_{(n+1)} \to B_\bullet$ defines an augmentation. If it does not, then we apply Lemma 2.10. This leads to a change in the $F_{ij}$, but since the $F_i$ are not affected, the properties of $h^\ell(B_{(n+1)}) \longrightarrow M^\ell$ are not affected.

Thus our inductive procedure works. Starting with $B_{(-1)} = A$, we let

$$B = \varinjlim_n B_{(n)}\,.$$

Since at every step of the induction $F_i$ and $F_{ij}$ are extensions of $f_i$ and $f_{ij}$, we get induced morphisms of $A$-algebras $f_i : B \to B_i$ and $f_{ij} : B \to B_{ij} \otimes \Omega_1$, satisfying $f_{ij}(0) = f_i$ and $f_{ij}(1) = f_j$. Therefore, we also get an induced morphism of $h^0(A)$-algebras $h^0(B) \to R$ and induced homomorphisms of $h^0(B)$-modules $h^\ell(B) \to M^\ell$, for all $\ell \leq 0$. All these are isomorphisms, by construction, and hence the (strictly commutative) square

$$
\begin{array}{ccc}
A & \longrightarrow & A_i \\
\downarrow & & \downarrow \\
B & \longrightarrow & B_i
\end{array}
$$

is 2-cartesian in $\mathfrak{R}_{\mathrm{qf}}^{\mathrm{op}}$. This proves that $A \to B$ is perfect, as this property is local in the étale topology on $A$.

To prove that we have an augmentation $f_\bullet : B \to B_\bullet$, we need to check that for all $i, j, k$ the path $f_{ik}^{-1} * f_{jk} * f_{ij}$ in $\pi_1\big(\mathrm{Hom}_A^\Delta(B, B_{ijk}), f_i\big)$ represents zero. But by construction, the image of this path in $\pi_1\big(\mathrm{Hom}_A^\Delta(B_{(n)}, B_{ijk}), f_i\big)$ represents zero, for all $n$. Thus we conclude, using the fact that

$$\pi_1 \mathrm{Hom}_A^\Delta(B, B_{ijk}) = \varprojlim_n \pi_1 \mathrm{Hom}_A^\Delta(B_{(n)}, B_{ijk})$$

which is Corollary I.4.12. $\square$

## 2.4 Strictifying truncated hypercubes

**Definition 2.12** A morphism $f : X \to Y$ in a 2-category $\mathfrak{S}$ is called a **fibration**, if for every 2-commutative diagram



there exists a lift $\eta'$ of $\eta$ to $X$, i.e., a diagram



such that $f \circ \eta' = \eta$.

Another way to say this is that the morphism of groupoids $X(T) \to Y(T)$ makes $X(T)$ into a fibered category over $Y(T)$, for all objects $T$ of $\mathfrak{S}$.

**Definition 2.13** Let $\mathfrak{S}$ be a 2-category with a distinguished class of 1-morphisms called $F$-morphism. We say that $\mathfrak{S}$ **has enough** $F$-morphisms, if for every morphism $X \to Y$ in $\mathfrak{S}$ there exists a strict factorization



where $X \to X'$ is 2-invertible and $X' \to Y$ is an $F$-morphism. We call any such factorization an $F$-**resolution** of $X \to Y$

**Proposition 2.14** *Let $\mathfrak{S}$ be a 2-category with a distinguished class of morphisms called $F$-morphism such that*

*(i) every F-morphism is a fibration,*

*(ii) strict fibered products exist in $\mathfrak{S}$, if at least one of the two participating morphisms is of type F,*

*(iii) A strict base change of an F-morphism is an F-morphism,*

*(iv) compositions of F-morphisms are F-morphisms.*

*Then if $U_\bullet$ is a strict truncated hypercube and $V_\bullet \to U_\bullet$ is a morphism of truncated hypercubes, then there exists a strictly commutative diagram*

$$
\begin{array}{ccc}
V_\bullet & \xrightarrow{\ \sim\ } & V'_\bullet \\
& \searrow & \downarrow \\
& & U_\bullet
\end{array}
$$

*where $V_\bullet \to V'_\bullet$ is a 2-invertible morphism of truncated hypercubes, and $V'_\bullet$, as well as $V'_\bullet \to U_\bullet$, is strict. Moreover, $V'_\bullet \to U_\bullet$ can be chosen such that all its structure 1-morphisms are F-morphisms.*

PROOF.    Start by choosing F-resolutions $V_i \to V'_i \to U_i$ of $V_i \to U_i$. Then, for every $(i,j) \in I^2$, replace the two morphisms $V_{ij} \to V'_i$ and $V_{ij} \to V'_j$ by 2-isomorphic ones, in such a way that the two squares

$$
\begin{array}{ccc}
V_{ij} \longrightarrow V'_i \\
\downarrow \qquad\quad \downarrow \\
U_{ij} \longrightarrow U_i
\end{array}
\qquad
\begin{array}{ccc}
V_{ij} \longrightarrow V'_j \\
\downarrow \qquad\quad \downarrow \\
U_{ij} \longrightarrow U_j
\end{array}
\tag{30}
$$

commute strictly.

Next, consider the strict fibered products

$$
\begin{array}{ccc}
P_{ij} & \longrightarrow & V'_i \times V'_j \\
\downarrow & & \downarrow \\
U_{ij} & \longrightarrow & U_i \times U_j
\end{array}
$$

We have canonical morphisms $V_{ij} \to P_{ij}$, which we F-resolve: $V_{ij} \to V'_{ij} \to P_{ij}$. We replace $V_{ij}$ by $V'_{ij}$. This preserves the strictness of the diagrams (30). It makes $V'_{ij} \to U_{ij}$ into F-morphisms.

Use the fact that $V'_{ij} \to U_{ij}$ is a fibration to replace $V_{ijk} \to V'_{ij}$ by a 2-isomorphic morphism making the square

$$
\begin{array}{ccc}
V_{ijk} & \longrightarrow & V'_{ij} \\
\downarrow & & \downarrow \\
U_{ijk} & \longrightarrow & U_{ij}
\end{array}
$$

strictly commutative.

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Next, consider the diagram

$$
\begin{array}{ccc}
V_{ijk} & \longrightarrow & V'_{ij} \\
\downarrow & \nearrow \Uparrow & \searrow \\
U_{ijk} \;\; \Swarrow \;\; V'_{ik} & \longrightarrow & V'_i \\
& \searrow & \downarrow \\
& U_{ik} & \longrightarrow U_i
\end{array}
\qquad (31)
$$

Both the square and the exterior hexagon in this diagram strictly commute. Using the fact that $V'_{ik} \to U_{ik} \times_{U_i} V'_i$ is a fibration, we may replace $V_{ijk} \to V'_{ik}$ by a 2-isomorphic morphism making the whole diagram (31) strictly commute.

Thus, we now have a diagram

$$
\begin{array}{ccc}
V_{ijk} & \longrightarrow & V'_{ij} \times V'_{ik} \\
\downarrow & \nearrow \Uparrow & \searrow \\
U_{ijk} \;\; \Swarrow \;\; V'_{jk} & \longrightarrow & V'_j \times V'_k \\
& \searrow & \downarrow \\
& U_{jk} & \longrightarrow U_j \times U_k
\end{array}
\qquad (32)
$$

in which, again, both the square and the exterior hexagon strictly commute. We exploit the fact that $V'_{jk} \to P_{jk}$ is a fibration, to make (32) strictly commute. At this point all squares in $V_\bullet$, as well as all squares in $V_\bullet \to U_\bullet$, which have $V_{ijk}$ as source, are strictly commutative.

The next step is to consider the strict fibered products

$$
\begin{array}{ccc}
P_{ijk} & \longrightarrow & V'_{ij} \times V'_{ik} \times V'_{jk} \\
\downarrow & & \downarrow \\
U_{ijk} & \longrightarrow & U_{ij} \times U_{ik} \times U_{jk}
\end{array}
$$

and to resolve the canonical morphism $V_{ijk} \to P_{ijk}$ by the composition $V_{ijk} \to V'_{ijk} \to P_{ijk}$. As above, we can strictify all squares in $V_\bullet$ and in $V_\bullet \to U_\bullet$ whose source is $V_{ijkl}$.

Finally, resolve $V_{ijkl} \to P_{ijkl}$, do finish the proof. $\square$

We will apply this proposition to $\mathfrak{R}^{\mathrm{op}}_{\mathrm{pf}}$, with '$F$-morphism' meaning 'perfect resolving morphism'.

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# 3 Differential graded schemes

## 3.1 Differential graded sheaves

We return to considering the 2-category $\mathfrak{S}$ with the étale topology, introduced in Section 1.5. Recall that we had defined a *differential graded sheaf* to be any sheaf on $\mathfrak{S}$.

If $B$ is a perfect resolving algebra, then it induces a representable presheaf $X_{\mathrm{qf}}$ over $\mathfrak{S}_{\mathrm{qf}}$. Let us restrict $X_{\mathrm{qf}}$ to $\mathfrak{S}$ via the the canonical embedding $\mathfrak{S} \to \mathfrak{S}_{\mathrm{qf}}$. We get a presheaf $X$ over $\mathfrak{S}$, which may not be representable (depending on the choice of $\mathfrak{S}$) if $B$ is not finite.

By Corollary 2.3, this presheaf $X$ over $\mathfrak{S}$ is a sheaf. We denote the differential graded sheaf $X$ by

$$\mathbb{S}\mathrm{pec}\, B\,.$$

If $B \to C$ is a morphism of perfect resolving algebras, then we get an induced morphism of differential graded sheaves

$$\mathbb{S}\mathrm{pec}\, C \longrightarrow \mathbb{S}\mathrm{pec}\, B\,.$$

In fact, we get a contravariant 2-functor

$$\mathbb{S}\mathrm{pec} : \mathfrak{R}_{\mathrm{pf}} \longrightarrow (\text{differential graded sheaves})\,.$$

**Lemma 3.1** *Let $B$ be a perfect resolving algebra and $U$ the object of $\mathfrak{S}_{\mathrm{pf}}$ given by $B$. Then there exists a finite covering family of $U_i \to U$ for the étale pretopology on $\mathfrak{S}_{\mathrm{pf}}$, such that every $U_i$ is given by a finite resolving algebra.*

PROOF. By Theorem I.3.8 there exist $g_i \in B^0$, such that the elementary open immersions $B \to B_{\{g_i\}}$ define a covering family and such that each $B_{\{g_i\}}$ is quasi-isomorphic to a finite resolving algebra. $\square$

**Corollary 3.2** *In case $\mathfrak{S}$ is contained in $\mathfrak{S}_{\mathrm{pf}}$, the canonical restriction 2-functor*

$$(\text{sheaves over } \mathfrak{S}_{\mathrm{pf}}) \longrightarrow (\text{sheaves over } \mathfrak{S})$$

*is fully faithful.*

PROOF. By Lemma 3.1, $\mathfrak{S}$ generates $\mathfrak{S}_{\mathrm{pf}}$. $\square$

**Corollary 3.3** *The contravariant 2-functor*

$$\mathbb{S}\mathrm{pec} : \mathfrak{R}_{\mathrm{pf}} \to (\text{differential graded sheaves})$$

*is fully faithful.*

PROOF. First let us consider the case that $\mathfrak{R}_{pf} \subset \mathfrak{S}$. By Yoneda's lemma, Proposition 1.20, we have a fully faithful 2-functor

$$\mathfrak{R}_{pf} \longrightarrow (\text{presheaves over } \mathfrak{S}).$$

By Corollary 2.3, this 2-functor maps into the subcategory of sheaves over $\mathfrak{S}$. So we have a fully faithful 2-functor

$$\mathfrak{R}_{pf} \longrightarrow (\text{sheaves over } \mathfrak{S}).$$

Now let us consider the case that $\mathfrak{S} \subset \mathfrak{R}_{pf}$. The above considerations applied to $\mathfrak{S}_{pf}$ give us a fully faithful 2-functor

$$\mathfrak{R}_{pf} \longrightarrow (\text{sheaves over } \mathfrak{S}_{pf}).$$

Composing with the fully faithful 2-functor of Corollary 3.2 finishes the proof. $\square$

## 3.2   Affine differential graded schemes

**Definition 3.4** A differential graded sheaf, which is isomorphic to $\mathbb{Spec}\, B$, for some perfect resolving algebra $B$, is called an **affine differential graded scheme**, or simply **affine**.

If an affine differential graded scheme is isomorphic to $\mathbb{Spec}\, B$, for a finite resolving algebra $B$, we call it **finite affine**, or simply **finite**.

**Note 3.5** By Corollary 3.3, the 2-functor $\mathbb{Spec}$ induces contravariant equivalences of 2-categories

$$\mathbb{Spec} : \mathfrak{R}_{pf} \longrightarrow (\text{affine differential graded schemes})$$
$$\mathbb{Spec} : \mathfrak{R}_{f} \longrightarrow (\text{finite affine differential graded schemes}).$$

**Example 3.6** Every complete intersection in affine space over $k$ gives rise to an affine differential graded scheme, which is well-defined up to isomorphism. This is because every complete intersection in affine space is defined by a regular sequence, and so the associated Koszul complex gives a differential graded algebra, which is determined by the complete intersection scheme up to quasi-isomorphism. For example, any finite extension field $K$ of $k$ may be considered as an affine differential graded scheme.

If $A$ is the usual affine coordinate ring of such a complete intersection, we commit the abuse of writing $\mathbb{Spec}\, A$ for this differential graded scheme.

**Lemma 3.7** *If $X$ and $Y$ are affine differential graded schemes, then so is $X \times Y$. If $X$ and $Y$ are finite, then so is $X \times Y$.*

PROOF. This follows from Proposition 1.38. $\square$

Let $f : X \to Y$ be a morphism between affine differential graded schemes, where $X \cong \mathbb{S}\mathrm{pec}\, B$ and $Y \cong \mathbb{S}\mathrm{pec}\, C$. Then, by Corollary 3.3, there exists a morphism of perfect resolving algebras $\phi : C \to B$, such that $\mathbb{S}\mathrm{pec}\, \phi = f$. The morphism $\phi$ is unique up to homotopy.

**Definition 3.8** The morphism of affine differential graded schemes $f : X \to Y$ is called **étale**, (an **open immersion**) if the corresponding morphism $\phi : C \to B$ of perfect resolving algebras is étale (an open immersion). (See Definitions I2.8 and I.2.13.)

**Lemma 3.9** *If $f : X \to Y$ is a morphism of affine differential graded schemes and $Y' \to Y$ is an étale morphism (an open immersion) of affine differential graded schemes, then the fibered product $X' = X \times_Y Y'$ is an affine differential graded scheme and $X' \to X$ is étale (an open immersion). If $X$, $Y$ and $Y'$ are finite, then so is $X'$.*

PROOF. This follows directly from Proposition 1.39. $\square$

**Lemma 3.10 (affine descent)** *Let $\mathfrak{X} \to U$ be a morphism of differential graded sheaves, where $U$ is affine. Assume that there exists an epimorphic family of morphisms of differential graded sheaves $U_i \to U$, where every $U_i$ is affine, and every morphism $U_i \to U$ is étale. Assume that for every $i$, the 2-fibered product $X_i = \mathfrak{X} \times_U U_i$ is affine. Then $\mathfrak{X}$ itself is affine.*

PROOF. Without loss of generality, the morphisms $U_i \to U$ are given by (étale) quasi-finite resolving morphisms $A \to A_i$ of perfect resolving algebras $A$, $A_i$. Because $(U_i \to U)$ is epimorphic, $\coprod_i \mathrm{Spec}\, h^0(A_i) \to \mathrm{Spec}\, h^0(A)$ is surjective. Define the strict cartesian hypercube $A \to A_\bullet$ in $\mathfrak{R}_{\mathrm{pf}}^{\mathrm{op}}$ as in Section 2.3. Let $U_\bullet = \mathbb{S}\mathrm{pec}\, A_\bullet$, so that $U_\bullet \to U$ is a (strict, cartesian) hypercube of affine differential graded schemes.

Form the fibered products $X_\bullet = \mathfrak{X} \times_U U_\bullet$ of differential graded sheaves. These form a 2-cartesian morphism

$$X_\bullet \to U_\bullet$$

of 2-cartesian truncated hypercubes of differential graded sheaves. By Lemma 3.9, the truncated hypercube $X_\bullet$ consists of affine differential graded schemes. Thus, via Corollary 3.3, we may choose a 2-cartesian truncated hypercube $B_\bullet$ of perfect resolving algebras, such that $\mathbb{S}\mathrm{pec}\, B_\bullet = X_\bullet$. The morphism $X_\bullet \to U_\bullet$ gives rise to a 2-cartesian morphism $A_\bullet \to B_\bullet$.

Now apply Proposition 2.14 to $A_\bullet \to B_\bullet$ in $\mathfrak{R}_{\mathrm{pf}}^{\mathrm{op}}$, with '$F$-morphism' meaning 'perfect resolving morphism', to show that we may assume, without loss of generality, that $B_\bullet$, as well as $A_\bullet \to B_\bullet$, is strict. Moreover, we may assume that $A_\bullet \to B_\bullet$ consists of perfect resolving morphisms. Thus we are now in the setup of Section 2.3, and from Theorem 2.11 we obtain a perfect resolving morphism $A \to B$ and a 2-cartesian morphism of hypercubes $[A \to A_\bullet] \to [B \to B_\bullet]$ in

$\mathfrak{R}^{\mathrm{op}}_{\mathrm{pf}}$. Applying $\mathbb{S}\mathrm{pec}$, we obtain a cartesian morphism of hypercubes $[X_\bullet \to X] \to [U_\bullet \to U]$ of affine differential graded schemes.

Then $X \cong \mathfrak{X}$ as sheaves on $\mathfrak{S}$, and so $\mathfrak{X}$ is representable by the perfect resolving algebra $B$. $\square$

**Remark** We cannot conclude that $\mathfrak{X}$ is finite, even if $U$, all $U_i$ and all $X_i$ are finite. A counterexample can be derived from Example I.3.11 or, more specifically, Example I.3.12. Let $A$ and $B$ be the differential graded algebras defined in Example I.3.12. Thus,

$$A = k[x, y,]\{\xi\}/d\xi = y^2 - 4(x^3 - x),$$

in the notation of Section I.2.5. Moreover, $B$ is a quasi-finite resolution of the differential graded algebra $h^0(A) \oplus L$. Here $L$ is a projective $h^0(A)$-module of rank one, which we put in degree -1. The differential on $h^0(A) \oplus L$ is zero. We get a counterexample if $L$ is non-trivial.

Suppose the elementary open immersion $A \to A_i$ trivializes $L$. Then $B_i = B \otimes_A A_i$ is quasi-isomorphic to a finite resolving algebra. Thus $A$, all $A_i$ and all $B_i$ are 'essentially' finite, but $B$ is not.

**Proposition 3.11** *A morphism of affine differential graded schemes is étale if and only if it is a categorically étale morphism of sheaves. It is an open immersion if and only if it is a monomorphism of sheaves (cf. Propositions 1.18 and 1.32).*

PROOF. Let our morphism be $\mathbb{S}\mathrm{pec}\, B' \to \mathbb{S}\mathrm{pec}\, B$. First we reduce to the case that $B \to B'$ is a resolving morphism. Then the categorically étale property translates into bijectivity of

$$\pi_1 \operatorname{Hom}^\Delta(B', A) \longrightarrow \pi_1 \operatorname{Hom}^\Delta(B, A),$$

for all resolving morphisms $B' \to A$. By Proposition I.4.18, this is equivalent to $B \to B'$ being étale.

For the second claim, we may assume that $\mathbb{S}\mathrm{pec}\, B' \to \mathbb{S}\mathrm{pec}\, B$ is étale. Then $\mathbb{S}\mathrm{pec}\, B' \to \mathbb{S}\mathrm{pec}\, B$ is an open immersion, if and only if the diagram

$$
\begin{array}{ccc}
\mathbb{S}\mathrm{pec}\, B' & \longrightarrow & \mathbb{S}\mathrm{pec}\, B' \\
\downarrow & & \downarrow \\
\mathbb{S}\mathrm{pec}\, B' & \longrightarrow & \mathbb{S}\mathrm{pec}\, B
\end{array}
$$

is 2-cartesian. This is the case if and only if

$$\pi_0 \operatorname{Hom}^\Delta(B', A) \to \pi_0 \operatorname{Hom}^\Delta(B, A)$$

is injective, for all $A$, which is the monomorphism property of $\mathbb{S}\mathrm{pec}\, B' \to \mathbb{S}\mathrm{pec}\, B$. $\square$

**Definition 3.12** An **affine Zariski cover** of an affine differential graded scheme $X$ is a collection of open immersions $U_i \to X$, where every $U_i$ is affine.

**Proposition 3.13** *Every affine differential graded scheme $X$ admits an affine Zariski cover $U_i \to X$, such that for every multi-index $(i_0, \ldots, i_p)$, with $p \geq 0$, the fibered product*

$$U_{i_0 \ldots i_p} = U_{i_0} \times_X \ldots \times_X U_{i_p}$$

*is finite.*

PROOF. Let $X = \mathbb{S}\mathrm{pec}\, B$. As in Lemma 3.1, we choose $g_i \in B^0$, such that the elementary open immersion $B \to B_{\{g_i\}}$ cover $X$, and such that each $B_{\{g_i\}}$ is quasi-isomorphic to a finite resolving algebra $A_i$.

The fibered product $\mathbb{S}\mathrm{pec}\, B_{\{g_i\}} \times_X \mathbb{S}\mathrm{pec}\, B_{\{g_j\}}$ is represented by $B_{\{g_i g_j\}}$, which is quasi-isomorphic to $A_{i\{g_j\}}$ and hence finite. Similarly for iterated fibered products. $\square$

## 3.3  Affine étale morphisms

**Definition 3.14** A morphism $\mathfrak{F} \to \mathfrak{G}$ of differential graded sheaves is called **affine étale** (an **affine open immersion**), if for every morphism $U \to \mathfrak{G}$, with $U$ affine, the fibered product $V = \mathfrak{F} \times_{\mathfrak{G}} U$ is affine and the morphism $V \to U$ is étale (an open immersion).

**Proposition 3.15** *A morphism $\mathfrak{F} \to \mathfrak{G}$ is affine étale (an affine open immersion) if and only if there exists an epimorphic family $U_i \to \mathfrak{G}$ of morphisms with (finite) affine $U_i$, such that for every $i$ the fibered product $V_i = \mathfrak{F} \times_{\mathfrak{G}} U_i$ is affine and the morphism $V_i \to U_i$ is étale (an open immersion).*

PROOF. This follows directly from Lemmas 3.9 and 3.10. $\square$

In particular, a morphism of affine differential graded schemes is étale if and only if it is affine étale. Similarly for open immersions.

**Proposition 3.16** *If $\mathfrak{F} \to \mathfrak{G}$ and $\mathfrak{F}' \to \mathfrak{G}'$ are affine étale morphisms of differential graded sheaves, then so is $\mathfrak{F} \times \mathfrak{F}' \to \mathfrak{G} \times \mathfrak{G}'$.*

PROOF. This follows from the fact the affine étale property is stable under composition and arbitrary base change. $\square$

**Note 3.17** Let $f : X \to Y$ be a morphism of affine differential graded schemes. Suppose there exists an epimorphic family of affine étale morphisms $U_i \to X$, where for all $i$ the composition $U_i \to Y$ is étale. Then $f$ is étale.

## 3.4  Differential graded schemes

**Definition 3.18** A differential graded sheaf $\mathfrak{X}$ is called a **differential graded scheme**, if there exists an epimorphic family of affine étale morphisms $U_i \to \mathfrak{X}$ such that each $U_i$ is affine.

Any such epimorphic family $U_i \to \mathfrak{X}$ is called an **affine étale cover** of $\mathfrak{X}$. If all $U_i \to \mathfrak{X}$ are affine open immersion, we speak of an **affine Zariski cover** of $\mathfrak{X}$.

**Remark**  It would be more accurate to call these objects 'differential graded algebraic spaces with affine diagonal', but we find that terminology too clumsy.

It seems likely that one can iterate this definition, and obtain more general objects, which would be differential graded schemes with weaker separation condition. At this point it is not clear how useful this would be, and so we call the above objects simply differential graded schemes, without a further qualifier.

**Proposition 3.19** *If $\mathfrak{X}$ is a differential graded scheme, then there exists an affine étale cover $U_i \to \mathfrak{X}$, where every $U_i$ is finite.*

PROOF. This follows directly from Proposition 3.13. $\square$

**Proposition 3.20** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be differential graded schemes. Then $\mathfrak{X} \times \mathfrak{Y}$ is a differential graded scheme.*

PROOF. Let $U_i \to \mathfrak{X}$ and $V_j \to \mathfrak{Y}$ be affine étale covers. Then $U_i \times V_j \to \mathfrak{X} \times \mathfrak{Y}$ is an affine étale cover of the product $\mathfrak{X} \times \mathfrak{Y}$. $\square$

**Lemma 3.21 (descent)** *Let $\mathfrak{F} \to \mathfrak{X}$ be a morphism of differential grade sheaves, where $\mathfrak{X}$ is a differential graded scheme. Let $U_i \to \mathfrak{X}$ be an affine étale cover, such that the fibered product $\mathfrak{V}_i = \mathfrak{F} \times_{\mathfrak{X}} U_i$, is a differential graded scheme, for all $i$. Then $\mathfrak{F}$ is a differential graded scheme.*

PROOF. Let $V_{ij} \to \mathfrak{V}_i$ be an affine étale cover of $\mathfrak{V}_i$, for all $i$. Since the affine étale property, as well as the epimorphism property are stable under base change and composition, it follows that $V_{ij} \to \mathfrak{F}$ is an affine étale cover. $\square$

**Definition 3.22** A differential graded scheme $\mathfrak{X}$ is of **amplitude** $N$, if for every affine étale morphism $\mathbb{S}\mathrm{pec}\, B \to \mathfrak{X}$, the perfect resolving algebra $B$ is of amplitude $N$.

If there exists an affine étale cover $\mathbb{S}\mathrm{pec}\, B_i \to \mathfrak{X}$, where for every $i$, the perfect resolving algebra $B_i$ is of amplitude $N$, then $\mathfrak{X}$ is of amplitude $N$.

## 3.5 Affine morphisms

**Definition 3.23** A morphism $\mathfrak{X} \to \mathfrak{Y}$ of differential graded schemes is called **affine**, if for every affine étale morphism $U \to \mathfrak{Y}$, with $U$ affine, the fibered product $V = \mathfrak{X} \times_{\mathfrak{Y}} U$ is affine.

**Proposition 3.24** *A morphism $\mathfrak{X} \to \mathfrak{Y}$ is affine if there exists an affine étale cover $U_i \to \mathfrak{Y}$, such that for every $i$ the fibered product $V_i = \mathfrak{X} \times_{\mathfrak{Y}} U_i$ is affine.*

PROOF. This follows from Lemmas 3.9 and 3.10. $\square$

**Proposition 3.25** *The diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ of a differential graded scheme $\mathfrak{X}$ is affine.*

PROOF. If $U_i \to \mathfrak{X}$ is an affine étale cover of $\mathfrak{X}$, then $U_i \times U_j$ is an affine étale cover of $\mathfrak{X} \times \mathfrak{X}$, and we have 2-cartesian diagrams

$$
\begin{array}{ccc}
U_{ij} & \longrightarrow & U_i \times U_j \\
\downarrow & & \downarrow \\
\mathfrak{X} & \longrightarrow & \mathfrak{X} \times \mathfrak{X}
\end{array}
$$

where $U_{ij} = U_i \times_{\mathfrak{X}} U_j$, which is affine, by the definition of affine étale morphism. $\square$

## 3.6 Étale morphisms

**Definition 3.26** A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of differential graded schemes is called **étale**, if for every morphism $U \to \mathfrak{Y}$, with $U$ affine, the fibered product $\mathfrak{V} = \mathfrak{X} \times_{\mathfrak{Y}} U$ is a differential graded scheme and for every affine étale morphism $V \to \mathfrak{V}$, with $V$ affine, the composition $V \to U$ is étale.

**Proposition 3.27** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of differential graded schemes. Suppose given an epimorphic family of morphisms $U_i \to \mathfrak{Y}$, with $U_i$ affine for all $i$. Suppose further that for every $i$ the fibered product $\mathfrak{V}_i = \mathfrak{X} \times_{\mathfrak{Y}} U_i$ is a differential graded scheme and that there exists an affine étale cover $V_{ij} \to \mathfrak{V}_i$ of $\mathfrak{V}_i$ such that the composition $V_{ij} \to U_i$ is étale, for all $j$. Then $f$ is étale*

PROOF. This is not difficult to prove using the techniques developed so far. In particular, use Lemma 3.21 and Note 3.17. $\square$

**Note** A morphism of differential graded schemes is affine and étale if and only if it is affine étale. An étale morphism with affine source is affine étale.

**Corollary 3.28** *If $\mathfrak{X} \to \mathfrak{Y}$ is a morphism of differential graded schemes and $\mathfrak{Y}' \to \mathfrak{Y}$ an étale morphism of differential graded schemes, then the fibered product $\mathfrak{X}' = \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ is a differential graded scheme and $\mathfrak{X}' \to \mathfrak{X}$ is étale.* $\square$

**Remark**   The question of the existence of more general fibered products in the 2-category of differential graded schemes is rather subtle. It is treated in detail in [3].

**Proposition 3.29** *Consider the 2-commutative diagram of differential graded schemes*

$$\mathfrak{X} \xrightarrow{\ f\ } \mathfrak{Y}$$
$$\ \ \ ^{h}\searrow \quad \swarrow^{g}$$
$$\mathfrak{Z}$$

*If $g$ is étale, then $f$ is étale if and only if $h$ is étale. If $f$ is an étale epimorphism, then $g$ is étale if and only if $h$ is étale.* □

**Proposition 3.30** *A morphism of differential graded schemes is étale if and only if it is categorically étale.*                              proof!

**Definition 3.31** An étale morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of differential graded schemes is called an **open immersion**, if it is a monomorphism of differential graded sheaves.

**Remark**   An affine étale morphism is an open immersion if and only if it is an affine open immersion. Open immersions are stable under base change.

**Proposition 3.32** *A morphism of differential graded schemes is an open immersion if and only if it is a monomorphism.*                              proof!

**Definition 3.33** If $\mathfrak{X}$ is a differential graded scheme, an **open subscheme** of $\mathfrak{X}$ is a full sub-2-category $\mathfrak{X}' \subset \mathfrak{X}$, such that $\mathfrak{X}'$ is itself a differential graded scheme and the inclusion morphism $\mathfrak{X}' \to \mathfrak{X}$ is an open immersion.

# 4 The basic 1-categorical invariants

For any differential graded sheaf $\mathfrak{F}$, the underlying 2-category $\mathfrak{F}$ is endowed with a topology in a canonical way. (A sieve is covering in $\mathfrak{F}$ if its image in $\mathfrak{S}$ is covering.) Thus every differential graded scheme $\mathfrak{X}$ has an associated 2-site, namely the 2-category $\mathfrak{X}$ itself, with this canonical topology.

**Definition 4.1** A **sheaf** over $\mathfrak{X}$, is a sheaf on this associated 2-site.

Write $\overline{\mathfrak{X}}$ for the 1-category associated to $\mathfrak{X}$. By the basic 1-categorical invariants of the differential graded scheme $\mathfrak{X}$ we mean certain sheaves of sets on $\mathfrak{X}$. Note that every sheaf of sets on $\mathfrak{X}$ comes in a unique and canonical way from a sheaf on $\overline{\mathfrak{X}}$. Thus the terminology.

**Remark** The rule $\mathfrak{X} \to \overline{\mathfrak{X}}$ defines a 2-functor from the 2-category of differential graded schemes to the 2-category of sites.

should it be topoi?

## 4.1 The associated graded structure sheaf

Let $\mathfrak{X}$ be a differential graded scheme. For an object $x$ of $\mathfrak{X}$, denote the image of $x$ under the structure 2-functor $\mathfrak{X} \to \mathfrak{S}$ by $A_x$.

**Definition 4.2** The **truncated structure sheaf** of $\mathfrak{X}$ is the sheaf of sets on $\mathfrak{X}$ defined by
$$x \longmapsto h^0(A_x).$$
We denote the truncated structure sheaf by $h^0(\mathcal{O}_{\mathfrak{X}})$. This is an abuse of notation, as we have not defined $\mathcal{O}_{\mathfrak{X}}$.

Note that for every morphism $U \to \mathfrak{X}$, with $U$ affine, $h^0(\mathcal{O}_{\mathfrak{X}})(U)$ is a finitely generated $k$-algebra.

The fact that $h^0(\mathcal{O}_{\mathfrak{X}})$ is a sheaf, follows directly from the definition of the étale topology.

Thus $h^0(\mathcal{O}_{\mathfrak{X}})$ is a sheaf of $k$-algebras on $\mathfrak{X}$.

**Definition 4.3** The $n$-th **higher structure sheaf** is the sheaf of sets on $\mathfrak{X}$ defined by
$$x \longmapsto h^n(A_x).$$
We denote the $n$-the higher structure sheaf by $h^n(\mathcal{O}_{\mathfrak{X}})$.

The direct sum
$$h^*(\mathcal{O}_{\mathfrak{X}}) = \bigoplus_n h^n(\mathcal{O}_{\mathfrak{X}})$$

is called the **associated graded structure sheaf** of $\mathfrak{X}$.

For every morphism $U \to \mathfrak{X}$ with $U$ affine, $h^n(\mathcal{O}_{\mathfrak{X}})(U)$ is a finitely generated $h^0(\mathcal{O}_{\mathfrak{X}})(U)$-module and $h^*(\mathcal{O}_{\mathfrak{X}})$ is a graded $h^0(\mathcal{O}_{\mathfrak{X}})$-algebra. Thus $h^n(\mathcal{O}_{\mathfrak{X}})$ is a coherent sheaf of modules and $h^*(\mathcal{O}_{\mathfrak{X}})$ a graded sheaf of algebras over the sheaf of $k$-algebras $h^0(\mathcal{O}_{\mathfrak{X}})$.

Of course, $h^n(\mathcal{O}_{\mathfrak{X}}) = 0$, for all $n > 0$.

**Remark 4.4** Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a morphism is differential graded schemes. Then there is a natural isomorphism (*sic!*) of sheaves of graded $k$-algebras

$$\phi^{-1}h^*(\mathcal{O}_{\mathfrak{Y}}) \longrightarrow h^*(\mathcal{O}_{\mathfrak{X}}) \,.$$

For example, for $\mathfrak{Y} = \mathbb{S}\mathrm{pec}\, k = \mathfrak{S}$, we get that $h^*(\mathcal{O}_{\mathfrak{X}})$ is the pullback of $h^*(\mathcal{O}_{\mathbb{S}\mathrm{pec}\, k})$ via the structure functor $\mathfrak{X} \to \mathfrak{S}$ (which is also clear from the definition). We will abbreviate $h^*(\mathcal{O}_{\mathbb{S}\mathrm{pec}\, k})$ by $h^*(\mathcal{O})$.

## 4.2 Higher tangent sheaves

We need some preliminaries concerning the naturality properties of $\underline{\mathrm{Der}}(B, A)$, for resolving algebras $B$, $A$. (For the notation $\underline{\mathrm{Der}}(B, A)$, see Section I1.5.)

Let $f : B \to A$ and $g : B \to A$ be morphisms of resolving algebras. Let $\theta : f \Rightarrow g$ be a homotopy, i.e., a morphism $\theta : B \to A \otimes \Omega_1$, such that $\partial_0 \theta = f$ and $\partial_1 \theta = g$.

$$B \underset{g}{\overset{f}{\Longrightarrow}} \Downarrow\theta\; A$$

Recall (Definition I.3.7), that $\theta$ induces a canonical isomorphism of $h^0(B)$-modules

$$h_\ell \underline{\mathrm{Der}}(B, {}_f A) \xrightarrow{\theta_*} h_\ell \underline{\mathrm{Der}}(B, {}_g A) \,.$$

Recall also, that $\theta_*$ depends only on the homotopy class of $\theta$, and is thus well-defined for a 2-isomorphism $\theta : f \Rightarrow g$ in $\mathfrak{R}$. Moreover, it is functorial for vertical composition of 2-morphisms: $\theta_* \eta_* = (\theta\eta)_*$.

We will require two further naturality properties of this induced canonical isomorphism. First, some more notation:

Suppose given a 2-commutative diagram

$$
\begin{array}{ccc}
B & \longrightarrow & A \\
 & \searrow{\scriptstyle\Swarrow_\eta} & \downarrow{\scriptstyle f} \\
 & & A'
\end{array}
$$

We denote the composition

$$h_\ell \underline{\mathrm{Der}}(B, A) \xrightarrow{f_*} h_\ell \underline{\mathrm{Der}}(B, {}_f A') \xrightarrow{\eta_*} h_\ell \underline{\mathrm{Der}}(B, A')$$

by $\eta(f_*)$.

**Proposition 4.5** *Consider a diagram*

$$B \longrightarrow A \overset{\eta \Uparrow \quad f}{\underset{\Downarrow \zeta \quad g}{\longrightarrow}} A' \tag{33}$$

*where the two morphisms from $B$ to $A'$ are equal. We get two homomorphisms of $h^0(B)$-modules*

$$h_\ell \underline{\mathrm{Der}}(B, A) \overset{\eta(f_*)}{\underset{\zeta(g_*)}{\rightrightarrows}} h_\ell \underline{\mathrm{Der}}(B, A') \ .$$

*These are equal, if there exists a 2-isomorphism*

$$A \overset{f}{\underset{g}{\Rightarrow \theta}} A'$$

*making Diagram (33) commute, i.e., such that $\eta = \zeta\theta$.*

PROOF. The proof is similar to the construction of the induced canonical isomorphism. $\square$

**Proposition 4.6** *Consider a diagram*

$$B \overset{f}{\underset{g}{\Rightarrow \theta}} B' \overset{p}{\longrightarrow} A \ .$$

*Then the induced diagram*

$$\begin{array}{ccc}
& & h_\ell \underline{\mathrm{Der}}(B, {}_f A) \\
& \overset{f^*}{\nearrow} & \\
h_\ell \underline{\mathrm{Der}}(B', A) & & \downarrow {}^{(p\theta)_*} \\
& \underset{g^*}{\searrow} & \\
& & h_\ell \underline{\mathrm{Der}}(B, {}_g A)
\end{array}$$

*commutes.*

PROOF. This proposition is a little more tricky. The problem is that $B' \otimes \Omega_1$ is not a resolving algebra. Thus it is not clear if $B' \otimes \Omega_1$ is cofibrant, and hence if $\underline{\mathrm{Der}}(B' \otimes \Omega_1, \cdot)$ is sufficiently well-behaved. Thus, instead of working with $\underline{\mathrm{Der}}(B' \otimes \Omega_1, \cdot)$, we use $\underline{\mathrm{Hom}}_{B' \otimes \Omega_1}(L_{B' \otimes \Omega_1}, \cdot)$. This requires a theory of the cotangent complex for non-resolving algebras, as developed in [7]. We omit the details of the rather lengthy diagram chases that conclude the proof. $\square$

We are now ready to define the higher tangent sheaves. Let $B$ be a perfect resolving algebra.

**Definition 4.7** The $\ell$-th **higher tangent sheaf** of $\mathbb{S}\mathrm{pec}\, B$ is the sheaf of sets on $\mathbb{S}\mathrm{pec}\, B$ defined by

$$(B \xrightarrow{x} A) \longmapsto h_\ell \, \underline{\mathrm{Der}}(B, A) = h_\ell(\Theta_B \otimes_B A)\,.$$

We denote the $\ell$-th higher tangent sheaf of $\mathbb{S}\mathrm{pec}\, B$ by $h_\ell(\Theta_{\mathbb{S}\mathrm{pec}\, B})$.

This definition gives rise to a presheaf, because of Proposition 4.5. The fact that $h_\ell(\Theta_{\mathbb{S}\mathrm{pec}\, B})$ is a sheaf follows from Corollary I.2.12.

**Remark** A morphism of perfect resolving algebras $f : B \to B'$, which gives rise to the morphism of differential graded schemes $\phi : \mathbb{S}\mathrm{pec}\, B' \to \mathbb{S}\mathrm{pec}\, B$, defines a canonical sheaf map

(marginal note:) here is one place where the correct definition of opposite 2-category is important

$$h_\ell(\Theta_{\mathbb{S}\mathrm{pec}\, B'}) \longrightarrow \phi^{-1} h_\ell(\Theta_{\mathbb{S}\mathrm{pec}\, B})\,. \tag{34}$$

Let $\theta : f \Rightarrow g$ be a homotopy between the morphisms $f, g : B \to B'$. Letting $\phi$ be the morphism of differential graded schemes induced by $f$ and $\psi$ the morphism of differential graded schemes induced by $g$, we get an induced 2-isomorphism $\eta : \psi \Rightarrow \phi$. The 2-isomorphism $\eta$ gives rise to a natural equivalence of functors $\eta^{-1} : \phi^{-1} \to \psi^{-1}$ and hence to a sheaf isomorphism

$$\eta^{-1} : \phi^{-1} h_\ell(\Theta_{\mathbb{S}\mathrm{pec}\, B}) \longrightarrow \psi^{-1} h_\ell(\Theta_{\mathbb{S}\mathrm{pec}\, B})\,.$$

The induced triangle of sheaves

$$
\begin{array}{ccc}
 & & \phi^{-1} h_\ell(\Theta_{\mathbb{S}\mathrm{pec}\, B}) \\
 & \nearrow & \\
h_\ell(\Theta_{\mathbb{S}\mathrm{pec}\, B'}) & & \downarrow {\scriptstyle \eta^{-1}} \\
 & \searrow & \\
 & & \psi^{-1} h_\ell(\Theta_{\mathbb{S}\mathrm{pec}\, B})
\end{array}
$$

commutes. This follows from Proposition 4.6.

Because of this, we may define $h_\ell(\Theta_U)$ for any affine differential graded scheme $U$, because the sheaf on $U$ pulled back via any isomorphism $U \to \mathbb{S}\mathrm{pec}\, B$, is independent of the choice of $B$ and $U \to \mathbb{S}\mathrm{pec}\, B$, at least up to canonical isomorphism.

Let $\mathfrak{X}$ be a differential graded scheme. Let $V \to \mathfrak{X}$ and $U \to \mathfrak{X}$ be étale morphisms, with $V$ and $U$ affine. Assume given a 2-commutative diagram of differential graded schemes

$$
\begin{array}{ccc}
V & \xrightarrow{\;f\;} & U \\
 & \searrow \;{\scriptstyle \nLeftarrow}\; & \downarrow \\
 & & \mathfrak{X}
\end{array}
\tag{35}
$$

we get an induced sheaf map

$$h_\ell(\Theta_V) \longrightarrow f^{-1}h_\ell(\Theta_U)\,,$$

which is an isomorphism, by Proposition I.1.37, and because $V \to U$ is necessarily étale.

Thus, as we let $U \to \mathfrak{X}$ vary over all étale morphisms with affine $U$, we get gluing data for a sheaf of sets $h_\ell(\Theta_{\mathfrak{X}})$ on $\mathfrak{X}$.

**Definition 4.8** The sheaf $h_\ell(\Theta_{\mathfrak{X}})$ is called the $\ell$-th **higher tangent sheaf** of $\mathfrak{X}$. The **associated graded tangent sheaf** of $\mathfrak{X}$ is the direct sum

$$h_*(\Theta_{\mathfrak{X}}) = \bigoplus_\ell h_\ell(\Theta_{\mathfrak{X}})\,.$$

The higher tangent sheaf $h_\ell(\Theta_{\mathfrak{X}})$ comes with isomorphisms

$$h_\ell(\Theta_U) \longrightarrow h_\ell(\Theta_{\mathfrak{X}})\,|\,U\,,$$

for every étale $U \to \mathfrak{X}$ with affine $U$. Any diagram (35) induces a commutative diagram

$$
\begin{array}{ccc}
h_\ell(\Theta_V) & \longrightarrow & f^{-1}h_\ell(\Theta_U) \\
\downarrow & & \downarrow \\
h_\ell(\Theta_{\mathfrak{X}})\,|\,V & \longrightarrow & f^{-1}\big(h_\ell(\Theta_{\mathfrak{X}})\,|\,U\big)
\end{array}
$$

of sheaves on $V$.

Every higher tangent sheaf $h_\ell(\Theta_{\mathfrak{X}})$ is a coherent $h^0(\mathcal{O}_{\mathfrak{X}})$-module. The associated graded tangent sheaf $h_*(\Theta_{\mathfrak{X}})$ is a sheaf of graded $h^*(\mathcal{O}_{\mathfrak{X}})$-modules.

Note that $h_\ell(\Theta_{\mathfrak{X}}) = 0$, for $\ell < -N$, if $\mathfrak{X}$ is of amplitude $N$.

**Remark** If $\mathbb{S}\mathrm{pec}\, B \to \mathfrak{X}$ is étale and $B \to A$ is an arbitrary morphism of perfect resolving algebras, then we have, by construction of $h_\ell(\Theta_{\mathfrak{X}})$, a canonical isomorphism of $h^0(B)$-modules

$$h_\ell\,\underline{\mathrm{Der}}(B,A) \overset{\sim}{\longrightarrow} h_\ell(\Theta_{\mathfrak{X}})(\mathbb{S}\mathrm{pec}\, A)\,.$$

For example, if $\phi : \mathbb{S}\mathrm{pec}\, K \to \mathfrak{X}$ is a $K$-valued point of $\mathfrak{X}$, then

$$h_\ell(\Theta_{\mathfrak{X}})(\phi) = h_\ell\,\underline{\mathrm{Der}}(B,K)\,,$$

for any affine étale neighbourhood $\mathbb{S}\mathrm{pec}\, K \to \mathbb{S}\mathrm{pec}\, B \to \mathfrak{X}$ of $\phi$.

**Definition 4.9** The **higher tangent spaces** of $\mathfrak{X}$ at the $K$-valued point $\phi$ of $\mathfrak{X}$ are the $K$-vector spaces $h_\ell(\Theta_{\mathfrak{X}})(\phi)$.

**Proposition 4.10** *If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of differential graded schemes, then we get an induced canonical homomorphism of sheaves of $h^0(\mathcal{O}_{\mathfrak{X}})$-modules*

$$h_\ell(\Theta_{\mathfrak{X}}) \longrightarrow f^{-1}h_\ell(\Theta_{\mathfrak{Y}}) . \tag{36}$$

*The morphism $f$ is étale, if and only if (36) is an isomorphism, for all $\ell$ (or for one fixed $\ell \geq 0$).*

*If $\phi : \operatorname{Spec} K \to \mathfrak{X}$ is a $K$-valued point of $\mathfrak{X}$, then $f$ induces a canonical homomorphism of $K$-vector spaces*

$$h_\ell(\Theta_{\mathfrak{X}})(\phi) \longrightarrow h_\ell(\Theta_{\mathfrak{Y}})\big(f(\phi)\big) , \tag{37}$$

*for all $\ell$. The morphism $f$ is étale, if and only if (37) is an isomorphism, for all $\ell$.* $\square$

### The relative case

Note that if $C' \to C \to B \to A$ is a composition of morphisms of resolving algebras, with $C$, $C'$ and $B$ perfect and $C \to B$ as well as $C' \to B$ resolving, then the canonical homomorphism of $h^0(A)$-modules

$$h_\ell \, \underline{\operatorname{Der}}_C(B, A) \longrightarrow h_\ell \, \underline{\operatorname{Der}}_{C'}(B, A)$$

is an isomorphism, if $C' \to C$ is étale.

Now assume given a morphism of differential graded schemes $\phi : \mathfrak{X} \to \mathfrak{Y}$.

**Definition 4.11** The $\ell$-th **relative higher tangent sheaf** of $\mathfrak{X} \to \mathfrak{Y}$, notation $h_\ell(\Theta_{\mathfrak{X}/\mathfrak{Y}})$, is defined in such a way that for every resolving morphism of perfect resolving algebras $C \to B$ and any 2-commutative diagram of differential graded schemes

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} C \\ \downarrow & \nearrow & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{Y} \end{array} \tag{38}$$

where $\operatorname{Spec} B \to \mathfrak{X}$ and $\operatorname{Spec} C \to \mathfrak{Y}$ are étale, we have a canonical isomorphism

$$h_\ell(\Theta_{B/C}) \longrightarrow h_\ell(\Theta_{\mathfrak{X}/\mathfrak{Y}}) \,|\, \operatorname{Spec} B \,,$$

where $h_\ell(\Theta_{B/C})$ is the sheaf on $\operatorname{Spec} B$ defined by

$$(B \to A) \longmapsto h_\ell \, \underline{\operatorname{Der}}_C(B, A) \,.$$

By the remark preceding the definition, $h_\ell(\Theta_{B/C})$ does not depend on the choice of $C$. Note also that the étale morphisms $\operatorname{Spec} B \to \mathfrak{X}$ admitting a factorization (38) are cofinal in the 2-category of all étale $\operatorname{Spec} B \to \mathfrak{X}$ and still cover $\mathfrak{X}$. Thus the fact that $h_\ell(\Theta_{\mathfrak{X}/\mathfrak{Y}})$ exists with the required properties is proved as in the absolute case.

**Remark 4.12** The morphism $\mathfrak{X} \to \mathfrak{Y}$ is étale, if and only if $h_\ell(\mathfrak{X}/\mathfrak{Y}) = 0$ for all $\ell$, or for on fixed $\ell \geq 0$.

**Remark 4.13** Given $C \to B$ and a diagram such as (38), for every morphism of resolving algebras $B \to A$ we have a short exact sequence of complexes of $A$-modules

$$0 \longrightarrow \underline{\mathrm{Der}}_C(B, A) \longrightarrow \underline{\mathrm{Der}}(B, A) \longrightarrow \underline{\mathrm{Der}}(C, A) \longrightarrow 0\,.$$

These give rise to natural long exact sequences of $h^0(\mathcal{O}_\mathfrak{X})$-modules

$$\ldots \longrightarrow h_\ell(\Theta_{\mathfrak{X}/\mathfrak{Y}}) \longrightarrow h_\ell(\Theta_\mathfrak{X}) \longrightarrow \phi^{-1}h_\ell(\Theta_\mathfrak{Y}) \longrightarrow h_{\ell-1}(\Theta_{\mathfrak{X}/\mathfrak{Y}}) \longrightarrow \ldots \quad (39)$$

Let

$$
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{\phi} & \mathfrak{Y} \\
& {\scriptstyle \kappa} \searrow \nearrow & \downarrow {\scriptstyle \psi} \\
& & \mathfrak{Z}
\end{array}
\qquad (40)
$$

be a 2-commutative diagram of differential graded schemes. By the naturality properties of (39), we have a 'long exact braid with four strands'

$$ (41) $$

## 4.3 Homotopy sheaves

The homotopy sheaves of a differential graded scheme are defined similarly to the higher tangent sheaves. First, they are defined for affine differential graded schemes and then they are glued with respect to the étale topology.

Let $B$ be a perfect resolving algebra and $\ell > 0$ and integer.

**Definition 4.14** The $\ell$-th **homotopy sheaf** of $\mathbb{Spec}\, B$, notation $\pi_\ell(\mathbb{Spec}\, B)$, is the sheaf of sets on $\mathbb{Spec}\, B$ defined by

$$(B \xrightarrow{x} A) \longmapsto \pi_\ell \mathrm{Hom}^\Delta(B, A)\,.$$

The fact that this defines a sheaf on $\mathbb{S}\mathrm{pec}\, B$ follows directly from descent theory, Theorem 2.1(i).

A morphism of perfect resolving algebras $f : B \to B'$ gives rise to a canonical sheaf map

$$\pi_\ell(\mathbb{S}\mathrm{pec}\, B') \longrightarrow \phi^{-1}\pi_\ell(\mathbb{S}\mathrm{pec}\, B)\,, \tag{42}$$

via the restriction map $\mathrm{Hom}^\Delta(B', A) \to \mathrm{Hom}^\Delta(B, A)$. For étale $f$, the sheaf map (42) is an isomorphism, by Proposition I.4.18. (Here we use the fact that $\ell > 0$.)

Let now $\mathfrak{X}$ be a differential graded scheme. As we let $\mathbb{S}\mathrm{pec}\, B \to \mathfrak{X}$ vary over all perfect resolving algebras $B$ and all étale morphisms to $\mathfrak{X}$, the various $\pi_\ell(\mathbb{S}\mathrm{pec}\, B)$ glue, via the gluing maps (42), to a sheaf of sets $\pi_\ell(\mathfrak{X})$ on $\mathfrak{X}$.

**Definition 4.15** The sheaf $\pi_\ell(\mathfrak{X})$ is called the $\ell$-th **homotopy sheaf** of the differential graded scheme $\mathfrak{X}$.

By construction, $\pi_\ell(\mathfrak{X})$ is endowed with a sheaf isomorphism

$$\pi_\ell(\mathbb{S}\mathrm{pec}\, B) \longrightarrow \pi_\ell(\mathfrak{X})\,|\,\mathbb{S}\mathrm{pec}\, B\,,$$

for all étale $\mathbb{S}\mathrm{pec}\, B \to \mathfrak{X}$. This isomorphism is compatible with the gluing isomorphisms (42). Thus, if $A$ is an arbitrary resolving algebra in $\mathfrak{R}$, endowed with a morphism $B \to A$, then

$$\pi_\ell(\mathfrak{X})(A) = \pi_\ell \mathrm{Hom}^\Delta(B, A)\,.$$

The sheaves $\pi_\ell(\mathfrak{X})$ are sheaves of groups; abelian, for $\ell \geq 2$.

check it's a sheaf

**Remark 4.16** Let $\mathfrak{A}ut\,\mathfrak{X}$ denote the sheaf of sets on $\mathfrak{X}$ given by

$$x \longmapsto \mathrm{Aut}(x)\,.$$

Here $\mathrm{Aut}(x)$ stands for the automorphism group of the object $x$ of $\mathfrak{X}$ inside the fiber $\mathfrak{X}_A$, where $x$ lies over $A \in \mathfrak{R}$. For $\mathfrak{X} = \mathbb{S}\mathrm{pec}\, B$, we have $\pi_1(\mathfrak{X})(x) = \mathrm{Aut}(x)$, for every object $x : B \to A$ of $\mathbb{S}\mathrm{pec}\, B$. Hence $\pi_1(\mathfrak{X}) = \mathrm{Aut}\,\mathfrak{X}$.

For every morphism of differential graded schemes $\phi : \mathfrak{Y} \to \mathfrak{X}$, we have an induced morphism $\mathfrak{A}ut\,\mathfrak{Y} \to \phi^{-1}\mathfrak{A}ut\,\mathfrak{X}$ of sheaves of sets on $\mathfrak{Y}$. In particular, for a morphism $\phi : \mathbb{S}\mathrm{pec}\, B \to \mathfrak{X}$, we get a canonical morphism $\pi_1(\mathbb{S}\mathrm{pec}\, B) \to \phi^{-1}\mathfrak{A}ut\,\mathfrak{X}$. These canonical morphisms glue to give a canonical morphism

$$\pi_1(\mathfrak{X}) \longrightarrow \mathfrak{A}ut\,\mathfrak{X}\,,$$

which is trivially an isomorphism.

If we choose pullbacks for the fibered category $\mathfrak{X} \to \mathfrak{S}$, we can identify the fiber $\mathfrak{X}_A$ of $\mathfrak{X}$ over the differential graded algebra $A$ in $\mathfrak{R}$, with the groupoid **Hom**$(\mathbb{S}\mathrm{pec}\, A, \mathfrak{X})$. Doing this we have

$$\pi_1(\mathfrak{X})(x) = \pi_1\,\mathbf{Hom}(\mathbb{S}\mathrm{pec}\, A, \mathfrak{X})\,,$$

for any $x : \mathbb{S}\mathrm{pec}\, A \to \mathfrak{X}$.

**Definition 4.17** Let $\pi_0(\mathfrak{X})$ denote the presheaf of pointed sets on $\mathfrak{X}$ defined by

$$\pi_0(\mathfrak{X})(x) = \pi_0(\mathfrak{X}_{A_x}),$$

where $A_x$ is the image of $x$ in $\mathfrak{S}$. Note that by Theorem 2.1 (i), for affine $\mathfrak{X}$, the presheaf $\pi_0(\mathfrak{X})$ is a sheaf.

**Example** Denote $\mathbb{S}\mathrm{pec}\, k[x]$, where $\deg x = 0$, by $\mathbb{A}^1$. Then we have $\pi_\ell(\mathbb{A}^1) = h^{-\ell}(\mathcal{O}_{\mathbb{A}^1})$. Thus, we have that $\pi_\ell(\mathbb{A}^1)$ is the pullback of $h^{-\ell}(\mathcal{O})$ via the structure functor $\mathbb{A}^1 \to \mathfrak{S}$, for all $\ell \geq 0$.

Let us call a morphism of differential graded schemes $\phi : \mathfrak{X} \to \mathbb{A}^1$ a **regular function** on $\mathfrak{X}$. Then for any regular function $\phi$ on the differential graded scheme $\mathfrak{X}$ we have

$$h^{-\ell}(\mathcal{O}_{\mathfrak{X}}) = \phi^{-1}\pi_\ell(\mathbb{A}^1),$$

for all $\ell \geq 0$.

We have a canonical map

$$\pi_0 \underline{\mathrm{Hom}}(\mathfrak{X}, \mathbb{A}^1) \longrightarrow \Gamma\big(\mathfrak{X}, h^0(\mathcal{O}_{\mathfrak{X}})\big),$$

which is bijective, if $\mathfrak{X}$ is affine.

**The relative case**

Just like the higher tangent sheaves, the homotopy sheaves also admit relative versions. Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of differential graded schemes and $\ell > 0$ an integer.

**Definition 4.18** The $\ell$-th **relative homotopy sheaf** of $\mathfrak{X}$ over $\mathfrak{Y}$, notation $\pi_\ell(\mathfrak{X}/\mathfrak{Y})$, is defined in such a way that for every resolving morphism of perfect resolving algebras $C \to B$ and any 2-commutative diagram of differential graded schemes (38), where $\mathbb{S}\mathrm{pec}\, B \to \mathfrak{X}$ and $\mathbb{S}\mathrm{pec}\, C \to \mathfrak{Y}$ are étale, we have a conical isomorphism

$$\pi_\ell(B/C) \longrightarrow \pi_\ell(\mathfrak{X}/\mathfrak{Y}) \,|\, \mathbb{S}\mathrm{pec}\, B,$$

where $\pi_\ell(B/C)$ is the sheaf on $\mathbb{S}\mathrm{pec}\, B$ defined by

$$(B \to A) \longmapsto \pi_\ell \mathrm{Hom}_C^\Delta(B, A).$$

Moreover, define the presheaf of pointed sets on $\mathfrak{X}$

$$\pi_0(\mathfrak{X}/\mathfrak{Y})$$

by defining $\pi_0(\mathfrak{X}/\mathfrak{Y})(x)$ to be $\pi_0$ of the fiber through $x$ of the morphism of groupoids $\mathfrak{X}_A \to \mathfrak{Y}_A$, where $A$ is the object of $\mathfrak{R}$ over which the object $x$ of $\mathfrak{X}$ lies.

**Proposition 4.19** *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of differential graded schemes and $r > 0$ and integer. The following are equivalent:*

*(i) $\pi_\ell(\mathfrak{X}/\mathfrak{Y}) = 0$, for all $\ell > 0$,*

*(ii) $\pi_r(\mathfrak{X}/\mathfrak{Y}) = 0$,*

*(iii) $\pi_\ell(\mathfrak{X}) \to \phi^{-1}\pi_\ell(\mathfrak{Y})$ is an isomorphism of sheaves of groups, for all $\ell > 0$,*

*(iv) $\pi_r(\mathfrak{X}) \to \phi^{-1}\pi_r(\mathfrak{Y})$ is an isomorphism of sheaves of groups.*

PROOF. Follows from Proposition I.4.18. $\square$

**Proposition 4.20 (Long exact homotopy sequence)** *There is a natural long sequence of presheaves on $\mathfrak{X}$*

$$\ldots \longrightarrow \pi_\ell(\mathfrak{X}/\mathfrak{Y}) \longrightarrow \pi_\ell(\mathfrak{X}) \longrightarrow \phi^{-1}\pi_\ell(\mathfrak{Y}) \xrightarrow{\partial} \pi_{\ell-1}(\mathfrak{X}/\mathfrak{Y}) \longrightarrow \ldots$$

$$\ldots \longrightarrow \phi^{-1}\pi_1(\mathfrak{Y}) \xrightarrow{\partial} \pi_0(\mathfrak{X}/\mathfrak{Y}) \longrightarrow \pi_0(\mathfrak{X}) \longrightarrow \pi_0(\mathfrak{Y}). \quad (43)$$

*This sequence gives rise to a long exact sequence of groups and pointed sets, when evaluated at an object $x : \mathbb{S}\mathrm{pec}\, A \to \mathfrak{X}$ of $\mathfrak{X}$, which admits a factorization*



*with étale $\mathbb{S}\mathrm{pec}\, C \to \mathfrak{Y}$. In particular, the part of (43) ending with $\phi^{-1}\pi_1(\mathfrak{Y})$ is an exact sequence of sheaves of groups on $\mathfrak{X}$.*

PROOF. Let $C \to B$ be a resolving morphism of perfect resolving algebras together with a 2-commutative diagram



$$(44)$$

with $\mathbb{S}\mathrm{pec}\, B \to \mathfrak{X}$ and $\mathbb{S}\mathrm{pec}\, C \to \mathfrak{Y}$ étale. Then for any morphism of resolving algebras $B \to A$, we get a fibration of spaces

$$\mathrm{Hom}^\Delta(B, A) \longrightarrow \mathrm{Hom}^\Delta(C, A)$$

with fiber $\mathrm{Hom}_C^\Delta(B, A)$. The associated long exact homotopy sequence, which is natural in $A$, gives rise to a long exact sequence of presheaves on $\mathbb{S}\mathrm{pec}\, B$

$$\ldots \longrightarrow \pi_\ell(B/C) \longrightarrow \pi_\ell(B) \longrightarrow \psi^{-1}\pi_\ell(C) \longrightarrow \ldots \longrightarrow \psi^{-1}\pi_1(C).$$

One checks that the various maps in this sequence glue, to give a sequence of sheaves on $\mathfrak{X}$

$$\ldots \longrightarrow \pi_\ell(\mathfrak{X}/\mathfrak{Y}) \longrightarrow \pi_\ell(\mathfrak{X}) \longrightarrow \phi^{-1}\pi_\ell(\mathfrak{Y}) \longrightarrow \ldots \longrightarrow \phi^{-1}\pi_1(\mathfrak{Y}) \,. \qquad (45)$$

By construction, this sequence is exact on the level of groups of sections over any object $x$ of $\mathfrak{X}$, lying over $A$ in $\mathfrak{R}$, such that $x : \mathbb{S}\mathrm{pec}\, A \to \mathfrak{X}$ factors through a diagram (44).

By construction, we have, for any object $x$ of $\mathfrak{X}$, an exact sequence of groups and pointed sets

$$\mathrm{Aut}(x) \longrightarrow \mathrm{Aut}\big(\phi(x)\big) \longrightarrow \pi_0(\mathfrak{X}/\mathfrak{Y})(x) \longrightarrow \pi_0(\mathfrak{X})(x) \longrightarrow \pi_0(\mathfrak{Y})\big(\phi(x)\big) \,.$$

(This is a general fact about morphisms of groupoids.) Thus we get an exact sequence of presheaves of groups and pointed sets on $\mathfrak{X}$

$$\mathfrak{A}ut(\mathfrak{X}) \longrightarrow \phi^{-1}\mathfrak{A}ut(\mathfrak{Y}) \longrightarrow \pi_0(\mathfrak{X}/\mathfrak{Y}) \longrightarrow \pi_0(\mathfrak{X}) \longrightarrow \phi^{-1}\pi_0(\mathfrak{Y}) \,.$$

By Remark 4.16, we have natural identifications $\pi_1(\mathfrak{X}) = \mathfrak{A}ut(\mathfrak{X})$ and $\pi_1(\mathfrak{Y}) = \mathfrak{A}ut(\mathfrak{Y})$, and so we can extend the sequence (45) three steps further to the right, as required. $\square$

**Remark 4.21** Given a 2-commutative diagram of differential graded schemes (40), then the various long exact homotopy sequences (43) are natural enough to give rise to a commutative long exact braid with four strands, similar to (41), except that it has a right end.

## 4.4 Differentials

Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of differential graded schemes.

**Proposition 4.22** *There exists an object $\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}}$ in the derived category of $h^0(\mathcal{O}_{\mathfrak{X}})$, together with natural isomorphisms*

$$\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}} \mid \mathbb{S}\mathrm{pec}\, B \xrightarrow{\;\sim\;} \overline{\Omega}_{B/C} \,,$$

*for every resolving morphism of perfect resolving algebras $C \to B$ and every 2-commutative diagram (44) with étale $\mathbb{S}\mathrm{pec}\, B \to \mathfrak{X}$ and $\mathbb{S}\mathrm{pec}\, C \to \mathfrak{Y}$. Here $\overline{\Omega}_{B/C}$ is the complex of sheaves of (finitely generated, free) $h^0(\mathcal{O}_{\mathbb{S}\mathrm{pec}\, B})$-modules defined by*

$$(B \xrightarrow{x} A) \longmapsto \overline{\Omega}_{B/C}(x) = \Omega_{B/C} \otimes_B h^0(A) \,.$$

*The complex $\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}}$ is perfect.*

PROOF. Note that $\Omega_{B/C} \otimes_B h^0(A) = \Omega_{B/C} \otimes_B h^0(B) \otimes_{h^0(B)} h^0(A)$. Hence $\overline{\Omega}_{B/C}$ is a presheaf of complexes of $h^0(\mathcal{O}_{\mathbb{S}\mathrm{pec}\, B})$-modules. Because every one of the complexes of sections of $\overline{\Omega}_{B/C}$ is finitely generated and free, $\overline{\Omega}_{B/C}$ is, in fact, a complex of sheaves of $h^0(\mathcal{O}_{\mathbb{S}\mathrm{pec}\, B})$-modules. Thus we have constructed an

object $\overline{\Omega}_{B/C}$ in the derived category of $h^0(\mathcal{O}_{\mathbb{S}\text{pec}\,B})$-modules which are bounded above and have coherent cohomology.

This construction gives rise to a functor from the (1-category associated to the) category of all $C \to B$ and Diagrams (44), to the derived category $h^0(\mathcal{O}_{\mathfrak{X}})$-modules, and by cohomological descent, we get the required object $\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}}$. $\square$

**Definition 4.23** The complex $\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}}$ is called the **cotangent complex** or the **complex of differentials** of $\mathfrak{X}$ over $\mathfrak{Y}$. Its dual is denoted by $\overline{\Theta}_{\mathfrak{X}/\mathfrak{Y}}$ and is called the **tangent complex** of $\mathfrak{X}$ over $\mathfrak{Y}$.

**Proposition 4.24** *Given a 2-commutative diagram (40) of differential graded schemes, we get induced distinguished triangles*

$$\phi^{-1}\overline{\Omega}_{\mathfrak{Y}/\mathfrak{Z}} \longrightarrow \overline{\Omega}_{\mathfrak{X}/\mathfrak{Z}} \longrightarrow \overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}} \longrightarrow \phi^{-1}\overline{\Omega}_{\mathfrak{Y}/\mathfrak{Z}}[1]$$

*and*

$$\overline{\Theta}_{\mathfrak{X}/\mathfrak{Y}} \longrightarrow \overline{\Theta}_{\mathfrak{X}/\mathfrak{Z}} \longrightarrow \phi^{-1}\overline{\Theta}_{\mathfrak{Y}/\mathfrak{Z}} \longrightarrow \overline{\Theta}_{\mathfrak{X}/\mathfrak{Y}}[1]$$

*in the derived category of $h^0(\mathcal{O}_{\mathfrak{X}})$. These distinguished triangles are natural in the sense that they give rise to commutative 'octahedra'. Let us only display the octahedron for $\overline{\Theta}$:*



**Definition 4.25** If $\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}}$ has perfect amplitude contained in $[-N, 0]$, we say that $\phi : \mathfrak{X} \to \mathfrak{Y}$ has **amplitude** $N$. If $\phi : \mathfrak{X} \to \mathfrak{Y}$ has amplitude $N$, we write $N = \text{amp}(\mathfrak{X}/\mathfrak{Y})$.

Note that this definition of amplitude agrees with the earlier one for the absolute case, Definition 3.22.

**Corollary 4.26** *We have*

$$\text{amp}(\mathfrak{X}) = \max\big(\text{amp}(\mathfrak{Y}), \text{amp}(\mathfrak{X}/\mathfrak{Y})\big),$$
$$\text{amp}(\mathfrak{X}/\mathfrak{Y}) = \max\big(\text{amp}(\mathfrak{Y}) + 1, \text{amp}(\mathfrak{X})\big),$$
$$\text{amp}(\mathfrak{Y}) = \max\big(\text{amp}(\mathfrak{X}/\mathfrak{Y}) - 1, \text{amp}(\mathfrak{X})\big).$$

PROOF. This follows from Proposition 4.24, see also Remark I.3.4. Note that we also have relative versions of these statements, with respect to a composition $\mathfrak{X} \to \mathfrak{Y} \to \mathfrak{Z}$. $\square$

**Proposition 4.27 (Spectral sequence)** *There is a natural convergent third quadrant spectral sequence of coherent $h^0(\mathcal{O}_{\mathfrak{X}})$-modules*

$$E_2^{p,q} = h^q(\mathcal{O}_{\mathfrak{X}}) \otimes_{h^0(\mathcal{O}_{\mathfrak{X}})} h^p(\overline{\Theta}_{\mathfrak{X}/\mathfrak{Y}}) \Longrightarrow h^{p+q}(\Theta_{\mathfrak{X}/\mathfrak{Y}}).$$

*If $\mathfrak{X} \to \mathfrak{Y}$ has amplitude $N$, then all terms of this spectral sequence with $p + q > N$ vanish.*

PROOF. Glue the spectral sequences from Proposition I.3.5 together. $\square$

**Locally free morphisms**

is this the correct name?

**Definition 4.28** Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of differential graded schemes. Suppose there exist affine étale covers $\mathbb{S}\mathrm{pec}\, B_i \to \mathfrak{X}$ and $\mathbb{S}\mathrm{pec}\, C_i \to \mathfrak{Y}$, resolving morphisms $C_i \to B_i$ and 2-commutative diagrams

$$
\begin{array}{ccc}
\mathbb{S}\mathrm{pec}\, B_i & \longrightarrow & \mathbb{S}\mathrm{pec}\, C_i \\
\downarrow & \nearrow\!\!\!\nearrow & \downarrow \\
\mathfrak{X} & \longrightarrow & \mathfrak{Y}
\end{array}
$$

where for every $i$, there exists a basis $(x_\nu)_{\nu \in I_i}$ for $B_i$ over $C_i$, such that $dx_\nu \in C_i$, for all $\nu \in I_i$. In this case we call $\phi$ **locally free**.

Note that for a locally free morphism of differential graded schemes $\mathfrak{X} \to \mathfrak{Y}$, the cotangent complex $\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}}$ has locally free cohomology sheaves over $h^0(\mathcal{O}_{\mathfrak{X}})$, and hence is locally isomorphic (in the derived category of $h^0(\mathfrak{X})$) to a finite complex of finitely generated free modules with zero differential.

**Example** Every étale morphism if locally free.

**Proposition 4.29** *For every morphism $\mathfrak{X} \to \mathfrak{Y}$ of differential graded schemes there exists an étale cover $\mathfrak{X}_i \to \mathfrak{X}$, such that each composition $\mathfrak{X}_i \to \mathfrak{Y}$ factors into finitely many locally free morphisms.*

PROOF. Use Proposition 3.19. $\square$

**Proposition 4.30** *There are natural isomorphisms of sheaves of sets on $\mathfrak{X}$*

$$\Xi_\ell : h_\ell(\Theta_{\mathfrak{X}/\mathfrak{Y}}) \longrightarrow \pi_\ell(\mathfrak{X}/\mathfrak{Y}),$$

*for all $\ell > 0$. For $\ell \geq 2$, these are isomorphisms of sheaves of abelian groups. If $\phi$ is locally free, then $\Xi_1$ is also an isomorphism of sheaves of groups.*
*To make the naturality properties of $\Xi_\ell$ more precise, let*

$$
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{\ \phi\ } & \mathfrak{Y} \\
& \kappa\searrow \ \nearrow & \downarrow \psi \\
& \mathfrak{Z} &
\end{array}
$$

66

be a 2-commutative diagram of differential graded schemes. Then we have induced commutative diagrams

$$
\begin{array}{ccccc}
h_\ell(\Theta_{\mathfrak{X}/\mathfrak{Y}}) & \longrightarrow & h_\ell(\Theta_{\mathfrak{X}/\mathfrak{Z}}) & \longrightarrow & \phi^{-1}h_\ell(\Theta_{\mathfrak{Y}/\mathfrak{Z}}) \\
\Big\downarrow{\scriptstyle\Xi_\ell} & & \Big\downarrow{\scriptstyle\Xi_\ell} & & \Big\downarrow{\scriptstyle\Xi_\ell} \\
\pi_\ell(\mathfrak{X}/\mathfrak{Y}) & \longrightarrow & \pi_\ell(\mathfrak{X}/\mathfrak{Z}) & \longrightarrow & \phi^{-1}\pi_\ell(\mathfrak{Y}/\mathfrak{Z})
\end{array}
$$

If, moreover, $\phi$ is locally free, then we also have a commutative diagram

$$
\begin{array}{ccc}
\phi^{-1}h_\ell(\Theta_{\mathfrak{Y}/\mathfrak{Z}}) & \xrightarrow{\;\delta\;} & h_{\ell-1}(\Theta_{\mathfrak{X}/\mathfrak{Y}}) \\
\Big\downarrow{\scriptstyle\Xi_\ell} & & \Big\downarrow{\scriptstyle\Xi_{\ell-1}} \\
\phi^{-1}\pi_\ell(\mathfrak{Y}/\mathfrak{Z}) & \xrightarrow{\;\partial\;} & \pi_{\ell-1}(\mathfrak{X}/\mathfrak{Y})
\end{array}
$$

in other words, we get a homomorphism from the long exact sequence of higher relative tangent sheaves to the long exact sequence of relative homotopy sheaves, both truncated at the transition from $\ell = 1$ to $\ell = 0$.

PROOF. This follows from the results of Section I.4.2. □

## 4.5   The associated algebraic space

**Proposition 4.31** *There exists a natural 2-functor*

$$
h^0 : (\text{dg-schemes}) \longrightarrow (\text{algebraic spaces})
$$

*from the 2-category of differential graded schemes to the 1-category of algebraic spaces over $k$, which satisfies the following properties:*
   *(i) For every perfect resolving algebra $B$ we have*

$$
h^0(\operatorname{Spec} B) = \operatorname{Spec} h^0(B)\,,
$$

   *(ii) étale morphisms get mapped to étale morphisms,*
   *(iii) any 2-cartesian diagram in which the two vertical morphisms are étale is mapped to a 1-cartesian diagram (with two vertical étale morphisms),*
   *(iv) any affine étale cover gets mapped to an affine étale cover.*
   *Moreover, for every differential graded scheme $\mathfrak{X}$, the algebraic space $h^0(\mathfrak{X})$ is locally of finite type over $k$ and has affine diagonal.*

**Definition 4.32** We call $h^0(\mathfrak{X})$ the algebraic space **associated** to the differential graded scheme $\mathfrak{X}$, or the **truncation** of $\mathfrak{X}$.

Let $\mathfrak{X}$ be a differential graded scheme and $\overline{\mathfrak{X}}$ its associated 1-site. Let $X = h^0(\mathfrak{X})$ be the associated algebraic space, which we consider as a fibered category

over the category of finite type affine $k$-schemes. Thus $X$ has an induced étale topology. Then we can use Proposition 4.31 to construct a natural functor

$$h^0 : \overline{\mathfrak{X}} \longrightarrow X \,,$$

which fits into a commutative diagram of categories

$$
\begin{array}{ccc}
\overline{\mathfrak{X}} & \xrightarrow{\ \ h^0\ \ } & X \\
\downarrow & & \downarrow \\
\mathfrak{S} & \xrightarrow{\ h^0\ } & \text{(finite type affine } k\text{-schemes)}
\end{array}
$$

The functor $h^0 : \overline{\mathfrak{X}} \to X$ is a continuous functor of categories endowed with Grothendieck topologies. Thus sheaves on $X$ pull back via $h^0$ to sheaves on $\overline{\mathfrak{X}}$. Let us denote this pull back functor by $\iota_*$. It has a left adjoint $\iota^{-1}$, which extends $h^0$ from representable sheaves to all sheaves:

or is it cocontinuous?

$$
\text{(sheaves on } \overline{\mathfrak{X}}) \underset{\iota_*}{\overset{\iota^{-1}}{\rightleftarrows}} \text{(sheaves on } X)
$$

Thus we have a morphism of topoi

$$\iota : \text{(sheaves on } X) \longrightarrow \text{(sheaves on } \overline{\mathfrak{X}})$$

and a morphism of sites

$$\iota : X \longrightarrow \overline{\mathfrak{X}} \,.$$

look up morphism of sites

This morphism of sites should be thought of as a globalization and a dualization of the natural morphism of differential graded algebras $A \to h^0(A)$. But note that we cannot think of the algebraic space $X$ as a differential graded scheme, unless it has perfect cotangent complex. (See [2].)

Note that $\iota^{-1}h^0(\mathcal{O}_{\mathfrak{X}}) = \mathcal{O}_X$. Because $\iota^{-1}$ is exact, we can pull back the higher structure sheaves, higher tangent sheaves, and the tangent and cotangent complex from $\overline{\mathfrak{X}}$ to $X$, simply by applying $\iota^{-1}$. All the exactness properties are preserved under this operation. We will use the notation $\cdot \otimes \mathcal{O}_X$, to denote the functor $\iota^{-1}$. For example,

$$\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}} \otimes \mathcal{O}_X = \iota^{-1}\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}} \,.$$

**Proposition 4.33** *A morphism $\mathfrak{X} \to \mathfrak{Y}$ of differential graded schemes if étale, if and only if $\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}}$ is acyclic and if and only if $\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}} \otimes \mathcal{O}_X$ is acyclic.* $\square$

**Proposition 4.34** *There is a one-to-one correspondence between the open subschemes of a differential graded scheme $\mathfrak{X}$ and the open subspaces of the associated algebraic space $X$.* $\square$

Define a *closed point* of the differential graded scheme $\mathfrak{X}$ to be an equivalence class of morphisms $\mathbb{S}\mathrm{pec}\,K \to \mathfrak{X}$, where $K$ is a finite extension field of $k$. The equivalence relation is generated by considering $\mathbb{S}\mathrm{pec}\,K \to \mathfrak{X}$ and $\mathbb{S}\mathrm{pec}\,L \to \mathfrak{X}$ equivalent, if there exists a 2-commutative diagram

$$
\begin{array}{ccc}
\mathbb{S}\mathrm{pec}\,K & \longrightarrow & \mathbb{S}\mathrm{pec}\,L \\
& \searrow \quad \Uparrow & \downarrow \\
& & \mathfrak{X}
\end{array}
$$

Let $|\mathfrak{X}|$ denote the set of all closed points of $\mathfrak{X}$. It is a topological space by calling a subset open if it is the set of all closed points of an open subscheme of $\mathfrak{X}$. We call $|\mathfrak{X}|$ the *Zariski topological space* associated to $\mathfrak{X}$.

If $|X|$ is the set of closed points of the algebraic space $X$ associated to $\mathfrak{X}$, endowed with the Zariski topology, then there is a homeomorphism $|X| \to |\mathfrak{X}|$.

**Definition 4.35** For a morphism of differential graded schemes $f : \mathfrak{X} \to \mathfrak{Y}$, we call $\mathrm{rk}\,\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}}$ the **relative dimension** of $\mathfrak{X}$ over $\mathfrak{Y}$.

The relative dimension is a locally constant, integer-valued function on $|\mathfrak{X}|$.

**Proposition 4.36** *A morphism of differential graded schemes is an isomorphism, if and only if it is étale and induces an isomorphism on truncations.*

PROOF. This follows from Corollary I.2.9. $\square$

### Obstruction theory

**Proposition 4.37** *Let $\mathfrak{X} \to \mathfrak{Y}$ be a morphism of differential graded schemes and $X \to Y$ its truncation. Then we have a canonical morphism in the derived category of $\mathcal{O}_X$*

$$
\alpha : \overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}} \otimes \mathcal{O}_X \longrightarrow L_{X/Y}\,,
$$

*where $L_{X/Y}$ is the relative cotangent complex of the morphism of algebraic spaces $X \to Y$. The morphism $\alpha$ is a relative obstruction theory for $X$ over $Y$ in the sense of [4].* $\square$

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of differential graded schemes of amplitude 1 and $X \to Y$ its truncation. Let $d$ be the relative dimension of $\mathfrak{X}$ over $\mathfrak{Y}$. For every pullback diagram of algebraic spaces

$$
\begin{array}{ccc}
U & \longrightarrow & V \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

we get an induced relative obstruction theory $\overline{\Omega}_{\mathfrak{X}/\mathfrak{Y}} \otimes \mathcal{O}_U \to L_{U/V}$ for $U$ over $V$, which is perfect, in the terminology of [4].

If $V$ is a variety, then this perfect obstruction theory defines a virtual fundamental class $f^![V] \in A_*(U)$. As we let $V$ vary, we get a bivariant class $f^! \in A_d(X \to Y)$, or, in other words, an *orientation* of $X$ over $Y$.

**Proposition 4.38** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{Z}$ be morphisms of differential graded schemes, both of amplitude 1. Let $h : \mathfrak{X} \to \mathfrak{Z}$ be isomorphic to the composition $g \circ f$. Then $h$ is also of amplitude 1 and $h^! = g^! \cdot f^!$.*

PROOF. This is a generalization of Proposition 7.5 of [4] on compatible obstruction theories. The present result can be proved using the results of [8], which improve on [loc. cit.]. $\square$

In particular, any differential graded scheme $\mathfrak{X}$ of amplitude 1 has a virtual fundamental class $[\mathfrak{X}] \in A_d(\mathfrak{X})$, where $d = \dim \mathfrak{X}$. If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of amplitude 1 between differential graded schemes of amplitude 1, then we have $f^![\mathfrak{Y}] = [\mathfrak{X}]$.

# References

[1] K. Behrend. Differential graded schemes I: Perfect resolving algebras. Preprint.

[2] K. Behrend. Differential graded schemes III: Quotients in 2-categories. In preparation.

[3] K. Behrend. Differential graded schemes IV: Fibered products. In preparation.

[4] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[5] I. Ciocan-Fontanine and M. Kapranov. Derived Quot schemes. *Ann. Sci. École Norm. Sup. (4)*, 34(3):403–440, 2001.

[6] P. Goerss and J. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Prog. Math.* Birkhäuser, Boston, 1999.

[7] V. Hinich. Homological algebra of homotopy algebras. *Comm. Algebra*, 25(10):3291–3323, 1997.

[8] B. Kim, A. Kresch, and T. Pantev. Functoriality in intersection theory and a conjecture of Cox, Katz, and Lee. *J. Pure Appl. Algebra*, 2002. To appear.

[9] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics No. 5. Springer-Verlag, New York, second edition, 1998.

[10] B. Toen and G. Vezzosi. Algebraic geometry over model categories. arXiv:math.AG/0110109.