

# On the de Rham Cohomology of Differential and Algebraic Stacks

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November 30, 2004

## Abstract

We introduce the notion of *cofoliation* on a stack. A cofoliation is a change of the differentiable structure which amounts to giving a full representable smooth epimorphism. Cofoliations are uniquely determined by their associated *Lie algebroids*.

Cofoliations on stacks arise from *flat connections* on groupoids. Connections on groupoids generalize connections on gerbes and bundles in a natural way. A flat connection on a groupoid is an integrable distribution of the morphism space compatible with the groupoid structure and complementary to both source and target fibres. A cofoliation of a stack determines the flat groupoid up to étale equivalence.

We show how a cofoliation on a stack gives rise to a refinement of the Hodge to De Rham spectral sequence, where the  $E_1$ -term consists entirely of vector bundle valued cohomology groups.

Our theory works for differentiable, holomorphic and algebraic stacks.

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# 1 Introduction

This paper concerns the Hodge to de Rham spectral sequence for stacks. By ‘stack’ we mean ‘smooth Artin stack’ in the differentiable, i.e.,  $C^\infty$ -category, the holomorphic category or the algebraic category over a field of characteristic zero.

## Manifolds

Recall the Hodge to de Rham spectral sequence of a manifold  $X$ :

$$E_1^{p,q} = H^q(X, \Omega^p) \implies H_{DR}^{p+q}(X). \quad (1)$$

The  $E_1$ -term consists of the *Hodge cohomology groups*. These are the cohomology groups of the vector bundles  $\Omega^p$ , the exterior powers of the cotangent bundle  $\Omega_X$ . The abutment is the de Rham cohomology of  $X$ .

In the differentiable case, the  $E_1$ -term of the spectral sequence (1) is just the de Rham complex. The same is true in the holomorphic case if  $X$  is Stein, or in the algebraic case if  $X$  is affine.

The existence of the spectral sequence (1) is almost trivial: by definition, the de Rham cohomology  $H_{DR}^n(X) = \mathbb{H}^n(X, (\Omega^\bullet, d))$  is equal to the hypercohomology of the de Rham complex, and the spectral sequence can be obtained for example from the double complex one gets if one calculates hypercohomology using Čech cochains.

Less trivial is the fact that  $H_{DR}^n(X)$  is equal to the ‘topological’ cohomology  $H^n(X)$  of  $X$ . In the differentiable and holomorphic case this is the (differentiable or holomorphic) Poincaré lemma. In the algebraic case, it is a theorem of Grothendieck that the algebraic de Rham cohomology  $H_{DR}^n(X, \mathbb{C}) = H^n(X^{\text{an}}, \mathbb{C})$ , the ‘topological’ cohomology of the associated analytic manifold.

Viewed in this light, the significance of (1) is that it expresses the topological invariants  $H^n(X)$  in terms of the ‘coherent invariants  $H^q(X, \Omega^p)$ . By ‘coherent’, we mean that the  $H^q(X, \Omega^p)$  are cohomology groups of coherent  $\mathcal{O}_X$ -modules (in fact vector bundles). Of course, only the *objects* on the  $E_1$ -level are coherent, the differential comes from non-coherent data, by which we simply mean that that the de Rham differential is not linear over functions (sections of  $\mathcal{O}_X$ ).

## Stacks

Any stack  $X$  admits groupoid presentations  $X_1 \rightrightarrows X_0$ . To any such groupoid presentation is associated a simplicial manifold  $X_\bullet$ . In good cases, the de Rham cohomology of  $X$  is defined as the total cohomology of the double complex  $\Gamma(X_\bullet, \Omega^\bullet)$ , with Čech and de Rham differentials (see, for example, [1]).

Alternatively, one may observe that there is a de Rham complex of big sheaves  $(\Omega_{\text{big}}^\bullet, d)$  on the big site  $X_{\text{big}}$  of the stack  $X$  and define  $H_{DR}^n(X)$  as the hypercohomology of the big site of  $X$  with values in this big de Rham complex. (Note: the big site is defined as the category of all manifolds  $U \rightarrow X$ , with

smooth structure map  $U \rightarrow X$ . No smoothness assumption on maps between various  $U/X$  is made. We use the étale topology on  $X_{\text{big}}$ .)

Either way, one obtains an  $E_1$ -spectral sequence

$$E_1^{p,q} = H^q(X_{\text{big}}, \Omega_{\text{big}}^p) \implies H_{DR}^{p+q}(X), \quad (2)$$

abutting to the de Rham cohomology. It is also not difficult to reduce the proof that  $H_{DR}(X)$  is equal to the ‘topological’ cohomology of  $X$  (i.e., the cohomology of the big topological site associated to  $X$ ) to the manifold case (see [ibid.]).

At first glance, the  $E_1$ -term of (2) does not look very coherent, as it involves big sheaves, which are by no means to be considered coherent (much less vector bundles).

One way to define the difference between big and small  $\mathcal{O}_X$ -modules is as follows: If  $\mathcal{F}$  is a sheaf of  $\mathcal{O}_X$ -modules on the big site of a stack  $X$ , then we get an induced sheaf of  $\mathcal{O}_{X_n}$ -modules  $\mathcal{F}_n$  on every manifold  $X_n$  partaking in the simplicial manifold associated to a groupoid presentation of  $X$ . Moreover, we get compatibility maps  $\pi^* \mathcal{F}_m \rightarrow \mathcal{F}_n$ , for every structure morphism  $\pi : X_n \rightarrow X_m$ . The sheaf  $\mathcal{F}$  is *coherent*, if every  $\mathcal{F}_n$  is a coherent  $\mathcal{O}_{X_n}$ -module and all  $\pi^* \mathcal{F}_m \rightarrow \mathcal{F}_n$  are isomorphisms. For example, the big sheaf  $\Omega_{\text{big}}$  induces  $\Omega_{X_n}$  on  $X_n$ . Hence  $\Omega_{\text{big}}$  is not coherent, since only the first of the two coherence conditions is satisfied. We also define a vector bundle on a stack to be a coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  all of whose components  $\mathcal{F}_n$  are vector bundles. Note that vector bundles have a well-defined rank, something that a big sheaf such as  $\Omega_{\text{big}}$  lacks.

The first indication that (2) is not as bad as it looks, is that for Deligne-Mumford stacks (in particular manifolds) we have  $H^q(X_{\text{big}}, \Omega_{\text{big}}^p) = H^q(X, \Omega^p)$ , and so (2) agrees with (1).

### The cotangent complex

The natural coherent analogue of the cotangent bundle of a manifold is the cotangent complex  $L_X$  of the stack  $X$ , which is an object of the derived category of  $\mathcal{O}_X$ -modules. Because of our smoothness assumption on all stacks, we can define it as the homomorphism of big sheaves  $L_X = [\Omega_{\text{big}} \rightarrow \Omega_{\text{big}/X}]$ . Here,  $\Omega_{\text{big}/X}$  is the big sheaf which induces on a smooth  $X$ -manifold  $U$  the sheaf  $\Omega_{U/X}$ .

Note that  $L_X$  is perfect, of perfect amplitude contained in  $[0, 1]$ . This means that  $L_X$  is locally quasi-isomorphic to the complex given by a homomorphism of vector bundles. In fact, if  $X_1 \rightrightarrows X_0$  is a groupoid presenting the stack  $X$ , then  $L_X|_{X_0} = [\Omega_{X_0} \rightarrow \Omega_{X_0/X}]$ . (Proof: for any  $U \rightarrow X_0$  we have  $[\Omega_{X_0}|_U \rightarrow \Omega_{X_0/X}|_U] \xrightarrow{\text{qis}} [\Omega_U \rightarrow \Omega_{U/X}]$ .) So once restricted to a presentation, the homomorphism of big sheaves ‘collapses’ to a homomorphism of vector bundles. (Note that  $\Omega_{X_0/X}$  can be identified with the conormal bundle of the identity section  $X_0 \rightarrow X_1$ , so that  $L_X|_{X_0}$  is the dual of the Lie algebroid of the groupoid  $X_1 \rightrightarrows X_0$ .)

The cotangent complex is certainly ‘coherent’ data, in the sense that  $L_X$  is an object of the derived category of  $\mathcal{O}_X$ -modules with coherent cohomology.

It is a remarkable fact, maybe first noticed by Teleman (see [7], for remarks on the equivariant case), that the natural homomorphism  $L_X \rightarrow \Omega_{\text{big}}$ , as well as its exterior powers, induce isomorphisms on cohomology:

$$\mathbb{H}^q(X, \Lambda^p L_X) = H^q(X, \Omega_{\text{big}}^p), \quad (3)$$

for all  $q, p$ . Thus, we can rewrite the above spectral sequence (2) as

$$E_1^{p,q} = \mathbb{H}^q(X, \Lambda^p L_X) \implies H_{DR}^{p+q}(X). \quad (4)$$

This looks much more like sequence (1) for manifolds. Moreover, the objects on the  $E_1$ -level have now been shown to be coherent data and the abutment is isomorphic to topological cohomology. We call (4) the Hodge to de Rham spectral sequence.

### This paper

So far the general story. We now ask ourselves if we can still ‘improve’ upon (4), i.e., if we might be able to construct a spectral sequence whose  $E_1$ -term actually consists of cohomology groups of  $X$  with values in vector bundles, as opposed to just hypercohomology groups with values in perfect complexes.

We are motivated by the case of a quotient stack  $X = [X_0/G]$ , for which the cotangent complex is (quasi-isomorphic to) a homomorphism of vector bundles, globally. In fact, consider the equivariant morphism of equivariant vector bundles  $\Omega_{X_0} \rightarrow \mathfrak{g}^\vee$  on  $X_0$ . Here,  $\mathfrak{g}^\vee$  is the trivial vector bundle associated to the dual of the Lie algebra of  $G$  with the adjoint representation,  $\Omega_{X_0}$  is the cotangent bundle of  $X_0$  with the  $G$ -action induced by the action of  $G$  on  $X_0$  and  $\Omega_{X_0} \rightarrow \mathfrak{g}^\vee$  is obtained by differentiating the various orbit maps  $G \rightarrow X_0$ . Because  $[\Omega_{X_0} \rightarrow \mathfrak{g}^\vee]$  is  $G$ -equivariant, it descends to a homomorphism of vector bundles  $[\Omega \rightarrow \Upsilon]$  on the quotient stack  $X$  and, in fact, the complex  $[\Omega \rightarrow \Upsilon]$  is quasi-isomorphic to the cotangent complex  $L_{[X/G]}$ . In other words, there is a distinguished triangle

$$L_X \longrightarrow \Omega \longrightarrow \Upsilon \longrightarrow L_X[1] \quad (5)$$

in the derived category of  $\mathcal{O}_X$ -modules. (Note that pulling back (5) via  $\pi : X_0 \rightarrow X$  gives the functoriality triangle  $\pi^* L_X \rightarrow L_{X_0} \rightarrow L_{X_0/X} \rightarrow \pi^* L_X[1]$ .)

Since  $\Lambda^p L_X = [\Omega^p \rightarrow \dots \rightarrow S^p \Upsilon]$ , one may ask if there exists an  $E_1$ -spectral sequence whose  $E_1$ -term consists of the various  $H^n(X, \Omega^{p-k} \otimes S^k \Upsilon)$  and which abuts to  $H_{DR}(X)$ :

$$E_1^{m,n} = \bigoplus_{p+k=m} H^n(X, \Omega^{p-k} \otimes S^k \Upsilon) \implies H_{DR}^{m+n}(X). \quad (6)$$

In the equivariant case, such a spectral sequence has, indeed, been constructed by Getzler [5]. Rewriting (6) in equivariant language it reads:

$$E_1^{m,n} = \bigoplus_{p+k=m} H_G^n(X_0, \Omega_{X_0}^{p-k} \otimes S^k \mathfrak{g}^\vee) \implies H_{DR,G}^{m+n}(X). \quad (7)$$

For example, if  $G$  is compact (or reductive, in the algebraic setting), then all higher  $H_G^n$  vanish and we obtain a double complex

$$\bigoplus_{p,k} \Gamma(X_0, \Omega_{X_0}^{p-k} \otimes S^k \mathfrak{g}^\vee)^G$$

computing the equivariant de Rham cohomology of  $X$ . This is, of course, nothing but the Cartan model.

In this article, we examine to what extent a spectral sequence (6) exists for general stacks. Note that (6) is also a generalization of (1). But, in contrast with (4), its  $E_1$ -term is computed in terms of vector bundles.

Moreover, one should think of (6) as a refinement of (4). In fact (whenever we can construct (6) at all), there exists an  $E_1$ -spectral sequence from (6) to (4). For fixed  $p$  it reduces on the  $E_1$ -level to the usual  $E_1$ -spectral sequence

$$E_1^{k,n} = H^n(X, \Omega^{p-k} \otimes S^k \Upsilon) \implies \mathbb{H}^{k+n}(X, \Lambda^p L_X) \quad (8)$$

of hypercohomology.

### Flat connections on groupoids

Let us describe what additional structure we need to have on our stack  $X$ , for us to construct the spectral sequence (6). Of course, we need a global resolution of the cotangent complex by vector bundles  $L_X = [\Omega \rightarrow \Upsilon]$ . This global resolution  $[\Omega \rightarrow \Upsilon]$  needs to be endowed with an extra structure, which we shall describe in the dual picture.

Giving a global resolution  $L_X = [\Omega \rightarrow \Upsilon]$  of the cotangent complex by vector bundles is equivalent to giving a surjective linear map  $a : E \rightarrow T_X$  from the vector bundle  $E = \Omega^\vee$  to the tangent stack  $T_X$  of  $X$ . We define  $N = \ker(E \rightarrow T_X)$ , which can also be constructed as the fibred product

$$\begin{array}{ccc} N & \longrightarrow & X \\ \phi \downarrow & \square & \downarrow 0 \\ E & \xrightarrow{a} & T_X. \end{array} \quad (9)$$

It turns out that  $N = \Upsilon^\vee$ .

We define a *pre-realization* of  $a : E \rightarrow T_X$  to be a presentation  $\pi : X_0 \rightarrow X$  of the stack  $X$ , together with a vector bundle map  $p : T_{X_0} \rightarrow E$  covering  $\pi$ , such that  $a \circ p$  is the canonical morphism  $T_{X_0} \rightarrow T_X$  (i.e., there is a given 2-isomorphism, etc.), and such that the rectangle marked with a box in

$$\begin{array}{ccccc} T_{X_0} & \xrightarrow{p} & E & \xrightarrow{a} & T_X \\ & \searrow & \square & \swarrow & \swarrow \\ & & X_0 & \xrightarrow{\pi} & X \end{array}$$

is cartesian, i.e., a pullback diagram. Such pre-realizations always exist.

It is easy to see that if  $X_1 \rightrightarrows X_0$  denotes the groupoid induced by the presentation  $X_0 \rightarrow X$  (which means that  $X_1 = X_0 \times_X X_0$ ), we get an induced subbundle  $E_1 \subset T_{X_1}$  defined by  $E_1 = T_{X_0} \times_E T_{X_0}$ . (Use the fact that  $T_{X_1} = T_{X_0} \times_{T_X} T_{X_0}$ .) Note also, that  $E_1 \rightrightarrows T_{X_0}$  is a subgroupoid of  $T_{X_1} \rightrightarrows T_{X_0}$ , and that  $E_1$  is complementary to both the source and target fibres of  $X_1 \rightrightarrows X_0$ .

We call such a distribution  $E_1 \subset T_{X_1}$  a **connection** on the groupoid  $X_1 \rightrightarrows X_0$ . The key question is whether or not the distribution  $E_1$  is integrable: if we can find a pre-realization of  $a : E \rightarrow T_X$  for which the associated distribution  $E_1 \subset T_{X_1}$  is integrable, we can construct the spectral sequence (6), at least in good cases, for example in the algebraic case if the diagonal of  $X$  is affine.

If the connection  $E_1 \subset T_{X_1}$  on a groupoid  $X_1 \rightrightarrows X_0$  is integrable as a distribution on  $X_1$ , we call the connection *flat*. A groupoid endowed with a flat connection is called a *flat groupoid*. The terminology is justified by the relation to (flat) connections on vector bundles and on gerbes.

Flat connections on groupoids were independently discovered by Tang [6].

## Cofoliations

In general, we have no means of comparing the spectral sequences (6) coming from different pre-realizations of  $a : E \rightarrow T_X$ . Thus we have to put additional structure on  $a : E \rightarrow T_X$ , to make the spectral sequence well-defined. We note that if the distribution  $E_1 \subset T_{X_1}$  is integrable, it induces a structure on  $a : E \rightarrow T_X$  comparable to that of *Lie algebroid*. If these induced Lie algebroid structures on  $a : E \rightarrow T_X$  are the same, two different pre-realizations give the same spectral sequence (6).

Thus we are led to study the Lie algebroid structures on  $a : E \rightarrow T_X$ , induced by flat groupoids pre-realizing  $a : E \rightarrow T_X$ . For lack of a better word<sup>1</sup>, we call these structures **cofoliations** on  $X$ . The spectral sequence (6) is thus an invariant of the cofoliation on  $X$ .

A cofoliation is analogous to a foliation, in that we may think of a cofoliation on  $X$  as changing the differentiable (or holomorphic or algebraic) structure on  $X$ . Except that the dimension goes up, instead of down. For example, let  $G$  be a Lie group and  $\tilde{G}$  the same group but with discrete differentiable structure. The quotient  $[G/\tilde{G}]$  is a cofoliation of the point. It gives the point a differentiable structure of dimension  $\dim G$ .

In general, for a stack  $X$ , a cofoliation  $\tilde{X} \rightarrow X$  is a change of differentiable structure making  $\tilde{X}$  a Deligne-Mumford stack and  $\tilde{X} \rightarrow X$  a full smooth representable epimorphism. (Analogously, a foliation of a manifold is a surjective immersion.)

Of course, this interpretation depends on accepting stacks with horrible diagonal as geometric. If one does not want to do this, nothing is lost: this is just a way of thinking. We do not actually use stacks such as  $[G/\tilde{G}]$  in this article.

<sup>1</sup>we are open to alternative suggestions!

## The de Rham complex

The spectral sequence (6) is constructed as follows: we start with a flat groupoid  $X_1 \rightrightarrows X_0$  representing and realizing the stack  $X$  with its cofoliation. We obtain, among other things, a global resolution  $\phi : \Omega \rightarrow \Upsilon$  of  $L_X$ . We have

$$\Lambda^p(\Omega \rightarrow \Upsilon) = \left( \bigoplus_{k=0}^p \Omega^{p-k} \otimes S^k \Upsilon, \phi \right).$$

Then we take the Čech complexes of  $\bigoplus_{p+k=m} \Omega^{p-k} \otimes S^k \Upsilon$ , for all  $m$ , associated to the covering  $X_0 \rightarrow X$ . Notation:

$$K^{p,k,n} = \Gamma(X_n, \Omega^{p-k} \otimes S^k \Upsilon).$$

We have the Čech differential  $\partial$  on  $K$ , raising  $n$  by 1. The  $\mathcal{O}_X$ -linear map  $\phi$  gives rise to a differential on  $K$  increasing  $k$  by 1. We also construct, using flat connections on the  $\Upsilon|_{X_n}$ , a differential  $d$  on  $K$ , which increases  $p$  by 1. Hence, the two differentials  $\phi$  and  $d$  both increase  $m$  by 1. We almost have, for every Čech degree  $n$ , a double complex  $(K^{\bullet,\bullet,n}, \phi + d)$ , except for the fact that  $\phi$  and  $d$  do not commute! To remedy the situation, we construct a homotopy operator  $\iota$  between the two compositions

$$(K^{p,k,\bullet}, \partial) \begin{array}{c} \xrightarrow{\phi \circ d} \\ \Downarrow \iota \\ \xrightarrow{d \circ \phi} \end{array} (K^{p+1,k+1,\bullet}, \partial).$$

Thus, after passing to Čech cohomology, the two differentials  $\phi$  and  $d$  commute, and we get, indeed, for every  $n$ , a double complex

$$(H^n(X, \Omega^\bullet \otimes S^\bullet \Upsilon), \phi + d).$$

The associated total complexes form the  $E_1$ -term of the spectral sequence (6).

In fact,  $\iota$  is a fourth differential on  $K^{\bullet,\bullet,\bullet}$ , and  $(\phi + \partial + d + \iota)$  is a differential. The cohomology of the total complex  $(K, \phi + \partial + d + \iota)$  is the de Rham cohomology of the stack  $X$  (thus the abutment of (6).) The complex  $(K, \phi + \partial + d + \iota)$  might be of independent interest. Even though it computes the same cohomology as the double complex  $\Gamma(X_\bullet, \Omega^\bullet)$  mentioned above, it is in some sense much smaller: it consists entirely of Čech cochains with values in vector bundles (not big sheaves).

## Acknowledgements

I would like to thank Ezra Getzler for useful discussions and for referring me to his paper [5], which inspired this research.

I would also like to thank Constantin Teleman for alerting me to Formula (3) and comments on an earlier version of this paper.

## 1.1 Notation

### General setup

We will be working in any of the following three categories: the  $C^\infty$ -category (also called the differentiable category), the holomorphic category, or the algebraic category, by which we mean the category of finite type smooth schemes over a field of characteristic zero. (The characteristic zero assumption is important, because we occasionally refer to the Lefschetz principle to reduce proofs to the holomorphic case.) We use the word ‘manifold’, to denote an object in any of these categories. A ‘smooth map’ is a morphism which is pointwise surjective on tangent spaces, an ‘immersion’ is a morphism which is pointwise injective on tangent spaces. An étale map is a smooth immersion. (In the differential and holomorphic case, an étale map is a local isomorphism.)

Let  $\mathbb{K}$  denote the real numbers if we are working in the  $C^\infty$ -category, the complex numbers if we are in the holomorphic category, or the ground field of characteristic zero if we are in the algebraic category.

A manifold  $X$  has a structure sheaf  $\mathcal{O}_X$ . (In the  $C^\infty$ -category this is the sheaf of  $\mathbb{R}$ -valued differentiable functions.) The tangent bundle of a manifold  $X$  is denoted by  $T_X$ , the cotangent bundle and its exterior powers by  $\Omega_X^p$ . All vector bundles are identified with their locally free sheaves of  $\mathcal{O}_X$ -modules.

### Groupoids and associated simplicial manifolds

A **groupoid** is a groupoid  $X_1 \rightrightarrows X_0$  in our underlying category whose source and target maps  $s, t : X_1 \rightarrow X_0$  are smooth. The identity section of a groupoid is usually denoted by  $\iota : X_0 \rightarrow X_1$ . If  $s$  and  $t$  are étale, we call the groupoid  $X_1 \rightrightarrows X_0$  *étale*. Every groupoid induces a ‘tangent groupoid’  $T_{X_1} \rightrightarrows T_{X_0}$ .

If  $\phi, \psi \in X_1$ , such that  $t(\phi) = s(\psi)$ , we write the composition as  $\phi * \psi$ . We also use notation  $p_1(\phi, \psi) = \phi$  and  $p_2(\phi, \psi) = \psi$ , as well as  $m(\phi, \psi) = \phi * \psi$ . Right multiplication by  $\psi$  is an isomorphism

$$R_\psi : t^{-1}(s(\psi)) \longrightarrow t^{-1}(t(\psi)),$$

with inverse  $R_{\psi^{-1}}$ . Similarly, left multiplication by  $\phi$  is an isomorphism

$$L_\phi : s^{-1}(t(\phi)) \longrightarrow s^{-1}(s(\phi)),$$

with inverse  $L_{\phi^{-1}}$ .

To any groupoid  $X_1 \rightrightarrows X_0$  is associated a simplicial manifold  $X_\bullet$ . We identify  $X_n$  with the set of composable arrows

$$x_0 \xrightarrow{\psi_1} \dots \xrightarrow{\psi_n} x_n$$

in  $X_1 \rightrightarrows X_0$ . For the structure maps of  $X_\bullet$  we use the following notation. For  $q = 0, \dots, n+1$  let

$$\hat{\pi}_q : X_{n+1} \longrightarrow X_n$$



be the projection leaving out the  $q$ -th object. In other words,

$$\begin{aligned} \hat{\pi}_q : (x_0 \xrightarrow{\psi_1} \dots \xrightarrow{\psi_{n+1}} x_{n+1}) &\longmapsto \\ &\longmapsto (x_0 \xrightarrow{\psi_1} \dots \rightarrow x_{q-1} \xrightarrow{\psi_q * \psi_{q+1}} x_{q+1} \rightarrow \dots \xrightarrow{\psi_{n+1}} x_{n+1}), \end{aligned}$$

with obvious modifications for  $q = 0, n + 1$ . For  $q = 0, \dots, n - 1$  we let

$$\iota_q : X_{n-1} \longrightarrow X_n$$

be the diagonal repeating the  $q$ -th object. In other words,

$$\iota_q : (x_0 \xrightarrow{\psi_1} \dots \xrightarrow{\psi_{n-1}} x_{n-1}) \longmapsto (x_0 \xrightarrow{\psi_1} \dots \rightarrow x_q \xrightarrow{\text{id}} x_q \rightarrow \dots \xrightarrow{\psi_{n-1}} x_{n-1}).$$

We also denote by  $\pi_q : X_n \rightarrow X_0$  the projection onto the  $q$ -th object:

$$\pi_q : (x_0 \xrightarrow{\psi_1} \dots \xrightarrow{\psi_n} x_n) \longmapsto x_q,$$

$q = 0, \dots, n$ . Finally, we have maps  $\pi_{qr} : X_n \rightarrow X_1$ , for  $0 \leq q < r \leq n$ , given by

$$\pi_{qr} : (x_0 \xrightarrow{\psi_1} \dots \xrightarrow{\psi_n} x_n) \longmapsto (x_q \xrightarrow{\psi_{q+1} * \dots * \psi_r} x_r),$$

## Stacks

In every one of our three categories we have a notion of stack. We will use the word *stack* for any stack over our base category associated to a groupoid  $X_1 \rightrightarrows X_0$  (which means that it is isomorphic to the stack of torsors under  $X_1 \rightrightarrows X_0$ ). Since we are only working with smooth objects  $X_0, X_1$ , all stacks are smooth. The stack associated to the tangent groupoid  $T_{X_1} \rightrightarrows T_{X_0}$  is called the *tangent stack*, denoted  $T_X$ .

If the stack  $X$  is given by the groupoid  $X_1 \rightrightarrows X_0$ , a *vector bundle* over  $X$  is any stack over  $X$ , induced by a groupoid  $E_1 \rightrightarrows E_0$  of vector bundles  $E_i$  over  $X_i$ , where both diagrams

$$\begin{array}{ccc} E_1 & \longrightarrow & E_0 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_0 \end{array}$$

are cartesian, i.e., pullback diagrams. Thus  $T_X$  is *not* a vector bundle over  $X$ , unless  $X$  is Deligne-Mumford, which means we can find an étale groupoid presenting  $X$ .

## The Lie algebroid of a groupoid

Let us fix our conventions concerning the Lie algebroid of a groupoid  $X_1 \rightrightarrows X_0$ . We denote by  $N$  the normal bundle of  $\iota : X_0 \rightarrow X_1$ . The tangent bundle  $T_{X_0}$ , as well as  $\iota^*T_t$  and  $\iota^*T_s$ , are subbundles of  $\iota^*T_{X_1}$ . Here  $T_t$  and  $T_s$  are the relative

tangent bundles of the maps  $s, t : X_1 \rightarrow X_0$ . In fact, we have two direct sum decompositions

$$\iota^*T_{X_1} = T_{X_0} \oplus \iota^*T_t \quad \text{and} \quad \iota^*T_{X_1} = T_{X_0} \oplus \iota^*T_s.$$

Any complement of  $T_{X_0}$  is automatically isomorphic to  $N$ . Thus there are two ways of thinking of  $N$  as a subbundle of  $\iota^*T_{X_1}$ . There are also two projections from  $\iota^*T_{X_1}$  onto  $T_{X_0}$ , one with kernel  $\iota^*T_t$  (denoted  $\eta_t$ ) and one with kernel  $\iota^*T_s$  (denoted  $\eta_s$ ). Composing, we get four maps  $N \rightarrow T_{X_0}$ , two of which vanish. Unfortunately, the other two are often not equal, but, rather, add up to zero. Thus we have to make up our mind, which of these two maps we declare to be the *canonical* map  $\phi : N \rightarrow T_{X_0}$ . (This map is also called the ‘anchor map’.) We make our choice as follows:

*We identify  $N$  with  $\iota^*T_t$  using the composition  $\iota^*T_t \rightarrow \iota^*T_{X_1} \rightarrow N$ ,*

and then apply  $\eta_s$ . This makes  $\phi$  essentially equal to  $\eta_s$ . We will also need to declare a ‘canonical’ isomorphism  $\iota^*T_s \xrightarrow{\sim} N$ . We choose the *negative* of the composition  $\iota^*T_s \rightarrow T_{X_1} \rightarrow N$ . Thus we have declared two canonical injections  $N \rightarrow \iota^*T_{X_1}$ . We denote the inclusion, with image  $\iota^*T_t$ , by  $\rho_s : N \rightarrow \iota^*T_{X_1}$ , and the other injection by  $\rho_t$ . The way we made our choices, we have  $\eta_t \circ \rho_t = \eta_s \circ \rho_s = \phi$  (and  $\eta_t \circ \rho_s = \eta_s \circ \rho_t = 0$ ).

Now that we have chosen  $\phi : N \rightarrow T_{X_0}$ , we can think of  $N$  as the relative tangent bundle of the map  $X_0 \rightarrow X$ , where  $X$  is the stack associated to  $X_1 \rightrightarrows X_0$ . To justify this, let us define identifications  $s^*N \xrightarrow{\sim} T_t$  and  $t^*N \xrightarrow{\sim} T_s$ . Fix a point  $\psi \in X_1$ . We have  $s^*N|_\psi = N|_{s(\psi)} = T_t|_{s(\psi)}$ . Then we compose with the derivative of right multiplication by  $\psi$ :

$$DR_\psi|_{s(\psi)} : T_t|_{s(\psi)} \longrightarrow T_t|_\psi$$

to obtain the identification  $s^*N = T_t$ . Similarly, we have  $t^*N|_\psi = N|_{t(\psi)} \cong T_s|_{t(\psi)}$ . Now composing with the derivative of left multiplication by  $\psi$ :

$$DL_\psi|_{t(\psi)} : T_s|_{t(\psi)} \longrightarrow T_s|_\psi,$$

we obtain the identification  $t^*N = T_s$ . It is important to notice that the two compositions

$$p_1^*t^*N \xrightarrow{\sim} p_1^*T_s \xrightarrow{\sim} T_m \quad \text{and} \quad p_2^*s^*N \xrightarrow{\sim} p_2^*T_t \xrightarrow{\sim} T_m$$

are equal. This implies that we have canonical identifications  $\pi_q^*N \cong T_{\pi_q}$ , for all  $q = 0, \dots, n$  on every  $X_n$ .

## 2 Flat Connections on Groupoids

### 2.1 Definition

Recall that an integrable distribution on a manifold  $X$  is a subbundle  $E \subset T_X$  of the tangent bundle such that  $E^\perp \subset \Omega_X$  generates an ideal in  $\bigoplus_p \Omega_X^p$  which

is preserved by the exterior derivative. Equivalently, the sheaf  $E$  is closed under the Lie bracket inside  $T_X$ .

**Definition 2.1** A **connection** on the groupoid  $X_1 \rightrightarrows X_0$  is a subbundle  $E \subset T_{X_1}$ , such that  $E \rightrightarrows T_{X_0}$  is a subgroupoid of  $T_{X_1} \rightrightarrows T_{X_0}$  and both diagrams

$$\begin{array}{ccc} E & \rightrightarrows & T_{X_0} \\ \downarrow & & \downarrow \\ X_1 & \rightrightarrows & X_0 \end{array}$$

are cartesian.

The connection  $E$  is **flat** or **integrable**, if  $E \subset T_{X_1}$  is an integrable distribution.

**Definition 2.2** A groupoid endowed with an integrable connection, will be called a **flat groupoid**.

**Remark** Flat connections on groupoids are called *étalifications* in [6].

**Remark 2.3** In the differential and holomorphic categories, the integrable distribution  $E$  admits integral submanifolds. Denote by  $\tilde{X}_1$  the union of the leaves of this foliation. Note that  $\tilde{X}_1$  has the same set of points as  $X_1$ , but a different differentiable (holomorphic) and topological structure. The canonical map  $\tilde{X}_1 \rightarrow X_1$  is a bijective immersion. The conditions on the flat connection  $E$  are equivalent to saying that  $\tilde{X}_1 \rightrightarrows X_0$  is an étale groupoid and that  $\tilde{X}_1 \rightarrow X_1$  is a morphism of groupoids.

We shall denote the groupoid  $E \rightrightarrows T_{X_0}$  also by  $E_1 \rightrightarrows E_0$ . As usual, we get a simplicial manifold  $E_\bullet$ , which is, in fact, a simplicial submanifold of  $T_{X_\bullet}$ . Note that  $E_\bullet$  is a vector bundle over  $X_\bullet$  (which  $T_{X_\bullet}$  is *not*). If  $E$  is a flat connection, the subbundle  $E_n \subset T_{X_n}$  is an integrable distribution on  $X_n$ , for all  $n \geq 0$ .

Let  $N_0 = N_{X_0/X_1}$  be the normal bundle to the identity of  $X_1 \rightrightarrows X_0$ . We always identify  $N_0$  with  $T_{X_0/X}$ , the relative tangent bundle of  $X_0$  over the stack  $X$  defined by the groupoid  $X_\bullet$ . The relative tangent bundle of  $\hat{\pi}_q : X_n \rightarrow X_{n-1}$ , is canonically identified with  $\pi_q^* N_0$ . Thus, for all  $q = 0, \dots, n$ , we have a canonical subbundle  $\pi_q^* N_0 \subset T_{X_n}$  (which is an integrable distribution).

If  $E$  is a connection, then for every  $q = 0, \dots, n$ , we have a direct sum decomposition

$$T_{X_n} = E_n \oplus \bigoplus_{r \neq q} \pi_r^* N_0. \quad (10)$$

In particular,  $T_{X_1} = E \oplus s^* N_0 = E \oplus t^* N_0$ . These splittings give rise to projections  $\omega : T_{X_1} \rightarrow s^* N_0$  and  $\tilde{\omega} : T_{X_1} \rightarrow t^* N_0$  and an isomorphism  $t^* N_0 \rightarrow s^* N_0$  compatible with the projections from  $T_{X_1}$ . Let us call  $\omega \in \Gamma(X_1, \Omega_{X_1} \otimes s^* N_0)$  the **differential form** of the connection  $E$ . Of course,  $E$  can be recovered from  $\omega$  as its kernel. Note that the isomorphism  $t^* N_0 \rightarrow s^* N_0$  is nothing but the restriction of  $\omega$ .

**Lemma 2.4** *The differential form  $\omega : T_{X_1} \rightarrow s^*N$  of a connection  $E$  on  $X_1 \rightrightarrows X_0$  satisfies:*

- (i)  $\omega|_{s^*N} = \text{id}_{s^*N}$ ,
- (ii)  $\iota^*\omega : T_{X_0} \rightarrow N$  satisfies  $\iota^*\omega = 0$ ,
- (iii) as homomorphisms  $T_{X_2} \rightarrow \pi_0^*N$ , we have  $\mu^*\omega = p_1^*\omega + \chi_{01}(p_1^*\omega)$ .

Using the decomposition (10) for  $q = q_1$  and  $q = q_2$ , where  $q_1 < q_2$ , gives us an isomorphism  $\pi_{q_2}^*N_0 \rightarrow \pi_{q_1}^*N_0$ , whose *negative* we shall denote by  $\chi_{q_1, q_2}$ . Thus we have an anti-commutative diagram

$$\begin{array}{ccc} T_{X_n} & \longrightarrow & \pi_{q_2}^*N_0 \\ & \searrow & \downarrow \chi_{q_1, q_2} \\ & & \pi_{q_1}^*N_0. \end{array}$$

Note that we may identify  $\chi_{q_1, q_2}$  with  $-\pi_{q_1, q_2}^*\omega$ .

The canonical projection  $T_{X_n} \rightarrow \pi_q^*T_{X_0}$  induces an isomorphism  $E_n \rightarrow \pi_q^*T_{X_0}$ , and hence, by composition, isomorphisms  $\psi_{q_1, q_2} : \pi_{q_2}^*T_{X_0} \rightarrow \pi_{q_1}^*T_{X_0}$ , for any  $q_1, q_2$ . Denoting the canonical homomorphism  $N_0 \rightarrow T_{X_0}$  by  $\phi$ , the diagrams

$$\begin{array}{ccc} \pi_{q_2}^*N_0 & \xrightarrow{\pi_{q_2}^*\phi} & \pi_{q_2}^*T_{X_0} \\ \chi_{q_1, q_2} \downarrow & & \downarrow \psi_{q_1, q_2} \\ \pi_{q_1}^*N_0 & \xrightarrow{\pi_{q_1}^*\phi} & \pi_{q_1}^*T_{X_0} \end{array}$$

commute. In other words, we have defined *descent data* for the homomorphism of vector bundles  $\phi : N_0 \rightarrow T_{X_0}$ . Let us denote the induced homomorphism of vector bundles on  $X$ , the stack associated to  $X_1 \rightrightarrows X_0$ , by  $\phi : N \rightarrow E$ . We call  $N \rightarrow E$  the **representative of the tangent complex** given by our connection. Let us denote the pullback of  $\phi : N \rightarrow E$  to  $X_n$  by  $\phi : N_n \rightarrow E_n$ .

**Remark** Another way to construct  $\phi : N \rightarrow E$  is as follows: the groupoid morphism  $E_\bullet \rightarrow T_{X_\bullet}$  gives rise to a linear epimorphism of  $X$ -stacks  $E \rightarrow T_X$ . Let  $N$  be the kernel, constructed as in Diagram (9). Then we have

$$T_X = [E/N],$$

the quotient (over  $X$ ) of the vector bundle  $E$  by the action of the vector bundle  $N$  by addition via  $\phi$ .

Duality between the tangent stack and the cotangent complex implies that we have constructed a distinguished triangle of complexes of  $\mathcal{O}_X$ -modules

$$L_X \longrightarrow E^\vee \longrightarrow N^\vee \longrightarrow L_X[1].$$

We will use notation  $\Omega = E^\vee$  and  $\Upsilon = N^\vee$ .

**Remark** A connection on  $X_\bullet$  is the same thing as descent data for the distinguished triangle

$$\pi^*L_X \longrightarrow L_{X_0} \longrightarrow L_{X_0/X} \longrightarrow \pi^*L_X[1]$$

inducing the trivial descent data on  $\pi^*L_X$ .

### More notation

Now, for every  $q$ , we have an identification  $N_n = \pi_q^*N_0$ , and hence  $n+1$  canonical ways of thinking of  $N_n$  as a subbundle of  $T_{X_n}$ . We denote these various embeddings by  $\rho_q : N_n \hookrightarrow T_{X_n}$ .

For every choice of  $0 \leq q < r \leq n$ , we also get a canonical way of making  $N_n$  into a quotient of  $T_{X_n}$ . In fact, take

$$T_{X_n} \rightarrow \pi_{qr}^*T_{X_1} \xrightarrow{\pi_{qr}^*\omega} \pi_q^*N = N_n.$$

Let us denote this quotient map by  $\omega_{qr} : T_{X_n} \rightarrow N_n$ . We have that  $\omega_{qr}(E_n) = \omega_{qr}(\pi_j^*N) = 0$ , for all  $j \neq q, r$ . Moreover,  $\omega_{qr}(\pi_q^*N) = \text{id}$ , and  $\omega_{qr}(\pi_r^*N) = -\text{id}$ . For  $q > r$ , let us define  $\omega_{qr} = -\omega_{rq}$ , and for  $q = r$ , set  $\omega_{qr} = 0$ .

We also have, via the various identifications  $E_n = \pi_q^*T_{X_0}$ , for every  $q$  a way of considering  $E_n$  as a quotient of  $T_{X_n}$ . Let us denote the corresponding quotient map by  $\eta_q : T_{X_n} \rightarrow E_n$ .

Note that for every  $q, r = 0, \dots, n$  the diagram

$$\begin{array}{ccc} N_n & \xrightarrow{\rho_q} & T_{X_n} \\ & \searrow \delta_{qr}\phi & \downarrow \eta_r \\ & & E_n \end{array} \quad (11)$$

commutes, where  $\delta_{qr}$  is the delta of Kronecker.

For future reference, let us also remark the commutativity of the diagram

$$\begin{array}{ccc} N_n & \xrightarrow{\phi} & E_n \\ & \searrow \sum_{q=0}^n \rho_q & \downarrow \\ & & T_{X_n} \end{array} \quad (12)$$

For the convenience of the reader, let us summarize the various maps we constructed:

Without a connection, we have canonical maps

$$\pi_q^*N \xrightarrow{\rho_q} T_{X_n} \xrightarrow{\eta_r} \pi_r^*T_{X_0},$$

but we use the connection to identify all  $\pi_q^*N$  with  $N_n$  and all  $\pi_r^*T_{X_0}$  with  $E_n$ . by adding the connection, we get maps

$$E_n \xrightarrow{\delta} T_{X_n} \xrightarrow{\omega_{qr}} N_n.$$

### A few functorial properties

**Definition 2.5** Let  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  be a morphism of groupoids. Suppose  $X_\bullet$  and  $Y_\bullet$  are endowed with connections  $E \subset T_{X_1}$  and  $F \subset T_{Y_1}$ . The morphism  $f$  is called **horizontal**, if the derivative  $Df_1 : T_{X_1} \rightarrow T_{Y_1}$  maps  $E$  into  $F$ .

**Remark** Given two horizontal morphisms  $f, g : X_\bullet \rightarrow Y_\bullet$ , there is an obvious notion of *horizontal natural transformation* from  $f$  to  $g$ .

A morphism of groupoids  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  is called *étale*, if both  $f_0 : X_0 \rightarrow Y_0$  and  $f_1 : X_1 \rightarrow Y_1$  are étale. Suppose  $X_\bullet$  and  $Y_\bullet$  are endowed with connections  $E$  and  $F$ , respectively. If  $f$  is étale and horizontal, then the diagram

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ T_{X_1} & \longrightarrow & T_{Y_1} \end{array} \quad (13)$$

is cartesian.

Conversely, given an étale morphism  $f : X_\bullet \rightarrow Y_\bullet$  of groupoids, any connection  $F$  on  $Y_\bullet$  induces a unique connection  $E$  on  $X_\bullet$  making  $f$  horizontal: define  $E$  as the fibred product (13). We write  $E = f^*F$ . If  $F$  is integrable, then so is  $E$ .

A morphism of groupoids  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  is called *cartesian*, if both diagrams

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \Downarrow & & \Downarrow \\ X_0 & \xrightarrow{f_0} & Y_0 \end{array}$$

are cartesian. If  $F$  is a connection on  $Y_\bullet$  and  $f$  is cartesian, we get an induced connection  $E$  on  $X_\bullet$  by setting  $E = F \times_{T_{Y_1}} T_{X_1}$ . If  $F$  is integrable, so is  $E$ .

## 2.2 The derived connection

A connection  $E \subset T_{X_1}$  on a groupoid  $X_1 \rightrightarrows X_0$  induces a connection  $\nabla$  on the vector bundle  $N_0$ , as follows. We define, for  $v \in \Gamma(X_0, T_{X_0})$  and  $\nu \in \Gamma(X_0, N_0)$  the covariant derivative of  $\nu$  with respect to  $v$  by

$$\nabla_v(\nu) = \iota^* \omega[\rho_s(s^* \nu), s^*(v)],$$

where  $[\cdot, \cdot]$  denotes the Lie bracket of vector fields on  $X_1$ . We identify  $T_{X_0}$  with  $E_0$ , so that  $s^*(v)$ , which is a section of  $E_1$ , is canonically a section of  $T_{X_1}$ . The map  $\rho_s$  is the inclusion map  $s^*N_0 \hookrightarrow T_{X_1}$ . Note that the ‘mirror’ formula

$$\tilde{\nabla}_v(\nu) = \iota^* \tilde{\omega}[\rho_t(t^* \nu), t^*(v)]$$

gives rise to the same connection,  $\tilde{\nabla} = \nabla$ . We call  $\nabla$  the **derived connection** associated to the connection  $E$ . If  $E$  is flat,  $\nabla$  is integrable.

**Remark 2.6** Note the following: given a smooth map of manifolds  $f : X \rightarrow Y$ , a distribution  $E \hookrightarrow T_X$ , such that  $E \xrightarrow{\sim} f^*T_Y$ , and a section  $\iota : Y \rightarrow X$  whose image is a leaf (i.e., parallel to  $E$ ), we get an induced connection on  $\iota^*T_{X/Y} = N_{Y/X}$ , where we think of  $Y$  as a submanifold of  $X$  via  $\iota$ .

Applying this principle to  $s : X_1 \rightarrow X_0$  with the identity section, we get a connection on  $N_0$ . We get the same connection applying this principle to  $t : X_1 \rightarrow X_0$ . This gives us a more geometric way of defining the derived connection  $\nabla$ .

**Remark 2.7** For every groupoid  $X_1 \rightrightarrows X_0$  we get an associated *derived* groupoid  $N \rightrightarrows X_0$ , which is simply the normal bundle to the identity section considered as a family of Lie groups. This process of deriving a groupoid is functorial and commutes with passing to tangent groupoids. Thus, if we derive the diagram

$$E \cdot \longrightarrow T_{X \cdot} \longrightarrow X \cdot$$

of groupoids, we obtain the sequence

$$\begin{array}{ccccc} N_{E_0/E_1} & \longrightarrow & T_N & \longrightarrow & N \\ \downarrow & & \downarrow & & \downarrow \\ E_0 & \longrightarrow & T_{X_0} & \longrightarrow & X_0 \end{array}$$

of vector bundles. In other words, we have a *derived* connection on the derived groupoid  $N \rightarrow X_0$ . A groupoid connection on a vector bundle considered as a groupoid is the same thing as a vector bundle connection in the traditional sense (see below). Thus, this process of deriving a groupoid induces a connection on the vector bundle  $N \rightarrow X_0$ . This connection is the derived connection defined above.

Pulling back to  $X_1$ , we get connections  $s^*\nabla$  on  $s^*N_0$  and  $t^*\nabla$  on  $t^*N_0$ . Note that, in general, the isomorphism  $\chi : t^*N_0 \rightarrow s^*N_0$  is not horizontal. Thus, the connection  $\nabla$  on  $N_0$  does not descend to a connection on  $N$  over the stack  $X$ .

Using  $\chi$  to identify  $s^*N_0$  with  $t^*N_0$ , we get two different flat connections on  $N_1$ , denoted  $s^*\nabla$  and  $t^*\nabla$ . Their difference  $\Psi_1 = s^*\nabla - t^*\nabla$  is a vector bundle homomorphism  $\Psi_1 : T_{X_1} \otimes N_1 \rightarrow N_1$ .

**Lemma 2.8** *For the same reason that  $\tilde{\nabla} = \nabla$ , we have that  $\Psi_1$  vanishes on  $E \otimes N_1$ : the two connections  $s^*\nabla$  and  $t^*\nabla$  agree on  $E \otimes N_1$ .  $\square$*

**Remark** The quotient  $T_{X_1}/E$  being canonically isomorphic to  $N_1$ , we see that  $\Psi_1$  induces a vector bundle homomorphism  $\Psi_1 : N_1 \otimes N_1 \rightarrow N_1$ . Restricting back (via  $\iota$ ) to  $X_0$ , we get a vector bundle homomorphism  $\Psi_0 : N_0 \otimes N_0 \rightarrow N_0$ . This pairing on  $N_0$  agrees with the Lie algebroid pairing on  $N_0$  if we restrict to covariantly constant sections of  $N_0$ .

**Remark 2.9** Passing to  $X_n$ , we get  $n + 1$  different connections on  $N_n$ , each one given by one of the identifications  $N_n = \pi_q^* N_0$ . In other words, all  $n + 1$  subbundles  $\eta_q : N_n \hookrightarrow T_{X_n}$  are endowed in a canonical way with an integrable connection. Let us denote the connection  $\pi_q^* \nabla$  on  $N_n$  by  $\nabla^q$ . Then, for  $0 \leq q < r \leq n$ , the difference  $\nabla^q - \nabla^r$  is equal to the composition

$$N \xrightarrow{\pi_{qr}^* \Psi_1} \pi_{qr}^* N^\vee \otimes N \xrightarrow{\omega_{qr}} \Omega_{X_n} \otimes N,$$

or, more succinctly,  $\nabla^q - \nabla^r = \omega_{qr}(\Psi)$ , which now holds for all  $q, r$ . The most important consequence of these considerations for us is that  $\nabla^q - \nabla^r$  takes values in  $E_n^\perp \otimes N$ , for all  $q, r$ .

**Remark** Note that  $s^* \Psi_0 = t^* \Psi_0 = \Psi_1$ , so that  $\Psi_0$  descends to  $X$ : there is a homomorphism of vector bundles  $N \otimes N \rightarrow N$  on  $X$ , making  $N$  a bundle of Lie algebras on the stack  $X$ . See also Remark 3.11.

## 2.3 Examples

**Example 2.10** An étale groupoid has a unique connection. It is integrable.

**Example 2.11** Let  $G$  be a Lie group, considered as a groupoid  $G \rightrightarrows *$ . There is a unique connection on this groupoid, namely  $0 \hookrightarrow T_G$ . It is integrable. Of course,  $N_0 = \mathfrak{g}$ , the Lie algebra of  $G$ . The derived connection is the unique (trivial) connection on the vector space  $\mathfrak{g}$ . The descent datum  $\chi : \mathfrak{g}_X \rightarrow \mathfrak{g}_X$  is the adjoint representation. It is not locally constant, unless  $G$  is abelian or discrete (or more generally, has an abelian connected component). This provides examples where the derived connection does not descend to the stack.

**Example 2.12** If  $X_1 \rightrightarrows X_0$  is a transformation groupoid  $G \times X \rightrightarrows X$ , letting  $E = T_{G \times X / G}$  defines the *canonical* flat connection.

**Example 2.13** For a manifold  $X$ , a connection on the groupoid  $X \times X \rightrightarrows X$  is the same thing as a trivialization of the tangent bundle  $T_X$  of  $X$ . (N.B. By a trivialization we mean trivialization up to choice of basis: more precisely, descent data of  $T_X$  to the point.)

The connection is flat if and only if the Lie bracket is constant. This means that there exists a Lie algebra  $\mathfrak{g}$  and an identification  $T_X = \mathfrak{g}_X$  such that for *constant* sections of  $T_X$  the Lie bracket as vector fields coincides with the Lie bracket as elements of  $\mathfrak{g}$ .

**Example 2.14** More generally, if  $f : X \rightarrow Y$  is a surjective submersion, and  $X_1 \rightrightarrows X_0$  is the associate *banal* groupoid given by  $X_0 = X$  and  $X_1 = X \times_Y X$ , a connection on  $X_1 \rightrightarrows X_0$  is the same thing as a triple  $(E, \epsilon, \psi)$ , where  $\epsilon : E \rightarrow T_Y$  is an epimorphism of vector bundles on  $Y$  and  $\psi : T_X \rightarrow E$  is a homomorphism of vector bundles covering  $f$ , identifying  $T_X$  with  $f^* E$  and such that  $\epsilon \circ \psi = Df$ .



Less formally, we may say that a connection is descent data for  $T_X$ , compatible with the map to  $f^*T_Y$ .

We will see later (Lemma 3.5) what flatness means in this context.

**Example 2.15** If  $F \rightarrow X$  is a vector bundle over a manifold  $X$  and we consider the groupoid  $X_1 \rightrightarrows X_0$  given by  $X_0 = X$  and  $X_1 = F$ , with the groupoid structure given by vector addition, a groupoid connection on  $X_1 \rightrightarrows X_0$  is the same thing as a vector bundle connection on  $F$ . Most directly, this can be seen by using the characterization of vector bundle connections in terms of horizontal distributions on the total space of  $F$ . Also, we may remark that  $F = N_{X_0/X_1}$  and the derived connection determines the groupoid connection. Integrability of the groupoid connection is equivalent to integrability of the vector bundle connection.

**Example 2.16** More generally, if  $X_1 \rightrightarrows X_0$  is a family of groups  $G \rightarrow X$ , a connection identifies infinitesimally close fibres of  $G \rightarrow X$  with each other as group schemes. For example, for a family of elliptic curves (in the holomorphic or algebraic context), this means that the  $j$ -invariant of the family is locally constant. In this example, the absence of infinitesimal automorphisms of elliptic curves implies that every connection is flat.

### Relation to connections on gerbes

Let  $X_1 \rightrightarrows X_0$  be an étale groupoid and  $R_1 \rightrightarrows X_0$  a groupoid with the same base space endowed with a morphism of groupoids  $R_\bullet \rightarrow X_\bullet$ . Assume that  $R_1 \rightarrow X_1$  is a surjective submersion. (This data induces a morphism of stacks  $\mathfrak{G} \rightarrow X$ , where  $\mathfrak{G}$  is a *gerbe* over the Deligne-Mumford stack  $X$ .) The kernel of  $R_1 \rightarrow X_1$  is a family of groups  $\mathcal{G} \rightarrow X_0$  over  $X_0$ . Any connection on  $R_\bullet$  induces one on  $\mathcal{G} \rightrightarrows X_0$ . (Note that  $X_1 \rightrightarrows X_0$  being étale, the diagonal  $X_0 \rightarrow X_1$  is an open immersion, hence the same is true for  $\mathcal{G} \subset X_1$ . The restriction of the connection from  $X_1$  to  $\mathcal{G}$  is a special case pulling back via étale morphisms, discussed above.)

Suppose that we have given an identification of  $\mathcal{G} \rightarrow X_0$  with a constant family of groups  $G \times X_0 \rightarrow X_0$ , where  $G$  is a Lie group, and that  $G \times X_0$  is central in  $R_1$ . In this case, the gerbe  $\mathfrak{G} \rightarrow X$  is a  $G$ -gerbe and  $R_\bullet \rightarrow X_\bullet$  is a  $G$ -central extension of groupoids. Note that  $R_1 \rightarrow X_1$  is now a principal  $G$ -bundle, the  $G$ -action being induced from the groupoid multiplication. Moreover, the product family  $G \times X_0 \rightrightarrows X_0$  has a canonical flat connection on it.

Assuming given a connection  $E$  on  $R_\bullet$ , which restricts to the canonical one on  $G \times X_0 \subset R_1$ , the compatibility of  $E$  with the groupoid multiplication implies the invariance of  $E$  under the  $G$ -structure. Thus,  $E$  is a bundle connection on the  $G$ -bundle  $R_1 \rightarrow X_1$ . Let  $\theta$  be the connection form (which may be identified with the form  $\omega$  of the groupoid connection). One checks that the Čech coboundary of  $\theta$  vanishes. Thus we have defined a *connective structure* on the groupoid central extension  $R_\bullet \rightarrow X_\bullet$ , and hence on the  $G$ -gerbe  $\mathfrak{G}$ . (See [2],

where the case of  $G = S^1$  is treated. See also the classic reference for connections on gerbes [4].)

If the groupoid connection  $E$  is flat, then the connective structure is flat (more precisely, we may take 0 as curving and then the curvature vanishes). Thus a flat connection on the groupoid  $R_\bullet$ , extending the canonical one on  $G \times X_0 \subset R_1$ , induces a flat connection on the  $G$ -gerbe  $\mathfrak{G}$ .

Conversely, given a  $G$ -gerbe  $\mathfrak{G} \rightarrow X$  over a Deligne-Mumford stack, we can find a groupoid central extension  $R_\bullet \rightarrow X_\bullet$  with kernel  $G \times X_0$ , inducing  $\mathfrak{G} \rightarrow X$ . We do this by choosing a local trivialization of  $\mathfrak{G}$  over an étale cover  $X_0 \rightarrow X$ . A connective structure on the  $G$ -gerbe  $\mathfrak{G}$  then defines a connection on the groupoid  $R_\bullet$ , which restricts to the canonical one on the kernel, if the local trivialization  $X_0 \rightarrow \mathfrak{G}$  giving rise to the central extension is compatible with the connective structure.

If  $\mathfrak{G}$  is endowed with a flat connection, then, at least in the differentiable and holomorphic context, we can locally trivialize  $\mathfrak{G}$  as a flat gerbe. Then the induced connection on the induced groupoid central extension  $R_\bullet \rightarrow X_\bullet$  will be flat. (If we use a local trivialization which respects only the connective structure on the flat gerbe to write  $\mathfrak{G}$  as a  $G$ -central extension, we get a connection on  $R_\bullet$  which is not flat. But we will get a covariantly closed 2-form  $B$  with values in  $N$  on  $X_0$ , whose Čech coboundary is equal to the curvature of the bundle connection on  $R_1$ . The form  $B$  represents the incompatibility of the local trivialization with the *curving*).

**Remark 2.17** It is conceivable, that, in general, there exists a notion of a *trivialization of the curvature* for a connection on a groupoid. This would be a 2-form  $B \in \Omega_{X_0}^2 \otimes N$ , such that  $t^*B - s^*B$  is the curvature of the distribution  $E_1 \subset T_{X_1}$ . (Recall that the curvature is a tensor  $\Lambda^2 E_1 \rightarrow T_{X_1}/E_1 = N_1$ .) Then the existence of a covariantly closed  $B$  might serve as a generalization or substitution of flatness of groupoid connections.

### 3 Cofoliations

Let us make a few remarks about Lie algebroids. Note that if  $a$  is surjective, its kernel is a vector bundle, and in fact a bundle of Lie algebras (i.e., the bracket is linear over functions).

**Definition 3.1** Let  $f : V \rightarrow U$  be a morphism of manifolds. Suppose  $F \rightarrow T_V$  and  $E \rightarrow T_U$  are Lie algebroids. We call a morphism  $\phi : F \rightarrow E$  a **homomorphism** of Lie algebroids covering  $f$ , if

- (i)  $\phi$  is a homomorphism of vector bundles covering  $f$ ,
- (ii)  $\phi$  commutes with the anchor maps,
- (iii) the brackets are compatible, which means that if  $s_0 \in \Gamma(U, E)$  and  $t_0 \in \Gamma(V, F)$  form a compatible pair of sections, i.e.,  $f^*(s_0) = \phi_*(t_0)$ , and  $s_1, t_1$  form another such pair, then  $[s_0, s_1]_E$  and  $[t_0, t_1]_F$  are compatible in the same way.

### 3.1 Cofoliations on Manifolds

Over a manifold  $X$ , an *integrable distribution* is essentially the same thing as a Lie algebroid  $a : E \rightarrow T_X$ , whose anchor map  $a$  is a subbundle. We call an integrable distribution a *foliation*, if there exist submanifolds integrating  $E \rightarrow T_X$ . Of course, in the differentiable or holomorphic context, every integrable distribution is a foliation, by the Frobenius integrability theorem. But in the algebraic category, there is a difference.

If, instead, the anchor of a Lie algebroid is surjective (as a homomorphism of vector bundles) we suggest calling the induced structure on  $X$  a *cofoliation*, at least if the analogues of integral submanifolds exist.

**Definition 3.2** Let  $a : E \rightarrow T_X$  be a Lie algebroid on the manifold  $X$ . A **local realization** of  $E$  is given by the following data:

- (i) a manifold  $U$ ,
- (ii) a smooth map  $\pi : U \rightarrow X$ ,
- (iii) a vector bundle homomorphism  $p : T_U \rightarrow E$  covering  $\pi$ .

This data is subject to the constraints:

- (iv) the diagram

$$\begin{array}{ccc}
 T_U & \xrightarrow{p} & E \\
 \downarrow & \square & \downarrow \\
 U & \xrightarrow{\pi} & X
 \end{array} \tag{14}$$

is cartesian,

- (v) the composition  $T_U \rightarrow E \rightarrow T_X$  is the derivative of  $\pi : U \rightarrow X$ ,

(vi) the map  $E \rightarrow \pi_* T_U$  induced by the cartesian diagram (14), respects brackets (in other words,  $p : T_U \rightarrow E$  is a homomorphism of Lie algebroids covering  $\pi : U \rightarrow X$ ).

A local realization is a **realization** (or *global realization*, if there is fear of confusion) if the structure map  $\pi : U \rightarrow X$  is surjective.

Note that the existence of global realizations implies surjectivity of the anchor map.

**Remark** If we were to ask that  $\pi$  be an immersion, instead of a submersion, we would get essentially the definition of integral submanifold.

**Definition 3.3** A **cofoliation** on the manifold  $X$  is given by a Lie algebroid  $a : E \rightarrow T_X$  with surjective anchor, which admits a global realization. An isomorphic Lie algebroid is considered to define the same cofoliation.

**Lemma 3.4** Let  $a : E \rightarrow T_X$  be a Lie algebroid and let  $(V, \rho, r)$  and  $(U, \pi, p)$  be two local realizations. Form the fibred products  $Z = V \times_X U$  and  $F = T_V \times_E T_U$ . Note that  $F \subset T_Z$  is a subbundle, as  $T_Z = T_V \times_{T_X} T_U$ .

Then the distribution  $F$  on  $Z$  is integrable.

PROOF. Use notation as in the diagram

$$\begin{array}{ccc}
 Z & \xrightarrow{s} & U \\
 q \downarrow & \searrow f & \downarrow \pi \\
 V & \xrightarrow{\rho} & X
 \end{array}$$

The fact that  $F \subset T_Z$  is closed under Lie bracket, can be checked locally, on a basis of  $F$ . Such a local basis may be pulled back from a local basis for  $E$  on  $X$ , as  $F = f^*E$ . Thus, it is sufficient to prove that  $f_*F \subset f_*T_Z$  is closed under the Lie bracket.

For the following, it is helpful to note that the commutative diagram of sheaves on  $X$

$$\begin{array}{ccc}
 f_*T_Z & \longrightarrow & f_*s^*T_U \\
 \downarrow & \square & \downarrow \\
 f_*q^*T_V & \longrightarrow & f_*f^*T_X
 \end{array} \tag{15}$$

is cartesian. Moreover, the diagram

$$\begin{array}{ccc}
 f_*F & \longrightarrow & f_*s^*T_U \\
 \downarrow & & \downarrow \\
 f_*q^*T_V & \longrightarrow & f_*f^*E
 \end{array} \tag{16}$$

is a commutative diagram of isomorphisms. There is a morphism from (16) to (15).

Now let  $e_1, e_2$  be two sections of  $E$  over  $X$ . We need to show that  $[f^*e_1, f^*e_2] \in f_*T_Z$  is contained in the subsheaf  $f_*F$ . For this, it suffices to prove that  $[f^*e_1, f^*e_2]$  maps to the same section of  $f_*f^*E$  under the two maps  $f_*T_Z \rightarrow f_*s^*T_U \rightarrow f_*f^*E$  and  $f_*T_Z \rightarrow f_*q^*T_V \rightarrow f_*f^*E$ . In fact, we claim that the image of  $[f^*e_1, f^*e_2]$  in  $f_*f^*E$  is equal to  $f^*[e_1, e_2]$ , under either of these maps.

Let us prove this for the composition  $f_*T_Z \rightarrow f_*s^*T_U \rightarrow f_*f^*E$ . Note that we have a commutative diagram

$$\begin{array}{ccccc}
 f_*T_Z & \longrightarrow & f_*s^*T_U & \longleftarrow & \pi_*T_U \\
 & & \downarrow & & \uparrow \\
 & & f_*f^*E & \longleftarrow & E
 \end{array}$$

Now under  $E \rightarrow \pi_*T_U$  the section  $[e_1, e_2]$  maps to  $[\pi^*e_1, \pi^*e_2]$ , because  $T_U \rightarrow E$  is a morphism of Lie algebroids. Now we note that  $[f^*e_1, f^*e_2]$  and  $[\pi^*e_1, \pi^*e_2]$  have same image in  $f_*s^*T_U$ , because  $T_Z \rightarrow T_U$  is a morphism of Lie algebroids, and we are done.  $\square$

**Lemma 3.5** *Let  $a : E \rightarrow T_X$  be a morphism of vector bundles. Let  $(U, \pi, p)$  be a triple satisfying items (i), (ii),  $\dots$ , (v) of Definition 3.2. Suppose that  $\pi$  is surjective. Form the banal groupoids  $U_1 = U \times_X U$  and  $E_1 = T_U \times_E T_U$ . Note that  $E_1$  is a connection on  $U_1 \rightrightarrows U$ .*

*The connection  $E_1$  is flat if and only if the subsheaf  $E \rightarrow \pi_* T_U$  is closed under the Lie bracket. If this is the case,  $E \rightarrow T_X$  is a Lie algebroid and  $(U, \pi, p)$  a global realization.*

PROOF. If  $E \rightarrow \pi_* T_U$  is closed under the Lie bracket,  $E \rightarrow T_X$  inherits the structure of a Lie algebroid and  $(U, \pi, p)$  is a realization. From Lemma 3.4 it follows that  $E_1$  is integrable. For the converse, note that the diagram

$$\begin{array}{ccc} f_* f^* E & \xleftarrow{s^*} & \pi_* T_U \\ q^* \uparrow & & \uparrow \\ \pi_* T_U & \xleftarrow{\quad} & E \end{array}$$

is cartesian, as  $\pi$  is surjective. (Here we have borrowed the notation  $s$  and  $q$  from the proof of Lemma 3.4.) Thus, to prove that  $E$  is closed under the Lie bracket inside  $\pi_* T_U$ , it is enough to prove that for sections  $e_1, e_2$  of  $E$  the bracket  $[\pi^* e_1, \pi^* e_2]$  maps to the same section of  $f_* f^* E$  under  $s^*$  and  $q^*$ . This is proved by reversing the argument of Lemma 3.4.  $\square$

This Lemma explains what flatness means for connections on banal groupoids, c.f. Example 2.14.

We also see that a flat connection on a banal groupoid  $U_1 \rightrightarrows U_0$  induces a cofoliation on the quotient space  $X$ . A cofoliation induces a flat connection on any banal groupoid coming from a local realization.

### Comparing realizations

Unlike the leaves of a foliation, realizations of cofoliations are not unique. Passing to an étale cover of  $U$ , we get another. We will now develop the substitute for uniqueness.

**Definition 3.6** Let  $(V, \rho, r)$  and  $(U, \pi, p)$  be two local realizations for the Lie algebroid  $a : E \rightarrow T_X$ . A **morphism** of realizations is an étale  $X$ -map  $f : V \rightarrow U$ , such that the induced diagram

$$\begin{array}{ccc} T_V & \xrightarrow{Df} & T_U \\ r \searrow & & \swarrow p \\ & E & \end{array}$$

commutes.

The local realizations of a fixed Lie algebroid  $E \rightarrow T_X$  form thus a category. In the differentiable and holomorphic context, this category is connected:

**Lemma 3.7** *Assume we are in the differentiable or holomorphic category. Let  $a : E \rightarrow T_X$  be a Lie algebroid. If  $U \rightarrow X$  and  $V \rightarrow X$  are global realizations of  $E$ , there exists a third global realization  $W \rightarrow X$  and morphisms of realizations  $W \rightarrow U$  and  $W \rightarrow V$ .*

PROOF. Simply form  $Z$  as in Lemma 3.4 and take  $W$  to be the union of enough local leaves of the foliation  $F$  to make  $W \rightarrow X$  surjective.  $\square$

**Remark** In fact, the category of local realizations behaves a lot like the category of local étale covers of a Deligne-Mumford stack by manifolds: it admits fibred products and disjoint unions. Any two objects have a product. (For the product, one has to accept the disjoint union of the leaves of the foliation  $F$  on  $Z$  as a manifold.)

**Remark** One can also take the groupoid with flat connection  $U_1 \rightrightarrows U_0$  induced by any global realization  $U_0 \rightarrow X$  of  $E$ , pass to the modified differentiable structure on  $U_1$ , denoted  $\tilde{U}_1$ , and take the associated ‘differentiable space’  $\tilde{X} = [U_0/\tilde{U}_1]$ . If one is willing to accept  $\tilde{X}$  as some kind of generalized manifold, it will serve as canonical universal realization. We could even think of  $\tilde{X}$  as  $X$  with a modified differentiable structure. This is in analogy to foliations, where the union of the leaves may be thought of as a different differentiable structure. But note that the dimension of  $\tilde{X}$  is equal to rank  $E$ , thus *larger* than the dimension of  $X$ .

### Pulling back cofoliations

Let  $a : E \rightarrow T_X$  be a cofoliation on the manifold  $X$ . Let  $Y \rightarrow X$  be a smooth map. Define  $E_{T_Y} = E \times_{T_X} T_Y$ . Note that  $E_{T_Y}$  is a vector bundle on  $Y$ , which comes with an epimorphism of vector bundles  $a_Y : E_{T_Y} \rightarrow T_Y$ . We will define on  $E_{T_Y}$  a Lie algebroid structure making it a cofoliation on  $Y$ .

For this, choose a global realization  $U \rightarrow X$  of  $E$ . Consider the fibred product  $V = U \times_X Y$ . We have a canonical morphism  $T_V \rightarrow E_{T_Y}$ , which satisfies Properties (i),(ii),...,(v) of Definition 3.2, as can be seen by contemplating the diagram

$$\begin{array}{ccccc}
 T_V & \longrightarrow & E_{T_Y} & \longrightarrow & T_Y \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & & V & \longrightarrow & Y \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 T_U & \longrightarrow & E & \longrightarrow & T_X \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & & U & \longrightarrow & X
 \end{array}$$

in which the parallelograms with horizontal edges are cartesian.

Let  $E_1$  be the flat connection induced on the banal groupoid  $U_1 = U \times_X U$ . Letting  $V_1 = V \times_Y V$ , we have a smooth map  $V_1 \rightarrow U_1$  and we let  $F_1$  be the pullback of the integrable distribution  $E_1$  via  $V_1 \rightarrow U_1$ , in other words,  $F_1 = E_1 \times_{T_{U_1}} T_{V_1}$ . Thus  $F_1$  is again an integrable distribution and, in fact, a flat connection on the banal groupoid  $[V_1 \rightrightarrows V]$ . By Lemma 3.5,  $E_{T_Y}$  is endowed with the structure of Lie algebroid over  $Y$ , such that  $V \rightarrow Y$  is a global realization.

**Lemma 3.8** *The Lie algebroid structure on  $E_{T_Y}$  is independent of the choice of the global realization  $U \rightarrow X$  for  $E$ .*

PROOF. In the differentiable and holomorphic case, this is easily checked using Lemma 3.7. In the algebraic case, one uses the Lefschetz principle to reduce to the case that the ground field is  $\mathbb{C}$ . Then the equality of two bracket operations can be checked on the underlying holomorphic manifold.  $\square$

**Definition 3.9** We call  $a_Y : E_{T_Y} \rightarrow T_Y$  with the Lie algebroid structure constructed above, the cofoliation on  $Y$  obtained by **pull-back** via  $Y \rightarrow X$ .

Note that  $E_{T_Y} \rightarrow E$  is a homomorphism of Lie algebroids covering  $Y \rightarrow X$ . Of course, pull-back is functorial.

### 3.2 Cofoliations on stacks

We now come to the definition of cofoliations on stacks.

Let  $X$  be a stack. Let  $E$  be a vector bundle over  $X$  and  $a : E \rightarrow T_X$  an  $X$ -morphism to the tangent stack. We call  $a$  *linear*, if for every smooth morphism  $U \rightarrow X$ , where  $U$  is a manifold, forming the fibred product

$$\begin{array}{ccc} E_{T_U} & \xrightarrow{a_U} & T_U \\ \downarrow & \square & \downarrow \\ E & \xrightarrow{a} & T_X \end{array}$$

we obtain a homomorphism of vector bundles  $a_U : E_{T_U} \rightarrow T_U$ . (Note that  $E_{T_U}$  is *not* the pullback of  $E$  to  $U$  via  $U \rightarrow X$ .)

**Definition 3.10** A **cofoliation** on the stack  $X$  is given by the following data:

- (i) a vector bundle  $E \rightarrow X$ ,
- (ii) a surjective linear map  $a : E \rightarrow T_X$ , (which means that  $a_U : E_{T_U} \rightarrow T_U$  is a surjective homomorphism of vector bundles, for all  $U$  as above),
- (iii) for every smooth  $U \rightarrow X$ , where  $U$  is a manifold, a bracket on  $\Gamma(U, E_{T_U})$ . This data is required to satisfy the constraints:
  - (iv) for every  $U$ , the bracket makes  $E_{T_U}$  a Lie algebroid on  $U$ , with anchor map  $a_U$ ,
  - (v) for any morphism  $V \rightarrow U$ , the induced morphism  $E_{T_V} \rightarrow E_{T_U}$  is a homomorphism of Lie algebroids covering  $V \rightarrow U$ ,

(vi) the cofoliation is *realizable*. By this we mean that there exists a presentation  $\pi : X_0 \rightarrow X$  of the stack  $X$ , together with a morphism of vector bundles  $p : T_{X_0} \rightarrow E$  covering  $\pi$ , such that  $p$  identifies  $T_{X_0}$  with  $\pi^*E$  and the composition  $a \circ p$  is (isomorphic to!) the canonical morphism  $T_{X_0} \rightarrow T_X$ . Moreover, we require that for every smooth  $U \rightarrow X$ , the fibred product  $U_0 = X_0 \times_X U$  is a realization of the Lie algebroid  $E_{T_U}$ .

**Remark** Because cofoliations on manifolds pull back via smooth maps, this definition of cofoliation on the stack  $X$  is equivalent to Definition 3.3 if  $X$  is a manifold. The only reason why we have to make a new definition for stacks is that we cannot characterize a cofoliation by the bracket on global sections (or even étale local sections) of  $E$ . We need sections over smooth maps  $U \rightarrow X$ .

**Remark 3.11** Note that Condition (v) implies that the bundle of Lie algebras  $\ker a_U$  pulls back to the bundle of Lie algebras  $\ker a_V$ . Thus we get an induced bundle of Lie algebras  $N$  over the stack  $X$ , which comes with a homomorphism  $N \rightarrow E$ . We may think of  $T_X$  as the quotient of  $E$  by  $N$ , where  $N$  acts on  $E$  by addition, through the homomorphism  $N \rightarrow E$ . Dually, we have a distinguished triangle

$$L_X \longrightarrow E^\vee \longrightarrow N^\vee \longrightarrow L_X[1] .$$

**Remark** Let  $X$  be a stack and fix a surjective linear map  $E \rightarrow T_X$ , where  $E$  is a vector bundle over  $X$ . Let  $\Xi_U^p$  be the  $p$ -th exterior power of the dual vector bundle of  $E_{T_U}$ . For fixed  $p$ , as  $U$  varies, the  $\Xi_U^p$  form a big sheaf  $\Xi^p$  on  $X$ . Then *all* the compatible brackets on the various  $E_{T_U}$  can also be encoded in *one* differential on the big sheaf  $\Xi^* = \bigoplus_p \Xi^p$ .

**Lemma 3.12** *Definition 3.6 and Lemma 3.7 carry over to stacks.*

PROOF. This can be proved by reducing to the manifold case. The main point is that one can check the integrability of a foliation smooth locally.  $\square$

### Relating cofoliations to flat groupoids

Every flat groupoid defines a cofoliation on the associated stack.

Let  $X$  be a stack and fix a surjective linear map  $E \rightarrow T_X$ , where  $E$  is a vector bundle over  $X$ . We may define a **pre-realization** of  $E \rightarrow T_X$  to be a presentation  $\pi : X_0 \rightarrow X$  of the stack  $X$  together with  $p : T_{X_0} \rightarrow E$ , such that  $p$  identifies  $T_{X_0}$  with  $\pi^*E$  and the composition  $a \circ p$  is the canonical morphism  $T_{X_0} \rightarrow T_X$ .

Any two pre-realizations  $X_0 \rightarrow X$  and  $Y_0 \rightarrow X$  give rise to a distribution  $F \subset T_{Z_0}$ , where  $Z_0 = X_0 \times_X Y_0$ . In particular, any pre-realization gives rise to a connection on the induced groupoid  $X_1 = X_0 \times_X X_0$ .

A pre-realization  $X_0 \rightarrow X$  gives rise to a cofoliation on  $X$  if the induced connection on  $X_1$  is integrable. Two such pre-realizations  $X_0 \rightarrow X$  and  $Y_0 \rightarrow X$



give rise to the same cofoliation on  $X$ , if and only if the induced distribution on  $Z_0 = X_0 \times_X Y_0$  is integrable.

In the differentiable or holomorphic context, if the flat groupoids  $X_\bullet$  and  $Y_\bullet$  induce the same cofoliation on  $X$ , there exists a third flat groupoid  $Z_\bullet$  with étale horizontal maps  $Z_\bullet \rightarrow X_\bullet$  and  $Z_\bullet \rightarrow Y_\bullet$ . The converse is always true.

**Example 3.13** Let  $\pi : \mathfrak{G} \rightarrow X$  be a gerbe over a Deligne-Mumford stack. Then one may consider cofoliations  $a : \pi^*T_X \rightarrow T_{\mathfrak{G}}$  whose anchor map satisfies  $T_\pi \circ a = \text{id}_{T_X}$ . (Here  $T_\pi : T_{\mathfrak{G}} \rightarrow T_X$  is the map induced by  $\pi$ .) These might serve as flat connections on the ‘general’ gerbe  $\mathfrak{G}$ . (By ‘general’ we mean not banded by any group  $G$ .) It should be interesting to compare this notion to the one defined by Breen and Messing [3].

**Remark 3.14** We did not investigate to what extent realizations always exist. It is conceivable, that there exists a ‘curvature vanishing’ condition that assures the existence of realizations, at least in the differentiable and holomorphic categories.

It is not difficult to prove that pre-realizations always exist. One might also be able to construct the spectral sequence (6) using pre-realizations, at least if this conjectural ‘curvature vanishing’ condition is satisfied. In that case, the de Rham differential  $d$  would certainly *not* square to zero on the level of co-chains.

This ‘curvature vanishing’ condition might be expressed as the closedness of a 2-form  $B$  as in Remark 2.17.

### Functoriality

**Definition 3.15** Let  $X$  and  $Y$  be stacks with cofoliations  $E \rightarrow T_X$  and  $F \rightarrow T_Y$ . A **morphism** of cofoliations is a pair  $(f, \phi)$  where  $f : X \rightarrow Y$  is a morphism of stacks, and  $\phi : E \rightarrow F$  is a morphism of vector bundles covering  $f$  such that

$$\begin{array}{ccc} E & \longrightarrow & T_X \\ \phi \downarrow & & \downarrow T_f \\ F & \longrightarrow & T_Y \end{array}$$

is 2-commutative. (So  $\phi$  involves two implicit 2-morphisms.) We also ask that there exist realizations  $X_0 \rightarrow X$  of  $E$  and  $Y_0 \rightarrow Y$  of  $F$  and a morphism  $f_0 : X_0 \rightarrow Y_0$ , making

$$\begin{array}{ccccc} T_{X_0} & \longrightarrow & E & \longrightarrow & T_X \\ \downarrow & \searrow & \downarrow & \swarrow & \downarrow \\ & X_0 & \longrightarrow & X & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ T_{Y_0} & \longrightarrow & F & \longrightarrow & T_Y \\ & \searrow & \downarrow & \swarrow & \downarrow \\ & Y_0 & \longrightarrow & Y & \end{array}$$

commute, and inducing a horizontal morphism of flat groupoids  $X_\bullet \rightarrow Y_\bullet$ .

**Example** Let  $f : X \rightarrow Y$  be a representable smooth morphism and  $F \rightarrow T_Y$  a cofoliation on  $Y$ , with realization  $Y_0 \rightarrow Y$ . Let  $X_0 = Y_0 \times_Y X$ . We get an induced cartesian morphism of groupoids  $X_\bullet \rightarrow Y_\bullet$  and so we can pull back the flat connection on  $Y_\bullet$  to a flat connection on  $X_\bullet$ , which defines a cofoliation  $E \rightarrow T_X$  on  $X$ . By Lemma 3.8, the cofoliation  $E$  is independent of the choice of the presentation  $Y_0$  of  $F$ .

The cofoliation  $F$  comes with an induced morphism of cofoliations  $E \rightarrow F$ . We say that  $E$  is obtained from  $F$  via *smooth pullback*. Notation:  $E = f^*F$ .

## 4 The De Rham Complex

### 4.1 The Čech complexes of the exterior powers of $[\Omega \rightarrow \Upsilon]$

**The trigraded vector space with its first differential  $\phi$**

Let  $X_1 \rightrightarrows X_0$  be a groupoid with flat connection  $E_\bullet \rightarrow X_\bullet$ . Let  $X$  be the associated stack and  $N \rightarrow E$  the induced representative of the tangent complex. Let us denote the dual of  $N \rightarrow E$  by

$$\Omega \xrightarrow{\phi} \Upsilon.$$

It is a homomorphism of vector bundles on  $X$ . We also use notation  $\Omega^p = \Lambda^p \Omega$  and  $\Upsilon^k = S^k \Upsilon$  (symmetric power). We consider the bigraded commutative  $\mathcal{O}_X$ -algebra

$$\bigoplus_{p \geq k \geq 0} \Omega^{p-k} \otimes \Upsilon^k, \quad (17)$$

where  $\Omega$  is in degree  $(1, 0)$  and  $\Upsilon$  in degree  $(1, 1)$ . We denote this bigraded algebra by  $L = \bigoplus L^{p,k}$ , where  $L^{p,k} = \Omega^{p-k} \otimes \Upsilon^k$ . The homomorphism  $\phi$  extends, in a unique way, to a graded derivation of degree  $(0, 1)$ , which is linear over  $\bigoplus_{k \geq 0} \Upsilon^k$ . Explicitly, this graded derivation is given by

$$\phi(\omega_1 \wedge \dots \wedge \omega_q \otimes \alpha_1 \dots \alpha_k) = \sum_{i=1}^q (-1)^{i+1} \omega_1 \wedge \dots \wedge \widehat{\omega}_i \wedge \dots \wedge \omega_q \otimes \phi(\omega_i) \cdot \alpha_1 \dots \alpha_k,$$

for  $\omega_i \in \Omega$  and  $\alpha_j \in \Upsilon$ .

Note that  $\phi^2 = 0$ , so that  $(L, \phi)$  is a differential bigraded sheaf of  $\mathcal{O}_X$ -algebras. Via the structure morphism  $\pi : X_n \rightarrow X$ , we pull back to any  $X_n$ .

We now introduce the trigraded  $\mathbb{K}$ -vector space  $K = \bigoplus_{p,k,n} K^{p,k,n}$  by

$$K^{p,k,n} = \Gamma(X_n, L^{p,k}) = \Gamma(X_n, \Omega^{p-k} \otimes \Upsilon^k).$$

Note that  $\phi$  induces a derivation of tridegree  $(0, 1, 0)$  on  $K$  by the formula  $\phi = (-1)^n \pi^* \phi$  and that  $\phi^2 = 0$ .

### The Čech differential

Since  $\hat{\pi}_q : X_{n+1} \rightarrow X_n$  and  $\iota_q : X_{n-1} \rightarrow X_n$  commute with the projections to  $X$ , we have for each  $q = 0, \dots, n+1$  a homomorphism

$$\hat{\pi}_q^* : \Gamma(X_n, \Omega^{p-k} \otimes \Upsilon^k) \longrightarrow \Gamma(X_{n+1}, \Omega^{p-k} \otimes \Upsilon^k)$$

and each  $q = 0, \dots, n-1$  a homomorphism

$$\iota_q^* : \Gamma(X_n, \Omega^{p-k} \otimes \Upsilon^k) \longrightarrow \Gamma(X_{n-1}, \Omega^{p-k} \otimes \Upsilon^k).$$

We can now define the Čech differential  $\partial : K \rightarrow K$  of degree  $(0, 0, 1)$  by

$$\partial = \sum_{q=0}^{n+1} (-1)^q \hat{\pi}_q^*.$$

The fact that  $\phi$  is defined over  $X$  implies the following:

**Lemma 4.1** *We have  $\partial^2 = 0$ ,  $\phi^2 = 0$  and  $(\partial + \phi)^2 = 0$ .  $\square$*

### The multiplicative structure

Fix  $n, m \geq 0$  and consider the two morphisms

$$\begin{aligned} s : X_{n+m} &\longrightarrow X_n \\ (x_0 \xrightarrow{\psi_1} \dots \xrightarrow{\psi_{n+m}} x_{n+m}) &\longmapsto (x_0 \xrightarrow{\psi_1} \dots \xrightarrow{\psi_n} x_n) \end{aligned}$$

and

$$\begin{aligned} t : X_{n+m} &\longrightarrow X_m \\ (x_0 \xrightarrow{\psi_1} \dots \xrightarrow{\psi_{n+m}} x_{n+m}) &\longmapsto (x_n \xrightarrow{\psi_{n+1}} \dots \xrightarrow{\psi_{n+m}} x_{n+m}). \end{aligned}$$

We define the *cup product* on  $K$  by

$$\begin{aligned} \Gamma(X_n, \Omega^{p-k} \otimes \Upsilon^k) \otimes \Gamma(X_m, \Omega^{p'-k'} \otimes \Upsilon^{k'}) &\longrightarrow \Gamma(X_{n+m}, \Omega^{p+p'-k-k'} \otimes \Upsilon^{k+k'}) \\ \omega \otimes \omega' &\longmapsto (-1)^{m(p-k)} s^*(\omega) \wedge t^*(\omega'). \end{aligned}$$

This definition makes  $K$  into a trigraded  $\mathbb{K}$ -algebra. Both  $\phi$  and  $\partial$  (and hence also  $\partial + \phi$ ) are graded derivations (of total degree 1) with respect to the cup product.

**Remark** This multiplicative structure is associative. It is not (graded) commutative.

### The double complex for fixed $p$

Fixing  $p$  and varying  $k$  and  $n$ , we obtain the total complex of a double complex  $(K^{p,\bullet,\bullet}, \partial, \phi)$ . If  $X_\bullet$  is sufficiently nice (all  $X_n$  are Stein in the holomorphic context, affine in the algebraic context), this total complex computes the hypercohomology of  $X_\bullet$  with values in  $\Lambda^p(\Omega \rightarrow \Upsilon)$ :

$$h^q(K^{p,\bullet,\bullet}, \partial + \phi) = \mathbb{H}^q(X_\bullet, \Lambda^p(\Omega \rightarrow \Upsilon)) = \mathbb{H}^q(X, \Lambda^p L_X),$$

where  $X$  is the stack associated to  $X_1 \rightrightarrows X_0$  and  $L_X$  its cotangent complex.

The cup product passes to  $\mathbb{H}(X, \Lambda^p L_X)$ , and is commutative on  $\mathbb{H}(X, \Lambda^p L_X)$ .

Filtering this double complex by  $k$  we obtain the spectral sequence (8) from the introduction.

## 4.2 The de Rham differential

### The alternating de Rham differential $d$

We consider an analogue of the bigraded algebra (17), living on  $X_n$ . In fact, let us define

$$L_{X_n} = \bigoplus_{p \geq k \geq 0} L_{X_n}^{p,k} = \bigoplus_{p \geq k \geq 0} \Omega_{X_n}^{p-k} \otimes \Upsilon_n^k. \quad (18)$$

In keeping with earlier notation, we denote by  $L_n$  the pullback of  $L$  to  $X_n$ . We will study how  $L_n$  and  $L_{X_n}$  relate to each other. Recall that, as  $n$  varies, all the  $L_n$  are pullbacks of each other. In other words,  $L$  is (componentwise) a vector bundle on the stack  $X$ . On the other hand, the  $L_{X_n}$  do not fit together so nicely. They only form a big sheaf on  $X$ .

Recall that  $E_n$  is a quotient of  $T_{X_n}$  in  $n+1$  different ways, the  $q$ -th quotient map being denoted  $\eta_q$ . Dually,  $\Omega_n$  is a subbundle of  $\Omega_{X_n}$  in  $n+1$  different ways. We denote the embedding dual to  $\eta_q$  again by  $\eta_q$ . The embedding  $E_n \hookrightarrow T_{X_n}$  corresponds to a quotient map  $\delta : \Omega_{X_n} \rightarrow \Omega_n$ . Of course, we have that  $\delta \circ \eta_q = \text{id}$ , for all  $q$ .

Taking alternating powers and tensoring with  $\Upsilon_n^k$ , we obtain an embedding  $\eta_q : \Omega_n^{p-k} \otimes \Upsilon_n^k \hookrightarrow \Omega_{X_n}^{p-k} \otimes \Upsilon_n^k$  and a quotient map  $\delta$  in the other direction. We still have  $\delta \circ \eta_q = \text{id}$ , for all  $q$ . We have thus constructed an algebra morphism  $\eta_q : L_n \rightarrow L_{X_n}$ , for every  $q$ , and another algebra morphism  $\delta : L_{X_n} \rightarrow L_n$ , which is a retraction of every  $\eta_q$ .

Now recall that we have the various embeddings  $\rho_q : N_n \rightarrow T_{X_n}$ . We denote the dual quotient maps by the same letter:  $\rho_q : \Omega_{X_n} \rightarrow \Upsilon_n$ . The homomorphism  $\rho_q$  extends in a unique way to a graded derivation of degree  $(0, 1)$  on  $L_{X_n}$ , which is linear over  $\bigoplus_{k \geq 0} \Upsilon_n^k$ . We denote this graded derivation again by  $\rho_q$ , for all  $q = 0, \dots, n$ .

**Lemma 4.2** *Letting  $\delta_{qr}$  denote the delta of Kronecker, the diagram*

$$\begin{array}{ccc} \Omega_n^{p-k} \otimes \Upsilon_n^k & \xrightarrow{\delta_{qr}\phi} & \Omega_n^{p-k-1} \otimes \Upsilon_n^{k+1} \\ \eta_r \downarrow & & \downarrow \eta_r \\ \Omega_{X_n}^{p-k} \otimes \Upsilon_n^k & \xrightarrow{\rho_q} & \Omega_{X_n}^{p-k-1} \otimes \Upsilon_n^{k+1} \end{array}$$

*commutes, for all  $q, r = 0, \dots, n$ . So does the diagram*

$$\begin{array}{ccc} \Omega_n^{p-k} \otimes \Upsilon_n^k & \xrightarrow{\phi} & \Omega_n^{p-k-1} \otimes \Upsilon_n^{k+1} \\ \delta \uparrow & & \uparrow \delta \\ \Omega_{X_n}^{p-k} \otimes \Upsilon_n^k & \xrightarrow{\sum_{q=0}^n \rho_q} & \Omega_{X_n}^{p-k-1} \otimes \Upsilon_n^{k+1} . \end{array}$$

PROOF. This follows immediately from Diagrams (11) and (12).  $\square$

Let  $\nabla : \Upsilon_0 \rightarrow \Omega_{X_0} \otimes \Upsilon_0$  be the derived connection. As remarked, on  $\Upsilon_n$ , we get  $n+1$  different induced connections  $\nabla^r : \Upsilon_n \rightarrow \Omega_{X_n} \otimes \Upsilon_n$ , for  $r = 0, \dots, n$ . The  $r$ -th connection is obtained by thinking of  $\Upsilon_n$  as  $\pi_r^* \Upsilon_0$ . To fix notation, let us focus on the 0-th connection  $\nabla^0$  and denote it by  $\nabla$ . In the end, our constructions will not depend on this choice.

We get induced integrable connections on all symmetric powers  $\Upsilon_n^k$  of  $\Upsilon_n$ :

$$\nabla : \Upsilon_n^k \longrightarrow \Omega_{X_n} \otimes \Upsilon_n^k .$$

Let us denote by

$$D : \Omega_{X_n}^p \otimes \Upsilon_n^k \longrightarrow \Omega_{X_n}^{p+1} \otimes \Upsilon_n^k$$

the associated covariant derivative, in other words, the differential of the de Rham complex of  $(\Upsilon_n^k, \nabla)$ . Note that  $D$  is a graded derivation of degree  $(1, 0)$  on the bigraded algebra  $L_{X_n}$ .

**Lemma 4.3** *The derivation  $D$  passes to the quotient bigraded algebra  $L_n$ .*

PROOF. This follows from the fact that  $E_n^\perp \subset \Omega_{X_n}$  generates an ideal preserved by the exterior derivative,  $E_n$  being an integrable distribution on  $X_n$ .  $\square$

We will denote the derivation  $D$  induces on  $L_n$  by

$$(-1)^n d : \Omega_n^{p-k} \otimes \Upsilon_n^k \longrightarrow \Omega_n^{p+1-k} \otimes \Upsilon_n^k .$$

Taking global sections, we finally arrive at the definition of

$$(-1)^n d : \Gamma(X_n, \Omega^{p-k} \otimes \Upsilon^k) \longrightarrow \Gamma(X_n, \Omega^{p+1-k} \otimes \Upsilon^k) .$$

Thus, we have defined  $d : K^{p,k,n} \rightarrow K^{p+1,k,n}$ . Note that  $d^2 = 0$ , which follows from  $D^2 = 0$ , which holds because  $\nabla$  is an integrable connection.

**Lemma 4.4** *Using any of the other connections  $\nabla^q$ , for  $q > 0$ , gives the same derivation on the quotient algebra  $L_n$ .*

PROOF. This follows immediately from Remark 2.9.  $\square$

**Corollary 4.5** *On  $\Gamma(X_n, L_n)$ , we have  $d\partial + \partial d = 0$ .*

PROOF. Let us denote the covariant derivative associated with the connection  $\nabla^q$  by  $D^q$ . Then we have the following relations:

$$\hat{\pi}_q^* D^0 = \begin{cases} D^1 \hat{\pi}_q^* & \text{if } q = 0, \\ D^0 \hat{\pi}_q^* & \text{if } q > 0. \end{cases}$$

The rest is a straightforward calculation, using that  $D^0$  and  $D^1$  both induce  $(-1)^n d$  on the quotient.  $\square$

**Corollary 4.6** *The map  $d : K \rightarrow K$  is a derivation with respect to the cup product of degree  $(1, 0, 0)$ .*

PROOF. This is a straightforward calculation, but it uses Lemma 4.4.  $\square$

**Remark** Unfortunately, we do not have  $d\phi + \phi d = 0$ . This necessitates the correction term  $\iota$ , defined next.

**The symmetric de Rham differential, or contraction,  $\iota$**

Consider a fixed  $X_n$  and define the *symmetric partial derivatives*

$$\mathfrak{L}_q : \Omega_{X_n}^p \otimes \Upsilon_n^k \longrightarrow \Omega_{X_n}^p \otimes \Upsilon_n^{k+1},$$

for  $q = 0, \dots, n$ , by

$$\mathfrak{L}_q = [\rho_q, D] = \rho_q D + D \rho_q.$$

Note that the commutator is considered to be a graded commutator, and both  $\rho_q$  and  $D$  being odd, we obtain the plus sign in the formula. (One may think of  $\mathfrak{L}_q$  as the covariant derivative of  $\rho_q$ .)

Define the *(total) symmetric derivative*

$$\mathfrak{L} : \Omega_{X_n}^p \otimes \Upsilon_n^k \longrightarrow \Omega_{X_n}^p \otimes \Upsilon_n^{k+1}, \quad \text{by} \quad \mathfrak{L} = \sum_{q=0}^n \mathfrak{L}_q.$$

Note that all  $\mathfrak{L}_q$ , as well as  $\mathfrak{L}$ , are bigraded derivations of degree  $(0, 1)$ .

**Lemma 4.7** *The derivations  $\mathfrak{L}$  and  $\mathfrak{L}_q$ , for  $q = 0, \dots, n$ , pass to the quotient algebra  $\delta : L_{X_n} \rightarrow L_n$ .*

PROOF. For  $\mathfrak{L}$ , the claim follows easily from previous results. Therefore, we only need to prove the claim for  $\mathfrak{L}_q$ , with  $q \geq 1$ .

Let us explain the proof in the differentiable case. Recall that we have distinguished the connection  $\nabla = \pi_0^* \nabla$ . Thus, to do computations, it is convenient to start from the direct sum decomposition

$$\Omega_{X_n} = \pi_0^* \Omega_0 \oplus \bigoplus_{j=1}^n \pi_j^* \Upsilon_0.$$

This is the dual of the direct sum decomposition (10) for  $q = 0$ . Around a given point of  $X_n$  we get an induced decomposition of  $X_n$  as a product of  $X_0$  with the  $\hat{\pi}_j$ -fibres, for  $j = 1, \dots, n$ . Since all  $\pi_j^* \Upsilon_0 = \Omega_{\hat{\pi}_j}$  are endowed with a flat connection, each of these fibres is a flat manifold. Choosing arbitrary coordinates  $\{y\}$  on  $X_0$  and flat coordinates  $\{x_j\}$  on the  $\hat{\pi}_j$ -fibre, we get a coordinate system  $\{y, x_1, \dots, x_n\}$  on  $X_n$  with the property that  $dx_j$  is a horizontal frame for  $\pi_j^* \Upsilon_0$ , for all  $j = 1, \dots, n$ . (Of course, each symbol  $x_j$  stands for an appropriate number of coordinates.)

Now it is easy to describe  $\mathfrak{L}_q$ , for  $q = 1, \dots, n$ . We start by remarking that  $\rho_q$  has been identified with the projection onto  $\pi_q^* \Upsilon_0$ . Since all our coordinate systems  $x_j$  are flat,  $\mathfrak{L}_q(dx_j) = 0$ , whether we think of  $dx_j$  as a section of  $\Omega_{X_n}$ , or of  $\Upsilon_n$ . Moreover,  $\mathfrak{L}_q(f) = \frac{\partial f}{\partial x_q} dx_q$ , for a function  $f$  on  $X_n$ . (Now it is also obvious, why we call the  $\mathfrak{L}_q$  partial derivatives.)

To prove the lemma, we need to show that

$$\mathfrak{L}_q : \Omega_{X_n} \longrightarrow \Omega_{X_n} \otimes \Upsilon_n$$

maps  $E_n^\perp \subset \Omega_{X_n}$  into  $E_n^\perp \otimes \Upsilon_n$ . We have identified  $E^\perp$  with  $\bigoplus_{j=1}^n \pi_j^* \Upsilon_0$ . We have

$$\mathfrak{L}_q \sum_{j=1}^n f_j dx_j = \sum_{j=1}^n dx_j \otimes \mathfrak{L}_q(f_j),$$

which finishes the proof.

The proof in the holomorphic case is analogous. The algebraic case then follows by appealing to the Lefschetz principle.

For future reference, let us remark, that from our description it follows that  $[\mathfrak{L}_q, \mathfrak{L}_j] = 0$ , for all  $j, q > 0$ .  $\square$

Because of this lemma it will cause no confusion if we denote the homomorphisms

$$\Omega_n^p \otimes \Upsilon_n^k \longrightarrow \Omega_n^p \otimes \Upsilon_n^{k+1}$$

induced by  $\mathfrak{L}$  and  $\mathfrak{L}_q$  by passing to the quotient, with the same symbols  $\mathfrak{L}$  and  $\mathfrak{L}_q$ . Moreover, had we used any other connection  $\nabla^r$  to define  $\mathfrak{L}_q$ , the resulting derivation would also pass to the quotient.

**Lemma 4.8** *Using any other connection  $\nabla^r$ , for  $r > 0$ , to define  $\mathfrak{L}_q$ , we get the same induced derivation on the quotient  $L_n$ .*

PROOF. Recall that  $\nabla^r - \nabla^0 = \omega_{r0}(\Psi)$ . Therefore, we need to show that  $[\omega_{r0}(\Psi), \rho_q]$  kills  $\Upsilon_n$  and maps  $\Omega_{X_n}$  into  $E_n^\perp \otimes \Upsilon_n$ . The latter is clear, because  $\omega_{r0}(\Psi)$  vanishes on  $\Omega_{X_n}$  and maps  $\Upsilon_n$  into  $E_n^\perp \otimes \Upsilon_n$ .

Now, if  $q \neq 0, r$ , then  $\rho_q \circ \omega_{r0}(\Psi) = 0$ . If  $q = 0$  or  $q = r$ , then  $\rho_q \circ \omega_{r0}(\Psi)$  is the symmetrization of  $\Psi$ . But because  $\Psi$  is antisymmetric, its symmetrization vanishes.  $\square$

**Corollary 4.9** *For the total symmetric derivative we have*

$$\delta \mathfrak{L} = (\phi d + d\phi)\delta.$$

PROOF. Modulo  $E_n^\perp$ , we have  $\mathfrak{L} = \sum \mathfrak{L}_q = \sum [\rho_q, D] = [\sum \rho_q, D] = (-1)^n (-1)^n [\phi, d]$ .  $\square$

Now consider for each  $q = 0, \dots, n-1$  the diagonal  $\iota_q : X_{n-1} \rightarrow X_n$ . The  $p$ -th exterior power of the canonical epimorphism  $\iota_q^* \Omega_{X_n} \rightarrow \Omega_{X_{n-1}}$  gives a map

$$\Gamma(X_n, \Omega_{X_n}^p \otimes \Upsilon_n^k) \xrightarrow{\iota_q^*} \Gamma(X_{n-1}, \iota_q^* \Omega_{X_n}^p \otimes \Upsilon_{n-1}^k) \longrightarrow \Gamma(X_{n-1}, \Omega_{X_{n-1}}^p \otimes \Upsilon_{n-1}^k).$$

Note that the diagram

$$\begin{array}{ccc} \Gamma(X_n, \Omega_{X_n}^{p-k} \otimes \Upsilon^k) & \xrightarrow{\iota_q^*} & \Gamma(X_{n-1}, \Omega_{X_{n-1}}^{p-k} \otimes \Upsilon^k) \\ \downarrow \delta & & \downarrow \delta \\ \Gamma(X_n, \Omega_{X_n}^{p-k} \otimes \Upsilon_n^k) & \xrightarrow{\iota_q^*} & \Gamma(X_{n-1}, \Omega_{X_{n-1}}^{p-k} \otimes \Upsilon_n^k) \end{array}$$

commutes, so there is no ambiguity in the notation  $\iota_q^*$ .

**Definition 4.10** We now define the **contraction**

$$\iota : \Gamma(X_n, \Omega_{X_n}^{p-k} \otimes \Upsilon^k) \longrightarrow \Gamma(X_{n-1}, \Omega_{X_{n-1}}^{p-k} \otimes \Upsilon^{k+1})$$

by

$$-\iota = \sum_{0 \leq i < j \leq n} (-1)^i \iota_i^* \mathfrak{L}_j.$$

Note that  $\iota : K^{p,k,n} \rightarrow K^{p+1,k+1,n-1}$  is of degree  $(1, 1, -1)$ .

**Proposition 4.11**  $[d, \iota] = 0$ .

PROOF. A simple calculation shows that  $[D, \mathfrak{L}_j] = 0$ , for all  $j$ . Since the derivative  $D$  commutes with restriction via  $\iota_j^*$ , it follows that  $D\iota = \iota D$ . The formula of the proposition follows by passing to the quotient.  $\square$

**Proposition 4.12**  $[\phi, \iota] = 0$ .



PROOF. It suffices to prove that  $[\phi, \mathfrak{L}_j] = 0$ , for all  $j > 0$ . Clearly,  $[\phi, \mathfrak{L}_j]$  vanishes on  $\bigoplus \Upsilon_n^k$ . So it is enough to check vanishing of  $[\phi, \mathfrak{L}_j]$  on a set of generators of  $\Omega_{X_n}$  over  $\mathcal{O}$ . Since only  $\mathfrak{L}_j$  for  $j > 0$  are involved, we may use coordinates as in the proof of Lemma 4.7. Because we are proving an identity on the quotient, it suffices to check the identity on the generators  $dy$ . Because all  $j$  involved are greater than zero, we know that  $\mathfrak{L}_j(dy) = 0$  and hence  $\phi \mathfrak{L}_j(dy) = 0$ . Also note that  $\phi(dy) = \pi_0^* \phi(dy)$ , and so  $\phi(dy)$  is constant in the  $x_j$ -directions. Hence  $\mathfrak{L}_j \phi(dy) = 0$ . Adding up, we get the required formula  $[\phi, \mathfrak{L}_j](dy) = 0$ .  $\square$

**Lemma 4.13** *For all  $q = 0, \dots, n-1$  and all  $j = 0, \dots, n-1$ , we have*

$$\rho_q \iota_j^* = \begin{cases} \iota_j^* \rho_q & \text{if } q < j, \\ \iota_j^* (\rho_q + \rho_{q+1}) & \text{if } q = j, \\ \iota_j^* \rho_{q+1} & \text{if } q > j. \end{cases}$$

PROOF. Let us first consider the cases  $q \neq j$ . Then we have the cartesian diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\hat{\pi}_{q'}} & X_{n-1} \\ \iota_j \uparrow & & \uparrow \iota_{j'} \\ X_{n-1} & \xrightarrow{\hat{\pi}_q} & X_{n-2} \end{array}$$

If  $q < j$ , then  $q' = q$  and  $j' = j-1$ . If  $q > j$ , then  $q' = q+1$  and  $j' = j$ . It shows that we have a commutative diagram

$$\begin{array}{ccc} \iota_j^* \Omega_{X_n} & \xrightarrow{\iota_j^* \rho_{q'}} & \iota_j^* \Omega_{\hat{\pi}_{q'}} \\ \downarrow & & \downarrow = \\ \Omega_{X_{n-1}} & \xrightarrow{\rho_q} & \Omega_{\hat{\pi}_q} \end{array}$$

which proves the formula for the case  $q \neq j$ . For the remaining case, notice that  $\sum \rho_q = \phi \delta$ , so that  $\sum \rho_q$  commutes with  $\iota_j$  and we have  $(\sum_{q=0}^{n-1} \rho_q) \iota_j^* = \iota_j^* \sum_{q=0}^n \rho_q$ . Combine this with the case  $q \neq j$  to finish.  $\square$

**Corollary 4.14** *For all  $q = 0, \dots, n-1$  and all  $j = 0, \dots, n-1$ , we have*

$$\mathfrak{L}_q \iota_j^* = \begin{cases} \iota_j^* \mathfrak{L}_q & \text{if } q < j, \\ \iota_j^* (\mathfrak{L}_q + \mathfrak{L}_{q+1}) & \text{if } q = j, \\ \iota_j^* \mathfrak{L}_{q+1} & \text{if } q > j. \end{cases}$$

**Proposition 4.15**  $\iota^2 = 0$ .

PROOF. This is a straightforward calculation using Corollary 4.14 and the fact that  $[\mathfrak{L}_q, \mathfrak{L}_j] = 0$  on  $X_n$ , if  $n \geq 2$ , which we remarked in the proof of Lemma 4.7.  $\square$

**Lemma 4.16** Consider  $\hat{\pi}_q : X_{n+1} \rightarrow X_n$ . Then for every  $j = 0, \dots, n+1$ , we have

$$\rho_j \hat{\pi}_q^* = \begin{cases} \hat{\pi}_q^* \rho_j & \text{if } j < q, \\ 0 & \text{if } j = q, \\ \hat{\pi}_q^* \rho_{j-1} & \text{if } j > q. \end{cases}$$

PROOF. The proof is analogous to the proof of Lemma 4.13, including the trick to reduce to the case  $j \neq q$ .  $\square$

**Corollary 4.17** We have

$$\mathfrak{L}_j \hat{\pi}_q^* = \begin{cases} \hat{\pi}_q^* \mathfrak{L}_j & \text{if } j < q, \\ 0 & \text{if } j = q, \\ \hat{\pi}_q^* \mathfrak{L}_{j-1} & \text{if } j > q. \end{cases}$$

PROOF. Comparing with the proof of Corollary 4.14, there is the added subtlety that  $\hat{\pi}_0^*$  does not commute with  $D$ . So, at least on  $L_{X_n}$ , we do not have  $\mathfrak{L}_1 \hat{\pi}_0^* = \hat{\pi}_0^* \mathfrak{L}_0$ . But because of Lemma 4.8, if we pass to the quotient  $L_n$ , we do have this required equality.  $\square$

**Proposition 4.18**  $[\partial, \iota] = -\mathfrak{L}$ .

PROOF. This is a straightforward calculation using Corollary 4.17.  $\square$

**Remark** Unfortunately,  $\iota : K \rightarrow K$  is not a derivation with respect to the cup product. Rather, we have the formula

$$\iota(\alpha \cup \beta) = \iota\alpha \cup \beta + (-1)^{\deg \alpha} \alpha \cup \iota\beta - I(\alpha) \cup \mathfrak{L}\beta.$$

Here  $I = \sum_i (-1)^i \iota_i^*$ . This follows easily from repeated applications of Corollary 4.17. The error term can be expressed as follows:

$$\begin{aligned} & -I\alpha \cup \mathfrak{L}\beta \\ & = I\alpha \cup \iota(\phi + \partial)\beta - (-1)^{\deg \alpha} I(\phi + \partial)\alpha \cup \iota\beta - (-1)^{\deg \alpha} (\phi + \partial)(I\alpha \cup \iota\beta). \end{aligned} \tag{19}$$

Thus, after passing to cohomology with respect to  $(\phi + \partial)$ , this error term vanishes.

**Corollary 4.19** Let us summarize. We have four maps on the complex  $K$  for which the following formulae hold:

1.  $\phi^2 = \partial^2 = d^2 = \iota^2 = 0$ .
2.  $[\phi, \partial] = [\phi, \iota] = [\partial, d] = [d, \iota] = 0$ .
3.  $[\phi, d] + [\partial, \iota] = 0$ .

$$4. (\phi + \partial)^2 = (d + \iota)^2 = [\phi + \partial, d + \iota] = 0.$$

$$5. (\phi + \partial + d + \iota)^2 = 0.$$

The degrees are as follows:

$$\deg \phi = (0, 1, 0) \quad \deg \partial = (0, 0, 1) \quad \deg d = (1, 0, 0) \quad \deg \iota = (1, 1, -1).$$

Finally,  $\phi$ ,  $\partial$  and  $d$  are derivations, whereas  $\iota$  is a derivation up to the error term (19).

We will use  $K$  to denote this trigraded  $\mathbb{K}$ -vector space with the differential  $(\phi + \partial + d + \iota)$ . If we need to include the flat groupoid  $(X_\bullet, E_\bullet)$  in the notation, we denote this complex by  $K(X_\bullet, E_\bullet)$ , or simply  $K(X_\bullet)$ .

**Proposition 4.20 (naturality)** *Given a horizontal morphism of flat groupoids  $f : X_\bullet \rightarrow Y_\bullet$ , we get an induced morphism  $f^* : K(Y_\bullet) \rightarrow K(X_\bullet)$ . This morphism respects the triple grading, the cup product and all four differentials  $\phi$ ,  $\partial$ ,  $d$  and  $\iota$ .  $\square$*

### 4.3 Conclusions

**Theorem 4.21** *Let  $X$  be a stack. If we are in the holomorphic category, assume that the diagonal of  $X$  is relatively Stein, if we are in the algebraic category, assume that  $X$  has affine diagonal.*

Any cofoliation on  $X$  gives rise to an  $E_1$ -spectral sequence

$$E_1^{m,n} = \bigoplus_{p+k=m} H^n(X, \Omega^{p-k} \otimes \Upsilon^k) \implies H_{DR}^{m+n}(X). \quad (20)$$

For every  $n$ , the term  $E_1^{\bullet,n}$  is a double complex, whose differentials are induced by  $\phi$  and  $d$ , respectively.

**Caveat (algebraic case)** *The techniques developed in this paper do not suffice to prove in the algebraic case that the spectral sequence (20) is an invariant of the cofoliation. Thus, in the algebraic case, it remains a conjecture that this spectral sequence does not depend on a chosen flat groupoid realization.*

**PROOF.** By our assumptions on the stack  $X$ , we can choose a flat groupoid  $X_\bullet$  realizing our cofoliation on  $X$ , where every  $X_n$  is a differentiable manifold, a Stein holomorphic manifold or a smooth affine variety, depending on the context. The purpose of this is so that we can use Čech cohomology of the simplicial manifold  $X_\bullet$  to compute cohomology of  $X$  with values in coherent sheaves over  $X$ .

Now we filter the complex  $K(X_\bullet)$  by the degree  $m = p + k$ . This gives rise to a spectral sequence abutting to the (total) cohomology of  $K(X_\bullet)$ , with the given  $E_1$ -term. We need to prove two things:

- (i) the cohomology of  $K(X_\bullet)$  is equal to the de Rham cohomology of  $X$ ,

(ii) the spectral sequence is independent of the flat groupoid  $X_\bullet$  chosen to realize the given cofoliation on  $X$ .

Let us prove (i). Note that the flat connection on  $X_1 \rightrightarrows X_0$  induces flat connections on every ‘shifted’ groupoid  $X_{n+1} \rightrightarrows X_n$ , by *cartesian pullback*. If we choose  $X_{n+1} \rightrightarrows X_n$  to have source and target maps  $\hat{\pi}_q$  and  $\hat{\pi}_r$ , then this flat connection is given by  $E_{n+1} \oplus \bigoplus_{j \neq q, r} \pi_j^* N \subset T_{X_{n+1}}$ .

Let us fix notation as follows:  $X_\bullet[n-1] = [X_{n+1} \rightrightarrows X_n]$ , with source and target maps  $\hat{\pi}_0$  and  $\hat{\pi}_1$ . Then we have a diagram

$$\dots \rightrightarrows X_\bullet[2] \rightrightarrows X_\bullet[1] \rightrightarrows X_\bullet[0] \longrightarrow X_\bullet$$

Of horizontal morphisms of flat groupoids. We get an augmented cosimplicial diagram

$$K(X_\bullet) \longrightarrow K(X_\bullet[0]) \rightrightarrows K(X_\bullet[1]) \rightrightarrows K(X_\bullet[2]) \rightrightarrows \dots$$

of morphisms of complexes. It is not difficult to check that if we pass to the alternating sums of the horizontal maps, we get a resolution of  $K(X_\bullet)$ :

$$K(X_\bullet) \longrightarrow K(X_\bullet[0]) \xrightarrow{\partial} K(X_\bullet[1]) \xrightarrow{\partial} K(X_\bullet[2]) \xrightarrow{\partial} \dots$$

Thus, if we assemble all  $K(X_\bullet[*])$  into a double complex with the Čech differential  $\partial$  we end up with a quasi-isomorphism of complexes

$$K(X_\bullet) \longrightarrow K(X_\bullet[*]). \quad (21)$$

Now each  $X_\bullet[n]$  is the banal groupoid associated to a surjective submersion  $X_{n+1} \rightarrow X_n$ . Thus we have a horizontal morphism  $X_\bullet[n] \rightarrow X_n$ , where  $X_n$  stands for the groupoid  $X_n \rightrightarrows X_n$  with its canonical flat connection. We get an induced quasi-isomorphism  $K(X_n) \rightarrow K(X_\bullet[n])$ , as both groupoids present the manifold  $X_n$ . But  $K(X_n)$  is the usual de Rham complex of  $X_n$ , so that we have, in fact, a quasi-isomorphism  $\Omega^\bullet(X_n) \rightarrow K(X_\bullet[n])$ . Assembling all these quasi-isomorphism together, we obtain another quasi-isomorphism

$$\Omega^\bullet(X_\bullet) \longrightarrow K(X_\bullet[*]).$$

Together with (21), we see that the cohomology of  $\Omega^\bullet(X_\bullet)$  and  $K(X_\bullet)$  are canonically isomorphic. Since  $\Omega^\bullet(X_\bullet)$  computes the de Rham cohomology of  $X$ , this completes the proof of (i).

To prove (ii), we start by noticing that an étale morphism of groupoids realizing the given cofoliation induces the identity on the  $E_1$ -term of our spectral sequence. Hence any two morphisms between the same two realizations induce the same canonical isomorphism of our spectral sequence. This proves (ii) in the differentiable and holomorphic case, by appealing to Lemma 3.12. In the algebraic case, one might prove independence of the realization by following the program outlined in Remark 3.14.  $\square$

**Proposition 4.22 (naturality)** *The spectral sequence (20) is natural for morphisms of cofoliations (see Definition 3.15). In particular, it commutes with smooth pullbacks, hence with étale localization.*

PROOF. Since every morphism of cofoliations is induced by a flat morphism of groupoids, we can use Proposition 4.20, to obtain an induced morphism of spectral sequences. We need to prove that this morphism of spectral sequences is independent of the realizations chosen. This is enough to check on the  $E_1$ -level. But on the  $E_1$ -level, any flat morphism of realizations induces the homomorphism  $H^n(Y, \Lambda^{p-k} F^\vee \otimes S^k M^\vee) \rightarrow H^n(X, \Lambda^{p-k} E^\vee \otimes S^k N^\vee)$  coming from  $(f, \phi) : E \rightarrow F$ . (Here  $M = \ker(F \rightarrow T_Y)$  and  $N = \ker(E \rightarrow T_X)$ . The other notation is taken from Definition 3.15.)  $\square$

**Proposition 4.23 (multiplicativity)** *The spectral sequence (20) is multiplicative, i.e., consists of differential graded  $\mathbb{K}$ -algebras and derivations.*

PROOF. Let  $\tilde{K} \subset K$  be the normalization of  $K$ . This is the subcomplex of elements vanishing under all pullback maps  $\iota_j$ . Nothing changes, except now  $\iota$  and hence  $(\phi + \partial + d + \iota)$  is a derivation. The proposition follows.  $\square$

**Remark** The spectral sequence (20) has more structure. For example, we can filter the whole spectral sequence using the degree  $p$ . In other words, we construct a spectral sequence of filtered algebras. Since the  $E_\nu$ -level is now a filtered complex, it has an associated spectral sequence. Its abutment is the  $E_\nu$ -term of the Hodge to De Rham spectral sequence (4).

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