Localization and Gromov-Witten Invariants

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Abstract

We explain how to apply the Bott residue formula to stacks of stable maps. This leads to a formula expressing Gromov-Witten invariants of projective space in terms of integrals over stacks of stable curves.

Contents

0 Introduction

The course is divided into three lectures. Lecture I is a short introduction to stacks. We try to give a few ideas about the philosophy of stacks and we give the definition of algebraic stack of finite type over a field. Our definition does not require any knowledge of schemes.

Lecture II introduces equivariant intersection theory as constructed by Edidin and Graham [5]. The basic constructions are explained in a rather easy special case. The localization property (in the algebraic context also due to Edidin-Graham [6]) is mentioned and proved for an example. We set up a general framework for using the localization property to localize integrals to the fixed locus, or subvarieties (substacks) containing the fixed locus.

In Lecture III we apply the localization formula to the stack of stable maps to $\mathbb{P}^r$. We deduce a formula giving the Gromov-Witten invariants of $\mathbb{P}^r$ (for any genus) in terms of integrals over stacks of stable curves $\overline{M}_{g,n}$. The proof given here is essentially complete, if sometimes sketchy. At the same time these lectures were given, Graber and Pandharipande [12] independently proved the same formula. Their approach is very different from ours. We avoid entirely the consideration of equivariant obstruction theories, on which [12] relies. The idea to use localization to compute Gromov-Witten invariants is, of course, due to Kontsevich (see [13], where the genus zero case is considered).

1 Lecture I: A short introduction to stacks

What is a variety?

We will explain Grothendieck's point of view that a variety is a functor.

Let us consider for example the affine plane curve $y^2 = x^3$. According to Grothendieck, the variety $y^2 = x^3$ is nothing but the 'system' of all solutions of the equation $y^2 = x^3$ in all rings. We restrict slightly and fix a ground filed k and consider instead of all rings only k-algebras of finite type (in other words quotients of polynomial rings in finitely many variables over k). So, following Grothendieck, we associate to every finitely generated k-algebra A, all solutions of $y^2 = x^3$ in A^2:

$$\begin{array}{rcl} h_V : (\text{f.g. } k\text{-algebras}) & \longrightarrow & (\text{sets}) \\ A & \longmapsto & \{(x,y) \in A^2 \mid y^2 = x^3\} \end{array}$$

Notice that h_V is actually a (covariant) functor: If $\phi : A \to B$ is a morphism of k-algebras and $(x, y) \in A^2$ satisfies $y^2 = x^3$, then $(\phi(x), \phi(y)) \in B^2$ satisfies $\phi(x)^2 = \phi(y)^3$. This makes precise what we mean by 'system' of solutions: We mean this functor. Grothendieck's point of view is that the variety $V \subset \mathbb{A}^2$ defined by $y^2 = x^3$ *is* this functor h_V. At least for affine varieties this is justified by the following corollary of *Yoneda's lemma.*

The (covariant) functor

$$\begin{array}{rcl} (\text{affine } k\text{-varieties}) & \longrightarrow & \mathrm{Funct}((\text{f.g. } k\text{-algebras}), (\text{sets})) \\ V & \longmapsto & h_V \end{array}$$

is fully faithful. Here Funct stands for the category of functors: objects are functors from (f.g. k-algebras) to (sets), morphisms are natural transformations. Because this functor is fully faithful we may think of (affine k-varieties) as a subcategory of $\mathrm{Funct}((\text{f.g. } k\text{-algebras}), (\text{sets}))$ and identify the variety V with the functor h_V.

Note 1 Given an affine variety V there are many ways to write it as the zero locus of a finite set of polynomials in some affine n-space. So one gets many functors h_V. This is not a problem, because all these functors are canonically isomorphic to the functor given by the affine coordinate ring $k[V]$ of V:

$$h_V(A) = \mathrm{Hom}_{k-\mathrm{alg}}(k[V], A)$$

For example, the affine coordinate ring of the curve $y^2 = x^3$ is $k[x, y]/(y^2 - x^3)$, and for every k-algebra A we have

$$\{(x, y) \in A^2 \mid y^2 = x^3\} = \mathrm{Hom}_{k-\mathrm{alg}}(k[x, y]/(y^2 - x^3), A).$$

Terminology: The functor h_V is the *functor represented by* V.

Once we have embedded the categroy (affine k-varieties) into $\mathrm{Funct}((\text{f.g. } k\text{-algebras}), (\text{sets}))$ we may enlarge the former inside the latter to get a larger category than (affine k-varieties), still consisting of 'geometric' objects.

For example, every finitely generated k-algebra A, reduced or not, gives rise to the functor

$$\begin{array}{rcl} h_{\mathrm{Spec}\, A} : (\text{f.g. } k\text{-algebras}) & \longrightarrow & (\text{sets}) \\ R & \longmapsto & \mathrm{Hom}_{k-\mathrm{alg}}(A, R) \end{array} .$$

The functor

$$\begin{array}{rcl} h_{\mathrm{Spec}} : (\text{f.g. } k\text{-algebras}) & \longrightarrow & \mathrm{Funct}((\text{f.g. } k\text{-algebras}), (\text{sets})) \\ A & \longmapsto & h_{\mathrm{Spec}\, A} \end{array}$$

is contravariant and fully faithful. This is Yoneda's lemma for the category (f.g. k-algebras). The above corollary of Yoneda's lemma follows from this and the equivalence of categories between affine k-varieties and their coordinate rings. Yoneda's lemma is completely formal and holds for every category in place of (f.g. k-algebras). The proof is a simple exercise in category theory.

In keeping with Grothendieck's philosophy of identifying a geometric object with the functor it represents, we write

$$\mathrm{Spec}\, A : (\text{f.g. } k\text{-algebras}) \longrightarrow (\text{sets})$$

for the functor $h_{\mathrm{Spec}\, A}$, and call it the *spectrum* of A. The full subcategory of $\mathrm{Funct}((\text{f.g. } k\text{-algebras}), (\text{sets}))$ consisting of functors isomorphic to functors of the form $\mathrm{Spec}\, A$ is called the *category of affine k-schemes of finite type*, denoted (aff/k).

To construct the functor h_V for a general k-variety V is a little tricky. Unless one knows scheme theory. Then it is easy, and we can do it for any k-scheme of finite type X:

$$\begin{array}{rcl} h_X : (\text{f.g. } k\text{-algebras}) & \longrightarrow & (\text{sets}) \\ A & \longmapsto & \mathrm{Hom}_{\text{schemes}}(\mathrm{Spec}\, A, X) \end{array}$$

It is then slightly less trivial than just Yoneda's lemma that one gets a (covariant) fully faithful functor

$$\begin{array}{rcl} h : (\text{f.t. } k\text{-schemes}) & \longrightarrow & \mathrm{Funct}((\text{f.g. } k\text{-algebras}), (\text{sets})) \\ X & \longmapsto & h_X \quad . \end{array}$$

(This is, in fact, part of what is known as *descent theory*.)

The largest subcategory of $\mathrm{Funct}((\text{f.g. } k\text{-algebras}), (\text{sets}))$ which still consists of 'geometric' objects is the category of finite type *algebraic spaces* over k. We will now describe this category (without using any scheme theory).

Algebraic spaces

First of all, to get a more 'geometric' picture, we prefer to think in terms of the category (aff/k) rather than the dual category (f.g. k-algebras).

Thus we replace Funct((f.g. k-algebras), (sets)) by the equivalent category $\mathrm{Funct}^*((\mathrm{aff}/k), (\mathrm{sets}))$, where Funct^* refers to the category of contravariant functors. Grothendieck calls $\mathrm{Funct}^*((\mathrm{aff}/k), (\mathrm{sets}))$ the category of *presheaves* on (aff/k).

We start by considering the covariant functor

$$\begin{array}{rcl} h : (\mathrm{aff}/k) & \longrightarrow & \mathrm{Funct}^*((\mathrm{aff}/k), (\mathrm{sets})) \\ X & \longmapsto & h_X \quad , \end{array}$$

where $h_X(Y) = \mathrm{Hom}_{k-\mathrm{schemes}}(Y, X) = \mathrm{Hom}_{k-\mathrm{alg}}(k[X], k[Y])$.

Note 2 The category (aff/k) containes fibered products (the dual concept in (f.g. k-algebras) is tensor product) and a final object $\mathrm{Spec}\, k$. The same is true for $\mathrm{Funct}^*((\mathrm{aff}/k), (\mathrm{sets}))$. Given a diagram

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathrm{Funct}^*((\mathrm{aff}/k), (\mathrm{sets}))$ the fibered product $W = X \times_Y Z$ is given by

$$\begin{array}{rcl} W(\mathrm{Spec}\, R) & = & X(\mathrm{Spec}\, R) \times_{Y(\mathrm{Spec}\, R)} Z(\mathrm{Spec}\, R) \\ & = & \{(x, z) \in X(\mathrm{Spec}\, R) \times Z(\mathrm{Spec}\, R) \mid \\ & & \quad f(\mathrm{Spec}\, R)(x) = g(\mathrm{Spec}\, R)(z) \in Y(\mathrm{Spec}\, R)\} \end{array}$$

A final object of $\mathrm{Funct}^*((\mathrm{aff}/k), (\mathrm{sets}))$ is the constant functor $\mathrm{Spec}\, R \mapsto \{\varnothing\}$. Here, of course, any one-element set in place of $\{\varnothing\}$ will do. Moreover, the functor h commutes with fibered products and final objects. One says that h is *left exact*.

Note 3 The category (aff/k) also contains direct sums (called *disjoint sums* in this context). If X and Y are affine k-schemes then their disjoint sum $Z = X \coprod Y$ has affine coordinate ring $A_Z = A_X \times A_Y$. Also, (aff/k) contains an initial object, the empty scheme, whose affine coordinate ring is the zero ring. We do not consider the corresponding notions in $\mathrm{Funct}^*((\mathrm{aff}/k), (\mathrm{sets}))$, the functor h does not commute with disjoint sums anyway.

Definition 4 Let X be an object of (aff/k) and $(X_i)_{i \in I}$ a family of objects *over* X (which means that each X_i comes endowed with a morphism $X_i \to X$). We call $(X_i)_{i \in I}$ a *covering* of X, if I is finite and the induced morphism $\amalg_{i \in I} X_i \to X$ is faithfully flat, i.e. flat and surjective.

Remark 5 This defines a *Grothendieck topology* on (aff/k).

Now that we have the notion of covering, we can define the notion of sheaf.

Definition 6 A *sheaf* on (aff/k) is an object X of $\mathrm{Funct}^*((\mathrm{aff}/k), (\mathrm{sets}))$ (i.e. a presheaf), satisfying the two sheaf axioms: Whenever $(U_i)_{i\in I}$ is a covering of an object U of (aff/k), we have

1. if $x, y \in X(U)$ are elements such that $x|U_i = y|U_i$, for all $i \in I$, then $x = y$, (Here $x|U_i$ denotes the image of x under $X(U) \to X(U_i)$.)

2. if $x_i \in X(U_i)$, $i \in I$, are given such that $x_i|U_{ij} = x_j|U_{ij}$, for all $(i, j) \in I \times I$, $(U_{ij} = U_i \times_U U_j)$ then there exists an element $x \in X(U)$ such that $x|U_i = x_i$, for all $i \in I$.

It is a basic fact from descent theory that for every (affine) k-scheme of finite type X, the functor h_X is a sheaf on (aff/k). The notion of covering in terms of faithful flatness is the most general notion of covering that makes this statement true.

Definition 7 An *algebraic space* (of finite type) over k is a sheaf X on (aff/k) such that

1. the diagonal $X \xrightarrow{\Delta} X \times X$ is quasi-affine,

2. there exists an affine scheme U and a smooth epimorphism $U \to X$.

Let us try to explain the meaning of *quasi-affine* and *smooth epimorphism* in this context. So let $f : X \to Y$ be an injective morphism of sheaves on (aff/k) (this means that for all objects U of (aff/k) the map $f(U) : X(U) \to Y(U)$ is injective). If U is an affine scheme and $U \to Y$ is a morphism and we form the fibered product

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathrm{Funct}^*((\mathrm{aff}/k), (\mathrm{sets}))$ then V is a subsheaf of U. Thus it makes sense to say that V is or is not a finite union of affine subschemes of U. Now the injection $f : X \to Y$ is called *quasi-affine*, if for all affine schemes U and for all morphisms $U \to Y$ (so equivalently for all elements of $Y(U)$) the pullback $V \subset U$ is a finite union of affine subschemes of U.

Now let X be a sheaf on (aff/k) such that the diagonal is quasi-affine. This implies that whenever we have two affine schemes U and V over X, then the fibered product $U \times_X V$ is a finite union of affine schemes. Now, in the situation of the above definition, the morphism $U \to X$ is called a smooth epimorphism, if for every affine scheme $V \to X$ the fibered product $U \times_X V$ can be covered by finitely many affine Zariski-open subschemes W_i such that for each i the morphism $W_i \to V$ is smooth and the induced morphism $\amalg W_i \to V$ is surjective.

Of course all k-varieties and k-schemes are algebraic k-spaces.

Definition 8 A *k-scheme* is an algebraic k-space X, which is locally in the Zariski-topology an affine scheme. This means that there exist affine k-schemes $U_1, \ldots, U_n$ and open immersions of algebraic spaces $U_i \to X$ such that $\coprod U_i \to X$ is surjective. (An open immersion of algebraic spaces $X \to Y$ is a morphism such that for every affine scheme $U \to Y$ the pullback $X \times_Y U \to U$ is an isomorphism onto a Zariski open subset.)

A *k-variety* is a k-scheme which is reduced and irreducible, which means that the U_i in the definition of scheme may be chosen reduced and irreducible with dense intersection.

One can prove that an algebraic space X is locally in the étale topology an affine scheme. This means that affine schemes $U_1, \ldots, U_n$ together with étale morphisms $U_i \to X$ can be found, such that $\coprod U_i \to X$ is an étale epimorphism. (The notion of étale epimorphism is defined as the notion of smooth epimorphism, above, using fibered products.)

Using such étale (or smooth) covers, one can do a lot of geometry on algebraic spaces. A vector bundle, for example, is a family of vector bundles E_i/U_i, together with gluing data $E_i|U_{i,j} \cong E_j|U_{i,j}$.

Groupoids

Definition 9 A *groupoid* is a category in which all morphisms are invertible.

Examples 10
1. Let X be a set. We think of X as a groupoid by taking X as set of objects and declaring all morphisms to be identity morphisms.
2. Let G be a group. We define the groupoid BG to have a single object with automorphism group G.
3. Let X be a G-set. Then we define the groupoid X_G to have set of objects X, and for two objects $x, y \in X$ we let $\mathrm{Hom}(x, y) = \{g \in G \mid$

$gx = y\}$. This groupoid is called the *transformation groupoid* given by the action of G on X.

4. Let $R \subset X \times X$ be an equivalence relation on the set X. Then we define an associated groupoid by taking as objects the elements of X and as morphisms the elements of R, where the element $(x, y) \in R$ is then a unique morphism from x to y.

We think of two groupoids as 'essentially the same' if they are equivalent as categories. We say that a groupoid is *rigid* if every object has trivial automorphism group, and *connected* if all objects are isomorphic. Every rigid groupoid is equal to the groupoid given by an equivalence relation. A groupoid is rigid if and only if it is equivalent to a groupoid given by a set as in Example 1, above. A groupoid is connected if and only if it is equivalent to a groupoid of type BG, for some group G. All these follow easily from the following well-known equivalence criterion.

Proposition 11 *Let $f : X \to Y$ be a morphism of groupoids (i.e. a functor between the underlying categories X and Y). Then f is an equivalence of categories if and only if f is fully faithful and essentially surjective.*

Remark 12 Groupoids form a 2-category. This means that the category of groupoids consists of

1. objects: groupoids
2. morphisms: functors between groupoids
3. 2-morphisms, or morphisms between morphisms: natural transformations between functors.

Note that this is a special type of 2-category, since all 2-morphisms are invertible. One should think of such a 2-category as a category where for any two objects X, Y the morphisms $\mathrm{Hom}(X, Y)$ form not a set but rather a groupoid.

Example 13 Another important example of a 2-category with invertible 2-morphisms is the (truncated) homotopy category:

1. objects: topological spaces
2. morphisms: continuous maps
3. 2-morphisms: homotopies up to reparametrization.

One may think of groupoids as generalized sets, or rather a common generalization of sets and groups. If we replace the category (sets) in the definition of algebraic space by the 2-category (groupoids), we get algebraic stacks. This is not a completely trivial generalization because of the complications arising from the fact that (groupoids) is a 2-category rather than a 1-category, like (sets).

We call a groupoid *finite*, if it has finitely many isomorphisms classes of objects and every object has a finite automorphism group. For a finite groupoid X we define its 'number of elements' by

$$\#(X) = \sum_x \frac{1}{\#\operatorname{Aut} x},$$

where the sum is taken over a set of representatives for the isomorphism classes.

Fibered products of groupoids

The fibered product is a construction that is not only basic for the theory of groupoids and stacks, but is also a good example of the philosophy of 2-categories.

Let

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a diagram of groupoids and morphisms. Then the fibered product $W = X \times_Y Z$ is the groupoids defined as follows: Objects of W are triples (x, ϕ, z), where $x \in \operatorname{ob} X$, $z \in \operatorname{ob} Z$ and $\phi : f(x) \to g(z)$ is a morphism in Y. A morphism in X from (x, ϕ, z) to (x', ϕ', z') is a pair (α, β), where $\alpha : x \to x'$ and $\beta : z \to z'$ are morphisms in X and Z, respectively, such that the diagram

$$\begin{tikzcd} f(x) \arrow[r, "\phi"] \arrow[d, "f(\alpha)"'] & g(z) \arrow[d, "g(\beta)"] \\ f(x') \arrow[r, "\phi'"] & g(z') \end{tikzcd}$$

commutes in Y.

The groupoid W comes together with two morphisms $W \to X$ and $W \to Z$ given by projecting onto the first and last components, respectively.

Moreover, W comes with a 2-morphism θ

$$
\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\theta \Nearrow"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\tag{1}
$$

making the diagram '2-commute', which just means that θ is an isomorphism from the composition $W \to X \to Y$ to the composition $W \to Z \to Y$. The 2-isomorphism θ is given by $\theta(x, \phi, z) = \phi$. It is a natural transformation by the very definition of W.

Example 14 If X, Y and Z are sets, then W is (canonically isomorphic to) the fibered product $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$ in the category of sets.

The 2-fibered product W satisfies a universal mapping property in the 2-category of groupoids. Namely, given any groupoid V with morphisms $V \to X$ and $V \to Z$ and a 2-isomorphism from $V \to X \to Y$ to $V \to Z \to Y$ (depicted in the diagram below by the 2-arrow crossing the dotted arrow), there exists a morphism $V \to W$ and 2-isomorphisms from $V \to X$ to $V \to W \to X$ and $V \to W \to Z$ to $V \to Z$ such that the diagram

$$
\begin{tikzcd}
V \arrow[dr, dotted, "\Nearrow" description] \arrow[drr, bend left, ""{name=U}] \arrow[ddr, bend right, ""{name=L}] & & \\
& W \arrow[r] \arrow[d] \arrow[dr, phantom, "\theta \Nearrow"] \arrow[u, phantom, "\Nearrow"] \arrow[l, phantom, "\Nearrow"] & Z \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

commutes, which amounts to a certain compatibility of the various 2-isomorphisms involved. (One should image this diagram as lying on the surface of a sphere.) The morphism $V \to W$ is unique up to unique isomorphism.

Whenever a diagram such as (1) satisfies this universal mapping property, we say that it is 2-cartesian (or just cartesian, because in a 2-category, 2-cartesian is the default value). In this case, W is equivalent to the fibered product constructed above.

If X is a G set, then we have two fundamental cartesian diagrams:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & pt \arrow[d] \\ X_G \arrow[r] & BG \end{tikzcd} \tag{2}$$

and

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "p"'] & X \arrow[d] \\ X \arrow[r] & X_G \end{tikzcd} \tag{3}$$

Here pt denotes the groupoid with one object and one morphism (necessarily the identity morphism of the object). If we write a set, we mean the set thought of as a groupoid. By σ and p we denote the action and the projection, respectively.

Diagram (3) is moreover 2-cocartesian[1]. Hence X_G satisfies the universal mapping property of a quotient of X by G in the category of groupoids. Note that in the category of sets the quotient set X/G satisfies the cocartesian property, but not the cartesian property (unless the action of G on X is free, in which case the set quotient X/G is equivalent to the groupoid quotient X_G). Thus quotients taken in the category of groupoids have much better properties than quotients taken in the category of sets. For example, we have

$$\#(X_G) = \frac{\#X}{\#G}$$

if X and G are finite.

Let X be a groupoid and let X_0 be the set of objects of X and X_1 the set of all morphisms of X. Let $s : X_1 \to X_0$ be the map associating with each morphism its source object, and $t : X_1 \to X_0$ the map associating with each morphism its target object. Then the diagram

$$\begin{tikzcd} X_1 \arrow[r, "t"] \arrow[d, "s"'] & X_0 \arrow[d, "\pi"] \\ X_0 \arrow[r, "\pi"] & X \end{tikzcd}$$

[1]The notion of 2-cocartesian is more subtle than one might be led to believe. The correct definition is not simply the dual notion to the 2-cartesian property explained above. It involves, instead of a square, a *cube*. For our purposes it is sufficient to remark that (3) is cocartesian with respect to test objects which are rigid groupoids, or even just sets. For such text objects, 2-cocartesian reduces to the usual notion of cocartesian.

is cartesian and cocartesian, where $\pi : X_0 \to X$ is the canonical morphism. Thus a groupoid may be thought of as the quotient of its object set by the action of the morphisms.

Algebraic stacks

We will subdivide the definition of algebraic stacks into three steps.

Prestacks

Prestacks are a generalization of presheaves (i.e. contravariant functors $(\mathrm{aff}/k) \to (\mathrm{sets})$).

Definition 15 A prestack is a (lax) contravariant functor $X : (\mathrm{aff}/k) \to (\mathrm{groupoids})$. This means that X is given by the data

1. for every affine k-scheme U a groupoid $X(U)$,
2. for every morphism of k-schemes $U \to V$ a morphism of groupoids $X(V) \to X(U)$,
3. for every composition of morphisms of k-schemes $U \to V \to W$ a natural transformation θ:

$$\begin{tikzcd} X(W) \arrow[r] \arrow[dr] & X(V) \arrow[d] \arrow[dl, phantom, "\stackrel{\theta}{\Rightarrow}"] \\ & X(U) \end{tikzcd}$$

(this means that θ is a natural transformation from the functor $X(W) \to X(U)$ to the composition of the functors $X(W) \to X(V) \to X(U)$).

This data is subject to the conditions

1. if $U \to U$ is the identity, then so is $X(U) \to X(U)$,
2. for each composition $U \to V \to W \to Z$ in (aff/k) a 2-cocycle condition expressing the compatibilities the various θ have to satisfy. Using the examples below as guide, this 2-cocycle condition is not difficult to write down. We leave this to the reader.

Examples 16 1. Each actual functor (presheaf) $(\mathrm{aff}/k) \to (\mathrm{sets})$ is a lax functor (prestack) $(\mathrm{aff}/k) \to (\mathrm{groupoids})$. All θ are identities in this case.

2. The following might be thought of as a prototype stack:

$$\begin{array}{rcl} \mathrm{Vect}_n : (\mathrm{aff}/k) & \longrightarrow & \text{(groupoids)} \\ U & \longmapsto & \text{(category of vector bundles of rank } n \text{ over } U \\ & & \text{with isomorphisms only)} \\ (U \to V) & \longmapsto & \text{pullback of vector bundles} \\ (U \to V \to W) & \longmapsto & \theta : \text{the canonical isomorphism of pullback} \\ & & \text{from } W \text{ to } U \text{ directly with pullback in two} \\ & & \text{steps via the intermediate } V. \end{array}$$

3. In this example all the θ are trivial again. Let G be an algebraic group over k and consider the functor

$$\begin{array}{rcl} \mathrm{pre}BG : (\mathrm{aff}/k) & \longrightarrow & \text{(groupoids)} \\ U & \longmapsto & B(G(U)) \\ (U \to V) & \longmapsto & \text{the morphism of groupoids} \\ & & B(G(V)) \to B(G(U)) \text{ induced by the} \\ & & \text{morphism of groups } G(V) \to G(U) \end{array}$$

Let us denote the category of contravariant lax functors from (aff/k) to (groupoids) by $\underline{\mathrm{Hom}}^*(\mathrm{aff}/k, \mathrm{groupoids})$. It is, of course, a 2-category. Its objects we have just defined. We leave it to the reader to explicate the morphisms and the 2-isomorphisms.

Given a lax functor X and an object x of the groupoid $X(U)$, where U is an affine k-scheme, we get an induced morphism

$$U \longrightarrow X$$

of lax functors (i.e., a natural transformation). We denote this morphism by the same letter:

$$x : U \to X.$$

The morphism x associates to $V \to U$ the pullback $x|V$.

A basic fact about $\underline{\mathrm{Hom}}^*(\mathrm{aff}/k, \mathrm{groupoids})$ is that it admits 2-fibered products, i.e. every diagram

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

can be completed to a cartesian diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\theta \nearrow"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

This is accomplished essentially by defining $W(U)$, for U an affine k-scheme, simply as the fibered product of $X(U)$ and $Z(U)$ over $Y(U)$.

Stacks

The notion of stacks generalizes the notion of sheaf on (aff/k).

Definition 17 A prestack $X : (\text{aff}/k) \to (\text{groupoids})$ is called a *stack* if it satisfies the following two stack axioms.

1. If U is an affine scheme and $x, y \in X(U)$ are objects of $X(U)$ then the presheaf

$$\begin{aligned} \text{Isom}(x,y) : (\text{aff}/U) &\longrightarrow (\text{sets}) \\ V &\longmapsto \text{Isom}(x|V, y|V) \end{aligned}$$

   is a sheaf on (aff/U).

2. X satisfies the *descent property*: Given an affine scheme U, with a cover (in the sense of Definition 4) $(U_i)_{i\in I}$, and given objects $x_i \in X(U_i)$, for all $i \in I$ and isomorphisms $\phi_{ij} : x_i|U_{ij} \to x_j|U_{ij}$, for all $(i,j) \in I \times I$, such that the (ϕ_{ij}) satisfy the obvious cocycle condition (for each $(i,j,k) \in I \times I \times I$), then there exists an object $x \in X(U)$ and isomorphisms $\phi_i : x_i \to x|U_i$, such that for all $(i,j) \in U_{ij}$ we have $\phi_j|U_{ij} \circ \phi_{ij} = \phi_i|U_{ij}$.

The data (x_i, ϕ_{ij}) is called a *descent datum* for X with respect to the covering (U_i); if (x, ϕ_i) exists, the descent datum is called *effective*. So the second stack axiom may be summarized by saying that every descent datum is effective.

Examples 18 1. Of course every sheaf is in a natural way a stack. Note how the stack axioms for presheaves reduce to the sheaf axioms.

2. The prestack Vect_n is a stack, since vector bundles satisfy the decent property.

3. The prestack preBG is not a stack. A descent datum for preBG with respect to the covering (U_i) of U is a Čech cocycle with values in G. It is effective if it is a boundary. Thus the Čech cohomology groups $H^1((U_i), G)$ contain the obstructions to preBG being a stack. Thus we let BG be the prestack whose groupoid of sections over $U \in (\text{aff}/k)$ is the category of principal G-bundles over U. This is then a stack. There is a general process associating to a prestack a stack, called *passing to the associated stack* (similar to sheafification). The stack BG is the stack associated to the prestack preBG.

Algebraic stacks

This notion generalizes the notion of algebraic space.

Definition 19 A stack $X : (\text{aff}/k) \to (\text{groupoids})$ is an *algebraic k-stack* if it satisfies

1. the diagonal $\Delta : X \to X \times X$ is representable and of finite type,
2. there exists an affine scheme U and a smooth epimorphism $U \to X$. Any such U is called a *presentation* of X.

The first property is a separation property. It can be interpreted in terms of the sheaves of isomorphisms occurring in the first stack axiom. It says that all these isomorphism sheaves are algebraic spaces of finite type. (The definition of representability is as follows. The morphism $X \to Y$ of stacks is *representable* if for all affine $U \to Y$ the base change $X \times_Y U$ is an algebraic space.)

The second property says that, locally, every stack is just an affine scheme. Thus one can do 'geometry' on an algebraic stack. For example, a vector bundle E over an algebraic stack X is a vector bundle E' on such an affine presentation U, together with gluing data over $U \times_X U$ (which is an algebraic space by the first property). For another example, an algebraic stack X is *smooth of dimension n*, if there exists a smooth presentation $U \to X$, where U is smooth of dimension $n + k$ and $U \to X$ is smooth of relative dimension k. (Smoothness of representable morphisms of stacks is defined 'locally', by pulling back to affine schemes, similarly to the case of algebraic spaces, above.) Note that according to this definition, negative dimensions make sense.

Examples 20 1. Of course, all algebraic spaces are algebraic stacks.

2. The stack Vect_n is algebraic. The isomorphism spaces are just twists of GL_n, and therefore algebraic. For a presentation, take $\operatorname{Spec} k \to \mathrm{Vect}_n$, given by the trivial vector bundle k^n over $\operatorname{Spec} k$. This is a smooth morphism of relative dimension n^2, since for any affine scheme U with rank n vector bundle E over U, the induced morphism $U \to \mathrm{Vect}_n$ pulls back to the bundle of frames of E, which is a principal GL_n-bundle, and hence smooth of relative dimension n^2. Note that this makes $Vect_n$ a smooth stack of dimension $-n^2$.

3. Let G be an algebraic group over k. To avoid pathologies assume that G is smooth (which is always the case if $\operatorname{char} k = 0$). Then BG is an algebraic stack. The proof of algebraicity is the same as for Vect_n, after all, Vect_n is isomorphic to BGL_n. Whenever P is a G-bundle over a scheme X, then we get an induced morphism $X \to BG$, giving rise to the cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\ X \arrow[r] & BG \end{tikzcd}$$

Therefore, $\operatorname{Spec} k \to BG$ is the universal G-bundle. Moreover, BG is smooth of dimension $-\dim G$.

4. If G is a (smooth) algebraic group acting on the algebraic space X, then we define an algebraic stack X/G as follows. For an affine scheme U, the groupoid $X/G(U)$ has as objects all pairs (P, ϕ), where $P \to U$ is a principal G-bundle and $\phi : P \to X$ is a G-equivariant morphism. One checks that X/G is an algebraic stack (for example, the canonical morphism $X \to X/G$ is a presentation) and that there are 2-cartesian diagrams

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & X/G \end{tikzcd} \tag{4}$$

and

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\ X/G \arrow[r] & BG \end{tikzcd} \tag{5}$$

2 Lecture II: Equivariant intersection theory

Intersection theory

For a k-scheme X let $A_*(X) = \bigoplus_k A_k(X)$, where $A_k(X)$ is the Chow group of k-cycles up to rational equivalence tensored with $\mathbb{Q}$. Readers not familiar with Chow groups may assume that the ground field is $\mathbb{C}$ and work with $A_k(X) = H_{2k}^{BM}(X^{\mathrm{an}})_{\mathbb{Q}}$ instead. Here X^{an} is the associated analytic space with the strong topology and BM stands for Borel-Moore homology, i.e. relative homology of a space relative to its one-point compactification. Everything works with this A_*, although the results are weaker.

Let also $A^*(X) = \bigoplus_k A^k(X)$ be the operational Chow cohomology groups of Fulton-MacPherson (see [9]), also tensored with $\mathbb{Q}$. If working with Borel-Moore homology as A_*, take $A^k(X) = H^{2k}(X^{\mathrm{an}})_{\mathbb{Q}}$, usual (singular) cohomology with $\mathbb{Q}$-coefficients.

The most basic properties of A^* and A_* are: $A^*(X)$ is a graded $\mathbb{Q}$-algebra, for every scheme X, and $A_*(X)$ is a graded $A^*(X)$-module, the operation being cap product

$$\begin{array}{rcl} A^k(X) \times A_n(X) & \longrightarrow & A_{n-k}(X) \\ (\alpha, \gamma) & \longmapsto & \alpha \cap \gamma \quad . \end{array}$$

Note that A^* and A_* exist more generally for Deligne-Mumford stacks. This was shown by A. Vistoli [16]. Deligne-Mumford stacks should be considered not too far from algebraic spaces or schemes (especially concerning their cohomological properties over $\mathbb{Q}$). Many moduli stacks (certainly all $\overline{M}_{g,n}(X, \beta)$) are of Deligne-Mumford type.

A Deligne-Mumford stack is an algebraic k-stack that is locally an affine scheme with respect to the *étale* topology. Thus a Deligne-Mumford stack X admits a presentation $p : U \to X$ (U affine) such that p is étale. This conditions implies, for example, that all automorphism groups are finite and reduced.

Equivariant theory

Let G be an algebraic group over k. To work *G-equivariantly* means to work in the category of algebraic G-spaces (i.e. algebraic k-spaces with G-action).

Now there is an equivalence of categories

$$\begin{array}{rcl} (\text{algebraic } G\text{-spaces}) & \longrightarrow & (\text{algebraic spaces } /BG) \\ X & \longmapsto & X/G \quad . \end{array} \tag{6}$$

Here (algebraic G-spaces) is the category of algebraic k-spaces with G-action and equivariant morphisms, (algebraic spaces $/BG$) is the category of algebraic stacks over BG which are representable over BG. So an object of (algebraic spaces $/BG$) is an algebraic stack X together with a representable morphism $X \to BG$. A morphism in (algebraic spaces $/BG$) from $X \to BG$ to $Y \to BG$ is an isomorphism class of pairs (f, η), where $f : X \to Y$ is a morphism of algebraic stacks and η a 2-morphism making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \arrow[dl, phantom, "\overset{\eta}{\Rightarrow}"] \\ & BG \end{tikzcd}$$

commute. The inverse of the functor (6) is defined using the construction of Diagram (5).

Defining equivariant Chow groups $A^*_G(X)$ and $A^G_*(X)$, for a G-space X, is equivalent to defining Chow groups $A^*(X/G)$ and $A_*(X/G)$ for stacks of the form X/G, i.e. quotient stacks.

If the quotient stack X/G is an algebraic space, then $A^G_*(X) = A_*(X/G)$ and $A^*_G(X) = A^*(X/G)$. In the general case, the construction is due to Edidin-Graham [5]. They proceed as follows. Assume that G is linear (and separable, to avoid certain pathologies in positive characteristic).

First define $A^G_p(X) = A_p(X/G)$ for p fixed. Choose a representation $G \to GL(V)$, such that there exists a G-invariant open subset U in the vector space V on which G acts freely (i.e. such that U/G is a space) and such that the complement $Z = V - U$ has codimension

$$\operatorname{codim}(Z, V) > \dim X - \dim G - p \quad .$$

The representation V of G associates to the principal G-bundle $X \to X/G$ a vector bundle over X/G. It is given by $X \times_G V = X \times V/G$, where G acts on $X \times V$ by $(x, v) \cdot g = (xg, g^{-1}v)$. It is not a space, but the open substack $X \times_G U \subset X \times_G V$ certainly is (the morphism $X \times_G U \to U/G$ is representable and U/G is already a space). Thus we have the following cartesian diagram.

$$\begin{tikzcd} X \times U \arrow[r, "\subset"] \arrow[d] & X \times V \arrow[r] \arrow[d] & X \arrow[d] \\ X \times_G U \arrow[r, "\subset"] & X \times_G V \arrow[r] & X/G \end{tikzcd}$$

The vertical maps are principal G-bundles, hence smooth epimorphisms. The inclusions on the left are open immersions with complement of codimension $> \dim X - \dim G - p$. The horizontal maps on the right are vector bundles of rank $\dim V$.

Having chosen V and $U \subset V$, we now define

$$A_p(X/G) = A_{p+\dim V}(X \times_G U),$$

which makes sense, because for a reasonable theory of Chow groups for quotient stacks we should have

$$A_p(X/G) = A_{p+\dim V}(X \times_G V),$$

since the Chow group of a vector bundle should be equal to the Chow group of the base, but shifted by the rank of the vector bundle, and

$$A_{p+\dim V}(X \times_G V) = A_{p+\dim V}(X \times_G U),$$

since the complement has dimension $\dim X \times_G Z < p + \dim V$, and cycles of dimension $< k$ should not affect A_k.

This definition is justified by giving rise to an adequate theory. For example, the definition is independent of the choice of V and $U \subset V$, as long as the codimension requirement is satisfied. This is proved by the 'double fibration argument', see [5].

As an example, let us work out what we get for $X/G = B\mathbb{G}_{\mathrm{m}}$. Consider the action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{A}^n$, given by scalar multiplication $\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^n \to \mathbb{A}^n$, $(t,x) \mapsto tx$. A principal bundle quotient exists for $U = \mathbb{A}^n - \{0\}$ and $Z = \{0\}$ has codimension n. Thus this representation is good enough to calculate $A_p(B\mathbb{G}_{\mathrm{m}})$ for $n > -1 - p \Longleftrightarrow p \geq -n$. Moreover, by definition, we have for all $p \geq -n$

$$A_p(B\mathbb{G}_{\mathrm{m}}) = A_{p+n}(\mathbb{P}^{n-1}).$$

In particular,

$$\begin{array}{rcl} A_p(B\mathbb{G}_{\mathrm{m}}) & = & 0, \quad \text{for all } p \geq 0 \\ A_{-1}(B\mathbb{G}_{\mathrm{m}}) & = & A_{n-1}(\mathbb{P}^{n-1}) \\ A_{-2}(B\mathbb{G}_{\mathrm{m}}) & = & A_{n-2}(\mathbb{P}^{n-1}), \quad \text{etc.} \end{array}$$

To see how these groups fit together for various n, let $n' \geq n$ and consider a projection $\mathbb{A}^{n'} \to \mathbb{A}^n$. This induces the projection with center $\ker(\mathbb{A}^{n'} \to$

$\mathbb{A}^n) = \mathbb{A}^{n'-n}$ from $\mathbb{P}^{n'-1}$ to $\mathbb{P}^{n-1}$.

$$\begin{tikzcd} U \arrow[r, "\subset"] \arrow[d] & \mathbb{P}^{n'-1} \\ \mathbb{P}^{n-1} & \end{tikzcd}$$

Here the vertical map is a vector bundle of rank $n' - n$ and the horizontal map is the inclusion of the complement of the center of projection $\mathbb{P}^{n'-n-1}$. Thus we have for all $p \geq -n$

$$A_{p+n}(\mathbb{P}^{n-1}) = A_{p+n+n'-n}(U) = A_{p+n'}(U) = A_{p+n'}(\mathbb{P}^{n'-1}).$$

So we have independence of $A_p(B\mathbb{G}_{\mathrm{m}})$ on the choice of n. This is a special case of the double fibration argument.

Under the identification $A_{p+n}(\mathbb{P}^{n-1}) = A_{p+n'}(\mathbb{P}^{n'-1})$ the hyperplane $[H]$ in $\mathbb{P}^{n-1}$ corresponds to the hyperplane $[H]$ in $\mathbb{P}^{n'-1}$. The same is true for all intersections $[H]^k$. We write $h = [H]$ and thus we have for all $k \in \mathbb{Z}$

$$A_k(B\mathbb{G}_{\mathrm{m}}) = \mathbb{Q}h^{-1-k},$$

where we agree that all negative powers of h are 0.

The equivariant cohomology groups $A_G^*(X) = A^*(X/G)$ are defined analogously to the usual A^*, namely by operating on $A_*^G(Y)$, for all equivariant $Y \to X$, where Y is a space (or equivalently all representable $Y \to X/G$, where Y is a stack).

In our example $B\mathbb{G}_{\mathrm{m}}$ we get $A^*(B\mathbb{G}_{\mathrm{m}}) = A^*_{\mathbb{G}_{\mathrm{m}}}(pt) = \mathbb{Q}[c]$, where c is the Chern class of the universal line bundle and is in degree $+1$. Whenever X is a $\mathbb{G}_{\mathrm{m}}$-space we get via the standard representation of $\mathbb{G}_{\mathrm{m}}$ a line bundle over $X/\mathbb{G}_{\mathrm{m}}$ (or equivalently an equivariant line bundle $X \times \mathbb{A}^1$ over X). The operation of $c \in A^*(B\mathbb{G}_{\mathrm{m}})$ on $A_*(X/\mathbb{G}_{\mathrm{m}})$ is through the Chern class of this line bundle. We have $c \cdot h^k = h^{k-1}$, and so we see that $A_*(B\mathbb{G}_{\mathrm{m}})$ is a free $A^*(B\mathbb{G}_{\mathrm{m}}) = \mathbb{Q}[c]$-module on $h^0 \in A_{-1}(B\mathbb{G}_{\mathrm{m}})$. We may think of h^0 as the fundamental class of $B\mathbb{G}_{\mathrm{m}}$ (it corresponds to $[\mathbb{P}^{n-1}]$ under any realization $A_{-1}(B\mathbb{G}_{\mathrm{m}}) = A_{n-1}(\mathbb{P}^{n-1})$.)

More generally, if T is an algebraic torus with character group M, then $A^*(BT) = \operatorname{Sym}_{\mathbb{Q}} M_{\mathbb{Q}} =: R_T$, canonically. (Note how c comes from the canonical character $\mathrm{id} : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$.) Moreover, $A_*(BT)$ is a free R_T-module of rank one on the generator $[BT]$ in degree $-\dim T$.

We shall be only interested in the case where the group $G = T$ is a torus. Then for all T-spaces X, we have that $A_T^*(X)$ is an R_T-algebra and

$A_*^T(X)$ is an R_T-module. Therefore, R_T is the natural ground ring to work over. As in the usual case (the non-equivariant case, where one passes from $A^*(pt) = \mathbb{Z}$ to $\mathbb{Q}$) we want to pass from R_T to its quotient field. However, so as to not loose the grading, we only localize at the multiplicative system of homogeneous elements of positive degree, and call the resulting ring Q_T. Then we may tensor all $A_T^*(X)$ and $A_*^T(X)$ with Q_T. Still better, though, is to first pass to the completion of R_T at the augmentation ideal, $\hat{R}_T$ and then invert the homogeneous elements of positive degree to obtain $\hat{Q}_T$.

Comparing equivariant with usual intersection theory

For a G-space X, there is a canonical morphism $X \to X/G$, which is smooth of relative dimension $\dim G$. It is, in fact, a principal G-bundle. Thus flat pullback defines a homomorphism $A_*^G(X) \to A_*(X)$ of degree $\dim G$. 'Usual' pullback defines $A_G^*(X) \to A^*(X)$ preserving degrees.

Lemma 21 *The top-dimensional map $A^G_{\dim X - \dim G}(X) \to A_{\dim X}(X)$ is an isomorphism.*

Proof By using the definitions, this reduces to proving that for a G-bundle of spaces, the top-dimensional Chow-groups agree. □

This isomorphism defines the fundamental class $[X_G]$ of X/G in $A^G_{\dim X - \dim G}(X)$.

Note 22 If one works with cohomology one gets a Leray spectral sequence

$$H_G^i(X, H^j(G)) \Longrightarrow H^{i+j}(X, \mathbb{Q}).$$

Localization

Let X be a T-space and $Y \subset X$ a closed T-invariant subspace such that on $U = X - Y$ the torus T acts without fixed points. Then we have the proper pushforward map

$$\iota_* : A_*^T(Y) \longmapsto A_*^T(X)$$

induced by the inclusion $\iota : Y \to X$.

Proposition 23 *After tensoring with Q_T*

$$\iota_* : A_*^T(Y) \otimes_{R_T} Q_T \longmapsto A_*^T(X) \otimes_{R_T} Q_T$$

is an isomorphism.

Proof Reduces the the case $Y = \varnothing$ and $X = U$, when the claim is that $A_*^T(X) \otimes_{R_T} Q_T = 0$. For details, see [6]. $\square$

Rather than studying the proof of this proposition, let us study an example.

Consider the torus $T = \mathbb{G}_{\mathrm{m}}{}^{n+1}$ and $M = \hat{T}$, with basis $\lambda_0, \ldots, \lambda_n$ and $A_T^*(pt) = A^*(BT) = R_T = \mathbb{Q}[\lambda_0, \ldots, \lambda_n]$. Let us denote the fundamental class of BT by $\mathfrak{t}$. Then we have $A_*^T(pt) = A_*(BT) = tR_T = t\mathbb{Q}[\lambda_0, \ldots, \lambda_n]$. Let $X = \mathbb{P}^n$ and consider the action of T on $\mathbb{P}^n$ given by

$$t \cdot \langle x_0, \ldots, x_n \rangle = \langle \lambda_0(t)x_0, \ldots, \lambda_n(t)x_n \rangle.$$

Take $Y = \{P_0, \ldots, P_n\}$, where $P_i = \langle 0, \ldots, 0, 1, 0, \ldots, 0 \rangle$, the 1 being in the ith position. Then localization (Proposition 23) says that

$$\iota_* : \bigoplus_{i=0}^n A_*^T(\{P_i\}) \otimes Q_T \longrightarrow A_*^T(\mathbb{P}^n) \otimes Q_T$$

is an isomorphism. Since everything is smooth, we may translate this into a statement about cohomology:

$$\iota_! : \bigoplus_{i=0}^n A_T^*(\{P_i\}) \otimes Q_T \longrightarrow A_T^*(\mathbb{P}^n) \otimes Q_T$$

is an isomorphism of degree $+n$.

To understand this isomorphism note that $\mathbb{P}^n/T \to BT$ is a $\mathbb{P}^n$-bundle, namely the projective bundle corresponding to the vector bundle E on BT given by the action of T on $\mathbb{A}^{n+1}$. Hence we have

$$\begin{aligned} A_T^*(\mathbb{P}^n) &= A^*(\mathbb{P}^n/T) \\ &= A^*(BT)[\xi]/\xi^{n+1} - c_1(E)\xi^n + \ldots + (-1)^{n+1}c_{n+1}(E) \\ &= \mathbb{Q}[\lambda_0, \ldots, \lambda_n][\xi]/\xi^{n+1} - \ldots + (-1)^{n+1}c_{n+1}(E). \end{aligned}$$

Now E is a sum of line bundles, each associated to one of the characters $\lambda_0, \ldots, \lambda_n$. Hence we have $c_i(E) = \sigma_i(\lambda_0, \ldots, \lambda_n)$, the symmetric function of degree i in $\lambda_0, \ldots, \lambda_n$. In other words,

$$\sum_{i=0}^{n+1} (-1)^i c_i(E)\xi^{n+1-i} = \prod_{i=0}^n (\xi - \lambda_i),$$

so that

$$A_T^*(\mathbb{P}^n) = \mathbb{Q}[\lambda_0, \ldots, \lambda_n, \xi] / \prod_{i=0}^n (\xi - \lambda_i).$$

Hence we have

$$\begin{aligned} A_T^*(\mathbb{P}^n) \otimes_{R_T} Q_T &= Q_T[\xi]/\prod_{i=0}^n (\xi - \lambda_i) \\ &= \prod_{i=0}^n Q_T[\xi]/(\xi - \lambda_i) \\ &= \prod_{i=0}^n Q_T \\ &= \prod_{i=0}^n A_T^*(P_i) \otimes_{R_T} Q_T, \end{aligned}$$

by the Chinese remainder theorem. This map

$$A_T^*(\mathbb{P}^n) \otimes_{R_T} Q_T \longrightarrow \prod_{i=0}^n A_T^*(P_i) \otimes_{R_T} Q_T$$

is of degree 0 and induced by ι^*. (Note that $\xi = c_1(\mathcal{O}(1))$ pulls back to λ_i at P_i, which is the character of the action of T on the fiber $\mathcal{O}(1)(P_i)$.) If we compose with

$$\prod_{i=0}^n A_T^*(P_i) \otimes Q_T \longrightarrow \prod_{i=0}^n A_T^*(P_i) \otimes Q_T$$

which is division by the tops Chern class of the tangent space (i.e. normal bundle) we get the inverse of the above map $\iota_!$. The tangent space $T_{\mathbb{P}^n}(P_i)$ has weights $(\lambda_j - \lambda_i)_{j \neq i}$ and so we divide by $\prod_{j \neq i}(\lambda_j - \lambda_i)$ in the ith component.

The residue formula

Let us return to the setup of Proposition 23. Moreover, assume that the inclusion $\iota : Y \to X$ is T-equivariantly the pullback of a regular immersion $\nu : V \to W$

$$\begin{tikzcd} Y \arrow[r, "\iota"] \arrow[d, "g"'] & X \arrow[d] \\ V \arrow[r, "\nu"] & W. \end{tikzcd} \tag{7}$$

Then we have the self intersection formula

$$\nu^! \iota_*(\alpha) = e(g^* N_{V/W})\alpha, \quad \text{for all } \alpha \in A_*^T(Y),$$

where e stands for the top Chern (i.e. Euler) class. So if $e(g^* N_{V/W}) \in A_T^*(Y) \otimes Q_T$ is invertible, we have

$$\alpha = \frac{\nu^! \iota_* \alpha}{e(g^* N)},$$

and we have identified the inverse of the localization isomorphism ι_*, namely

$$\frac{1}{e(g^* N)} \nu^! : A_*^T(X) \otimes Q_T \longrightarrow A_*^T(Y) \otimes Q_T.$$

That $e(g^* N)$ is invertible, is in practise easily verified, one just has to check that the weights of $g^* N$ at the fixed points of X under T are non-zero. If X is smooth and $\iota = \nu$, then it is a theorem that these weights are always non-zero and so $e(N)$ is always invertible.

Let us from now assume that $e(g^* N)$ is, indeed, invertible in $A_*^T(Y) \otimes Q_T$. Then we have for all $\beta \in A_*^T(X)$

$$\beta = \iota_* \frac{\nu^! \beta}{e(g^* N)}.$$

If X is smooth and $\iota = \nu$, we will want to apply this to $[X_T] \in A_*^T(X)$:

$$[X_T] = \iota_* \frac{[Y_T]}{e(N_{Y/X})}.$$

So if $\alpha \in A_T^*(X)$ we have

$$\alpha[X_T] = \iota_* \frac{\iota^*(\alpha)[Y_T]}{e(N_{Y/X})}$$

in $A_*^T(X)$.

Now assume that X is moreover proper. Then $X/T \to BT$ is proper and proper pushforward gives a homomorphism $\deg^T : A_*^T(X) \otimes Q_T \to A_*^T(pt) \otimes Q_T = \mathrm{t}Q_T$ and we get

$${}^T\!\!\int_X \alpha := \deg^T(\alpha[X_T]) = \deg^T\Big(\frac{\iota^*(\alpha)[Y_T]}{e(N_{Y/X})}\Big) = {}^T\!\!\int_Y \frac{\iota^*(\alpha)}{e(N_{Y/X})},$$

an equation in $A_*^T(pt) \otimes_{R_T} Q_T = \mathrm{t}Q_T$.

Now consider the cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & pt \arrow[d] \\ X/T \arrow[r] & BT. \end{tikzcd}$$

Since flat pullback commutes with proper pushforward, we get an induced commutative diagram

$$
\begin{tikzcd}
A_*(X) \arrow[r, "\deg"] & \mathbb{Q} \\
A_*^T(X) \arrow[u] \arrow[r, "\deg^T"] & \mathfrak{t}\mathbb{Q}[\lambda_0, \dots, \lambda_n], \arrow[u, "\theta"']
\end{tikzcd}
\tag{8}
$$

where the homomorphism $\theta : \mathfrak{t}\mathbb{Q}[\lambda_0, \dots, \lambda_n] \to \mathbb{Q}$ is given by sending $\mathfrak{t}$ to 1 and the λ_i to 0. Diagram (8) fits into the larger diagram

$$
\begin{tikzcd}
A_*(X) \arrow[r, "\deg"] & \mathbb{Q} \\
A_*^T(X) \arrow[u] \arrow[r, "\deg^T"] \arrow[d] & \mathfrak{t}R_T \arrow[u, "\theta"'] \arrow[d] \\
A_*^T(X) \otimes Q_T \arrow[r, "\deg^T"] & \mathfrak{t}Q_T \\
A_*^T(Y) \otimes Q_T. \arrow[u, "\iota_*", "\cong"'] \arrow[ur, "\deg^T"'] &
\end{tikzcd}
\tag{9}
$$

Corollary 24 (Residue Formula) *1. Assume X is smooth and $\iota = \nu$. If $a \in A^{\dim X}(X)$ comes from $\alpha \in A_T^{\dim X}(X)$, then*

$$
{}^T\!\!\int_Y \frac{\iota^*(\alpha)}{e(N_{Y/X})} \in \mathfrak{t}Q_T
$$

is contained in the submodule $\mathfrak{t}\mathbb{Q}$ and we have

$$
\int_X a = \deg a[X] = \theta \deg^T \alpha[X_T] = \theta\, {}^T\!\!\int_Y \frac{\iota^*(\alpha)}{e(N_{Y/X})}.
$$

The θ in this formula only serves to remove the factor of $\mathfrak{t}$.

2. General case. Assume $\beta \in A_*^T(X)$. Write b for the corresponding element of $A_*(X)$. Let $\alpha \in A_T^*(X)$ and write a for the corresponding element of $A^*(X)$. Then if $\deg \beta - \deg \alpha = -\dim T$, then $\deg b - \deg a = 0$ and

$$
\int_b a = \theta\, {}^T\!\!\int_\beta \alpha = \theta \deg^T \alpha \cap \beta = \theta \deg^T \alpha \cap \iota_* \frac{\nu^! \beta}{e(g^* N)} \tag{10}
$$

$$
= \theta \deg^T \left(\iota^*(\alpha) \cap \frac{\nu^! \beta}{e(g^* N)} \right) = \theta\, {}^T\!\!\int_{\nu^! \beta} \frac{\iota^* \alpha}{e(g^* N)}.
$$

Again, this is to be interpreted to mean that

$$ {}^T\!\!\int_{\nu^! \beta} \frac{\iota^* \alpha}{e(g^* N)} \in \mathfrak{t} Q_T $$

is contained in $\mathfrak{t}\mathbb{Q}$ *and after removing* $\mathfrak{t}$ *we get* $\int_b a$.

Proof This is just a simple diagram chase using (9) and keeping track of degrees. □

Remark 25 1. Evaluating the rational function of degree zero $\theta\, {}^T\!\!\int_Y \frac{\iota^*(\alpha)}{e(N_{Y/X})}$ at an element $\mu \in M^\vee$ corresponds to restricting the action of T to the corresponding one-parameter subgroup. For a generic one-parameter subgroup the fixed locus of T and of the one-parameter subgroup will be the same and the denominator of ${}^T\!\!\int_Y \frac{\iota^*(\alpha)}{e(N_{Y/X})}$ will not vanish at μ. Then ${}^T\!\!\int_Y \frac{\iota^*(\alpha)}{e(N_{Y/X})}$ can be calculated by evaluating at μ. This is also how one evaluates in practise.

2. The standard way to ensure that a comes from α is to take polynomials in Chern classes of equivariant vector bundles.

3. Assume that Y is the fixed locus. Then $A^T_*(Y) = A_*(Y) \otimes_{\mathbb{Q}} A_*(BT)$ and $A^*_T(Y) \supset A^*(Y) \otimes_{\mathbb{Q}} R_T$. If $\iota^*(\alpha) \in R_T \subset A^*_T(Y)$, then

$$ {}^T\!\!\int_{\nu^! \beta} \frac{\iota^* \alpha}{e(g^* N)} = \iota^*(\alpha)\, {}^T\!\!\int_{\nu^! \beta} \frac{1}{e(g^* N)} $$

by the projection formula.

4. Also, if T acts trivially on Y, and $N_{Y/X}$ has a filtration with line bundle quotients L_i, then $e(N_{Y/X}) = \prod_i (c(L_i) + \lambda_i)$, where $c(L_i) \in A^*(Y)$ is the Chern class of L_i and $\lambda_i \in R_T$ the weight of T on L_i. This gives a very explicit form of the Bott residue formula.

Example 26 Let T operate on $\mathbb{P}^1$, in such a way that 0 and ∞ are the fixed points of T. Let E be an equivariant vector bundle on $\mathbb{P}^1$. Then $E(0)$ and $E(\infty)$ are representations of T. Let $\lambda_1 \ldots, \lambda_r$ be the weights of T on $E(0)$ and $\mu_1 \ldots, \mu_r$ the weights of T on $E(\infty)$. Also, let ω be the character through which T acts on $\mathbb{P}^1$, i.e. $t \cdot 1 = \omega(t)$. Assume that $H^1(\mathbb{P}^1, E) = 0$. Then we can calculate the weights of T on $H^0(\mathbb{P}^1, E)$ by equivariant Riemann-Roch: Let $\alpha_1, \ldots, \alpha_n$ be these weights. Then we have (apply Riemann-Roch to $\mathbb{P}^1/T \xrightarrow{\pi} BT$):

$$ \operatorname{ch}(H^0(\mathbb{P}^1, E)) = \deg^T(\operatorname{ch}(E) \operatorname{td}(T_\pi) \cap [\mathbb{P}^1_T]) $$

or

$$\sum_{i=1}^{n} e^{\alpha_i} = \frac{\mathrm{ch}(E(0))\,\mathrm{td}(T_{\mathbb{P}^1}(0))}{c_1(T_{\mathbb{P}^1}(0))} + \frac{\mathrm{ch}(E(\infty))\,\mathrm{td}(T_{\mathbb{P}^1}(\infty))}{c_1(T_{\mathbb{P}^1}(\infty))},$$

by localization. Now since $\mathrm{td}(x) = \frac{x}{1-e^{-x}}$ and the weight of T on $T_{\mathbb{P}^1}(0)$ is ω and on $T_{\mathbb{P}^1}(\infty)$ is $-\omega$, we get

$$\sum_{i=1}^{n} e^{\alpha_i} = \frac{\mathrm{ch}(E(0))}{1-e^{-\omega}} + \frac{\mathrm{ch}(E(\infty))}{1-e^{\omega}}$$

or

$$\sum_{i=1}^{n} e^{\alpha_i} = \frac{\sum e^{\lambda_j}}{1-e^{-\omega}} + \frac{\sum e^{\mu_j}}{1-e^{\omega}}$$

in $\hat{Q}_T$. Note that we have uncapped with $[BT]$.

This determines the α_i uniquely. Useful to calculate the α_i in this context is the formula (which holds for all $a, b \in \mathbb{Z}$)

$$\frac{e^{a\omega}}{1-e^{\omega}} + \frac{e^{b\omega}}{1-e^{-\omega}} = \sum_{n=a}^{b} e^{n\omega},$$

where for $a \geq b+1$ we set $\sum_{n=a}^{b} e^{n\omega} = -\sum_{i=b+1}^{a-1} e^{i\omega}$.

3 Lecture III: The localization formula for Gromov-Witten invariants

Using the localization formula is one of the most useful methods we have to calculate Gromov-Witten invariants, besides the WDVV-equations (i.e. the associativity of the quantum product) and its analogues for higher (but still very low) genus. The idea of applying the Bott formula in this context is due to Kontsevich [13]. It has been used by Givental [11] to verify the predictions of Mirror symmetry for complete intersections in toric varieties.

If the variety we are interested in has finitely many fixed points under a torus action, the Bott formula reduces the calculation of its Gromov-Witten invariants to a calculation on $\overline{M}_{g,n}$ and a combinatorial problem. In this lecture we will treat the case of projective space $\mathbb{P}^r$.

Let the ground field be of characteristic 0. Let $\overline{M}_{g,n}(\mathbb{P}^r, d)$ denote the stack of stable maps of degree d to $\mathbb{P}^r$, whose source is a genus g curve with n marked points. For an affine k-scheme U the groupoid

$$\overline{M}_{g,n}(\mathbb{P}^r, d)(U)$$

is the groupoid of such stable maps parameterized by U. These are diagrams

$$\begin{array}{ccc} C & \xrightarrow{f} & \mathbb{P}^r \\ {\scriptstyle\pi}\big\downarrow & & \\ U & & \end{array}$$

where $\pi : C \to U$ is a family of prestable curves with n sections and f is a family of maps of degree d, such that the stability condition is satisfied (see, for example, [13], [14], [10], [4], [2]). Evaluation at the n marks defines a morphism

$$\mathrm{ev} : \overline{M}_{g,n}(\mathbb{P}^r, d) \longrightarrow (\mathbb{P}^r)^n.$$

Gromov-Witten invariants are the induced linear maps

$$\begin{aligned} A^*(\mathbb{P}^r)^{\otimes n} &\longrightarrow \mathbb{Q} \\ a_1 \otimes \ldots \otimes a_n &\longmapsto \int_{[\overline{M}_{g,n}(\mathbb{P}^r,d)]} \mathrm{ev}^*(a_1 \otimes \ldots a_n). \end{aligned}$$

For $g > 0$ the cycle $[\overline{M}_{g,n}(\mathbb{P}^r, d)]$ is the 'virtual fundamental class' of $\overline{M}_{g,n}(\mathbb{P}^r, d)$ (see [2], [3], [1] or [15]). This is a carefully constructed cycle giving rise to a consistent theory of Gromov-Witten invariants (i.e., a so-called cohomological field theory, [14]). The usual fundamental cycle is, in general, not even in the correct degree, as $\overline{M}_{g,n}(\mathbb{P}^r, d)$ may have higher dimension than expected, because of the presence of obstructions.

Now consider the torus $T = {\mathbb{G}_{\mathrm{m}}}^{r+1}$ with character group M, whose canonical generators are denoted $\lambda_0 \ldots, \lambda_n$. Then $R_T = \mathbb{Q}[\lambda_0, \ldots, \lambda_r]$ and $Q_T \subset \mathbb{Q}(\lambda_0, \ldots, \lambda_r)$. The torus T acts on $\mathbb{P}^r$ by

$$\begin{aligned} T \times \mathbb{P}^r &\longrightarrow \mathbb{P}^r \\ (t, \langle x_0, \ldots, x_r \rangle) &\longmapsto \langle \lambda_0(t) x_0, \ldots, \lambda_r(t) x_r \rangle. \end{aligned}$$

We get an induced action of T on $\overline{M}_{g,n}(\mathbb{P}^r, d)$: given $t \in T(U)$ and

$$\begin{array}{ccc} C & \xrightarrow{f} & \mathbb{P}^r \\ {\scriptstyle\pi}\big\downarrow & & \\ U & & \end{array}$$

in $\overline{M}_{g,n}(\mathbb{P}^r,d)(U)$ we define $t\cdot(C,f)=(C,t\circ f)$, where $(C,t\circ f)$ stands for

$$\begin{tikzcd} C \arrow[r, "(\pi,f)"] \arrow[d, "\pi"'] & U\times\mathbb{P}^r \arrow[r, "t"] & \mathbb{P}^r \\ U & & \end{tikzcd}$$

We leave it as an exercise, to turn this into an action of the group $T(U)$ on the groupoid $\overline{M}_{g,n}(\mathbb{P}^r,d)(U)$, i.e., actions on the morphism and object sets compatible with all the groupoid structure maps. Compatibility under change of U gives the action of the algebraic group T on the algebraic stack $\overline{M}_{g,n}(\mathbb{P}^r,d)$.

The same general arguments that allow the construction of the virtual fundamental class of $\overline{M}_{g,n}(\mathbb{P}^r,d)$ give rise to an equivariant virtual fundamental class $[\overline{M}_{g,n}(\mathbb{P}^r,d)_T]\in A^T_*(\overline{M}_{g,n}(\mathbb{P}^r,d))$, which pulls back to the usual virtual fundamental class $[\overline{M}_{g,n}(\mathbb{P}^r,d)]\in A_*(\overline{M}_{g,n}(\mathbb{P}^r,d))$. We shall apply Formula (10) with $\beta=[\overline{M}_{g,n}(\mathbb{P}^r,d)_T]$ and $b=[\overline{M}_{g,n}(\mathbb{P}^r,d)]$.

If $\alpha_1,\ldots,\alpha_n\in A^*_T(\mathbb{P}^r)$ and $a_1,\ldots,a_n$ are the corresponding classes in $A^*(\mathbb{P}^r)$, the induced Gromov-Witten invariants are given by

$$\begin{aligned} &\int_{[\overline{M}_{g,n}(\mathbb{P}^r,d)]} \mathrm{ev}^*(a_1\otimes\ldots\otimes a_n) \\ &\quad= \theta \;{}^T\!\!\!\int_{\nu^![\overline{M}_{g,n}(\mathbb{P}^r,d)_T]} \frac{\iota^*\,\mathrm{ev}^*(\alpha_1\otimes\ldots\otimes\alpha_n)}{e(g^*N)}, \end{aligned} \tag{11}$$

at least if the α_i are homogeneous and $\sum_{i=1}^n \deg\alpha_i=\deg[\overline{M}_{g,n}(\mathbb{P}^r,d)]$.

To apply this formula, we need to construct a cartesian T-equivariant diagram such as (7):

$$\begin{tikzcd} Y \arrow[r, "\iota"] \arrow[d, "g"'] & \overline{M}_{g,n}(\mathbb{P}^r,d) \arrow[d] \\ V \arrow[r, "\nu"] & W \end{tikzcd} \tag{12}$$

where ν is a regular closed immersion and Y contains all the fixed points of T in $\overline{M}_{g,n}(\mathbb{P}^r,d)$. Of course, we get the best results if we take Y as small as possible, namely equal to the fixed locus of T on $\overline{M}_{g,n}(\mathbb{P}^r,d)$. The point of view of more general Y is still useful, because it lets us decompose the problem into several steps. We pass successively to smaller Y until we reach the fixed locus. The regular immersions $\nu : V \to W$ will be chosen at each step in such a way that we can keep track of $\nu^![\overline{M}_{g,n}(\mathbb{P}^r, d)_T]$, i.e., we can follow what happens to the virtual fundamental class.

As we shall see, the fixed locus can be described in terms of stacks of stable curves $\overline{M}_{g,n}$. Thus Formula (11) reduces the computation of Gromov-Witten invariants to a computation on various $\overline{M}_{g,n}$. Since the fixed locus has many components, the combinatorics turn out to be non-trivial. Moreover, the integrals one has to evaluate on $\overline{M}_{g,n}$ are non-trivial, too. Still, this approach has been very successful in determining Gromov-Witten invariants. (See [11], [13], [12] or [7], [8] for more details.)

We shall next determine the fixed locus. The connected components of the fixed locus are indexed by marked modular graphs (τ, d, γ). Thus the right hand side of (11) is a sum over all marked modular graphs (τ, d, γ) involved. We can treat the fixed components given by different marked graphs separately, i.e., we determine for each (τ, d, γ) the classes $\nu^![\overline{M}_{g,n}(\mathbb{P}^r, d)_T]$ and $\frac{1}{e(g^*N)}$ restricted to the fixed locus component given by (τ, d, γ). Then we have

$$\begin{aligned} &\int_{[\overline{M}_{g,n}(\mathbb{P}^r,d)]} \mathrm{ev}^*(a_1 \otimes \ldots \otimes a_n) \qquad (13) \\ &= \theta \sum_{(\tau,d,\gamma)} {}^T\!\!\int_{\nu^![\overline{M}_{g,n}(\mathbb{P}^r,d)_T]_{(\tau,d,\gamma)}} \frac{\iota^* \mathrm{ev}^*(\alpha_1 \otimes \ldots \otimes \alpha_n)}{e(g^*N)_{(\tau,d,\gamma)}}. \end{aligned}$$

The fixed locus

Recall that *modular graphs* are the graphs that give the degeneracy type of prestable marked curves. They consist of a set of vertices V_τ, a set of flags F_τ (which can either be tails or pair up to edges), and non-negative integer markings of the vertices, giving the vertices a genus. Tails are denoted S_τ and edges E_τ. The set of flags connected with the vertex v is denoted $F_\tau(v)$. A vertex is *stable* if its genus is at least 2, its genus is one and its valence (the number of flags it bounds) is at least 1 or its genus is 0 and its valence at least 3. The stabilization τ^s of a modular graph is obtained by contracting all edges containing unstable vertices. The set of vertices of the stabilization is equal to the set of stable vertices. For details, see [4].

Let (τ, d, γ) be a marked modular graph of the following type.

1. τ: a modular graph which is connected and whose stabilization τ^s is not empty. Moreover, the genus $g(\tau)$ is equal to g and the set of tails S_τ is $S_\tau = \{1, \ldots, n\}$.

2. $d : V_\tau \to \mathbb{Z}_{\geq 0}$ a marking of the vertices by 'degrees', such that

   (a) $d(v) = 0$, for every stable vertex $v \in V_\tau^s$,

   (b) $\sum_{v \in V_\tau} d(v) = d$.

   Note that we use the same letter d for the marking of the graph τ and the total degree of the graph.

3. γ consists of three maps:

   (a) $\gamma : V_\tau^s \to \{P_0, \dots, P_r\}$, where $P_i = \langle 0, \dots, 1, \dots, 0\rangle$, the 1 being in the i-th position; so γ associates to every stable vertex of τ a fixed point of T on $\mathbb{P}^r$,

   (b) $\gamma : V_\tau^u \to \{L_{ij} \mid 0 \leq i < j \leq r\}$, where $L_{ij} = \langle 0, \dots, x, \dots, y, \dots, 0\rangle$ and x is in the i-th, y in the j-th position; so γ associates to every unstable vertex a one-dimensional orbit closure,

   (c) $\gamma : F_\tau \to \{P_0, \dots, P_r\}$; so γ associates to every flag a fixed point.

These data are subject to the following list of compatibility requirements:

1. Every edge has an unstable vertex, i.e., no edge connects stable vertices,

2. γ is constant on edges,

3. if v is a stable vertex then $\gamma(v) = \gamma(i)$, for all $i \in F_\tau(v)$,

4. if v is an unstable vertex then

   (a) $\gamma(i) \in \gamma(v)$, for all $i \in F_\tau(v)$,

   (b) all $\gamma(i)$, for $i \in F_\tau(v)$ are distinct.

Fix such a marked modular graph (τ, d, γ). The following stacks will be important in what follows:

1. $\overline{M}(\tau^s) = \prod_{v \in V_\tau^s} \overline{M}_{g(v), F_\tau(v)}$,

2. $\overline{M}(\mathbb{P}^r, \tau, d)$, which is defined as the fibered product

$$\begin{tikzcd} \overline{M}(\mathbb{P}^r, \tau, d) \arrow[r] \arrow[d] & \prod_{v \in V_\tau} \overline{M}_{g(v), F_\tau(v)}(\mathbb{P}^r, d(v)) \arrow[d] \\ (\mathbb{P}^r)^{E_\tau} \arrow[r, "\Delta"] & (\mathbb{P}^r \times \mathbb{P}^r)^{E_\tau}, \end{tikzcd} \tag{14}$$

   where the vertical maps are evaluation maps,

3. $\overline{M}(\mathbb{P}^r, \tau, d; \gamma)$, which is the substack of $\overline{M}(\mathbb{P}^r, \tau, d)$ defined by requiring that $f_{\partial(i)}(x_i) = \gamma(i) \in \{P_0, \dots, P_r\}$, for all $i \in F_\tau$. Here $\partial(i)$ is the vertex incident with i, $f_{\partial(i)}$ is the stable map indexed by this vertex and x_i is the mark of the source curve of $f_{\partial(i)}$ indexed by i. Clearly, $\overline{M}(\mathbb{P}^r, \tau, d; \gamma)$ is a closed substack of $\overline{M}(\mathbb{P}^r, \tau, d)$.

Stacks of type $\overline{M}(\mathbb{P}^r, \tau, d)$ are studied in great detail in [4]. Given a collection $(C_v, x_i, f_v)_{v \in V_\tau, i \in F_\tau}$, representing an element of $\overline{M}(\mathbb{P}^r, \tau, d)$, we can associate a stable map in $\overline{M}_{g,n}(\mathbb{P}^r, d)$ by gluing, for every edge $\{i_1, i_2\}$ of τ, the curves $C_{\partial(i_1)}$ and $C_{\partial(i_2)}$ by identifying x_{i_1} with x_{i_2}. Doing this in families defines the morphism

$$\overline{M}(\mathbb{P}^r, \tau, d) \longrightarrow \overline{M}_{g,n}(\mathbb{P}^r, d). \tag{15}$$

In general, a morphism such as (15), giving rise to a boundary component of $\overline{M}_{g,n}(\mathbb{P}^r, d)$ is only a finite morphism. But because of the special nature of (τ, d) in our context, (15) is actually a finite étale morphism followed by a closed immersion. More precisely:

Proposition 27 *Let* $\mathrm{Aut}(\tau, d)$ *be the subgroup of the automorphism group of the modular graph* τ *preserving the degrees* d*. Then* $\mathrm{Aut}(\tau, d)$ *acts on* $\overline{M}(\mathbb{P}^r, \tau, d)$ *and (15) induces a closed immersion*

$$\overline{M}(\mathbb{P}^r, \tau, d)/Aut(\tau, d) \longrightarrow \overline{M}_{g,n}(\mathbb{P}^r, d).$$

Proof One has to prove that any stable map in $\overline{M}_{g,n}(\mathbb{P}^r, d)$ of degeneracy type (τ, d) or worse, can be written uniquely (up to $\mathrm{Aut}(\tau, d)$) as the result of gluing a collection $(C_v, x_i, f_v)_{v \in V_\tau, i \in F_\tau}$. This is true because every stable vertex has degree 0 and no edge connects stable vertices. □

Next, we shall construct a morphism $\overline{M}(\tau^s) \to \overline{M}(\mathbb{P}^r, \tau, d)$. Let $(C_v, x_v)_{v \in V_\tau^s}$ be a collection of stable marked curves, $x_v = (x_i)_{i \in F_\tau(v)}$, in other words, a k-valued point of $\overline{M}(\tau^s)$. Then produce a collection of stable maps as follows:

1. for $v \in V_\tau$ a stable vertex, let $f_v : C_v \to \mathbb{P}^r$ by the constant map to $\gamma(v) \in \{P_0, \dots, P_r\}$,

2. for $v \in V_\tau$ unstable, let $C_v = \mathbb{P}^1$ and f_v be

$$\begin{aligned} f_v : \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 = \gamma(v) \subset \mathbb{P}^r \\ z &\longmapsto z^{d(v)}. \end{aligned}$$

   Then put marks on $C_v = \mathbb{P}^1$: for each $i \in F_\tau(v)$ let $x_i \in C_v$ be equal to $0 = \langle 1, 0 \rangle$ or $\infty = \langle 0, 1 \rangle$, in the unique way such that $f_v(x_i) = \gamma(i)$.

This defines $(C_v, x_v, f_v)_{v\in V_\tau}$, an element of $\overline{M}(\mathbb{P}^r, \tau, d)(k)$. Again, this can be done in families and we obtain the desired morphism $\overline{M}(\tau^s) \to \overline{M}(\mathbb{P}^r, \tau, d)$.

This morphism is also a finite étale covering followed by a closed immersion:

Proposition 28 *Let $\mu = \prod_{v\in V_\tau^u} \mu_{d(v)}$, where $\mu_{d(v)}$ is the cyclic group of $d(v)$-th roots of 1. Let μ act trivially on $\overline{M}(\tau^s)$. Then we have a closed immersion*

$$\overline{M}(\tau^s)/\mu \longrightarrow \overline{M}(\mathbb{P}^r, \tau, d). \tag{16}$$

$\square$

We can say more, because, in fact, the group $\operatorname{Aut}(\tau, d)$ acts on the morphism (16). More precisely,

Proposition 29 *The semidirect product $G = \mu \rtimes \operatorname{Aut}(\tau, d)$ acts on $\overline{M}(\tau^s)$ and (16) induces a closed immersion*

$$\overline{M}(\tau^s)/G \longrightarrow \overline{M}(\mathbb{P}^r, \tau, d)/\operatorname{Aut}(\tau, d).$$

$\square$

Putting Propositions 27 and 29 together, we obtain the composition

$$\begin{tikzcd} \overline{M}(\tau^s)/G \arrow[r] \arrow[rr, bend left, "\Phi_{(\tau,d,\gamma)}"] & \overline{M}(\mathbb{P}^r, \tau, d)/\operatorname{Aut}(\tau, d) \arrow[r] & \overline{M}_{g,n}(\mathbb{P}^r, d), \end{tikzcd} \tag{17}$$

which is a closed immersion.

Proposition 30 *Consider the group $T(k)$ acting on the set of isomorphism classes of $\overline{M}_{g,n}(\mathbb{P}^r, d)(k)$. An element of this set is fixed if and only if it is in the image of $\Phi_{(\tau,d,\gamma)}(k)$, for some marked modular graph (τ, d, γ) as described above.* $\square$

In this sense, the image of $\coprod \Phi$ is the fixed locus of $\overline{M}_{g,n}(\mathbb{P}^r, d)$. Thus we are justified in calling the image of $\Phi_{(\tau,d,\gamma)}$ the *fixed component* indexed by (τ, d, γ). But if we endow $\overline{M}(\tau^s)/G$ with the trivial action of T, then $\Phi_{(\tau,d,\gamma)}$ is not T-equivariant. To make it so, we have to pass to a larger torus.

Consider the character group $M \subset M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q}$ and let $\widetilde{M} = M + \sum_{v\in V_\tau^u} \frac{1}{d(v)}\lambda_v \subset M_\mathbb{Q}$, where λ_v is the character of T through which T acts on $\mathbb{P}^1 = L_{ij} = \gamma(v)$. Let $\widetilde{T}$ be the torus with character group $\widetilde{M}$. We have a finite homomorphism $\widetilde{T} \to T$. We can view passing from T to $\widetilde{T}$ as a way to make the character λ_v divisible by $d(v)$.

The torus $\widetilde{T}$ acts on $\overline{M}_{g,n}(\mathbb{P}^r, d)$ through $\widetilde{T} \to T$. We can now construct a 2-isomorphism θ in the diagram

$$\begin{tikzcd}
\widetilde{T} \times \overline{M}(\tau^s) \arrow[r, "\mathrm{id} \times \Phi"] \arrow[d, "\mathrm{proj}"'] \arrow[dr, phantom, "\Downarrow\theta"] & \widetilde{T} \times \overline{M}_{g,n}(\mathbb{P}^r, d) \arrow[d, "\mathrm{action}"] \\
\overline{M}(\tau^s) \arrow[r, "\Phi"] & \overline{M}_{g,n}(\mathbb{P}^r, d).
\end{tikzcd}$$

Let us describe θ on k-valued points. We need to define a natural transformation. So for each $(t, (C_v, x_v))$ of $\widetilde{T}(k) \times \overline{M}(\tau^s)(k)$ we need to define a morphism $\theta : t \cdot \Phi(C_v, x_v) \to \Phi(C_v, x_v)$. Using notation as above, we have $\Phi((C_v, x_v)_{v \in V_\tau^s}) = (C_v, x_v, f_v)_{v \in V_\tau}$ and $t \cdot \phi(C_v, x_v) = (C_v, x_v, t \circ f_v)$. Then

1. for $v \in V_\tau^s$, we let $\theta_v : C_v \longrightarrow C_v$ be the identity,
2. for $v \in V_\tau^u$, we let $\theta_v : C_v = \mathbb{P}^1 \longrightarrow C_v = \mathbb{P}^1$ be given by

$$z \longmapsto \frac{\lambda_v}{d(v)}(t) z,$$

which fits into the commutative diagram

$$\begin{tikzcd}
C_v \arrow[r, "f_v = (\,\cdot\,)^{d(v)}"] \arrow[d, "\theta_v"'] & \gamma(v) \arrow[r, "\subset"] \arrow[d, "\lambda_v"] & \mathbb{P}^r \arrow[d, "t"] \\
C_v \arrow[r, "f_v = (\,\cdot\,)^{d(v)}"] & \gamma(v) \arrow[r, "\subset"] & \mathbb{P}^r.
\end{tikzcd}$$

Thus it is better to think of the image of $\Phi_{(\tau,d,\gamma)}$ as a fixed component of $\widetilde{T}$, rather than T, acting on $\overline{M}_{g,n}(\mathbb{P}^r, d)$.

Going back to Diagram (12), we can now say what Y is. We shall use $Y = \coprod_{(\tau,d,\gamma)} \overline{M}(\tau^s)/G_{(\tau,d)}$. The integrals on $\overline{M}(\tau^s)/G_{(\tau,d)}$ will be evaluated on $\overline{M}(\tau^s)$. This leads to the correction factor

$$\frac{1}{\# G_{(\tau,d)}} = \frac{1}{\# \mathrm{Aut}(\tau, d)} \prod_{v \in V_\tau^u} \frac{1}{d(v)}.$$

We shall next show how to obtain regular immersions $\nu : V \to W$ as in Diagram (12). As mentioned above, we can treat each fixed component separately. We will proceed in several steps, corresponding to the following factorization of Φ:

$$\overline{M}(\tau^s)/G \xrightarrow{\text{III.}} \overline{M}(\mathbb{P}^r,\tau,d;\gamma)/A \xrightarrow{\text{II.}} \overline{M}(\mathbb{P}^r,\tau,d)/A \xrightarrow{\text{I.}} \overline{M}_{g,n}(\mathbb{P}^r,d),$$

where $A = \operatorname{Aut}(\tau,d)$. For each step we shall construct a suitable ν and then determine $\nu^![\overline{M}_{g,n}(\mathbb{P}^r,d)_T]$ and $\frac{1}{e(g^*N)}$.

The first step

We use the following diagram for (12):

$$\begin{tikzcd}
\overline{M}(\mathbb{P}^r,\tau,d)/\operatorname{Aut}(\tau,d) \arrow[r] \arrow[d, "g"'] & \overline{M}_{g,n}(\mathbb{P}^r,d) \arrow[d] \\
\mathfrak{M}(\tau)/\operatorname{Aut}(\tau,d) \arrow[r, "\nu"] & \mathfrak{M}_{g,n}
\end{tikzcd}$$

We note that this diagram is not cartesian, but $\overline{M}(\mathbb{P}^r,\tau,d)/\operatorname{Aut}(\tau,d)$ is open and closed in the cartesian product. Since we are only interested in the (τ,d,γ)-component of the fixed locus at the moment, this is sufficient. Here $\mathfrak{M}_{g,n}$ stands for the (highly non-separated) Artin stack of prestable curves of genus g with n marks. Moreover,

$$\mathfrak{M}(\tau) = \prod_{v\in V_\tau} \mathfrak{M}_{g(v),F_\tau(v)}$$

and the morphism $\mathfrak{M}(\tau) \to \mathfrak{M}_{g,n}$ is given by gluing according to the edges of τ. The vertical maps are given by forgetting the map, retaining the prestable curve, without stabilizing. The diagram is $\widetilde{T}$-equivariant, if we endow $\mathfrak{M}(\tau)$ and $\mathfrak{M}_{g,n}$ with the trivial $\widetilde{T}$-action. We also note that ν is not a closed immersion, but certainly a regular local immersion (for this terminology see [16]), which is sufficient for our purposes.

It is a general fact about virtual fundamental classes, used in the proof of the WDVV-equation, that the Gysin pullback along ν preserves virtual fundamental classes:

$$\nu^![\overline{M}_{g,n}(\mathbb{P}^r,d)_T] = [\overline{M}(\mathbb{P}^r,\tau,d)/Aut(\tau,d)].$$

(One way to define the virtual fundamental class of $\overline{M}(\mathbb{P}^r,\tau,d)$ is to set it equal to the Gysin pullback via Δ of the product of virtual fundamental classes in Diagram 14.)

The normal bundle of $\mathfrak{M}(\tau)$ in $\mathfrak{M}_{g,n}$ splits into a direct sum of line bundles, one summand for each edge of τ. For the edge $\{i_1, i_2\}$, the normal line bundle is

$$x_{i_1}^*(\omega^\vee) \otimes x_{i_2}^*(\omega^\vee),$$

where x_{i_1} and x_{i_2} are the sections of the universal curves corresponding to the flags i_1 and i_2 of τ and ω is the relative dualizing sheaf of the universal curve, whose dual, $\omega^\vee$ is the relative tangent bundle. We use notation c_i for the Chern class of the line bundle $x_i^*(\omega)$ on $\mathfrak{M}(\tau)$. Then

$$\frac{1}{e(g^*N)} = \prod_{\{i_1,i_2\}\in E_\tau} \frac{1}{-c_{i_1} - c_{i_2}}. \tag{18}$$

The second step

Instead of considering $\overline{M}(\mathbb{P}^r, \tau, d; \gamma)/\operatorname{Aut}(\tau, d) \to \overline{M}(\mathbb{P}^r, \tau, d)/\operatorname{Aut}(\tau, d)$, we shall consider

$$\overline{M}(\mathbb{P}^r, \tau, d; \gamma) \longrightarrow \overline{M}(\mathbb{P}^r, \tau, d). \tag{19}$$

We call an edge (flag, tail) of τ *stable*, if it meets a stable vertex. Otherwise, we call it unstable. We shall need to consider stacks of the following type:

$$\overline{M}_{0,S}(\mathbb{P}^r, d; \gamma(S)),$$

where S is a finite set (we only consider the cases that S has 1 or 2 elements) and $\gamma : S \to \{P_0, \ldots, P_n\}$ is a map. The stack $\overline{M}_{0,S}(\mathbb{P}^r, d; \gamma(S)) \subset \overline{M}_{0,S}(\mathbb{P}^r, d)$ is the closed substack of stable maps f, defined by requiring that $f(x_i) = \gamma(i)$, for all $i \in S$.

Lemma 31 *For $\#S \leq 2$, the stack $\overline{M}_{0,S}(\mathbb{P}^r, d; \gamma(S))$ is smooth of the expected dimension $\dim \overline{M}_{0,S}(\mathbb{P}^r, d) - r\#S$.*

Proof This follows from $H^1(C, f^*T_{\mathbb{P}^r}(-x_1 - x_2)) = 0$, for a stable map $f : C \to \mathbb{P}^r$ in $\overline{M}_{0,S}(\mathbb{P}^r, d; \gamma(S))$. □

Note that we have

$$\overline{M}(\mathbb{P}^r, \tau, d; \gamma) = \prod_{v \in V_\tau^s} \overline{M}_{g(v),F_\tau(v)} \times \prod_{v \in V_\tau^u} \overline{M}_{0,F_\tau(v)}(\mathbb{P}^r, d(v); \gamma F_\tau(v)),$$

and in particular, that $\overline{M}(\mathbb{P}^r, \tau, d; \gamma)$ is smooth of the 'expected' dimension

$$\sum_{v \in V_\tau^s} \dim \overline{M}_{g(v),F_\tau(v)} + \sum_{v \in V_\tau^u} \dim \overline{M}_{g(v),F_\tau(v)}(\mathbb{P}^r, d(v)) - r\#F_\tau^u.$$

Now the morphism (19) fits into the $\widetilde{T}$-equivariant cartesian diagram

$$\begin{tikzcd}
\overline{M}(\mathbb{P}^r,\tau,d;\gamma) \arrow[r] \arrow[d, "g"] \arrow[dd, bend right, "g_1"'] & \overline{M}(\mathbb{P}^r,\tau,d) \arrow[r] \arrow[d] & \prod_{v\in V_\tau}\overline{M}_{g(v),F_\tau(v)}(\mathbb{P}^r,d(v)) \arrow[d, "e\times p"] \\
(\mathbb{P}^r)^{F^s_\tau} \arrow[r, "\nu"] \arrow[d] & (\mathbb{P}^r)^{E_\tau}\times(\mathbb{P}^r)^{S_\tau}\times(\mathbb{P}^r)^{V^s_\tau} \arrow[r, "\Delta\times\mathrm{id}"] & (\mathbb{P}^r)^{F_\tau}\times(\mathbb{P}^r)^{V^s_\tau} \arrow[d, "q"] \\
pt \arrow[rr, "\nu_1"] & & (\mathbb{P}^r)^{F^u_\tau}\times(\mathbb{P}^r)^{V^s_\tau}.
\end{tikzcd} \tag{20}$$

The morphism $e\times p$ is the product of the evaluation morphism

$$e:\prod_{v\in V_\tau}\overline{M}_{g(v),F_\tau(v)}(\mathbb{P}^r,d(v))\longrightarrow(\mathbb{P}^r)^{F_\tau}$$

and the projection

$$\prod_{v\in V_\tau}\overline{M}_{g(v),F_\tau(v)}(\mathbb{P}^r,d(v)) = \\ \prod_{v\in V^u_\tau}\overline{M}_{0,F_\tau(v)}(\mathbb{P}^r,d(v))\times\prod_{v\in V^s_\tau}\left(\overline{M}_{g(v),F_\tau(v)}\times\mathbb{P}^r\right)\stackrel{p}{\longrightarrow}(\mathbb{P}^r)^{V^s_\tau}.$$

The morphism $\Delta\times\mathrm{id}$ is the product of the diagonal

$$(\mathbb{P}^r)^{E_\tau}\stackrel{\Delta}{\longrightarrow}(\mathbb{P}^r\times\mathbb{P}^r)^{E_\tau}=(\mathbb{P}^r)^{F_\tau-S_\tau}$$

and the identity on $(\mathbb{P}^r)^{S_\tau}\times(\mathbb{P}^r)^{V^s_\tau}$. The square to the upper right of (20) is just a base change of the defining square of $\overline{M}(\mathbb{P}^r,\tau,d)$. The morphism ν is the product of the identity

$$(\mathbb{P}^r)^{F^s_\tau}\longrightarrow(\mathbb{P}^r)^{E^s_\tau}\times(\mathbb{P}^r)^{S^s_\tau}$$

and the morphism

$$\gamma:pt\longrightarrow(\mathbb{P}^r)^{E^u_\tau}\times(\mathbb{P}^r)^{S^u_\tau}\times(\mathbb{P}^r)^{V^s_\tau},$$

induced by the marking γ on the graph (τ,d). The morphism g is given by evaluation at the points corresponding to stable flags and is, in fact, constant. The morphism q projects out the factors corresponding to stable flags. Finally, ν_1 is given, again, by γ.

The stack in the upper right corner of (20) is smooth, but not of the 'expected' dimension. It has a virtual fundamental class given by

$$\left(\prod_{v\in V^s_\tau} e\big(H(v)^\vee \boxtimes T_{\mathbb{P}^r}(\gamma(v))\big)\right)[\prod_{v\in V_\tau} \overline{M}_{g(v),F_\tau(v)}(\mathbb{P}^r, d(v))_T]. \qquad (21)$$

Here $H(v)$ is the 'Hodge bundle' corresponding to the vertex v. If $\pi_v : C_v \to \overline{M}_{g(v),F_\tau(v)}$ is the universal curve, then $H(v) = \pi_{v*}(\omega_{C_v})$, where ω_{C_v} is the relative dualizing sheaf.

It is part of the general compatibilities of virtual fundamental classes that (21) pulled back via $(\Delta \times \mathrm{id})^!$ gives the virtual fundamental class of $\overline{M}(\mathbb{P}^r, \tau, d)$. Now because there is no excess intersection in the lower rectangle of (20), we get the same class in $\overline{M}(\mathbb{P}^r, \tau, d; \gamma)$ by pulling back (21) in two steps via $(\Delta \times \mathrm{id})^!$ and $\nu^!$ or in one step via $\nu_1^!$. Thus

$$\nu^![\overline{M}(\mathbb{P}^r, \tau, d)_T] = \nu_1^!(\text{the class (21)}).$$

But by Lemma 31, the big (total) square in (20) has no excess intersection either. Thus $\nu_1^!(\text{the class (21)})$ is equal to

$$\begin{aligned}
&\nu^![\overline{M}(\mathbb{P}^r, \tau, d)_T] \\
&= \left(\prod_{v\in V^s_\tau} e\big(H(v)^\vee \boxtimes T_{\mathbb{P}^r}(\gamma(v))\big)\right)[\overline{M}(\mathbb{P}^r, \tau, d; \gamma)_T] \\
&= \prod_{v\in V^s_\tau}\prod_{i\neq\gamma(v)} (\lambda_i - \lambda_{\gamma(v)})^{g(v)}\, c_t(H(v))|_{t=\frac{1}{\lambda_i-\lambda_{\gamma(v)}}}\, [\overline{M}(\mathbb{P}^r, \tau, d; \gamma)_T].
\end{aligned}$$

Because the morphism g in (20) is constant, $g^*(N)$ is constant and so $e(g^*(N))$ is just the product of the weights of T on g^*N. Thus

$$\begin{aligned}
&\frac{1}{e(g^*N)} \qquad (22)\\
&= \prod_{j\in E^u_\tau} \frac{1}{e\big(T_{\mathbb{P}^r}(\gamma(j))\big)} \prod_{j\in S^u_\tau} \frac{1}{e\big(T_{\mathbb{P}^r}(\gamma(j))\big)} \prod_{j\in V^s_\tau} \frac{1}{e\big(T_{\mathbb{P}^r}(\gamma(v))\big)} \\
&= \prod_{j\in E^u_\tau\cup S^u_\tau}\prod_{i\neq\gamma(j)} \frac{1}{\lambda_i-\lambda_{\gamma(j)}} \prod_{v\in V^s_\tau}\prod_{i\neq\gamma(v)} \frac{1}{\lambda_i-\lambda_{\gamma(v)}}
\end{aligned}$$

The third step

We shall consider the morphism

$$\overline{M}(\tau^s)/\mu \longrightarrow \overline{M}(\mathbb{P}^r, \tau, d; \gamma) = \overline{M}(\tau^s) \times \prod_{v\in V^u_\tau} \overline{M}_{0,F_\tau(v)}(\mathbb{P}^r, d(v); \gamma F_\tau(v)),$$

which we may insert into the $\widetilde{T}$-equivariant cartesian diagram of smooth stacks without excess intersection

$$
\begin{tikzcd}
\overline{M}(\tau^s)/\mu \arrow[r] \arrow[d, "g"'] & \overline{M}(\tau^s) \times \prod_{v \in V_\tau^u} \overline{M}_{0,F_\tau(v)}(\mathbb{P}^r, d(v); \gamma F_\tau(v)) \arrow[d] \\
\prod_{v \in V_\tau^u} B\mu_{d(v)} \arrow[r, "\nu"] & \prod_{v \in V_\tau^u} \overline{M}_{0,F_\tau(v)}(\mathbb{P}^r, d(v); \gamma F_\tau(v)).
\end{tikzcd}
\tag{23}
$$

It follows that $\nu^![\overline{M}(\mathbb{P}^r, \tau, d; \gamma)_T] = [\overline{M}(\tau^s)/\mu]$.

To calculate the normal bundle of ν, factor ν into $\#V_\tau^u$ morphisms and thus reduce to considering the morphism

$$B\mu_{d(v)} \longrightarrow \overline{M}_{0,F_\tau(v)}(\mathbb{P}^r, d(v); \gamma F_\tau(v)).$$

To fix notation, let us consider a positive integer d and

$$B\mu_d \longrightarrow \overline{M}_{0,2}(\mathbb{P}^r, d; P_0, P_1) \tag{24}$$

(the case of v having valence 1 we leave to the reader). The stack

$$\overline{M}_{0,2}(\mathbb{P}^r, d; P_0, P_1) \subset \overline{M}_{0,2}(\mathbb{P}^r, d)$$

is defined by requiring the image of the first marked point to be $P_0 \in \mathbb{P}^r$ and the image of the second marked point to be $P_1 \in \mathbb{P}^r$.

The particular stable map

$$
\begin{array}{rcl}
f : \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 = L_{01} \subset \mathbb{P}^r \\
z & \longmapsto & z^d
\end{array}
\tag{25}
$$

(where $x_1 = 0$ and $x_2 = \infty$ are the marks on $\mathbb{P}^1$) is the unique fixed point of $\widetilde{T}$ on $\overline{M}_{0,2}(\mathbb{P}^r, d; P_0, P_1)$ and gives rise to the morphism (24).The normal bundle to (24) is the tangent space to (25) in $\overline{M}_{0,2}(\mathbb{P}^r, d; P_0, P_1)$ and hence equal to

$$H^0(\mathbb{P}^1, f^*T_{\mathbb{P}^r}(-0-\infty)) \;/\; H^0(\mathbb{P}^1, T_{\mathbb{P}^1}(-0-\infty)). \tag{26}$$

We calculate the weights of $H^0(\mathbb{P}^1, f^*T_{\mathbb{P}^r}(-0-\infty))$ and $H^0(\mathbb{P}^1, T_{\mathbb{P}^1}(-0-\infty))$ using Example 26.

Let (α_i) denote the weights of $\widetilde{T}$ on $H^0(\mathbb{P}^1, f^*T_{\mathbb{P}^r}(-0-\infty))$. The torus $\widetilde{T}$ acts on $\mathbb{P}^1$ via the character $\omega = \frac{\lambda_1-\lambda_0}{d}$. We also need the weights of $f^*T_{\mathbb{P}^r}(-0-\infty)(0)$ and $f^*T_{\mathbb{P}^r}(-0-\infty)(\infty)$. To calculate these, note that $T_{\mathbb{P}^r}(P_0)$ has weights $(\lambda_i-\lambda_0)_{i\neq 0}$ and $T_{\mathbb{P}^r}(P_1)$ has weights $(\lambda_i-\lambda_1)_{i\neq 1}$. The same holds after applying f^*. Twisting by (-0) and $(-\infty)$ changes the weights by $T_{\mathbb{P}^1}(0)$ and $T_{\mathbb{P}^1}(\infty)$, respectively. But $T_{\mathbb{P}^1}(0)$ has weight $\frac{\lambda_1-\lambda_0}{d}$ and $T_{\mathbb{P}^1}(\infty)$ has weight $\frac{\lambda_0-\lambda_1}{d}$. Thus the weights of $f^*T_{\mathbb{P}^r}(-0-\infty)$ are $(\lambda_i-\lambda_0-\omega)_{i\neq 0}$ at (0) and $(\lambda_i-\lambda_1+\omega)_{i\neq 1}$ at (∞). Then by Example 26 we have

$$\begin{aligned}
\sum e^{\alpha_i} &= \frac{1}{1-e^{-\omega}}\sum_{i\neq 0} e^{\lambda_i-\lambda_0-\omega} + \frac{1}{1-e^{\omega}}\sum_{i\neq 1} e^{\lambda_i-\lambda_1+\omega} \\
&= 1 + \sum_{\text{all } i} e^{\lambda_i-\lambda_0}\left(\frac{e^{-\omega}}{1-e^{-\omega}} + \frac{e^{\omega+\lambda_0-\lambda_1}}{1-e^{\omega}}\right) \\
&= 1 + \sum_{i} e^{\lambda_i-\lambda_0}\left(\frac{e^{(1-d)\omega}}{1-e^{\omega}} + \frac{e^{-\omega}}{1-e^{-\omega}}\right) \\
&= 1 + \sum_{i} e^{\lambda_i-\lambda_0}\sum_{n=1}^{d-1} e^{-n\omega} \\
&= 1 + \sum_{i=0}^{r}\sum_{\substack{n+m=d\\ n,m\neq 0}} e^{\lambda_i-\frac{n}{d}\lambda_1-\frac{m}{d}\lambda_0},
\end{aligned}$$

by the ‘useful formula’ mentioned in Exercise 26.

Similarly, $H^0(\mathbb{P}^1, T_{\mathbb{P}^1}(-0-\infty))$ is one-dimensional and has weight 0, so that the weights of (26) are

$$\left(\lambda_i - \frac{n}{d}\lambda_1 - \frac{m}{d}\lambda_0\right)_{\substack{i=0,\dots,r\\ n+m=d\\ n,m>0}}.$$

We deduce that for the normal bundle N of the morphism ν in (23) we have

$$\frac{1}{e(g^*N)} = \tag{27}$$

$$\prod_{\substack{v\in V_\tau^u\\ |v|=2\\ \gamma(v)=L_{ab}}} \prod_{i=0}^{r} \prod_{\substack{n+m=d(v)\\ n,m\neq 0}} \frac{1}{\lambda_i - \frac{n}{d}\lambda_a - \frac{m}{d}\lambda_b}$$

$$\prod_{\substack{v\in V_\tau^u\\ |v|=1\\ \gamma(v)=L_{ab}}} \left(\prod_{\substack{n+m=d(v)\\ n,m\neq 0}} \frac{1}{\lambda_a - \frac{n}{d}\lambda_a - \frac{m}{d}\lambda_b} \prod_{\substack{n+m=d(v)\\ n\neq 0,1}} \frac{1}{\lambda_b - \frac{n}{d}\lambda_a - \frac{m}{d}\lambda_b} \right.$$

$$\left. \prod_{\substack{i=0\\ i\neq a,b}}^{r} \prod_{\substack{n+m=d(v)\\ n\neq 0}} \frac{1}{\lambda_i - \frac{n}{d}\lambda_a - \frac{m}{d}\lambda_b} \right).$$

Conclusion

We have now completed the computation of the right hand side of (13). We have

$$\begin{aligned} &\nu^![\overline{M}_{g,n}(\mathbb{P}^r, d)_T]_{(\tau,d,\gamma)} \\ &= \prod_{v\in V_\tau^s} \prod_{i\neq\gamma(v)} (\lambda_i - \lambda_{\gamma(v)})^{g(v)}\ c_t(H(v))|_{t=\frac{1}{\lambda_i-\lambda_{\gamma(v)}}}\ [\overline{M}(\tau^s)/G_{(\tau,d)}] \end{aligned}$$

and $1/e(g^*N)_{(\tau,d,\gamma)}$ is the product of the three contributions (18), (22) and (27). When pulling back the contribution (18), which is

$$\prod_{\{i_1,i_2\}\in E_\tau} \frac{1}{-c_{i_1} - c_{i_2}},$$

to $\overline{M}(\tau^s)$, we replace $-c_i$, for an unstable flag $i \in F_\tau$ by the weight of $\widetilde{T}$ on $T_{\mathbb{P}^1}(x_i)$. This weight is $\frac{\lambda_j - \lambda_i}{d}$, where $\{i,j\}$ is the edge containing i.

Thus we finally arrive at the localization formula for Gromov-Witten invariants of $\mathbb{P}^r$. Our graph formalism is well-suited for our derivation of the formula. To actually perform calculations, it is more convenient to translate our formalism into the simpler graph formalism introduced by Kontsevich [13]. But this, of course, just amounts to a reindexing of our sum.

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